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'Stringy' Newton–Cartan gravity

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Abstract

We construct a 'stringy' version of Newton–Cartan gravity in which the concept of a Galilean observer plays a central role. We present both the geodesic equations of motion for a fundamental string and the bulk equations of motion in terms of a gravitational potential which is a symmetric tensor with respect to the longitudinal directions of the string. The extension to include a nonzero cosmological constant is given. We stress the symmetries and (partial) gaugings underlying our construction. Our results provide a convenient starting point to investigate applications of the AdS/CFT correspondence based on the non-relativistic 'stringy' Galilei algebra.

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1. Introduction

Einstein's special relativity is based on an equivalence between frames that are connected to each other by the Poincaré symmetries, consisting of translations and Lorentz transformations in a D-dimensional spacetime[3]. The extension to general relativity can be viewed as the gauge theory of these Poincaré transformations where the constant parameters of the different transformations have been promoted to arbitrary functions of the spacetime coordinates x^μ $(\mu = 0, 1, \ldots, D-1)$. This leads to a theory invariant under general coordinate transformations. In general relativity, the curvature of spacetime is described by an invertible metric function $g_{\mu\nu}(x)$ which is symmetric in the spacetime indices and which replaces the Minkowski metric $\eta_{\mu\nu}$ of flat spacetime corresponding to special relativity. The equations of motion for the metric function are given by the well-known Einstein's equations of motion which are basically a set of second-order differential equations for $g_{\mu\nu}(x)$ with the energy–momentum tensor as a source term. The equation of motion of a particle moving in a curved

[3] Since our arguments do not depend on the dimension, we keep the dimension D of spacetime arbitrary.

spacetime is given by the geodesic equation corresponding to that spacetime. All equations transform covariantly with respect to general coordinate transformations.

One of the observations underlying general relativity is that an observer in a local 'free-falling' frame does not experience any gravitational force. Consequently, the equation of motion of a particle in such a frame describes a straight line corresponding to motion with a constant velocity. These equations of motion transform covariantly under the Poincaré symmetries of special relativity. Indeed, locally, general relativity coincides with special relativity corresponding to $g_{\mu\nu}(x) = \eta_{\mu\nu}$.

To apply general relativity in practical situations, it is often convenient to consider the Newtonian limit which is defined as the limit of small velocities $v \ll c$ with respect to the speed of light c, and a slowly varying and weak gravitational field. The Newtonian limit is not the unique non-relativistic limit of general relativity. It is a specific limit which is based on the assumption that particles are the basic entities and it further makes the additional assumption of a slowly varying and weak gravitational field. In this work, we will encounter different limits which are based on strings or, more general, branes, as the basic objects, and which do not necessarily assume a slowly varying and weak gravitational field.

Taking the Newtonian limit, there is a universal time t and there is only equivalence between frames that are connected to each other by the Galilei symmetries, consisting of (space and time) translations, boost transformations and $(D-1)$-dimensional spatial rotations. Like in general relativity, an observer in a free-falling frame does not experience any gravitational force. All free-falling frames are connected to each other by the Galilei symmetries. For practical purposes, it is convenient to consider not only free-falling frames but to include all frames corresponding to a so-called Galilean observer [1, 2]. These are all frames that are accelerated, with arbitrary (time-dependent) acceleration, with respect to a free-falling frame. An example of a frame describing a Galilean observer with constant acceleration [3] is the one attached to the Earth's surface, thereby ignoring the rotation of the Earth. Newton showed that in the constant-acceleration frames the gravitational force is described by a time-independent scalar potential $\Phi(x^i)$ $(i = 1, \ldots, D-1)$, the so-called Newton potential. In frames with time-dependent acceleration, the potential becomes an arbitrary function $\Phi(x)$ of the spacetime coordinates. A noteworthy difference between general relativity and Newtonian gravity is that while in general relativity *any* observer can locally in spacetime use a general coordinate transformation to make the metric flat, in Newtonian gravity only the Galilean observers can use an acceleration to make the Newton potential disappear. The Newton potential deforms the free motion of a particle and is itself described by a Poisson equation with the mass density $\rho(x)$ as a source term, and it takes over the role played by the metric function in general relativity. In the Newtonian limit, the Newton potential is contained in the time–time component of $g_{\mu\nu}(x)$, and the potential term in the geodesic equation is given by the spacetime–time component of the Christoffel symbol.

The equations of motion corresponding to a Galilean observer are invariant under the so-called acceleration-extended Galilei symmetries. This corresponds to an extension of the Galilei symmetries in which the (constant) space translations and boost transformations have been gauged resulting into a theory which is invariant under arbitrary time-dependent spatial translations[4]. The gravitational potential can be viewed as the 'background gauge field' necessary to realize these time-dependent translations. Starting from a free particle in a Newtonian spacetime, there are now two ways to derive the equations of motion for a Galilean observer from a gauging principle. If one is only interested in the physics observed by a Galilean observer, it is sufficient to gauge the constant space translations by promoting

[4] The group of acceleration-extended Galilei symmetries is also called the Milne group [4].

the corresponding (constant) parameters to arbitrary functions of time. This automatically includes the gauging of the boost transformations. The equation of motion of a particle is then obtained by deforming the free equation of motion with the background gravitational potential $\Phi(x)$ such that the resulting equation is invariant under the acceleration-extended Galilei symmetries. The Poisson equation of $\Phi(x)$ can be obtained by realizing that it is the only equation, of second order in the spatial derivatives, that is invariant under the acceleration-extended Galilei symmetries.

