Properties of double field theory

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Properties of Double Field Theory

Van Swinderen Institute PhD series 2016

ISBN: 978-90-367-8949-3 (printed version)
ISBN: 978-90-367-8950-9 (electronic version)

The work described in this thesis was performed at the Van Swinderen Institute for Particle Physics and Gravity of the University of Groningen, the University of Buenos Aires and the Instituto de Astronomía y Física del Espacio (CONICET-UBA) and supported by the Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET) and an Ubbo Emmius sandwich scholarship from the University of Groningen.

Printed by Grafimedia-RUG

university of groningen

Properties of Double Field Theory

PhD thesis

to obtain the degree of PhD at the
University of Groningen
on the authority of the
Rector Magnificus Prof. E. Sterken
and in accordance with
the decision by the College of Deans.

This thesis will be defended in public on

Tuesday 21 June 2016 at 09:00 hours

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1
Introduction

Our current approach to understanding interactions between particles makes use of quantum field theories. The particles are generally assumed to be point-like and the interactions occur in events of space-time. It often happens that infinities appear in the calculations when computing scattering amplitudes. In a sense, this is due to the point-like property assumed. The most successful quantum field theory describing the dynamics of the elementary particles is known as the Standard Model. It is a gauge theory that describes the strong and electro-weak forces through the gauge group $SU(3) \times SU(2) \times U(1)$. Its degree of predictability has been tested to a high level of accuracy, at least up to the current 13 TeV energy scale provided by the LHC in proton collisions. However, from the theoretical point of view, the model suffers from several drawbacks. For instance, the Standard Model has several free parameters to be adjusted by experiments, and there is no natural explanation about the origin of the gauge structures. More importantly, it does not incorporate gravitational force.

On the other hand, General Relativity is the current theory that describes gravity. It is a classical theory based on the equivalence principle and the covariance principle under general coordinate transformations. It has been tested to incredible accuracy from an experimental point of view. These include solar-system experiments, pulsar-timing measurements, the deflection of light, the old problem of the perihelion advance of Mercury, and very recently the experimental confirmation of gravitational waves [1]. Despite the fact that General Relativity provides an elegant description of space-time and matter at macroscopic scales, it loses predictability at the microscopy level described by quantum (field) theory.

We know there are circumstances where the understanding of physical phenomena requires a consistent framework between gravity and the quantum theory. Common examples are the description of space-time and matter in short-time scales after the Big Bang singularity, or short-distances near singularities inside black holes. Thus, in order to have a complete and unified description of nature at every scale, a consistent theory of quantum gravity is required.

String theory appeared in the late 1960s as an attempt to describe the strong nuclear force. The basic objects of this model were one-dimensional (extended) objects in space-time, that is, relativistic strings. The idea was that the vibrational modes of these strings represent different (bosonic) hadrons. Back then, the theory of Quantum Chromodynamics was being developed and was soon recognized as the correct theory of strong interactions, setting aside the string models.

However, in the 1980s, it was realized that it makes more sense to think of string theory as a potential unified theory of quantum gravity rather than a theory of the strong force. This new insight was possible thanks to the identification of several remarkable properties:

- A massless spin two particle (graviton) appears in the spectrum of the theory.
- The discovery of an anomaly cancellation mechanism which allows for the gauge groups $SO(32)$ and $E_8 \times E_8$. These groups are big enough so as to contain $SU(3) \times SU(2) \times U(1)$ and for parity violation (required by the electro-weak force).
- String Theory requires supersymmetry in order to account for fermions and to eliminate the tachyonic modes.
- The consistency of (super) string theory requires 10 space-time dimensions.
- The theory's only input is the string tension T. The dimensionless string coupling g_s is determined through the expectation value of a scalar (dilaton) field.

The prediction of the space-time dimension is a remarkable feature of the theory. One way to deal with the extra-dimensions is to assume that they curl up to form a small compact space in order to avoid detection at low energies. This has revived the concept of Kaluza-Klein compactifications and generalizations of it. Another important feature is that interactions in string theory are not point-wise but occur in an extended region of space-time, thus, spreading out the usual point particle divergences that are inherent in perturbative quantum field theory calculations. Furthermore, the perturbation theory of string theory is UV finite: no need for renormalization, contrary to the usual non-renormalizability problems of General Relativity. The recognition of the strings as fundamental building blocks of nature requires them to be roughly the size of the Planck-length $\ell_P = \sqrt{\hbar G/c^3} \approx 10^{-33}$cm, which is the natural scale involving the fundamental constants of gravity and quantum field theory. At such a small scale, a direct detection of a string is not possible with our current detectors. The only possibility for experimental tests is through indirect observation. For example, if the standard model is to arise out of string theory, the masses for the known particles should arise from small symmetry breaking effects of the massless states of string theory [2].

So far, string theory has provided us with numerous insights about quantum gravity, relationships between gravity and gauge theories, the nature of space-time, and more. All these remarkable discoveries within string theory have been possible due to the existence of several dualities. Probably, the most famous one is the AdS/CFT correspondence [3]: a gravitational theory defined on an anti-de Sitter background is "dual" vis-à-vis a conformal field theory in one dimension less. Here, "dual" means that the two theories are meant to describe the same physics. This conjecture has boosted the non-perturbative understanding of string theory, since the usefulness of the correspondence relies on the fact that, when one of the theories is strongly coupled, the other (dual) theory is weakly coupled.

There is another important duality, which is the key concept of this thesis, and it is called *T-duality*. It has a prominent role in understanding the internal structure of string theory, and relating seemingly different backgrounds and regimes of validity. The simplest example is to consider a string propagating in $M \times S^1$, where M is Minkowski space-time in, say, $D = 9$ dimensions, and S^1 is a circle with radius R. We will see in Chapter (2) that the mass spectrum of the theory is invariant under the interchange of $R \to \alpha' \hbar^2 c^2/R$, provided that the momentum and winding numbers are also interchanged[1]. The winding number tells us how many times the string winds around the compact dimension. This invariance implies that the dynamics in a space-time with radius R is equivalent to the one with the inverse radius. In this sense, T-duality is teaching us that strings probe space-time in a very different way compared to point particles. This result is possible thanks to the extended nature of the string. If the compact space-time is a n-torus instead of a circle, the T-duality becomes enhanced to the action of the $O(n, n, \mathbb{Z})$ group. When one considers string (or supergravity) theory in a more general target space with isometric directions, a target symmetry that maps two (dual) backgrounds into each other arises. The relationship between

[1]The tension T of the string is sometimes written as $T = 1/(2\pi\alpha')$ ($\hbar = c = 1$), where α' is called the Regge-slope parameter for historical reasons.

the backgrounds has come to be known as "Buscher's rules" [4, 5].

String dualities reveal intriguing relationships among different theories. In total, there are five consistent superstring theories in $D = 10$; these are: Type IIA, type IIB, Heterotic $E_8 \times E_8$, Heterotic $SO(32)$, and Type I. The Type II and Heterotic theories are theories of closed strings, while Type I contains both closed and open strings. The Type II and Heterotic theories have a common bosonic subsector called the Neveu-Schwarz (NS) sector[2], which contains the metric, an antisymmetric 2-form known as Kalb-Ramond field, and a scalar known as dilaton. All these theories are related by dualities. While T-duality establishes the physical equivalence of theories defined on dual backgrounds with very different geometries, another duality called S-duality relates the strong and weak coupling limits of dual theories. It is known that T-duality relates type IIA with type IIB and relates both Heterotic theories. On the other hand, S-duality relates type IIB with itself under a weak-strong coupling inversion, and also relates type IIA with a new 11-dimensional theory called M-theory. It is conjectured that M-theory, whose classical limit is 11-dimensional supergravity, should be the full theory of strings and branes. Finally, U-duality (a combination of T and S-duality) has been conjectured to be a symmetry of the full M-theory.

Since most of these duality-symmetries do not appear in a manifest way, it is convenient to construct duality covariant models that give an effective description of the low-energy states of the string and their interactions. In some approaches, duality invariance is achieved through an enlargement of the coordinate space [6–9]. The idea of implementing T-duality as a manifest symmetry was first considered by M. Duff [10, 11] and A. Tseytlin [12, 13], and further developed by W. Siegel [14, 15]. Only recently, after the works by C. Hull, B. Zwiebach, and O. Hohm [16–19], has a field theory defined on a double space been built and named Double Field Theory (DFT). Prior geometric aspects of the "double geometry," intimately related to generalized geometry [20–23] were worked in [6, 7, 24–27]. More generally, other models that try to make the U-duality a manifest symmetry have been considered very recently; most of them enlarge the coordinate space even more [28–33], [8, 9, 34–42], [43].

Double Field Theory (DFT) is a field theory, which makes the T-duality group of string theory manifest. A T-duality symmetric field theory should incorporate, on an equal footing, winding and momentum variables. In toroidal compactifications of string theories, the compact momentum modes are dual to compact coordinates x^a, $a = 1, ..., n$. For the winding modes, a new set of coordinates $\tilde{x}_a$ (conjugate variables) must be taken into account and incorporated as variable

[2] It is actually called the NS-NS sector for reasons that will be explained in Chapter (2). For simplicity, we will just call it the NS-sector.

in DFT; thus DFT is defined in a doubled space, and hence its name. As mentioned before, when toroidal compactifications are involved, the T-duality group becomes $O(n,n,\mathbb{Z})$. Instead, the DFT we are going to consider has a global $O(D,D,\mathbb{R})$ symmetry and use coordinates $X^M = (\tilde{x}_i, x^i)$, $M = 1,\cdots,2D$ and $i = 0,\cdots,D-1$, which are not all necessarily compact. The fundamental fields of the theory are a symmetric $O(D,D)$[3] tensor $\mathcal{H}_{MN}$, called the "generalized metric" and a scalar field d, called the "generalized dilaton." The matrix $\mathcal{H}_{MN}$ encodes the usual space-time metric g_{ij} and the Kalb-Ramond field b_{ij}. When this theory is reduced to the usual x-space, that is, only coordinates x^i are allowed, it reduces to the common string sector known as the Neveu-Schwarz (NS) sector. Despite the beauty of incorporating x and $\tilde{x}$ coordinates in a symmetrical way, the DFT we are considering is a restricted theory. This means that the consistency of the theory is up to the so-called "strong constraint." This condition implies that the fields effectively depend on half of the coordinates [18]. Nevertheless, the theory still has a formal $O(D,D)$ symmetry without specifying any subset of the $(D+D)$-coordinates. The strong constraint is of the form:

$$\partial_i \tilde{\partial}^i (\cdots) = 0. \tag{1.1}$$

Here, the derivatives are with respect to $(\tilde{x}_i, x^i)$ and the dots represent any arbitrary product of fields and gauge parameters.

DFT is important in its own right in order to understand the geometry that is probed by the strings. But another interesting point of view is to use DFT to understand supergravity flux compactifications. The low-energy effective descriptions of string theories are provided by supergravity theories. Nevertheless, not every supergravity theory has a well-defined string origin, and, in general, they are gauged or massive-deformed supergravities. Gauged supergravities are characterized by constant parameters called "gaugings" that gauge some subgroup of the global symmetry group. These theories also possess a non-trivial potential for the scalars of the theories. There also exist massive-deformed supergravities, which have constant parameters (mass parameters) that do not come from any gauge procedure, for instance, $D = 10$ Roman's supergravity [44]. In lower dimensions, some of these gaugings can be understood as coming from fluxes [45–47] of the higher-dimensional string fields of low-energy description. This means that some field-strengths of the effective theory, with indices in the internal (compact) directions, have a non-trivial background value. When the fluxes come from components of these field strengths (or even some metric components), they are

[3] We will often write $O(D,D,\mathbb{R})$ simply as $O(D,D)$.

called geometric fluxes. If one reduces string theory on toroidal backgrounds, the resulting lower dimensional supergravity theory contains moduli, *i.e.* scalars that are not stabilized by any potential. It is of phenomenological interest to construct models, in which such moduli are stabilized, since as far as we know there are no massless scalars in nature. This is in general achieved by introducing fluxes, which generate a lower dimensional gauged supergravity that might possess a non-trivial scalar potential in order to stabilize the moduli. In addition, the incorporation of fluxes provides a mechanism to break supersymmetry.

The string dualities, like U-duality, appear as continuous global symmetries of the supergravity theories. The way a gauging is mapped to another gauging by duality is encoded in the so-called "embedding tensor" [48], which means that the constant parameter that identifies the gauging can be formally considered as a tensor of the global symmetry group. One thing in particular that can happen is that a flux (a gauging) is mapped by duality to a gauging whose higher-dimensional origin is unknown. These gaugings are called "non-geometric" fluxes [49]. The non-geometric nature of these fluxes mimics the fact that the string dualities themselves cannot be simply understood in terms of geometric isometries. If one only considers the Neveu-Schwarz supergravity sector, the global symmetry group of the lower dimensional effective actions turns out to be $O(n, n, \mathbb{R})$. In this case, the $O(n, n, \mathbb{R})$ group acts on the embedding tensor which encodes the fluxes, and mix them by $O(n, n, \mathbb{R})$ rotations. Thus, taking into account that $O(n, n, \mathbb{R})$ appears as a global symmetry group, it is expected that DFT plays a prominent role in flux compactifications, and can shed some light on the higher-dimensional origin of non-geometric fluxes or the massive-deformations. In fact, such an interpretation has been achieved in the so-called "Flux Formulation" of DFT [50–53].

In addition to the NS-sector, the ten dimensional type II supergravities have a set of p-form gauge fields, C_1 and C_3 for type IIA or C_0, C_2 and C_4 for IIB, belonging to what is known as the R-R sector. When the RR-fields are incorporated in DFT [54–57], the reduction to x-space gives massless IIA or IIB supergravity. As mentioned before, there is another ten dimensional supergravity theory known as Romans' ten dimensional supergravity. This theory is a deformation of the massless Type IIA by virtue of a mass parameter. It is possible to obtain this theory from DFT at the cost of relaxing the strong constraint [58]. The key observation was that the gauge invariance of the RR-sector of DFT and closure of the gauge algebra requires only a weak form of the constraint known as the weak constraint (we will introduce it in Chapter (3)). Configurations that are compatible with this weak form of the constraint would allow for fields with a particular linear dependence on the dual coordinates, namely $\chi(x, \tilde{x}) = \chi_0(x) + m\tilde{x}$, where χ is some field. This particular dependence leads to massive (Romans) Type IIA theory where the constant m is a mass parameter. This relaxation can be interpreted as a Scherk-Schwarz (SS) reduction ansatz [59]. Scherk-Schwarz reductions are a generalization of the usual Kaluza-Klein reduction, the difference being that the fields are allowed to depend on the internal directions of the compact space (contrary to the Kaluza-Klein reduction). But they do so in such a way that all the internal dependence gets factorized out of the action as a volume factor. The factorization, in general, takes the form of the structure constants of Lie algebras. The Scherk-Schwarz ansatz plays an important role as a proposal for relaxing the constraint in the Neveu-Schwarz sector of DFT. It was observed that compactifications of the DFT action lead to quadratic constraints on the gaugings that were stronger than the usual ones from gauged supergravities. These stronger conditions were intimately related to the possibility of relaxing the strong constraint at least in an internal compact space. Indeed, it was shown in [50–52] that "generalized Scherk-Schwarz" reductions of DFT allow the strong constraint to be relaxed and produces gauged supergravity in lower dimensions. The criteria for the conditions, under which the NS-sector of the DFT can undergo a relaxation of the strong constraint, were formalized in [53].

Not only is DFT capable of providing a higher-dimensional description of non-geometric fluxes but it might also provide a unified $O(D, D)$ description for the electro-magnetic dual potentials of them. We will do this in Chapter (6). Let us briefly review the concept of electro-magnetic duality. It is known that electro-magnetic duality in four dimensions is a symmetry of the full set of *sourceless* Maxwell's equations in Minkowski space. The full set of equations can be described as

$$\partial_a F^{ab} = 0, \quad \partial_{[a} F_{bc]} = 0. \tag{1.2}$$

These are the usual Maxwell's equations, plus the Bianchi identity, respectively, when written in terms of the usual electric and magnetic fields (E, B). The Bianchi identity implies that locally $F_{ab} = 2\partial_{[a} A_{b]}$, where A_a, is a 1-form potential. Then the first equation becomes an equation of motion for A_a. At the level of the equations (1.2), we could now perform a field redefinition given by $F_{ab} = (1/2)\epsilon_{ab}{}^{cd}\tilde{F}_{cd}$, where ϵ_{abcd} is the permutation symbol. The field redefinition allows us to re-obtain the same set of equations with the substitution $F \to \tilde{F}$, as long as the notion of Bianchi identity and equation of motion is interchanged. In other words, to solve for $\tilde{F}$ in terms of a potential $\tilde{A}$, the first equation in terms of F must become a Bianchi identity for $\tilde{F}$, and the Bianchi identity for F must become an equation of motion for $\tilde{F}_{ab} = 2\partial_{[a}\tilde{A}_{b]}$. This is the electro-magnetic duality of the sourceless Maxwell's equations[4].

This remarkable duality allows for the idea of magnetic monopoles: The electric field of an electric point-charge can be interpreted as being the field of a magnetic monopole in the dual theory. However, the magnetic monopole is not well defined, since it also contains the so-called "Dirac strings." Dirac has shown that it is possible to have a consistent quantum mechanical theory of electric and magnetic monopoles as long as both charges satisfy the Dirac quantization condition $q_e q_m = 2\pi n \epsilon_0 \hbar c^2$ with $n \in \mathbb{Z}$ (ϵ_0 is the vacuum permittivity). If a single monopole exists in the universe, then all electric charges are quantized, and also the other way around. The charge of the magnetic monopole is quantized in inverse units of the electric charge. So we see from the Dirac quantization condition that electro-magnetic duality plays an important role in understanding strong and weak coupling behavior: For fixed n when the electric charge (regarded as a coupling constant) is small/large, the magnetic charge (regarded as a coupling constant) becomes large/small. The magnetic monopoles are interpreted as non-perturbative states from the electric theory point of view. It is straightforward to generalize the electro-magnetic duality to more dimensions and for (abelian) p-forms. For instance, in D-dimensions, a p-form is dual to a $(D-2-p)$-form. In light-cone gauge, the dualization is easy to understand [60,61]. Take, for instance, a 1-form in D-dimensions: In light-cone gauge the relevant components A_i will be transformed in the vector representation of the little group $SO(D-2)$, $i = 1, \cdots, D-2$. Thus, the Hodge-map acting on A_i will take the vector representation to a physically equivalent $(D-3)$-index irreducible representation, that is, $\tilde{A}_{i_1,\cdots,i_{D-3}} = \epsilon_{i_1,\cdots,i_{D-3}i_{D-2}} A^{i_{D-2}}$. The Hodge-dual map is a powerful tool that maps physically equivalent irreducible representations of the little group. The main problem of dualizing in light-cone gauge is that we are loosing covariance. In general, to perform a covariant dualization for arbitrary representations is not an easy task. Instead of p-forms, we can try to perform the same analysis with other irreducible representations of the little group. We can take, for instance, a graviton, which is represented by a traceless symmetric tensor h_{ij} (in physical gauge). Then we can dualize it in the first index to obtain a different representation. The new object $D^{i_1,\cdots,i_{D-3}}{}_m = \epsilon^{i_1,\cdots,i_{D-2}} h_{i_{D-2}m}$ is a mixed-symmetry tensor [62] corresponding to the $(D-3,1)$ Young tableau of $SO(D-2)$[5]. For instance, in four dimensions, $D^{i,j} = D^{j,i}$ and $D^i{}_i = 0$, so the dual field $D^{i,j}$ has the same properties as the graviton. In $D = 10$, it is

[4]This is the simplest form of duality invariance in $D = 4$. There is a bigger $SL(2,\mathbb{R})$ symmetry that leaves the equations invariant.

[5]This means a Young tableau with two columns. The first column has $(D-3)$ rows and the second column has one.

a $(7,1)$ mixed-symmetry field. While at the linearized level there is a straightforward procedure for dualizing the graviton, there are strong no-go theorems stating that, at the non-linear level, some new ingredients are needed [63, 64]. Similar to the magnetic monopoles, there exist solutions, known as Kaluza-Klein monopoles, that couple electrically to the dual graviton.

It is well known that, in D-dimensions, particles (0-branes) couple to 1-forms (for instance, electric particles to the Maxwell potential). The natural generalization, then, is that p-branes (p-dimensional extended objects in space-time) couple to $p+1$-forms. String theory possesses such p-forms in its spectrum. In particular, the so-called D-branes couple to the RR-potentials of string theory. In the common NS-sector the Kalb-Ramond, 2-form b_2 can be dualized to a 6-form D_6 in $D = 10$. The strings couple electrically to b_2, and the object that couples electrically to the D_6 is known as the NS5-brane [65]. Therefore, from the point of view of the full (non-perturbative) string or M-theory, neither the 2- nor the 6-form is more fundamental, suggesting that a democratic formulation, in which they appear on equal footing, is more appropriate. Remarkably, taking into account further dualities or symmetries of string theory, such as T-duality, this then implies that even more fields of a more exotic nature are needed. For instance, under the T-duality group the 2-form transforms into the metric. Thus, when dualizing b_2 into D_6, the T-duality covariance requires that we also dualize the graviton into the dual graviton. That is why DFT is also a natural framework for studying how the dual fields are related by $O(D,D)$ transformations. In general, mixed-symmetry tensor fields will appear when we consider the full U-duality of string theory. Upon reduction of 11-dimensional supergravity, several exceptional groups arise as global symmetries. It was conjectured in [66] that the infinite-dimensional E_{11} algebra is the U-duality symmetry of M-theory[6]. The interesting feature is that E_{11} predicts not only the usual field contents but also their respective dual fields on an equal footing. In particular, it predicts fields in mixed-symmetry Young-tableau representations like the dual graviton. All in all, since T-duality or U-duality predicts mixed-symmetry potentials, it provides evidence that there are some branes in string theory [75–77] that have a more exotic nature than the usual ones, which couple to p-forms. We call these branes "exotic branes."

We stress that this thesis is mainly concerned with the standard version of DFT, which only incorporates the metric, Kalb-Ramond, and dilaton fields. Extensions that include, for instance, heterotic vector fields were done in [78] and with Yang-Mills symmetries in [79]. The inclusion of R-R forms was performed

[6] The E_{11} program has been developed in [66–74].

in [54–57], and the inclusion of fermions in a supersymmetric fashion was done in [80–82]. The connection with $O(D,D)$ covariant world-sheet theories was established in [83–88]. We will not comment on α' corrections to the DFT action, but the reader can take a look at [89–94], [95,96]. A review of DFT can be found in [97].

1.1 Outline of the thesis

This thesis is based on the publications [53, 98, 99]. More specifically, Chapter (4) is based on [53], Chapter (5) is based on [98], and Chapter (6) is based on [99]. Complementary material for the other chapters is taken from books or publications that will be cited in the course of the thesis. The organization is as follows. In Chapter (2), we will introduce some background material about string theory in order for the reader to understand the idea behind Double Field Theory in subsequent chapters. There are excellent and nice books on string theory [2, 100–105] that expand the introduction of this thesis in more detail and cover the basics we will show in this chapter. We begin by reviewing the bosonic string. We will perform the quantization of the string in the light-cone gauge in order to obtain the spectrum and the field content of the theory. Then we will move on to talk about the string propagating in a compact background: first the circle and then toroidal reduction. The statement of T-duality, winding number, $O(D,D)$ group, and dual coordinates will be introduced.

In Chapter (3), we will introduce the basic concepts of Double Field theory. We will motivate the construction of DFT and its properties. The action, generalized metric, strong constraint, and gauge invariance will be briefly discussed. The chapter will end with a review of the frame field formulation of DFT and will present the tools needed to work in the Flux Formulation of DFT. The reader can then choose to move on to other chapters.

In Chapter (4), we will motivate and expand the Flux Formulation of DFT. The goal of this chapter is to explore to what extent one can deal with the gauge consistency constraints of DFT without imposing the strong constraint. We will rely on a geometric construction in order to build geometric quantities like the generalized Ricci scalar. The generalized Ricci scalar constructed in this chapter is basically the DFT Lagrangian with terms that violate the strong constraint but are essential in order to make contact with gauged supergravity in lower dimensions. This is the only chapter where the strong constraint is not assumed a priori but instead we find the consistency constraints that will supersede it. In Subsection 4.1.2, we will introduce the generalized fluxes, which are higher-dimensional fields that upon compactification give rise to the usual constant fluxes (i.e., the gaugings). In terms of them, the action, equations of motion, and gauge consistency constraints are found. In Section 4.2, the novel notions of stringy differential geometry are adapted to hold beyond the strong constraint. We will analyze some identities satisfied by the generalized fluxes known as generalized Bianchi identities. We also present a first order formulation of DFT in the presence of sources. A more systematic first order formulation will be done in Chapter (6) at the linearized level.

In Chapter (5), we will study the usual chain of solutions known as NS5-KK5-Q5 (or 5^2_2) and R5 in the DFT language. We argue that the R5-brane solution has a $\tilde{x}$-dependence rendering it locally non-geometric from the usual supergravity point of view. We justify this by using a generalized Scherk-Schwarz reduction ansatz of DFT. We will end this chapter by commenting about the dual fields of these branes, and we will analyze, schematically, a duality relationship between the fluxes and mixed-symmetry potentials. Although the argument relies on assuming isometric directions, we will not require this in Chapter (6). Nevertheless, it is instructive to understand the duality relationships first in this way.

In Chapter (6), we will construct the dual theory of Double Field Theory, which we call Dual Double Field Theory. This theory incorporates all the mixed-symmetry potentials mentioned in the previous chapter in a $O(D, D)$ covariant way. First, we will introduce the usual electro-magnetic duality between the standard fields of the NS-sector. Then we will move on to consider a dualization procedure proposing a first-order action in the Flux Formulation and also in a geometric formulation using the spin connection. We will end by showing how the Dual Double Field Theory reduces to the standard dualization for the NS-fields and formalizes the duality relationships argued in Chapter (5).

Finally, in Chapter (7) we will present the conclusions and outlook of the thesis.

2
Preliminaries

2.1 Bosonic String

Our intention in this chapter is to present the basic results that will be relevant, for our purposes, for the rest of this thesis. We will introduce the bosonic string, its mode expansion and spectrum. We will omit many details, which can be found elsewhere in the literature. We will mainly follow [2,100–102]. When we quantize the string, we will focus on the light-cone quantization to obtain the spectrum and the field content. Finally, we will end by analyzing the string moving in a compact space, and how the notion of T-duality and winding modes arises. These notions are the basic concepts to be used in the following chapters.

The action describing a string moving in Minkowski space-time is essentially the integral of the infinitesimal area spanned by the string as it moves through space-time. It is called the Nambu-Goto action ($\hbar = c = 1$):

$$S_{NG} = -\frac{1}{2\pi\alpha'}\int\sqrt{(\partial_\tau X^\mu \partial_\sigma X_\mu)^2 - (\partial_\tau X^\mu)^2(\partial_\sigma X^\mu)^2}d\tau d\sigma. \tag{2.1}$$

Here, the world-sheet swept out by the string (in analogy to the world-line of a particle) is parametrized by coordinates (τ, σ) being τ a temporal coordinate and σ a spatial one. The derivatives ∂_τ and ∂_σ are with respect to these coordinates. The map between the world-sheet and the target space is made through the target coordinates:

$$X^\mu = X^\mu(\tau, \sigma), \quad \mu = 0, \cdots, D-1, \tag{2.2}$$

and the space-time indices μ are raised and lower with the D-dimensional Minkowski metric $\eta_{\mu\nu}$. The factor in front of the Nambu-Goto action is the string tension as was mentioned in the Introduction. For quantization convenience it is better to consider the Polyakov form of the action, which is classically equivalent to the Nambu-Goto one. The Polyakov action takes the form

$$S_P = -\frac{1}{4\pi\alpha'}\int d\tau d\sigma\sqrt{-h}h^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu\eta_{\mu\nu}. \tag{2.3}$$

Here $h^{\alpha\beta}(\tau,\sigma)$ is a dynamical variable and is known as the world-sheet metric. It has Lorentzian signature $(-,+)$. The index α represents τ or σ.

The Polyakov action has a number of symmetries. It has global Poincare invariance on the target space given by $\delta X^\mu = a^\mu{}_\nu X^\nu + b^\mu$ where $a_{\mu\nu} = -a_{\nu\mu}$ and b^μ are the parameters of the Poincare group. The action also has two local invariances given by reparametrizations of the world-sheet coordinates $(\tau,\sigma)\to(\tau'(\tau,\sigma),\sigma'(\tau,\sigma))$ (X^μ are massless world-sheet scalars) and a Weyl symmetry $h_{\alpha\beta}\to e^{2\omega(\tau,\sigma)}h_{\alpha\beta}$ for arbitrary ω.

The variation of the Polyakov action with respect to $h_{\alpha\beta}$ defines the energy-momentum tensor $T_{\alpha\beta}$ which is conserved due to reparametrization invariance. The Weyl symmetry has the important consequence of the vanishing of the trace of the energy-momentum tensor, i.e. $T_{\alpha\beta}h^{\alpha\beta} = 0$. The equation of motion for X^μ is found to be:

$$-\frac{1}{\sqrt{-h}}\partial_\alpha\left(\sqrt{-h}h^{\alpha\beta}\partial_\beta X^\mu\right) = 0. \tag{2.4}$$

These equations of motion should be supplemented with boundary conditions:

$$\int_{-\infty}^{+\infty} d\tau\sqrt{-h}X'_\mu\delta X^\mu|_{\sigma=0}^{\sigma=\pi} = 0, \tag{2.5}$$

where we have chosen for convenience $0\leq\sigma\leq\pi$ and $-\infty\leq\tau\leq+\infty$. The vanishing of the boundary conditions and consistency of D-dimensional Poincare invariance give rise to closed and open strings. The closed strings are the one that satisfy

$$X^\mu(\tau,\sigma) = X^\mu(\tau,\sigma+\pi). \tag{2.6}$$

The endpoints of the string are joined to form a (closed) loop. The open strings are defined to be:

$$X'^\mu|_{\sigma=0} = X'^\mu|_{\sigma=\pi} = 0. \tag{2.7}$$

These are called Neumann boundary conditions and the interpretation is that the endpoints of the string are free to move. It is possible to mix with Dirichlet boundary conditions:

$$X^{\mu}|_{\sigma=0} = X_0^{\mu} \quad \text{and} \quad X^{\mu}|_{\sigma=\pi} = X_{\pi}^{\mu}, \tag{2.8}$$

for some $\mu = 1, \cdots, D-1$. A priori, these conditions break translation invariance, but the correct interpretation demands that the endpoints of the string are attached to dynamical objects called D-branes.

2.1.1 Gauge-fixing

In this subsection we are going to calculate the string spectrum. We will do it in the light-cone gauge. For this, we will introduce the mode expansion for the string, the Virasoro conditions and the Hamiltonian.

When there are no topological obstructions it is possible to fully gauge-fix the world-sheet metric by using the reparametrization invariance and Weyl rescaling. The metric takes the form:

$$h_{\alpha\beta} = \eta_{\alpha\beta} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.9}$$

This is the conformal gauge. In this gauge, the action and equation of motion for X^{μ} get the simpler form:

$$S = \frac{T}{2} \int d^2\sigma \Big(\partial_{\tau} X^{\mu} \partial_{\tau} X_{\mu} - \partial_{\sigma} X^{\mu} \partial_{\sigma} X_{\mu} \Big), \tag{2.10}$$

$$\Big(\frac{\partial^2}{\partial \sigma^2} - \frac{\partial^2}{\partial \tau^2} \Big) X^{\mu} = 0. \tag{2.11}$$

After gauge-fixing, we need to impose as an additional constraint that $T_{\alpha\beta} = 0$, which in the conformal gauge takes the form :

$$T_{\tau\sigma} = \dot{X} \cdot X' = 0, \quad T_{\tau\tau} = T_{\sigma\sigma} = \frac{1}{2} \left(\dot{X}^2 + X'^2 \right) = 0, \tag{2.12}$$

or in a more compact form:

$$(\dot{X} \pm X')^2 = 0. \tag{2.13}$$

Here, $\dot{X}$ and X' denotes derivative with respect to τ and σ, respectively. These constraints are known as the Virasoro conditions. The mode expansion for the closed string that solves (2.11) is given by:

$$X^{\mu}(\tau, \sigma) = X_R^{\mu}(\tau - \sigma) + X_L^{\mu}(\tau + \sigma), \tag{2.14}$$

where

$$X_R^\mu = \frac{1}{2}x^\mu + \frac{1}{2}\ell_s^2 p^\mu(\tau-\sigma) + \frac{i}{2}\ell_s \sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-2in(\tau-\sigma)}, \tag{2.15}$$

$$X_L^\mu = \frac{1}{2}x^\mu + \frac{1}{2}\ell_s^2 p^\mu(\tau+\sigma) + \frac{i}{2}\ell_s \sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}_n^\mu e^{-2in(\tau+\sigma)}. \tag{2.16}$$

Here x^μ is a constant, p^μ is the total momentum of the string and $\ell_s = \sqrt{2\alpha'}$ is known as the string length scale. The coefficients α_n^μ ($\tilde{\alpha}_n^\mu$) are known as right (left)-modes and satisfy:

$$\alpha_{-n}^\mu = *(\alpha_n^\mu) \text{ and } \tilde{\alpha}_{-n}^\mu = *(\tilde{\alpha}_n^\mu). \tag{2.17}$$

For the open string we have the following expansion:

$$X_L^\mu = x^\mu + \ell_s^2 p^\mu \tau + i\ell_s \sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-in\tau}\cos(n\sigma). \tag{2.18}$$

It is convenient to define the zero modes for the closed and open string as follows:

$$\text{(closed)}\ \ \alpha_0^\mu = \tilde{\alpha}_0^\mu = \frac{1}{2}\ell_s p^\mu, \qquad \text{(open)}\ \ \alpha_0^\mu = \ell_s p^\mu. \tag{2.19}$$

The Hamiltonian is defined in the usual way and takes the following form in the conformal gauge:

$$H = \int (\dot{X}_\mu P^\mu - \mathcal{L})d\sigma = \frac{T}{2}\int (\dot{X}^2 + X'^2)d\sigma, \tag{2.20}$$

where $P^\mu = \delta S/\delta \dot{X}_\mu$ is the conjugate variable to X^μ and $\mathcal{L}$ the Lagrangian density. In terms of the mode expansion the closed string has the following Hamiltonian:

$$H = \frac{1}{2}\sum_{n=-\infty}^{+\infty}\Big(\alpha_{-n}\cdot\alpha_n + \tilde{\alpha}_{-n}\cdot\tilde{\alpha}_n\Big), \tag{2.21}$$

the dot represent contraction in Minkowski indices. For the open string we have:

$$H = \frac{1}{2}\sum_{n=-\infty}^{+\infty}\Big(\alpha_{-n}\cdot\alpha_n\Big). \tag{2.22}$$

The Virasoro conditions have the following form:

$$0 = (\dot{X} - X')^2 = 2\ell_s^2 \sum_{m=-\infty}^{+\infty} L_m e^{-2im(\tau-\sigma)}, \tag{2.23}$$

$$0 = (\dot{X} + X')^2 = 2\ell_s^2 \sum_{m=-\infty}^{+\infty} \tilde{L}_m e^{-2im(\tau+\sigma)}. \tag{2.24}$$

The L_m ($\tilde{L}_m$) are known as Virasoro generators which play an important role at the quantum level and they are given in terms of Fourier coefficients by:

$$L_m = \frac{1}{2} \sum_{n=-\infty}^{+\infty} \alpha_{m-n} \cdot \alpha_n \quad \text{and} \quad \tilde{L}_m = \frac{1}{2} \sum_{n=-\infty}^{+\infty} \tilde{\alpha}_{m-n} \cdot \tilde{\alpha}_n. \tag{2.25}$$

At the classical level, it is easy to see that given the definition of Virasoro generators, the Hamiltonian for the string takes the following form:

$$\text{(closed)} \;\; \frac{1}{2} H = L_0 + \tilde{L}_0, \quad \text{(open)} \;\; H = L_0. \tag{2.26}$$

Classically, the Virasoro generators vanish for every $m \in \mathbb{Z}$ since the energy-momentum tensor $T_{\alpha\beta}$ has been imposed to be zero in the conformal gauge. In particular, $L_0 = 0$ ($\tilde{L}_0 = 0$) so the Hamiltonian turns out to be equal to zero. We can calculate the mass of the string using the relativistic relation between the mass and the momentum $M^2 = -p_\mu p^\mu$ (p_μ being the total momentum of the string). Then, using the definition in terms of α_0 (and $\tilde{\alpha}_0$) we can calculate the mass:

$$\text{(closed)} \qquad M^2 = \frac{2}{\alpha'} \sum_{n=1} \Big(\alpha_{-n} \cdot \alpha_n + \tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n \Big), \tag{2.27}$$

$$\text{(open)} \qquad M^2 = \frac{1}{\alpha'} \sum_{n=1} \alpha_{-n} \cdot \alpha_n. \tag{2.28}$$

2.1.2 Light-cone Quantization

As usual, in order to quantize the theory, the Poisson brackets between X^μ and P^μ are promoted to commutators satisfying

$$[X^\mu(\tau, \sigma'), P^\nu(\tau, \sigma)] = i\eta^{\mu\nu} \delta(\sigma - \sigma'), \tag{2.29}$$

where in the right-hand side appears the Dirac delta function. Using the mode expansion for X^μ the following commutators are obtained:

$$[\alpha_m^\mu, \alpha_n^\nu] = [\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^\nu] = m\eta^{\mu\nu}\delta_{m+n,0} \quad \text{and} \quad [\alpha_m^\mu, \tilde{\alpha}_n^\nu] = 0. \tag{2.30}$$

It is easy to show that the algebra of the modes is the same as the algebra of creation and annihilation operators. In this case the α_{-m}^μ operators play the role of creation ones and we may define the ground state $|0, k^\mu\rangle$ as:

$$\alpha_m^\mu |0, k^\mu\rangle = 0 \quad m > 0, \quad p^\mu |0, k^\mu\rangle = k^\mu |0, k^\mu\rangle. \tag{2.31}$$

In the conformal gauge, the temporal components in the commutation relation produce a negative sign due to the Minkowski signature. This causes states of negative norm. In order to decouple the ghost states from the theory it is required that the physical states are the ones that are annihilated by half of the Virasoro generators:

$$L_m |\phi\rangle = 0, \quad m \geq 0, \tag{2.32}$$

where $|\phi\rangle$ is a physical (on-shell) state. Note that the only generators which have normal ordering ambiguity are L_0 and $\tilde{L}_0$, which are redefined as:

$$L_0 = \frac{1}{2}\alpha_0^2 + \sum_{n=1} \alpha_{-n} \cdot \alpha_n - a, \tag{2.33}$$

and similar for $\tilde{L}_0$[1]. The requirement of $m \geq 0$ instead of $m \in \mathbb{Z}$ in (2.32) is compatible with the quantum Virasoro algebra:

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}m(m^2 - 1)\delta_{m+n,0}. \tag{2.34}$$

The central charge c is equal to the space-time dimension. If we take the difference between L_0 and $\tilde{L}_0$ the normal ordering constant cancels out:

$$\left(L_0 - \tilde{L}_0\right)|\phi\rangle = 0. \tag{2.35}$$

This equation is known as the *level-matching condition* and will play an important role in T-duality and Double Field Theory. In terms of number operators the above equation is equal to:

$$\left(N - \tilde{N}\right)|\phi\rangle = 0, \tag{2.36}$$

[1] We take the normal ordering constants to be equal, i.e. $a = \tilde{a}$.

where $N = \sum_{n=1} \alpha_{-n} \cdot \alpha_n$ and similar for $\tilde{N}$. Due to the constant coming from normal ordering, the mass operator is shifted by a constant and we end up with:

$$\text{(open)} \quad \alpha' M^2 = N - 2a, \tag{2.37}$$

$$\text{(closed)} \quad \alpha' M^2 = 4(N - a) = 4(\tilde{N} - a) = 2(N + \tilde{N} - 2a). \tag{2.38}$$

In the covariant quantization we quantize the theory and impose the constraints as operator equations acting on the Fock space of the theory. The temporal components in the commutation relation contain a negative sign due to the Minkowski signature and cause states of negative norm to appear. It can be proved that the Virasoro constraints decouple the ghosts states for the critical values of $a = 1$ and $D = 26$. In this sense, the consistency of string theory predicts the space-time dimension to be $D = 26$. In the light cone quantization we first solve the Virasoro constraints to determine the space of physically distinct solutions and then quantize the system arriving at the same conclusion that $a = 1$ and $D = 26$. In the supersymmetric version of the string, it can be proven that the consistency of the theory requires $D = 10$ instead, as we will mention in the last section of this chapter.

The light-cone gauge can be motivated as follows. There is a residual symmetry after fixing the conformal gauge. The conformal gauge is still preserved under reparametrizations that generate a Weyl transformation:

$$\partial^\alpha \xi^\beta + \partial^\beta \xi^\alpha = \Lambda \eta^{\alpha\beta}. \tag{2.39}$$

Here, ξ is an infinitesimal parameter and Λ is associated with a Weyl transformation. It can be shown that this equation is solved by gauge parameters that depend on light-cone worldsheet coordinates $\sigma^\pm = \tau \pm \sigma$. This implies that we can define new coordinates $\tilde{\tau}$ and $\tilde{\sigma}$ such that $\tilde{\tau}$ is an arbitrary solution of the wave equation and $\tilde{\sigma}$ got fixed (up to a constant) once $\tilde{\tau}$ is determined. We would like to use this extra freedom to gauge-fix one of the target coordinates and to solve the Virasoro constraints. To achieve this purpose, it is convenient to use light-cone coordinates

$$X^\pm = \frac{1}{\sqrt{2}}\left(X^0 \pm X^{D-1}\right), \quad X^i \quad (i = 1, \cdots, D-2). \tag{2.40}$$

The coordinates X^i are called transverse coordinates. Then we can choose to gauge-fix $X^+(\tau, \sigma) = x^+ + \ell_s^2 p^+ \tau$ (we dropped the tildes). It is possible to use the Virasoro constraints to determine X^- in terms of transverse oscillators. In light-cone coordinates, the Virasoro constraints are:

$$\dot{X}^- \pm X'^- = \frac{1}{2p^+ l_s^2}\left(\dot{X}^i \pm X'^i\right)^2. \tag{2.41}$$

This equation enables us to rewrite the α_n^- oscillators in terms of the transverse ones (i.e. α_n^i). In summary, in the light-cone gauge, X^+ and X^- are completely fixed except for their zero modes ($\alpha_0^+ = \ell_s p^+$, $\alpha_0^- = \ell_s p^-$). Due to this non-covariant gauge choice, the Fock space is free of negative norm states, but Lorentz invariance must be checked. In fact, in light-cone gauge, it can be shown that the vanishing of the commutation relations of the Lorentz generators imposes $a = 1$ and $D = 26$.

2.1.3 Spectrum

Once the values $a = 1$ and $D = 26$ are fixed, we can easily determine the spectrum of the theory in the light-cone gauge since all states have positive norm. After solving for α_n^- in terms of the transverse oscillators, the open string mass operator in the light-cone gauge takes the form:

$$\alpha' M^2 = N - a, \qquad \text{with } \; N = \sum_{i=1}^{D-2}\sum_{n=1}^{\infty} \alpha_{-n}^i \alpha_n^i. \tag{2.42}$$

For $N = 0$ the ground state provides a tachyon $\alpha' M^2|0\rangle = -|0\rangle$. This is a sign of instability in the bosonic theory (in the superstring it is possible to eliminate the tachyon). For $N = 1$ there is a vector boson $\alpha_{-1}^i|0\rangle$ with $M^2 = 0$. We could have infered the value of a by demanding preservation of the Lorentz group at quantum level: the state $\alpha_{-1}^i|0\rangle$ belongs to an irreducible representation of $SO(D-2)$ so it should be massles, fixing the value to $a = 1$. Then, by performing a ζ-function regularization in the actual computation of the normal ordering constant we would have fixed the value of D to be $D = 26$ [2]. For $N \geq 2$ we have massive states.

For the closed string the mass operator takes the form:

$$\frac{1}{2}\alpha' M^2 = N + \tilde{N} - 2. \tag{2.43}$$

[2] We have $a = -\frac{1}{2}(D-2)\sum_{n=1}^{\infty} n$. Using the Riemann ζ-function $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ we could find the value of D using the value $s = -1$ and $a = 1$.

Again, N and $\tilde{N}$ are similarly defined as in (2.42). For $N = 0$ the ground state is also a tachyon with mass $\alpha' M^2 = -4$. For $N = \tilde{N} = 1$ we have the state $\alpha^i_{-1}\tilde{\alpha}^j_{-1}|0\rangle$. Under $SO(D-2)$ we can decompose this state into a singlet called 'dilaton', a symmetric traceless tensor which represents the 'graviton', and an antisymmetric 2-tensor (sometimes called the Kalb-Ramond field in the supergravity language). In the space-time representation we represent these fields as:

$$h_{ij}(x) \;\; \text{(graviton)}, \quad b_{ij}(x) \;\; \text{(Kalb-Ramond)}, \quad d(x) \;\; \text{(dilaton)}. \tag{2.44}$$

Up to field redefinitions, these field represent the common sector of the string supergravities and we will call it the Neveu-Schwarz (NS) sector.

2.1.4 Compactification and T-duality

We have seen that string theory predicts the dimension of space-time to be equal to $D = 26$ for the bosonic case. In superstring theory the dimension of space-time turns out to be equal to $D = 10$. In both cases, a mechanism to make contact with the physical 4-dimensional world is needed. This mechanism is known as 'compactification'. In the simplest case it is assumed that the extra dimensions are curled up into small circles so as to evade detection at low energies. This particular compactification is called toroidal compactification or Kaluza-Klein compactification (i.e. the internal manifold parametrized by the extra dimensions forms a torus). There exist compactifications with more complicated internal spaces involved, like Calabi-Yau manifolds, orbifolds, etc. In the following, we will study string theory assuming one of the space-time coordinates to be compactified into a circle while the other ones are uncompactified. We impose the circle condition on, say, coordinate X^{25}:

$$X^{25}(\sigma + \pi, \tau) = X^{25}(\sigma, \tau) + 2\pi R\omega, \qquad \omega \in \mathbb{Z} \tag{2.45}$$

where ω is called the *winding number* and counts how many times the string is wrapped around the circle. The mode expansion is the same as before except for the coordinate X^{25} (due to the condition (2.45)):

$$\begin{aligned} X^{25}(\sigma, \tau) = x^{25} + 2\alpha' p^{25}\tau + 2R\omega\sigma + \frac{i\sqrt{\alpha'}}{\sqrt{2}} \sum_{n \neq 0} \frac{1}{n} \alpha^{25}_n e^{-2in(\tau - \sigma)} + \\ \frac{i\sqrt{\alpha'}}{\sqrt{2}} \sum_{n \neq 0} \frac{1}{n} \tilde{\alpha}^{25}_n e^{-2in(\tau + \sigma)}. \end{aligned} \tag{2.46}$$

Since the coordinate X^{25} is compactified, the momentum along this direction takes on the discrete values:

$$p^{25} = \frac{m}{R} \qquad m \in \mathbb{Z}. \tag{2.47}$$

The number m is known as the Kaluza-Klein excitation number. We can split X^{25} in left and right components $X^{25}(\tau,\sigma) = X^{25}_L(\tau+\sigma) + X^{25}_R(\tau-\sigma)$. In doing so we need to define the following zero modes:

$$\sqrt{2\alpha'}\alpha_0^{25} = \alpha'\frac{m}{R} - \omega R \quad \text{and} \quad \sqrt{2\alpha'}\tilde{\alpha}_0^{25} = \alpha'\frac{m}{R} + \omega R. \tag{2.48}$$

Also, in splitting in left and right movers there is enough freedom to add a new constant $\tilde{x}_{25}$ such that $x_L^{25} = \frac{1}{2}(x^{25} + \tilde{x}_{25})$ and $x_R^{25} = \frac{1}{2}(x^{25} - \tilde{x}_{25})$. The contributions to the operators L_0 and $\tilde{L}_0$ come from every dimension including the 25. We take the point of view of an observer living in the lower-dimensional theory and we define the mass square as following:

$$M^2 = -\sum_{\mu=0}^{24} p_\mu p^\mu. \tag{2.49}$$

Then, from the equations $L_0 = 0$ and $\tilde{L}_0 = 0$ $(a = 1)$ the above definition takes the form:

$$\frac{1}{2}\alpha' M^2 = (\tilde{\alpha}_0^{25})^2 + 2\tilde{N} - 2 = (\alpha_0^{25})^2 + 2N - 2, \tag{2.50}$$

where N is defined as $N = \sum_{n=1} \alpha_{-n} \cdot \alpha_n$ and similar for $\tilde{N}$. After adding and subtracting the above equation we obtain the following equations:

$$N - \tilde{N} = m\omega, \tag{2.51}$$

$$\alpha' M^2 = \alpha' \left[\left(\frac{m}{R}\right)^2 + \left(\frac{\omega R}{\alpha'}\right)^2 \right] + 2\tilde{N} + 2N - 4. \tag{2.52}$$

The condition (2.51) shows how the level-matching condition gets modified in the presence of compact coordinates. If the winding is zero the string behaves like a particle (in a space-time with one compact direction) and the symmetry between the right and left movers is restored. If the Kaluza-Klein excitation number is zero the string still has energy due to the winding number as can be seen in (2.52). This is because it costs energy to wrap the string around the compactified direction. Now suppose we take the radius to infinity in (2.52) (decompactification limit), then, we see that the momentum modes go to a continuous limit while the winding modes get infinitely massive. On the other hand, in the compactification limit of taking the radius to zero we see a swapped behavior. The winding modes go to a continous and the momentum modes get heavier. This behavior allows us to reinterpret the result and claim the winding mode is the momentum mode and the radius is the inverse of R. This is the statement of *T-duality*. In mathematical terms, the theory is invariant under the interchange:

$$R \leftrightarrow \tilde{R} = \frac{\alpha'}{R}, \quad m \leftrightarrow \omega. \tag{2.53}$$

This duality symmetry can be easily checked on equations (2.51) and (2.52). But more generally, it can be proven to be a symmetry of the full theory including interactions. The T-duality transformation maps two (equivalent) theories one with radius R and the other (the T-dual theory) with radius $\tilde{R} = \alpha'/R$. There exists a self-dual radius $R = \sqrt{\alpha'}$. The T-duality maps this radius to itself. The self-dual radius implies there exists a notion of a minimum distance in a circle compactification: the theories defined on a radius smaller than the self-dual radius are equivalent to theories defined on radius larger than the self-dual radius.