In case one is interested in not only the physics as experienced by a Galilean observer but also by other observers, corresponding to, e.g., rotating frames, it is convenient to first gauge *all* symmetries of the Newtonian theory. One thus ends up with a gravitational theory invariant under much more symmetries than the acceleration-extended Galilean symmetries. This procedure was described in [2], and somewhat differently in [5]. The gauging contains an additional subtlety with respect to the relativistic case. In the relativistic case, both the equations of motion and the Lagrangian leading to the equations of motion are invariant under the Poincaré symmetries. This is different from the Newtonian case. It turns out that although the equations of motion are invariant under the Galilei symmetries, the corresponding Lagrangian is only invariant under boosts up to a total time derivative. This leads to a central extension of the Galilei algebra, containing an extra so-called central charge generator Z, which is called the Bargmann algebra [6].[5] The gauging procedure, in order to be well defined, must be applied to the Bargmann algebra. Once one decides to restrict to a Galilean observer, with flat spatial directions, one must impose as a kinematical constraint that the curvature with respect to the spatial rotations vanishes. It should be stressed that one is not forced to impose this curvature constraint, and one could stay more general and try to solve the resulting theory of gravity for a curved transverse space. But if one does restrict to a flat transverse space and a Galilean observer, the gauging procedure as described in [5] leads to a geometrical reformulation of non-relativistic gravity called Newton–Cartan gravity [8]. In this reformulation, the trajectory of a particle is described by a geodesic in a curved so-called Newton–Cartan spacetime. Such a spacetime is described by a (non-invertible) temporal metric $\tau_{\mu\nu}$ and a spatial metric $h^{\mu\nu}$, which both are covariantly constant. Through projective relations, one can also define the 'inverses' $\tau^{\mu\nu}$ and $h_{\mu\nu}$ of these metrics. The equations of motion are defined in terms of the (singular) metric and Christoffel symbols of the Newton–Cartan spacetime. A noteworthy feature is that metric compatibility does not define the Christoffel symbols *uniquely* in terms of (derivatives of) the temporal and spatial metrics. To make contact with a Galilean observer, one imposes a set of gauge-fixing conditions which restrict the symmetries to the acceleration-extended Galilei ones. The expected equations of motion in terms of a gravitational potential $\Phi(x)$ then follow. The (derivative of the) gravitational potential emerges as the spacetime–time component of the Christoffel symbol.

It is natural to extend the above ideas from particles to strings. This will give us information about the gravitational forces as experienced by a non-relativistic string instead of a particle. Although the symmetries involved are different, the ideas are the same as in the particle case discussed above. The starting point in this case is a string moving in a flat Minkowski background. Taking the non-relativistic limit leads to the action for a non-relativistic string [9–11] that is invariant under a 'stringy' version of the Galilei symmetries. The transformations involved, which will be specified later, are similar to the particle case except that now not only time but also the spatial direction along the string plays a special role. This leads to an $M_{1,1}$ foliation of spacetime. Again, the Lagrangian is only invariant up to a total derivative (in the

[5] Alternatively, one may construct an invariant Lagrangian at the expense of introducing an additional coordinate. One thus ends up with a higher dimensional realization of the Bargmann algebra in which the central charge transformation corresponds to a translation in the extra direction [7].

world-sheet coordinates), and hence, we obtain an extension of the 'stringy' Galilei algebra which involves two additional generators Z_a and $Z_{ab} = -Z_{ba}$ $(a = 0, 1)$.[6] Due to the extra index structure, these generators provide general extensions rather than central extensions of the stringy Galilei algebra [12].

Any two free-falling frames are connected by a stringy Galilei transformation. A 'stringy' Galilean observer is now defined as an observer with respect to any frame that is accelerated, with arbitrary (time- and longitudinal-coordinate-dependent) acceleration, with respect to a free-falling frame. The corresponding acceleration-extended 'stringy' Galilei symmetries are obtained by gauging the translations in the spatial directions transverse to the string by promoting the corresponding parameters to arbitrary functions of the world-sheet coordinates. These transformations involve the constant transverse translations and the stringy boost transformations, which are linear in the world-sheet coordinates.

Again, there are two ways to obtain the equations of motion for a stringy Galilean observer. The first way is to start from the string in a Minkowski background and gauge the transverse translations. In the string case, this requires the introduction of a background gravitational potential $\Phi_{\alpha\beta}(x) = \Phi_{\beta\alpha}(x)$ $(\alpha = 0, 1)$, as was also pointed out in [13]. This is a striking difference with general relativity where, independent of whether particles or strings are the basic objects, one always ends up with the same metric function $g_{\mu\nu}(x)$. This is related to the fact that in the non-relativistic case spacetime is a foliation and that the dimension of the foliation space depends on the nature of the basic object (particles or strings). Alternatively, one gauges the full deformed stringy Galilei algebra and imposes a set of kinematical constraints, like in the particle case. The equation of motion for $\Phi_{\alpha\beta}(x)$ can be obtained by requiring that it is of second order in the transverse spatial derivatives and invariant under the acceleration-extended stringy Galilei transformations. In the string case, one requires that both the curvature of spatial rotations transverse to the string and the curvature of rotations among the foliation directions vanishes. This leads to a flat foliation corresponding to an $M_{1,1}$ foliation of spacetime as well as to flat transverse directions. One next introduces the equations of motion making use of the (non-invertible) temporal and spatial metrics and Christoffel symbols corresponding to the stringy Newton–Cartan spacetime. To make contact with a stringy Galilean observer, one imposes gauge-fixing conditions which reduce the symmetries to the acceleration-extended stringy Galilei ones. As expected, the two approaches lead to precisely the same expression for the equation of motion of a fundamental string and of the gravitational potential $\Phi_{\alpha\beta}(x)$ itself. The (derivative of the) latter emerges as a transverse–longitudinal–longitudinal component of the Christoffel symbol.

In order to study applications of the AdS/CFT correspondence based on the symmetry algebra corresponding to a non-relativistic string, it is necessary to include a (negative) cosmological constant Λ. It is instructive to first discuss the particle case. In the relativistic case, this means that the Poincaré algebra gets replaced by an anti-de Sitter (AdS) algebra corresponding to a particle moving in an AdS background. It is well known that one cannot obtain general relativity with a (negative) cosmological constant by gauging the AdS algebra in the same way that one can obtain general relativity by gauging the Poincaré algebra. The (technical) reason for this is that one cannot find a set of (so-called conventional) curvature constraints whose effect is to convert the translation transformations into general coordinate transformations and, at the same time, to make certain gauge fields to be dependent on others; see, e.g., [14]. However, we are lucky. It turns out that when taking the non-relativistic limit of a particle moving in an AdS background, which is a Λ-deformation of the Minkowski background, one ends up with a non-relativistic particle action which is a particular case of

[6] Our notation and conventions can be found in appendix A.

the non-relativistic particle action for a Galilean observer with zero cosmological constant but with the following nonzero value of the gravitational potential:

$$\Phi(x^i) = -\tfrac{1}{2}\Lambda x^i x^j \delta_{ij}, \tag{1.1}$$

where $\{x^i\}$ are the transverse coordinates. The action is invariant under the so-called Newton–Hooke symmetries which are a Λ-deformation of the Galilei symmetries. All Newton–Hooke symmetries can be viewed as particular time-dependent transverse translations. Therefore, when gauging the transverse translations, it does not matter whether one gauges the Galilei or Newton–Hooke symmetries, in both cases one ends up with the same theory but with a different interpretation of the potential. When gauging the Galilei symmetries, one interprets the potential $\Phi(x)$ as a purely gravitational potential $\phi(x)$, i.e. $\Phi(x) = \phi(x)$. On the other hand, when gauging the Newton–Hooke symmetries, one writes $\Phi(x)$ as the sum of a purely gravitational potential $\phi(x)$ and a Λ-dependent part:

$$\Phi(x) = \phi(x) - \tfrac{1}{2}\Lambda x^i x^j \delta_{ij}. \tag{1.2}$$

In both cases, turning off gravity amounts to setting $\phi(x) = 0$. For $\Lambda = 0$, this implies $\Phi(x) = 0$ but for $\Lambda \neq 0$ this implies $\Phi(x^i) = \frac{1}{2}\Lambda x^i x^j \delta_{ij}$. These different conditions lead to different surviving symmetries: (centrally extended) Galilei symmetries for $\Lambda = 0$ versus (centrally extended) Newton–Hooke symmetries [15, 16] for $\Lambda \neq 0$.

It is now a relatively straightforward task to generalize the above discussion to a string moving in an AdS background. Taking the non-relativistic limit of a string moving in such a background leads to a non-relativistic action that is invariant under a stringy version of the Newton–Hooke symmetries [17, 18]. Note that this action is Λ-deformed in two ways: (i) there is a Λ-dependent potential term in the action like in the particle case and (ii) the foliation metric is deformed from $M_{1,1}$ ($\Lambda = 0$) to AdS_2 ($\Lambda \neq 0$). The latter deformation, which leads to an AdS_2 foliation of spacetime, is trivial in the particle case. All stringy Newton–Hooke symmetries can be viewed as particular world-sheet-dependent transverse translations. It is therefore sufficient to gauge the symmetries for the case $\Lambda = 0$ only, which amounts to gauging the stringy Galilei symmetries. In the second stage, one obtains the $\Lambda \neq 0$ case by a different interpretation of the potential $\Phi_{\alpha\beta}(x)$ and by replacing the flat foliation space by an AdS_2 spacetime. To be concrete, in analogy to the particle case, we gauge the stringy Galilei symmetries only and, next, write the background potential $\Phi_{\alpha\beta}(x)$, which is needed for this gauging, as

$$\Phi_{\alpha\beta}(x) = \phi_{\alpha\beta}(x) + \tfrac{1}{4}\Lambda\, x^i x^j\, \delta_{ij}\tau_{\alpha\beta}, \tag{1.3}$$

where $\phi_{\alpha\beta}(x)$ is the purely gravitational potential and $\tau_{\alpha\beta}$ is an AdS_2 metric. At the same time, we have replaced the flat foliation by an AdS_2 space leading to an AdS_2 foliation of spacetime[7].

In this way, it is a relatively simple manner to obtain the geodesic equations of motion for a fundamental string in a cosmological background and to derive the equations of motion for the potential $\Phi_{\alpha\beta}(x)$ itself. We will give the explicit expressions in the second part of this paper.

This work is organized as follows. In section 2, we review, as a warming-up exercise, the particle case for zero cosmological constant. The gauging of the Bargmann algebra, i.e. the centrally extended Galilei algebra, will only be discussed at the level of the symmetries; for full details, we refer to [5]. In section 3, we derive the relevant expressions for the string

[7] When gauging the full (deformed) stringy Galilei symmetries, one of the kinematical constraints which have to be imposed in order to restrict to a stringy Galilean observer, for $\Lambda \neq 0$, is that the curvature corresponding to rotations among the longitudinal directions is proportional to Λ. This leads to a flat foliation for $\Lambda = 0$ but an AdS_2 foliation for $\Lambda \neq 0$.

case. In particular, we discuss the gauging of the full (deformed) stringy Galilei symmetries. The extension to a nonzero cosmological constant will be discussed in section 4 using the observations mentioned above. In this section, we will present explicit expressions for the equation of motion for a non-relativistic fundamental string in a cosmological background and the equations of motion for the potential $\Phi_{\alpha\beta}(x)$. These two equations together describe the dynamics of 'stringy' Newton–Cartan gravity as observed by a 'stringy' Galilean observer. The potential application of this theory to the AdS/CFT correspondence based on the non-relativistic Newton–Hooke algebra will be briefly discussed in section 5.

2. The particle case

Our starting point is the action describing a particle of mass m moving in a D-dimensional Minkowski spacetime, i.e. $\Lambda = 0$, with the metric $\eta_{\mu\nu}$ $(\mu = 0, 1, \ldots, D-1)$:

$$S = -m \int \mathrm{d}\tau \sqrt{-\eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}. \tag{2.1}$$

Here, τ is the evolution parameter parametrizing the worldline and the dot indicates differentiation with respect to τ. We have taken the speed of light to be $c = 1$. This action is invariant under worldline reparametrizations. The Lagrangian, defined by $S = \int L\,\mathrm{d}\tau$, is invariant under the Poincaré transformations with parameters $\lambda^\mu{}_\nu$ (Lorentz transformations) and ζ^μ (translations):

$$\delta x^\mu = \lambda^\mu{}_\nu x^\nu + \zeta^\mu. \tag{2.2}$$

Following [11, 17], we take the non-relativistic limit by rescaling the longitudinal coordinate $x^0 \equiv t$ and the mass m with a parameter ω and taking $\omega \gg 1$:

$$x^0 \to \omega x^0, \quad m \to \omega m, \quad \omega \gg 1. \tag{2.3}$$

This rescaling is such that the kinetic term remains finite. This results into the following action:

$$S \approx -m\omega^2 \int \dot{x}^0 \left(1 - \frac{\dot{x}^i \dot{x}_i}{2\omega^2 (\dot{x}^0)^2}\right) \mathrm{d}\tau, \quad i = 1, \ldots, D-1. \tag{2.4}$$