It is useful to introduce a *dual coordinate* operator:

$$\tilde{X}_{25}(\tau, \sigma) = X_L(\tau + \sigma) - X_R(\tau - \sigma), \tag{2.54}$$

explicitly it takes the form

$$\begin{aligned}\tilde{X}_{25}(\sigma, \tau) = \tilde{x}_{25} + 2R\omega\tau + 2\alpha' p^{25}\sigma - \frac{i\sqrt{\alpha'}}{\sqrt{2}} \sum_{n\neq 0} \frac{1}{n} \alpha_n^{25} e^{-2in(\tau-\sigma)} +& \\ \frac{i\sqrt{\alpha'}}{\sqrt{2}} \sum_{n\neq 0} \frac{1}{n} \tilde{\alpha}_n^{25} e^{-2in(\tau+\sigma)}.&\end{aligned} \tag{2.55}$$

From this coordinate we can see that the constant $\tilde{x}_{25}$ enters as the main constant for $\tilde{X}_{25}$ playing the same role as x^{25} in (2.46). More importantly, now $\omega R/\alpha'$ has the interpretation of momentum whereas p^{25} the one of winding. The constant x^{25} was a coordinate living in a space of radius R with conjugate momenta p^{25}. Now we can interpret the constant $\tilde{x}_{25}$ as a coordinate living in a space of radius $\tilde{R} = \alpha'/R$ with conjugate momenta $\omega R/\alpha'$. This suggests that the theory can be described in terms of $\tilde{X}_{25}$ with a compact radius $\tilde{R} = \alpha'/R$ instead of X^{25} with radius R. Thus, T-duality is the manifestation of $X^{25} \to \tilde{X}_{25}$. It turns out that this is the case. It can be shown that the theory remains invariant when described in terms of the dual variables [100]. In the next chapter we will present Double

Field Theory and the idea is basically to build a field theory based on coordinates $(\tilde{x}, x)$, thus, doubling the coordinates of space-time and realizing T-duality as a manifest symmetry.

The massless spectrum of the compactified theory with $m = \omega = 0$ and $N = \tilde{N} = 1$ is composed of the lower dimensional metric, Kalb-Ramond, dilaton, and two (abelian) vector fields, giving rise to $U(1) \times U(1)$. These are the usual Kaluza-Klein states when reducing from D to $D-1$. For special values of the radius, like the self-dual one there is an 'enhancement of symmetry'. This means that for this special radius there will appear extra gauge fields in such away that they can be combined with the former two abelian ones to form an $SU(2)\times SU(2)$ non-abelian gauge group[3].

2.1.5 Mode expansion and level-matching condition for toroidal background

In this section we will follow [16, 107] very closely. We consider now a space-time with n compact directions, this means it is a product of d-dimensional Minkowski space with a n-torus: $\mathbb{R}^{d-1,1} \times T^n$ $(D = d + n = 26)$. We will denote the space-time coordinates as $X^i = (X^\mu, X^a)$ with μ the uncompactified coordinates and a the compact ones.

We take $\eta_{\mu\nu}$ as the Minkowski metric and G_{ab} as the internal torus-metric. The non-trivial boundary conditions on the coordinates are:

$$X^\mu(\tau, \sigma + 2\pi) = X^\mu(\tau, \sigma), \tag{2.56}$$

$$X^a(\tau, \sigma + 2\pi) = X^a(\tau, \sigma) + 2\pi W^a \qquad W^a \in \mathbb{Z}, \tag{2.57}$$

where we take $0 \leq \sigma \leq 2\pi$ and W^a is the winding number on the a-direction. For simplicity we take $W^i = (0, W^a)$. We consider also a constant background of the Kalb-Ramond field B_{ab}. The world-sheet action describing the string on this background takes the form:

$$S = -\frac{1}{4\pi}\int_0^{2\pi} d\sigma \int_{-\infty}^{\infty} d\tau \left(\eta^{\alpha\beta}\partial_\alpha X^i \partial_\beta X^j G_{ij} + \epsilon^{\alpha\beta}\partial_\alpha X^i \partial_\beta X^j B_{ij}\right), \tag{2.58}$$

where

$$\epsilon^{\tau\sigma} = -1, \quad G_{ij} = \begin{pmatrix} \eta_{\mu\nu} & 0 \\ 0 & G_{ab} \end{pmatrix}, \quad B_{ij} = \begin{pmatrix} 0 & 0 \\ 0 & B_{ab} \end{pmatrix}, \quad G^{ij}G_{jk} = \delta^i_k \tag{2.59}$$

[3] The reader can take a look at [106] for a study of the self-dual radius in the DFT context. In this thesis we will exclude such specific states.

G and B are $D \times D$ matrices.
The mode expansion for the string takes the form:

$$X^i(\sigma,\tau) = X^i_L(\tau+\sigma) + X^i_R(\tau-\sigma), \tag{2.60}$$

$$X^i_L(\tau+\sigma) = \frac{1}{2}x^i + \frac{1}{\sqrt{2}}\tilde{\alpha}^i_0(\tau+\sigma) + \text{oscillators}(\tilde{\alpha}), \tag{2.61}$$

$$X^i_R(\tau-\sigma) = \frac{1}{2}x^i + \frac{1}{\sqrt{2}}\alpha^i_0(\tau-\sigma) + \text{oscillators}(\alpha), \tag{2.62}$$

$$X^i = x^i + \frac{1}{\sqrt{2}}(\tilde{\alpha}^i_0 + \alpha^i_0)\tau + \frac{1}{\sqrt{2}}(\tilde{\alpha}^i_0 - \alpha^i_0)\sigma + \text{oscillators}. \tag{2.63}$$

Here we have taken $\alpha' = 1$. When imposing the conditions (2.56) and (2.57) we obtain:

$$\frac{1}{\sqrt{2}}(\tilde{\alpha}^i_0 - \alpha^i_0) = W^i \tag{2.64}$$

The conjugate momentum to X^i turns out to be:

$$P_i = \frac{\delta S}{\delta \dot{X}^i} = \frac{1}{2\pi}\left(G_{ij}\dot{X}^j + B_{ij}X'^j\right). \tag{2.65}$$

The total momentum of the string is obtained by integrating the above expresions:

$$p_i = \int_0^{2\pi} P_i d\sigma = G_{ij}\frac{1}{\sqrt{2}}(\tilde{\alpha}^j_0 + \alpha^j_0) + B_{ij}\frac{1}{\sqrt{2}}(\tilde{\alpha}^j_0 - \alpha^j_0). \tag{2.66}$$

We want to obtain an expression for $\tilde{\alpha}^i_0$ and α^i_0 in terms of the winding and total momentum. By adding and subtracting (2.64) and (2.66) we obtain:

$$\tilde{\alpha}^i_0 = \frac{1}{\sqrt{2}}G^{ij}(p_j + E_{kj}W^k), \tag{2.67}$$

$$\alpha^i_0 = \frac{1}{\sqrt{2}}G^{ij}(p_j - E_{jk}W^k), \tag{2.68}$$

where $E_{jk} = G_{jk} + B_{jk}$. The momentum p_a is quantized as in the circle case (see eq. (2.47)), while the non-compact momentum is equal to $\tilde{\alpha}_{0\mu} = \alpha_{0\mu} = \frac{1}{\sqrt{2}}p_\mu$. As usual in quantum mechanics, we can write the total momentum operator as $p_i = \frac{1}{i}\frac{\partial}{\partial x^i}$ so in analogy we may define $w^i = \frac{1}{i}\frac{\partial}{\partial \tilde{x}_i}$. As stressed before in the circle case, T-duality allows us to interpret the winding number as a momentum number in the T-dual description. The variable $\tilde{x}$ is defined as the conjugate variable to the winding. Acting with the metric G_{ij} to (2.67) and (2.68) we obtain:

$$\tilde{\alpha}_{0i} = -\frac{i}{\sqrt{2}}\left(\frac{\partial}{\partial x^i} + E_{ki}\frac{\partial}{\partial \tilde{x}_k}\right) \equiv \frac{-i}{\sqrt{2}}\tilde{D}_i, \tag{2.69}$$

$$\alpha_{0i} = -\frac{i}{\sqrt{2}}\left(\frac{\partial}{\partial x^i} - E_{ik}\frac{\partial}{\partial \tilde{x}_k}\right) \equiv \frac{-i}{\sqrt{2}}D_i, \tag{2.70}$$

where we define $\tilde{D}^i \equiv G^{ij}\tilde{D}_j$ and $D^i \equiv G^{ij}D_j$.
Using the definitions of D_i, $\tilde{D}_i$ and E_{ij} we can reobtain the level-matching condition:

$$\begin{aligned} 0 = L_0 - \tilde{L}_0 \quad &= \quad N - \tilde{N} - \frac{1}{4}(D_i G^{ij} D_j - \tilde{D}_i G^{ij}\tilde{D}_j) \\ &= \quad N - \tilde{N} + \partial_i \tilde{\partial}^i. \end{aligned} \tag{2.71}$$

The usual massless states ($N = \tilde{N} = 1$) are e_{ij} with a symmetric and antisymmetric part (i.e. graviton and Kalb-Ramond field) and a scalar d. These must satisfy the level matching condition:

$$\partial_k \tilde{\partial}^k e_{ij}(\tilde{x}, x) = \partial_k \tilde{\partial}^k d(\tilde{x}, x) = 0. \tag{2.72}$$

The level-matching condition acting on the fields will be called the weak constraint in the DFT language. And when the constraint is allowed to act on any product of fields it will be called the strong constraint.

2.1.6 $O(n, n, \mathbb{Z})$ transformations: T-duality group

The string when compactified on a circle gives rise to the T-duality symmetry. In a toroidal compactification with constant backgrounds the T-duality group gets enlarged to the $O(n, n, \mathbb{Z})$ group. This is what we will discuss at next.

The level-matching condition (2.71), in terms of p and w is $N - \tilde{N} = p_i w^i$. By defining a vector v of $2D$ components and integer entries as:

$$v = \begin{pmatrix} w^i \\ p_i \end{pmatrix}, \tag{2.73}$$

the level-matching condition reads:

$$N - \tilde{N} = \frac{1}{2}v^t \eta v \quad \text{with} \quad \eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{2.74}$$

The Hamiltonian of the theory is given by:

$$H = \int_0^{2\pi} d\sigma \hat{H} = \frac{1}{2} v^t \mathcal{H}(E) v + N + \tilde{N} + \cdots \tag{2.75}$$

where $\hat{H}$ is the Hamiltonian density and the dots represents terms which are not relevant to our $O(n, n, \mathbb{Z})$ discussion. The matrix $\mathcal{H}$ is given by:

$$\mathcal{H}(E) = \begin{pmatrix} G - BG^{-1}B & BG^{-1} \\ -G^{-1}B & G^{-1} \end{pmatrix}, \tag{2.76}$$

where $E = G+B$. In the DFT language, this matrix will be called the generalized metric. Its inverse is equal to

$$\eta \mathcal{H} \eta = \mathcal{H}^{-1}. \tag{2.77}$$

The mass operator is again defined as the one measured by a spectator living in the lower dimensional non-compact dimensions

$$M^2 = (w^a \; p_a) \begin{pmatrix} (G - BG^{-1}B)_{ab} & (BG^{-1})_a{}^b \\ -(G^{-1}B)^a{}_b & (G^{-1})^{ab} \end{pmatrix} \begin{pmatrix} w^b \\ p_b \end{pmatrix} + 4(N + \tilde{N} - 2) \tag{2.78}$$

We need to require that the physics of the system does not change. If we require that the theory is invariant under the transformation

$$v = O^t v', \tag{2.79}$$

where O is an invertible integer-valued matrix we have:

$$v^t \eta v = v'^t O \eta O^t v' \quad \Longrightarrow \quad \eta = O \eta O^t. \tag{2.80}$$

In other words, the matrices O generate the group $O(D, D, \mathbb{Z})$ and they are the ones that leave the theory invariant. Strictly speaking, $O(n, n, \mathbb{Z})$ is the physical T-duality group, but we have formally extended to $O(D, D, \mathbb{Z})$. In particular, the Hamiltonian must remain invariant. This requires that there must be a change in the background fields $E \to E'$. From (2.75) we demand that:

$$v'^t \mathcal{H}(E') v' = v^t \mathcal{H}(E) v = v'^t O \mathcal{H}(E) O^t v' \quad \Longrightarrow \quad \mathcal{H}(E') = O \mathcal{H}(E) O^t. \tag{2.81}$$

Remarkably, in terms of E the transformation on the background is given by

$$E' = (aE + b)(cE + d)^{-1}, \tag{2.82}$$

where a, b, c, d, are $D \times D$ blocks of O:

$$O = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in O(D, D, \mathbb{Z}). \tag{2.83}$$

Here E represents a fixed background, but in the original construction of DFT, a dynamical background $\mathcal{E}$ was used. We will comment more about this in the next chapter. As we mentioned in the introduction, DFT is based on the group $O(D, D, \mathbb{R})$ instead of $O(D, D, \mathbb{Z})$. The restricted DFT we will consider makes contact with the standard NS-supergravity to be introduced in the next section. It is known that the classical group that leaves the NS-action invariant when performing toroidal reductions is the continuous $O(n, n, \mathbb{R})$ group. In this sense, it is natural to expect that the restricted DFT has this continuous symmetry instead of the discrete one.

2.2 Supergravity as a low energy limit

The non-linear sigma model describing a string propagating in a general background is described by:

$$S = -\frac{1}{4\pi\alpha'} \int d\sigma d\tau \Big(\sqrt{h} h^{\alpha\beta} \partial_\alpha X^i \partial_\beta X^j g_{ij}(X) + \epsilon^{\alpha\beta} \partial_\alpha X^i \partial_\beta X^j b_{ij}(X) + \alpha' \sqrt{h} \phi(X) \bar{R}(h) \Big), \tag{2.84}$$

Here, $\bar{R}$ is the Ricci scalar of the world-sheet. Notice that the background fields involved (i.e. the metric g_{ij}, Kalb-Ramond b_{ij} and dilaton ϕ) are the massless fields appearing in the closed string spectrum[4] (see Subsection (2.1.3)). The second term represents the electric coupling of the string to the Kalb-Ramond 2-form, meaning that the string is charged under this field.

It is essential that the sigma model is locally scale invariant. The consistency of the quantum system requires that conformal invariance should be preserved (to decouple the ghost from the spectrum). Classically, the last term breaks the Weyl-symmetry. To preserve the conformal symmetry at the quantum level it is required that the trace of the world-sheet energy-momentum tensor vanishes. Schematically, in the conformal gauge the trace takes the form:

$$\langle T^\alpha{}_\alpha \rangle = \beta^\phi \sqrt{h} \bar{R} + \frac{1}{\alpha'} \beta^g_{ij} \sqrt{h} h^{\alpha\beta} \partial_\alpha X^i \partial_\beta X^j + \frac{1}{\alpha'} \beta^b_{ij} \epsilon^{\alpha\beta} \partial_\alpha X^i \partial_\beta X^j, \tag{2.85}$$

[4]Strictly speaking, the identification of the background fields with the string modes is ambiguous and usually field redefinitions are involved.

where β^{ϕ}, β^{g}_{ij} and β^{b}_{ij} are local functionals of the $g_{ij}(X)$, $b_{ij}(X)$ and $\phi(X)$. In this non-linear sigma model, the background fields can be interpreted as (local) coupling functions of the system, thus, we can study the β-functions of the model by taking a Taylor expansion of the coupling functions around a classical solution. In fact, the β-functions are precisely β^{ϕ}, β^{g}_{ij} and β^{b}_{ij}. Thus, the theory is conformally invariant only when these β-functions vanish. The classical symmetry breakdown given by the coupling ϕ is compensated by a one-loop α'-order correction in g_{ij} and b_{ij}. To lowest order in α' and in the string coupling g_s and when the dimension of space-time is equal to $D = 26$ (or $D = 10$ in the superstring case) the β-functions are of the form [108]:

$$\beta^{g}_{ij} = \alpha'\Big(R_{ij} + 2\nabla_i\nabla_j\phi - \frac{1}{4}H_{ikl}H_j{}^{kl}\Big) + \mathcal{O}(\alpha'^2), \tag{2.86}$$

$$\beta^{b}_{ij} = \alpha'\Big(\frac{1}{2}e^{2\phi}\nabla^k(e^{-2\phi}H_{kij})) + \mathcal{O}(\alpha'^2), \tag{2.87}$$

$$\beta^{\phi} = -\frac{\alpha'}{2}\Big(\nabla^2\phi - \partial_i\phi\partial^i\phi + \frac{1}{4}R - \frac{1}{48}H^2\Big) + \mathcal{O}(\alpha'^2), \tag{2.88}$$

Here R_{ij} is the space-time Ricci scalar. ∇ is the usual covariant derivative with Levi-civita connection and H is the field strength of b given by $H_{ijk} = 3\partial_{[i}b_{jk]}$. In the Taylor expansion it is assumed that the radius of curvature of space-time is much larger that ℓ_s, this means that we are taking a low-energy (long-distance) approximation. The low-energy approximation is no longer valid for a radius comparable to ℓ_s.

Remarkably, the vanishing of the β-functions can be viewed as the equations of motion for the background fields coming from the action:

$$S_{NS} = \frac{1}{2k_0^2}\int d^Dx\sqrt{-g}e^{-2\phi}\Big(R + 4(\partial\phi)^2 - \frac{1}{12}H^2\Big). \tag{2.89}$$

This is the low-energy effective action of the common sector (Neveu-Schwarz sector) of the closed oriented string theories. The overall factor of $e^{-2\phi}$ is related with a tree-level expansion in the string coupling parameter $g_s = e^{\phi_0}$ where ϕ_0 is the vacuum expectation value of ϕ. The metric g_{ij}[5] is known as the string-frame metric and by performing a conformal rescaling we can take it to the more familiar Einstein-frame metric:

$$g_{ij} = e^{\frac{4(\phi-\phi_0)}{D-2}}g_{Eij}. \tag{2.90}$$

[5] We are using the signature $(-,+,\cdots,+)$.

In the Einstein-frame the action turns out to be:

$$S_{(E)NS} = \frac{1}{2k^2}\int d^D x\sqrt{-g_E}\Big(R_E - \frac{4}{D-2}(\partial\phi)^2 - \frac{1}{12}e^{\frac{-8(\phi-\phi_0)}{D-2}}H^2\Big). \tag{2.91}$$

Here $k^2 = G_N/8\pi$ where G_N is the D-dimensional Newton's constant and the relation with k_0 is of the form $k^2 = k_0^2 e^{2\phi_0}$. This means that we can reabsorb the coefficient in front of the action, and since there is no potential to fix the vacuum expectation value, the constant is arbitrary, i.e. it is not fixed by the equations of motion.

The allowed backgrounds in which the string can propagate are then just solutions of the classical field equations. For instance, the string itself appears as a possible background solution to the equations of motion. It is known as the fundamental string (F1) solution. In the string frame the fundamental string solution extended in the z coordinate in $D = 10$ is given by:

$$ds^2 = -H^{-1}\Big(dt^2 - dz^2\Big) + \delta^{mn}dy_m dy_n, \tag{2.92}$$

$$e^{-2\phi} = e^{-2\phi_0}H_{F1}, \quad b_{tz} = \pm(H_{F1}^{-1} - 1), \quad H_{F1} = 1 + \frac{h_{F1}}{(\sqrt{\delta^{mn}y_m y_n})^6}. \tag{2.93}$$

Here $m = 1, \cdots, 8$ and δ^{mn} is the Kronecker delta. The harmonic function H_{F1} associated with this solution depends on the transverse coordinates y and h_{F1} is a constant. The $\pm$ sign in b has to do with the positive or negative charge of the string. Another important solution is the NS5-brane solution. Since the Kalb-Ramond is a 2-form, it is equally valid to consider the dual field which is a 6-form and we call it $D_{i_1,\cdots,i_6}$.[6] More on dualization will be said in the final chapter. This 6-form couples naturally to a 5-brane. So, it is expected that a 5-brane solution magnetically charged with respect to the Kalb-Ramond field exists. Indeed, this is the case. The NS5-brane solution in the string frame is given by:

$$ds^2 = -dt^2 + d\boldsymbol{x}_5^2 + H_{N5}\delta^{mn}dy_m dy_n, \tag{2.94}$$

$$e^{-2\phi} = e^{-2\phi_0}H_{N5}^{-1}, \quad 3\partial_{[m}b_{np]} = \pm\tilde{\epsilon}_{mnpq}\sqrt{-g}g^{qs}\partial_s\left(\ln H_{N5}\right), \tag{2.95}$$

$$H_{N5} = 1 + \frac{h_{N5}}{(\sqrt{\delta^{mn}y_m y_n})^2}. \tag{2.96}$$

[6] Depending on the context, we will sometimes denote the Kalb-Ramond field and the electric-magnetic dual as B_2 and B_6 respectively.

This time the transverse directions are $m = 1, \cdots, 4$. The coordinates t and $\boldsymbol{x}_5 = (x^1, \cdots, x^5)$ are the world-volume coordinates. The symbol $\tilde{\epsilon}_{mnpq}$ is the permutation symbol in four dimensions. We note that the Kalb-Ramond field takes non-trivial values along the spatial transverse directions. The NS5-brane is magnetically charged with respect to the Kalb-Ramond field.

In Chapter (5) we will use T-duality to construct new backgrounds. These backgrounds are all related by T-duality transformations, so they will be easily implemented in the DFT language. For instance, we can start with the smeared NS5-brane along some transverse direction (meaning that the harmonic function does not depend on some particular coordinate) and we can apply an $O(1,1)$ transformation to produce a new background called Kaluza-Klein monopole (KK5). This is a purely gravitational 5-brane solution. We can repeat the process to the KK5 and end up with another solution called the Q5-brane or 5^2_2 solution. This solution is said to be an exotic brane because it has non-trivial monodromy properties when one circles around the brane. We will comment more in Chapter (5). Similarly, the F1 solution gets mapped under T-duality to a solution known as PP-wave[7]. The idea of implementing T-duality is easily done in DFT and is equivalent to applying the so-called Busher's rules. These rules can be obtained by applying a Kaluza-Klein reduction to the NS-action. The procedure would be the following: Start with the NS-action (2.89) and assume the space-time is $M^{D-1} \times S^1$, with the circle S^1 parametrized with the coordinate $x \sim x + 2\pi R$. Assume all the fields are independent of coordinate x (in other words, we are only keeping the KK zero-mode in the Fourier decomposition of the fields). Then, propose the following reduction ansatz for the fields:

$$\hat{g}_{\hat{i}\hat{j}} = \begin{pmatrix} g_{ij} + k^2 A_i A_j & k^2 A_i \\ k^2 A_j & k^2 \end{pmatrix}, \tag{2.97}$$

$$\hat{\phi} = \phi + \frac{1}{2}\ln k, \tag{2.98}$$

$$\hat{b}_{ij} = b_{ij} - A_{[i}B_{j]}, \quad \hat{b}_{ix} = B_i. \tag{2.99}$$

The hatted quantities denote indices and fields referring to the higher-dimensional D space-time while un-hatted ones refer to the $(D-1)$-dimensional ones. After plugging this ansatz for the fields inside (2.89) we end up with a $(D-1)$-dimensional action with fields g_{ij}, b_{ij}, A_i, B_i, k and ϕ. This lower-dimensional

[7]There are several T-duality related solutions. For simplicity, in this thesis we will focus only in the ones related to the NS5-brane solution.

action has the following symmetry:

$$A_i \leftrightarrow B_i, \qquad k \to k^{-1}. \tag{2.100}$$

This is precisely the T-duality symmetry discussed in the previous sections. The Kaluza-Klein vector field A_i coming from the metric is electrically sourced by particles which come from states carrying momentum in the internal x direction. The vector B_i is known as the winding vector since it is sourced by particles which come from states carrying winding momentum. The interchange of $k \to k^{-1}$ is just the interchange of the radius of compactification. This symmetry implies that starting with a higher-dimensional action with a scalar $k' = k^{-1}$, KK vector $A'_i = B_i$ and winding vector $B'_i = A_i$, it will reduce to the same lower-dimensional action that comes from a higher-dimensional one with fields (k, A, B). The relation between the higher-dimensional fields under this prescription gives precisely the Buscher's rules [4, 5]:

$$\hat{g}'_{ij} = \hat{g}_{ij} - \frac{1}{\hat{g}_{xx}} \left(\hat{g}_{ix}\hat{g}_{jx} - \hat{b}_{ix}\hat{b}_{jx} \right) \,, \tag{2.101}$$

$$\hat{b}'_{ij} = \hat{b}_{ij} + \frac{2}{\hat{g}_{xx}} \hat{g}_{[i|x|}\hat{b}_{j]x} \,, \tag{2.102}$$

$$\hat{g}'_{ix} = \frac{\hat{b}_{ix}}{\hat{g}_{xx}} \qquad \hat{b}'_{ix} = \frac{\hat{g}_{ix}}{\hat{g}_{xx}} \qquad \hat{g}'_{xx} = \frac{1}{\hat{g}_{xx}} \,, \tag{2.103}$$

$$\hat{\phi}' = \hat{\phi} - \frac{1}{2} \ln \hat{g}_{xx} \quad . \tag{2.104}$$

2.3 Brief comment about the superstring

For completion, we would like to comment about the different superstrings, in particular the type II superstrings. It is worth stressing that this thesis mainly concerns with the (common) bosonic sector of the type II superstrings.

The bosonic string does not incorporate fermions and a tachyon appears in the spectrum. Both issues can be solved by considering supersymmetry. There are two formalisms to incorporate supersymmetry: The Ramond-Neveu-Schwarz (RNS) approach and the Green-Schwarz approach (GS). The RNS approach consists of introducing supersymmetry at the world-sheet level by coupling to the sigma-model 2-component Majorana spinors ψ^μ that transform as space-time vectors. The Green-Schwarz approach (GS) consists of introducing space-time fermions. Both formalisms are equally valid at least in $D = 10$ Minkowski space-time. We briefly comment the RNS approach following [101]. In the conformal gauge one can couple to the bosonic string model a 2-component Majorana spinor and the resulting action has global supersymmetry. The equations of motion allow for a mode expansion of the fermionic fields and one can use canonical quantization to obtain the spectrum. In the case of the open superstring, the boundary conditions allow for two sectors known as Ramond (R) and Neveu-Schwarz (NS). The former gives rise to space-time fermions while the later gives space-time bosons. In the case of the closed string it is possible to impose the Ramond or Neveu-Schwarz conditions on the right- and left-movers separately. The pairing gives rise to four different sectors: NS-NS, NS-R, R-NS and R-R. The NS-NS and R-R sectors contain space-time bosons and the others contain fermions. Using the Noether method, one can find the energy-momentum tensor associated with Poincaré translations and the supercurrent tensor associated with global supersymmetry. In the superstring we also have negative-norm states that should be decouple from the theory. One can then proceed as in the bosonic case, where there, a residual (conformal) symmetry would be used to fix a light-cone gauge. In this case, there is a superconformal symmetry that allows to fix the gauge. One then has to impose the vanishing of the super-Virasoro constraints, which are the vanishing of the components of the supercurrent and the energy-momentum tensor. These components must vanish in order to eliminate the negative-norm states. In the light-cone gauge, the physical excitations are obtained by acting on the ground states with trasverse creation modes of the bosonic and fermionic oscillators. The Lorentz invariance is maintained only when $D = 10$ and the normal ordering constants take the values $a_{NS} = \frac{1}{2}$, $a_R = 0$. Nevertheless, the spectrum contains a tachyon unless a truncation on the states of the theory is performed. This is achieved by introducing a so-called GSO-projector. This operator eliminates the tachyon of the theory and restores space-time supersymmetry. In the case of the closed superstring, two different theories can be obtained depending on whether the GSO-parity of the left- and right-movers of the R-sector is equal or not. This has to do with the fact that the GSO-projector depends on the chirality of the ground states of the R-sector. When the chirality of the ground states of the left- and right-movers of this sector coincides it gives rise to the 'type IIB superstring'. When the chirality of the ground states is opposite it gives rise to 'type IIA superstring'. The massless states of the type IIB superstring are given by:

$$
\begin{aligned}
\text{NS-NS}: &\quad h_{ij}\ \ \text{(graviton)},\ \ b_{ij}\ \ \text{(2-form)},\ \ d\ \ \text{(dilaton)},\\
\text{NS-R}+\text{R-NS}: &\quad \zeta_i^{\alpha}\ \ \text{(gravitinos)},\ \ \chi^{\alpha}\ \ \text{(dilatinos)},\\
\text{R-R}: &\quad C^{(0)}\ \ \text{(scalar)},\ \ C^{(2)}_{ij}\ \ \text{(2-form)},\ \ C^{(4)}_{ijkl}\ \ \text{(4-form)}.
\end{aligned}
\tag{2.105}
$$

Here ζ_i^{α} , $\alpha = 1, 2$ represents two gravitinos with the same chirality. χ^{α} represents two spin (1/2) fermions called dilatinos. The R-R fields of type IIB are a scalar $C^{(0)}$, a 2-form $C^{(2)}$ and a 4-form $C^{(4)}$ with self-dual field strength. In the 'type IIA superstring', the massless states in the NS-NS sector are the same as in type IIB. In the NS-R and R-NS sector the field content is also the same as before except that the gravitinos have opposite chirality. The R-R sector of type IIA, however, is different:

$$
\text{R-R}: \quad C^{(1)}_i\ \ \text{(1-form)},\ \ C^{(3)}_{ijk}\ \ \text{(3-form)}.
\tag{2.106}
$$

The R-R fields are a 1-form gauge potential $C^{(1)}$ and a 3-form $C^{(3)}$.

The type II supergravities are the low-energy effective descriptions of the type II superstring theories. The common sector is the NS-NS sector which contains the graviton, the Kalb-Ramond field and the dilaton. For simplicity, we are calling this sector the NS-sector. There are three more consistent superstring theories: Heterotic $SO(32)$ and $E_8 \times E_8$ and type I superstring. The Heterotic supergravities also share the same common sector plus vector supermultiplets with their respective gauge groups. The type II and Heterotic ones are theories of closed oriented strings. This means that the world-sheet is orientable. From type IIB, it is possible to obtain type I superstring by restricting to states that are invariant under a world-sheet parity reversal (orientation reversal). Thus, the type I is an unoriented theory and causes the elimination of the Kalb-Ramond field of the common sector.

2.4 Summary

In this chapter we reviewed the bosonic string, its mode expansion, and the spectrum of the theory. We showed that the spectrum consists of a graviton, an antisymmetric 2-form known as the Kalb-Ramond field in the supergravity language, and a scalar field called dilaton. These fields are the relevant ones that will appear in DFT. We then analyzed the bosonic string, when some directions are compact. We saw that the spectrum is invariant under the T-duality symmetry, which interchanges R with $1/R$, and also the winding and momentum modes. The theory is equally described in terms of dual coordinates $\tilde{X}$, which are conjugate to the winding. We saw that, when toroidal compactifications are performed, the T-duality gets enhanced to the $O(D, D, \mathbb{Z})$ group. We commented that the low energy limit of the bosonic string is described by a supergravity whose bosonic content is described by the massless state of the string, and we introduced the F1 and NS5 solutions. Finally, we briefly introduced the superstring, in particular, the type II superstring theories and their field contents. In the next chapter, we will introduce DFT, which treats x and $\tilde{x}$ on an equal footing in order to realize the T-duality group as a manifest symmetry.

3
Double Field Theory

3.1 Introduction

We saw in the previous chapter that T-duality is a symmetry of string theory. In fact, it is a symmetry of string field theory [109], which is the second-quantized version of string theory, contrary to the first-quantized version of the last chapter. String field theory treats momenta and winding on an equal footing. This implies that, when Fourier transforms the component fields to position space, they would depend on space-time coordinates conjugate to momentum and also on the coordinates conjugate to winding [16]. The component fields are defined on $M \times T^{2n}$, where M is $(D-n)$-dimensional Minkowski space-time, and T^{2n} is a doubled torus. In order to have a better understanding of how T-duality comes about at the space-time level, Double Field Theory was born [16–19]. Motivated by closed string field theory, the idea [16] is to build a field theory, which depends on both x (standard coordinates) and $\tilde{x}$ (coordinates conjugate to winding), thus doubling the dimension of the space-time and making T-duality a symmetry of the theory.[1]

Some important works on earlier versions of Double Field Theory (DFT) include that of Duff [10], Tseytlin [12, 13] and Siegel [14, 15]. In [10], a target

[1]In this thesis the dimension of the space-time should be regarded as $D = 26$, as in the bosonic string, or $D = 10$, as in the superstring. However, we will keep our results free from restrictions to a specific dimension, unless otherwise stated.

space with the double of coordinates was considered in order to capture a duality rotation between the equations of motion and the Bianchi identity for the usual coordinates. The full set of equations would be given an $O(n,n)$-covariant form in this extended target space but an $O(n,n)$ invariant sigma-model could not be found. In [12,13] a string world-sheet action in which the string coordinate x and its dual $\tilde{x}$ are treated on an equal footing was proposed. The T-duality would be realized as a symmetry of this world-sheet action. The price to paid was the lack of manifestly local Lorentz invariance. In [14, 15], a field theory for gravity and axion, using independent left and right vielbeins with local $GL(D)$ symmetry, was constructed in an $O(n,n)$ covariant way. Also, a new type of geometrical structure including covariant derivatives and a new Lie derivative were introduced which are closely related to the $O(D,D)$ geometrical structure of DFT. More recently, in [6] a doubled world-sheet action with manifest T-duality was considered. The torus fibers were doubled from T^n to T^{2n} and $2n$ local coordinates were defined. Although, the number of fibers coordinates was doubled, a self-duality constraint would be imposed in order to halve the degrees of freedom. To make contact with the conventional formulation, one needs to choose a splitting of T^{2n} into a physical T^n.

In this chapter, we will motivate the construction of DFT and introduce the basic properties. In Section (3.3), we will introduce the algebra of the gauge transformations of DFT and the C-bracket. In Section (3.4), we will introduce the generalized metric and the geometric form of the DFT action. We also introduce the concept of the generalized Lie derivative. In the last Section (3.5), we will introduce the frame fields of DFT and the Flux-form of the DFT action.

3.2 Preliminaries to Double Field Theory

We have seen that the physical fields must satisfy the level-matching condition:

$$L_0 - \tilde{L}_0 = N - \tilde{N} - p_a \omega^a = 0. \tag{3.1}$$

For simplicity, the theory restricts to fields with $M^2 = 0$, where M is the D-dimensional mass, and satisfy $N = \tilde{N} = 1$:

$$h_{ij}(x^\mu, x^a, \tilde{x}_a), \quad b_{ij}(x^\mu, x^a, \tilde{x}_a) \;\text{ and }\; d(x^\mu, x^a, \tilde{x}_a). \tag{3.2}$$

Here x^μ are the $(D-n)$-dimensional Minkowski coordinates and $(\tilde{x}_a, x^a)$ are the periodic coordinates for T^{2n} $(a = 1 \cdots n)$. The index $i = (\mu, a)$ ranges between $i = 0, \cdots, D-1$, so there is no doubling in the tensor indices. These fields are the graviton, Kalb-Ramond, and dilaton when there is no dependence on tilde coordinates. At the component level, the level matching condition is equivalent to:

$$p_a\omega^a|\Phi\rangle = 0 \to \partial_a\tilde{\partial}^a\Phi(x^\mu, x^a, \tilde{x}_a) = 0. \tag{3.3}$$

Where Φ is, for instance, any of the massless fields (3.2). The gauge parameters must also be annihilated by $\partial_a\tilde{\partial}^a$. We call this constraint the *weak constraint*. Using string field theory arguments, the first version of DFT was built in [16]. It is a theory with gauge invariance up to cubic order in fluctuations of the massless fields and at most two derivatives. Schematically[2], it is of the form:

$$S = \int dx d\tilde{x} \Big[\mathcal{L}^{(2)}\Big(D, \bar{D}, e_{ij}, d\Big) + \mathcal{L}^{(3)}\Big(D, \bar{D}, e_{ij}, d\Big)\Big]. \tag{3.4}$$

We can think of this action as representing the dynamics of fluctuations $e_{ij} = h_{ij} + b_{ij} + \mathcal{O}^2(h, b)$ and d around a constant background $E_{ij} = G^{(0)}_{ij} + B^{(0)}_{ij}$. Here $\bar{D}$ and D are derivatives that depend on the constant background E and were defined in (2.69) and (2.70). For simplicity, we leave aside the discussion of field redefinitions between the fields coming from string field theory and the supergravity fields. It can be proven that this action is T-duality invariant, in the sense of being invariant under the action of $O(D, D, \mathbb{Z})$ on the fields. In fact, $O(D, D, \mathbb{Z})$ can act on the fields *regardless* of isometric directions. It also possesses a discrete $\mathbb{Z}_2$ symmetry:

$$e_{ij} \to e_{ji}, \quad D \leftrightarrow \bar{D} \ \text{ and } \ d \to d. \tag{3.5}$$

This symmetry is a consequence of the orientation invariance of the closed string theory. At this stage, the dual coordinates ($\tilde{x}$) in Double Field Theory are needed to represent physical degrees of freedom and they are not an artifact of the theory. We should stress that the action requires the fields and gauge parameters to be constrained by $\partial_a\tilde{\partial}^a\Phi = 0$, where Φ represents the fields or gauge parameters. In general, products of fields do not satisfy this constraint but, in principle, this can be remedied by introducing projectors on products of fields. Products of fields appear for instance in the gauge transformations or even in the action, although up to cubic order they are not needed in the action. The problem is that the introduction of these projectors make the computations very cumbersome[3]. This idea changes with the application of the *strong constraint*.

[2] We do not display this action and the transformation properties of these fields because we are not going to use them. The reader can find them at [16].

[3] For a proposal of DFT using the weak constraint the reader can look at [110].

The construction of the theory at higher orders is rather involved (because of the projectors involved in the theory). To proceed then, a major simplification is achieved by restricting on a subsector of the full Double Field Theory, that is, to implement that all the fields (and gauge parameters) and arbitrary *products* of them should be annihilated by

$$\partial_i\tilde{\partial}^i(\cdots) = 0. \tag{3.6}$$

The dots represent any *arbitrary* product of fields and gauge parameters in contrast to the weak constraint[4]. The constraint (3.6) is known as the *strong constraint*. Strictly speaking, DFT should be interpreted as a theory living in $\mathbb{R}^{D-n-1,1}\times T^{2n}$ so only $O(n,n,\mathbb{Z})$ acts on the double torus, preserving the toroidal conditions on the coordinates. But the restricted DFT we are going to consider (i.e. DFT plus strong constraint) allows to formally extend $O(n,n,\mathbb{Z})$ to $O(D,D,\mathbb{R})$ acting on a formal $\mathbb{R}^{2D}$. The strong constraint is so restrictive that the following can be proven [18]: given a set of fields that satisfy the strong constraint, there exists an $O(D,D)$ frame with coordinates $(\tilde{x}_i,x^i)$ such that the fields depend only on x^i. In other words, upon using the strong constraint the theory is not truly doubled.

The strong constraint allows the gauge algebra to close (off-shell and to all orders) and to construct a background independent action to all orders in the fields [17, 18, 107]. The form of the action is:

$$\begin{aligned}S = \int dx d\tilde{x} e^{-2d}\Big[&-\frac{1}{4}g^{ik}g^{jl}\mathcal{D}^p\mathcal{E}_{kl}\mathcal{D}_p\mathcal{E}_{ij}+\frac{1}{4}g^{kl}(\mathcal{D}^j\mathcal{E}_{ik}\mathcal{D}^i\mathcal{E}_{jl}+\bar{\mathcal{D}}^j\mathcal{E}_{ki}\bar{\mathcal{D}}^i\mathcal{E}_{lj})\\ &+\mathcal{D}^i d\bar{\mathcal{D}}^j\mathcal{E}_{ij}+\bar{\mathcal{D}}^i d\mathcal{D}^j\mathcal{E}_{ji}+4\mathcal{D}^i d\mathcal{D}_i d\Big].\end{aligned} \tag{3.7}$$

Remarkably, it was shown that expanding this action up to cubic order would resemble the one in (3.4). The field $\mathcal{E}$ can be splitted in a symmetric and antisymmetric part $\mathcal{E}_{ij}=g_{ij}+b_{ij}$[5]. The derivatives now depend on the full background and are defined as $\mathcal{D}_i=\partial_i-\mathcal{E}_{ik}\tilde{\partial}^k$ and $\bar{\mathcal{D}}_i=\partial_i+\mathcal{E}_{ki}\tilde{\partial}^k$. Indices are raised with g^{ij}. As before, the action is invariant under the $\mathbb{Z}_2$ symmetry and under the gauge transformations:

$$\begin{aligned}\delta\mathcal{E}_{ij} =& \mathcal{D}_i\tilde{\xi}_j-\bar{\mathcal{D}}_j\tilde{\xi}_i+(\xi^i\partial_i+\tilde{\xi}_i\tilde{\partial}^i)\mathcal{E}_{ij}+\\ &+\mathcal{D}_i\xi^k\mathcal{E}_{kj}+\bar{\mathcal{D}}_j\xi^k\mathcal{E}_{ik},\end{aligned} \tag{3.8}$$

[4] For restricted fields, i.e. fields that depend on half of the coordinates, the products of them satisfy the weak constraint but the converse is not true.

[5] g_{ij} and b_{ij} are not infinitesimal here and when do not depend on $\tilde{x}$-coordinates they are the usual metric and Kalb-Ramond field of the NS-sector.

$$\delta d = -\frac{1}{2}(\partial_i \xi^i + \tilde{\partial}^i \tilde{\xi}_i) + (\xi^i \partial_i + \tilde{\xi}_i \tilde{\partial}^i) d. \tag{3.9}$$

The gauge parameters are $\tilde{\xi}_i$ and ξ^i, both depending on x and $\tilde{x}$. The action is invariant under $O(D,D)$ although is not written in a manifestly $O(D,D)$ form yet (we will introduce the manifest $O(D,D)$ action in Section (3.4)). Very often we will use the notation $\tilde{\partial}^i = 0$, meaning with this the particular case of fields not depending on $\tilde{x}_i$ (equivalently, we say 'when reducing to x-space'). Having said this, if we set $\tilde{\partial}^i = 0$ in (3.8) it reduces to

$$\delta \mathcal{E}_{ij} = \mathcal{L}_\xi \mathcal{E}_{ij} + 2\partial_{[i}\tilde{\xi}_{j]}, \tag{3.10}$$

which is the conventional standard form for diffeomorphisms with infinitesimal parameter ξ^i acting on g_{ij} and b_{ij} plus usual 2-form gauge transformation with gauge parameter $\tilde{\xi}_i$. The transformation of the quantity e^{-2d}, when $\tilde{\partial}^i = 0$, is equal to that of a scalar density. Moreover, if we assume that no field in the action (3.7) depends on $\tilde{x}$ coordinates, the resulting reduced action, up to a boundary term, turns out to be the usual NS-supergravity (2.89):

$$S_{NS} = \int dx \sqrt{-g} e^{-2\phi} \Big(R + 4(\partial\phi) - \frac{1}{12} H^2 \Big). \tag{3.11}$$

This is the action describing the fields of the Neveu-Schwarz (NS) sector of string theory in the string frame (see Section (2.2)). In obtaining this action, the DFT dilaton was redefined to match the usual scalar density appearing in the string frame:

$$\sqrt{-g} e^{-2\phi} = e^{-2d}. \tag{3.12}$$

3.3 Gauge algebra and C-bracket

It is desirable to introduce a more covariant $O(D,D)$ notation. We choose a basis for $O(D,D)$ such that the invariant metric takes an off-diagonal form:

$$\eta_{MN} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{3.13}$$

where 1 is the D-dimensional Kronecker-delta. We can encode coordinates and derivatives into $O(D,D)$ expressions:

$$X^M = \begin{pmatrix} \tilde{x}_i \\ x^i \end{pmatrix}, \quad \partial_M = \begin{pmatrix} \tilde{\partial}^i \\ \partial_i \end{pmatrix}. \tag{3.14}$$

The invariant metric η is convenient for raising and lowering $O(D,D)$ indices $M = 1, \cdots, 2D$. The strong constraint on the fields takes the form:

$$\partial_M \partial^M (\cdots). \tag{3.15}$$

Again, the dots represents any arbitrary product of fields and gauge parameters. Gauge parameters can be assembled together into an $O(D,D)$ vector

$$\xi^M = \begin{pmatrix} \tilde{\xi}_i \\ \xi^i \end{pmatrix}. \tag{3.16}$$

The action (3.7) is invariant under T-duality. More precisely, it is invariant under non-linear $O(D,D)$ tranformations

$$\mathcal{E}'(X') = (a\mathcal{E}(X) + b)(c\mathcal{E} + d)^{-1}, \tag{3.17}$$

$$d'(X') = d(X), \quad X' = OX, \tag{3.18}$$

where $O \in O(D,D)$ has the form:

$$O = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{3.19}$$

Note that (3.17) is the full dynamical version of (2.82). The matrices a, b, c, d are $D \times D$ blocks. The action on the coordinates is described as

$$X'^M = O^M{}_N X^N. \tag{3.20}$$

The closure of the gauge transformations is governed by the C-bracket which is given by[6]:

$$[\xi_1, \xi_2]^M_{(C)} = 2\xi^N_{[1} \partial_N \xi^M_{2]} - \xi_{Q[1} \partial^M \xi^Q_{2]}. \tag{3.21}$$

The gauge algebra with the C-bracket has already appeared in [14, 15] and it is related to the Courant bracket which has been prominent in the mathematics literature [20–23]. Selecting $\tilde{\partial}^i = 0$ in (3.21) reduces to the Courant bracket:

$$[\xi_1, \xi_2]_{(C)i} = 2\xi^j_{[1} \partial_j \tilde{\xi}_{2]i} - \left(\xi^j_{[1} \partial_i \tilde{\xi}_{2]j} - \tilde{\xi}_{[2j} \partial_i \xi^j_{1]} \right). \tag{3.22}$$

The Courant-bracket can be understood as a skew-symmetric bracket for sections defined on $(T \oplus T^\star)(M)$ where $T(M)$ and $T^\star(M)$ represent the tangent and cotangent bundle. The C-bracket reduces to the Courant-bracket for parameters

[6] We use the convention $[A, B] = \frac{1}{2}(AB - BA)$.

independent of $\tilde{x}$, thus the formal sum of the DFT gauge parameters $\tilde{\xi}$ and ξ are playing the role of a section in $(T \oplus T^{\star})(M)$. In this sense, we can think of the C-bracket as an $O(D,D)$ "covariantization" of the Courant-bracket. The Courant-bracket appears naturally in a theory with diffeomorphisms and b-field gauge transformations [107, 111]. When calculating the closure of the gauge algebra an ambiguity arises due to an exact 1-form that enters in the gauge parameter of b. More precisely

$$\delta b = \mathcal{L}_{\xi} b + d\tilde{\xi}, \tag{3.23}$$

is invariant under $\tilde{\xi} \to \tilde{\xi} + d\sigma$ and σ is a scalar. Here, $\mathcal{L}_{\xi}$ is the usual Lie derivative and d the exterior derivative. This ambiguity allows one to deform the bracket of the gauge algebra allowing to obtain the Courant-bracket[7]. The gauge symmetry is reducible, meaning that there are trivial gauge transformations that leave the gauge transformations invariant and they take the form:

$$\Sigma^M = \partial^M \chi. \tag{3.24}$$

There is an intimate relationship between this redundant gauge symmetry and the vanishing of the Jacobiator of the C-bracket when acting on fields. It is known that the Jacobiator of the Courant-bracket fails to vanish by a derivative of a quantity known as the Nijenhuis operator. In a similar way, the Jacobiator of the C-bracket fails to vanish. However, this does not present a problem for the realization of the symmetry algebra acting on fields. The reason being that the Jacobiator of the C-bracket fails to vanish up to a term of the form (3.24) so the symmetry algebra can be safely realized. DFT leads to a symmetry algebra that is not a Lie algebra and the reason might be that it has inherited part of the homotopy-Lie algebra structure of the string field theory [16].

3.4 Standard Double Field Theory action

It is possible to go even further with the $O(D,D)$ covariantization [19]. It is well known that when analyzing toroidal compactifications of string and supergravity theories an $O(D,D)$ symmetric matrix appears (as we have seen in Section (2.1.6)). It will be referred to as the generalized metric:

$$\mathcal{H}_{MN} = \begin{pmatrix} g^{ij} & -g^{ik}b_{kj} \\ b_{ik}g^{kj} & g_{ij} - b_{ik}g^{kl}b_{lj} \end{pmatrix}. \tag{3.25}$$

[7] In fact one obtains a κ-bracket and the Courant-bracket is the one with $\kappa = 1$. With κ different from 1 is not possible to find an $O(D,D)$ covariant extension of it [107].

Here g and b are the same as before $\mathcal{E}_{ij} = g_{ij} + b_{ij}$. The generalized metric satisfies the following properties

$$\mathcal{H}_{MN} = \mathcal{H}_{NM}, \quad \mathcal{H}^{MP}\mathcal{H}^{PN} = \delta^M{}_N, \tag{3.26}$$

where $\mathcal{H}^{MN} = \eta^{MP}\eta^{NQ}\mathcal{H}_{PQ}$ is the inverse of $\mathcal{H}_{MN}$. This implies that $\mathcal{H}$ is actually a (symmetric) $O(D,D)$ element. Without specifying a parametrization, $\mathcal{H}_{MN}$ can be thought of as a constrained field that satisfies that its inverse is equal to $\mathcal{H}^{MN} = \eta^{MP}\eta^{NQ}\mathcal{H}_{PQ}$. Since we are dealing with an $O(D,D)$ invariant theory it is natural to ask if this theory can be reformulated in terms of the generalized metric and generalized dilaton d. The answer turns out to be affirmative and the action (3.7), up to a boundary term, takes the manifestly $O(D,D)$ form[8]:

$$S_{DFT} = \int dX e^{-2d}\mathcal{R}, \tag{3.27}$$

$$\text{with} \;\; \mathcal{R} \equiv 4\mathcal{H}^{MN}\partial_M\partial_N d - \partial_M\partial_N\mathcal{H}^{MN} - 4\mathcal{H}^{MN}\partial_M d\partial_N d \tag{3.28}$$

$$+ 4\partial_M\mathcal{H}^{MN}\partial_N d + \frac{1}{8}\mathcal{H}^{MN}\partial_M\mathcal{H}^{KL}\partial_N\mathcal{H}_{KL} \tag{3.29}$$

$$- \frac{1}{2}\mathcal{H}^{MN}\partial_M\mathcal{H}^{KL}\partial_K\mathcal{H}_{NL}. \tag{3.30}$$

The quantity $\mathcal{R}$ is known as the 'generalized Ricci scalar' in analogy with the usual Ricci scalar of General Relativity and we will talk more about it in the next chapter. Note that in DFT we raise and lower indices with the invariant $O(D,D)$ metric η instead of the generalized metric $\mathcal{H}$. It is worth stressing that the non-linear $O(D,D)$ transformations acting on $\mathcal{E}$ (eq. (3.17)) translate into tensorial $O(D,D)$ transformations on $\mathcal{H}$

$$h^P{}_M h^Q{}_N \mathcal{H}'_{PQ}(X') = \mathcal{H}_{MN}(X), \tag{3.31}$$

$$X'^M = h^M{}_N X^N \;\; \text{with} \;\; h \in O(D,D). \tag{3.32}$$

The non-linear gauge transformations (3.8) take a simpler (linear) and geometric form in terms of $\mathcal{H}$:

$$\delta_\xi\mathcal{H}^{MN} = \xi^P\partial_P\mathcal{H}^{MN} + (\partial^M\xi_P - \partial_P\xi^M)\mathcal{H}^{PN} + (\partial^N\xi_P - \partial_P\xi^N)\mathcal{H}^{MP}. \tag{3.33}$$

For the dilaton we just get from (3.9)

$$\delta_\xi d = -\frac{1}{2}\partial_M\xi^M + \xi^M\partial_M d. \tag{3.34}$$

[8] In fact, it is not necessary to integrate by parts to get a manifestly $O(D,D)$ covariant action. It can be shown that each term of (3.7) can be described in terms of $\mathcal{H}$, d, η and ∂_M.