The first term on the right-hand side, which is a total derivative, can be canceled by coupling the particle to a constant background gauge field A_μ by adding a term

$$S_I = m \int A_\mu \dot{x}^\mu \mathrm{d}\tau, \tag{2.5}$$

and choosing $A_0 = \omega^2$ and $A_i = 0$ [9]. The effect of this cancelation is that only states charged under the gauge field have finite energy [9]. Because this A_μ can be written as a total derivative, the associated field-strength vanishes, such that no dynamics for the background gauge field is introduced. The limit $\omega \to \infty$ then yields the following non-relativistic action:

$$S = \frac{m}{2} \int \frac{\dot{x}^i \dot{x}^j \delta_{ij}}{\dot{x}^0} \mathrm{d}\tau. \tag{2.6}$$

This action is invariant under worldline reparametrizations and the following Galilei symmetries:

$$\delta x^0 = \zeta^0, \quad \delta x^i = \lambda^i{}_j x^j + v^i x^0 + \zeta^i, \tag{2.7}$$

where $\left(\zeta^0, \zeta^i, \lambda^i{}_j, v^i\right)$ parametrize a (constant) time translation, space translation, spatial rotation and boost transformation, respectively. The equations of motion corresponding to the action (2.6) are[8]

$$\ddot{x}^i = \frac{\ddot{x}^0}{\dot{x}^0}\dot{x}^i. \tag{2.8}$$

It turns out that the non-relativistic Lagrangian (2.6) is invariant under boosts only up to a total τ-derivative:

$$\delta L = \frac{\mathrm{d}}{\mathrm{d}\tau}(mx^i v^j \delta_{ij}). \tag{2.9}$$

This leads to a modified Noether charge giving rise to a centrally extended Galilei algebra containing an extra so-called central charge generator Z; see, e.g., [19, 20]. This centrally extended Galilei algebra is called the Bargmann algebra [6].

The above results apply to free-falling frames without any gravitational interactions. Such frames are connected to each other via the Galilei symmetries (2.7). We now wish to extend these results to include frames that apply to a Galilean observer, i.e. that are accelerated with respect to the free-falling frames, with arbitrary (time-dependent) acceleration. As explained in section 1, we can do this via two distinct gauging procedures. The first procedure is convenient if one is only interested in the physics experienced by a Galilean observer. In that case it is sufficient to gauge the transverse translations by replacing the constant parameters ζ^i by arbitrary time-dependent functions $\zeta^i \rightarrow \xi^i(x^0)$. Applying this gauging to the action (2.6) leads to the following gauged action containing the gravitational potential $\Phi(x)$:[9]

$$S = \frac{m}{2}\int \mathrm{d}\tau \left(\frac{\dot{x}^i\dot{x}^j\delta_{ij}}{\dot{x}^0} - 2\dot{x}^0\Phi(x)\right). \tag{2.10}$$

The action (2.10) is invariant under worldline reparametrizations and the acceleration-extended symmetries (we write x^0 as t from now on),

$$\delta t = \zeta^0, \quad \delta x^i = \lambda^i{}_j x^j + \xi^i(t), \tag{2.11}$$

provided that the 'background gauge field' $\Phi(x)$ transforms as follows:

$$\delta\Phi(x) = -\frac{1}{\dot{t}}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\dot{\xi}_i}{\dot{t}}\right)x^i + \partial_0 g(t). \tag{2.12}$$

The second term with the arbitrary function $g(t)$ represents a standard ambiguity in any potential describing a force and gives a boundary term in the action (2.10). This action leads to the following modified equation of motion describing a particle moving in a gravitational potential:

$$\ddot{x}^i + (\dot{t})^2\delta^{ij}\partial_j\Phi(x) = \frac{\ddot{t}}{\dot{t}}\dot{x}^i. \tag{2.13}$$

Notice how (2.12) and (2.13) simplify if one takes the static gauge

$$t = \tau, \tag{2.14}$$

for which $\dot{t} = 1$ and $\ddot{t} = 0$. Using this static gauge, we see that for *constant* accelerations $\ddot{\xi}^i$ = constant, it is sufficient to introduce a time-independent potential $\Phi(x^i)$ but that for

[8] One can check that the equation of motion for $\{x^0\}$ and $\{x^i\}$ corresponding to the action (2.6) are not independent; the first can be derived from the latter. When we will include gravity in (2.6) via the worldline-reparametrization-invariant coupling $\dot{x}^0\Phi(x)$, see equation (2.10), this will again be the case.

[9] Note that $\Phi(x)$ is a background field representing a set of coupling constants from the worldline point of view. Since these coupling constants also transform, we are dealing not with a 'proper' symmetry but with a 'pseudo' or 'sigma-model' symmetry; see, e.g., [21, 22].

time-dependent accelerations we need a potential $\Phi(x)$ that depends on both the time and the transverse spatial directions.

The equation of motion of $\Phi(x)$ itself is easily obtained by requiring that it is of second order in spatial derivatives and invariant under the acceleration-extended Galilei symmetries (2.11) and (2.12). Since the variation of $\Phi(x)$, see equation (2.12), contains an arbitrary function of time and is linear in the transverse coordinate, it is clear that the unique second-order differential operator satisfying this requirement is the Laplacian $\Delta \equiv \delta^{ij}\partial_i\partial_j$. Requiring that the source term is provided by the mass density function $\rho(x)$, which transforms as a scalar with respect to (2.11), this leads to the following Poisson equation:

$$\triangle\Phi(x) = V_{D-2}G\rho(x), \tag{2.15}$$

where we have introduced Newton's constant G for dimensional reasons, and V_{D-2} is the volume of a $(D-2)$-dimensional sphere.

The second gauging procedure is relevant if one is interested in describing the physics in more frames than the set of accelerated ones. In that case one needs to gauge *all* the symmetries of the Bargmann algebra. This gauging has been described in [5]. We will not repeat the full procedure here but explain the basic points and concentrate on the symmetries involved. The starting point is the Bargmann algebra which consists of time and space translations, spatial rotations, boosts and central charge transformations. In the gauging procedure, one associates a gauge field with each of the symmetries (for our index notation, see appendix A):

$$\begin{aligned} \tau_\mu &: \text{ time translations},\\ e_\mu{}^{a'} &: \text{ space translations},\\ \omega_\mu{}^{a'\underline{0}} &: \text{ boosts},\\ \omega_\mu{}^{a'b'} &: \text{ spatial rotations},\\ m_\mu &: \text{ central charge transformations}. \end{aligned} \tag{2.16}$$

Furthermore, the constant parameters describing the transformations are promoted to arbitrary functions of the spacetime coordinates $\{x^\mu\}$:

$$\begin{aligned} \tau(x^\mu) &: \text{ time translations},\\ \zeta^{a'}(x^\mu) &: \text{ space translations},\\ \lambda^{a'\underline{0}}(x^\mu) &: \text{ boosts},\\ \lambda^{a'b'}(x^\mu) &: \text{ spatial rotations},\\ \sigma(x^\mu) &: \text{ central charge transformations}. \end{aligned} \tag{2.17}$$

Besides these transformations, all gauge fields transform under general coordinate transformations with parameters $\xi^\mu(x^\mu) = (\xi^0(x^\mu), \xi^i(x^\mu))$. As a first step in the gauging procedure, one imposes a set of so-called conventional constraints on the curvatures of the gauge fields. The purpose of these constraints is twofold. First of all, it has the effect that the time and space translations become equivalent to general coordinate transformations modulo the other symmetries of the algebra [23]. This can be seen from the following identity, which relates the general coordinate transformation of a gauge field $B_\mu{}^A$ to its curvature $R_{\mu\lambda}{}^A$ and the other gauge transformations in the theory with field-dependent parameters:

$$\delta_{gct}(\xi^\lambda)B_\mu{}^A + \xi^\lambda R_{\mu\lambda}{}^A - \sum_{\{C\}}\delta\big(\xi^\lambda B_\lambda{}^C\big)B_\mu{}^A = 0. \tag{2.18}$$

Second, the conventional constraints enable one to solve for the gauge fields $\omega_\mu{}^{a'\underline{0}}$ and $\omega_\mu{}^{a'b'}$ in terms of the other ones [5]:

$$\omega_\mu{}^{a'b'} = 2\mathrm{e}^{\rho[a'}\partial_{[\rho}e_{\mu]}{}^{b']} - \mathrm{e}^{\rho a'}\mathrm{e}^{\nu b'}e_\mu{}^{c'}\partial_{[\nu}e_{\rho]}{}^{c'} - \tau_\mu\mathrm{e}^{\rho[a'}\omega_\rho{}^{b']\underline{0}}, \tag{2.19}$$

$$\omega_\mu{}^{a'\underline{0}} = \mathrm{e}^{\nu a'}\partial_{[\mu}m_{\nu]} + \mathrm{e}^{\nu a'}\tau^\rho e_\mu{}^{b'}\partial_{[\nu}e_{\rho]}{}^{b'} + \tau_\mu\tau^\nu\mathrm{e}^{\rho a'}\partial_{[\nu}m_{\rho]} + \tau^\nu\partial_{[\mu}e_{\nu]}{}^{a'}. \tag{2.20}$$

The same constraints have a third effect, namely that the gauge field τ_μ of time translations can be written as the spacetime derivative of an arbitrary function $f(x)$:

$$\tau_\mu = \partial_\mu f(x). \tag{2.21}$$

At this point, the symmetries of the theory are the general coordinate transformations plus the boosts, spatial rotations and central charge transformations, all with parameters that are the arbitrary functions of the spacetime coordinates.

The gauge fields τ_μ and $e_\mu{}^{a'}$ of time and spatial translations are identified as the (singular) temporal and spatial vielbeins. One may also introduce their inverses (with respect to the temporal and spatial subspaces) τ^μ and $\mathrm{e}^\mu{}_{a'}$:

$$\begin{aligned} &e_\mu{}^{a'}\mathrm{e}^\mu{}_{b'} = \delta^{a'}_{b'}, \qquad e_\mu{}^{a'}\mathrm{e}^\nu{}_{a'} = \delta^\nu_\mu - \tau_\mu\tau^\nu, \qquad \tau^\mu\tau_\mu = 1, \\ &\tau^\mu e_\mu{}^{a'} = 0, \qquad \tau_\mu \mathrm{e}^\mu{}_{a'} = 0. \end{aligned} \tag{2.22}$$

The spatial and temporal vielbeins define spatial and temporal metrics as follows:

$$\begin{aligned} &\tau_{\mu\nu} = \tau_\mu\tau_\nu, \qquad\qquad \tau^{\mu\nu} = \tau^\mu\tau^\nu, \\ &h_{\mu\nu} = e_\mu{}^{a'}e_\nu{}^{b'}\,\delta_{a'b'}, \qquad h^{\mu\nu} = \mathrm{e}^\mu{}_{a'}\mathrm{e}^\nu{}_{b'}\,\delta^{a'b'}. \end{aligned} \tag{2.23}$$

A Γ-connection can be introduced by assuming the vielbein postulates:

$$\partial_\mu e_\nu{}^{a'} - \omega_\mu{}^{a'b'}e_\nu{}^{b'} - \omega_\mu{}^{a'\underline{0}}\tau_\nu - \Gamma^\rho_{\nu\mu}e_\rho{}^{a'} = 0, \qquad \partial_\mu\tau_\nu - \Gamma^\lambda_{\nu\mu}\tau_\lambda = 0. \tag{2.24}$$

These vielbein postulates state that τ_μ is covariantly constant, whereas $e_\mu{}^{a'}$ is not[10], and can be uniquely solved for the Γ-connection, giving

$$\Gamma^\rho_{\nu\mu} = \tau^\rho\partial_{(\mu}\tau_{\nu)} + \mathrm{e}^\rho{}_{a'}\big(\partial_{(\mu}e_{\nu)}{}^{a'} - \omega_{(\mu}{}^{a'b'}e_{\nu)}{}^{b'} - \omega_{(\mu}{}^{a'\underline{0}}\tau_{\nu)}\big), \tag{2.25}$$

where the dependent fields $\omega_\mu{}^{a'b'}$ and $\omega_\mu{}^{a'\underline{0}}$ are given by (2.19) and (2.20). If we plug in these explicit solutions, one obtains

$$\begin{aligned} &\Gamma^\rho_{\nu\mu} = \tau^\rho\partial_{(\mu}\tau_{\nu)} + \tfrac{1}{2}h^{\rho\sigma}(\partial_\nu h_{\sigma\mu} + \partial_\mu h_{\sigma\nu} - \partial_\sigma h_{\mu\nu}) + h^{\rho\sigma}K_{\sigma(\mu}\tau_{\nu)}, \\ &K_{\mu\nu} = 2\partial_{[\mu}m_{\nu]}. \end{aligned} \tag{2.26}$$