These gauge transformations in DFT can be interpreted as a new kind of Lie derivative, called 'generalized Lie derivative', acting on the fields of the theory. So, in a sense, DFT can be defined in a more geometric way by introducing curvatures and connections which transform with respect to the generalized Lie derivative. We will introduce this geometric formulation in the next chapter. The generalized Lie derivative can be defined on $O(D,D)$ arbitrary tensors as follows [19, 112–114]:

$$\mathcal{L}_\xi A_M = \xi^P \partial_P A_M + (\partial_M \xi^P - \partial^P \xi_M) A_P. \tag{3.35}$$

We see that the last term is unusual with respect to the standard Lie derivative acting on 1-forms. The invariant $O(D,D)$ metric enters non-trivially acting on the gauge parameter. In fact, it is important to note that $\mathcal{L}_\xi \eta_{MN} = 0$ along arbitrary vectors, which is not possible from the point of view of ordinary diffeomorphisms acting on a constant 2-tensor. The generalized Lie derivative also vanishes acting on the Kronecker delta $\delta_M{}^N$. The generalized Lie derivative satisfies the Leibniz rule so the product of generalized tensors is again a generalized tensor. This implies that

$$\mathcal{L}_\xi A^M = \eta^{MN} \mathcal{L}_\xi A_N. \tag{3.36}$$

The generalized Lie derivative vanishes when the gauge parameter is trivial $\xi^M = \partial^M \chi$. A scalar is defined in the usual way as $\mathcal{L}_\xi \Phi = \xi^M \partial_M \Phi$. The commutator of generalized Lie derivatives closes according to the C-bracket when the strong constraint holds:

$$[\mathcal{L}_{\xi_1}, \mathcal{L}_{\xi_2}] = \mathcal{L}_{-[\xi_1,\xi_2]_{(C)}}. \tag{3.37}$$

Given the terminology of generalized tensor, we can say that the exponential of the DFT dilaton and the generalized Ricci scalar transform as a scalar density and a scalar respectively:

$$\delta e^{-2d} = \partial_M (\xi^M e^{-2d}), \qquad \delta \mathcal{R} = \xi^M \partial_M \mathcal{R}. \tag{3.38}$$

From these transformations it is easy to see that the DFT action is gauge invariant. Although there exist finite generalized coordinate transformations defined in DFT, we will not comment on them. The reader can take a look at for example [113–118].

Now the DFT action should be regarded as a theory with fundamental fields $(\mathcal{H}, d)$ instead of $(\mathcal{E}_{ij}, d)$. We can derive the equations of motion in terms of $\mathcal{H}$ and d but we will leave this for the next chapter when we introduce a more geometric description of DFT.

3.5 Frame fields in Double Field Theory

Similar as in General Relativity, we can introduce frame fields instead of working with the generalized metric. In DFT the frame field formulation has been worked out in [15, 19, 53, 119]. The fundamental fields in the frame formulation of DFT are the generalized vielbein $E_A{}^M$ and the generalized dilaton d. The indices $M = 1, \cdots, 2D$ are interpreted as curved indices and $A = 1, \cdots, 2D$ are flat (tangent space) indices. The vielbein transforms from the right under global $G = O(D,D)$ transformations and has a local $H = O(D-1,1) \times O(D-1,1)$ action from the left:

$$E'_A{}^M(X') \;=\; O^M{}_N\, E_B{}^N(X)\, h_A{}^B(X)\;, \quad X'^M \;=\; O^M{}_N X^N\;, \tag{3.39}$$

where $O \in G$ and $h \in H$. The generalized vielbein transforms under generalized coordinate transformations like a generalized vector A^M. In the original frame formulation of DFT the subgroup $H = O(D-1,1) \times O(D-1,1)$ is embedded canonically, indicated by the index split of the doubled Lorentz indices $A = (a, \bar{a})$, $a, \bar{a} = 0, \ldots, D-1$, under which the flattened metric is assumed to be diagonal,

$$\mathcal{G}_{AB} \;\equiv\; E_A{}^M E_B{}^N \eta_{MN} \;\equiv\; 2\,\mathrm{diag}(-\eta_{ab}, \eta_{\bar{a}\bar{b}})\;, \tag{3.40}$$

where η_{ab} and $\eta_{\bar{a}\bar{b}}$ are two copies of the flat D-dimensional Lorentz metric diag$(-+\cdots+)$, and the relative sign between them is so that the overall signature is compatible with the (D,D) signature of η_{MN}. The generalized metric can be obtained by rotating with the generalized vielbein:

$$\mathcal{H}_{MN} = E^A{}_M E^B{}_N \mathcal{H}_{AB}, \tag{3.41}$$

where $\mathcal{H}_{AB} = 2\,\mathrm{diag}(\eta_{ab}, \eta_{\bar{a}\bar{b}})$ is the invariant H metric embedded canonically with respect to the indices $A = (a, \bar{a})$.

A different but equivalent form of the constraint is given by choosing the flattened metric so that it takes the same form as the $O(D,D)$ metric,

$$\eta_{AB} \;\equiv\; \mathcal{E}_A{}^M \mathcal{E}_B{}^N \eta_{MN} \;=\; \begin{pmatrix} 0 & \delta^a{}_b \\ \delta_a{}^b & 0 \end{pmatrix}, \tag{3.42}$$

where we denote the frame field by $\mathcal{E}_A{}^M$ to indicate that it satisfies a different constraint. Due to this constraint, $\mathcal{E}_A{}^M$ is a proper $O(D,D)$ group element. The flat indices split as $A = ({}^a, {}_a)$ and, therefore, in this formalism one has to carefully distinguish between upper and lower indices. The tangent space indices are raised and lowered with η_{AB} or $\mathcal{G}_{AB}$, depending on the formalism.

In the formalism based on (3.42), we define the $O(D-1,1) \times O(D-1,1)$ invariant metric

$$S_{AB} \equiv \begin{pmatrix} s^{ab} & 0 \\ 0 & s_{ab} \end{pmatrix}, \tag{3.43}$$

where s_{ab} and s^{ab} are again two copies of the flat D-dimensional Lorentz metric with signature $(-,+,\cdots,+)$, in terms of which the generalized metric can be written as

$$\mathcal{H}_{MN} = \mathcal{E}^A{}_M \, \mathcal{E}^B{}_N \, S_{AB} \, . \tag{3.44}$$

The different metrics in curved and flat spaces are summarized in Table (3.1). Since the generalized metric encodes the metric g and the b-field it should be possible to describe them in terms of the frame field. Indeed, a possible parameterization, leading to the standard form of the generalized metric is given by

$$\mathcal{E}^A{}_M = \begin{pmatrix} e_a{}^i & e_a{}^k b_{ki} \\ 0 & e^a{}_i \end{pmatrix}, \tag{3.45}$$

where $e^a{}_i$ is a D-dimensional vielbein of the metric $g_{ij} = e^a{}_i s_{ab} e^b{}_j$. Other parametrizations will be useful in the next chapter in the context of non-geometric fluxes [120–123]. We note that the constraint (3.42), and all differential identities that follow from it, are invariant under *local* G transformations (denoted $G_L \sim G$) acting on the vielbein from the left. That is, $h \in G_L$ satisfies $\eta_{AB} = h_A{}^C \eta_{CD} h_B{}^D$. However, the action and dynamical equations are only invariant under the subgroup $H \subset G_L$, i.e. under transformations satisfying in addition $S_{AB} = h_A{}^C S_{CD} h_B{}^D$. When compared with standard supergravity, one of the $O(1, D-1)$ factors of H reproduces the local D-dimensional Lorentz symmetry and the other is used to choose a triangular form for the vielbein as in (3.45).

The generalized vielbein allows us to rewrite DFT in terms of G-singlets only (this has been done in [14, 15, 19, 119]). For this purpose we define the flat derivative $\mathcal{D}_A = \mathcal{E}_A{}^M \partial_M$ and the *Weitzenböck* connection

$$\Omega_{ABC} = \mathcal{D}_A \mathcal{E}_B{}^M \mathcal{E}_{CM} = -\Omega_{ACB} \, , \tag{3.46}$$

where the antisymmetry follows from (3.42). With this object we define the 'generalized fluxes' [50–52]:

$$\mathcal{F}_{ABC} = 3\Omega_{[ABC]} \, , \tag{3.47}$$

$$\mathcal{F}_A = \Omega^B{}_{BA} + 2\mathcal{D}_A d \, , \tag{3.48}$$

	Gen. vielbein	Gen. metric	$O(D,D)$ metric
Curved space		$\mathcal{H}_{MN}$	$\eta_{MN} = \begin{pmatrix} 0 & \delta^i{}_j \\ \delta_i{}^j & 0 \end{pmatrix}$
Flat space	$E^A{}_M$	$\mathcal{H}_{AB} = 2\begin{pmatrix} \eta_{ab} & 0 \\ 0 & \eta_{\bar{a}\bar{b}} \end{pmatrix}$	$\mathcal{G}_{AB} = 2\begin{pmatrix} -\eta_{ab} & 0 \\ 0 & \eta_{\bar{a}\bar{b}} \end{pmatrix}$
Flat space	$\mathcal{E}^A{}_M$	$S_{AB} = \begin{pmatrix} s^{ab} & 0 \\ 0 & s_{ab} \end{pmatrix}$	$\eta_{AB} = \begin{pmatrix} 0 & \delta^a{}_b \\ \delta_a{}^b & 0 \end{pmatrix}$

Table 3.1
Comparison of the different metrics used in DFT. In order to switch between curved and flat indices one has to rotate the indices with the corresponding vielbein.

which will be further explained in the next chapter. They play an important role in DFT flux compactifications. The dynamics of the NS sector of DFT is described by an action that can be written in a compact form (up to total derivatives) in terms of a scalar function of the generalized vielbein and dilaton as

$$S = \int dX \; e^{-2d} \; \mathcal{R}(\mathcal{E}, d) \,, \tag{3.49}$$

where

$$\mathcal{R} = S^{AB}\left(2\mathcal{D}_A\mathcal{F}_B - \mathcal{F}_A\mathcal{F}_B\right) + \mathcal{F}_{ABC}\mathcal{F}_{DEF}\left[\frac{1}{4}S^{AD}\eta^{BE}\eta^{CF} - \frac{1}{12}S^{AD}S^{BE}S^{CF}\right]. \tag{3.50}$$

This action is equivalent to the standard DFT action when written in terms of $\mathcal{H}$ (and d), up to boundary terms. Here, the vielbein appears only through $\mathcal{D}_A$, $\mathcal{F}_{ABC}$ and $\mathcal{F}_A$. When the parameterization (3.45) is chosen, and the strong constraint is imposed in the global frame in which the dual coordinate dependence vanishes, this action reduces to the usual NS action of supergravity. When the theory is expressed in terms of these quantities we will refer to it as the 'Flux Formulation of Double Field Theory'. Under an infinitesimal G_L-transformation parameterized by $\Lambda_A{}^B$, with $\Lambda_{AB} = -\Lambda_{BA}$, the vielbein transforms as

$$\delta\mathcal{E}_A{}^M = \Lambda_A{}^B\mathcal{E}_B{}^M \ . \tag{3.51}$$

Referring to definitions (3.46), (3.47) and (3.48) we obtain the variations

$$\begin{aligned} \delta_\Lambda\Omega_{ABC} &= \mathcal{D}_A\Lambda_{BC} + \Lambda_A{}^D\Omega_{DBC} + \Lambda_B{}^D\Omega_{ADC} + \Lambda_C{}^D\Omega_{ABD} \ , && (3.52)\\ \delta_\Lambda\mathcal{F}_{ABC} &= 3\left(\mathcal{D}_{[A}\Lambda_{BC]} + \Lambda_{[A}{}^D\mathcal{F}_{BC]D}\right) \ , && (3.53)\\ \delta_\Lambda\mathcal{F}_A &= \mathcal{D}^B\Lambda_{BA} + \Lambda_A{}^B\mathcal{F}_B \ . && (3.54) \end{aligned}$$

For H-transformations, the parameters also satisfy $\Lambda_{\check{A}B} = \Lambda_{A\check{B}}$, where we introduced the notation

$$\Lambda_{\check{A}B} \equiv S_A{}^C\Lambda_{CB} \ . \tag{3.55}$$

The generalized fluxes satisfy Bianchi identities (BI). They can be modified in the presence of sources and when the strong constraint is not imposed as we will see in Chapter (4). These identities are:

$$\begin{aligned} \mathcal{D}_{[A}\mathcal{F}_{BCD]} - \frac{3}{4}\mathcal{F}_{[AB}{}^E\mathcal{F}_{CD]E} &= 0,\\ \mathcal{D}^C\mathcal{F}_{CAB} + 2\mathcal{D}_{[A}\mathcal{F}_{B]} - \mathcal{F}^C\mathcal{F}_{CAB} &= 0,\\ \mathcal{D}^A\mathcal{F}_A - \frac{1}{2}\mathcal{F}^A\mathcal{F}_A + \frac{1}{12}\mathcal{F}^{ABC}\mathcal{F}_{ABC} &= 0. \end{aligned} \tag{3.56}$$

They play a prominent role in constructing the Dual Double Field Theory action in Chapter (6).

3.6 Summary

In this chapter, we introduced Double Field Theory. We motivated the idea and introduced an action with the double of coordinates $(\tilde{x}, x)$ that is invariant under $O(D,D)$ transformations. A background-independent form of this action was possible due to the strong constraint (3.6). This constraint allows a closed gauge algebra to be derived. The closure of the algebra is governed by the C-bracket (3.21), which is a generalization of the Courant bracket introduced in generalized geometry. We introduced the generalized metric (3.25) and the geometric form of the DFT action in (3.27). This form of the action has manifest $O(D,D)$ covariance. In the last section, we introduced the frame formulation of DFT and explained the properties of the generalized vielbein (3.39). This allowed us to introduce the generalized fluxes (3.47) and (3.48), and we presented the Flux Formulation of DFT, which will be developed in the next chapter.

4 Fluxes and Geometry

4.1 Flux formulation of DFT

4.1.1 Introduction

The idea of this chapter is to expand more on the motivation for a Flux Formulation of DFT and to explore to what extent one can deal with the gauge consistency constraints without imposing the strong constraint. We will show that a geometric construction of DFT, using the Flux Formulation, naturally leads to a generalized Ricci scalar that contains strong constraint-violating terms. These terms were shown to be essential for making contact with gauged supergravity in lower dimensions, and hence the name Flux Formulation [50–52].

We have seen that DFT is usually supplemented *ad hoc* with a differential constraint on fields and gauge parameters, named strong constraint or sometimes called section condition. It effectively un-doubles the double coordinate dependence, and implies that locally DFT is a reformulation of supergravity. Given the coordinates of the double space X^M, $M = 1, \ldots, 2D$, and the corresponding derivatives $\partial_M = \partial/\partial X^M$, the constraint states that

$$\eta^{MN}\partial_M\partial_N \cdots = 0 \ , \qquad \eta^{MN} = \begin{pmatrix} 0 & \delta_i{}^j \\ \delta^i{}_j & 0 \end{pmatrix} \ , \tag{4.1}$$

where η^{MN} is the $O(D,D)$ invariant metric, $i, j = 1, \ldots, D$ and the dots stand for arbitrary (products of) fields and gauge parameters. For instance, generalized diffeomorphisms in the double-space then reduce to usual standard diffeomorphisms and two-form gauge transformations. In fact, gauge invariance and closure of the gauge algebra lead to a set of differential constraints that restrict the theory and, in particular, these constraints are satisfied when the strong constraint is enforced. We will obtain later in this chapter those constraints.

The first step towards a relaxation of the strong constraint was implemented in the Ramond-Ramond sector [58]. For the Neveu-Schwarz sector, it was shown in [50–52] that closure of the algebra of generalized diffeomorphisms and gauge invariance of the action of DFT give rise to a set of constraints that are not in one to one correspondence with the strong constraint. Although they imply that DFT is a restricted theory, solutions that violate the strong constraint are thus allowed.

Scherk-Schwarz (SS) compactifications [59] provide a scenario where fields and gauge parameters are restricted: given a background defined by a duality twist, the fields and gauge parameters must *accommodate* to it, and can no longer be generic. This means the following. When a SS reduction ansatz is proposed, the fields and gauge parameters acquire a particular dependence on the internal space which can be parametrized by a matrix called twist matrix. This twist matrix must satisfy certain properties in order to make the reduction possible [59]. The part of the fields that do not depend on the internal coordinates can be interpreted as perturbations around the (twisted) background and correspond to the dynamical degrees of freedom of the effective action, which is a gauged supergravity [97]. When the restricted fields are inserted into the consistency constraints of DFT, the duality twist generates gaugings (including the so-called non-geometric gaugings [49, 124, 125]) that arrange in the form of the quadratic constraints of gauged supergravities [50]. Then, under a SS reduction, the constraints of DFT are in one to one correspondence with the constraints of gauged supergravity. U-duality invariant scenarios exhibit the same behavior, [43, 126, 127]. The quadratic constraints were completely solved in some particular gauged supergravities in [128], where it was shown that the duality orbits of non-geometric fluxes are only generated through truly doubled duality twists.

We summarize the fields of the Flux Formulation:

- The fields of the theory, namely the generalized dilaton $d(X)$ and vielbein ${\mathcal{E}_A}^M(X)$, which turns flat indices $A, B, \ldots$ into curved ones $M, N, \ldots$, are arranged in "dynamical" fluxes defined as:

$$\mathcal{F}_{ABC} = 3\Omega_{[ABC]}\,, \tag{4.2}$$

$$\mathcal{F}_A = {\Omega^B}_{BA} + 2\mathcal{D}_A d\,, \tag{4.3}$$

where

$$\Omega_{ABC} = \mathcal{D}_A \mathcal{E}_B{}^N \mathcal{E}_{CN} \, , \tag{4.4}$$

and we have introduced a planar derivative $\mathcal{D}_A = \mathcal{E}_A{}^M \partial_M$. The fluxes $\mathcal{F}_{ABC}$ and $\mathcal{F}_A$ are thus non-constant in contrast to the usual fluxes. Therefore, we will often say they are dynamical. The different components of $\mathcal{F}_{ABC}$ correspond to the standard geometric (H_{abc} and $\tau_{ab}{}^c$) and non-geometric ($Q_a{}^{bc}$ and R^{abc}) fluxes upon compactification. More specifically, it was realized [50–52] that when they are constant and the indices refer to the internal group $O(6,6)$, they can be identified with the electric gauging parameters f_{ABC} and ξ_A, or *fluxes* entering the embedding tensor. This was done by comparing compactifications of DFT with $\mathcal{N} = D = 4$ gauged supergravity. Moreover, the different components of these dynamical fluxes correspond to covariant derivatives of scalars, curvature of the gauge fields, and other covariant combinations that appear in the effective action. This is similar to the constructions of [120–122, 129, 130], where ten-dimensional actions with their associated differential geometries were built in terms of field dependent quantities related to the non-geometric fluxes.

- Some consistency constraints take the form of generalized quadratic constraints, and involve the following Bianchi identities (BI) for the dynamical fluxes when the strong constraint is not assumed:

$$\begin{aligned} \mathcal{D}_{[A}\mathcal{F}_{BCD]} - \frac{3}{4}\mathcal{F}_{[AB}{}^E\mathcal{F}_{CD]E} &= \mathcal{Z}_{ABCD} \, , \\ \mathcal{D}^C\mathcal{F}_{CAB} + 2\mathcal{D}_{[A}\mathcal{F}_{B]} - \mathcal{F}^C\mathcal{F}_{CAB} &= \mathcal{Z}_{AB} \, , \\ \mathcal{D}^A\mathcal{F}_A - \frac{1}{2}\mathcal{F}^A\mathcal{F}_A + \frac{1}{12}\mathcal{F}^{ABC}\mathcal{F}_{ABC} &= \mathcal{Z}, \end{aligned} \tag{4.5}$$

where

$$\begin{aligned} \mathcal{Z}_{ABCD} &\equiv -\frac{3}{4}\Omega_{E[AB}\Omega^E{}_{CD]} \, , \\ \mathcal{Z}_{AB} &\equiv \left(\partial^M\partial_M\mathcal{E}_{[A}{}^N\right)\mathcal{E}_{B]N} - 2\Omega^C{}_{AB}\mathcal{D}_C d \, , \\ \mathcal{Z} &\equiv -2\mathcal{D}^A d\,\mathcal{D}_A d + 2\partial^M\partial_M d + \frac{1}{4}\Omega^{ABC}\Omega_{ABC} \end{aligned} \tag{4.6}$$

Upon SS compactifications, the constraints lead to the quadratic constraints for the constant electric bosonic gaugings of half-maximal gauged supergravity. $\mathcal{Z}_{ABCD}$, $\mathcal{Z}_{AB}$ and $\mathcal{Z}$ vanish under the strong constraint (4.1), but more generally the full set of constraints admits truly double configurations.

Let us emphasize that the strong constraint can be imposed on all of the results, which would then reduce to then known results in the literature[1].

We would like to stress that we do not assume a SS form of fields and gauge parameters: we simply list the consistency constraints of the theory that appear through the computations, and show that in particular they admit truly doubled solutions of the SS form. Other compactification scenarios might provide new solutions to the constraints. Interestingly, the expressions (4.5) appear all along the many computations in this chapter. They arise when analyzing closure of the gauge transformations, covariance of the generalized fluxes (which in turn implies gauge invariance of the action), invariance of the action under double Lorentz transformations, covariance of the generalized Riemann and Ricci tensors, and they also show up in the BI for the generalized Riemann tensor.

4.1.2 Double Field Theory in Flux Formulation

We have mentioned that $\mathcal{H}_{MN}$ satisfies the constraint

$$\mathcal{H}_{MP}\eta^{PQ}\mathcal{H}_{QN} = \eta_{MN} \ . \tag{4.7}$$

In particular, a possible parameterization is the following

$$\mathcal{H}_{MN} = \begin{pmatrix} g^{ij} & -g^{ik}B_{kj} \\ B_{ik}g^{kj} & g_{ij} - B_{ik}g^{kl}B_{lj} \end{pmatrix} , \tag{4.8}$$

but we will see later that other parametrizations more suitable in the context of non-geometric fluxes are more natural. Given these objects, an invariant action under the gauge and global transformations can be found,

$$S = \int dX e^{-2d}\mathcal{R}(\mathcal{H}, d) \ , \tag{4.9}$$

with

$$\begin{aligned} \mathcal{R} \ \equiv \ & 4\,\mathcal{H}^{MN}\partial_M\partial_N d - \partial_M\partial_N\mathcal{H}^{MN} + 4\partial_M\mathcal{H}^{MN}\,\partial_N d - 4\,\mathcal{H}^{MN}\partial_M d\,\partial_N d \\ & - \frac{1}{2}\mathcal{H}^{MN}\partial_M\mathcal{H}^{KL}\,\partial_K\mathcal{H}_{NL} + \frac{1}{8}\,\mathcal{H}^{MN}\partial_M\mathcal{H}^{KL}\,\partial_N\mathcal{H}_{KL} + \Delta_{SC}\mathcal{R}\,, \end{aligned} \tag{4.10}$$

[1]Besides (4.5) there are additional BI associated to the quadratic constraints of the maximal theory, which arise upon completing the NS-NS action with the Ramond-Ramond (RR) sector $\not\nabla\mathcal{G} = \mathcal{Z}_{RR}$ where $\mathcal{G}$ contains the information on RR forms, and $\not\nabla$ is a generalized Dirac operator. Interestingly, when analyzing the RR sector of the theory, $\mathcal{Z}$ will appear as part of the consistency constraints [53].

where $\Delta_{SC}\mathcal{R}$ stands for terms that vanish under (4.1) and were not included in [19]. This action reduces to the standard supergravity action for the NS-NS sector when $\mathcal{H}_{MN}$ is parameterized as in (4.8) and the strong constraint (4.1) is enforced in a frame in which $\tilde{\partial}^i = 0$[2].

In the frame formulation of DFT, the generalized metric is written in terms of a generalized vielbein $\mathcal{E}^A{}_M$ as $\mathcal{H}_{MN} = \mathcal{E}^A{}_M S_{AB} \mathcal{E}^B{}_N$. A possible parameterization, leading to (4.8) is given by (3.45). In this chapter, the indices in $H = O(D-1,1) \times O(D-1,1)$ are always raised and lowered with the flat counterpart of the G-metric

$$\eta_{AB} = \mathcal{E}_A{}^M \mathcal{E}_B{}^N \eta_{MN} = \begin{pmatrix} 0 & \delta^a{}_b \\ \delta_a{}^b & 0 \end{pmatrix} . \tag{4.11}$$

We recall that the last equality is verified by the parameterization (3.45), but for a generic doubled vielbein this gauge choice is a constraint forcing $\mathcal{E}_A{}^M$ to be an element of G itself. The additional degrees of freedom contained in the vielbein compared to those in $\mathcal{H}_{MN}$ are then un-physical due to the new local symmetry H. Throughout this chapter, we will generally not make use of any particular parameterization but rather consider the vielbein as a constrained field satisfying (4.11).

The idea in the Flux Formulation is to rewrite DFT in terms of the generalized fluxes (4.2), (4.3) and (4.4). After a straight-forward calculation, the dynamics of the NS sector of DFT is described by an action that can be written in a compact form (up to total derivatives) in terms of a scalar function of the generalized vielbein and dilaton as follows

$$S = \int dX \; e^{-2d} \; \mathcal{R}(\mathcal{E}, d) \,, \tag{4.12}$$

where

$$\begin{aligned} \mathcal{R} \;\; = \;\; & S^{AB} \left(2\mathcal{D}_A \mathcal{F}_B - \mathcal{F}_A \mathcal{F}_B\right) + \mathcal{F}_{ABC} \mathcal{F}_{DEF} \left[\frac{1}{4} S^{AD} \eta^{BE} \eta^{CF} - \frac{1}{12} S^{AD} S^{BE} S^{CF}\right] \\ & -2D^A \mathcal{F}_A + \mathcal{F}^A \mathcal{F}_A - \frac{1}{6} \mathcal{F}^{ABC} \mathcal{F}_{ABC} \,. \end{aligned} \tag{4.13}$$

The first line of this action is the same as (3.49) and the second line includes many strong constraint-violating terms, some of which were added *by hand* in [50–52], and of course were absent in the original formulation of DFT. For instance, a

[2] We recall that by $\tilde{\partial}^i = 0$ we mean a field configuration where every field is independent of $\tilde{x}$-coordinates.

term proportional to $\mathcal{F}^{ABC}\mathcal{F}_{ABC}$ should be added to the action (4.9) to recover the scalar potential of half-maximal gauged supergravity. When this term is non-vanishing, its effect is to add a piece to the dilaton potential, which is indispensable to reproduce duality orbits of non-geometric fluxes. After an integration by parts, the action (4.12) takes the form of the scalar potential of the bosonic electric sector of half-maximal gauged supergravity [131] when the fluxes are identified with the constant electric gaugings [50].

The second line in (4.13) identically vanishes under the strong constraint. These terms are covariant under the global and local symmetries. Here, we construct all the terms of the generalized Ricci scalar (4.13) systematically. We will do so closely following the guidelines of [15, 111, 112, 119, 132, 133]: we will introduce connections to covariantize the derivatives under the gauge symmetries of the theory and then impose a set of conditions on them, such as vanishing of the generalized torsion and compatibility with the dynamical degrees of freedom and the $O(D, D)$ metric. Although only some projections of the connection are determined, a notion of generalized Riemann tensor can be introduced which, upon traces and projections, leads to a fully determined generalized Ricci tensor (whose flatness determines the equations of motion) and the generalized Ricci scalar (that defines the action (4.12)). The procedure followed here does not assume a priori the strong constraint (this was also done in the U-duality case in [43], and also in a different geometric construction of DFT [134]). We find that the strong constraint-violating terms appearing in the generalized Ricci scalar (second line of (4.13)) are those introduced in [50–52] plus others that are needed to guarantee gauge invariance up to the consistency constraints.

Comparing (4.13) with (4.10) we see that the missing strong constraint-like terms read

$$\Delta_{SC}\mathcal{R} = \frac{1}{2}(S_{AB} - \eta_{AB})\partial_M \mathcal{E}^A{}_P \partial^M \mathcal{E}^B{}_Q \eta^{PQ} + 4\partial_M d \partial^M d - 4\partial_M \partial^M d \ . \tag{4.14}$$

The first line in (4.13) is invariant under a $\mathbb{Z}_2$ symmetry reproducing the $B \to -B$ symmetry of supergravity. This symmetry acts at the same time on the left and on the right of the vielbein by an $O(2D)$ transformation

$$\mathbb{Z} = \begin{pmatrix} \mathbb{I} & \\ & -\mathbb{I} \end{pmatrix} , \qquad \mathcal{E} \to \mathbb{Z}\mathcal{E}\mathbb{Z} \ . \tag{4.15}$$

Since $\mathbb{Z}\eta\mathbb{Z} = -\eta$, only terms involving an even number of contractions with η are invariant. The second line in (4.13) instead breaks the $\mathbb{Z}_2$ symmetry. It was shown in [50–52], based on the results of [135–137], that its presence forbids an embedding of the effective action of DFT into $\mathcal{N} = 8$ supergravity in four dimensions. In order to truncate $\mathcal{N} = 8 \to 4$ in four-dimensions, a $\mathbb{Z}_2$ symmetry is imposed, and only the invariant terms are kept. It is therefore to be expected that such a symmetry is related to the one mentioned here. Actually, let us mention that the quadratic constraints of gauged supergravities are automatically solved by the strong constraint (4.1). The second line in (4.13) can be recast as

$$\begin{aligned} \mathcal{Z} &= \mathcal{D}^A \mathcal{F}_A - \frac{1}{2}\mathcal{F}^A \mathcal{F}_A + \frac{1}{12}\mathcal{F}^{ABC}\mathcal{F}_{ABC} \\ &= -2\mathcal{D}^A d\, \mathcal{D}_A d + 2\partial^M \partial_M d + \frac{1}{4}\Omega^{ABC}\Omega_{ABC}, \end{aligned} \tag{4.16}$$

and written in this way, it is easy to see that it vanishes upon using the strong constraint.

4.1.3 Gauge symmetries and constraints

We have mentioned in the last chapter that under an infinitesimal G_L-transformation parameterized by ${\Lambda_A}^B$, with $\Lambda_{AB} = -\Lambda_{BA}$, the vielbein transforms as

$$\delta {\mathcal{E}_A}^M = {\Lambda_A}^B {\mathcal{E}_B}^M \ . \tag{4.17}$$

Then, up to boundary terms we find that the action transforms as

$$\delta_\Lambda S = \int dX e^{-2d} {\Lambda_A}^C (\eta^{AB} - S^{AB}) \mathcal{Z}_{BC} \, , \tag{4.18}$$

where

$$\begin{aligned} \mathcal{Z}_{AB} &= \mathcal{D}^C \mathcal{F}_{CAB} + 2\mathcal{D}_{[A}\mathcal{F}_{B]} - \mathcal{F}^C \mathcal{F}_{CAB} \\ &= \left(\partial^M \partial_M \mathcal{E}_{[A}{}^N\right) \mathcal{E}_{B]N} - 2{\Omega^C}_{AB}\mathcal{D}_C d \ . \end{aligned} \tag{4.19}$$

Notice that this vanishes under the strong constraint (4.1), but more generally H-invariance only requires the following minimal constraint

$$(\delta_{[A}{}^C - S_{[A}{}^C)\mathcal{Z}_{B]C} = 0 \, . \tag{4.20}$$

Here the S contribution comes from the first line in (4.13) and the η term from the second line. Notice that invariance of the full action requires this term to vanish, but if $\mathcal{Z}_{AB}$ is requested to vanish entirely as a constraint, then the action splits in two sectors (the first and second line in (4.13)) both being invariant under all the symmetries independently (up to $\mathcal{Z}_{AB} = 0$). This allows some freedom to fix the relative coefficient between both sectors, but we believe that this coefficient would be fixed as in (4.13) due to supersymmetry, since it is the one required to match half-maximal supergravity in four dimensions [50], [128].

Generalized diffeomorphisms are generated by infinitesimal parameters $\xi^M = \mathcal{E}_A{}^M \lambda^A$ in the fundamental representation of G that take the form

$$\begin{aligned} \delta_\xi d = & \xi^M \partial_M d - \frac{1}{2}\partial_M \xi^M \\ = & \frac{1}{2}\lambda^A \mathcal{F}_A - \frac{1}{2}\mathcal{D}_A \lambda^A \,, \\ \delta_\xi \mathcal{E}^A{}_M = & \xi^P \partial_P \mathcal{E}^A{}_M + \left(\partial_M \xi^P - \partial^P \xi_M\right) \mathcal{E}^A{}_P \\ = & \mathcal{E}_{BM}\left(2\mathcal{D}^{[B}\lambda^{A]} + \mathcal{F}^{AB}{}_C \lambda^C\right) . \end{aligned} \tag{4.21}$$

This further implies

$$\delta_\xi \mathcal{F}_{ABC} = \lambda^D \mathcal{D}_D \mathcal{F}_{ABC} + 4\mathcal{Z}_{ABCD}\lambda^D + 3\mathcal{D}_D \lambda_{[A}\Omega^D{}_{BC]} \,, \tag{4.22}$$

$$\begin{aligned} \delta_\xi \mathcal{F}_A = & \lambda^D \mathcal{D}_D \mathcal{F}_A + \mathcal{Z}_{AB}\lambda^B + \mathcal{F}^B \mathcal{D}_B \lambda_A \\ & - \mathcal{D}^B \mathcal{D}_B \lambda_A + \Omega^C{}_{AB}\mathcal{D}_C \lambda^B \,, \end{aligned} \tag{4.23}$$

where

$$\mathcal{Z}_{ABCD} = \mathcal{D}_{[A}\mathcal{F}_{BCD]} - \frac{3}{4}\mathcal{F}_{[AB}{}^E \mathcal{F}_{CD]E} = -\frac{3}{4}\Omega_{E[AB}\Omega^E{}_{CD]} \,, \tag{4.24}$$

and $\mathcal{Z}_{AB}$ was defined in (4.19). Again, the failure of $\mathcal{F}_{ABC}$ and $\mathcal{F}_A$ to transform as scalars implies that DFT is a restricted theory and can only be consistently defined for a subset of fields and gauge parameters that ensure gauge invariance and closure. The quantity (4.24) also vanishes if (4.1) is imposed, but demanding that $\mathcal{F}_{ABC}$ and $\mathcal{F}_A$ transform as scalars only requires a relaxed version of the strong constraint

$$\begin{aligned} 4\mathcal{Z}_{ABCD}\lambda^D + 3\mathcal{D}_D \lambda_{[A}\Omega^D{}_{BC]} &= 0 \,, \\ \mathcal{Z}_{AB}\lambda^B + \mathcal{F}^B \mathcal{D}_B \lambda_A - \mathcal{D}^B \mathcal{D}_B \lambda_A + \Omega^C{}_{AB}\mathcal{D}_C \lambda^B &= 0 \,. \end{aligned} \tag{4.25}$$

We will now show that both, invariance of the action under H-transformations (4.20) and generalized diffeomorphisms (4.25) follow from closure.

Consider a gauge transformation for a generic tensorial density V^M of weight $\omega(V)$

$$\delta_\xi V^M = \xi^P \partial_P V^M + \left(\partial^M \xi_P - \partial_P \xi^M\right) V^P + \omega(V)\partial_P \xi^P V^M \,, \tag{4.26}$$

the equations (4.21) are then recovered for $\omega(e^{-2d}) = 1$ and $\omega(\mathcal{E}) = 0$. These transformations define the so-called C-bracket

$$[\xi_1, \xi_2]_{\mathrm{C}}^M = \frac{1}{2}\left(\delta_{\xi_1}\xi_2 - \delta_{\xi_2}\xi_1\right)^M = 2\xi_{[1}^N \partial_N \xi_{2]}^M - \xi_{[1}^P \partial^M \xi_{2]P} = \mathcal{E}_A{}^M \left([\lambda_1, \lambda_2]_{\mathrm{C}}^A + \mathcal{F}_{BC}{}^A \lambda_1^B \lambda_2^C\right) . \quad (4.27)$$

Generically, the commutator of two transformations of an arbitrary vector V^M is not a transformation, but differs as

$$[\delta_{\xi_1}, \delta_{\xi_2}]\, V^M = \delta_{[\xi_1,\xi_2]_{\mathrm{C}}} V^M - F^M(\xi_1, \xi_2, V)\,, \quad (4.28)$$

where

$$F^M(\xi_1, \xi_2, V) = \xi_{[1}^Q \partial^P \xi_{2]Q} \partial_P V^M + 2\partial_P \xi_{[1Q} \partial^P \xi_{2]}^M V^Q + \omega(\xi_3) \xi_{[1}^Q \partial_P \partial^P \xi_{2]Q} V^M . \quad (4.29)$$

This indicates that the gauge transformation of a tensor is not automatically a tensor, and that the vanishing of its failure (denoted as Δ_ξ) must be imposed as a constraint

$$\Delta_{\xi_1} \delta_{\xi_2} V^M = 0\ . \quad (4.30)$$

The vanishing of F^M in (4.28) then follows from (4.30). We will refer to (4.30) as the *closure* constraints. Notice that in particular they imply

$$\begin{aligned} \Delta_{\xi_1} \mathcal{F}_{AB}{}^C &= \mathcal{E}^C{}_M \Delta_{\xi_1} \delta_{\mathcal{E}_A} \mathcal{E}_B{}^M = 0 \\ \Delta_{\xi_1} \mathcal{F}_A &= -e^{2d} \Delta_{\xi_1} \delta_{\mathcal{E}_A} e^{-2d} = 0 \end{aligned} \quad (4.31)$$

and then they guarantee that the dynamical fluxes transform as scalars under generalized diffeomorphisms, guaranteeing in turn the gauge invariance of the action, i.e. closure implies (4.25). Also, notice that due to closure

$$\mathcal{Z}_{ABCD} = \Delta_{E_A} \mathcal{F}_{BCD} = 0\ , \qquad \mathcal{Z}_{AB} = \Delta_{E_A} \mathcal{F}_B = 0 \quad (4.32)$$

and then H-invariance of the action (4.20) is also guaranteed by closure.

Summarizing, closure requires the imposition of constraints (4.30) that guarantee gauge invariance of the action, i.e. closure implies (4.20) and (4.25). There are further constraints arising from their gauge transformed. Since they are known to admit solutions beyond the strong constraint in the Neveu-Schwarz sector, let us now briefly review them [50–52]. In the next section we will deal with geometry, and new constraints will arise, which are also satisfied by these solutions.

Scherk-Schwarz solutions

All the constraints above are solved by restricting the fields as

$$\mathcal{E}_A{}^M(X) = \widehat{E}_A{}^I(x)U_I{}^M(Y) \ , \qquad d = \widehat{d}(x) + \lambda(Y) \ , \tag{4.33}$$

and the gauge parameters as

$$\xi^M(X) = \lambda^A(x)\widehat{E}_A{}^I(x)U_I{}^M(Y) \ . \tag{4.34}$$

Here we have used the following notation for the coordinate dependence $X = (\tilde{x}, \tilde{y}; x, y)$, $Y = (\tilde{y}, y)$. So, while the Y coordinates are double and play the roll of internal coordinates in a SS compactification, the x coordinates correspond to the un-doubled external space-time directions (the hats indicate dependence on x only). This ansatz satisfies all the constraints, when $U(Y)$, which is an element of $O(D, D)$ called duality twist matrix, is *constrained* to satisfy

- $(U_I{}^M - \delta_I{}^M)\partial_M \widehat{g} = 0$
- $f_{IJK} = 3\tilde{\Omega}_{[IJK]} = const.$, $\qquad \tilde{\Omega}_{IJK} = U_I{}^M \partial_M U_J{}^N U_{KN}$
- $f_I = \tilde{\Omega}^J{}_{JI} + 2U_I{}^M\partial_M\lambda = 0$
- the quadratic constraints of half-maximal supergravity [131]

$$f_{H[IJ}f^H{}_{KL]} = 0 \ . \tag{4.35}$$

Moreover, the first, third and fourth conditions can be further relaxed through the introduction of a warp factor in order to account for gaugings in the fundamental representation of $O(D, D)$, but here we introduce this ansatz for simplicity. It was shown in [128] that all the possible solutions to (4.35) can be reached by means of proper selections of duality twist matrices. Some solutions (the duality orbits of non-geometric fluxes) *require* truly double twist matrices, i.e. depending on both y and $\tilde{y}$ in such a way that the strong constraint is violated, and no T-duality can be performed to get rid of the dual coordinate dependence.

Of course, there might be other solutions to these constraints, perhaps associated to other kind of compactifications. Let us emphasize that this ansatz *contains* the usual decompactified strong constrained case. In fact, taking $U = 1$, $\lambda = 0$ and the coordinates x^i taking values $i = 1, \ldots, D$, one obtains the usual situation analyzed in the literature. From the point of view of this ansatz, this is just a particular limit in which all the compact directions are decompactified.

For these SS configurations all the consistency constraints are satisfied. In fact, it can be checked that

$$\mathcal{Z}_{ABCD} = 0 \ , \qquad \mathcal{Z}_{AB} = 0 \ , \tag{4.36}$$

and also relations of the form

$$\partial_M \lambda^A \partial^M \lambda^B = 0 \ , \quad \partial_M \partial^M \lambda^A = 0 \ , \quad \Omega^D{}_{AB} \mathcal{D}_D \lambda^C = 0 \ , \tag{4.37}$$

hold as well. Notice also that now the set of generalized diffeomorphisms has been reduced to a residual subgroup broken by the background. The SS ansatz can be thought of as a fixed background U, with perturbations $\widehat{E}$ around it, such that when this is plugged in the action and equations of motion one obtains an effective action for the perturbations. The compactification to four-dimensions was shown to reproduce the electric sector of half-maximal gauged supergravity.

Under a SS reduction, the dynamical fluxes become

$$\mathcal{F}_{ABC} = \widehat{F}_{ABC} + f_{IJK} \widehat{E}_A{}^I \widehat{E}_B{}^J \widehat{E}_C{}^K \ , \qquad \widehat{F}_{ABC} = 3 \widehat{\Omega}_{[ABC]} \ , \tag{4.38}$$

$$\mathcal{F}_A = \widehat{\Omega}^B{}_{BA} + 2 \widehat{E}_A{}^I \partial_I \widehat{d} \ , \tag{4.39}$$

where

$$\widehat{\Omega}_{ABC} = \widehat{E}_A{}^I \partial_I \widehat{E}_B{}^J \widehat{E}_{CJ} \ . \tag{4.40}$$

We see that these configurations are purely x-dependent, and all the truly double dependence has accommodated into the constant gaugings. This is in fact a generic feature of SS compactifications: covariant tensors with planar indices only depend on external coordinates.

We now continue without assuming this particular form of the fields and gauge parameters, but we stress that this ansatz also solves the forthcoming constraints in Section 4.2.

4.1.4 Equations of motion

The equations of motion of the DFT action (4.9) (without the terms we denoted $\Delta_{SC}\mathcal{R}$) were derived and analyzed in [18, 19, 138] (see [139] for an analysis of boundary terms). Here we obtain the equations of motion of the action (4.12).

The variations of the objects are given by

$$\delta_{\mathcal{E}} \Omega_{ABC} = \mathcal{D}_A \Delta_{BC} + \Delta_A{}^D \Omega_{DBC} + \Delta_B{}^D \Omega_{ADC} + \Delta_C{}^D \Omega_{ABD} \ , \tag{4.41}$$

$$\delta_{\mathcal{E}} \mathcal{F}_{ABC} = 3 \left(\mathcal{D}_{[A} \Delta_{BC]} + \Delta_{[A}{}^D \mathcal{F}_{BC]D} \right) \ , \tag{4.42}$$

$$\delta_{\mathcal{E}} \mathcal{F}_A = \mathcal{D}^B \Delta_{BA} + \Delta_A{}^B \mathcal{F}_B \ , \tag{4.43}$$

$$\delta_d \mathcal{F}_A = 2 \mathcal{D}_A \, \delta d \ , \tag{4.44}$$

and these in turn translate into variations of the action (4.12) given by

$$\delta_{\mathcal{E}} S = \int dX e^{-2d}\, \mathcal{G}^{AB} \Delta_{AB} \ , \tag{4.45}$$

$$\delta_d S = \int dX e^{-2d}\, \mathcal{G} \delta d \ , \tag{4.46}$$

where

$$\Delta_{AB} = \delta \mathcal{E}_A{}^M \mathcal{E}_{BM} = -\Delta_{BA} \ . \tag{4.47}$$

The antisymmetric property comes from the constraint $\mathcal{E}_A{}^M \mathcal{E}_{BM} = \eta_{AB}$. The equations of motion are then

$$\mathcal{G}^{[AB]} = 0 \ , \tag{4.48}$$

$$\mathcal{G} = 0 \ , \tag{4.49}$$

where

$$\begin{aligned} \mathcal{G}^{[AB]} &= 2(S^{D[A} - \eta^{D[A})\mathcal{D}^{B]}\mathcal{F}_D + (\mathcal{F}_D - \mathcal{D}_D)\breve{\mathcal{F}}^{D[AB]} + \breve{\mathcal{F}}^{CD[A}\mathcal{F}_{CD}{}^{B]} \\ &= \mathcal{Z}^{AB} + 2S^{D[A}\mathcal{D}^{B]}\mathcal{F}_D + (\mathcal{F}_D - \mathcal{D}_D)\breve{\mathcal{F}}^{D[AB]} + \breve{\mathcal{F}}^{CD[A}\mathcal{F}_{CD}{}^{B]} \ , \end{aligned} \tag{4.50}$$

$$\mathcal{G} = -2\mathcal{R} \, . \tag{4.51}$$

Here, we have introduced the notation

$$\breve{\mathcal{F}}^{ABC} = \breve{S}^{ABCDEF}\ \mathcal{F}_{DEF} \, , \qquad \check{\mathcal{F}}^{ABC} = \breve{\mathcal{F}}^{ABC} + \mathcal{F}^{ABC} \ , \tag{4.52}$$

where

$$\begin{aligned} \breve{S}^{ABCDEF} &= \frac{1}{2} S^{AD}\eta^{BE}\eta^{CF} + \frac{1}{2}\eta^{AD} S^{BE}\eta^{CF} + \frac{1}{2}\eta^{AD}\eta^{BE} S^{CF} \\ &\quad -\frac{1}{2} S^{AD} S^{BE} S^{CF} - \eta^{AD}\eta^{BE}\eta^{CF} \\ &= \check{S}^{ABCDEF} - \eta^{AD}\eta^{BE}\eta^{CF} \ . \end{aligned} \tag{4.53}$$

The operator $\breve{S}$ defines an involutive map $\breve{S}^2 = 1$, so $-\breve{S}/2$ is a projector.

In the next section, these equations of motion will be re-obtained from a generalized notion of Ricci flatness.

4.2 Geometry, connections and curvature

It was shown [15, 111, 112, 114, 119, 132, 133] that the action and equations of motion of DFT can be obtained from traces and projections of a generalized Riemann tensor. The construction goes beyond Riemannian geometry because it is based on the generalized rather than the standard Lie derivative. Then, the notions of connections, torsion and curvature have to be generalized and many interesting features arise in this framework. For example, it turns out that the vanishing of the torsion and the compatibility conditions do not completely determine the connections and curvatures but only fix some of their projections. The strong constraint was always assumed in these constructions. In this section we re-examine these generalized objects without imposing the strong constraint, but only assumed the closure constraints discussed in the previous section, plus new ones arising here. Our route will closely follow that of [112].

4.2.1 Generalized connections

We begin by defining a covariant derivative acting on tensors with curved and/or planar indices as

$$\nabla_M V_A{}^K = \partial_M V_A{}^K + \Gamma_{MN}{}^K V_A{}^N - \omega_{MA}{}^B V_B{}^K \ , \tag{4.54}$$

where $\Gamma_{MN}{}^K$ is a Christoffel connection, and $\omega_{MA}{}^B$ a spin connection. The forthcoming list of conditions are imposed to restrict these connections in a similar way as in Riemannian geometry. The list is ordered in such a way that each item assumes the previous ones.

- **Compatibility with the generalized frame**. Covariant constancy of $\mathcal{E}_A{}^N$

  $$\nabla_M \mathcal{E}_A{}^N = 0 \ , \tag{4.55}$$

  relates the Christoffel, spin and Weitzenböck connections

  $$\Gamma_{ML}{}^N = -\Omega_{ML}{}^N + \mathcal{E}^A{}_L \mathcal{E}_B{}^N \omega_{MA}{}^B \ . \tag{4.56}$$

  Since the Weitzenböck connection is fully determined by the generalized frame, this condition simply relates the Christoffel and spin connections.

- **Compatibility with the $O(D,D)$ invariant metric**. Given the covariant constancy of the generalized frame, covariant constancy of the metric η^{MN} can be equally cast as

  $$\nabla_M \eta^{NP} = 0 \quad \Longleftrightarrow \quad \nabla_M \eta^{AB} = 0 \, , \tag{4.57}$$

which in turn implies

$$\Gamma_{MNP} = -\Gamma_{MPN} \quad \Longleftrightarrow \quad \omega_{MAB} = -\omega_{MBA} \; . \tag{4.58}$$

- **Compatibility with the generalized metric**. Covariant constancy of the generalized metric

$$\nabla_M \mathcal{H}_{NK} = 0 \quad \Longleftrightarrow \quad \nabla_M S_{AB} = 0 \, , \tag{4.59}$$

implies that

$$\partial_M \mathcal{H}_{NK} - \Gamma_{MN}{}^P \mathcal{H}_{PK} - \Gamma_{MK}{}^P \mathcal{H}_{NP} = 0 \iff \omega_{M\check{A}\check{B}} = -\omega_{M\check{B}\check{A}} \; . \tag{4.60}$$

Here we used the check notation for indices contracted with the planar generalized metric (3.55).