The Riemann tensor can be obtained, using the vielbein postulates, from the curvatures of the spin connection fields:

$$R^\mu{}_{\nu\rho\sigma}(\Gamma) = -\mathrm{e}^\mu{}_{a'}R_{\rho\sigma}{}^{a'b'}(M'')e_{\nu b'} - \mathrm{e}^\mu{}_{a'}R_{\rho\sigma}{}^{a'\underline{0}}(M')\tau_\nu. \tag{2.27}$$

At this stage, the independent gauge fields are given by $\{\tau_\mu, e_\mu{}^{a'}, m_\mu\}$. The dynamics of the Newton–Cartan point particle is now described by the following action [1]:

$$L = \frac{m}{2}\left(\frac{h_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}{\tau_\rho\dot{x}^\rho} - 2m_\mu\dot{x}^\mu\right). \tag{2.28}$$

Alternatively, this action can be written as

$$L = \frac{m}{2}N^{-1}\dot{x}^\mu\dot{x}^\nu(h_{\mu\nu} - 2m_\mu\tau_\nu) \tag{2.29}$$

with $N \equiv \tau_\mu\dot{x}^\mu$.

The first term in this Lagrangian can be seen as the covariantization of the Lagrangian of (2.6) with the Newton–Cartan metrics $h_{\mu\nu}$ and τ_μ. The presence of the central charge gauge field m_μ represents the ambiguity when trying to solve the Γ-connection in terms of the (singular) metrics of Newton–Cartan spacetime. The Lagrangian (2.28) is quasi-invariant under the gauged Bargmann algebra; under Z-transformations $\delta m_\mu = \partial_\mu\sigma$, the Lagrangian (2.28) transforms as a total derivative, while for the other transformations, the Lagrangian is

[10] Remember that $\nabla_\rho h^{\mu\nu} = 0$ and $\nabla_\rho h_{\mu\nu} \neq 0$.

invariant. In fact, the $m_\mu \dot{x}^\mu$ term in (2.28) is needed in order to render the action invariant under boost transformations which transform both the spatial metric $h_{\mu\nu}$ and the central charge gauge field m_μ as follows:

$$\delta h_{\mu\nu} = 2\lambda^{a'}{}_{\underline{0}} e_{(\mu}{}^{a'} \tau_{\nu)}, \qquad \delta m_\mu = \lambda^{a'}{}_{\underline{0}} e_\mu{}^{a'}. \tag{2.30}$$

Varying the Lagrangian (2.28) gives, after a lengthy calculation[11], the geodesic equation

$$\ddot{x}^\mu + \Gamma^\mu_{\nu\rho} \dot{x}^\nu \dot{x}^\rho = \frac{\dot{N}}{N} \dot{x}^\mu. \tag{2.31}$$

Here, $N \equiv \tau_\mu \dot{x}^\mu = \dot{f}$, which in adapted coordinates becomes $N = \dot{t}$, and the Γ-connection is given by (2.25). The geodesic equation (2.31) can be regarded as the covariantization of (2.13).

Unlike the particle dynamics, the gravitational dynamics cannot be obtained from an action in a straightforward way; see, e.g., [7]. The equation describing the dynamics of Newton–Cartan spacetime may be written in terms of the Ricci tensor of the Γ-connection as follows:

$$R_{\mu\nu}(\Gamma) = V_{D-2} G \rho \tau_{\mu\nu}. \tag{2.32}$$

To make contact with the equations for a Galilean observer, derived in the first gauging procedure, one must impose the kinematical constraint that the curvature corresponding to the $(D-1)$-dimensional spatial rotations equals zero:

$$R_{\mu\nu}{}^{a'b'}(M'') = 0. \tag{2.33}$$

Here, M'' refers to the generators of spatial rotations. It should be stressed that one is not forced to impose this curvature constraint, and one could stay more general and try to solve the resulting theory of gravity for a curved transverse space. We will see that the constraint (2.33) can be considered as an ansatz for the transverse Newton–Cartan metric $h^{\mu\nu}$ to be flat. It is also convenient to choose the so-called *adapted* coordinates in which the function $f(x)$ in equation (2.21) is set equal to the time or foliation coordinate $t : f(x) = t$. This reduces the general coordinate transformations to constant time translations and spatial translations with an arbitrary spacetime-dependent parameter.

The kinematical constraint (2.33) enables us to do two things. First, we can now choose a flat Cartesian coordinate system in the $(D-1)$ spatial dimensions, because the transverse space is flat as can be seen from equation (2.27):[12]

$$R^i{}_{jkl}(\Gamma) = 0. \tag{2.34}$$

The solution (2.19) implies that the spatial components $\omega_i{}^{a'b'}$ of the gauge field of spatial rotations is zero in such a coordinate system, which expresses the fact that the transverse Christoffel symbols vanish:

$$\Gamma^i_{jk} \sim \delta^i_{a'} \delta^j_{b'} \omega_k{}^{a'b'} = 0. \tag{2.35}$$

This choice of coordinates restricts the spatial rotations to those that have a time-dependent parameter only. Second, due to the same kinematical constraint (2.33), the time component $\omega_0{}^{a'b'}$ of the same gauge field is a pure gauge; $R_{\mu\nu}{}^{a'b'}(M'')$ is the field strength of an $SO(D-1)$ gauge theory and contains only $\omega_\mu{}^{a'b'}$, as can be seen from (B.6). As such, the constraint

[11] Some details are given in appendix C.

[12] Note that equation (2.34) already follows from the equations of motion (2.32) in the case of $D = 4$, because in three dimensions a vanishing Ricci tensor implies a vanishing Riemann tensor.

(2.33) allows one to gauge fix $\omega_\mu{}^{a'b'}$ to zero[13], and this restricts the spatial rotations to having constant parameters only. Through (2.25), one can show that this implies

$$\Gamma^i_{0j} \sim \delta^i_{a'}\delta^j_{b'}\omega_0{}^{a'b'} = 0. \tag{2.36}$$

The same choice of a Cartesian coordinate system also restricts the spatial translations to having only time-dependent parameters. This reduces the symmetries acting on the spacetime coordinates to the acceleration-extended Galilei symmetries given in equation (2.11). The central charge transformations now only depend on time and do not act on the spacetime coordinates. The vielbein postulate tells us that the only remaining connection component Γ^i_{00} can be written as $\Gamma^i_{00} = \partial^i\Phi(x)$, where

$$\Phi(x) = m_0(x) - \tfrac{1}{2}\delta_{ij}\tau^i(x)\tau^j(x) + \partial_0 m(x). \tag{2.37}$$

Here, m_0 and $\partial_i m$ are the time component and spatial gradient components of the extension gauge field m_μ, and τ^i are the space components of the inverse temporal vielbein τ^μ. Using the transformation properties of Γ^i_{00}, one can show that $\Phi(x)$, defined by equation (2.37), indeed transforms like in equation (2.12) under the acceleration-extended Galilei symmetries[14].

One can show that after gauge fixing the Newton–Cartan symmetries to the acceleration-extended Galilei symmetries, as described above, the Lagrangian (2.28) reduces to

$$L = \frac{m}{2}\left(\frac{\delta_{ij}\dot{x}^i\dot{x}^j}{\dot{x}^0} + \dot{x}^0(\delta_{ij}\tau^i\tau^j - 2m_0 - 2\partial_0 m)\right), \tag{2.38}$$

where a boundary term has been discarded[15]. Upon comparison with the action (2.10), this again identifies the potential as in (2.37). Note that the $\tau^i\dot{x}^i$ terms cancel, reflecting the choice of gauge (2.36) and indicating that this particular reference frame is non-rotating. Similarly, equation (2.32) reduces in this gauge to the Poisson equation (2.15).

As expected, having the same symmetries, the equations of motion (2.31) and (2.32) reduce to precisely the equations of motion (2.13) and (2.15), we obtained in the first gauging procedure.

3. From particles to strings

We now consider instead of particles of mass m strings with tension T moving in a D-dimensional Minkowski spacetime, with metric $\eta_{\mu\nu}$ ($\mu = 0, 1, \ldots, D-1$). The action describing the dynamics of such a string is given by the Nambu–Goto action (we take $c = 1$)[16]

$$S = -T\int \mathrm{d}^2\sigma\sqrt{-\gamma}, \tag{3.1}$$

where $\sigma^{\bar{\alpha}}$ ($\bar{\alpha} = 0, 1$) are the world-sheet coordinates and γ is the determinant of the induced world-sheet metric $\gamma_{\bar{\alpha}\bar{\beta}}$:

$$\gamma_{\bar{\alpha}\bar{\beta}} = \partial_{\bar{\alpha}}x^\mu\partial_{\bar{\beta}}x^\nu\eta_{\mu\nu}. \tag{3.2}$$

[13] Explicitly, one can write $R_{\mu\nu}{}^{a'b'}(M'') = 2D_{[\mu}\omega_{\nu]}{}^{a'b'}$ and $\delta\omega_\mu{}^{a'b'} = D_\mu\lambda^{a'b'}$, where D_μ is the gauge covariant derivative. Putting $R_{\mu\nu}{}^{a'b'}(M'') = 0$ imposes the constraint $\omega_\mu{}^{a'b'} = D_\mu f^{a'b'}$ on the gauge field for some $f^{a'b'}$. Performing then a gauge transformation on $\omega_\mu{}^{a'b'}$ and choosing the gauge parameter to be $\lambda^{a'b'} = -f^{a'b'}$, the result follows.

[14] The fact that Φ transforms with the double time derivative of ξ^i shows that it indeed transforms as a component of the Γ-connection.

[15] We have made use of the fact that, because $x^\mu = x^\mu(\tau)$, the τ-derivative of a general function $f(x)$ can be written as $\dot{f}(x) = \dot{x}^0\partial_0 f(x) + \dot{x}^i\partial_i f(x)$, which in the static gauge becomes $\dot{f}(x) = \partial_0 f(x) + \dot{x}^i\partial_i f(x)$.

[16] Alternatively, one can consider the Polyakov action. This case has been considered in [17].

The action (3.1) is invariant under world-sheet reparametrizations. Like in the particle case, the Lagrangian corresponding to this action is invariant under Poincaré transformations in the target spacetime; see equation (2.2).

Following [11, 17], we take the non-relativistic limit by rescaling the longitudinal coordinate $x^\alpha = (x^0 \equiv t, x^1)$ with a parameter ω and taking $\omega \gg 1$:[17]

$$x^\alpha \to \omega x^\alpha, \quad \omega \gg 1. \tag{3.3}$$

This results into the following action ($i = 2, \ldots, D-1$):

$$S \approx -T\omega^2 \int d^2\sigma \sqrt{-\bar{\gamma}} \left(1 + \frac{1}{2\omega^2} \bar{\gamma}^{\bar{\alpha}\bar{\beta}} \partial_{\bar{\alpha}} x^i \partial_{\bar{\beta}} x^j \delta_{ij}\right), \tag{3.4}$$

where $\bar{\gamma}_{\bar{\alpha}\bar{\beta}}$ is the pull-back of the longitudinal metric $\eta_{\alpha\beta}$:

$$\bar{\gamma}_{\bar{\alpha}\bar{\beta}} = \partial_{\bar{\alpha}} x^\alpha \partial_{\bar{\beta}} x^\beta \eta_{\alpha\beta}. \tag{3.5}$$

Unlike the world-sheet metric (3.2), the pull-back used in (3.5) is given by a 2×2 matrix, and as such is invertible. This means that the inverse metric $\bar{\gamma}^{\bar{\alpha}\bar{\beta}}$ can be explicitly given: it is the pull-back of the longitudinal inverse metric $\eta^{\alpha\beta}$,

$$\bar{\gamma}^{\bar{\alpha}\bar{\beta}} = \partial_\alpha \sigma^{\bar{\alpha}} \partial_\beta \sigma^{\bar{\beta}} \eta^{\alpha\beta}, \tag{3.6}$$

such that $\bar{\gamma}^{\bar{\alpha}\bar{\beta}} \bar{\gamma}_{\bar{\beta}\bar{\epsilon}} = \delta^{\bar{\alpha}}_{\bar{\epsilon}}$.