- **Covariance under generalized diffeomorphisms**. The spin connection is requested to transform covariantly under generalized diffeomorphisms

$$\delta_\xi \omega_{AB}{}^C = \xi^P \partial_P \omega_{AB}{}^C \, . \tag{4.61}$$

Through vielbein compatibility we then have

$$\Delta_\xi \Gamma_{MNP} = -\Delta_\xi \Omega_{MNP} = 2\partial_M \partial_{[N}\xi_{P]} - \partial_Q \xi_M \Omega^Q{}_{NP} \, , \tag{4.62}$$

where we define Δ_ξ as the failure of an expression to transform covariantly.

- **Covariance under double Lorentz transformations**. Under local H transformations, we demand that $\nabla_M V_A{}^K$ transforms as a Lorentz vector. This implies that

$$\delta_\Lambda \Gamma_{MN}{}^K = 0 \; , \tag{4.63}$$

and

$$\delta_\Lambda \omega_{MA}{}^B = \partial_M \Lambda_A{}^B + \omega_{MC}{}^B \Lambda_A{}^C - \omega_{MA}{}^C \Lambda_C{}^B \; . \tag{4.64}$$

- **Vanishing generalized torsion**. The standard definition of torsion turns out to be non-covariant under generalized diffeomorphisms. Then, one has to resort to a generalized definition [111]

$$(\delta_\xi^\nabla - \delta_\xi) V^M = \mathcal{T}_{QP}{}^M \xi^Q V^P , \tag{4.65}$$

where V^M is a vector and δ^∇ is the generalized gauge transformation with ∂_M replaced by ∇_M. This definition yields

$$\mathcal{T}_{QP}{}^M = 2\Gamma_{[QP]}{}^M - \Gamma^M{}_{PQ} \, . \tag{4.66}$$

Combined with compatibility with the $O(D,D)$ metric, one finds that

$$\mathcal{T}_{MNK} = 3\Gamma_{[MNK]} \quad \Longleftrightarrow \quad \mathcal{T}_{ABC} = 3\omega_{[ABC]} - \mathcal{F}_{ABC} \,, \tag{4.67}$$

and then setting the torsion to zero, we obtain

$$\Gamma_{[MNK]} = 0 \quad \Longleftrightarrow \quad \mathcal{F}_{ABC} = 3\omega_{[ABC]} \,. \tag{4.68}$$

Note that this condition is consistent with the transformation properties of $\mathcal{F}_{ABC}$ under generalized diffeomorphisms provided the gauge consistency constraints hold. The antisymmetrization of the spin connection (which is requested to be covariant) coincides with the dynamical fluxes, which were also requested to be covariant. It then follows from the constraints that the generalized torsion is covariant as well.

- **Compatibility with the generalized dilaton**. Demanding partial integration in the presence of the dilaton measure e^{-2d}:

$$\int e^{-2d} W \nabla_M U^M = - \int e^{-2d} U^M \nabla_M W \,, \tag{4.69}$$

one finds

$$\Gamma_{PM}{}^P = -2\partial_M d \quad \Longleftrightarrow \quad \omega^B{}_{BA} = \mathcal{F}_A \,. \tag{4.70}$$

Again we find consistency in requiring that the spin connection is covariant, because its trace is related to the dynamical fluxes which are covariant as well.

It was shown that these constraints only determine some projections of the connections, leaving undetermined pieces which cannot be identified with the physical degrees of freedom. Still, some projections of a generalized Riemann tensor reproduce the action and equations of motion. In some cases [132,133] some further projections on the connection are requested to vanish in order to eliminate the undetermined part. However, in these cases the derivative is only covariant under particular projections and then full covariance is lost. In [134] the connection was chosen to be equal to the Weitzenböck connection, and then the spin connection vanishes. The advantage of the construction of [134] is that the connection is simple and determined but the torsion (4.67) is non-vanishing and equals the antisymmetric part of the Weitzenböck connection. This torsion carries the dynamics of the system and, interestingly, the strong constraint can be relaxed in this formulation as well.

By constructing a generalized Riemann tensor without assuming the strong constraint we will show that this provides a systematic way of obtaining the full action (4.12), and equations of motion (4.48).

Notice that due to the above requirements, the derivative of the spin connection is required to transform as a tensor under generalized diffeomorphisms

$$\Delta_\xi \partial_M \omega_{AB}{}^C = \partial^P \xi_M \partial_P \omega_{AB}{}^C = 0\,. \tag{4.71}$$

Moreover, due to (4.62) we have an additional constraint from covariance of the covariant derivative

$$\Delta_\xi \nabla_M V_N = \Delta_\xi \left[\partial_M V_N - \Gamma_{MNP} V^P\right] = 0\,, \tag{4.72}$$

which can be recast in the form

$$\partial_P \xi_M \partial^P V_N + \partial_P \xi_M \Omega^P{}_{NQ} V^Q = 0\,. \tag{4.73}$$

We now have new constraints, for the vectors, gauge parameters and connections, like (4.71) and (4.73), that arise by demanding that this geometric construction is consistent with a relaxation of the strong constraint. Notice that these constraints are not requested for consistency of the theory and only some projections of them are physical (due to the undetermined components of the connection). In any case, as strong as they look, they are all satisfied once again by the SS solutions of Subsection 4.1.3. In fact, as we explained in that subsection, in the SS scenario the covariant objects in planar indices only depend on the external coordinates, and then it is easy to see that (4.71) is satisfied in a SS reduction where the gauge parameters take the form (4.34). As for (4.73), notice that the strong constraint terms of the form $\Omega^Q{}_{MN}\Omega_{QRS}$ cancel, so it is also satisfied by the SS ansatz. Then, these new constraints are also solved by truly double SS reductions, but more generally might be solved by other truly double configurations.

4.2.2 Generalized curvature

The usual Riemann tensor in planar indices (i.e., rotated with the vielbein)

$$R_{ABC}{}^D = 2\left(\mathcal{D}_{[A}\omega_{B]C}{}^D - \Omega_{[AB]}{}^E \omega_{EC}{}^D - \omega_{[A|C}{}^E \omega_{|B]E}{}^D\right)\,, \tag{4.74}$$

is not a scalar under generalized diffeomorphisms (even if the strong constraint were imposed) because the Weitzenböck connection is not covariant. However, following the steps of [15, 111, 112, 114, 132, 133] one can extend this definition in order to covariantize it[3]. Consider for example the following modified curvature

$$\begin{aligned}\hat{R}_{ABCD} &= R_{ABCD} - \Omega^{E}{}_{AB}\omega_{ECD} \\ &= 2\mathcal{D}_{[A}\omega_{B]CD} - \mathcal{F}_{AB}{}^{E}\omega_{ECD} - 2\omega_{[A|C}{}^{E}\omega_{|B]ED} \ .\end{aligned} \tag{4.75}$$

An extra term is included in order to promote the Weitzenböck connection to a generalized flux, which is covariant. This expression is now a scalar under generalized diffeomorphisms. With the addition of the new term in (4.75), the G_L covariance has now been compromised. In order to restore it we further extend the definition as

$$\mathcal{R}_{ABCD} = \hat{R}_{ABCD} + \hat{R}_{CDAB} + \omega^{E}{}_{AB}\,\omega_{ECD} \ , \tag{4.76}$$

which is also a scalar under generalized diffeomorphisms. Of course, we are expecting that G_L or H invariance is achieved only up to strong constraint violating terms, because so is the action (4.18). A quick computation shows that

$$\Delta_\Lambda \mathcal{R}_{ABCD} = \mathcal{D}_E\Lambda_{BA}\Omega^{E}{}_{CD} + \mathcal{D}_E\Lambda_{DC}\Omega^{E}{}_{AB} \ , \tag{4.77}$$

so if one pretends a fully covariant Riemann tensor, this must be set to zero. In particular, under a SS reduction Λ_{AB} would depend on external coordinates only, and this constraint would be automatically satisfied.

Rotating all indices with the generalized vielbein, and using (4.56), the generalized Riemann tensor in curved indices can be cast in the form

$$\mathcal{R}_{MNKL} = \hat{\mathcal{R}}_{MNKL} - \Omega_{QMN}\Omega^{Q}{}_{KL} \ , \tag{4.78}$$

where

$$\begin{aligned}\hat{\mathcal{R}}_{MNKL} &= R_{MNKL} + R_{KLMN} + \Gamma_{QMN}\Gamma^{Q}{}_{KL} \ , \\ R_{MNKL} &= 2\partial_{[M}\Gamma_{N]KL} + 2\Gamma_{[M|QL}\Gamma_{|N]K}{}^{Q} \ .\end{aligned} \tag{4.79}$$

Here, $\hat{\mathcal{R}}_{MNKL}$ is the generalized Riemann tensor found in [112]. We see that the difference between (4.78) and (4.79) is a strong constraint-violating term which does not vanish with our assumptions of closure constraints. This extra factor was also considered in [43], where the first geometric construction with

[3] Imposing the vanishing of its failure to transform covariantly as a new constraint, is not an option. We are assuming that all the constraints of DFT are solved by $\tilde{\partial}^i = 0$, so that there is always a limit that makes contact with supergravity.

a relaxed strong constraint was built in the U-duality case. The generalized Riemann tensor (4.78) enjoys the same symmetry properties of the usual one, namely $\mathcal{R}_{MNKL} = \mathcal{R}_{([MN][KL])}$.

We now want to consider traces and projections of the generalized Riemann tensor to get a generalized Ricci tensor and scalar. For instance, imposing (4.68) and (4.70), we obtain

$$\mathcal{R}_{AB}{}^{AB} = -4\mathcal{Z} \ , \tag{4.80}$$

where $\mathcal{Z}$ was defined in (4.16). This vanishes upon using the strong constraint, but here it gives rise to some of the strong constraint-violating terms in the action. On the other hand, contractions with S (or $\mathcal{H}$) give the same answer

$$\mathcal{R}_{\breve{A}\breve{B}}{}^{AB} = -4\mathcal{Z} \ . \tag{4.81}$$

Thus, we are led to consider traces of the generalized Riemann tensor with mixed S^{AC} and η^{BD} contractions. After imposing conditions (4.58), (4.60), (4.68), (4.70), all the undetermined parts of the connection drop out from (4.76) and one gets

$$\mathcal{R}_{\breve{A}B}{}^{AB} = -2\mathcal{R} - 4\mathcal{Z} \ . \tag{4.82}$$

In order to combine these results we introduce the projectors

$$\begin{aligned} P_M{}^N &= \frac{1}{2}\left(\delta_M{}^N - \mathcal{H}_M{}^N\right) \quad \text{or} \quad P_A{}^B = \mathcal{E}_A{}^M \mathcal{E}^B{}_N P_M{}^N = \frac{1}{2}\left(\delta_A{}^B - S_A{}^B\right) , \\ \bar{P}_M{}^N &= \frac{1}{2}\left(\delta_M{}^N + \mathcal{H}_M{}^N\right) \quad \text{or} \quad \bar{P}_A{}^B = \mathcal{E}_A{}^M \mathcal{E}^B{}_N \bar{P}_M{}^N = \frac{1}{2}\left(\delta_A{}^B + S_A{}^B\right) . \end{aligned} \tag{4.83}$$

Using the results (4.80), (4.81) and (4.82) we see that the unique combination giving the full generalized Ricci scalar in terms of projectors is

$$\mathcal{R} = \frac{1}{4} P^{AC} P^{BD} \mathcal{R}_{ABCD} \ , \tag{4.84}$$

where $\mathcal{R}$ was defined in (4.13).[4]

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[4] Other combinations give

$$\bar{P}^{MK} P^{NL} \mathcal{R}_{MNKL} = P^{MK} \bar{P}^{NL} \mathcal{R}_{MNKL} = 0, \tag{4.85}$$

$$\bar{P}^{MK} \bar{P}^{NL} \mathcal{R}_{MNKL} = -4\mathcal{R} - 16\mathcal{Z}. \tag{4.86}$$

Note the difference between acting with PP and $\bar{P}\bar{P}$ on $\mathcal{R}_{MNKL}$ when the strong constraint is relaxed.

Also, the completely antisymmetric part of $\mathcal{R}_{ABCD}$ only involves the anti-symmetric parts of the connection. Imposing (4.68) and (4.70) again, we obtain from (4.76) an algebraic BI for the generalized Riemann tensor

$$\mathcal{R}_{[ABCD]} = \frac{4}{3}\mathcal{D}_{[A}\mathcal{F}_{BCD]} - \mathcal{F}_{[AB}{}^{E}\mathcal{F}_{CD]E} = \frac{4}{3}\mathcal{Z}_{ABCD} \, . \tag{4.87}$$

Identities like this, and many others are extensively discussed in [112].

4.2.3 Generalized Ricci flatness

The full action (4.12) can be written as

$$S = \frac{1}{4}\int dX \; e^{-2d} \; P^{MK} P^{NL} \mathcal{R}_{MNKL} \; , \tag{4.88}$$

and its variation with respect to the vielbein $\mathcal{E}$ gives

$$\delta_{\mathcal{E}} S = \frac{1}{4}\int dX \; e^{-2d} \; \Big(2(\delta_{\mathcal{E}} P^{MK}) P^{NL} \mathcal{R}_{MNKL} + P^{MK} P^{NL} \delta_{\mathcal{E}} \mathcal{R}_{MNKL} \Big) \, . \tag{4.89}$$

The projectors satisfy $P^2 = P$, $\bar{P}^2 = \bar{P}$, $P + \bar{P} = 1$ and $P\bar{P} = 0$, and we require that the shifted ones $P' = P + \delta_{\mathcal{E}} P$ (or $\bar{P}'$) also obey these relations. This implies that

$$\delta_{\mathcal{E}} P^{MK} = P^{M}{}_{R} \delta_{\mathcal{E}} P^{RL} \bar{P}^{K}{}_{L} + \bar{P}^{M}{}_{L} \delta_{\mathcal{E}} P^{LR} P_{R}{}^{K} \; . \tag{4.90}$$

Also, by definition we have

$$\delta_{\mathcal{E}} P^{RL} = -\frac{1}{2}\left(\delta \mathcal{E}_{A}{}^{R} S^{AB} \mathcal{E}_{B}{}^{L} + \mathcal{E}_{A}{}^{R} S^{AB} \delta \mathcal{E}_{B}{}^{L} \right) \; , \tag{4.91}$$

and inserting this information in the first term of (4.90) we find

$$2(\delta_{\mathcal{E}} P^{MK}) P^{NL} \mathcal{R}_{MNKL} = -4 \Delta_{AC} \; P^{BC} \bar{P}^{DA} P^{EF} \mathcal{R}_{BEDF} \, , \tag{4.92}$$

where we used (4.47). Recalling (4.78), the second term of (4.89) is

$$\begin{aligned} \int dX \; e^{-2d} \; P^{MK} P^{NL} \delta_{\mathcal{E}} \mathcal{R}_{MNKL} = & \\ = \int dX \; e^{-2d} \; P^{MK} P^{NL} \delta_{\mathcal{E}} (\hat{\mathcal{R}}_{MNKL} - \Omega_{QMN} \Omega^{Q}{}_{KL}) \; . & \end{aligned} \tag{4.93}$$

The infinitesimal variation of $\hat{\mathcal{R}}_{MNKL}$ with respect to $\mathcal{E}$ can be computed by first varying with respect to Γ

$$\delta_{\mathcal{E}} \hat{\mathcal{R}}_{MNKL} = 2 \nabla_{[M} \delta_{\mathcal{E}} \Gamma_{N]KL} + 2 \nabla_{[K} \delta_{\mathcal{E}} \Gamma_{L]MN} \; . \tag{4.94}$$

Inserting this variation into (4.93), the projectors pass through the covariant derivative (since $\nabla\eta = \nabla\mathcal{H} = 0$) and we get a total derivative, due to the dilaton compatibility condition. The second term of (4.93) gives

$$\begin{aligned}\int dX\ e^{-2d}\ P^{MK}P^{NL}\delta_{\mathcal{E}}(\Omega_{QMN}\Omega^{Q}{}_{KL}) &= \\ &= -2\int dX\ e^{-2d}\ \Delta_{AC}P^{AE}P^{CF}\mathcal{Z}_{EF}\ .\end{aligned} \tag{4.95}$$

Putting all together, we obtain

$$\begin{aligned}\delta_{\mathcal{E}}S &= \frac{1}{4}\int dX\ e^{-2d}\ \Delta_{AC}P^{BC}\bar{P}^{AD}(-4P^{EF}\mathcal{R}_{BEDF} - 2\mathcal{Z}_{BD}) \\ &= \int dX\ e^{-2d}\ \Delta_{AC}\ \mathcal{G}^{[AC]}\ .\end{aligned} \tag{4.96}$$

Then the equations of motion are

$$\mathcal{G}^{[AC]} = P^{B[A}\bar{P}^{C]D}\left(P^{EF}\mathcal{R}_{BEDF} + \frac{1}{2}\mathcal{Z}_{BD}\right) = 0\,, \tag{4.97}$$

which match those found in (4.48).

It might seem surprising at first sight that this form of generalized Ricci flatness is governed by an antisymmetric tensor. We recall however that there is a remarkable property of the projections with P and $\bar{P}$

$$\begin{array}{ccccc} P_M{}^R\bar{P}_N{}^S K_{RS} = 0 & \Rightarrow & P_{[M}{}^R\bar{P}_{N]}{}^S K_{RS} = 0 & \Rightarrow & P_Q{}^M P_{[M}{}^R\bar{P}_{N]}{}^S K_{RS} = 0 \\ \Uparrow & & \Updownarrow & & \Downarrow \\ P_Q{}^M P_{(M}{}^R\bar{P}_{N)}{}^S K_{RS} = 0 & \Leftarrow & P_{(M}{}^R\bar{P}_{N)}{}^S K_{RS} = 0 & \Leftarrow & P_M{}^R\bar{P}_N{}^S K_{RS} = 0 \end{array} \tag{4.98}$$

Namely, the symmetric and antisymmetric pieces contain the same information. Then, it is possible to define a *symmetric* generalized Ricci tensor, whose flatness gives the equations of motion as well

$$\mathcal{R}^{AC} = P^{B(A}\bar{P}^{C)D}\left(P^{EF}\mathcal{R}_{BEDF} + \frac{1}{2}\mathcal{Z}_{BD}\right) = 0\ . \tag{4.99}$$

4.3 Bianchi identities and Sources

In the previous sections we have identified three quantities (4.19), (4.24) and (4.16) that vanish upon using the strong constraint (4.1):

$$\mathcal{Z}_{ABCD} = -\frac{3}{4}\Omega_{E[AB}\Omega^{E}{}_{CD]}\,, \tag{4.100}$$

$$\mathcal{Z}_{AB} = \left(\partial^M\partial_M\mathcal{E}_{[A}{}^{N}\right)\mathcal{E}_{B]N} - 2\Omega^{C}{}_{AB}\mathcal{D}_C d\,, \tag{4.101}$$

$$\mathcal{Z} = -2\mathcal{D}^A d\,\mathcal{D}_A d + 2\partial^M\partial_M d + \frac{1}{4}\Omega^{ABC}\Omega_{ABC}\,. \tag{4.102}$$

They appeared when analyzing the symmetries, constraints and equations of motion. Interestingly, these quantities can be written purely in terms of fluxes and their derivatives. They lead to the following duality orbits of *generalized BI* for all the fluxes

$$\mathcal{D}_{[A}\mathcal{F}_{BCD]} - \frac{3}{4}\mathcal{F}_{[AB}{}^{E}\mathcal{F}_{CD]E} = \mathcal{Z}_{ABCD}\,, \tag{4.103}$$

$$\mathcal{D}^C\mathcal{F}_{CAB} + 2\mathcal{D}_{[A}\mathcal{F}_{B]} - \mathcal{F}^C\mathcal{F}_{CAB} = \mathcal{Z}_{AB}\,, \tag{4.104}$$

$$\mathcal{D}^A\mathcal{F}_A - \frac{1}{2}\mathcal{F}^A\mathcal{F}_A + \frac{1}{12}\mathcal{F}^{ABC}\mathcal{F}_{ABC} = \mathcal{Z}\,. \tag{4.105}$$

Before we analyze these Bianchi identities it is instructive to recall how the Bianchi identities are sourced. In the usual description of supergravity, magnetic sources appear as defects in the Bianchi identities of the field strengths of the theory. For instance, for an NS5-brane one has

$$dH = T_{NS5}\,\delta_4\ , \tag{4.106}$$

where δ_4 is a delta function four-form based on the brane's world-volume, with legs in the directions transverse to the world-volume. In this picture the three-form cannot be defined globally from the two-form gauge field. Adding a Lagrange multiplier six-form, the sourceless Bianchi identity follows as an equation of motion from

$$S = \int\left(-\frac{1}{2}\star H\wedge H - B_6\wedge dH\right)\ , \tag{4.107}$$

where the three-form is now treated as independent of B_2 and one has two first-order equations of motion. Adding to this action a Wess-Zumino coupling on the NS5-brane world-volume

$$S_{WZ} = T_{NS5}\int_{\mathcal{W}_6}\pi_{\mathcal{W}_6}(B_6) = T_{NS5}\int\delta_4\wedge B_6\ , \tag{4.108}$$

one precisely recovers the Bianchi identity for the three-form in presence of a NS5-brane, as the equation of motion of B_6. One can then integrate H out and express the dynamics in terms of B_6 solely[5].

Since $dH = 0$ is contained in our generalized Bianchi identities (as we will see when we split the indices) and since $dH \neq 0$ when a NS5-brane is present, the generalized Bianchi identities cannot hold as such when sources are present. We propose that a flux configuration in the presence of some extended objects satisfies

$$\mathcal{D}_{[A}\mathcal{F}_{BCD]} - \frac{3}{4}\mathcal{F}_{[AB}{}^{E}\mathcal{F}_{CD]E} = \mathcal{J}_{ABCD} \ , \tag{4.109}$$

$$\mathcal{D}^{C}\mathcal{F}_{CAB} + 2\mathcal{D}_{[A}\mathcal{F}_{B]} - \mathcal{F}^{C}\mathcal{F}_{CAB} = \mathcal{J}_{AB} \ , \tag{4.110}$$

$$\mathcal{D}^{A}\mathcal{F}_{A} - \frac{1}{2}\mathcal{F}^{A}\mathcal{F}_{A} + \frac{1}{12}\mathcal{F}^{ABC}\mathcal{F}_{ABC} = \mathcal{J} \ , \tag{4.111}$$

$$\tag{4.112}$$

where $\mathcal{J}_{\dots}$ represent currents for these (postulated) extended objects. We want to stress that, since the quantities $\mathcal{Z}_{\dots}$ enter the Bianchi identities on the same footing as the currents $\mathcal{J}_{\dots}$, it seems that one has *a-priori* the option to describe an extended object either by a source term $\mathcal{J}_{\dots} \neq 0$ or by a strong constraint-violating solution with $\mathcal{Z}_{\dots} \neq 0$. For non-vanishing currents, the fluxes cannot be given any longer in terms of the vielbein and dilaton.

4.3.1 Relation to standard fluxes

The fluxes $\mathcal{F}_{ABC}$ encode the standard T-dual fluxes. This can be seen by splitting the indices as

$$\mathcal{F}_{abc} = H_{abc} \ , \quad \mathcal{F}^{a}{}_{bc} = \tau_{bc}{}^{a} \ , \quad \mathcal{F}^{ab}{}_{c} = Q_{c}{}^{ab} \ , \quad \mathcal{F}^{abc} = R^{abc} \, . \tag{4.113}$$

Notice that being defined with planar indices these fluxes are T-duality invariant, but after a rotation with the generalized vielbein, they obey the usual T-duality chain

$$H_{ijk} \ \xleftrightarrow{T_k} \ \tau_{ij}{}^{k} \ \xleftrightarrow{T_j} \ Q_{i}{}^{jk} \ \xleftrightarrow{T_i} \ R^{ijk} \tag{4.114}$$

where T-dualities are defined by

$$(T_l)^{N}{}_{M} = \delta^{N}{}_{M} - \delta^{N,l}\delta_{M,l} - \delta^{N,l+D}\delta_{M,l+D} + \delta^{N,l+D}\delta_{M,l} + \delta^{N,l}\delta_{M,l+D} \, . \tag{4.115}$$

[5]This is due to the linear nature of this action. When non-linearities are present, for instance like the Chern-Simons term of eleven-dimensional supergravity, one can in general not get rid of the electric potential.

Splitting in components equation (4.100) we find

$$\begin{aligned}
\mathcal{D}_{[a}H_{bcd]} - \frac{3}{2}H_{e[ab}\tau_{cd]}{}^{e} &= \mathcal{Z}_{abcd} \ , \\
3\mathcal{D}_{[a}\tau_{bc]}{}^{d} - \mathcal{D}^{d}H_{abc} + 3\tau_{[ab}{}^{e}\tau_{c]e}{}^{d} - 3Q_{[a}{}^{de}H_{bc]e} &= \mathcal{Z}_{abc}{}^{d} \ , \\
2\mathcal{D}_{[a}Q_{b]}{}^{cd} + 2\mathcal{D}^{[c}\tau_{ab}{}^{d]} - \tau_{ab}{}^{e}Q_{e}{}^{cd} - H_{abe}R^{ecd} + 4Q_{[a}{}^{e[c}\tau_{b]e}{}^{d]} &= \mathcal{Z}_{ab}{}^{cd} \ , \\
3\mathcal{D}^{[a}Q_{d}{}^{bc]} - \mathcal{D}_{d}R^{abc} + 3Q_{e}{}^{[ab}Q_{d}{}^{c]e} - 3\tau_{de}{}^{[a}R^{bc]e} &= \mathcal{Z}^{abc}{}_{d} \ , \\
\mathcal{D}^{[a}R^{bcd]} - \frac{3}{2}R^{e[ab}Q_{e}{}^{cd]} &= \mathcal{Z}^{abcd} \ .
\end{aligned} \tag{4.116}$$

From equation (4.101) we get

$$\begin{aligned}
\mathcal{D}^{c}H_{abc} + \mathcal{D}_{c}\tau_{ab}{}^{c} + 2\mathcal{D}_{[a}\mathcal{F}_{b]} - \mathcal{F}^{c}H_{abc} - \mathcal{F}_{c}\tau_{ab}{}^{c} &= \mathcal{Z}_{ab} \ , \\
\mathcal{D}^{c}\tau_{ca}{}^{b} + \mathcal{D}_{c}Q_{a}{}^{bc} + \mathcal{D}_{a}\mathcal{F}^{b} - \mathcal{D}^{b}\mathcal{F}_{a} - \mathcal{F}^{c}\tau_{ca}{}^{b} - \mathcal{F}_{c}Q_{a}{}^{bc} &= \mathcal{Z}_{a}{}^{b} \ , \\
\mathcal{D}_{c}R^{abc} + \mathcal{D}^{c}Q_{c}{}^{ab} + 2\mathcal{D}^{[a}\mathcal{F}^{b]} - \mathcal{F}_{c}R^{abc} - \mathcal{F}^{c}Q_{c}{}^{ab} &= \mathcal{Z}^{ab} \ ,
\end{aligned} \tag{4.117}$$

and equation (4.102) reads in components

$$\mathcal{D}^{a}\mathcal{F}_{a} + \mathcal{D}_{a}\mathcal{F}^{a} - \mathcal{F}^{a}\mathcal{F}_{a} + \frac{1}{6}H_{abc}R^{abc} + \frac{1}{2}\tau_{ab}{}^{c}Q_{c}{}^{ab} = \mathcal{Z} \ . \tag{4.118}$$

It is possible to allow the vielbein to be parametrized in full generality as an $O(D,D)$ element, and we call it *extended-parameterization*

$$\mathcal{E}^{A}{}_{M} = \begin{pmatrix} e_{a}{}^{k} & e_{a}{}^{j}B_{jk} \\ e^{a}{}_{j}\beta^{jk} & e^{a}{}_{k} + e^{a}{}_{i}\beta^{ij}B_{jk} \end{pmatrix} , \tag{4.119}$$

in terms of a D-dimensional vielbein $e_{a}{}^{i}$, a two-form B_{ij} and an antisymmetric bi-vector β^{ij}. For this parameterization the generalized metric takes the form

$$\mathcal{H}_{MN} = \begin{pmatrix} g^{ij} - \beta^{im}g_{mn}\beta^{nj} & (g^{ik} - \beta^{im}g_{mn}\beta^{nk})B_{kj} - \beta^{im}g_{mj} \\ B_{ik}(\beta^{km}g_{mn}\beta^{nj} - g^{kj}) + g_{im}\beta^{mj} & \begin{array}{c} g_{ij} - B_{ik}(g^{kl} - \beta^{km}g_{mn}\beta^{nl})B_{lj} \\ + g_{im}\beta^{mn}B_{nj} + B_{im}\beta^{mn}g_{nj} \end{array} \end{pmatrix} . \tag{4.120}$$

With the extended parametrization, the generalized vielbein and generalized metric are over parametrized and some of the fields are unphysical. We will come back to this point later (see comment below (4.127)). The fluxes match those computed in [50–52], namely

$$
\begin{aligned}
\mathcal{F}_{abc} &= 3\left[\nabla_{[a}B_{bc]} - B_{d[a}\tilde{\nabla}^{d}B_{bc]}\right] , \\
\mathcal{F}_{ab}{}^{c} &= 2\Gamma_{[ab]}{}^{c} + \tilde{\nabla}^{c}B_{ab} + 2\Gamma^{mc}{}_{[a}B_{b]m} + \beta^{cm}\mathcal{F}_{mab} , \\
\mathcal{F}_{c}{}^{ab} &= 2\Gamma^{[ab]}{}_{c} + \partial_{c}\beta^{ab} + B_{cm}\tilde{\partial}^{m}\beta^{ab} + 2\mathcal{F}_{mc}{}^{[a}\beta^{b]m} - \mathcal{F}_{mnc}\beta^{ma}\beta^{nb} , \\
\mathcal{F}^{abc} &= 3\left[\beta^{[\underline{a}m}\nabla_{m}\beta^{\underline{bc}]} + \tilde{\nabla}^{[a}\beta^{bc]} + B_{mn}\tilde{\nabla}^{n}\beta^{[ab}\beta^{c]m} + \beta^{[\underline{a}m}\beta^{\underline{b}n}\tilde{\nabla}^{\underline{c}]}B_{mn}\right] + \\
&\quad + \beta^{am}\beta^{bn}\beta^{cl}\mathcal{F}_{mnl} ,
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{F}_{a} &= -\tilde{\nabla}^{c}B_{ac} + \Gamma^{cd}{}_{a}B_{dc} - \Gamma_{ca}{}^{c} + 2B_{ac}\tilde{\nabla}^{c}d + 2\nabla_{a}d , \\
\mathcal{F}^{a} &= -\Gamma^{ca}{}_{c} - \tilde{\nabla}^{d}\beta^{ac}B_{cd} - \Gamma^{da}{}_{e}\beta^{ec}B_{cd} - \beta^{ac}\tilde{\nabla}^{d}B_{cd} + 2\tilde{\nabla}^{a}d \\
&\quad +2\beta^{ac}B_{ce}\tilde{\nabla}^{e}d + 2\beta^{ac}\nabla_{c}d - \nabla_{c}\beta^{ac} + \Gamma_{cd}{}^{a}\beta^{dc} ,
\end{aligned}
$$

where we have used the following relations and definitions

$$
e_{a}{}^{i}e^{a}{}_{j} = \delta^{i}_{j} , \qquad e_{a}{}^{i}e^{b}{}_{i} = \delta^{b}_{a} , \quad B_{ab} = e_{a}{}^{i}e_{b}{}^{j}B_{ij} , \qquad \beta^{ab} = e^{a}{}_{i}e^{b}{}_{j}\beta^{ij} ,
$$

$$
\partial_{a} = e_{a}{}^{i}\partial_{i} , \qquad \tilde{\partial}^{a} = e^{a}{}_{i}\tilde{\partial}^{i} ,
$$

$$
\begin{aligned}
\nabla_{a}B_{bc} &= \partial_{a}B_{bc} - \Gamma_{ab}{}^{d}B_{dc} - \Gamma_{ac}{}^{d}B_{bd} , \quad \tilde{\nabla}^{a}B_{bc} = \tilde{\partial}^{a}B_{bc} + \Gamma^{ad}{}_{b}B_{dc} + \Gamma^{ad}{}_{c}B_{bd} , \\
\nabla_{a}\beta^{bc} &= \partial_{a}\beta^{bc} + \Gamma_{ad}{}^{b}\beta^{dc} + \Gamma_{ad}{}^{c}\beta^{bd} , \quad \tilde{\nabla}^{a}\beta^{bc} = \tilde{\partial}^{a}\beta^{bc} - \Gamma^{ab}{}_{d}\beta^{dc} - \Gamma^{ac}{}_{d}\beta^{bd} ,
\end{aligned}
$$

and

$$
\Gamma_{ab}{}^{c} = e_{a}{}^{i}\partial_{i}e_{b}{}^{j}e^{c}{}_{j} , \qquad \Gamma^{ab}{}_{c} = e^{a}{}_{i}\tilde{\partial}^{i}e^{b}{}_{j}e_{c}{}^{j} . \tag{4.121}
$$

After imposing the strong constraint and selecting the frame $\tilde{\partial}^{i} = 0$, the fluxes (4.121) agree with those obtained in [140, 141], namely

$$
\begin{aligned}
\mathcal{H}_{abc} &= 3\left[\partial_{[a}B_{bc]} + f_{[ab}{}^{d}B_{c]d}\right] \equiv 3\nabla_{[a}B_{bc]} , \\
\mathcal{F}_{ab}{}^{c} &= f_{ab}{}^{c} - \mathcal{H}_{abm}\beta^{mc} , \\
\mathcal{Q}_{c}{}^{ab} &= \partial_{c}\beta^{ab} + 2f_{cm}{}^{[a}\beta^{mb]} + \mathcal{H}_{cmn}\beta^{ma}\beta^{nb} , \\
\mathcal{R}^{abc} &= 3\left[\beta^{[am}\partial_{m}\beta^{bc]} + f_{mn}{}^{[a}\beta^{bm}\beta^{c]n}\right] - \mathcal{H}_{mnp}\beta^{ma}\beta^{nb}\beta^{pc} ,
\end{aligned}
\tag{4.122}
$$

where $f_{ab}{}^{c} = 2\Gamma_{[ab]}{}^{c}$. Applying the same restrictions on (4.116), the resulting equations exactly match the Bianchi identity derived in [140] (recall that the right hand sides of (4.116) vanish when the strong constraint is imposed).

The fluxes (4.122) were shown to be the coefficients of the following Roytenberg algebra:

$$
\begin{aligned}
[e_a, e_b] &= \mathcal{F}_{ab}{}^{c} e_c + \mathcal{H}_{abc} e^c \,, \\
\left[e_a, e^b\right] &= \mathcal{Q}_a{}^{bc} e_c - \mathcal{F}_{ac}{}^{b} e^c \,, \\
\left[e^a, e^b\right] &= \mathcal{Q}_c{}^{ab} e^c + \mathcal{R}^{abc} e_c \,,
\end{aligned} \tag{4.123}
$$

obtained as a Courant algebroid on basis sections $\{e_a, e^b\} \in TM \oplus T^*M$ in [140–142]. And they also determine the Jacobiators

$$
\begin{aligned}
\mathrm{Jac}(e_a, e_b, e_c) &= \frac{1}{2}\mathcal{D}\mathcal{H}_{abc} \,, \\
\mathrm{Jac}(e_a, e_b, e^c) &= \frac{1}{2}\mathcal{D}\mathcal{F}_{ab}{}^{c} \,, \\
\mathrm{Jac}(e_a, e^b, e^c) &= \frac{1}{2}\mathcal{D}\mathcal{Q}_a{}^{bc} \,, \\
\mathrm{Jac}(e^a, e^b, e^c) &= \frac{1}{2}\mathcal{D}\mathcal{R}^{abc} \,,
\end{aligned} \tag{4.124}
$$

with $\mathcal{D} = d^H + d^H_\beta$, d^H and d^H_β being the H-twisted de Rham and Poisson differentials respectively, which hold up to the Bianchi identity. (see [140] for details).

Here we notice that DFT provides a natural framework containing these structures covariantly. Indeed, a covariant expression encoding the algebra (4.123) follows from the C-bracket of generalized vielbeins:

$$
\left[\mathcal{E}_A{}^M, \mathcal{E}_B{}^N\right]^{(C)}_P = \mathcal{F}_{ABC}\mathcal{E}^C{}_P \,, \tag{4.125}
$$

and the cyclic sum of double C-brackets gives:

$$
\left[[\mathcal{E}_A{}^M, \mathcal{E}_B{}^N]^{(C)}, \mathcal{E}_C{}^P\right]^{(C)}_Q + \text{cyclic} = -4\mathcal{Z}_{ABCE}\mathcal{E}^E{}_Q + \frac{1}{2}\mathcal{D}_E\mathcal{F}_{ABC}\mathcal{E}^E{}_Q \,, \tag{4.126}
$$

precisely the covariant generalization of (4.124).

4.3.2 Geometric versus non-geometric frames

It is also interesting to note that when the strong constraint is imposed on the fields, (4.5) becomes the Bianchi identities of [49, 124, 125], [143, 144] for constant fluxes, and those of [140] for non-constant fluxes. They span T-duality orbits of

Bianchi identities, containing $\partial_{[i}H_{jkl]} = 0$ as a particular representative. These identities are known to be sourced by localized branes (see for example [145,146]), like the NS5-brane. More generally, we have here duality orbits of the Bianchi identities for non-geometric fluxes that can be related to more exotic T-fold-like objects with non-trivial monodromies, such as the Q5-brane also knows as 5^2_2 brane [75, 76, 147], or the R-branes [148], or a bound states of them, etc. The Q5-brane is said to be globally non-geometric (it is globally ill-defined) and the R5-brane is said to be locally non-geometric, meaning that a local supergravity description is lacking. We will discuss more about them in the next chapter. In this section we define two 'frames' in which geometric and non-geometric backgrounds are best described. As we mentioned before, the vielbein can be parametrized in terms of a D-dimensional vielbein $e_a{}^i$, a two-form B_{ij} and an antisymmetric bi-vector β^{ij}

$$\mathcal{E}^A{}_M = \begin{pmatrix} e_a{}^k & e_a{}^j B_{jk} \\ e^a{}_j \beta^{jk} & e^a{}_k + e^a{}_i \beta^{ij} B_{jk} \end{pmatrix} . \tag{4.127}$$

Given that the generalized vielbein belongs to the coset G/H, defined in this way it is over-parameterized. Only D^2 degrees of freedom are physical, while the remaining $D(D-1)$ can be removed through a gauge choice. For example, for the *geometric* configurations defined in terms of a B-field and a metric, it is better to remove the β-dependence through a $O(1, D-1)^2$ transformation. On the other hand, there are *non-geometric* configurations for which it is better to remove the B-field, and describe the background in terms of β. The Bianchi identities and fluxes obtained from the Flux Formulation of DFT encode in a natural way the non-geometric structures found in β-supergravity, which is a convenient redefinition of supergravity to analyze the non-geometric backgrounds (see [120–123, 149] and [150]).

Geometric frame

The geometric frame corresponds to the gauge choice $\beta^{ij} = 0$ and $\tilde{\partial}^i = 0$. The generalized metric reads

$$\mathcal{H}_{MN} = \begin{pmatrix} g^{ij} & g^{ik} B_{kj} \\ -B_{ik} g^{kj} & g_{ij} - B_{ik} g^{kl} B_{lj} \end{pmatrix} . \tag{4.128}$$

This is the frame usually considered for geometric descriptions of supergravity backgrounds described in terms of a B-field and a metric. The corresponding three-form H_{ijk} and the geometric flux $\tau_{ij}{}^k$ in curved and planar indices read

$$\begin{aligned} \mathcal{H}_{abc} &= 3\left[\partial_{[a}B_{bc]} + f_{[ab}{}^d B_{c]d}\right] , \\ \mathcal{F}_{ab}{}^c &= f_{ab}{}^c , \qquad \mathcal{Q}_c{}^{ab} = 0 , \qquad \mathcal{R}^{abc} = 0 , \end{aligned} \tag{4.129}$$

and

$$\begin{aligned} H_{ijk} &= e^a{}_i e^b{}_j e^c{}_k \mathcal{H}_{abc} = 3\partial_{[i}B_{jk]} , \\ \tau_{ij}{}^k &= e^a{}_i e^b{}_j e_c{}^k \mathcal{F}_{ab}{}^c = 2\Gamma_{[ij]}{}^k , \qquad \Gamma_{ij}{}^k = \partial_i e_a{}^k e^a{}_j , \\ Q_i{}^{jk} &= 0 , \qquad R^{ijk} = 0 , \end{aligned} \tag{4.130}$$

respectively. The dilaton flux can be written as

$$f_i = e^a{}_i \mathcal{F}_a = 2\partial_i\phi + \tau_{ij}{}^j . \tag{4.131}$$

The only non-trivial Bianchi identities from the previous section then read

$$\partial_{[i}H_{jkl]} = \mathcal{J}_{ijkl} , \tag{4.132}$$

$$-3R^l{}_{[ijk]} = \nabla_{[i}\tau_{jk]}{}^l + \tau_{[ij}{}^m \tau_{k]m}{}^l = \mathcal{J}_{ijk}{}^l , \tag{4.133}$$

$$2R_{[ij]} + 4\partial_{[i}\partial_{j]}\phi = \nabla_k \tau_{ij}{}^k + 2\partial_{[i}f_{j]} = \mathcal{J}_{ij} , \tag{4.134}$$

where the $\mathcal{J}$ are only non-trivial on the world-volume of sources, as we will see later. Notice that $\mathcal{J}_{ij}$ sources a dilaton-like Bianchi identity $df_i = 0$.

Non-geometric frame

On the other hand, one can also define a non-geometric frame taking $B_{ij} = 0$, for instance, to study the non-geometric Q-background and set $\tilde{\partial}^i = 0$. The generalized metric the reads

$$\mathcal{H}_{MN} = \begin{pmatrix} g^{ij} - \beta^{im} g_{mn} \beta^{nj} & -\beta^{im} g_{mj} \\ g_{im}\beta^{mj} & g_{ij} \end{pmatrix} . \tag{4.135}$$

This frame was also considered in the context of DFT, and a differential geometry was considered for this frame in [120]. The fluxes in planar indices read

$$\begin{aligned} \mathcal{H}_{abc} &= 0 , \qquad \mathcal{F}_{ab}{}^c = f_{ab}{}^c , \\ \mathcal{Q}_c{}^{ab} &= \partial_c \beta^{ab} + 2 f_{cm}{}^{[a} \beta^{mb]} , \\ \mathcal{R}^{abc} &= 3\left[\beta^{[am}\partial_m \beta^{bc]} + f_{mn}{}^{[a}\beta^{bm}\beta^{c]n}\right] , \end{aligned} \tag{4.136}$$

while in curved indices they take the form

$$
\begin{aligned}
H_{ijk} &= 0 \, , \qquad \tau_{ij}{}^{k} = 2\Gamma_{[ij]}{}^{k} \, , \\
Q_i{}^{jk} &= e^a{}_i e_b{}^j e_c{}^k \mathcal{Q}_a{}^{bc} = \nabla_i \beta^{jk} + 2\tau_{li}{}^{[j}\beta^{k]l} \, , \\
R^{ijk} &= e_a{}^i e_b{}^j e_c{}^k \mathcal{R}^{abc} = 3\beta^{[\underline{i}l}\nabla_l \beta^{\underline{jk}]} \, . \qquad (4.137)
\end{aligned}
$$

We will analyze the specific Q5-brane and R5-brane solutions inside DFT in the next chapter.

4.3.3 Towards a first order formulation of DFT

We would now like to see if a first-order formulation of DFT is available in order to formulate couplings to magnetic objects from a dynamical perspective. Here we perform a dualization procedure assuming certain properties in the presence of sources $\mathcal{J}$ and $\mathcal{J}_{ABCD}$. In Chapter (6) we will dualize DFT without sources using the full set of Bianchi identities at the linearized level. For simplicity, we assume through this section that the strong constraint terms $\mathcal{Z}_{\dots}$ are vanishing. Since we will assume non-vanishing currents $\mathcal{J}_{\dots}$, the fluxes cannot be given any longer in terms of the vielbein and the dilaton. We can however introduce deviation terms and write them as

$$
\begin{aligned}
\mathcal{F}_{ABC} &= f_{ABC}(\mathcal{E}) + \Theta_{ABC} \, , \qquad (4.138) \\
\mathcal{F}_{A} &= f_{A}(\mathcal{E}, d) + \Theta_{A} \, , \qquad (4.139) \\
& \qquad (4.140)
\end{aligned}
$$

where $f_{ABC} = 3\Omega_{[ABC]}$ and $f_A = 2\mathcal{D}_A + \Omega^B{}_{BA}$. Plugging these general expressions in the sourced Bianchi identity yields

$$
\begin{aligned}
\nabla^f_{[A}\Theta_{BCD]} - \frac{3}{4}\Theta_{[AB}{}^E\Theta_{CD]E} &= \mathcal{J}_{ABCD} \, , \qquad (4.141) \\
2\nabla^f_{[A}\Theta_{B]} + \left(\mathcal{D}^C - f^C\right)\Theta_{CAB} + \Theta^C\Theta_{CAB} &= \mathcal{J}_{AB} \, , \qquad (4.142) \\
\left(\mathcal{D}^A - f^A\right)\Theta_A - \frac{1}{2}\Theta^A\Theta_A + \frac{1}{12}\left(2f^{ABC} + \Theta^{ABC}\right)\Theta_{ABC} &= \mathcal{J} \, , \qquad (4.143)
\end{aligned}
$$

where the connection in the pseudo-covariant derivative

$$
\nabla^f_A \Theta_B = \mathcal{D}_A\Theta_B - \omega_{AB}{}^C\Theta_C \, , \qquad (4.144)
$$

satisfies the following conditions

$$
\begin{aligned}
\omega_{[AB]C} &= \frac{1}{2} f_{ABC} \, , \qquad (4.145) \\
\omega^B{}_{BA} &= f_A \, . \qquad (4.146)
\end{aligned}
$$

Let us note that the vanishing of the currents does not imply in principle the vanishing of the deviation terms, but instead yields complex non-linear differential equations.

Following the previous reasoning employed for coupling the NS5-brane to the three-form, we introduce an antisymmetric Lagrange multiplier 4-tensor B^{ABCD} imposing the first Bianchi identity as its equation of motion, and consider the fluxes as independent variables. This field will be called D_{ABCD} at the linearized level in Chapter (6). The modified action reads

$$\begin{aligned} S' = \int dX e^{-2d} \Big[& 2\mathcal{D}^{\check{A}}\mathcal{F}_A - \mathcal{F}^{\check{A}}\mathcal{F}_A + \frac{1}{6}\breve{\mathcal{F}}^{ABC}\mathcal{F}_{ABC} - 2\mathcal{J} \\ & + B^{ABCD}\left(\mathcal{D}_A\mathcal{F}_{BCD} - \frac{3}{4}\mathcal{F}_{AB}{}^{E}\mathcal{F}_{CDE} - \mathcal{J}_{ABCD}\right)\Big] \\ & + S_{loc}\left(\mathcal{E}, d\right), \end{aligned} \tag{4.147}$$

where we used the check notation (3.55) to indicate that indices are contracted with the planar generalized metric, and we defined (see (4.53))

$$\breve{\mathcal{F}}^{ABC} = \breve{S}^{ABCDEF}\mathcal{F}_{DEF} \,. \tag{4.148}$$

The fluxes $\mathcal{F}_{ABC}$ and $\mathcal{F}_A$ are now treated as independent variables, the vielbein then enters the action only through derivatives $\mathcal{D}_A$ and possibly the additional local action S_{loc}. Note also that (4.111) has been used to rewrite the flux terms that vanish in the standard case when the strong constraint holds. Varying with respect to the various fields yields

$$\delta\mathcal{F}_A \;:\; \mathcal{F}_A = f_A \,, \tag{4.149}$$

$$\delta\mathcal{F}_{ABC} \;:\; \breve{\mathcal{F}}^{ABC} = 3\left(\left(\mathcal{D}_D - f_D\right)B^{DABC} + \frac{3}{2}\mathcal{F}_{DE}{}^{A}B^{DEBC}\right) \,, \tag{4.150}$$

$$\delta B^{ABCD} \;:\; \mathcal{D}_{[A}\mathcal{F}_{BCD]} - \frac{3}{4}\mathcal{F}_{[AB}{}^{E}\mathcal{F}_{CD]E} = \mathcal{J}_{ABCD} \,, \tag{4.151}$$

$$\delta\mathcal{E}_A{}^M \;:\; 2\mathcal{D}^{[A}\mathcal{F}_C S^{B]C} + B^{CDE[A}\mathcal{D}^{B]}\mathcal{F}_{CDE} = \mathcal{G}_{loc}^{[AB]} \,, \tag{4.152}$$

$$\delta d \;:\; 2\mathcal{D}^{\check{A}}\mathcal{F}_A - \mathcal{F}^{\check{A}}\mathcal{F}_A + \frac{1}{6}\breve{\mathcal{F}}^{ABC}\mathcal{F}_{ABC} = 2\mathcal{J} - S_{loc} + \frac{1}{2}\frac{\delta S_{loc}}{\delta d} \tag{4.153}$$

where the Bianchi identity (4.151) has already been used to simplify the dilaton equation of motion (4.153). The equation of motion for $\mathcal{F}_A$ (4.149) automatically sets it to the standard value $f_A = 2\mathcal{D}_A d + \Omega^B{}_{BA}$. Let us note that it is not clear that this action gives the correct equations of motion for dynamical fluxes in the presence of sources, but must only be considered as a first step toward such a description. Imposing by hand the relation $\mathcal{F}_{ABC} = f_{ABC}$, the source $\mathcal{J}_{ABCD}$ has to vanish due to (4.151) and (4.150) can be rewritten as

$$\breve{\mathcal{F}}^{ABC} = 3\nabla^f_D B^{DABC} \ . \tag{4.154}$$

Taking another divergence of this equation, we obtain

$$\nabla^f_C \breve{\mathcal{F}}^{CAB} = -3\nabla^f_C \nabla^f_D B^{CDAB} = B^{CDE[A}\mathcal{D}^{B]}\mathcal{F}_{CDE} \ , \tag{4.155}$$

where we dropped strong constraint-violating terms in the last equality. Combining with (4.152), one then recovers the standard equations for DFT

$$2\mathcal{D}^{[A}\mathcal{F}_C S^{B]C} + \nabla^f_C \breve{\mathcal{F}}^{CAB} = \mathcal{G}^{[AB]}_{loc} \ , \tag{4.156}$$

up to the local source term $\mathcal{G}^{[AB]}_{loc}$ and up to strong constraint-vanishing terms. Using (4.149) and the assumption $\mathcal{F}_{ABC} = f_{ABC}$, the dilaton equation of motion is then also recovered from (4.153)

$$\mathcal{R} = 2\mathcal{J} - S_{loc} + \frac{1}{2}\frac{\delta S_{loc}}{\delta d} \ , \tag{4.157}$$

again up to source and strong constraint-vanishing terms. It would be interesting to pursue this study with, for instance, other Lagrange multipliers to take into account all possible sources.

4.4 Summary

In this chapter, we expanded on the Flux Formulation of Double Field Theory, in which geometric and non-geometric fluxes are space-time dependent. Gauge consistency imposes a set of quadratic constraints on the generalized fluxes, which can be solved by truly double configurations. The constraints are related to generalized Bianchi Identities for (non-) geometric fluxes in the double space, sourced by (exotic) branes. Following previous constructions, we then obtained generalized connections, torsion, and curvatures compatible with the consistency conditions. The strong constraint-violating terms needed to make contact with gauged supergravities, systematically arise in this formulation.

5 Exotic Branes

5.1 Introduction

Our plan for this chapter is to use DFT to study the T-duality chain of related branes using $O(D, D)$ transformations. We will study the chain containing, as a representative, the NS5-brane solution, following a route similar to [151–153]. This gives us the whole chain of non-geometric fluxes in $D = 7$, which are related to the mixed symmetry tensors. We will clarify how to use the supergravity frame in DFT to extract the usual T-duality rules. The R5-brane depends on a dual coordinate from the point of view of the supergravity frame but is consistent with a generalized Scherk-Schwarz reduction of DFT[1] [50–52].

The simplest example of a flux is the Scherk-Schwarz (SS) reduction of IIB supergravity to nine dimensions, in which the RR scalar C acquires a linear dependence on the internal coordinate x^9, *i.e.* $C = C(x^p) + mx^9$, where m is a constant and $p = 0, ..., 8$. This ansatz leads to a consistent truncation to $D = 9$, because C only occurs in the IIB action via derivatives, and the resulting nine-dimensional theory is a gauged supergravity. In nine dimensions, the only T-duality symmetry is the one exchanging IIA and IIB supergravity. Therefore, one expects that what T-duality does in this case is to provide a IIA supergravity origin for

[1]This chapter is based on [98] and also contains some notes that were worked out as a preliminary version of it (based on [154]).

the same nine-dimensional gauging. This is just the dimensional reduction of Romans' massive IIA theory [44][2].

In the NS-sector one can perform SS reductions down to any dimension and the constant fluxes or gaugings that appear in the lower dimensional actions can be encoded in a embedding tensor θ_{MNP}, with $M = 1, \ldots, 2n$, belonging to the three-index completely antisymmetric representation of $\mathrm{SO}(n,n)$. The components of this tensor under $\mathrm{SL}(n,\mathbb{R})$ representations give the well-known chain of NS fluxes [49]

$$\theta_{MNP} \;\to\; H_{abc} \quad f_{ab}{}^{c} \quad Q_a{}^{bc} \quad R^{abc} \quad . \tag{5.1}$$

H_{abc} is known as the H-flux coming from the field strenght of the Kalb-Ramond. $f_{ab}{}^{c}$ denotes a flux coming from the metric. Both of them are known as geometric fluxes. The $Q_a{}^{bc}$ and R^{abc} fluxes are both non-geometric, but their non-geometric nature is very different: while the former can be written as $Q_a{}^{bc} = \partial_a \beta^{bc}$, which is a SS reduction of a suitable combination of the NS 2-form B_{ab} and the metric called β^{ab} (the same β field that appeared in the previous chapter), there is no possible geometric interpretation for the latter within supergravity. This is similar to what happens to the RR flux in nine dimensions, which cannot be obtained geometrically from the IIA massless theory. The difference is that in the case of the RR flux, as discussed above, the T-dual origin of the flux arises in terms of a deformation of the ten-dimensional IIA theory, which is the massive IIA theory of Romans. However, there is no equivalent to the Romans theory in the NS sector. Hence, it is impossible to have a higher-dimensional origin of a purely R^{abc} flux within supergravity.

For instance, starting from $D = 10$ the H_{abc} flux arises for the first time in seven dimensions ($H_{789} = \partial_7 B_{89}$). The field B_{89} has a linear dependence on x^7 and the flux can be seen as a SS reduction from $D = 8$ to $D = 7$ along the x^7 coordinate. By performing a T-duality along, say, x^9, the flux is mapped to $f_{78}{}^{9}$ as (5.1) shows. This comes from a SS reduction of the metric components. From the $D = 10$ point of view the background fields $g_{\mu\nu}$, $B_{\mu\nu}$, and ϕ ($\mu = 0, \cdots, 9$) are related by the well known Buscher's rules (see Section (2.2)).

If one performs a further T-duality, say along x^8, this leads to a $Q_7{}^{89}$ flux, which arises as a SS reduction for the ten dimensional field $\beta^{\mu\nu}$ which is defined in β-supergravity [120–123] for this specific compactification as follows:

$$\beta^{\mu\nu} = -((g - Bg^{(-1)}B)^{-1})^{\mu\sigma} B_{\sigma\rho} g^{\rho\nu} \quad . \tag{5.2}$$

[2]In this chapter we will refer to type IIA supergravity as either the massless type IIA supergravity coming from type IIA superstring theory or the massive Romans' supergravity.

In particular, in $D = 8$ this gives

$$\beta^{89} = -\frac{B_{89}}{\det g + (B_{89})^2} \quad , \tag{5.3}$$

where $\det g$ is the determinant of the metric in the 8 and 9 directions. Although T-duality implies the presence of the R^{789} flux, performing a further T-duality along x^7 is problematic because the field β^{89} has a linear dependence on x^7. This is the reason why this flux is dubbed purely 'non-geometric'.

One can understand the same non-geometric properties as arising by considering the branes that are sources for these fluxes. In particular, the brane that sources the ${Q_7}^{89} = \partial_7\beta^{89}$ flux is the so-called 5^2_2-brane smeared along the x_7 direction. This brane is an exotic brane since when one circles around the brane in transverse space the metric does not come back to the same point [75, 76]. The nontrivial monodromy can be understood as a shift in the β-field, that in the 5^2_2 background takes the simple form

$$\beta^{89} = -\frac{B_{89}}{g_{88}g_{99} + (B_{89})^2} \quad . \tag{5.4}$$

In the H-flux background (meaning H_{abc}-flux), the patches are connected through gauge transformations of the two-form, and in the f-background the transition functions are diffeomorphisms. More generally, the Q-background makes use of the T-duality group (which is in a sense, the same action of a shift in the β-field), and is therefore called a T-fold [6, 7, 24–27].

When smearing the 5^2_2 along x^7 one obtains a harmonic function that is linear in the only remaining transverse direction, while for consistency the field β^{89} must acquire a linear dependence on x^7. As before, the question is how can one perform a T-duality along x^7.

Double field theory (DFT) provides an approach to deal with this issue. In DFT, all fields depend on x^μ, the usual space-time coordinates and $\tilde{x}_\mu$, the winding coordinates. In DFT, T-duality swaps x and $\tilde{x}$, which implies that the SS ansatz $\beta^{89} = \beta^{89}(x) + {Q_7}^{89}x^7$ corresponding to the Q-flux is mapped to $\tilde{\beta}^{89} = \tilde{\beta}^{89}(x) + R^{789}\tilde{x}_7$. In the supergravity frame, *i.e.* the frame where all the fields depend on x only, the Q-flux ansatz satisfies the strong constraint. But after performing a T-duality to obtain the R-flux, the dual background necessarily will depend on a dual coordinate, thus violating the supergravity frame. The dual coordinate dependence in the R-flux ansatz is actually compatible with a generalized SS reduction of DFT, in the sense that reductions on both standard and dual internal coordinates are allowed [50–53]. Exactly the same applies for the corresponding domain-wall solutions: if one performs a T-duality along x^7 on the smeared 5^2_2-brane solution discussed above in DFT, one obtains a so-called R5-brane, which is a domain wall in seven dimensions with β depending linearly on $\tilde{x}^7$. This is analogous to what happens in the RR sector. In that case the ansatz $C = C(x) + mx^9$ is mapped to $C_\mu = C_\mu(x) + \delta_{\mu 9} m\tilde{x}_9$ [58] (C_μ being the 1-form of massless type IIA supergravity) when one performs a T-duality along x^9. The difference with the previous case is that in the case of the RR sector the violation of the strong constraint leads to a well-defined ten-dimensional theory, which is Romans' massive IIA supergravity theory. In the case of the NS sector, instead, such a violation will not lead to a consistent theory in ten dimension (or in nine and eight, for that matter). This result is the DFT equivalent of the statement that in the case of the RR fluxes the massive deformation corresponds to a massive theory in ten dimensions, while in the case of the NS fluxes such a massive theory does not exist in dimension higher than seven.

5.1.1 Frame choices of the section condition and T-duality rules

In the frame $\tilde{\partial}^\mu \Phi = 0$ (we call it supergravity frame), where we mean the fields do not depend on $\tilde{x}$, the DFT action reduces to the usual NS action (3.11)[3]. One can actually reduce the DFT action in any frame that satisfies the strong constraint. In particular, reducing in the frame $\partial_\mu \Phi = 0$ one gets the NS-action after a field redefinition that takes the form of an $O(D, D)$ transformation that interchanges x with $\tilde{x}$ (and also interchanging contravariant indices with covariant ones in order to be consistent with the lower index of the $\tilde{x}$-dependence of the frame). This redefinition works because it takes the same form of an $O(D, D)$ transformations. When a frame is chosen the $O(D, D)$ covariance of the theory is broken. However, we can still use the DFT formalism to implement $O(D, D)$ transformations that preserve the chosen frame. This is a convenient framework to obtain in a faster way Busher's rules when isometric directions are assumed.

The section condition:

$$\partial_M \partial^M C = 0, \quad \partial_M A \partial^M B = 0, \tag{5.5}$$

where A, B and C are arbitrary fields, is manifestly invariant under $O(D, D)$ transformation:

$$\partial'_M \partial'^M C' = 0, \quad \partial'_M A' \partial'^M B' = 0. \tag{5.6}$$

[3] In previous chapters the space-time indices were denoted by i, j, k here they are denoted μ, ν, ρ.

In the usual supergravity frame:

$$\tilde{\partial}^{\mu} A(X^M) = 0. \tag{5.7}$$

Now lets consider which $O(D,D)$ transformations leave the frame invariant:

$$\tilde{\partial}^{\mu} A(X^M) = 0 \rightarrow \tilde{\partial}'^{\mu} A'(X'^M) = 0. \tag{5.8}$$

The $O(D,D)$ transformations are:

$$X'^M = O^M{}_N X^N, \quad O^M{}_N = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad X^M = \begin{pmatrix} \tilde{x}_{\mu} \\ x^{\mu} \end{pmatrix}. \tag{5.9}$$

$O^M{}_N$ belongs to $O(D,D)$ and it satisfies:

$$a^t c + c^t a = 0, \quad b^t d + d^t b = 0, \quad a^t d + c^t b = 1. \tag{5.10}$$

We apply and $O(D,D)$ transformation to the derivative of a field, in this case without lost of generality a scalar field:

$$\partial'_M A'(X') = (O^{-1})^N{}_M \partial_N A(X) \tag{5.11}$$

We see that (5.8) implies conditions on $O^M{}_N$:

$$\begin{aligned} \tilde{\partial}'^{\mu} A'(X') &= d^{\mu}{}_{\nu} \tilde{\partial}^{\nu} A(x^{\rho}) + c^{\mu\nu} \partial_{\nu} A(x^{\rho}) = c^{\mu\nu} \partial_{\nu} A(x^{\rho}) \\ \partial'_{\mu} A'(X') &= b_{\mu\nu} \tilde{\partial}^{\nu} A(x^{\rho}) + a_{\mu}{}^{\nu} \partial_{\nu} A(x^{\rho}) = a_{\mu}{}^{\nu} \partial_{\nu} A(x^{\rho}) \end{aligned} \tag{5.12}$$

One possibility is require $c = 0$ and due to (5.10) we see that $d = (a^{-1})^t$. Then:

$$O^M{}_N = \begin{pmatrix} a & b \\ 0 & (a^{-1})^t \end{pmatrix} \tag{5.13}$$

which is the "geometric subgroup" $GL(D) \ltimes \Lambda_2$. Another possibility is to consider a non-invertible c different from the zero matrix such that for every field A results in $\tilde{\partial}^{\mu} A'(X') = 0$. The only possibility seems to be then when there are isometric directions for the fields. In this case, the factorized T-dualities leave the constraint (5.8) invariant. These are of the form[4]:

$$\mathcal{O} = -\begin{pmatrix} 1 - e_a & -e_a \\ -e_a & 1 - e_a \end{pmatrix}, \quad (e_a)_{\mu\nu} \equiv \delta_{a\mu} \delta_{a\nu}, \quad (\mu, \nu = 1, ..., D). \tag{5.14}$$

[4] We will follow the conventions as in [55] for the factorized T-dualities.

where a is the isometric direction. It is straightforward to extract the Buscher rules from the transformed generalized metric $\mathcal{H}$ (see eq. (3.31)):

$$\mathcal{H}'(x) = \mathcal{O}^{-1t}\mathcal{H}\mathcal{O}(x), \quad d'(x) = d(x). \tag{5.15}$$

Note the same x^ρ dependence on both sides of the equalities, since $\mathcal{O}$ is a T-duality matrix of $O(D,D)$ acting on isometric directions. Using (5.14) and the geometric form of the generalized metric (4.128), we get the following Buscher's rules from (5.15):

$$g'_{ij} = g_{ij} - \frac{1}{g_{aa}}\left(g_{ia}g_{ja} - B_{ia}B_{ja}\right) \tag{5.16}$$

$$B'_{ij} = B_{ij} + \frac{2}{g_{aa}} g_{[i|a|}B_{j]a} \tag{5.17}$$

$$g'_{ia} = -\frac{B_{ia}}{g_{aa}}, \quad B'_{ia} = -\frac{g_{ia}}{g_{aa}}, \quad g'_{aa} = \frac{1}{g_{aa}}, \tag{5.18}$$

$$e^{-2d'(X')} = e^{-2d(X)} \to \sqrt{-g'}e^{-2\phi'} = \sqrt{-g}e^{-2\phi}. \tag{5.19}$$

Here $x^\mu = (x^i, x^a)$ and the dependence of the metric and Kalb-Ramond field are respectively $g_{\mu\nu}(x^i)$ and $B_{\mu\nu}(x^i)$. It is easy to show that when applying a T-duality in a non-isometric direction, the constraint (5.8) is not preserved.

5.2 NS5-KK5-Q5-R5 branes, DFT and T-duality

In this section we analyze how to move between the solutions related by T-duality starting with the NS5-brane. The chain of solutions is known as the NS5, KK5, 5^2_2 and R5 chain. It is worth stressing that since all these solutions are related by $O(D,D)$ transformations, then it is enough to prove that one of them is a solution to the DFT equations of motion to argue the whole chain represents several solutions. The NS5 and KK5-brane solutions were proved to be solutions of the DFT equations of motion in [151, 152] and in exceptional field theory in [153]. In what follows, we will repeat some of their steps.

We can formally apply $O(D,D)$ transformations at the DFT level that is equivalent, after reduction to $(D-n)$-dimensions, to apply $O(n,n)$ transformations at the level of the lower dimensional gauged supergravity (our case of interest will be $n=3$). In section (5.3.1) we are going to do this procedure, first at the lower dimensional level, we apply $O(3,3)$ rotations relating the reduced generalized metric of the different solutions. In particular, in order to go from the Q-flux to the R-flux. We will compare the same procedure from the higher dimensional point of view (at the DFT level). We will see that the T-duality from the Q5 to the R5 brane in the higher dimensional theory corresponds to the same T-duality between the Q-flux and the R-flux in the lower dimensional theory. This requires the introduction of a dual coordinate dependence on the R5 solution. This T-duality is what we call 'Generalized (or massive) T-duality' [24] since it is performed along a non-isometric direction. We recall that Buscher's rules are always meant to apply when one assumes isometric directions on the background. However, DFT allows for this generalized T-duality. We will justify this by stressing that the dual coordinate dependence of the R5 brane is allowed thanks to the generalized Scherk-Schwarz procedure [52] down to $D = 7$.

5.2.1 NS5-brane

We start by describing the (smeared) NS5-brane along the y^4 direction in DFT. We use coordinates $X^M = (\tilde{x}_a, \tilde{y}_i, \tilde{y}_4; x^a, y^i, y^4)$, $i = 1,2,3$. The following generalized metric is a solution of the DFT equations of motion [151, 152]:

$$\mathcal{H}^{MN} = \begin{pmatrix} \eta_{ab} & 0 & 0 & 0 & 0 & 0 \\ 0 & H(\delta_{ij} + \frac{A_i A_j}{H^2}) & 0 & 0 & 0 & H^{-1}A_i \\ 0 & 0 & H + \frac{A^2}{H} & 0 & -H^{-1}A_i & 0 \\ 0 & 0 & 0 & \eta^{ab} & 0 & 0 \\ 0 & 0 & -H^{-1}A_i & 0 & H^{-1}\delta^{ij} & 0 \\ 0 & H^{-1}A_i & 0 & 0 & 0 & H^{-1} \end{pmatrix}, \tag{5.20}$$

$$e^{-2d} = \sqrt{-g}e^{-2\phi} = He^{-2\phi_0}. \tag{5.21}$$

Here $A^2 = A_i A_j \delta^{ij}$ and

$$\eta_{ab}dx^a dx^b = -dt^2 + d\boldsymbol{x}_5^2 = -dt^2 + dx_1^2 + \cdots + dx_5^2. \tag{5.22}$$

The coordinates $(t, x_1, \cdots, x_5)$ represents the world-volume directions. It is easy to extract from the generalized metric the space-time metric, Kalb-Ramond field and the string dilaton. The expresions are:

$$ds^2 = -dt^2 + d\boldsymbol{x}_{(5)}^2 + H\left(dy_4^2 + d\boldsymbol{y}_{(3)}^2\right), \tag{5.23}$$

$$B_{iy^4} = A_i, \tag{5.24}$$

$$e^{-2\phi} = H^{-1} e^{-2\phi_0}. \tag{5.25}$$

Here $d\boldsymbol{y}^2_{(3)} = dy_1^2 + dy_2^2 + dy_3^2$. The solution described above is just the NS5-brane smeared along the y^4 direction. The smearing of a solution in a particular direction allows one to get rid of some particular coordinate dependence. Usually, the harmonic function H depends on all transverse directions y^i and y^4. The delocalization in y^4 implies that the standard field strength of the NS5-brane $H_{mnp} = 3\partial_{[m}B_{np]} = \epsilon_{mnp}{}^q \partial_q \ln H(y^y, y^4)$ with $m = i, y^4 = 1, ..., 4$ is fully described in terms of the standard vector field of the monopole:

$$H_{ijk} = 0, \quad H_{ijy^4} = 2\partial_{[i}B_{j]y^4} = 2\partial_{[i}A_{j]}. \tag{5.26}$$

5.2.2 KK5-brane

We apply and $O(D, D)$ factorized T-duality that interchanges y^4 with $\tilde{y}_4$ to the previous solution as shown in (5.15). Since (5.20) does not depend on y^4 the new solution will not depend on $\tilde{y}_4$. Hence, the new solution is reducible in the new supergravity frame $\tilde{\partial}'^\mu = 0$[5]. The new DFT solution after applying the $O(D, D)$ rotation (and dropping the primes of the transformed fields and coordinates) is:

$$\mathcal{H}^{MN} = \begin{pmatrix} \eta_{ab} & 0 & 0 & 0 & 0 & 0 \\ 0 & H(\delta_{ij} + \frac{A_i A_j}{H^2}) & H^{-1}A_i & 0 & 0 & 0 \\ 0 & H^{-1}A_i & H^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \eta^{ab} & 0 & 0 \\ 0 & 0 & 0 & 0 & H^{-1}\delta^{ij} & -A_i H^{-1} \\ 0 & 0 & 0 & 0 & -A_i H^{-1} & H(1 + \frac{A^2}{H^2}) \end{pmatrix} \tag{5.27}$$

Extracting the supergravity backgrounds we obtain the Kaluza-Klein (KK5-brane) solution:

$$ds^2 = \eta_{ab} dx^a dx^b + H\delta_{ij} dy^i dy^j + H^{-1}(dy^4 + A_i dy^i)^2, \quad \phi = \phi_0. \tag{5.28}$$

where

$$H(y^i) = 1 + \frac{h}{r^2}, \quad r^2 = \delta_{ij} y^i y^j, \quad \text{and} \quad \nabla \times \boldsymbol{A} = \nabla H. \tag{5.29}$$

[5]We may reduce the new solution in the frame $\tilde{\partial}'^i \Phi = 0$, $\partial_{y'^4}\Phi = 0$. This gives back the NS5-brane. As said before, we stick to the supergravity frame in which $\tilde{\partial}'^\mu = 0$, $\mu = (i, a)$ with a representing isometric directions.

We have extracted the string dilaton through the transformation law of d:

$$e^{-2d} = e^{-2d'} = \sqrt{-g'}e^{-2\phi'} \rightarrow \phi' = \phi_0, \quad (\det(g'_{\mu\nu}) = -H^2). \tag{5.30}$$

5.2.3 Q5-brane

We can take the generalized metric of the KK monopole and perform an $O(D, D)$ transformation that sends $y^3 \leftrightarrow \tilde{y}_3$. In order to compactify y^3 we array infinite monopoles along the y^3 direction at intervals of $2\pi R_3$:

$$H = 1 + \sum_{m\in\mathbb{Z}} \frac{h}{\sqrt{\rho^2 + (y^3 - 2\pi R_3 m)}} \longrightarrow h_0 + \sigma \ln(\frac{\mu}{\rho}). \tag{5.31}$$

where $\sigma = h/\pi R_3$, $\rho^2 = \delta_{ij}y^i y^j$ with $i, j \neq 3$ (ρ is a radial coordinate) and μ is a renormalization scale. The factor $h_0(\Lambda)$ is a quantity which diverges when the cutoff Λ tends to infinity. When H is given by (5.31) then the vector potential takes the form $A = A_3 dy^3 = -\sigma\theta dy^3$ (θ is the angular coordinate in the (y^1, y^2)-plane). Now taking this into account, when performing an $O(D, D)$ transformation that sends $y^3 \leftrightarrow \tilde{y}_3$ we get the following generalized metric $\mathcal{H}^{MN}$:

$$\begin{pmatrix} \eta_{ab} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & H\delta_{ij} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & H^{-1} & 0 & 0 & 0 & 0 & -H^{-1}A_3 \\ 0 & 0 & 0 & H^{-1} & 0 & 0 & H^{-1}A_3 & 0 \\ 0 & 0 & 0 & 0 & \eta^{ab} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & H^{-1}\delta^{ij} & 0 & 0 \\ 0 & 0 & 0 & H^{-1}A_3 & 0 & 0 & H + \frac{A_3^2}{H} & 0 \\ 0 & 0 & -H^{-1}A_3 & 0 & 0 & 0 & 0 & H + \frac{A_3^2}{H} \end{pmatrix} \tag{5.32}$$

Then the metric, Kalb-Ramond field and dilaton turn out to be the Q5-brane also known as 5^2_2 background:

$$ds^2 = H\delta_{ij}dy^i dy^j + \frac{H}{H^2 + A_3^2}(d\tilde{z}^2 + d\tilde{y}_3^2) + \eta_{ab}dx^a dx^b, \tag{5.33}$$

$$\begin{gathered} B_{zy^3} = -B_{y^3 z} = \frac{A_3}{H^2 + A_3^2} \\ e^{-2d} = \sqrt{-g}e^{-2\phi} = He^{-2\phi_0} \Rightarrow e^{-2\phi} = \frac{H^2 + A_3^2}{H}e^{-2\phi_0}. \end{gathered} \tag{5.34}$$

It is more convenient to express these fields using the β-basis as defined in (4.135):

$$
\begin{aligned}
ds^2 &= \tilde{g}_{ij} dx^i dx^j \\
&= \eta_{ab} dx^a dx^b + H \left((dy^1)^2 + (dy^2)^2 \right) + H^{-1} \left((dy^4)^2 + (dy^3)^2 \right),
\end{aligned} \tag{5.35}
$$

$$
\beta^{y^3 y^4} = -A_3. \tag{5.36}
$$

In this basis, the non-geometric aspect (like monodromy) of this T-fold is understood as β-shifts, turning the T-fold into a geometric object. From the DFT point of view, even in the standard basis, the generalized metric is well defined and the T-fold aspect can be understood as a generalized coordinate transformation [115].

5.2.4 R5-brane

We look for the solution which we call R5-brane that should be obtained, at least in principle, by performing another T-duality along one of the two remaining transverse directions, that we choose to be the direction y^2. We fix an isometry for the harmonic function H along y^2, that is (dropping infinite constants),

$$
H(y^1, y^2) \to m|y^1|, \tag{5.37}
$$

m being a massive parameter. The smearing process brings consequences to the graviphoton ($\nabla H = \nabla \times \boldsymbol{A}$):

$$
\partial_1 H = \partial_2 A_3, \tag{5.38}
$$

that is

$$
A_3 = m y^2. \tag{5.39}
$$

In other words, an isometric KK-reduction cannot be done and Buscher's rules cannot be applied in the usual way. However, after smearing the Q5-brane we apply and $O(D,D)$ transformation that interchanges y^2 with $\tilde{y}_2$, since the smeared Q5-brane solution depends on y^2 this a generalized T-duality. As a consequence, the generalized metric $\mathcal{H}^{MN}$ of the R5 is:

$$\begin{pmatrix}
\eta_{ab} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & H & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & H^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & H^{-1} & 0 & 0 & 0 & 0 & 0 & -\frac{m\tilde{y}_2}{H} \\
0 & 0 & 0 & 0 & H^{-1} & 0 & 0 & 0 & \frac{m\tilde{y}_2}{H} & 0 \\
0 & 0 & 0 & 0 & 0 & \eta^{ab} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & H^{-1} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & H & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{m\tilde{y}_2}{H} & 0 & 0 & 0 & H+\frac{m^2\tilde{y}_2^2}{H} & 0 \\
0 & 0 & 0 & -\frac{m\tilde{y}_2}{H} & 0 & 0 & 0 & 0 & 0 & H+\frac{m^2\tilde{y}_2^2}{H}
\end{pmatrix} \tag{5.40}$$

For the fundamental fields we get:

$$G = \begin{pmatrix}
\eta_{ab} & 0 & 0 & 0 & 0 \\
0 & H & 0 & 0 & 0 \\
0 & 0 & H^{-1} & 0 & 0 \\
0 & 0 & 0 & H(H^2+m^2\tilde{y}_2^2)^{-1} & 0 \\
0 & 0 & 0 & 0 & H(H^2+m^2\tilde{y}_2^2)^{-1}
\end{pmatrix}, \tag{5.41}$$

$$B_{34} = -\frac{m\tilde{y}_2}{H^2+m^2\tilde{y}_2^2}. \tag{5.42}$$

Or in the β-basis:

$$\tilde{g}_{mn} = \begin{pmatrix}
\eta_{ab} & 0 & 0 & 0 & 0 & \\
0 & H & 0 & 0 & 0 & 0 \\
0 & 0 & H^{-1} & 0 & 0 & \\
0 & 0 & 0 & H^{-1} & 0 & \\
0 & 0 & 0 & 0 & H^{-1} &
\end{pmatrix}, \tag{5.43}$$

with

$$\tilde{\beta}^{34} = -m\tilde{y}_2. \tag{5.44}$$

We will see in the next section that the flux for this solution is

$$R^{234} = \tilde{\partial}^2\tilde{\beta}^{34} = -m. \tag{5.45}$$

This is a new (DFT) brane solution which depends on the non-geometric coordinate $\tilde{y}_2$ and is obtained from Q5 by making (formally) use of the standard Buscher's rules together with the relevant transformation:

$$y^2 \to \tilde{y}_2. \tag{5.46}$$

This sounds a bit unnatural due to the absence of an isometry but it is allowed in DFT. We argue that it is consistent with a generalized Scherk-Schwarz reduction of DFT down to $D = 7$ as we will see in the next section.

5.3 Reduction to $D = 7$, Fluxes and Dual potentials

5.3.1 Reduction to $D = 7$ and GDFT

We now want to understand the T-duality chain in lower dimensions. We can compactify the NS-action down to $D = 7$ dimensions. The details for such a compactification are well described in [97] and we refer to it for more information. We will use the results of [97] for the case $D = 7$ with an ansatz compatible with the chain of solutions. The reduced action can be found to be:

$$S_7 = \int d^7X\sqrt{|\hat{g}_7|}e^{-2\bar{\phi}}\Big(\hat{R}_7 + 4(\partial\bar{\phi})^2 - \frac{1}{8}\partial_1\hat{\mathcal{M}}_{AB}\partial^1\hat{\mathcal{M}}^{AB} + \mathcal{V}\Big). \tag{5.47}$$

The 7-dimensional coordinates are $x^\mu = (t, x^1, \cdots, x^5, y^1)$. The field $\bar{\phi} = \phi - \frac{1}{4}\log(\det g_{mn})$ is the lower dimensional dilaton. $\mathcal{M}_{AB}$ is

$$\hat{\mathcal{M}}^{AB} = \begin{pmatrix} \hat{g}_{ab} - \hat{B}_{ac}\hat{g}^{cd}\hat{B}_{db} & \hat{B}_{ac}\hat{g}^{cb} \\ -\hat{g}^{ac}\hat{B}_{cb} & \hat{g}^{ab} \end{pmatrix} \quad (a, b = 2, 3, 4). \tag{5.48}$$

And the scalar potential is of the form $\mathcal{V} = -\frac{1}{12}f_{AC}{}^E f_{BD}{}^F\hat{\mathcal{M}}^{AB}\hat{\mathcal{M}}^{CD}\hat{\mathcal{M}}_{EF} = -\frac{1}{2}m^2H^{-3}$. H is the harmonic function, $\hat{\mathcal{M}}_{AB}$ is known as the reduced generalized metric and depends only on y^1. The only non-zero components of the constants $f_{AB}{}^C$ are:

$$f_{abc} \equiv H_{abc} = 3\left(\partial_{[a}v_{bc]} + f_{[ab}{}^d v_{c]d}\right) \tag{5.49}$$

$$f_{ab}{}^c = 2u_{[a}{}^m\partial_m u_{b]}{}^n u^c{}_n \tag{5.50}$$

This action possesses $O(3,3)$ global invariance imposing that the constant fluxes $f_{AB}{}^{C}$ transform as tensors[6]. For specific solutions, like the NS5 we find that $H = my^1$ and the gravi-photon $A_3 = my^2$. Then, using the $O(3,3)$ coordinates $A = (\tilde{y}_2, \tilde{y}_3, \tilde{y}_4, y^2, y^3, y^4)$, we find for the NS5:

$$u^a{}_m = \delta^a{}_m, \quad v_{mn} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -A_3 \\ 0 & A_3 & 0 \end{pmatrix} \tag{5.51}$$

and

$$\mathcal{M}^{AB}_{(\mathrm{NS5})} = \begin{pmatrix} H & 0 & 0 & 0 & 0 & 0 \\ 0 & H & 0 & 0 & 0 & 0 \\ 0 & 0 & H & 0 & 0 & 0 \\ 0 & 0 & 0 & H^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & H^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & H^{-1} \end{pmatrix}. \tag{5.52}$$

The T-dual scalar potential gives the following:

$$\mathcal{V} = -\frac{1}{2}(H_{234})^2 \hat{g}^{22}\hat{g}^{44}\hat{g}^{33}, \tag{5.53}$$

with

$$H_{234} = \partial_2 v_{34} = -m. \tag{5.54}$$

For the KK-monopole we find:

$$u^a{}_m = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & A_3 & 1 \end{pmatrix}, \quad v_{mn} = 0, \tag{5.55}$$

and

$$\mathcal{M}^{AB}_{(\mathrm{KK5})} = \begin{pmatrix} H & 0 & 0 & 0 & 0 & 0 \\ 0 & H & 0 & 0 & 0 & 0 \\ 0 & 0 & H^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & H^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & H^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & H \end{pmatrix}. \tag{5.56}$$

The potential gets the form:

$$\mathcal{V} = -\frac{1}{2}(f_{23}{}^{4})^2 \hat{g}^{22}\hat{g}^{33}\hat{g}_{44}. \tag{5.57}$$

[6]Technically, when the gaugings are turned on the global symmetry is broken. So they should be interpreted either as different configurations of the theory related by $O(3,3)$ or for fixed $f_{AB}{}^{C}$ the fields should be redefined according to an $O(3,3)$ rotation, see [97] for more details.

with:

$$f_{23}{}^{4} = u_{2}{}^{2}\partial_{2}u_{3}{}^{4}u^{4}{}_{4} = -m. \tag{5.58}$$

These results are correct since there is an $O(3,3)$ rotation in direction y^4 in order to go from H_{234} to $f_{23}{}^{4}$. The form of the potential suggests that for the remaining $O(3,3)$ rotations in directions y^3, y^2 would give respectively:

$$\mathcal{V} = -\frac{1}{2}(f_{2}{}^{34})^{2}\hat{g}^{22}\hat{g}_{33}\hat{g}_{44}, \tag{5.59}$$

and:

$$\mathcal{V} = -\frac{1}{2}(f^{234})^{2}\hat{g}_{22}\hat{g}_{33}\hat{g}_{44}. \tag{5.60}$$

In fact we can perform the remaining $O(3,3)$ rotations along y^3 and y^2 on the reduced generalized metric, we obtain respectively:

$$\mathcal{M}^{AB}_{(\mathrm{Q5})} = \begin{pmatrix} H & 0 & 0 & 0 & 0 & 0 \\ 0 & H^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & H^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & H^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & H & 0 \\ 0 & 0 & 0 & 0 & 0 & H \end{pmatrix}, \tag{5.61}$$

$$\mathcal{M}^{AB}_{(\mathrm{R5})} = \begin{pmatrix} H^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & H^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & H^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & H & 0 & 0 \\ 0 & 0 & 0 & 0 & H & 0 \\ 0 & 0 & 0 & 0 & 0 & H \end{pmatrix}. \tag{5.62}$$

In the following section we will show that actually $f_{2}{}^{34} = Q_{2}{}^{34} = \partial_{2}\beta^{34}$ and $f^{234} = R^{234} = \tilde{\partial}^{2}\beta^{34}$, representing the fluxes for the Q5 and the R5 solution respectively. Thus, we have found the well-known T-duality chain of fluxes [49]:

$$H_{234} \to f_{23}{}^{4} \to Q_{2}{}^{34} \to R^{234}. \tag{5.63}$$

The gaugings H_{abc} and $f_{ab}{}^{c}$ are given in terms of the twists $u^{a}_{m}(y^2)$, $v_{ab}(y^2)$ by the equations (5.49) and (5.50). This dependence on the twist cannot afford to give the fluxes Q and R from a higher dimensional point of view. This is another reason why Q and R are called non-geometric fluxes. We will find these non-geometric gaugings by compactifying DFT using generalized Scherk-Schwarz compactification.

5.3.2 Generalized Scherk-Schwarz reduction and the Q5, R5 branes

We would like to perform a generalized Scherk-Schwarz reduction for the Q5 and R5 brane. As said before, most of the details can be found in [97] and also in [50–53] and we will simply use the results. In order to carry on a SS reduction we need to rotate the DFT coordinates $(\tilde{x}_\mu, x^\mu)$ to $\mathbb{X}^M = (X^i, \mathbb{Y}^A)$ with $X^i = (\tilde{x}_i, x^i)$, $\mathbb{Y}^A = (\tilde{y}_m, y^m)$. The coordinates X^i are called the double external coordinates and $\mathbb{Y}^A$ are the double internal coordinates. The idea is to write the fields in a form compatible with a SS reduction, i.e. in such a way that the twist matrices $U^N{}_M(Y)$ get factorized out of the DFT action. For this, we should rotate the invariant $O(D, D)$ metric and the generalized metric according to [97]:

$$\tilde{\eta} = R\eta R^T, \quad \tilde{\mathcal{H}}^{-1} = R\mathcal{H}^{-1}R^T, \quad R = \begin{pmatrix} \delta_i{}^j & 0 & 0 & 0 \\ 0 & 0 & \delta_m{}^n & 0 \\ 0 & \delta_i{}^j & 0 & 0 \\ 0 & 0 & 0 & \delta_m{}^n \end{pmatrix}, \quad R^{-1} = R^T. \tag{5.64}$$

$$\tilde{\mathcal{H}}_{MN}(\mathbb{X}) = U^I{}_M(Y)\hat{H}_{IJ}(X)U^J{}_N(Y) \tag{5.65}$$

$$\hat{H}_{IJ} = \begin{pmatrix} \hat{g}^{ij} & 0 & 0 \\ 0 & \hat{g}_{ij} & 0 \\ 0 & & \mathcal{M}_{AB} \end{pmatrix} \tag{5.66}$$

$$U^I{}_M = \begin{pmatrix} \delta^i{}_j & 0 & 0 \\ 0 & \delta_i{}^j & 0 \\ 0 & 0 & U^A{}_M \end{pmatrix} \tag{5.67}$$

This is the form we need to study the fluxes coming from the chain of solutions. We apply the rotation (5.64) to (NS5) to extract (5.66) and we obtain:

$$\hat{g}_{ij} = \begin{pmatrix} \eta_{ab} & 0 \\ 0 & H \end{pmatrix}, \tag{5.68}$$

this is a domain wall in $D = 7$ dimensions. In fact, our chain of solutions reduce to the same domain wall. After rotating the generalized metric we can extract

$U^A{}_M$ and the reduced generalized metric for the NS5-brane:

$$U^A{}_M = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -A_3 \\ 0 & 0 & 1 & 0 & A_3 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathcal{M}^{AB}_{(\mathrm{NS5})} = \begin{pmatrix} H & 0 & 0 & 0 & 0 & 0 \\ 0 & H & 0 & 0 & 0 & 0 \\ 0 & 0 & H & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{H} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{H} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{H} \end{pmatrix}. \tag{5.69}$$

An important point is to recognize that this is in agreement with (5.52). For the KK5-brane we obtain:

$$U^A{}_M = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -A_3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & A_3 & 1 \end{pmatrix}, \quad \mathcal{M}^{AB}_{(\mathrm{KK5})} = \begin{pmatrix} H & 0 & 0 & 0 & 0 & 0 \\ 0 & H & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{H} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{H} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{H} & 0 \\ 0 & 0 & 0 & 0 & 0 & H \end{pmatrix}. \tag{5.70}$$

In agreement with (5.56). So far, for the NS5 and the KK5, the twist matrix $U^A{}_M$ is given by a "the geometric" parameterization:

$$U^A{}_M = \begin{pmatrix} u_a{}^m & u_a{}^n v_{nm} \\ 0 & u^a{}_m \end{pmatrix}. \tag{5.71}$$

The form of the twist matrix encodes 'geometric solutions' and must be compared to (5.51) and (5.55). This upper triangular form of $U^A{}_M$ is only preserved by the geometric subgroup of $O(3,3)$. At next we will see that $U^A{}_M$, for the Q5 and R5 branes, will require a different parametrization.

Now we take the Q5 parametrized with respect to the β-basis variables and we perform the rotation (5.64), we obtain:

$$\hat{g}_{ij} = \hat{\hat{g}}_{ij}, \tag{5.72}$$

where $\hat{g}_{ij}$ is the domain wall (5.68) and

$$U^A{}_M = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -A_3 & 0 & 1 & 0 \\ 0 & A_3 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathcal{M}^{AB}_{(\text{Q5})} = \begin{pmatrix} H & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{H} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{H} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{H} & 0 & 0 \\ 0 & 0 & 0 & 0 & H & 0 \\ 0 & 0 & 0 & 0 & 0 & H \end{pmatrix}. \tag{5.73}$$

For the R5 brane, using the β-basis, we rotate again and extract the following $U^A{}_M$ to find the reduced generalized metric:

$$U^A{}_M = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -m\tilde{y}_2 & 0 & 1 & 0 \\ 0 & m\tilde{y}_2 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathcal{M}^{AB}_{(\text{R5})} = \begin{pmatrix} \frac{1}{H} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{H} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{H} & 0 & 0 & 0 \\ 0 & 0 & 0 & H & 0 & 0 \\ 0 & 0 & 0 & 0 & H & 0 \\ 0 & 0 & 0 & 0 & 0 & H \end{pmatrix}. \tag{5.74}$$

If we compare these results to (5.61) and (5.62) it implies that the β-basis is the right one to perform a generalized SS reduction of the non-geometric Q5 and R5 branes. The difference now is the parametrization of $U^A{}_M$, they both belong to the non-geometric parametrization given by:

$$U^A{}_M = \begin{pmatrix} u_a{}^m & 0 \\ u^a{}_n \beta^{nm} & u^a{}_m \end{pmatrix}. \tag{5.75}$$

In full generality, it is possible to parametrize the twist matrix as:

$$U^A{}_M = \begin{pmatrix} u_a{}^m & u_a{}^n v_{nm} \\ u^a{}_n \beta^{nm} & u^a{}_m + u^a{}_n \beta^{np} v_{pm} \end{pmatrix}. \tag{5.76}$$

With this general twist matrix, the reduction can afford geometric and non-geometric gaugings to be turned on simultaneously[7]. Moreover since this twist matrix allows for configurations that depend explicitly on $\tilde{y}_m$, it implies that the higher dimensional origin is locally non-geometric. In fact, it has been shown in [52, 53] that the gaugings

$$f_{ABC} = 3\tilde{\Omega}_{[ABC]}, \qquad \tilde{\Omega}_{ABC} = U_A{}^M \partial_M U_B{}^N U_{CN}. \tag{5.77}$$

[7] We did not considered this possibility of turning on simultaneously geometric and non-geometric gaugings since we are interested in single brane solutions.

satisfy quadratic constraints which are weaker versions of the strong constraint. The components of ${f_{AB}}^C$ for our particular solutions now become:

$$\begin{aligned}
&(\mathrm{NS5}) \quad f_{234} \equiv H_{234} = \partial_2 v_{34} = -m, \\
&(\mathrm{KK5}) \quad {f_{23}}^4 = {u_2}^2 \partial_2 {u_3}^4 u^4{}_4 = -m, \\
&(\mathrm{Q5}) \quad {f_2}^{34} \equiv {Q_2}^{34} = \partial_2 \beta^{34} = -m, \\
&(\mathrm{R5}) \quad f^{234} \equiv R^{234} = \tilde{\partial}^2 \beta^{34} = -m.
\end{aligned} \tag{5.78}$$

The metric and Kalb-Ramond field given by (5.41) and (5.42) or in the β-basis given by (5.43) and (5.44) give the correct result in $D = 7$ dimensions.

We would like to finish this section by commenting on the duality orbits of (exotic) branes. The NS5-brane and KK5-monopole lie along the same geometric T-duality orbit and we have seen that they are not sufficient to span the full duality orbit since the Q5 and R5 - branes are required by T-duality [49]. We have commented in the previous chapter that in the presence of sources the Bianchi identity locally breakdown on the world-volume, so we can use the duality orbits of the Bianchi identities to speculate about brane orbits. By performing a T-duality along some non-isometric direction, the solution will depend on a dual coordinate, and its geometric interpretation breaks down even locally, from a D-dimensional perspective. In DFT, this is not a problem, given that the notion of T-duality is generalized and allows for such kind of transformations. Given that the equations of motion are T-duality invariant, the configuration obtained in this way will automatically solve them, for instance, the usual chain of solutions NS5-KK5-5^2_2-R5 that gives rise to the lower-dimensional fluxes $H \to f \to Q \to R$. This chain, however, corresponds to a particular representative of the orbit containing the geometric branes, i.e. the NS5 and KK5 branes. In this sense, by construction, the non-geometric branes can be T-dualized back to geometric objects. More interesting is to determine if there exist truly non-geometric bound states of branes, belonging to truly non-geometric orbits. These cannot be T-dualized to a frame in which the configuration becomes geometric. A possibility is to consider bound states combining the presence of geometric and non-geometric branes, such that under T-dualities their roles get exchanged, but non-geometry is conserved. A first step in this direction was nicely achieved in [148], were intersections of Q and R-branes were analyzed. Non-geometric duality orbits were addressed for fluxes in [128]. There, it was shown that genuine non-geometric orbits exist for fluxes, in which all types of gaugings H, τ, Q and R are turned on simultaneously, and there is no T-duality frame in which any of them vanish.

For such configurations the strong constraint must necessarily be relaxed, and it would be nice to explore whether this situation is reproduced by branes as well.

As mentioned before, the duality orbits of Bianchi identities and their associated sources $\mathcal{J}_{ABCD}$, $\mathcal{J}_{AB}$ and $\mathcal{J}$ (see § (4.3)), in principle, allows for bound-states of (non) geometric branes. However, we stress that this is mere speculation. A compel treatment of the duality orbits would require a full examination of the Bianchi identities, with the strong constraint relaxed, and the understanding of the electric-fields that couple to the branes (for instance, the B_6 potential of the NS5-brane). In the next section, we will motivate what kind of potentials would couple to the NS-branes and in Chapter (6) we construct a dual theory of DFT. This dual theory allows to treat in a $O(D, D)$ covariant way the electric-fields that would couple to the NS-branes. Although the construction of the dual theory assumes the strong constraint, we believe that the formulation of the 'Dual Double Field Theory' with a relaxed strong constraint could give rise to the possibility of describing bound states of exotic branes that can not be described in supergravity.

5.4 Duality relations

The analysis of [58] shows that the Romans mass parameter can be thought as the 0-form field strength G_0 of the 1-form C_1 in doubled space, *i.e.* $G_0 = \tilde{\partial}^\mu C_\mu$. On the other hand, the democratic formulation of the RR fields implies that in IIA supergravity one can introduce a 9-form C_9 whose field strength G_{10} is the Hodge dual of the Romans mass parameter G_0. The special thing about this duality relation is that it maps a non-geometric configuration for the 1-form C_1 to a fully geometric configuration for C_9. In general, in any dimension D one can introduce $(D-1)$-form potentials which are dual to the embedding tensor. From a group-theoretical point of view, it can be shown that all such $(D-1)$-forms in maximal supergravity theories should come from mixed-symmetry potentials that arise in the decomposition of the adjoint representation of E_{11} [66] corresponding to the ten-dimensional IIA and IIB theories [68]. These mixed-symmetry potentials can be divided into three sets:

- The actual fields of the ten-dimensional theory, that are the metric, the scalars and all the forms (electric and magnetic duals), together with the 'dual graviton', which is a mixed-symmetry potential in the (7,1) Young-Tableau representation;
- Mixed-symmetry potentials with one set of eight antisymmetric indices, *i.e.* in (8,...) Young Tableaux representations;

- Mixed-symmetry potentials with one set of nine antisymmetric indices (the RR 9-form C_9 is a special case in this set, because it has nine antisymmetric indices but it is not a mixed-symmetry potential).

The full list of mixed-symmetry potentials that give rise to the $(D-1)$ form dual to the NS embedding tensor θ_{MNP} was given in [155]. In this chapter we first want to expand in this direction.

In particular, we will show that

- The geometric fluxes H and f are dual to potentials belonging to the first set;

- The locally geometric flux Q is dual to a potential belonging to the second set;

- The non-geometric flux R is dual to a potential belonging to the third set.

The first correspondence between fluxes and mixed-symmetry potentials is straightforward. In order to understand the second correspondence, one can use the observation [67] that the mixed-symmetry fields in the second set can be thought of as generalised duals of the standard supergravity fields [156–158]. Therefore, they do not correspond to new fields and one can expect that they are dual to redefinitions of the supergravity fields depending on the standard coordinates. The mixed-symmetry fields in the third set are instead fields that do not satisfy any generalized duality relation in ten dimensions, they arise as deformation parameters only when they are reduced to $(D-1)$-forms. In this sense, the RR 9-form is an exception because it is already a form in ten dimensions, which is the dual counterpart of the statement that the violation of the strong constraint in the RR sector discussed in [58] is still consistent within DFT. Let us be more specific. We have mentioned that we can consider the mass of the Romans theory as the dual of the 10-form field strength of the 9-form RR potential C_9. Similarly, the embedding tensor θ_{MNP} is dual in any dimension D to a $(D-1)$-form potential $D_{D-1,MNP}$. Starting from $n \geq 3$ or, equivalently, $D \leq 7$, the duality relation (neglecting the contribution from any other field) has the schematic form

$$\frac{1}{\sqrt{|g|}}\epsilon^{\mu_1\dots\mu_D}\partial_{\mu_1}D_{\mu_2\dots\mu_D,MNP} = \mathcal{M}_M{}^Q\mathcal{M}_N{}^R\mathcal{M}_P{}^S\theta_{QRS} \quad , \tag{5.79}$$

where $\mathcal{M}$ parametrises the coset $\mathrm{O}(n,n)/[\mathrm{O}(n)\times\mathrm{O}(n)]$ and can be thought of as the DFT generalised metric $\mathcal{H}$ of $\mathrm{O}(n,n)$ with G and B only dependent on the D-dimensional spacetime coordinates so $M = 1, \cdots, 2n$ and $\mu = 0, \cdots, D-1$.

These $\mathcal{M}$'s are needed to have a duality relation that transforms covariantly under $\mathrm{O}(n,n)$.

For instance, if the higher-dimensional theory lives in $D = 10$ one can split the lower dimensional duality relation (5.79) into four different 7-dimensional relations, one for each flux, as follows:

$$
\begin{aligned}
D_6 \; &: \qquad \frac{1}{\sqrt{|g|}}\epsilon^{\mu_1\ldots\mu_7}\partial_{\mu_1}D_{\mu_2..\mu_7}{}^{abc} = \mathcal{M}^{ad}\mathcal{M}^{be}\mathcal{M}^{cf}H_{def} \quad ,\\
D_{7,1} \; &: \qquad \frac{1}{\sqrt{|g|}}\epsilon^{\mu_1\ldots\mu_7}\partial_{\mu_1}D_{\mu_2..\mu_7}{}^{ab}{}_c = \mathcal{M}^{ad}\mathcal{M}^{be}\mathcal{M}_{cf}f_{de}{}^f \quad ,\\
D_{8,2} \; &: \qquad \frac{1}{\sqrt{|g|}}\epsilon^{\mu_1\ldots\mu_7}\partial_{\mu_1}D_{\mu_2..\mu_7}{}^a{}_{bc} = \mathcal{M}^{ad}\mathcal{M}_{be}\mathcal{M}_{cf}Q_d{}^{ef} \quad ,\\
D_{9,3} \; &: \qquad \frac{1}{\sqrt{|g|}}\epsilon^{\mu_1\ldots\mu_7}\partial_{\mu_1}D_{\mu_2..\mu_7,abc} = \mathcal{M}_{ad}\mathcal{M}_{be}\mathcal{M}_{cf}R^{def} \quad .
\end{aligned}
\tag{5.80}
$$

The indices a, b, c run from $1, \cdots, n$. The field $D_{(6),MNP}$ arises from the dimensional reduction of the ten-dimensional mixed-symmetry fields [155] [8]

$$
D_6 \quad D_{7,1} \quad D_{8,2} \quad D_{9,3} \quad . \tag{5.81}
$$

In general, each potential in (5.81), after reduction to D dimensions, is dual to each of the fluxes listed in (5.1). For instance, the 6-form D_6 is dual to the H-flux H_{abc} because by reduction one gets a $(D-1)$-form $D_{(D-1),a_1\ldots a_{n-3}}$ which is equivalent to $D_{(D-1)}{}^{abc}$ as a representation of $\mathrm{SL}(n,\mathbb{R})$. The same applies to the other mixed-symmetry fields given in (5.81) when applying the D-dimensional epsilon tensor. This means that we could consider an $O(n,n)$ 4-index antisymmetric tensor to encode the $SL(d,\mathbb{R})$ duals of the mixed-symmetry potentials:

$$
D_{MNPQ} : D_6 \quad D_{7,1} \quad D_{8,2} \quad D_{9,3} \quad . \tag{5.82}
$$

This will be the key field for the next chapter to perform a DFT dualization. For the time being, it is instructive to think about how in principle the above duality relations could be naively uplifted to ten dimensions, given that all the fields involved are ten-dimensional fields. For the first relation this is obvious since it does not involve a mixed-symmetry potential. Indeed, because the left-hand side can be uplifted to $\epsilon^{\mu_1\cdots\mu_{10}}\partial_{\mu_1}D_{\mu_2..\mu_7}$ and one ends up with the duality relation

[8] In these expressions we denote with $D_{p,q}$ a field with indices in the Young tableau representation with two columns, one of length p and one of length q. For instance, this means that the $D_{7,1}$ field has eight indices in total, seven of which are totally antisymmetric and such that antisymmetrising all eight indices one obtains zero.

between B_2 and D_6 in ten dimensions. The second relation is only consistent if the lower index c denotes an isometric direction. This means that one gets $\epsilon^{\mu_1 \cdots \mu_{10}} \partial_{\mu_1} D_{\mu_2..\mu_8,c}$ where one of the ten indices $\mu_1 ... \mu_{10}$ are parallel to c, but the field does not depend on x^c. Similarly, in the other two cases the lower indices bc and abc correspond to isometric directions. In particular, in the last case one ends up with $\epsilon^{\mu_1 \cdots \mu_{10}} \partial_{\mu_1} D_{\mu_2..\mu_{10},abc}$ which means that also three of the μ indices of the field must coincide with the isometric indices abc. Remarkably, the requirement of isometric direction will be dropped out in the next chapter. Basically the key ingredient to avoid the isometric assumption is to add extra fields to the duality relations. This will turn out to be the case in the next chapter when we consider as the main field of the dual theory D_{MNPQ} together with an antisymmetric 2-tensor D_{MN} and a singlet D.

If one considers the uplift to ten dimensions of the duality relations in the way discussed above, these D-fields always depend on the standard coordinates, with the exception of those corresponding to the isometric directions. The characteristics of the fluxes on the right-hand side of the duality relations is thus mapped to the index properties of the corresponding mixed-symmetry potential. In particular, the global non-geometric nature of the Q flux is translated into the fact that the dual potential is a mixed-symmetry potential with a set of eight antisymmetric indices, while the local non-geometric nature of the R flux corresponds to the fact that the dual potential is in this case a mixed-symmetry potential with a set of nine antisymmetric indices. The incompatibility of the R flux with the strong constraint in DFT (*i.e.* the fact that it corresponds to a SS reduction in $\tilde{x}$) is equivalent to the impossibility of writing a consistent coupling to the mixed-symmetry potential $D_{9,3}$ in ten-dimensional supergravity. The difference with the Romans case is that there one has a 9-form potential C_9 that is not a mixed-symmetry field. This implies that C_9 is a well defined potential in ten-dimensions, and its field strength is dual to the Romans mass parameter.

5.5 Summary

We considered the NS fluxes and studied them from the DFT point of view and the lower dimensional point of view. The analysis of the R5 solution showed that a dual coordinate dependence on the solution allows for a non-geometric description of the R-flux. We then pointed out the relationship between the fluxes and the mixed-symmetry potentials, and argued that the dual fields to the Q and R-flux in $D = 10$ could be interpreted as an (8,3) and (9,3) mixed-symmetry tensor fields. We also introduced an $O(n,n)$ antisymmetric field, that is, D_{MNPQ}, that encodes the mixed-symmetry potentials. In the next chapter, we will construct a dual theory for the fluxes, and the dual fields will be encoded in D_{MNPQ}.

6

Dual Double Field Theory

6.1 Introduction

In this chapter, we will perform the dualization of Double Field Theory at the linearized level, thereby capturing, in particular, the dual potential B_6 and the dual graviton (plus dilaton) in a T-duality covariant way. While we are restricting ourselves to the free, quadratic theory, we believe that our results provide important pointers for the full non-linear theory. The construction of the non-linear theory would be necessary in order to describe the world-volume dynamics (in a way that is compatible with T-duality) of several branes. In other words, the branes that are related by T-dualities couple to some potentials, which, in turn, should be related by T-duality as well. For instance, we have seen that the NS5-brane is related by T-duality to the KK5-brane. The NS5-brane couples electrically to a B_6 potential, and it is expected that the KK5-brane will couple to the dual graviton [159]. Then, one expects the B_6 and the dual graviton to be related by T-duality. In general, from T-duality one would expect the appearance of mixed-Young tableaux fields as exotic duals of the usual gauge potentials [61, 156], [77, 155, 160], [157, 158], together with associated "exotic" branes [75, 76] like the Q5-brane. We find that the expected fields are indeed described by the Dual DFT.

The duality transformations relating a field strength to its Hodge-dual, interchanging Bianchi identities and field equations, are ubiquitous in gauge theory, supergravity and string theory. For instance, the electromagnetic duality in four dimensions is essential for the S-duality of $\mathcal{N} = 4$ super-Yang-Mills theories. As we mentioned before, in order to define the world-volume dynamics of certain branes, it is necessary to replace some of the standard p-form gauge potentials of string theory by their duals. Remarkably, taking into account further dualities or symmetries of string theory, such as T-duality (or U-duality), implies that even more fields of a more exotic nature are needed [28–33]. In DFT the Kalb-Ramond field and the metric are part of an irreducible object, the generalized metric. One then would expect that the B_6, the dual graviton plus other exotic duals are part of an irreducible object. The results of the previous chapter suggest that all dual fields can be organized into a 4-index antisymmetric tensor under $O(D,D)$. A first attempt to introduce a 4-index antisymmetric tensor into the DFT action in the presence of sources was performed in Subsection (4.3.3). In Section (5.4) we argued that the duality relation between the embedding tensor θ_{MNP} of lower dimensional supergravity and a $(D-1)$-potential $D_{(D-1),MNP}$ could be uplifted to higher dimensions by introducing mixed-symmetry dual potentials (in particular, relating the so-called Q- and R-fluxes to $(8,2)$ and $(9,3)$ mixed-symmetry tensor fields, respectively) and that these mixed-symmetry potentials can be encoded in an antisymmetric 4-index tensor of $O(D,D)$.

Starting from the linearization of the DFT action, written in terms of the linearized frame field that reads $h_{AB} = -h_{BA}$, we will apply the standard procedure of obtaining the dual theory, introducing Lagrange multiplier fields that impose the Bianchi identities for the generalized fluxes. This naturally leads to a 4-index antisymmetric field D_{ABCD}, but also to a field D_{AB} in the antisymmetric 2-tensor representation and a field D in the singlet representation. In the following all these fields are called 'D-fields'. In this chapter the D-fields carry flat indices $A, B = 1, \ldots, 2D$ under the doubled local Lorentz group $O(D-1,1) \times O(D-1,1)$. The incorporation of D_{AB} and D is essential for the dualization of the theory. We will see that upon reducing to the physical, 'undoubled' space-time and breaking $O(D,D)$ to $GL(D)$, the dual theory and its fields can be matched precisely with what one should expect for the dualization of the 'component' fields (i.e. without employing the DFT formalism). However, the D-fields contain extra fields that are *not* pure gauge under the (double) local Lorentz symmetry (unlike, say, the antisymmetric part of the linearized vielbein in Einstein gravity). Hence, it seems that to formulate a duality- and gauge-invariant dual theory, extra fields are required comparing to the standard dual fields of supergravity. This match requires a careful analysis of so-called 'exotic' dualizations [61, 156], in which, for instance, the Kalb-Ramond 2-form B_2 is not dualized into a 6-form in $D = 10$ but into a gauge field with $(8,2)$ Young tableau symmetry. The precise dynamical implementation of such dualizations has only been investigated quite recently, in the work of Boulanger et. al. [161–163]. One of the novel features of such dualizations is that an action only exists provided extra fields are included which, however, are non-propagating and nicely fit into the spectrum of representations determined before by independent methods. Given the necessity of extra fields for exotic dualizations, we will see that we encounter extra fields in the dualization of DFT (which, again, do not upset the counting of degrees of freedom).

The rest of this chapter is organized as follows. To set the stage for the dualization of DFT, in Section (6.2) we review the standard dualization of p-form gauge potentials and the graviton at the linearized level. Moreover, we work out the dualization in 'string frame', i.e., in gravity plus dilaton, which shows some important differences to the dualization in Einstein frame. In Section (6.3) we turn to 'exotic' dualizations, and we discuss in detail the dualization of the Kalb-Ramond 2-form to a $(D-2,2)$ potential plus extra fields. In Section (6.4) we perform the dualization for linearized DFT. The geometric content of the dual DFT action is discussed in Section (6.5). In Section (6.6) we compare the DFT results with the component results and find precise agreement.

6.2 Standard Dualizations

6.2.1 p-form dualization

As a warm-up we start by recalling the dualization of the electromagnetic field in four dimensions. Starting with the Maxwell action

$$S[A] \;=\; -\frac{1}{4}\int d^4x\, F_{ab}\, F^{ab}\;, \tag{6.1}$$

where $F_{ab} = 2\partial_{[a}A_{b]}$, one moves to a first-order formulation where F_{ab} is an independent field, and the Bianchi identity is imposed by introducing a Lagrange multiplier $\tilde{A}_a$,

$$S[A,F] \;=\; \int d^4x\Big(-\frac{1}{4}F_{ab}F^{ab}+\frac{1}{2}\epsilon^{abcd}\tilde{A}_a\partial_bF_{cd}\Big)\;. \tag{6.2}$$

This action is gauge invariant under $\delta\tilde{A}_a = \partial_a\Lambda$, $\delta F_{ab} = 0$. Varying with respect to $\tilde{A}_a$ one obtains the Bianchi identity $\partial_{[a}F_{bc]} = 0$, which can be solved in terms of the Maxwell potential, giving back the original Maxwell theory. Conversely, one can solve for F in terms of $\tilde{A}_a$ to obtain the duality relation

$$F_{ab} = \frac{1}{2}\epsilon_{ab}{}^{cd}\tilde{F}_{cd}\;, \tag{6.3}$$

where $\tilde{F}_{ab} = 2\partial_{[a}\tilde{A}_{b]}$ is the dual field strength. Insertion into the action leads to the dual Maxwell action for $\tilde{F}_{ab}$.

By applying the same procedure in any dimension and for any p-form potential A_p, one obtains a dual $(D-p-2)$-form potential $\tilde{A}_{D-p-2}$, whose gauge paramenter is a $(D-p-3)$-form,

$$\delta\tilde{A}_{D-p-2} = d\Lambda_{D-p-3}\ . \tag{6.4}$$

In order to set the stage for the comparison with the dualization in DFT, we will often consider the Hodge duals of the potential $\tilde{A}_{D-p-2}$ and the gauge parameter Λ_{D-p-3}. The corresponding field is denoted by $\tilde{A}_{p+2}$ and the parameter by Λ_{p+3}. The field strengths and the gauge variation then take the divergence form

$$F^{a_1\ldots a_{p+1}} \ = \ \partial_b\tilde{A}^{ba_1\ldots a_{p+1}}\ , \qquad \delta\tilde{A}^{a_1\ldots a_{p+2}} = \partial_a\Lambda^{aa_1\ldots a_{p+2}}\ , \tag{6.5}$$

while the corresponding first-order action reads

$$S[\tilde{A}_{p+2},F_{p+1}] \ = \ \int d^Dx\Big(-\frac{1}{2\cdot(p+1)!}F_{a_1\ldots a_{p+1}}F^{a_1\ldots a_{p+1}}+\tilde{A}^{a_1\ldots a_{p+2}}\partial_{a_1}F_{a_2\ldots a_{p+2}}\Big)\ . \tag{6.6}$$

For instance, consider a 2-form b_2 in D dimensions with field strength $H_{abc} = 3\partial_{[a}b_{bc]}$. Starting from the standard action

$$S[b] \ = \ -\frac{1}{12}\int d^Dx\, H_{abc}\, H^{abc}\,, \tag{6.7}$$

we pass to a first order action with a fully antisymmetric 4-tensor D^{abcd} and 3-form H_{abc} as independent fields,

$$S[D,H] \ = \ \int d^Dx\Big(-\frac{1}{12}H_{abc}H^{abc} + D^{abcd}\partial_aH_{bcd}\Big)\ . \tag{6.8}$$

The equation for D^{abcd} gives the Bianchi identity for H, $\partial_{[a}H_{bcd]} = 0$, while the equation for H gives the duality relation

$$-\tfrac{1}{6}H^{abc} \ = \ \partial_dD^{dabc}\ . \tag{6.9}$$

The action and field equations are invariant under the gauge transformation

$$\delta D^{abcd} = \partial_e\Sigma^{eabcd}\,, \tag{6.10}$$

where Σ^{eabcd} is completely antisymmetric. The more familiar form of the duality relation is obtained by passing to the Hodge-dual $(D-4)$-form

$$\tilde{D}_{a_1\ldots a_{D-4}} \ \equiv \ \tfrac{1}{4!}\,\epsilon_{a_1\ldots a_{D-4}b_1\ldots b_4}\,D^{b_1\ldots b_4}\ , \tag{6.11}$$

in terms of which (6.9) reduces to the standard duality relation between the $(D-3)$-form field strength of this $(D-4)$-form potential and H. Alternatively, defining

$$\tilde{H}_{abc} \;\equiv\; -2\eta_{ad}\eta_{be}\eta_{cf}G^{def} \;\equiv\; -6\eta_{ad}\eta_{be}\eta_{cf}\partial_g D^{gdef}\,, \tag{6.12}$$

the above duality relation reads $H_{abc} = \tilde{H}_{abc}$. The 'field strength' G^{abc} in the above equation will appear naturally in Section (6.6). The equations of motion and Bianchi identity for the dual field are then swapped with respect to the original variables:

$$\text{(E.o.M)} \quad \partial_{[a}\tilde{H}_{bcd]} \;=\; 0\,, \tag{6.13}$$

$$\text{(B.I)} \qquad \partial_a\tilde{H}^{abc} \;=\; 0\,. \tag{6.14}$$

6.2.2 The dual graviton

We now repeat the same analysis for the dual of the D-dimensional graviton at the linearised level. We write the linearized Einstein-Hilbert action for the vielbein fluctuation $h_{a|b}$ (including the antisymmetric part, as indicated by the bar) as

$$S_{\text{EH}}[h] \;=\; \int d^Dx\,\left[f_{ab}{}^b f^{ac}{}_c - \tfrac{1}{2}f_{abc}f^{acb} - \tfrac{1}{4}f_{abc}f^{abc}\right]\,, \tag{6.15}$$

with the linearized coefficients of anholonomy,

$$f_{ab}{}^c \;=\; 2\partial_{[a}h_{b]|}{}^c\,. \tag{6.16}$$

These quantities satisfy the Bianchi identity

$$\partial_{[a}f_{bc]}{}^d \;=\; 0\,, \tag{6.17}$$

while the field equations obtained by variation with respect to h are

$$\partial^c f_{c(ab)} + \partial_{(a}f_{b)c}{}^c - \eta_{ab}\,\partial^c f_{cd}{}^d \;=\; 0\,. \tag{6.18}$$

We now pass to a first order action by adding the Lagrange multiplier $D^{abc}{}_d \equiv D^{[abc]}{}_d$ to impose the Bianchi identity,

$$S[f,D] \;=\; \int d^Dx\left(f_{ab}{}^b f^{ac}{}_c - \tfrac{1}{2}f_{abc}f^{acb} - \tfrac{1}{4}f_{abc}f^{abc} + 3D^{abc}{}_d\,\partial_a f_{bc}{}^d\right)\,. \tag{6.19}$$

Varying with respect to $D^{abc}{}_d$ and $f_{ab}{}^c$, respectively, gives

$$\partial_{[a}f_{bc]}{}^d \;=\; 0, \tag{6.20}$$

$$-\tfrac{1}{2}f^{ab}{}_c - f^{[a}{}_c{}^{b]} - 2\delta_c{}^{[a}f^{b]}{}_d{}^d \;=\; 3\,\partial_d D^{dab}{}_c\,.$$

The first equation implies locally that f takes the form (6.16). The second equation is then the duality relation between the graviton, contained in $h_{a|}{}^{b}$, and the dual graviton, contained in $D^{abc}{}_{d}$. From this duality relation we may recover the original (linearized) Einstein equations (6.18) by acting with ∂_a and using that the right-hand side gives zero by the 'Bianchi identity' $\partial_d\partial_a D^{dab}{}_c \equiv 0$.

Conversely, we can express the theory in terms of the dual variables. We first note that in terms of the 'field-strength' for the dual graviton,

$$G_a{}^{bc} \;\equiv\; 3\,\partial_d D^{dbc}{}_a \;, \tag{6.21}$$

the duality relation is equivalent to

$$f_{ab}{}^{c} \;=\; 2\,G_{[ab]}{}^{c} - \frac{2}{D-2}G_d{}^{d}{}_{[a}\delta_{b]}{}^{c} \;=\; 6\,\partial_e D^{e}{}_{[b}{}^{c}{}_{a]} - \frac{6}{D-2}\,\partial_e D^{e}{}_{d[a}{}^{d}\,\delta_{b]}{}^{c}\,, \tag{6.22}$$

where we reinserted the explicit potentials in the last step. Inserting now this expression for f in terms of D into (6.19) one obtains the dual action for D.

Let us discuss the physical content of the dual theory in a little more detail. To this end we decompose

$$D^{abc}{}_{d} \;=\; D^{(tr)abc}{}_{d} \;+\; 3\,\delta_d{}^{[a}D'^{bc]} \;, \tag{6.23}$$

where $D^{(tr)abc}{}_{d}$ is traceless and $D'^{ab} = \frac{1}{(D-2)}D^{abc}{}_{c}$ is the trace part. In order to further elucidate the representation content, consider the 'Hodge-dual' field

$$\tilde{D}_{a_1\ldots a_{D-3}|b} \;\equiv\; \tfrac{1}{6}\,\epsilon_{a_1\ldots a_{D-3}cde}\,D^{cde}{}_{b}\;, \tag{6.24}$$

whose irreducible $GL(D)$ representations are given by

$$(D-3)\;\otimes\;\Box\;=\;(D-3,1)\;\oplus\;(D-2)\;. \tag{6.25}$$

It is easy to see that the traceless potential $D^{(tr)abc}{}_{d}$ in eq. (6.23) corresponds to the $(D-3,1)$ mixed Young-tableau representation, while D'^{ab} corresponds to the totally antisymmetric $(D-2)$. It turns out that the totally antisymmetric representation is pure gauge. Indeed, the gauge invariance of the linearized Einstein-Hilbert action (6.15) under diffeomorphisms and local Lorentz transformations

$$\delta h_{a|b} = \partial_a\xi_b - \Lambda_{ab}\,, \tag{6.26}$$

elevates to a gauge invariance of the master action (6.19), acting on the fields (6.23) as

$$\delta_\Lambda D^{(tr)abc}{}_{d} \;=\; 0\,, \qquad \delta_\Lambda D'^{ab} \;=\; \tfrac{1}{3}\,\Lambda^{ab}\;. \tag{6.27}$$

Due to this Stückelberg invariance, the field D'^{ab} drops out of the action upon insertion of (6.22) into (6.19), leaving a two-derivative action for the physical dual graviton in the $(D-3,1)$ Young tableau representation.[1] The D-fields also possess gauge transformations that leave the 'field strength' $G_a{}^{bc}$ and hence the action and duality relations invariant,

$$\delta_\Sigma D^{abc}{}_d \;=\; \partial_e \Sigma^{eabc}{}_d \,, \tag{6.28}$$

with the parameter $\Sigma^{eabc}{}_d = \Sigma^{[eabc]}{}_d$ (that could be decomposed into traceless and trace part in order to obtain the gauge transformations of $D^{(tr)abc}{}_d$ and D'^{ab}).

6.2.3 Dual graviton and dilaton

We now consider the dual graviton and dilaton together at the linearized level. We first consider a canonically coupled scalar (i.e. in Einstein frame) with Lagrangian $\mathcal{L} = R - \frac{1}{2}(\partial\varphi)^2$. We thus add to the linearized action (6.19) the first-order action[2]

$$S[f_a^{(E)}, D^{(E)ab}] \;=\; \int d^D x \Big(-\frac{1}{2} f^{(E)a} f_a^{(E)} + D^{(E)ab} \partial_a f_b^{(E)} \Big) \,, \tag{6.29}$$

where the antisymmetric $D^{(E)ab}$ is the Lagrange multiplier whose equation of motion yields the Bianchi identity

$$\partial_{[a} f_{b]}^{(E)} \;=\; 0 \,. \tag{6.30}$$

This implies locally $f_a^{(E)} = \partial_a \varphi$, from which we recover upon reinsertion into (6.29) the original scalar theory. Alternatively, varying with respect to $f_a^{(E)}$ gives the duality relation

$$f^{(E)a} \;=\; \partial_b D^{(E)ab} \,, \tag{6.31}$$

and eliminating $f_a^{(E)}$ accordingly from (6.29) yields the theory for the dual dilaton $D^{(E)ab}$ (or, equivalently, for the $(D-2)$-form potential). The action and duality relations are invariant under the gauge tranformation

$$\delta D^{(E)ab} = \partial_c \Sigma^{cab} \,, \tag{6.32}$$

[1] It should be emphasized that while the standard Einstein-Hilbert action can be written entirely in terms of the symmetric $h_{(ab)}$ and the dual action entirely in terms of the irreducible $(D-3,1)$, the dualization requires the presence of an antisymmetric part, either $h_{[ab]}$ or D'^{ab}, since in the master action (6.19) or the duality relations (6.20), local Lorentz invariance allows us to set only one to zero, not both.

[2] The superscript E refers to the Einstein-frame.

where Σ^{abc} is fully antisymmetric.

The above dualization of a scalar was completely decoupled from the dualization of gravity. For dualization in string frame, however, this picture changes significantly in that it will be the trace of the field $D^{abc}{}_d$ (c.f. the previous subsection) that becomes the dual dilaton, while the analogue of $D^{(E)ab}$ will be pure gauge, transforming with a shift under local Lorentz transformations.

We start from the action with Lagrangian $\mathcal{L} = e^{-2\phi}(R + 4(\partial\phi)^2)$ for the graviton-dilaton sytem, whose linearization yields

$$S[h_{a|b}, \phi] \;=\; \int d^D x \left(f^a f_a - \frac{1}{4} f_{ab}{}^c f^{ab}{}_c - \frac{1}{2} f_{ab}{}^c f^a{}_c{}^b \right) , \tag{6.33}$$

where

$$f_a \;\equiv\; f_{ab}{}^b + 2\,\partial_a\phi \;, \tag{6.34}$$

with the coefficients of anholonomy defined in (6.16). Varying with respect to the vielbein and the dilaton, respectively, one obtains the equations of motion

$$\partial_c f^c{}_{(ab)} + \partial_{(a} f_{b)} \;=\; 0 \;, \qquad \partial_a f^a \;=\; 0 \;. \tag{6.35}$$

The $f_{ab}{}^c$ and f_a satisfy the following Bianchi identities:

$$\partial_c f_{ab}{}^c + 2\,\partial_{[a} f_{b]} \;=\; 0 \;, \qquad \partial_{[a} f_{bc]}{}^d \;=\; 0 \;. \tag{6.36}$$

As before, we can pass to a first-order action with Lagrange multipliers $D^{abc}{}_d = D^{[abc]}{}_d$ and $D'^{ab} = D'^{[ab]}$ imposing the Bianchi identities,

$$\begin{aligned} S[f_a, f_{ab}{}^c, D, D'] \;=\; \int d^D x \Big(& f^a f_a - \frac{1}{4} f_{ab}{}^c f^{ab}{}_c - \frac{1}{2} f_{ab}{}^c f^a{}_c{}^b \\ & + 3D^{abc}{}_d\, \partial_a f_{bc}{}^d + D'^{ab}\big(\partial_c f_{ab}{}^c + 2\partial_a f_b\big)\Big). \end{aligned} \tag{6.37}$$

Varying with respect to the fundamental fields $D^{abc}{}_d$, D'^{ab}, f_a and $f_{ab}{}^c$, respectively, one obtains

$$\partial_{[a} f_{bc]}{}^d \;=\; 0, \tag{6.38}$$

$$\partial_c f_{ab}{}^c + 2\partial_{[a} f_{b]} \;=\; 0, \tag{6.39}$$

$$f^a \;=\; \partial_b D'^{ba}, \tag{6.40}$$

$$\frac{1}{2} f^{ab}{}_c - \frac{1}{2} f^a{}_c{}^b + \frac{1}{2} f^b{}_c{}^a \;=\; 3\partial_e D^{eab}{}_c + \partial_c D'^{ab} \;. \tag{6.41}$$

It is straightforward to see, using the Poincaré lemma, that the general solution of the first two equations, (6.38) and (6.39), give back (6.16) and (6.34), which upon back-substitution into the action gives the string frame action (6.33) for dilaton plus gravity. The final two equations above, (6.40) and (6.41), are duality relations, which allow us to recover the second-order equations of motion (6.35) as integrability conditions. To this end we act on (6.40) with ∂_a, which by the Bianchi identity $\partial_a \partial_b D'^{ab} \equiv 0$ implies

$$\partial_a f^a \;=\; 0 \quad \Leftrightarrow \quad \partial_a f^a{}_b{}^b + 2\,\partial^a \partial_a \phi \;=\; 0\,, \tag{6.42}$$

in agreement with the dilaton field equation (the second equation in (6.35)). In order to obtain the first equation in (6.35) we act with ∂_d on (6.41) to obtain

$$-\frac{1}{2}\partial_a f^{ab}{}_c - \frac{1}{2}\partial_a f^a{}_c{}^b + \frac{1}{2}\partial_a f^b{}_c{}^a \;=\; \partial_c \partial_a D'^{ab} \;=\; \partial_c f^b\,, \tag{6.43}$$

using in the last step the second duality relation (6.40). After lowering the index b and symmetrizing in (b,c), equation (6.43) becomes equivalent to the first equation in (6.35). Note that the antisymmetric combination in (b,c) is zero by the Bianchi identity (6.36). Thus, we have correctly recovered the equation of motion for the graviton.

We can also solve eqs. (6.40) and (6.41) for f_a and $f_{ab}{}^c$ in terms of the D-fields. Back-substitution into (6.37) then yields the dual theory, which we analyze now in a little more detail. Defining the dual field strengths

$$G_a{}^{bc} \;\equiv\; 3\,\partial_e D^{ebc}{}_a + \partial_a D'^{bc}\,, \qquad g^a \;\equiv\; \partial_b D'^{ba}\,, \tag{6.44}$$

we find

$$f_{ab}{}^c \;=\; g_{ab}{}^c \;\equiv\; 2G_{[ab]}{}^c\,, \qquad f_a \;=\; g_a\,, \tag{6.45}$$

where we introduced $g_{ab}{}^c$ and g_a for convenience. The equations of motion and Bianchi identities for the dual system are then

$$\text{B.I's:} \begin{cases} \partial_c g^c{}_{(ab)} + \partial_{(a} g_{b)} = 0\,, \\[2mm] \partial_c g_{ab}{}^c + 2\partial_{[a} g_{b]} = 0\,, \\[2mm] \partial_a g^a = 0\,. \end{cases} \qquad \text{E.o.M's:} \; \Big\{ \; \partial_{[a} g_{bc]}{}^d = 0\,. \tag{6.46}$$

In order to further analyze the content of these equations it is useful to decompose D as follows

$$D^{abc}{}_d \;=\; D^{(tr)abc}{}_d + 3\,\delta_d{}^{[a} \bar{D}^{bc]}\,, \tag{6.47}$$

where $D^{(tr)abc}{}_d$ is traceless and $\bar{D}^{ab} = \bar{D}^{[ab]}$ the trace part. The equations of motion for the components then read

$$\partial_e \partial_{[a} D^{(tr)e}{}_b{}^d{}_{c]} \ = \ 0 \ , \qquad \partial_c \partial_{[a} \bar{D}^c{}_{b]} \ = \ 0 \ . \tag{6.48}$$

Note that the D'^{ab}'s dropped out, which means that they are subject to a Stückelberg symmetry. From the duality relations (6.45) and the split (6.47) it is easy to obtain the usual duality relation between the dilaton and the dual dilaton:

$$f_a - f_{ab}{}^b = g_a - g_{ab}{}^b \ \Rightarrow \ 2\partial_a \phi = -3(D-2)\partial_c \bar{D}^c{}_a. \tag{6.49}$$

We observe that D'^{ab} disappears and the field $\bar{D}^{ab}$ (the trace of $D^{abc}{}_d$) is the dual dilaton, which is the opposite of the situation in Einstein frame.

We close this subsection by discussing the gauge transformations for the D-fields. The duality relations (6.40) and the master action are invariant under local Lorentz transformations with D'^{ab} and $D^{abc}{}_d$ transforming as

$$\delta_\Lambda D^{abc}{}_d \ = \ 0 \ , \qquad \delta_\Lambda D'^{ab} \ = \ \Lambda^{ab} \ . \tag{6.50}$$

With respect to the decomposition (6.47) this implies in particular $\delta_\Lambda D^{(tr)abc}{}_d = 0$ and $\delta_\Lambda \bar{D}_{ab} = 0$, implying that the physical dual graviton and the dual dilaton are invariant. The D-fields also possess gauge transformations that leave the 'field strengths' $G_a{}^{bc}$ and g^a (and hence the duality relations and action) invariant,

$$\delta_\Sigma D^{abc}{}_d \ = \ \partial_e \Sigma^{eabc}{}_d + \partial_d \Sigma^{abc} \ , \qquad \delta_\Sigma D'^{ab} \ = \ -3\partial_e \Sigma^{eab} \ . \tag{6.51}$$

The gauge parameters satisfy $\Sigma^{eabc}{}_d = \Sigma^{[eabc]}{}_d$ and $\Sigma^{abc} = \Sigma^{[abc]}$. One may decompose into traceless and trace parts in order to read off the transformations for $D^{(tr)abc}{}_d$ and $\bar{D}^{ab}$.

6.3 Exotic Dualization of Kalb-Ramond field

In this section we will discuss the dualization of a 2-form gauge potential ('the B-field') into an exotic mixed Young tableaux fields. We first review general aspects of such mixed Young tableaux gauge fields and then turn to a master action that can be used to dualize the B-field into such a tensor, provided extra fields are included. These fields are quite unusual in that they are not auxiliary (they cannot be eliminated algebraically) nor pure gauge, yet they do not add to the propagating degrees of freedom.

6.3.1 Generalities of $(D-2,2)$ Young tableau gauge fields

We consider a gauge field in the $(D-2,2)$ Young diagram representation:

$$B_{a_1\dots a_{D-2},bc} \;\equiv\; B_{[a_1\dots a_{D-2}],bc} \;\equiv\; B_{a_1\dots a_{D-2},[bc]}\;, \qquad B_{[a_1\dots a_{D-2},b]c} \;\equiv\; 0\,. \quad (6.52)$$

There are two types of gauge parameters, $\mu \in (D-3,2)$ and $\lambda \in (D-2,1)$, acting as[3]

$$\begin{aligned}\delta B_{a_1\dots a_{D-2},bc} &= (D-2)\,\partial_{[a_1}\mu_{a_2\dots a_{D-2}],bc} \\ &\qquad + \partial_{[\underline{b}}\lambda_{a_1\dots a_{D-2},\underline{c}]} + \tfrac{1}{2}(D-2)\,\partial_{[a_1}\lambda_{|bc|a_2\dots a_{D-3},a_{D-2}]}\;.\end{aligned} \quad (6.53)$$

These gauge transformations preserve the algebraic constraints on B. We can define a gauge invariant curvature, starting from the first-order generalized Christoffel symbol

$$\Gamma_{a_1\dots a_{D-1},bc} \;\equiv\; (D-1)\,\partial_{[a_1}B_{a_2\dots a_{D-1}],bc}\;, \quad (6.54)$$

which is invariant under μ transformations and satisfies the Bianchi identities

$$\Gamma_{[a_1\dots a_{D-1},b]c} \;=\; 0\;, \qquad \partial_{[a_1}\Gamma_{a_2\dots a_D],bc} \;=\; 0\,. \quad (6.55)$$

As common for Young tableaux fields with more than one column, this first-order object is not fully gauge invariant (it is analogous to the Christoffel symbols), because under λ transformations we have

$$\delta_\lambda\Gamma_{a_1\dots a_{D-1},bc} \;=\; (D-1)\,\partial_{[\underline{b}}\partial_{[a_1}\lambda_{a_2\dots a_{D-1}],\underline{c}]}\;. \quad (6.56)$$

A fully gauge invariant curvature is the Riemann-like tensor obtained by taking another derivative and antisymmetrizing over three indices,

$$\mathcal{R}_{a_1\dots a_{D-1},bcd} \;\equiv\; 3\,\partial_{[\underline{b}}\Gamma_{a_1\dots a_{D-1},\underline{cd}]}\;. \quad (6.57)$$

This Riemann tensor satisfies the Bianchi identities

$$\mathcal{R}_{[a_1\dots a_{D-1},b]cd} \;=\; 0\;, \qquad \partial_{[a_1}\mathcal{R}_{a_2\dots a_D],bcd} \;=\; 0\;, \quad (6.58)$$

and hence lives in the $(D-1,3)$ Young diagram representation.

Naively, one would now impose the Einstein-type field equations that set to zero the generalized Ricci tensor ${\mathcal{R}_{a_1\dots a_{D-2}}}^{d}{}_{,bcd}$, but it turns out that a theory with these field equations is actually topological. To see this note that these field

[3] We sometimes underline indices in order to indicate which indices participate in an antisymmetrization.

equations imply vanishing of the double-trace of the Riemann tensor, which by the equivalence

$$\mathcal{R}_{a_1\dots a_{D-3}}{}^{bc}{}_{,a_{D-2}bc} \;=\; 0 \quad \Leftrightarrow \quad \epsilon_{a_1\dots a_{D-3}}{}^{cde}\,\epsilon_{a_{D-2}}{}^{b_1\dots b_{D-1}}\,\mathcal{R}_{b_1\dots b_{D-1},cde} \;=\; 0\;, \tag{6.59}$$

implies vanishing of the full Riemann tensor and hence that the field is pure gauge. However, we can impose weaker field equations that do lead to propagating degrees of freedom, setting to zero the triple-trace of the Riemann tensor,

$$\mathcal{R}_{a_1\dots a_{D-4}}{}^{bcd}{}_{,bcd} \;=\; 0\;. \tag{6.60}$$

Note that these are the same number of equations as for the conventional dual of a 2-form ($D-p-2=D-4$), but now these are equations for the $(D-2,2)$ gauge field. Such dualities have been discussed by Hull in [61, 156]. This also proves that there can be no action principle implying (6.60) for the $(D-2,2)$ gauge field alone — simply because the variation with respect to the $(D-2,2)$ field would yield more equations. However, one can write an action that implies this field equation at the cost of introducing more fields (that are not pure gauge), which also serves as a master action proving the equivalence with the standard 2-form action, as we will now discuss.

6.3.2 Master action

In order to construct this master action we follow [161–163] and write the standard action for the Kalb-Ramond field up to total derivatives as

$$S[b] \;=\; -\tfrac{1}{12}\int d^Dx\, H^{abc}H_{abc} \;=\; -\tfrac{1}{4}\int d^Dx\,\big(\partial^a b^{bc}\,\partial_a b_{bc} - 2\,\partial_a b^{ab}\,\partial^c b_{cb}\big)\;. \tag{6.61}$$

We can then replace it by the first-order action

$$S[Q,D] \;=\; \int d^Dx\big(-\tfrac{1}{4}\,Q^{a|bc}Q_{a|bc} + \tfrac{1}{2}\,Q_{a|}{}^{ab}Q^{c|}{}_{cb} - \tfrac{1}{2}\,D^{ab|cd}\,\partial_a Q_{b|cd}\big)\;, \tag{6.62}$$

where the fields have the symmetries

$$Q_{a|bc} \;=\; -Q_{a|cb}\;, \qquad D_{ab|cd} \;=\; -D_{ba|cd} \;=\; -D_{ab|dc}\;. \tag{6.63}$$

Note that, as usual for master actions, these fields do not live in irreducible representations. The above action may seem like a rather unnatural rewriting of a 2-form theory, but we will see in Section (6.6) that, in the appropriate sector, DFT reproduces precisely such an action. This action is invariant under the gauge transformations

$$\begin{aligned} \delta Q_{a|bc} &= \partial_a K_{bc}\,, \qquad K_{ab} \;\equiv\; 2\partial_{[a}\tilde{\xi}_{b]}\,, \\ \delta D_{ab,cd} &= \partial^e \Sigma_{eab|cd} + 4\,\eta_{[a[\underline{c}}K_{b]\underline{d}]}\,, \end{aligned} \tag{6.64}$$

where $\Sigma_{abc|de} \equiv \Sigma_{[abc]|de} \equiv \Sigma_{abc|[de]}$.

Let us now verify the equivalence with the second-order action. We vary with respect to D and Q, respectively, to obtain

$$\begin{aligned} \partial_{[a}Q_{b]|cd} &= 0 \quad \Rightarrow \quad Q_{a|bc} \;=\; \partial_a b_{bc}\,, \\ \partial^d D_{da|bc} &= Q_{a|bc} - \eta_{ab}Q^d{}_{|dc} + \eta_{ac}Q^d{}_{|db}\,. \end{aligned} \tag{6.65}$$

Reinserting the solution of the first equation into the action we recover the original second-order action (6.61). Equivalently, at the level of the equations of motion, we can act on the second equation with ∂^a to obtain the Bianchi identity

$$0 \;=\; \partial^a\partial^d D_{da|bc} \;=\; \partial^a Q_{a|bc} - \partial_b Q^a{}_{|ac} + \partial_c Q^a{}_{|ab} \;=\; \partial^a\left(\partial_a b_{bc} - \partial_b b_{ac} + \partial_c b_{ab}\right)\,, \tag{6.66}$$

which becomes the standard second-order equation for b_{ab}. Thus, the first-order action is on-shell equivalent to the second-order action.

In order to determine the dual theory, we have to use the second equation in (6.65) (the duality relation) and solve for Q in terms of D,

$$Q_{a|bc} \;=\; \partial^d D_{da|bc} - \frac{2}{D-2}\,\eta_{a[\underline{b}}\,\partial^d D_{de|}{}^e{}_{\underline{c}]}\,, \tag{6.67}$$

which upon reinsertion into (6.62) yields the dual action for D,

$$\mathcal{L} \;=\; \frac{1}{4}\,\partial_a D^{ab|cd}\,\partial^e D_{eb|cd} - \frac{1}{2(D-2)}\,\partial_a D^{ab|}{}_b{}^c\,\partial^d D_{de|}{}^e{}_c\,. \tag{6.68}$$

Variation with respect to D yields the second-order equation

$$\partial_{[a}\partial^e D_{|e|b]|cd} - \frac{2}{D-2}\partial_{[a}\left(\eta_{b][c}\,\partial^e D_{ef|}{}^f{}_{d]}\right) \;=\; 0\,, \tag{6.69}$$

which is equivalent to the result obtained from (6.67) by taking a curl and using the Bianchi identity $\partial_{[a}Q_{b]|cd} = 0$.

In the remainder of this section, we will analyze the dual theory in a little more detail. We decompose D into its irreducible representations:

$$D_{ab|cd} \;:\quad \ydiagram{1,1} \otimes \ydiagram{1,1} \;=\; \ydiagram{1,1,1,1} \oplus \widetilde{\ydiagram{2,1,1}} \oplus \widetilde{\ydiagram{2,2}} \oplus \ydiagram{1,1} \oplus \ydiagram{2}\,, \tag{6.70}$$

where we decomposed at the right-hand side into traceless tableaux (indicated by a tilde) and the trace parts. Thus, the decomposition reads

$$D_{ab|cd} \;=\; \tilde{D}_{ab|cd} + 4\,\eta_{[a[\underline{c}}\,\widehat{C}_{b]|\underline{d}]}\;, \tag{6.71}$$

where $\tilde{D}$ is fully traceless, corresponding to the first three representations in (6.70), and $\widehat{C}_{a|b}$ is a general 2-tensor (with antisymmetric and symmetric parts), corresponding to the last two representations.

We will now show that the duality relations imply the correct equations for the $(D-2,2)$ field. In order to simplify the index manipulations we specialize to $D=4$, which shows already all essential features, and for which the conventional dual to the B-field is a scalar and the exotic dual is a $(2,2)$ tensor. In this case we can decompose D as

$$D_{ab|cd} \;=\; \tfrac{1}{2}\epsilon_{ab}{}^{ef}B_{ef,cd} + 4\eta_{[a[\underline{c}}C_{b]|\underline{d}]} - 2\eta_{c[a}\eta_{b]d}C\;, \tag{6.72}$$

where B is the 'epsilon-dual' form of the traceless $\tilde{D}$ in (6.70) and hence lives in the $(2,2)$ Young tableau. Moreover, we have redefined the general 2-tensor for later convenience,

$$C_{a|b} \;\equiv\; \widehat{C}_{a|b} - \tfrac{1}{2}\eta_{ab}\widehat{C}\;. \tag{6.73}$$

The Σ gauge symmetries can be decomposed as follows

$$\Sigma_{abc|ef} \;\equiv\; \epsilon_{abc}{}^{d}\tilde{\Sigma}_{ef|d}\,, \qquad \tilde{\Sigma}_{ab|c}\;:\; \ydiagram{1,1} \otimes \ydiagram{1} \;=\; \ydiagram{1,1,1} \oplus \ydiagram{2,1}\;, \tag{6.74}$$

so that we can write

$$\tilde{\Sigma}_{ab|c} \;=\; \lambda_{ab,c} + \epsilon_{abcd}\,\xi^{d}\;, \tag{6.75}$$

where $\lambda \in (2,1)$ and ξ is a new vector gauge parameter. Applying the gauge transformations (6.64) to (6.72) and using this decomposition of the gauge parameter one finds the following gauge transformations for the component fields:

$$\begin{aligned}\delta B_{ab,cd} \;&=\; \partial_{[a}\lambda_{|cd|,b]} + \partial_{[c}\lambda_{|ab|,d]}\;,\\ \delta C_{a|b} \;&=\; 2\partial_{[a}\tilde{\xi}_{b]} - \partial_b\xi_a + \tfrac{1}{4}\epsilon_a{}^{cde}\partial_c\lambda_{de,b}\;.\end{aligned} \tag{6.76}$$

The transformation in the first line is precisely the expected gauge transformation of a $(2,2)$ gauge field, c.f. (6.53), while the symmetry parametrized by μ in (6.53) trivializes in $D=4$ because there is no $(1,2)$ Young tableau. Note that the extra field $C_{a|b}$ transforms under the gauge symmetry parametrized by $\lambda_{ab,c}$. The duality relation (6.67) in terms of B and C reads

$$Q_{b|cd} \;=\; -\tfrac{1}{3!}\epsilon_b{}^{efg}\,\Gamma_{efg,cd} \;+\; 2\partial_{[c}C_{b|d]}\;, \tag{6.77}$$

with the generalized Christoffel symbol (6.54). It is an instructive exercise to verify the gauge invariance of this equation: Under 'b-field gauge transformations' with parameter $\tilde{\xi}_a$ the left- and right-hand sides are *not* invariant, but their respective variations precisely cancel. The right-hand side is manifestly invariant under the ξ_a transformations, while under λ transformations the variations of the two terms on the right-hand side cancel.

We next show that the duality relation implies as integrability condition the desired field equation for the $(2,2)$ field. To this end we act on (6.77) with $\epsilon^{abij}\partial_a$, for which the left-hand side gives zero, and one obtains

$$0 \;=\; -\partial_a\Gamma^{aij}{}_{,cd} + 2\epsilon^{abij}\partial_a\partial_{[c}C_{b|d]}\;. \tag{6.78}$$

Now summing over i,c and j,d, the second term depending on C drops out, leaving

$$0 \;=\; -\partial_a\Gamma^{acd}{}_{,cd} \;\equiv\; -\partial_{[a}\Gamma^{acd}{}_{,cd]} \qquad\Leftrightarrow\qquad \mathcal{R}^{abc}{}_{,abc} \;=\; 0\;. \tag{6.79}$$

Thus, we obtained the expected field equation (6.60) for $D=4$, which proves that the $(2,2)$ gauge field propagates the single degree of freedom of the b-field in $D=4$.

6.3.3 Dual action

Let us finally determine and analyze the Lagrangian in terms of the dual fields, obtained by substituting (6.72) into (6.68),

$$\begin{aligned}\mathcal{L}[B,C] \;=\; &-\tfrac{1}{24}\,\Gamma^{abc,de}\,\Gamma_{abc,de} - \tfrac{1}{3!}\,\epsilon^{abcd}\,\Gamma_{bcd,}{}^{ef}\,\partial_e C_{a|f}\\ &+\tfrac{1}{2}\,\partial^a C^{b|c}\,\partial_a C_{b|c} - \tfrac{1}{2}\,\partial^c C^{a|b}\,\partial_b C_{a|c} - \tfrac{1}{2}\,\partial^c C^{a|b}\,\partial_a C_{c|b}\\ &+\partial_a C^{a|b}\,\partial_b C - \tfrac{1}{2}\,\partial^a C\,\partial_a C\;.\end{aligned} \tag{6.80}$$

It is amusing to write this in a slightly more geometric form by defining the generalized 'Einstein tensor'

$$\mathcal{G}_{a|b} \;\equiv\; \tfrac{1}{2}\big(-\Box C_{a|b} + \partial^c\partial_a C_{c|b} + \partial^c\partial_b C_{a|c} - \partial_a\partial_b C + \eta_{ab}(\Box C - \partial^c\partial^d C_{c|d})\big)\;, \tag{6.81}$$

which satisfies the Bianchi identities

$$\partial^a \mathcal{G}_{a|b} \;=\; \partial^b \mathcal{G}_{a|b} \;=\; 0\;, \tag{6.82}$$

and in terms of which the action reads

$$\mathcal{L} \;=\; -\tfrac{1}{24}\,\Gamma^{abc,de}\,\Gamma_{abc,de} - \tfrac{1}{3!}\,\epsilon^{abcd}\,\Gamma_{bcd,}{}^{ef}\,\partial_e C_{a|f} + C^{a|b}\mathcal{G}_{a|b}(C)\;. \tag{6.83}$$

Note that decomposing C into symmetric and antisymmetric parts, $C_{a|b} = s_{ab} + a_{ab}$, with $s_{ab} \equiv s_{(ab)}$, $a_{ab} \equiv a_{[ab]}$, the generalized Einstein tensor becomes

$$\mathcal{G}_{a|b}(s,a) \;=\; G_{ab}(s) - \tfrac{1}{2}\,\partial^c h_{cab}(a)\;, \tag{6.84}$$

in terms of the standard 3-form curvature $h_{abc} \equiv 3\partial_{[a}a_{bc]}$ and the (linearized) Einstein tensor $G_{ab} = R_{ab} - \frac{1}{2}R\eta_{ab}$, where

$$R_{ab} \;\equiv\; \partial_a \gamma_c{}^c{}_{,b} - \partial^c \gamma_{ca,b}\;, \qquad \gamma_{ab,c} \;\equiv\; \tfrac{1}{2}(\partial_a s_{bc} + \partial_b s_{ac} - \partial_c s_{ab})\;. \tag{6.85}$$

The above Lagrangian can then be written as

$$\begin{aligned}\mathcal{L} \;=\; &-\tfrac{1}{24}\,\Gamma^{abc,de}\,\Gamma_{abc,de} + \tfrac{1}{12}\,\epsilon^{abcd}\,\Gamma_{bcd,}{}^{ef}\,h_{aef} - \tfrac{1}{6}\,\epsilon^{abcd}\,\Gamma_{bcd,}{}^{ef}\,\gamma_{ae,f}\\ &+ s^{ab}G_{ab}(s) + \tfrac{1}{6}\,h^{abc}h_{abc}\;.\end{aligned} \tag{6.86}$$

Curiously, one obtains the conventional (linearized) Einstein-Hilbert term for s_{ab} plus the standard kinetic term for a_{ab}, both multiplied by an overall factor of -2. These wrong-sing kinetic terms for a 'graviton' and a 'Kalb-Ramond field' naively would lead one to conclude that this theory propagates a ghost-like spin-2 mode and (in $D = 4$) a scalar mode. However, since the action is not diagonal and since these fields are subject to larger gauge symmetries parameterized by $\lambda_{ab,c}$, there is no conflict with the equivalence to a single scalar mode, which is guaranteed by the construction from a master action.

As a consistency check, let us verify that this action indeed implies the expected field equation for the $(2,2)$ field. Varying (6.83) with respect to $B_{ab,cd}$ and $C_{a|b}$, respectively, yields

$$\begin{aligned}\partial^e\Gamma_{e\langle ab,cd\rangle} - R^{\star}_{\langle ab,cd\rangle}(C) \;&=\; 0\;,\\ \mathcal{G}_{a|b}(C) + \tfrac{1}{12}\,\epsilon_{acde}\,\partial_f\Gamma^{cde,f}{}_b \;&=\; 0\;,\end{aligned} \tag{6.87}$$

where $\langle\ \rangle$ denotes the projection onto the $(2,2)$ Young diagram representation,[4] and we defined the analogue of the linearized Riemann tensor for $C_{a|b}$ and its dualization

$$R_{abcd}(C) \;\equiv\; 4\,\partial_{[\underline{c}}\,\partial_{[a}\,C_{b]|\underline{d}]}\;, \qquad R^{\star}_{ab,cd}(C) \;\equiv\; \tfrac{1}{2}\,\epsilon_{ab}{}^{ef}\,R_{ef,cd}(C)\;. \tag{6.89}$$

This Riemann tensor satisfies the Bianchi identity $R_{[abcd]} = 0$, which in turn implies that the double trace of $R^{\star}_{ab,cd}$ vanishes (note, however, that $R_{[abc]d}$ generally is non-zero because C carries an antisymmetric part). As a consequence, taking the double trace of the first equation in (6.87), the C dependent term drops out, implying the required field equation $\mathcal{R}^{abc}{}_{,abc} = 0$, precisely as in (6.79). The $(2,2)$ projection of the dual Riemann tensor in (6.87) plays a role analogous to the Weyl tensor in Einstein gravity, where it is left undetermined by the field equations and hence encodes the propagating graviton degrees of freedom. Here, on the contrary, the tensor $R^{\star}_{\langle ab,cd\rangle}$ is fully determined by the $(2,2)$ gauge potential, in agreement with the non-propagating nature of $C_{a|b}$.

6.4 Dualizations in Linearized DFT

In this section we discuss the relations between dual and standard fields in Double Field Theory (DFT), using linearized DFT in the frame formulation [14, 15, 19, 53, 119]. We will add Lagrange multipliers (denoted as D-fields in the following) to the linearized DFT action in order to enforce the Bianchi identities. This will allow us to obtain duality relations between the conventional fields and the D-fields and, as integrability conditions, second order differential equations.

6.4.1 Linearized DFT in frame formulation

We summarize the tools that will be needed in this chapter (see § (3.5)). The fundamental fields in the frame formulation of DFT are the generalized vielbein $E_A{}^M$ or $\mathcal{E}_A{}^M$ and the generalized dilaton d. In a possible frame formulation of DFT the subgroup $H = O(D-1,1)\times O(D-1,1)$ is embedded canonically, indicated by the index split of the doubled Lorentz indices $A = (a,\bar{a})$, $a,\bar{a} = 0,\ldots,D-1$, under which the flattened metric is assumed to be diagonal,

$$\mathcal{G}_{AB} \;\equiv\; E_A{}^M E_B{}^N \eta_{MN} \;\equiv\; 2\,\mathrm{diag}(-\eta_{ab},\eta_{\bar{a}\bar{b}})\;, \tag{6.90}$$

[4] Explicitly, acting on a tensor $X_{ab|cd}$ that is antisymmetric in each index pair, this projector reads

$$X_{\langle ab|cd\rangle} \;\equiv\; \tfrac{1}{4}\big(X_{ab|cd} + X_{cd|ab} + \tfrac{1}{2}X_{ac|bd} - \tfrac{1}{2}X_{bc|ad} - \tfrac{1}{2}X_{ad|bc} + \tfrac{1}{2}X_{bd|ac}\big)\;. \tag{6.88}$$

where η_{ab} and $\eta_{\bar{a}\bar{b}}$ are two copies of the flat D-dimensional Lorentz metric diag$(-+\cdots+)$. Another possibility is given by choosing the flattened metric so that it takes the same form as the $O(D,D)$ metric,

$$\eta_{AB} \;\equiv\; \mathcal{E}_A{}^M \mathcal{E}_B{}^N \eta_{MN} \;=\; \begin{pmatrix} 0 & \delta^a{}_b \\ \delta_a{}^b & 0 \end{pmatrix}, \tag{6.91}$$

The flat indices split as $A = ({}^a, {}_a)$. The tangent space indices are raised and lowered with η_{AB} or $\mathcal{G}_{AB}$, depending on the formalism. In the formalism based on (6.91), we define the $O(D-1,1)\times O(D-1,1)$ invariant metric

$$S_{AB} \;\equiv\; \begin{pmatrix} \eta^{ab} & 0 \\ 0 & \eta_{ab} \end{pmatrix}, \tag{6.92}$$

where η_{ab} and η^{ab} are again two copies of the flat Lorentz metric[5]. In the following we use the perturbation theory for both formalisms, with frame fields subject to either (6.90) or (6.91), because each is more convenient for different purposes. In the remainder of this section we discuss the formalism based on (6.91), using the conventions of [53], while the formalism based on (6.90) will be discussed and applied in Section (6.5).

Let us now discuss the perturbation theory in this frame-like formalism, whose details have been developed in [164] for flat and curved backgrounds. Here we consider perturbations around a constant background, writing

$$\mathcal{E}_A{}^M \;=\; \bar{\mathcal{E}}_A{}^M + h_A{}^B\,\bar{\mathcal{E}}_B{}^M\,. \tag{6.93}$$

The constraint (6.91), which requires $\mathcal{E}_A{}^M$ to be $O(D,D)$ valued, implies to first order in the fluctuation $h_{AB} + h_{BA} = 0$. We thus assume h_{AB} to be antisymmetric. [6] Moreover, in the following we denote the linearization of the dilaton by d and its background value by $\bar{d}$. The linearized theory is naturally written in terms of generalized coefficients of anholonomy, also known as generalized fluxes, which are defined as

$$\mathcal{F}_{ABC} \;=\; 3\,\mathcal{D}_{[A}h_{BC]}\,, \qquad \mathcal{F}_A \;=\; \mathcal{D}^B h_{BA} + 2\,\mathcal{D}_A d\,, \tag{6.94}$$

with the flattened (doubled) derivative

$$\mathcal{D}_A \;\equiv\; \bar{\mathcal{E}}_A{}^M\partial_M\,. \tag{6.95}$$

[5] We will make an abuse of notation in this chapter by calling s_{ab} $(s^{ab}) \to \eta_{ab}$ (η^{ab}). This is to facilitate the comparison of the supergravity results with the DFT results in Section (6.6).

[6] Note, however, that beyond first order this relation gets modified.

Note that in DFT we impose the 'strong constraint' $\partial^M X\,\partial_M Y = \partial^M\partial_M X = 0$ for any fields X, Y, which then implies $\mathcal{D}^A\mathcal{D}_A = 0$ acting on arbitrary objects (which we will sometimes abbreviate as $\mathcal{D}^2 = 0$). It is then easy to verify that the above coefficients of anholonomy satisfy the Bianchi identities

$$\begin{aligned} \mathcal{D}_{[A}\mathcal{F}_{BCD]} &= 0\,, \\ \mathcal{D}^C\mathcal{F}_{CAB} + 2\,\mathcal{D}_{[A}\mathcal{F}_{B]} &= 0\,, \\ \mathcal{D}^A\mathcal{F}_A &= 0\,. \end{aligned} \tag{6.96}$$

Conversely, it is straightforward to prove, using the Poincaré lemma and the strong constraint $\mathcal{D}^A\mathcal{D}_A = 0$, that the general solution of these equations is given by (6.94).

Let us now turn to the linearized DFT action, which takes the form

$$S_{DFT} \;=\; \int d^{2D}X\, e^{-2\bar{d}}\left(S^{AB}\mathcal{F}_A\mathcal{F}_B + \frac{1}{6}\,\breve{\mathcal{F}}^{ABC}\mathcal{F}_{ABC}\right)\,, \tag{6.97}$$

where $\breve{\mathcal{F}}^{ABC}$ is defined as:

$$\breve{\mathcal{F}}^{ABC} \;\equiv\; \breve{S}^{ABCDEF}\mathcal{F}_{DEF}\,, \tag{6.98}$$

with the short-hand notation

$$\breve{S}^{ABCDEF} = \frac{1}{2}S^{AD}\eta^{BE}\eta^{CF} + \frac{1}{2}\eta^{AD}S^{BE}\eta^{CF} + \frac{1}{2}\eta^{AD}\eta^{BE}S^{CF} - \frac{1}{2}S^{AD}S^{BE}S^{CF}\,. \tag{6.99}$$

The tensors $\breve{S}$ and S satisfy the following identities:

$$\breve{S}_{ABC}{}^{GHI}\breve{S}_{GHI}{}^{DEF} \;=\; \delta_A{}^D\delta_B{}^E\delta_C{}^F\,, \qquad S_A{}^B S_B{}^C \;=\; \delta_A{}^C. \tag{6.100}$$

The action (6.97) is invariant under infinitesimal generalized diffeomorphisms (with the generalized coefficients of anholonomy being invariant to first order) and local double Lorentz transformations $\delta_\Lambda h_{AB} = \Lambda_{AB}$, with infinitesimal parameter Λ_{AB} satisfying

$$\Lambda_{AB} = -\Lambda_{BA}\,, \qquad S_A{}^C\Lambda_{CB} = S_B{}^C\Lambda_{AC}\,. \tag{6.101}$$

In fact, the local Lorentz group leaves invariant the two metrics (6.91) and (6.92), which defines an $O(D-1,1)\times O(D-1,1)$ subgroup of $O(D,D)$. Under these doubled Lorentz transformations, the coefficients of anholonomy transform as

$$\delta_\Lambda\mathcal{F}_{ABC} \;=\; 3\,\mathcal{D}_{[A}\Lambda_{BC]}\,, \qquad \delta_\Lambda\mathcal{F}_A \;=\; \mathcal{D}^B\Lambda_{BA}\,. \tag{6.102}$$

The equations of motion following from the linearized DFT action (6.97) for h_{AB} and d, respectively, are given by

$$2\mathcal{D}^{[B}\mathcal{F}_A\, S^{C]A} + \mathcal{D}_A\breve{\mathcal{F}}^{ABC} \;=\; 0\,, \tag{6.103}$$

$$2S^{AB}\mathcal{D}_B\mathcal{F}_A \;=\; 0\,. \tag{6.104}$$

6.4.2 Master action and duality relations

We now pass to a first-order or master action as in previous sections, promoting $\mathcal{F}_A$ and $\mathcal{F}_{ABC}$ to independent fields and introducing (totally antisymmetric) Lagrange multipliers D^{ABCD}, D^{AB} and D that enforce the Bianchi identities. The action thus reads

$$\begin{aligned} S \;=\; \int dX\, e^{-2\bar{d}}\Big[&S^{AB}\mathcal{F}_A\mathcal{F}_B + \frac{1}{6}\,\breve{\mathcal{F}}^{ABC}\mathcal{F}_{ABC} \\ &+ D^{ABCD}\,\mathcal{D}_A\mathcal{F}_{BCD} + D^{AB}\big(\mathcal{D}^C\mathcal{F}_{CAB} + 2\,\mathcal{D}_A\mathcal{F}_B\big) + D\,\mathcal{D}^A\mathcal{F}_A\Big]\,. \end{aligned} \tag{6.105}$$

Varying with respect to the fundamental fields D^{ABCD}, D^{AB}, D, $\mathcal{F}_{ABC}$ and $\mathcal{F}_A$, respectively, we obtain the field equations

$$\mathcal{D}_{[A}\mathcal{F}_{BCD]} \;=\; 0\,, \tag{6.106}$$

$$\mathcal{D}^C\mathcal{F}_{CAB} + 2\,\mathcal{D}_{[A}\mathcal{F}_{B]} \;=\; 0\,, \tag{6.107}$$

$$\mathcal{D}^A\mathcal{F}_A \;=\; 0\,, \tag{6.108}$$

$$\breve{\mathcal{F}}^{ABC} \;=\; 3\,\big(\mathcal{D}_D D^{DABC} + \mathcal{D}^{[A}D^{BC]}\big)\,, \tag{6.109}$$

$$2S^{AB}\mathcal{F}_B - 2\,\mathcal{D}_B D^{BA} - \mathcal{D}^A D \;=\; 0\,. \tag{6.110}$$

With the first three equations we recover the Bianchi identities, which can be solved as in (6.94), giving back the original (linearized) DFT. The last two equations (6.109) and (6.110) can then be interpreted as the duality relations. From these we may obtain the original second-order linearized DFT equations as integrability conditions. To this end, we act on eq. (6.109) with $\mathcal{D}_A$ and obtain

$$\mathcal{D}_A\breve{\mathcal{F}}^{ABC} \;=\; -2\,\mathcal{D}^{[B}\mathcal{D}_A D^{|A|C]}\,, \tag{6.111}$$

where we have used $\mathcal{D}_{[A}\mathcal{D}_{B]} = 0$ and the strong constraint $\mathcal{D}^2 = 0$. Now we can use (6.110) in order to eliminate $\mathcal{D}_A D^{AC}$ on the right-hand side, which gives back the linearized field equation (6.103). Similarly, by acting on eq. (6.110) with $\mathcal{D}_A$ and using the strong constraint one obtains the linearized dilaton equation of motion (6.104).

Let us now discuss the gauge symmetries in the dual formulation. First, the duality relations and master action are invariant under the following gauge transformations:

$$\begin{aligned}\delta D^{ABCD} &= \mathcal{D}_E \Sigma^{EABCD} + \mathcal{D}^{[A}\Sigma^{BCD]} \, , \\ \delta D^{AB} &= \mathcal{D}^{[A}\Sigma^{B]} + \frac{3}{4}\,\mathcal{D}_E \Sigma^{EAB} \, , \\ \delta D &= \mathcal{D}_A \Sigma^A \, ,\end{aligned} \tag{6.112}$$

where $\Sigma^{ABCDE} = \Sigma^{[ABCDE]}$ and $\Sigma^{ABC} = \Sigma^{[ABC]}$. The D-fields also transform under double Lorentz transformations. Using (6.102) in the above duality relations, one finds

$$\delta_\Lambda D_{ABCD} = 0 \, , \qquad \delta_\Lambda D_{AB} = -S^E{}_{[A}\Lambda_{B]E} \, , \qquad \delta_\Lambda D = 0 \, . \tag{6.113}$$

6.4.3 Dual DFT

Let us now investigate the equations of motion for the theory in terms of the dual D-fields. These are obtained from the Bianchi identities (6.96) and the duality relations (6.109)–(6.110). First, we need to solve the duality relations for the coefficients of anholonomy in terms of the dual D-fields, which yields, using eq. (6.100),

$$\mathcal{F}_{ABC} = 3\,\check{S}_{ABC}{}^{DEF}(\mathcal{D}^G D_{GDEF} + \mathcal{D}_{[D}D_{EF]}) \, , \tag{6.114a}$$

$$\mathcal{F}_A = S_A{}^B(\mathcal{D}^C D_{CB} + \frac{1}{2}\,\mathcal{D}_B D) \, . \tag{6.114b}$$

Inserting these into the Bianchi identities (6.96), we obtain

$$0 = \check{S}_{[ABC|EFG|}\,\mathcal{D}_{D]}\Big(\mathcal{D}_H D^{HEFG} + \mathcal{D}^{[E}D^{FG]}\Big) \, , \tag{6.115}$$

$$\begin{aligned} 0 = {} & 3\,\check{S}_{CABDEF}\mathcal{D}^C\Big(\mathcal{D}_G D^{GDEF} + \mathcal{D}^{[D}D^{EF]}\Big) \\ & + 2\,S_{C[B}\mathcal{D}_{A]}\Big(\mathcal{D}_D D^{DC} + \frac{1}{2}\,\mathcal{D}^C D\Big) \, ,\end{aligned} \tag{6.116}$$

$$0 = S_{AB}\mathcal{D}^A\mathcal{D}_C D^{CB} + \frac{1}{2}\,S^{AB}\mathcal{D}_A\mathcal{D}_B D \, . \tag{6.117}$$

In order to illuminate further these equations for the dual D-fields, let us introduce the following field strengths:

$$G^{ABC} \equiv 3\left(\mathcal{D}_D D^{DABC} + \mathcal{D}^{[A} D^{BC]}\right), \tag{6.118}$$

and

$$G^{A} \equiv \mathcal{D}_B D^{BA} + \frac{1}{2}\mathcal{D}^{A} D\,, \tag{6.119}$$

which are invariant under the Σ-transformations (6.112). In terms of these field strengths the duality relations take the following simpler form:

$$\begin{aligned} \mathcal{F}^{ABC} &= \breve{S}^{ABCDEF}\, G_{DEF}, \\ \mathcal{F}^{A} &= S^{AB}\, G_B\,. \end{aligned} \tag{6.120}$$

Finally, defining $\mathcal{G}_{ABC} \equiv \breve{S}_{ABC}{}^{DEF} G_{DEF}$ and $\mathcal{G}_A \equiv S_A{}^B G_B$, the second-order equations (6.115)–(6.117) for the dual fields take exactly the same form as the Bianchi identities for the original fields. Our final form of duality relations between fluxes and dual fluxes is then

$$\begin{aligned} \mathcal{F}_{ABC} &= \mathcal{G}_{ABC}\,, \\ \mathcal{F}_{A} &= \mathcal{G}_{A}\,. \end{aligned} \tag{6.121}$$

The set of equations for the original and dual system is summarized in Table 6.1.

	DFT	Dual DFT
E.o.M's	- - - $2\mathcal{D}^{[B}\mathcal{F}_A\, S^{C]A} + \mathcal{D}_A \breve{\mathcal{F}}^{ABC} = 0$ $2S^{AB}\mathcal{D}_A\mathcal{F}_B = 0$	$\mathcal{D}_{[D}\mathcal{G}_{ABC]} = 0$ $\mathcal{D}^C\mathcal{G}_{CAB} + 2\mathcal{D}_{[A}\mathcal{G}_{B]} = 0$ $\mathcal{D}^A\mathcal{G}_A = 0$
B.I's	$\mathcal{D}_{[D}\mathcal{F}_{ABC]} = 0$ $\mathcal{D}^C\mathcal{F}_{CAB} + 2\mathcal{D}_{[A}\mathcal{F}_{B]} = 0$ $\mathcal{D}^A\mathcal{F}_A = 0$	- - - $2\mathcal{D}^{[A}\mathcal{G}_C S^{B]C} + \mathcal{D}_C\breve{\mathcal{G}}^{CAB} = 0$ $2S^{AB}\mathcal{D}_A\mathcal{G}_B = 0$

TABLE 6.1
Comparison of equations of motion and Bianchi identities between DFT and dual DFT.

6.5 Geometric form of dual DFT action

In this section we elaborate on the geometric form of the dual DFT action. We first present a master action in terms of connections that, in a sense, is complementary to that presented in Section (6.4), but which leads to equivalent results. Finally, we determine the dual action and write it in a geometric form that is completely analogous to the dual action for the exotic duals discussed in Section (6.3).

6.5.1 DFT action in connection form

In order to define the master action in a (semi-)geometric form, let us first review the linearized frame-like geometry of DFT, based on a frame field $E_A{}^M$, where the flat indices split as $A = (a, \bar{a})$. Since the frame field is subject to (6.90), expansion about a constant background,

$$E_A{}^M \;=\; \bar{E}_A{}^M - h_A{}^B \bar{E}_B{}^M \;, \tag{6.122}$$

leads to the following first-order constraints on the fluctuations

$$h_{a\bar{b}} \;=\; -h_{\bar{b}a} \;, \qquad h_{ab} \;\equiv\; h_{[ab]} \;, \qquad h_{\bar{a}\bar{b}} \;\equiv\; h_{[\bar{a}\bar{b}]} \;. \tag{6.123}$$

The first field is physical, encoding the symmetric metric fluctuation and the antisymmetric b-field fluctuation. The final two fields are pure gauge with respect to the local $O(D-1,1)\times O(D-1,1)$ tangent space symmetry. Indeed, defining $\partial_A \equiv \langle E_A{}^M\rangle\partial_M$, the linearized gauge transformations can be written as

$$\delta h_{AB} \;=\; \partial_A\xi_B - \partial_B\xi_A + \Lambda_{AB} \;, \tag{6.124}$$

where $\Lambda_{AB} = \mathrm{diag}(\Lambda_{ab}, \Lambda_{\bar{a}\bar{b}})$, and therefore

$$\begin{aligned} \delta h_{a\bar{b}} &= \partial_a\xi_{\bar{b}} - \partial_{\bar{b}}\xi_a \;, \\ \delta h_{ab} &= 2\,\partial_{[a}\,\xi_{b]} + \Lambda_{ab} \;, \\ \delta h_{\bar{a}\bar{b}} &= 2\,\partial_{[\bar{a}}\,\xi_{\bar{b}]} + \Lambda_{\bar{a}\bar{b}} \;, \end{aligned} \tag{6.125}$$

while the dilaton transforms as

$$\delta d \;=\; -\tfrac{1}{2}(\partial_a\xi^a + \partial_{\bar{a}}\xi^{\bar{a}}) \;. \tag{6.126}$$

From (6.125) we infer that h_{ab} and $h_{\bar{a}\bar{b}}$ can be gauged away. The spin connection components of the linearized theory read

$$
\begin{aligned}
\omega_{a\bar{b}\bar{c}} &= -2\,\partial_{[\bar{b}}h_{|a|\bar{c}]} + \partial_a h_{\bar{b}\bar{c}}\ , \\
\omega_{\bar{a}bc} &= 2\,\partial_{[b}h_{c]\bar{a}} + \partial_{\bar{a}} h_{bc}\ , \\
\omega_a &\equiv \omega_{ba}{}^{b} = \partial^b h_{ab} + \partial^{\bar{b}} h_{a\bar{b}} + 2\partial_a d\ , \\
\omega_{\bar{a}} &\equiv \omega_{\bar{b}\bar{a}}{}^{\bar{b}} = -\partial^b h_{b\bar{a}} + \partial^{\bar{b}} h_{\bar{a}\bar{b}} + 2\partial_{\bar{a}} d\ , \\
\omega_{[abc]} &= \partial_{[a}h_{bc]}\ , \\
\omega_{[\bar{a}\bar{b}\bar{c}]} &= \partial_{[\bar{a}}h_{\bar{b}\bar{c}]}\ .
\end{aligned}
\tag{6.127}
$$

These objects indeed transform as connections for the doubled local Lorentz symmetry:

$$
\begin{aligned}
\delta\omega_{a\bar{b}\bar{c}} &= \partial_a \Lambda_{\bar{b}\bar{c}}\ , \qquad & \delta\omega_{\bar{a}bc} &= \partial_{\bar{a}}\Lambda_{bc}\ , \\
\delta\omega_a &= \partial^b \Lambda_{ab}\ , \qquad & \delta\omega_{\bar{a}} &= \partial^{\bar{b}}\Lambda_{\bar{a}\bar{b}}\ .
\end{aligned}
\tag{6.128}
$$

In particular, the connections are fully invariant under generalized diffeomorphisms. The above connections satisfy the Bianchi identities

$$
\begin{aligned}
\partial_a\omega^a + \partial_{\bar{a}}\omega^{\bar{a}} &= 0\ , \\
\partial^{\bar{a}}\omega_{\bar{a}bc} - 2\,\partial_{[b}\,\omega_{c]} + 3\,\partial^a\omega_{[abc]} &= 0\ , \\
\partial^{a}\omega_{a\bar{b}\bar{c}} - 2\,\partial_{[\bar{b}}\,\omega_{\bar{c}]} + 3\,\partial^{\bar{a}}\omega_{[\bar{a}\bar{b}\bar{c}]} &= 0\ , \\
\partial^c\omega_{\bar{b}ac} - \partial^{\bar{c}}\omega_{a\bar{b}\bar{c}} + \partial_a\omega_{\bar{b}} - \partial_{\bar{b}}\omega_a &= 0\ , \\
\partial_{[a}\,\omega_{|\bar{d}|bc]} - \partial_{\bar{d}}\,\omega_{[abc]} &= 0\ , \\
\partial_{[\bar{a}}\,\omega_{|d|\bar{b}\bar{c}]} - \partial_{d}\,\omega_{[\bar{a}\bar{b}\bar{c}]} &= 0\ , \\
\partial_{[a}\,\omega_{b]\bar{c}\bar{d}} + \partial_{[\bar{c}}\,\omega_{\bar{d}]ab} &= 0\ , \\
\partial_{[a}\,\omega_{bcd]} &= 0\ , \\
\partial_{[\bar{a}}\,\omega_{\bar{b}\bar{c}\bar{d}]} &= 0\ .
\end{aligned}
\tag{6.129}
$$

This is a rather extensive list of identities, but except for the first one they are all consequences of the algebraic Bianchi identity for the full Riemann tensor, $\mathcal{R}_{[ABC]D} = 0$, see [112, 114, 132, 133], and are also equivalent to (6.96).

We now give invariant curvatures in order to define the dynamics of linearized DFT. There is a linear generalized Riemann tensor,

$$\mathcal{R}_{ab,\bar{c}\bar{d}} \;\equiv\; \partial_{[a}\,\omega_{b]\bar{c}\bar{d}} - \partial_{[\bar{c}}\,\omega_{\bar{d}]ab} \;=\; -4\,\partial_{[a}\partial_{[\bar{c}}\,h_{b]\bar{d}]}\;, \tag{6.130}$$

which, however, does not have a non-linear completion. The linearized (generalized) Ricci tensor (which is *not* the trace of the above Riemann tensor) reads

$$\mathcal{R}_{a\bar{b}} \;\equiv\; -\partial^{c}\omega_{\bar{b}ac} + \partial_{\bar{b}}\,\omega_{a} \;\equiv\; -\partial^{\bar{c}}\omega_{a\bar{b}\bar{c}} + \partial_{a}\omega_{\bar{b}}\;, \tag{6.131}$$

where the equivalence of the two definitions follows from the fourth Bianchi identity in (6.129). The explicit expression in components reads

$$\mathcal{R}_{a\bar{b}} \;=\; \Box h_{a\bar{b}} - \partial_{a}\partial^{c}h_{c\bar{b}} + \partial_{\bar{b}}\partial^{\bar{c}}h_{a\bar{c}} + 2\partial_{a}\partial_{\bar{b}}d\;, \tag{6.132}$$

where $\Box \equiv \partial^{a}\partial_{a} \equiv -\partial^{\bar{a}}\partial_{\bar{a}}$. As it should be, the pure gauge degrees of freedom dropped out. Also note that there are differential Bianchi identities relating (6.130) to (6.131),

$$\partial^{\bar{c}}\mathcal{R}_{ab,\bar{c}\bar{d}} \;=\; -2\,\partial_{[a}\mathcal{R}_{b]\bar{d}}\;, \qquad \partial^{a}\mathcal{R}_{ab,\bar{c}\bar{d}} \;=\; 2\,\partial_{[\bar{c}}\mathcal{R}_{|b|\bar{d}]}\;. \tag{6.133}$$

The linearized scalar curvature is

$$\mathcal{R} \;\equiv\; -\partial^{a}\omega_{a} \;\equiv\; \partial^{\bar{a}}\omega_{\bar{a}} \;=\; -2\Box d - \partial^{a}\partial^{\bar{b}}h_{a\bar{b}}\;, \tag{6.134}$$

where we have given the explicit component expression in the last step. Finally, the linearized DFT action in terms of the connections reads

$$\begin{aligned}\mathcal{L}^{(2)}_{\rm DFT} \;=\; \frac{1}{2}\big(&\,\omega^{a\bar{b}\bar{c}}\omega_{a\bar{b}\bar{c}} + 3\,\omega^{[\bar{a}\bar{b}\bar{c}]}\omega_{[\bar{a}\bar{b}\bar{c}]} + 2\,\omega^{\bar{a}}\omega_{\bar{a}}\\ &- \omega^{\bar{a}bc}\omega_{\bar{a}bc} - 3\,\omega^{[abc]}\omega_{[abc]} - 2\,\omega^{a}\omega_{a}\big)\;,\end{aligned} \tag{6.135}$$

whose general variation reads $\delta\mathcal{L} = 4\delta h^{a\bar{b}}\,\mathcal{R}_{a\bar{b}} - 8\delta d\,\mathcal{R}$. Let us note that, upon inserting (6.127), the two lines in the above action actually give the same result, by virtue of the strong constraint and the relative sign between them, but for our present purposes this action is convenient because it treats barred and unbarred indices on the same footing.

6.5.2 Master action

We now give a first-order master action that can be used to define the dual theory and in which the connections are promoted to independent fields, in analogy to previous sections. Apart from that, the approach is complementary to that used in previous sections in that the dual fields do not enter the master action as Lagrange multipliers but rather emerge upon 'solving' the field equations by reinterpreting them as Bianchi identities. This approach is of course fully equivalent to that used before (the difference being whether the fields or their duals enter the master action that serves as the starting point), but it is reassuring to confirm explicitly that both procedures give the same result.

We now treat the connections as independent fields and replace the linearized DFT action (6.135) by the first-order action

$$\begin{aligned}\mathcal{L}^{(1)}_{\rm DFT} \;=\; &-\tfrac{1}{2}\omega^{a\bar{b}\bar{c}}\omega_{a\bar{b}\bar{c}} + \omega^{a\bar{b}\bar{c}}\big(-2\partial_{\bar{b}}h_{a\bar{c}} + \partial_a h_{\bar{b}\bar{c}}\big) - \tfrac{3}{2}\,\omega^{[\bar{a}\bar{b}\bar{c}]}\omega_{[\bar{a}\bar{b}\bar{c}]} + 3\;\omega^{[\bar{a}\bar{b}\bar{c}]}\partial_{\bar{a}}h_{\bar{b}\bar{c}}\\ &-\omega^{\bar{a}}\omega_{\bar{a}} + 2\,\omega^{\bar{a}}(-\partial^b h_{b\bar{a}} + \partial^{\bar{b}}h_{\bar{a}\bar{b}} + 2\partial_{\bar{a}}d)\\ &+\tfrac{1}{2}\omega^{\bar{a}bc}\omega_{\bar{a}bc} - \omega^{\bar{a}bc}\big(2\partial_b h_{c\bar{a}} + \partial_{\bar{a}}h_{bc}\big) + \tfrac{3}{2}\,\omega^{[abc]}\omega_{[abc]} - 3\,\omega^{[abc]}\partial_a h_{bc}\\ &+\omega^a\omega_a - 2\,\omega^a\big(\partial^{\bar{b}}h_{a\bar{b}} + \partial^b h_{ab} + 2\partial_a d\big)\;.\end{aligned} \tag{6.136}$$

The field equations for the ω determine them in terms of the physical fields as given in (6.127), so that reinserting into the action we recover (6.135). On the other hand, varying with respect to d, $h_{a\bar{b}}$, h_{ab} and $h_{\bar{a}\bar{b}}$, respectively, we obtain

$$\begin{aligned}\partial_{\bar{a}}\omega^{\bar{a}} - \partial_a\omega^a \;&=\; 0\;,\\ -\partial_{\bar{c}}\,\omega^{a\bar{b}\bar{c}} - \partial_c\,\omega^{\bar{b}ac} + \partial^a\omega^{\bar{b}} + \partial^{\bar{b}}\omega^a \;&=\; 0\;,\\ \partial_{\bar{a}}\omega^{\bar{a}bc} + 3\,\partial_a\omega^{[abc]} - 2\,\partial^{[b}\,\omega^{c]} \;&=\; 0\;,\\ \partial_a\omega^{a\bar{b}\bar{c}} + 3\,\partial_{\bar{a}}\omega^{[\bar{a}\bar{b}\bar{c}]} - 2\,\partial^{[\bar{b}}\,\omega^{\bar{c}]} \;&=\; 0\;.\end{aligned} \tag{6.137}$$

Expressing ω in terms of the physical fields, the first two equations give the DFT equations $\mathcal{R}_{a\bar{b}} = 0$ and $\mathcal{R} = 0$, while the last two equations are the second and third Bianchi identity in (6.129).

In order to determine the dual theory we interpret now all four of the equations (6.137) as Bianchi identities and solve them in terms of dual fields. We proceed hierarchically, starting with the first equation, which can be solved as

$$\begin{aligned}\omega^{\bar{a}} \;&=\; \partial_{\bar{b}}D^{\bar{a}\bar{b}} + \partial_b D^{b\bar{a}} + \partial^{\bar{a}}D\;,\\ \omega^a \;&=\; \partial_b D^{ab} + \partial_{\bar{b}}D^{a\bar{b}} - \partial^a D\;,\end{aligned} \tag{6.138}$$

with $D^{\bar{a}\bar{b}}$ and D^{ab} antisymmetric, $D^{b\bar{a}}$ unconstrained and a singlet D. This result can be obtained as follows. First, the non-singlet terms follow from the standard

Poincaré lemma, writing the equation as $\partial_A \Omega^A = 0$ for $\Omega^A \equiv (-\omega^a, \omega^{\bar{a}})$, which implies $\Omega^A = \partial_B D^{AB}$ for antisymmetric D^{AB}, whose components give the above D fields. The only subtlety is that the derivatives are subject to the strong constraint, which allows for the singlet term that drops out by $\partial^{\bar{a}}\partial_{\bar{a}} = -\partial^a\partial_a$. Thus, (6.138) is the general solution of the first equation in (6.137).

Next, we turn to the second equation in (6.137), where we can eliminate ω^a and $\omega^{\bar{a}}$ according to (6.138). We first solve the equation for the special case that all these D fields are zero:

$$-\partial_{\bar{c}}\,\omega^{a\bar{b}\bar{c}} - \partial_c\,\omega^{\bar{b}ac} \;=\; 0\,. \tag{6.139}$$

This is solved by

$$\begin{aligned}\omega^{a\bar{b}\bar{c}} \;&=\; \partial_{\bar{d}} D^{\bar{b}\bar{c}\bar{d},a} + \partial_d D^{da,\bar{b}\bar{c}}\,,\\ \omega^{\bar{b}ac} \;&=\; \partial_d D^{cda,\bar{b}} + \partial_{\bar{d}} D^{ac,\bar{b}\bar{d}}\,,\end{aligned} \tag{6.140}$$

where the D fields are antisymmetric in each group of similar indices. Including now the trace connections we need to solve the inhomogeneous equation

$$\partial_{\bar{c}}\,\omega^{a\bar{b}\bar{c}} + \partial_c\,\omega^{\bar{b}ac} \;=\; \partial^a\partial_{\bar{c}}D^{\bar{b}\bar{c}} + \partial^a\partial_c D^{c\bar{b}} + \partial^{\bar{b}}\partial_c D^{ac} + \partial^{\bar{b}}\partial_{\bar{c}} D^{a\bar{c}}\,, \tag{6.141}$$

where we note that the singlet D dropped out. This equation is solved by

$$\begin{aligned}\omega^{a\bar{b}\bar{c}} \;&=\; \partial^a D^{\bar{b}\bar{c}} + 2\,\partial^{[\bar{b}} D^{|a|\bar{c}]}\,,\\ \omega^{\bar{b}ac} \;&=\; \partial^{\bar{b}} D^{ac} + 2\,\partial^{[a} D^{c]\bar{b}}\,,\end{aligned} \tag{6.142}$$

which can be verified by employing the strong constraint again. Thus, the general solution is given by the sum of (6.140) and (6.142),

$$\begin{aligned}\omega^{a\bar{b}\bar{c}} \;&=\; \partial_{\bar{d}} D^{\bar{b}\bar{c}\bar{d},a} + \partial_d D^{da,\bar{b}\bar{c}} + \partial^a D^{\bar{b}\bar{c}} + 2\,\partial^{[\bar{b}} D^{|a|\bar{c}]}\,,\\ \omega^{\bar{b}ac} \;&=\; \partial_d D^{cda,\bar{b}} + \partial_{\bar{d}} D^{ac,\bar{b}\bar{d}} + \partial^{\bar{b}} D^{ac} + 2\,\partial^{[a} D^{c]\bar{b}}\,.\end{aligned} \tag{6.143}$$

Finally, we solve the last two equations in (6.137). Inserting (6.138) and (6.143) determines $\omega^{[abc]}$ up to solutions of $\partial_a\omega^{[abc]} = 0$, which by the Poincaré lemma are given by $\partial^d D_{abcd}$ for a new totally antisymmetric tensor D_{abcd}. Applying the same reasoning to $\omega^{[\bar{a}\bar{b}\bar{c}]}$ introduces the new field $D_{\bar{a}\bar{b}\bar{c}\bar{d}}$, and we finally find for the connections in terms of the dual fields,

$$\begin{aligned}
\omega^{a\bar{b}\bar{c}} &= \partial_{\bar{d}} D^{\bar{b}\bar{c}\bar{d},a} + \partial_d D^{da,\bar{b}\bar{c}} + \partial^a D^{\bar{b}\bar{c}} + 2\,\partial^{[\bar{b}} D^{|a|\bar{c}]} \,, \\
\omega^{\bar{a}bc} &= \partial_d D^{bcd,\bar{a}} + \partial_{\bar{d}} D^{bc,\bar{a}\bar{d}} + \partial^{\bar{a}} D^{bc} + 2\,\partial^{[b} D^{c]\bar{a}} \,, \\
\omega^{\bar{a}} &= \partial_{\bar{b}} D^{\bar{a}\bar{b}} + \partial_b D^{b\bar{a}} + \partial^{\bar{a}} D \,, \\
\omega^{a} &= \partial_b D^{ab} + \partial_{\bar{b}} D^{a\bar{b}} - \partial^a D \,, \\
\omega_{[abc]} &= \partial_{[a} D_{bc]} - \partial^d D_{abcd} - \tfrac{1}{3}\,\partial^{\bar{d}} D_{abc,\bar{d}} \,, \\
\omega_{[\bar{a}\bar{b}\bar{c}]} &= \partial_{[\bar{a}} D_{\bar{b}\bar{c}]} - \partial^{\bar{d}} D_{\bar{a}\bar{b}\bar{c}\bar{d}} - \tfrac{1}{3}\,\partial^{d} D_{\bar{a}\bar{b}\bar{c},d} \,.
\end{aligned} \tag{6.144}$$

For the reader's convenience we summarize here the dual D fields:

$$\begin{gathered}
D\,, \qquad D_{ab}\,, \qquad D_{\bar{a}\bar{b}}\,, \qquad D_{a\bar{b}}\,, \\
D_{abc\bar{d}}\,, \quad D_{\bar{a}\bar{b}\bar{c}d}\,, \quad D_{ab\bar{c}\bar{d}}\,, \quad D_{abcd}\,, \quad D_{\bar{a}\bar{b}\bar{c}\bar{d}}\,.
\end{gathered} \tag{6.145}$$

Comparing with the list of Bianchi identities (6.129) we infer that the D fields and Bianchi identities are in one-to-one correspondence. Thus, these fields could be used as Lagrange multipliers to impose the Bianchi identities, confirming the equivalence with the master action procedure discussed in Section (6.4).

We now turn to the dual gauge symmetries that leave (6.144) invariant and thus describe the redundancies between the D fields. For the two-index fields one finds

$$\begin{aligned}
\delta_\Sigma D_{ab} &= 2\,\partial_{[a}\Sigma_{b]} + \partial^c \Sigma_{abc} + \partial^{\bar{c}} \Sigma_{ab,\bar{c}} \,, \\
\delta_\Sigma D_{\bar{a}\bar{b}} &= -2\,\partial_{[\bar{a}}\Sigma_{\bar{b}]} + \partial^{\bar{c}} \Sigma_{\bar{a}\bar{b}\bar{c}} + \partial^{c} \Sigma_{\bar{a}\bar{b},c} \,, \\
\delta_\Sigma D_{a\bar{b}} &= \partial_a \Sigma_{\bar{b}} - \partial_{\bar{b}} \Sigma_a + \partial^c \Sigma_{ca,\bar{b}} + \partial^{\bar{c}} \Sigma_{\bar{c}\bar{b},a} \,, \\
\delta_\Sigma D &= \partial_a \Sigma^a + \partial_{\bar{a}} \Sigma^{\bar{a}} \,.
\end{aligned} \tag{6.146}$$

Note that the dual diffeomorphism parameters Σ_a and $\Sigma_{\bar{a}}$ act on these fields in exactly the same way as the original diffeomorphism parameters ξ_a and $\xi_{\bar{a}}$ act on h_{ab}, $h_{\bar{a}\bar{b}}$, $h_{a\bar{b}}$ and d. For the four-index field we find

$$\begin{aligned}
\delta_\Sigma D^{\bar{a}\bar{b}\bar{c},d} &= \partial_{\bar{e}} \Sigma^{\bar{a}\bar{b}\bar{c}\bar{e},d} - \partial^d \Sigma^{\bar{a}\bar{b}\bar{c}} + 3\,\partial^{[\bar{a}}\,\Sigma^{\bar{b}\bar{c}],d} \,, \\
\delta_\Sigma D^{ab,\bar{c}\bar{d}} &= 2\,\partial^{[a}\,\Sigma^{|\bar{c}\bar{d}|,b]} - 2\,\partial^{[\bar{c}}\,\Sigma^{|ab|,\bar{d}]} \,, \\
\delta_\Sigma D^{abc,\bar{d}} &= \partial_e \Sigma^{abce,\bar{d}} - \partial^{\bar{d}} \Sigma^{abc} + 3\,\partial^{[a} \Sigma^{bc],\bar{d}} \,, \\
\delta_\Sigma D_{abcd} &= \partial^e \Sigma_{abcde} + \tfrac{4}{3}\,\partial_{[a}\Sigma_{bcd]} - \tfrac{1}{3}\,\partial^{\bar{e}} \Sigma_{abcd,\bar{e}} \,, \\
\delta_\Sigma D_{\bar{a}\bar{b}\bar{c}\bar{d}} &= \partial^{\bar{e}} \Sigma_{\bar{a}\bar{b}\bar{c}\bar{d}\bar{e}} + \tfrac{4}{3}\,\partial_{[\bar{a}}\Sigma_{\bar{b}\bar{c}\bar{d}]} - \tfrac{1}{3}\,\partial^{e} \Sigma_{\bar{a}\bar{b}\bar{c}\bar{d},e} \,.
\end{aligned} \tag{6.147}$$

It can be verified by a straightforward computation that these transformations leave (6.144) invariant. Finally, in order for (6.144) to transform under local Lorentz transformations as required by (6.128), the D fields need to transform as

$$\delta_\Lambda D^{ab} \;=\; \Lambda^{ab}\,, \qquad \delta_\Lambda D^{\bar a\bar b} \;=\; \Lambda^{\bar a\bar b}\,. \tag{6.148}$$

Thus, exactly as for h_{ab} and $h_{\bar a\bar b}$, these fields are pure gauge.

6.5.3 Geometric action for dual DFT fields

Let us now insert (6.144) into the master action (6.136) in order to obtain the action for the dual D fields. The terms involving the original fields drop out because these fields enter linearly, multiplying constraints that have been solved in terms of the D fields. The second-order action therefore reads

$$\begin{aligned}\mathcal{L}^{(2)}_{\rm DFT} \;=\; -\frac{1}{2}\big(\,&\omega^{a\bar b\bar c}\omega_{a\bar b\bar c} + 3\,\omega^{[\bar a\bar b\bar c]}\omega_{[\bar a\bar b\bar c]} + 2\,\omega^{\bar a}\omega_{\bar a}\\ &- \omega^{\bar a bc}\omega_{\bar a bc} - 3\,\omega^{[abc]}\omega_{[abc]} - 2\,\omega^{a}\omega_{a}\big)\,,\end{aligned} \tag{6.149}$$

with the connections given by (6.144). This takes precisely the same form as (6.135), except that the overall sign has changed. The computation is simplified by using that the dependence of ω on $D_{a\bar b}$, D_{ab}, $D_{\bar a\bar b}$ and D is precisely analogous to the expressions in terms of the original fields, up to the following identifications,

$$D_{a\bar b} \;\to\; h_{a\bar b}\,, \quad D_{ab} \;\to\; -h_{ab}\,, \quad D_{\bar a\bar b} \;\to\; h_{\bar a\bar b}\,, \quad D \;\to\; 2d\,, \tag{6.150}$$

and an overall sign for the connections with unbarred Lie algebra indices, which is irrelevant since the connections enter the action quadratically. A direct computation yields the explicit form of the dual Lagrangian,

$$\begin{aligned}\mathcal{L} \;=\; &-\tfrac{1}{2}\,\partial_{\bar d}D^{\bar b\bar c\bar d,a}\,\partial^{\bar e}D_{\bar b\bar c\bar e,a} - \partial_{\bar d}D^{\bar b\bar c\bar d,a}\,\partial^{e}D_{ea,\bar b\bar c} - \tfrac{1}{2}\,\partial_{d}D^{da,\bar b\bar c}\,\partial^{e}D_{ea,\bar b\bar c}\\ &+\tfrac{1}{2}\,\partial_{d}D^{bcd,\bar a}\,\partial^{e}D_{bce,\bar a} + \partial_{d}D^{bcd,\bar a}\,\partial^{\bar e}D_{bc,\bar a\bar e} + \tfrac{1}{2}\,\partial_{\bar d}D^{bc,\bar a\bar d}\,\partial^{\bar e}D_{bc,\bar a\bar e}\\ &-\tfrac{3}{2}\,\partial_{\bar d}D^{\bar a\bar b\bar c\bar d}\,\partial^{\bar e}D_{\bar a\bar b\bar c\bar e} - \partial_{\bar d}D^{\bar a\bar b\bar c\bar d}\,\partial^{e}D_{\bar a\bar b\bar c,e} - \tfrac{1}{6}\,\partial_{d}D^{\bar a\bar b\bar c,d}\,\partial^{e}D_{\bar a\bar b\bar c,e}\\ &+\tfrac{3}{2}\,\partial_{d}D^{abcd}\,\partial^{e}D_{abce} + \partial_{d}D^{abcd}\,\partial^{\bar e}D_{abc,\bar e} + \tfrac{1}{6}\,\partial_{\bar d}D^{abc,\bar d}\,\partial^{\bar e}D_{abc,\bar e}\\ &- D^{ab,\bar c\bar d}\,\mathcal{R}_{ab,\bar c\bar d}(D_{a\bar b}) - \mathcal{L}^{(2)}_{\rm DFT}(D_{a\bar b},D)\,.\end{aligned} \tag{6.151}$$

Note that in the last line we encounter the standard linearized DFT Lagrangian $\mathcal{L}^{(2)}$, but for $D_{a\bar b}$ and D, with the 'wrong' overall sign, in complete analogy to the mixed Young tableau action discussed in Section (6.3). Also in perfect analogy to that discussion is that this wrong-sign kinetic term does not indicate the presence of ghosts, for the action is not diagonal. Rather, the off-diagonal term is proportional to the linearized Riemann tensor (6.130), but expressed in terms of $D_{a\bar{b}}$. Thus, the Σ_a and $\Sigma_{\bar{a}}$ transformations are manifest symmetries of this action, while the invariance under the remaining dual diffeomorphisms (6.147) can be verified by a direct computation. Also note that the fields $D_{\bar{a}\bar{b}}$ and D_{ab} dropped out, as it should be in view of the Stückelberg-type Lorentz invariance (6.148).

We close this section by discussing two of the D-field equations, because they exhibit an intriguing structure. Varying with respect to $D_{a\bar{b}}$ and D we obtain

$$\mathcal{R}_{a\bar{b}}(D) \;=\; \partial^c\partial^{\bar{d}}D_{ac,\bar{b}\bar{d}}\,, \qquad \mathcal{R}(D) \;=\; 0\,. \tag{6.152}$$

For the first equation neither the left-hand side nor the right-hand side are dual diffeomorphism invariant under transformations with parameter $\Sigma_{ab,\bar{c}}$ and $\Sigma_{\bar{a}\bar{b},c}$, but they precisely cancel against each other. The field equation for $D_{ab,\bar{c}\bar{d}}$ reads

$$\mathcal{R}_{ab\bar{c}\bar{d}} \;=\; S_{ab\bar{c}\bar{d}}\,, \tag{6.153}$$

where we defined

$$S_{ab\bar{c}\bar{d}} \;\equiv\; \partial_{[a}\partial^{e}D_{|e|b],\bar{c}\bar{d}}+\partial_{[\bar{c}}\,\partial^{\bar{e}}D_{|ab|,\bar{d}]\bar{e}}+\partial_{[a}\partial^{\bar{e}}D_{|\bar{c}\bar{d}\bar{e}|,b]}+\partial_{[\bar{c}}\,\partial^{e}D_{|abe|,\bar{d}]}\;. \tag{6.154}$$

Thus, intriguingly, the equation takes the form of a second-order duality relation, relating the (linearized) Riemann tensor to a 'dual' Riemann tensor. As above, both sides are not separately invariant under dual diffeomorphisms with parameter $\Sigma_{ab,\bar{c}}$ and $\Sigma_{\bar{a}\bar{b},c}$, but the full equation of course is, as it should be and as may be verified by a quick computation.

6.6 Comparison of results

In Section (6.4) we have shown that at the linearized level the DFT equations and Bianchi identities for the fluxes $\mathcal{F}_{ABC}$ and $\mathcal{F}_A$ arise from first order duality equations given, for instance, in eq. (6.121), relating these fluxes to the dual fluxes $\mathcal{G}_{ABC}$ and $\mathcal{G}_A$. The dual fluxes are defined in terms of the field strengths G_{ABC} of the dual potentials (the D-fields) in eqs. (6.118) and (6.119). The field equations and Bianchi identities for the fields and the dual fields are listed in Table 6.1. The aim of this section is to show that if one restricts all DFT fields to only depend on x, *i.e.* if one sets $\tilde{\partial}^\mu\Phi=0$ for any DFT field Φ, one recovers the previous results of dualization: the standard dualities between the 2-form and the $(D-4)$-form and between the graviton (plus dilaton) and the mixed-symmetry $(D-3,1)$ potential discussed in Section (6.2), and the exotic duality between the 2-form and the mixed-symmetry $(D-2,2)$ potential discussed in Section (6.3).

The dual potentials introduced in Section (6.4) are D_{ABCD}, D_{AB} and D. Upon breaking $O(D-1,1)\times O(D-1,1)$ to the diagonal subgroup, the field D_{ABCD} can be decomposed as

$$D_{ABCD} \;\rightarrow\; D^{abcd} \quad D^{abc}{}_{d} \quad D^{ab}{}_{cd} \quad D^{a}{}_{bcd} \quad D_{abcd}\;, \tag{6.155}$$

while the field D_{AB} decomposes as

$$D_{AB} \;\rightarrow\; D^{ab} \quad D^{a}{}_{b} \quad D_{ab}\;. \tag{6.156}$$

When reducing to x-space we use, by a slight abuse of notation, the same symbols for the components of the DFT D-fields and the supergravity D-fields. The identification uses the ordering of the indices as given above to match the results of the previous sections. The same applies for the components of G_{ABC}. We make an exception, in the following subsection, for the identification of the components of D_{AB} and D with the ones in x-space:

$$\begin{aligned} D^{ab} &\;\rightarrow\; D'^{ab}\,, \\ D^{a}{}_{b} &\;\rightarrow\; D'^{a}{}_{b}\,, \qquad\qquad D \;\rightarrow\; D''\;, \\ D_{ab} &\;\rightarrow\; D'_{ab}\;, \end{aligned} \tag{6.157}$$

the convention being that x-space fields carrying a prime can be gauged or redefined away.

If one inserts the above identifications into eq. (6.105), one recovers the first order actions of Section (6.2). In particular, the fields D^{abcd}, $D^{abc}{}_{d}$ and D'^{ab} are precisely the potentials that were introduced in that section when we performed the standard dualization for the 2-form and the graviton plus dilaton system. This requires that, in x-space, the fields $D'^{a}{}_{b}$ and D'' can be redefined away in terms of single and double traces of $D^{ab}{}_{cd}$. We will also argue that $D^{a}{}_{bcd}$, D_{abcd} and D_{ab} cannot be realized in x-space frame.

One can also recover the duality relations for each field by performing the decomposition directly in the duality relation (6.121). We first identify the components of $\mathcal{F}_{ABC}$ in x-space as:

$$\begin{aligned} \mathcal{F}_{ABC} &= \{H_{abc}, f_{ab}{}^{c}, Q_{a}{}^{bc}, R^{abc}\}\;, \\ \mathcal{F}_{A} &= \{f_a, Q^a\}\;, \end{aligned} \tag{6.158}$$

which at this stage are just labels for the components of the $\mathcal{F}$ flux. As we will see, H_{abc}, $f_{ab}{}^c$ and f_a play the same role as in in Section (6.2), and we will discuss later $Q_a{}^{bc}$, Q^a and R^{abc}, which are related to non-geometric fluxes. Note that because of the presence of the tensor $\breve{S}_{ABCDEF}$ in the definition of $\mathcal{G}_{ABC}$ in terms of G_{ABC}, eq. (6.121) relates a given component of $\mathcal{F}_{ABC}$ to different components of G_{ABC} and thereby to different components of the dual potentials. This has to be understood as follows: if one turns on a particular component of the flux $\mathcal{F}_{ABC}$, eq. (6.121) still gives equations for all the dual potentials. The equations for the dual potentials dual to the vanishing fluxes will furnish algebraic relations among the different components of G_{ABC}, and after reinserting these relations into the duality relation for the non-vanishing fluxes one finds that this is dual to a specific component of G_{ABC} suitably antisymmetrized. This will also be discussed in each case in the remainder of this section, which is organized as follows. In the first subsection we will show how from DFT one recovers the standard dualizations of Section (6.2), while in the second subsection we will show how the exotic dualization of Section (6.3) is also contained in DFT. Finally, in the third subsection we will briefly discuss the remaining dual fields, which are related to non-geometric fluxes such as the R-flux.

6.6.1 Standard duality relations for the 2-form and graviton plus dilaton

The truncation of the action given in eq. (6.105) to x-space with only either the H-flux or the f-flux turned on straightforwardly reproduces the field theory analysis of Section (6.2). In the case of the H-flux, only the component D^{abcd} of D_{ABCD} appears in the action, and one immediately recovers eq. (6.8). In the case of the f-flux, one turns on only the component $D^{abc}{}_d$ in D_{ABCD} and D'^{ab} in D_{AB} to recover precisely the action in eq. (6.37). The analysis performed in Section (6.2) showed that D'^{ab} is pure gauge while $D^{abc}{}_d$ describes both the dual of the graviton and the dual of the dilaton.

As anticipated at the beginning of this section, a more careful analysis is required if one wants to perform the truncation at the level of the duality relations. In the case of the H-flux, the duality relation (6.121) simply gives $H_{abc} = \mathcal{G}_{abc}$, with the other components of $\mathcal{G}_{ABC}$ vanishing. In terms of G_{ABC} this gives

$$H_{abc} = \mathcal{G}_{abc} = \frac{1}{2}\eta_{cf}G_{ab}{}^f - \frac{1}{2}\eta_{be}G_{ac}{}^e + \frac{1}{2}\eta_{ad}G_{bc}{}^d - \frac{1}{2}\eta_{ad}\eta_{be}\eta_{cf}G^{def} \quad . \quad (6.159)$$

In this equation both G^{abc} and $G_{ab}{}^c$ occur, but one has to take into account also the equation for the vanishing dual flux $\mathcal{G}_a{}^{bc}$, which gives

$$0 = \mathcal{G}_a{}^{bc} = \frac{1}{2}\eta^{be}G_{ae}{}^c - \frac{1}{2}\eta^{cf}G_{af}{}^b - \frac{1}{2}\eta_{ad}\eta^{be}\eta^{cf}G_{ef}{}^d + \frac{1}{2}\eta_{ad}G^{dbc}\,, \tag{6.160}$$

implying the algebraic relation

$$G_{ab}{}^c = -\eta_{be}\eta_{ad}G^{dec}\,. \tag{6.161}$$

Upon inserting this relation into eq. (6.159) one obtains

$$\mathcal{G}_{abc} = -2\eta_{ad}\eta_{be}\eta_{cf}G^{def}\,, \tag{6.162}$$

which is in agreement with (6.12), identifying $\mathcal{G}_{abc} = \tilde{H}_{abc}$ and using the definition of G^{abc} given in eq. (6.118).

We now perform the same analysis for the graviton-dilaton system. Turning on only the fluxes $f_{ab}{}^c$ and f_a in eq. (6.121) we must recover eq. (6.45), where $\mathcal{G}_{ab}{}^c$ is identified with $g_{ab}{}^c$ and $\mathcal{G}_a$ with g_a. In terms of G_{ABC}, one has

$$\mathcal{G}_{ab}{}^c = \frac{1}{2}\eta^{cf}G_{abf} - \frac{1}{2}\eta_{ad}\eta_{be}\eta^{cf}G_f{}^{de} - \frac{1}{2}\eta_{ad}G_b{}^{dc} + \frac{1}{2}\eta_{be}G_a{}^{ec}\,. \tag{6.163}$$

The two components G_{abc} and $G_a{}^{bc}$ that occur in this equation are related by the condition that the dual flux $\mathcal{G}^{abc}$ vanishes, which yields the relation

$$0 = -\frac{1}{2}\eta^{ad}\eta^{be}\eta^{cf}G_{def} + \frac{1}{2}\eta^{cf}G_f{}^{ab} - \frac{1}{2}\eta^{be}G_e{}^{ac} + \frac{1}{2}\eta^{ad}G_d{}^{bc}\,. \tag{6.164}$$

Inserting this into eq. (6.163) one obtains

$$\mathcal{G}_{ab}{}^c = \eta_{be}G_a{}^{ec} - \eta_{ad}G_b{}^{dc}\,, \tag{6.165}$$

which precisely reproduces eq. (6.45) by using (6.118). It is also straightforward to show that $\mathcal{G}_a$ coincides with g_a defined in (6.45) after using eq. (6.119).

6.6.2 Q-flux dualization from DFT

We now consider the truncation to x-space of the DFT dualization for the Q-flux component in (6.158) and show that it reproduces the exotic dualization of the 2-form discussed in Section (6.3). We start from the first order action (6.105), specialized to the Q-flux components, and reduce to x-space,

$$\begin{aligned} S[Q,D] \;=\; \int \mathrm{d}^D x\,\Big(&Q^aQ_a - \frac{1}{4}Q_a{}^{bc}Q^a{}_{bc} - \frac{1}{2}Q_a{}^{bc}Q_b{}^a{}_c \\ &+ 3D^{ab}{}_{cd}\,\partial_a Q_b{}^{cd} + 2D^a{}_b(\partial_c Q_a{}^{bc} + \partial_a Q^b) + D\,\partial_a Q^a\Big)\,, \end{aligned} \tag{6.166}$$

where the fields $D^{ab}{}_{cd} \equiv D^{[ab]}{}_{[cd]}$, $D^a{}_b$, D and Q^a, $Q_a{}^{bc}$ are independent, and we dropped the primes relative to (6.157). The field equations for the D-fields read

$$\begin{aligned} \partial_{[a}Q_{b]}{}^{cd} &= 0\,, \\ \partial_c Q_a{}^{bc} + \partial_a Q^b &= 0\,, \\ \partial_a Q^a &= 0\,, \end{aligned} \tag{6.167}$$

which are the Bianchi identities (6.96), reduced to x-space and specialized to the components $Q_a{}^{bc}$ and Q^a. The solution of these equations is

$$Q_a{}^{bc} = \partial_a \beta^{bc}\,, \qquad Q^a = \partial_b \beta^{ba} + \text{constant}\,, \tag{6.168}$$

and we will see in the following that the constant term is irrelevant. The field equations for Q^a and $Q_a{}^{bc}$ yield the duality relations

$$\begin{aligned} 2Q_a &= 2\partial_b D^b{}_a + \partial_a D\,, \\ -\frac{1}{2}Q^a{}_{bc} - \frac{1}{2}Q_b{}^a{}_c + \frac{1}{2}Q_c{}^a{}_b &= 3\partial_e D^{ea}{}_{bc} - 2\partial_{[b}D^a{}_{c]}\,, \end{aligned} \tag{6.169}$$

which are equivalent to the duality relations following from (6.109) and (6.110) upon specializing to the Q-fluxes.

Comparing with the master action discussed in Section (6.3), we observe that here we have Lagrange multiplier fields, $D_a{}^b$ and D, which have no analogues in that previous analysis, but we will now show that these fields are irrelevant. We first note that (6.166) is invariant under the gauge transformations with local parameter χ

$$\delta_\chi D = \chi\,, \qquad \delta_\chi D^a{}_b = -\tfrac{1}{2}\,\chi\,\delta^a{}_b\,, \qquad \delta_\chi D^{ab}{}_{cd} = -\tfrac{1}{3}\,\chi\,\delta^{[a}{}_c\,\delta^{b]}{}_d\,, \tag{6.170}$$

with $\delta_\chi Q = 0$. These act as a Stückelberg symmetry on D. Thus, we can gauge this field to zero.[7] Equivalently, we can express the action directly in terms of the gauge invariant objects

$$\widehat{D}^a{}_b \equiv D^a{}_b + \tfrac{1}{2}\,D\,\delta^a{}_b\,, \qquad \widehat{D}^{ab}{}_{cd} \equiv D^{ab}{}_{cd} + \tfrac{1}{3}\,D\,\delta^{[a}{}_c\,\delta^{b]}{}_d\,, \tag{6.171}$$

which yields

$$\begin{aligned} S[Q,D] = \int \mathrm{d}^D x \Big(&Q^a Q_a - \frac{1}{4}Q_a{}^{bc}Q^a{}_{bc} - \frac{1}{2}Q_a{}^{bc}Q_b{}^a{}_c \\ &+ 3\widehat{D}^{ab}{}_{cd}\,\partial_a Q_b{}^{cd} + 2\widehat{D}^a{}_b\big(\partial_c Q_a{}^{bc} + \partial_a Q^b\big)\Big)\,. \end{aligned} \tag{6.172}$$

[7]Note that this gauge invariance cannot be realized in the $O(D,D)$ covariant formalism of DFT, for it acts on the trace part of $D^a{}_b$ and the double trace part of $D^{ab}{}_{cd}$. There are no analogous traces of the covariant and fully antisymmetric fields D_{AB} and D_{ABCD}.

As expected, the singlet D field dropped out. The field $\widehat{D}^a{}_b$ cannot be eliminated similarly by a gauge symmetry. Rather, its own field equation yields the second of the Bianchi identites in (6.167), and back-substituting their solution (6.168) into the action (6.172) gives the free Kalb-Ramond action for the b-field, which at the linearized level is equivalent to the 'β-supergravity' for the bi-vector field $\beta^{ab} \equiv b^{ab}$ (with the indices raised by the flat Minkowski metric) [120–123, 149]. Note, in particular, that the constant term in (6.168) contributes to the Lagrangian only an irrelevant constant and a total derivative term. Therefore, it is physically equivalent to set the constant to zero, in which case $Q^a = Q_b{}^{ba}$ and the second and third Bianchi identity are no longer independent but are traces of the first one. Summarizing, on-shell the above action is equivalent to the same action with $Q^a = Q_b{}^{ba}$ and with the only Lagrange multiplier being $\widehat{D}^{ab}{}_{cd}$, enforcing the first Bianchi identity in (6.167). This action is then manifestly equivalent to the master action (6.62) discussed in Section (6.3).[8] Thus, we have shown that in the Q-flux sector the DFT dualization reduces to the exotic dualization of the B-field into a mixed-symmetry potential with a $(D-2, 2)$ Young tableau.

6.6.3 The R-flux

We now consider the R-flux contribution of (6.158) in the truncation of the master action (6.105) to x-space. The action reduces to

$$S \;=\; \int \mathrm{d}^D x (D^a{}_{bcd}\, \partial_a R^{bcd} + D'_{ab}\, \partial_c R^{cab}) \;, \tag{6.173}$$

where $D^a{}_{bcd} = D^a{}_{[bcd]}$ and $D'_{ab} = D'_{[ab]}$. Note that the field D'_{ab} can be absorbed into the trace of $D^a{}_{bcd}$. The equations for the dual potentials in this case simply imply that R^{abc} has to be constant and hence that in this sector the fields carry no degrees of freedom. This is consistent with the form of the R-flux in x-space at the non-linear level:

$$R^{abc} \;=\; 3\, \beta^{[a|e|}\partial_e \beta^{bc]} \;, \tag{6.174}$$

whose linearization vanishes for vanishing β background. The duality then implies that the dual flux $\mathcal{G}^{abc}$ also vanishes.

Finally, let us also note that the field D_{abcd} disappears from the action in x-space since it couples to a Bianchi identity for the R-flux that explicitly contains a derivative $\tilde{\partial}^\mu$ with respect to the dual coordinate. The field D_{abcd} can be written as a $(10, 4)$ gauge field in $D = 10$ by using the epsilon tensor, as can be deduced by writing its gauge transformation from eq. (6.112) and keeping only x derivatives.

[8] Note that the third Q^2 term in (6.172) is absent in (6.62), but upon eliminating Q both actions agree up to total derivatives, which is sufficient for the equivalence as master actions.

On the other hand, in an $O(D,D)$ frame in which we take all the fields to depend only on the coordinates $\tilde{x}$, D_{abcd} would become the 'standard' dual of the field β since the R-flux takes the form

$$R^{abc} \;=\; 3\,\tilde{\partial}^{[a}\beta^{bc]}\;, \tag{6.175}$$

which plays precisely the same role as the H-flux in x-space. An analogous inversion of roles also holds for all other fields, as is guaranteed by the $O(D,D)$ invariance of the action (6.105). We summarize this in Table 6.2:

x-space			$\tilde{x}$-space		
b_{ab}	$\leftrightarrow$	D^{abcd}	β^{ab}	$\leftrightarrow$	D_{abcd}
$h_{a\|}{}^{b}$	$\leftrightarrow$	$D^{abc}{}_{d}$	$h^{a\|}{}_{b}$	$\leftrightarrow$	$D^{a}{}_{bcd}$
β^{ab}	$\leftrightarrow$	$D^{ab}{}_{cd}$	b_{ab}	$\leftrightarrow$	$D^{ab}{}_{cd}$

Table 6.2

Dual fields for the Kalb-Ramond field, vielbein fluctuation and β-field in x and $\tilde{x}$-space.

6.7 Summary

We presented the dual formulation of Double Field Theory at the linearized level. This is a classically equivalent theory describing the duals of the dilaton, the Kalb-Ramond field, and the graviton in an $O(D,D)$ covariant way. In agreement with previous proposals, the resulting theory encodes fields in mixed-Young-tableau representations, combining them into an antisymmetric 4-tensor under $O(D,D)$. When comparing with the dualizations of the component fields, we found that there are extra fields, which are not all pure gauge. The need for these additional fields is analogous to a similar phenomenon for “exotic" dualizations recently done in the literature.

7 Conclusions and Outlook

This thesis has focused on the study of Double Field Theory, a field theory which makes the T-duality group of string theory manifest. The key feature of DFT is that it is defined on a double space, parametrized by coordinates $(\tilde{x}_i, x^i)$, where x are the usual space-time coordinates and $\tilde{x}$ are the variables conjugate to the winding of the string.

In the first three chapters, we provided the basic elements necessary to understand DFT: We reviewed the bosonic string and showed how the graviton, an antisymmetric 2-form, and a scalar field arise in its spectrum. These fields are the ones involved in the DFT considered in this thesis. We showed how the notion of T-duality and winding modes arises, when we consider a string moving in a compact space. We commented about the low-energy description of the bosonic string and explained that the low-energy picture is represented by the NS-supergravity theory. We then moved on to introduce the (restricted) DFT action and exhibited its symmetry properties. We commented that the geometry behind DFT is a generalization of the usual differential geometry, and it is intrinsically related to generalized geometry. The main tools for a frame formulation of DFT were presented, and it was shown how the DFT action can be written in the Flux Formulation.

In Chapter (4), we expanded on the Flux Formulation of DFT initially developed in the works of [50–53]. This is a rewriting of the usual DFT action that is suited for compactification purposes. In this chapter, we introduced the generalized fluxes. They are the natural fields living in a higher dimensional theory, which, upon compactification, give rise to the constant fluxes in lower dimensions. In this formulation, the gauge consistency constraints of the theory take the form of generalized quadratic constraints for the fluxes, which are known to admit solutions that violate the strong constraint. Building on previous constructions for a geometric formulation of DFT, we have computed connections and curvatures on the double space, under the assumption that covariance is achieved up to the generalized quadratic constraints, rather than the strong constraint. Interestingly, this procedure gives rise to all the strong constraint-violating terms in the action (4.13) that are needed to make contact with the scalar potential of $\mathcal{N} = 4$ gauged supergravity in $D = 4$ dimensions. This completes the original formulation of the theory [16] by incorporating missing terms that allow for duality orbits of non-geometric fluxes. Also, the consistency constraints were shown to be related to generalized Bianchi identities that break down on the world-volume of (exotic) branes [75, 76]. The duality orbits of these Bianchi identities deserve further investigation. For example, in [77, 155, 165–168] the universal T-duality representations for branes in different dimensions were classified. The objects that belong to the usual NS-chain are indeed related to the Bianchi identities discussed here. More generally, the quadratic constraints arising in U-duality invariant constructions should be sourced by U-duality orbits of branes. It would be interesting to incorporate source terms in the action in a T-duality invariant way, such that the source terms appear naturally in the consistency constraints of the theory.

By now, there is plenty of evidence that the strong constraint or section condition can be relaxed [50–52], [126, 127, 134], [43], [58], [169–173]. However, it is not yet clear whether a relaxation of the strong constraint in DFT describes a trustable limit of string theory. In any case, transcending supergravity opens the door to seeking new truly double solutions to the equations of motion or their associated supersymmetric killing-spinor equations. The T-duality invariance of the theory allows new T-fold-like solutions to be built, like those of [148], but more generally a relaxed strong constraint would allow solutions to be found, which lack a local interpretation, from a supergravity point of view, in any global frame. By now, the only known solutions to the minimal constraints are either when fields do not depend on half of the coordinates or (a more relaxed version) when fields assume a Scherk-Schwarz form. We believe that other kinds of compactifications will lead to new possibilities.

In Chapter (5), we studied how the solutions of the NS sector can be described in DFT. In particular, we studied the NS5-KK5-Q5 (5^2_2) and R5 brane chain of solutions. We showed that, in order to go from the Q5 to the R5 brane, we are forced to perform a T-duality along a non-isometric direction. However, as was mentioned in this chapter, DFT allows for this kind of possibility. The R5 brane becomes a DFT solution with an explicit dual coordinate dependence, and becomes, under Scherk-Schwarz reduction, a domain wall in $D = 7$, exactly like the other branes do. These branes, under compactification, produce the usual constant fluxes given by the chain (5.1), so they live in the same T-dual geometric orbit. In the final section, using schematic arguments, we considered how a duality relationship between a $(D-1)$-form and the embedding tensor of the lower dimensional supergravities in any dimension can be rewritten in terms of mixed-symmetry potentials, on one side, and fluxes, on the other. In particular, we considered the NS fluxes, and we showed that the geometric fluxes are dual to mixed-symmetry potentials with eight antisymmetric indices, while the non-geometric fluxes are dual to mixed-symmetry potentials with nine antisymmetric indices. In these relationships, the mixed-symmetry potentials depend on the normal coordinates, and the non-geometric nature of the flux translates to the impossibility of coupling the potential in supergravity consistently. It is worth stressing that, for this duality relationship to work, isometric directions were assumed. This was no longer the case in Chapter (6). The potentials listed in (5.81) were proposed as coming from a DFT field D_{MNPQ} with four antisymmetric indices. This is in agreement with results coming from decomposing the E_{11} Kac-Moody algebra in representations of $SO(10,10)$ in the E_{11}-program [66–71]. This field can be considered to be the dual field to the generalized metric $\mathcal{H}_{MN}$ of DFT.

From a phenomenological point of view, the non-geometric duality orbits (which necessarily violate the strong constraint) are the most interesting ones, since they favor moduli stabilization and dS vacua, evading the many no-go theorems for geometric fluxes [174–179]. From a four-dimensional perspective, the effect of the strong constraint is to eliminate the orbits that give rise to vacua with desirable phenomenological features. Despite the fact that in this chapter we have analyzed a geometric orbit of fluxes, it is important to note that a higher dimensional description of the R-flux in DFT is possible thanks to a specific $\tilde{x}$-coordinate dependence. It is natural to speculate that truly non-geometric duality orbits might admit a realization with a dependence on $(\tilde{x}, x)$ such that one cannot get rid of the dual coordinates' dependence by performing T-duality transformations. We believe that not only a better understanding of the strong constraint is required to understand non-geometric duality orbits but also a better understanding of the generalized Bianchi identities and of the (mixed-symmetry) potentials that couple to the (exotic) branes. As a natural extension, one can also consider how the analysis of the duality relationships performed in this chapter can be extended to fluxes that are sourced by domain walls that are more and more non-perturbative in string theory. All such domain walls, and their corresponding mixed-symmetry potentials, have been classified [180].

In Chapter (6) we have determined the dualization of double field theory at the linearized level, which captures, in addition to the conventional dual fields in $D = 10$ string theory (the 6-form dual to the Kalb-Ramond 2-form and the 8-form dual to the dilaton), fields in mixed-Young tableau representations, such as the dual of the graviton and an exotic dual of the 2-form, plus additional fields. We have called the resulting theory "Dual Double Field Theory." The dual fields can be organized into a totally antisymmetric 4-tensor under the T-duality group $O(D, D)$. The idea for this field and mixed-symmetry potentials came from the work developed in Section (5.4). But it turns out that defining an $O(D, D)$ covariant master action (and, consequently, an action for the dual fields) requires extra fields. A careful analysis shows, however, that reducing the dual DFT to the physical space-time yields precisely the expected dual theories. In particular, we analyzed the exotic dualization of the 2-form, following the strategy introduced in [161–163]. This is illuminating, because it shows that, besides the dual $(D - 2, 2)$ gauge potential, extra fields are needed. In general, there are strong no-go theorems implying, under rather mild assumptions, that there is no non-linear action for a mixed-Young tableau field that is invariant under the linear gauge symmetries [63, 64]. In the $O(D, D)$ covariant framework analyzed here, this problem presents itself in a quite different fashion. Because of the coupling to extra fields, the no-go theorem is not applicable, and hence it may well be that there is a consistent non-linear deformation of the dual DFT action (6.151). For instance, this would require finding a non-linear extension of the field equation (6.153). Another reason to be optimistic about the existence of a non-linear extension is that, in "exceptional field theory" (the extension of DFT to U-duality groups), dual graviton components are already encoded at the non-linear level [28–33], which is achieved by means of additional (compensator) fields. The detailed formulation of these theories is somewhat different, however, in that they require a split of coordinates and indices so that the mixed-Young tableau nature of the dual graviton is no longer visible. Therefore, the precise relationship between the dual formulation presented here and that implicit in [28–33] remains to be established. Once this has been achieved and/or the full non-linear form of the dual DFT has been constructed, we would have a full duality covariant formulation of the low-energy dynamics of the type II strings in terms of all fields and their duals, both for the RR sector, for which this was established a while ago [54], and the NS sector. The construction of such a theory could be very important for the description of various types of (exotic) branes. Indeed, exotic branes are non-perturbative string states that are electrically charged with respect to mixed-symmetry potentials.

The branes that are charged under the D potentials discussed in this chapter have tensions that scale like g_s^{-2} in string frame. While the NS5-brane is charged under the standard potential D^{abcd}, the KK monopole, the Q-brane, and the R-brane are charged under the mixed-symmetry potentials $D^{abc}{}_d$, $D^{ab}{}_{cd}$, and $D^a{}_{bcd}$, respectively. The Q-brane solution [76] is locally geometric, while the R-brane does not admit a geometric description. This is clearly in agreement with our findings, namely that one can write down a duality relationship in x space at the linearized level for $D^{ab}{}_{cd}$ but not for $D^a{}_{bcd}$. Actually, one should also consider non-geometric objects that are charged under the potential D_{abcd}. Upon dimensional reduction, this would give rise to space-filling branes with the same scaling of the tension. These branes have been classified in [155]. In general, these branes do not have any solution in supergravity, but their existence is crucial, for instance, in orientifold models. The 1/2-BPS branes with tension g_s^{-2} satisfy specific "wrapping rules" [167, 168]: the number of p-branes in D dimensions is given by the number of $p+1$-branes in $D+1$ dimensions plus twice the number of p-branes in $D+1$ dimensions. This means that these branes "double," when they do not wrap the internal cycle. As far as the $(D-5)$-branes, the $(D-4)$-branes, and the $(D-3)$-branes are concerned, this is expected from the fact that such branes are magnetically dual to the fundamental string, fundamental particles, and fundamental instantons, respectively. Therefore, for these branes the wrapping rules are simply the dual of the wrapping rules for fundamental strings, which see a doubled circle, and thus double, when they wrap. The fact that all the potentials associated with these branes enter the DFT duality relationships discussed in this chapter explains why the $(D-2)$ and $(D-1)$-branes with tension proportional to g_s^{-2} also satisfy the same wrapping rules, although they are not dual to propagating fields in x space.

The classification of 1/2-BPS branes in string theory was extended to branes with tension scaling like g_s^{-3} in the string frame [167]. Such branes are charged with respect to mixed-symmetry potentials that are magnetically dual to the P-fluxes (a prototype of a P-flux is the S-dual of the Q-flux). In [98], it was observed that all such potentials can be collected in the field $E_{MN,\dot{\alpha}}$ in the tensor-spinor representation of $\mathrm{SO}(10,10)$. It could be very interesting to write down a linearized DFT duality relationship for such a field, precisely as we did for the D fields in this chapter.

DFT has opened up the possibility of obtaining a better picture of how strings probe space-time. It has naturally provided us with an extended space-time that has information about purely stringy effects, like the winding, and, with it, the notion of a new geometry beyond the usual one of General Relativity. Despite the fact that true DFT coming from string field theory remains to be constructed, we can learn a lot from either its restricted version or its less restrictive ones under generalized Scherk-Schwarz reductions. DFT has proved to be an excellent tool for flux compactifications, and, in this way, it might shed some light on how to obtain desirable phenomenological vacua coming from non-geometric compactifications. We have shown that it is the right framework for understanding non-geometric duality orbits of (exotic) branes, and the dynamics of their mixed-symmetry potentials in a T-dual covariant way. A lot of work remains to be done, but certainly DFT has turned out to be a good guide in our quest for a better understanding of the symmetries of string theory.

List of Publications

- E. A. Bergshoeff, O. Hohm, V. A. Penas and F. Riccioni, *Dual Double Field Theory*, accepted for publication in JHEP, arXiv:1603.07380.
- E. A. Bergshoeff, V. A. Penas, F. Riccioni and S. Risoli, *Non-geometric fluxes and mixed-symmetry potentials*, JHEP **1511** (2015) 020, arXiv:1508.00780.
- D. Geissbühler, D. Marqués, C. Núñez and V. A. Penas, *Exploring Double Field Theory*, JHEP **1306** (2013) 101, arXiv:1304.1472.
- S. Iguri, V. Penas, *Duality phases and halved maximal D=4 supergravities*, Phys. Rev. D 87, 085004 (2013), arXiv:1303.0272
- S. Iguri, C. Núñez and V. A. Penas, *Compactificación de cuerdas con flujos y estabilización de módulos*, (Anales de Física (RNF 2012) - AFA).

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Samenvatting

Kwantumveldentheorie en algemene relativiteit zijn de theorieën die we tegenwoordig gebruiken om het gedrag van materie op microscopische schaal te begrijpen en ruimtetijd op grote schaal. Ondanks het succes van beide theorieën afzonderlijk ontbreekt een consistente theorie die beide beschrijft. De theorie die tot nu toe het dichtst hierbij in de buurt komt is snarentheorie. Snarentheorie is een poging om kwantumzwaartekracht te beschrijven met een model van eendimensionaal uitgestrekte objecten (snaren). De snaren zijn van dezelfde ordegrootte als de Planck-lengte $\ell_P = \sqrt{\hbar G/c^3} \approx 10^{-33}$cm, wat de natuurlijke schaal is waarin de fundamentele constanten van kwantummechanica en zwaartekracht in voorkomen. Het meest algemeen wordt M-theorie voorgesteld als de ultieme theorie van snaren en membranen. Snarentheorie heeft enkele opmerkelijke eigenschappen: Het heeft een massaloze spin-twee deeltje (graviton) in het spectrum van de theorie; de ontdekking van een anomale —annuleringen — mechanisme waardoor ijkgroepen die groot genoeg zijn om $SU(3) \times SU(2) \times U(1)$ en om pariteit schending te bevatten (nodig voor de elektrozwakke kracht); snarentheorie heeft supersymmetrie nodig om fermionen te bevatten en om tachionische excitaties te voorkomen die voorkomen in het bosonische geval; (super) snarentheorie heeft tien ruimtetijd dimensies nodig om consistent te zijn; de enige input is de snaarspanning T; de dimensiloze snaarkoppeling g_s is bepaald dor de verwachtingswaarde van een scalair veld (dilaton).

Tot nu toe heeft snarentheorie ons meerdere inzichten gegeven in kwantumzwaartekracht, relaties tussen zwaartekracht en ijktheorieën, de oorsprong van ruimtetijd en meer. Al deze markante ontdekkingen binnen de snarentheorie zijn mogelijk door het bestaan van enkele dualiteiten. In het bijzonder, de dualiteit die van belang is in dit proefschrift is *T-dualiteit*. Het eenvoudigste voorbeeld om deze dualiteit te begrijpen is door een snaar te beschouwen die propageert in $M \times S^1$, waar M is Minkowski ruimtetijd in zeg, $D = 9$ dimensies, en S^1 is een cirkel met radius R. Men kan laten zien (Hoofdstuk(2))dat het massaspectrum van de theorie invariant is onder de transformatie $R \to \alpha' \hbar^2 c^2 / R$, onder voorwaarde dat de momentum en winding getallen ook omgewisseld worden. Het windinggetal vertelt ons hoe vaak een snaar om een compacte dimensie heen is gewikkeld. Als de compacte ruimtetijd een n-torus is in plaats van een cirkel, dan wordt de T-dualiteit versterkt tot de actie van de $O(n, n, \mathbb{Z})$ groep.

In totaal zijn er vijf consistente supersnaar theorieën in $D = 10$ dimensies; deze zijn: Type IIA, type IIB, $E_8 \times E_8$, Heterotisch $SO(32)$, en type I. De type II en Heterotische theorien bevatten alleen gesloten snaren, terwijl type I open en gesloten snaren bevat. De type II en Heterotic theorien hebben een gemeenschappelijke bosonische sub sector die de Neveu-Schwarz (NS) sector wordt genoemd, deze bevat de metriek, een antisymmetrische 2-form die bekend staat als Kalb-Ramond veld, en een scalair veld dat dilaton wordt genoemd. Al deze theorieën zijn gerelateerd door dualiteiten. Bijvoorbeeld, T-dualiteit relateert type IIA met type IIB en relateert beide Heterotische theorieën. Aan de andere kant, S-dualiteit relateert type IIB met zichzelf onder elektrozwakke koppeling inversie, en ook relateert het type IIA met 11-dimensionale superzwaartekracht.

Omdat de meeste dualiteit symmetrieën zich niet manifesteren, is het handig om een dualiteit covariant model te construeren dat een effectieve beschrijving geeft van de lage energie excitaties van een snaar. In sommige aanpakken is deze dualiteit invariantie bewerkstelligd door de coördinaatruimte te vergroten. [6–9] Pas recentelijk, na het werk van C. Hull, B. Zwiebach, and O. Hohm [16–19]is een veldentheorie geconstrueerd op een dubbele ruimtetijd. Deze theorie wordt "Double Field Theory" (Dubbele velden theorie)(DFT) genoemd.

Dubbele velden theorie (DFT) is een veldentheorie die gebruik maakt van de manifestatie van de T-dualiteit van groepen snaren. In toroidale compactificaties zijn de compacte momentum excitaties duaal aan compacte coördinaten x^a, $a = 1, ..., n$. Voor de winding modes moet een nieuwe set coördinaten $\tilde{x}_a$ (geconjugeerde variabelen) in acht, en opgenomen worden als variabelen in DFT; dus DFT is gedefinieerd in twee ruimtes, vandaar de naam. De DFT die in dit proefschrift beschouwd wordt heeft een globale $O(D, D, \mathbb{R})$ symmetrie en maakt gebruik van coördinaten $X^M = (\tilde{x}_i, x^i)$, $M = 1, \cdots, 2D$ waar $i = 0, \cdots, D-1$ die niet noodzakelijk allen compact zijn. De fundamentele velden van de theorie zijn een $O(D, D)$ tensor $\mathcal{H}_{MN}$ die de gegeneraliseerde metriek wordt genoemd en een scalair veld d, genoemd het algemene dilaton. Wanneer de theorie gereduceerd wordt tot de bekende x-ruimte, dat wil zeggen, alleen coördinaten x^i mogen voorkomen. Het reduceert naar de bekende snarensector die bekend staat als de NS sector. De DFT die we beschouwen is een beperkte theorie. Dit betekent de theorie consistent is tot op een zogenaamde sterke voorwaarde, dat wil zeggen $\partial_i \tilde{\partial}^i (\cdots) = 0$. De afgeleides worden genomen met betrekking tot $(\tilde{x}_i, x^i)$ en de punten representeren een willekeurig product van velden en ijkparameters. Deze conditie impliceert dat de velden effectief beschreven worden door de helft van de coördinaten [18].

Een interessant gezichtspunt van DFT is supergravitatie fluxcompactificaties. De laag energetische effectieve beschrijving van snarentheorieën worden mogelijk gemaakt door superzwaartekracht. Desalniettemin heeft niet elke superzwaartekracht theorie een goed gedefinieerde snarentheoretische oorsprong, en, in het algemeen, zijn zij geijkte of massieve gedeformeerde superzwaartekracht theorieën. Geijkte superzwaartekracht theorieën worden gekarakteriseerd door constante parameters die ijkingen worden genoemd, die een subgroep van de globale symmetriegroep ijken. In lagere dimensies kunne enkele van deze ijkingen begrepen worden als afkomstig van fluxen [45–47] of van hoger dimensionale snaar velden van een laag energetische beschrijving. Dit betekent dat sommige veldsterktes van de effectieve theorie, met indices in de interne (compacte) richtingen, een niet triviale achtergrond waarde hebben. Deze fluxen worden geometrische fluxen genoemd. De manier waarop een ijking naar een andere ijking wordt gestuurd door de globale dualiteitgroep is vastgelegd in de zogenaamde inbedding tensor [48]. Een ding in het bijzonder dat kan gebeuren is dat een flux (een ijking) door een dualiteit rotatie naar een ijking wiens hoger dimensionale oorsprong onbekend is wordt gestuurd. Deze ijkingen worden niet-geometrische fluxen genoemd [49]. Als men alleen de NS sector beschouwd dan blijkt de globale symmetriegroep van de lager-dimensionale effectieve actie $O(n, n, \mathbb{R})$ te zijn. Dit is waarom verwacht wordt dat DFT een prominente rol speelt in fluxcompactificaties en een hoger-dimensionale oorsprong kan zijn van niet-geometrische fluxen. In feite wordt dit bewerkstelligd in het "Flux-Formulatie" van DFT [50–53]. Een sleutelelement in de Flux-Formulatie is de sterke voorwaarde [50–52] af te zwakken, en een manier waarop dit gedaan wordt is door Scherk-Schwarz (SS) reducties te maken (dat wil zeggen, reducties die op een specifieke manier van de interne coördinaten afhangen).

DFT is niet alleen in staat om ons van een hoger-dimensionale beschrijving van niet-geometrische fluxen te voorzien maar kan ons ook voorzien van een geünificeerde $O(D, D)$ beschrijving van de elektromagnetische duale potentiaal van de fluxen.

In de gebruikelijke NS-sector kan de Kalb-Ramond 2-vorm b_2 gedualiseerd worden naar een 6-vorm D_6 in $D = 10$. De snaren koppelen elektrisch aan de b_2, en het voorwerp dat zich elektrisch aan de D_6 koppelt is de zogenaamde NS5-braan [65]. Vandaar, vanuit het perspectief van de volledige (niet-perturbatieve) snaar of M-theorie, is noch de 2- of de 6-vorm van nature fundamenteler, wat suggereert dat een democratische formulering, waarin ze beide op gelijke voet voorkomen, van meer toepassing is. Als men verdere dualiteiten en symmetriën van snaartheorie in gedachte neemt, zoals T-dualiteit, impliceert dit dat meerdere velden van meer exotische aard vereist zijn. Bijvoorbeeld, onder de T-dualiteit groep transformeert de 2-vorm naar de metriek. Hierdoor, wanneer b_2 naar D_6 gedualiseerd wordt, vereist de T-dualiteit covariantie dat we ook de graviton naar een "duale graviton"dualiseren. Dit is waarom DFT ook een natuurlijk raamwerk vormt voor het bestuderen hoe duale velden gerelateerd zijn onder $O(D, D)$ transformaties.

Deze scriptie is gebaseerd op de publicaties [53, 98, 99]. Specifiek, hoofdstuk

(4) is gebaseerd op [53], hoofdstuk (5) is gebaseerd op [98] en hoofdstuk (6) is gebaseerd op [99]. Bijkomend materiaal voor de andere hoofdstukken is gebasseerd op boeken en publicaties en zullen door de loop van de scriptie worden geciteerd.

In de eerste drie hoofdstukken voorzien wij de basiselementen die noodzakelijk zijn om DFT te begrijpen. Wij introduceren de bosonische snaar en zijn spectrum. Wij hebben aangetoond hoe het begrip T-dualiteit en windingsfuncties naar voren komen. Wij zijn toen verder gegaan naar het introduceren van de (gerestricteerde) DFT actie en hebben deze zijn symmetrie-eigenschappen laten zien.

In hoofdstuk (4), hebben wij de flux formulering van DFT uitgeweid. Dit is een herschrijving van de gebruikelijke DFT actie die geschikt is voor compactificatie doeleinden. In dit hoofdstuk introduceren wij de gegeneraliseerde fluxen, namelijk de hoger-dimensionale velden, waaruit wanneer gecompactificeerd, de constante fluxen in lagere dimensies uit voortvloeien. Wij hebben de connecties en krommingen berekend op de dubbele ruimte, onder de aanname dat covariantie wordt volbracht tot aan zwakkere versies van de sterke voorwaarde. Deze procedure levert alle sterke voorwaarde-schendende termen op die noodzakelijk zijn om contact te maken met de scalaire potentiaal van $\mathcal{N} = 4$ geëikte superzwaartekracht in $D = 4$ dimensies. Op dit moment, bestaat er voldoende bewijs dat de sterke voorwaarde of doorsnede voorwaarde verzwakt kan worden [50–52], [126, 127, 134], [43], [58], [169–173]. Het is echter nog niet duidelijk of of een verzwakking van de sterke voorwaarde in DFT een betrouwbaar limiet van de snaartheorie beschrijft. Een verzwakte voorwaarde zou kunnen toestaan dat oplossingen worden gevonden die een lokale interpretatie missen in een globaal stelsel, vanuit een superzwaartekracht perspectief. Tegen deze tijd zijn de enige bekende oplossingen van de minimale voorwaarden wanneer velden onafhankelijk zijn van de helft van de coördinaten of (een zwakkere variant) wanneer velden een Scherk-Schwarz vorm aannemen.

In hoofdstuk (5), bestudeerde wij hoe de oplossingen van de NS sector beschreven kunnen worden in DFT. In het bijzonder bestudeerde wij hoe de NS5-KK5-Q5 (5^2_2) en R5 braan keten van oplossingen. Wij toonde aan dat om van de Q5 naar de R5 braan te gaan we genoodzaakt zijn om en T-dualiteit transformatie uit te voeren langs een niet-isometrische richting op deze manier verscheen duale coordinaat afhankelijkheid. In de laatste sectie, op basis van schematische argumenten, beschouwde wij hoe een dualiteit relatie tussen een $(D - 1)$-vorm en de inbeddende tensor van de lager dimensionale superzwaartekrachtmodellen. Wij toonde aan dat de geometrische fluxen duaal zijn aan gemengde-symmetrie potentialen met acht antisymmetrische indices, terwijl de niet-geometrische fluxen duaal zijn aan gemengde-symmetrie potentialen met negen antisymmetrische indices. Het is het waard om te benadrukken dat, om deze dualiteit relatie te verwezenlijken het bestaan van isometrische richtingen aangenomen werd. De potentialen die in (5.81) opgenoemd zijn werden voorgesteld als zijnde komende van een DFT veld D_{MNPQ} met vier antisymmetrische indices en zouden duaal zijn aan de gegeneraliseerde metriek $\mathcal{H}_{MN}$ van DFT.

Vanuit een fenomenologisch perspectief, zijn de niet-geometrische dualiteit orbitalen (die noodzakelijkerwijs de sterke voorwaarde schenden) de meest interessante, aangezien zij voorkeur geven aan modulus stabilisatie en dS vacua en daarmee de vele no-go theorema's voor geometrische fluxen ontwijken [174–179]. Het is natuurlijk om te speculeren dat werkelijke niet-geometrische dualiteit orbitalen mogelijk een realisatie toestaan met een afhankelijkheid van $(\tilde{x}, x)$ zodanig dat men niet van de duale coordinaat afhankelijkheid af kan komen door middel van T-dualiteit transformaties. Als een natuurlijke uitbreiding, zou men ook kunnen beschouwen hoe de analyse van de dualiteit relatie zoals uitgevoerd in dit hoofdstuk uitgebreid kan worden naar fluxen die geput worden uit domeinwanden die reeds meer niet-perturbatief worden in snaartheorie, zoals die geclassificeerd in [180].

In hoofdstuk (6) hebben wij vastgesteld dat de dualisatie van DFT op het gelineariseerde niveau die beschrijvingen geven voor, in toevoeging tot de conventionele duale velden in $D = 10$ snaartheorie (de 6-vorm duaal aan de Kalb-Ramond 2-vorm en de 8-vorm duaal aan de dilaton), velden in gemengde Young-tableau representaties, zoals de duaal van de graviton en een exotische duaal van de 2-vorm, plus bijkomende velden. We hebben de resulterende theorie "Dual Double Field Theory"genoemd. De duale velden kunnen gereorganiseerd worden tot een antisymmetrische 4-tensor onder de T-dualiteit groep $O(D, D)$. Een zorgvuldige analyse toont echter aan dat het reduceren van de duale DFT naar de fysieke ruimte-tijd precies de verwachte duale theorie oplevert. No-go theorema's voor het construeren van een actie voor exotische velden zijn niet van toepassing in deze opzetting. Waardoor het mogelijk is dat er een consistente niet-lineaire deformatie van de duale DFT actie bestaat (6.151). Een volledig niet-lineaire vorm van de duale DFT zou toestaan dat er een volledig covariante formulering van de lage energie dynamica van type II snaren bestaan in termen van alle velden en hun dualen.

DFT heeft de mogelijkheid geopend om een beter beeld te verkrijgen van hoe de snaren ruimte-tijd aantasten. Het heeft ons op natuurlijke een uitgebreide ruimte-tijd aangereikt die over informatie beschikt van pure snaareffecten, zoals winding en daarmee de notie van een nieuwe geometrie voorbij de gebruikelijke geometrie van algemene relativiteit. DFT heeft zichzelf bewezen als een uitstekend hulpmiddel voor flux compactificatie, en kan op deze manier mogelijk licht schijnen op een manier om wenselijke fenomenologische vacua te verkrijgen die voort komen uit niet-geometrische compactificaties. We hebben aangetoond dat het het juiste raamwerk vormt voor het begrijpen van niet-geometrische dualiteit orbitalen van (exotische) branen en de dynamica van hun gemengde-symmetrie potentialen op een T-dualiteit covariante manier. Er blijft nog veel werk over, maar het is duidelijk dat DFT een goede gids bleek te zijn op onze tocht naar een beter begrip van de symmetriën van snaartheorie.

Acknowledgments

I would like to express my gratitude to many people who have helped me to complete my PhD project. Certainly, trying to fulfill such an endeavor is not an one person task and the constant support of my colleagues, friends and family was always indispensable to me. This acknowledgment chapter was much bigger and with more details, but in order to avoid cluttering, I had to shorten it. Some people might imagine how long could have been.

First of all, I want to thank both of my supervisors, Eric Bergshoeff and Carmen Núñez. Let me do it in chronological order. Carmen, in the first three years of my PhD at the IAFE you gave me solid information and introduced the beautiful subjects of String Theory and Double Field Theory to me. You gave me the opportunity to perform my first steps in research in your group. I owe a debt of gratitude to you in helping me to overcome some bureaucratic problems with the work-place at the beginning of my PhD. Also, I want to thank you for encouraging me to travel to Groningen to develop my skills and to gain experience from abroad. Eric, here at RUG, I finished my last two years of PhD under your scientific guidance. I want to thank you for having given to me the opportunity to join your research group. I really appreciate and thank for all of the physics discussions we have had. I have learned from you the importance of spending time in reviewing the basics and analyzing the simplest models before tackling the difficult ones. Also thank you for the economic support while being here.

I would like to express my gratitude to my reading committee, Gaston Giribet, Mariana Graña and Rien van de Weijgaert for carefully reading my manuscript and useful comments.

Second, I would like to thank the institutes and people who I have worked with. I would like to express my gratitude to the University of Buenos Aires (UBA), the department of physics of FCEN-UBA and IAFE. A special mention should be given to IAFE, where my first two papers were developed and published with Sergio Iguri, Diego Marques and Carmen. Diego and Sergio, I will remember the nice dinner meetings at your places. I learned from both of you a lot of math-physics, supergravity, DFT and how the mechanism of the academic world works, which has helped me a lot. Together with Carmen and Yago, you make me feel part of the "DFT porteño group". I would like to express my gratitude to my other institute, the Van Swinderen Institute-University of Groningen. At the Van Swinderen Institute I have met a lot of wonderful people and nice collaborators. I would like to thank Fabio Riccioni who has had infinite patience in explaining some facts of group theory to me. I also want to thank Fabio for his kind hospitality during my stay at Rome and to have introduced the people from

Tor Vergata (like Gianfranco Pradisi) to me. Stefano, I will never forget to go for lunch at ‘La Tana Sarda’ and walking around during the night in Mainz and all the fun we had. I would like to thank Olaf Hohm for contributing his expertise to our final project. I want to thanks Diederik Roest for always being willing to reply to my questions. Thanasis, thank you for your amazing support, kind advices and all the new mathematical structures that I learned from you. Perhaps, you should leave the door open to avoid constant knocking. I will remember the Christmas gathering at your place and the going outs.

Third, I would like to thank my friends from both institutes. In both institutes I have found wonderful, nice, fun and talented people who I have spent very nice moments during my PhD. From UBA: Andrés, Germán, Javier, Jean Paul, Marcelo, Mario, Maxi, Yago. Mario, I really appreciate how much you did for me to help me moving in here and all the nice talks and drinks. Yago, I will remember the very funny trip to Sao Paulo and the karaoke bar-kiosko moment with Emilio. Maxi, thank you for all the funny talks at IAFE, and the guitar grunge sessions at your place. Javier and Jessica, I hope we can share more time with you in Bariloche. Jean Paul and Marce, thank you for all the nice talks we have had, specially the physics-philosophy-religion ones. Andrés, I will recall the nice trip to trieste. And also thank you, together with Marce, for helping me with conicet several times.

From Groningen: Aditya and Pulastya, Arunesh, Ana, Blaise, Dries, Gideon, Gökhan, Inge, Jan Willem, Julian, Marco, Marwa, Marija, Martina, Mehmet, Lorena, Luca (B), Luca (R), Remko, Sandra, Shankhadeep, Souvik, Thanasis, Tiago, Thomas, Yihao. Marija, Ana and Lorena, thank you for helping me in different things. Lore you always helped me a lot even from far away. Ana thank you for the nice talks and dinner at your place. Blaise and Sandra, we enjoyed a lot in going to your place. Blaise, we should have started playing aoeII much before but I enjoyed a lot together with Jan. Dries and Gideon thank you very much for writing part of the Samenvatting, I really appreciate. Thanks Luca (R) for the E11 lessons and the walks at Groningen. Thanks Luca (B), Martina and Marco for the nice moments. Gökhan, thank you for all the nice conversations we have had, I will recall the key-words "since you are best friend", "chiquitita" and "chamusho", you must admit that our office was the best one. Inge and Jan, thank you very much for inviting us to your place may times and cooking together. I will recall all the funny and non-sense moments and the nice ‘snert’ soup made by Inge. Jan, thank you very much for writing part of my Samenvatting and to help me in a lot of different things while being here. Julian, thanks for taking on the heavy burden of being my paranymph (together with Luciana), I will recall the funny moments by bike, bars, Dragon Ball karaoke and much more. Thomas, I will remember the South Park plus pizza at your place, about Biarroyo, the journeys to Leuven and Mainz, our thoughts about non-sense that make sense and much more.

A mis amigos de la vida y mi familia, me ayudaron muchisimo durante my PhD: Juan Manuel, Ricki, Mauro, Martin, Fede, Diego, Hernán. Gracias por las charlas sobre física, vida extraterreste, teorias de conspiración y mucho más. Valoro muchisimo el apoyo y la motivación que me dieron, charlas por whatssap y facebook (Dieguito y los msjs de voz), y pasar tiempo en aoeII y SCII (Martincho y Mauro). Gracias Mauro y Ricki por venir a Groningen a saludarme y Hernán por los llamados. Germán gracias por Stanco, me entretuvo bastante estando aca. Finalmente, a mi querida familia: Mis papas Elisa y Victor, mi abue Victor, mi pequeña hermana Flo, mis tios Lidia, Tito, Nina y Emilio. Millones de gracias Ma, Pa, Flo y abue por siempre estar conmigo, apoyarme, darnos ayuda a mi y a Lucy en todo lo que necesitamos y por proveerme de tanta inspiración para seguir mi sueño. Siempre están ahi cada vez que necesito, con las cosas de la vida y hasta con conicet tambien. A mis tios de españa que pasamos unas hermosas fiestas juntos. Muchisimas gracias a mis suegros Mónica, Horacio a Mecha y Maxi, Bety, Roberto, Aurora. Su ayuda para mi y Luciana fue de muchisima importancia para venir aqui a Groningen y miles de gracias por apoyarme en física y siempre darnos su aliento.

Quiero agradecer particularmente a Lucy, mi hermosa esposa que hizo las mejores cakes en Groningen y la he embarcado en un mundo de locuras, estres, diversión, alegría y mucho amor. Sin tu ayuda y compromiso durante estos años no hubiera podido realizar semejante osadía, te amo.