The divergent term on the right-hand side of equation (3.4) is a total world-sheet derivative [11]. This can be seen by using the identity $\eta_{[\beta[\alpha}\eta_{\gamma]\delta]} = -\frac{1}{2}\epsilon_{\beta\delta}\epsilon_{\alpha\gamma}$, which holds in two dimensions and in which $\epsilon_{\alpha\gamma}$ is the two-dimensional epsilon symbol. This allows one to write

$$\sqrt{-\bar{\gamma}} = \partial_{\bar{\alpha}} \left(\tfrac{1}{2} \epsilon^{\bar{\alpha}\bar{\gamma}} \epsilon_{\alpha\gamma} x^\alpha \partial_{\bar{\gamma}} x^\gamma\right). \tag{3.7}$$

The divergent term can be canceled by coupling the string to a constant background 2-form potential $B_{\mu\nu}$ through the following Wess–Zumino term:

$$S_I = T \int d^2\sigma \, \epsilon^{\bar{\alpha}\bar{\beta}} \partial_{\bar{\alpha}} x^\mu \partial_{\bar{\beta}} x^\nu B_{\mu\nu}, \tag{3.8}$$

and choosing the constant field components $B_{\mu\nu}$ such that

$$B_{\alpha\beta} = \tfrac{1}{2}\omega^2 \epsilon_{\alpha\beta}, \quad B_{i\alpha} = B_{ij} = 0. \tag{3.9}$$

The resulting field strength of $B_{\mu\nu}$ is zero, similar to the particle case. The limit $\omega \to \infty$ of the sum of (3.4) and (3.8) then leads to the following non-relativistic action:

$$S = -\frac{T}{2} \int d^2\sigma \sqrt{-\bar{\gamma}} \left(\bar{\gamma}^{\bar{\alpha}\bar{\beta}} \partial_{\bar{\alpha}} x^i \partial_{\bar{\beta}} x^j \delta_{ij}\right). \tag{3.10}$$

This action is invariant under world-sheet reparametrizations and the following 'stringy' Galilei symmetries:

$$\delta x^\alpha = \lambda^\alpha{}_\beta x^\beta + \zeta^\alpha, \quad \delta x^i = \lambda^i{}_j x^j + \lambda^i{}_\beta x^\beta + \zeta^i, \tag{3.11}$$

where $(\zeta^\alpha, \zeta^i, \lambda^i{}_j, \lambda^i{}_\alpha, \lambda^\alpha{}_\beta)$ parametrize a (constant) longitudinal translation, transverse translation, transverse rotation, 'stringy' boost transformation and longitudinal rotation, respectively. As for the point particle, the equations of motion for the longitudinal and transverse components are not independent. The equations of motion for $\{x^i\}$ corresponding to the action (3.10) are given by

$$\partial_{\bar{\alpha}} \left(\sqrt{-\bar{\gamma}} \bar{\gamma}^{\bar{\alpha}\bar{\beta}} \partial_{\bar{\beta}} x^i\right) = 0. \tag{3.12}$$

[17] Note that, unlike the particle case, the parameter T does not get rescaled.

The non-relativistic Lagrangian defined by (3.10) is invariant under a stringy boost transformation only up to a total world-sheet divergence:

$$\delta L = \partial_{\bar{\alpha}} \left(-T\sqrt{-\bar{\gamma}} \frac{\partial \sigma^{\bar{\alpha}}}{\partial x^{\alpha}} \lambda_i{}^{\alpha} x^i \right), \tag{3.13}$$

where (3.6) has been used. This leads to a modified Noether charge giving rise to an extension of the stringy Galilei algebra containing two extra generators: Z_a and Z_{ab} $(a = (0, 1))$ [12]. The corresponding extended stringy Galilei algebra is given in appendix B.

We now wish to connect to the physics as experienced by a 'stringy' Galilean observer by gauging the translations in the spatial directions transverse to the string. In this procedure, we replace the constant parameters ζ^i by functions $\xi^i(x^\alpha)$ depending only on the longitudinal coordinates. Applying this gauging to the non-relativistic action (3.10) leads to the following gauged action containing a gravitational potential $\Phi_{\alpha\beta}$:

$$S = -\frac{T}{2} \int \mathrm{d}^2\sigma \sqrt{-\bar{\gamma}} \left(\bar{\gamma}^{\bar{\alpha}\bar{\beta}} \partial_{\bar{\alpha}} x^i \partial_{\bar{\beta}} x^j \delta_{ij} - 2\eta^{\alpha\beta} \Phi_{\alpha\beta} \right). \tag{3.14}$$

This action can be compared with the point particle action (2.10).[18] The string action (3.14) is invariant under world-sheet reparametrizations and the acceleration-extended stringy Galilei symmetries [12]:

$$\delta x^\alpha = \lambda^\alpha{}_\beta x^\beta + \zeta^\alpha, \qquad \delta x^i = \lambda^i{}_j x^j + \xi^i(x^\alpha). \tag{3.15}$$

The local transverse translations are only realized provided that the background potentials $\Phi_{\alpha\beta}$ transform as follows:

$$\delta \Phi_{\alpha\beta} = -\frac{1}{2\sqrt{-\bar{\gamma}}} \eta_{\alpha\beta} \partial_{\bar{\alpha}} \left(\sqrt{-\bar{\gamma}} \bar{\gamma}^{\bar{\alpha}\bar{\beta}} \partial_{\bar{\beta}} \xi_i \right) x^i + \nabla_{(\alpha} g_{\beta)}(x^\epsilon), \tag{3.16}$$

for arbitrary $g_\beta(x^\epsilon)$. Equation (3.16) is the string analogue of equation (2.12). The action (3.14) leads to the following modified equations of motion for the transverse coordinates $\{x^i\}$:

$$\partial_{\bar{\alpha}} \left(\sqrt{-\bar{\gamma}} \bar{\gamma}^{\bar{\alpha}\bar{\beta}} \partial_{\bar{\beta}} x^i \right) + \sqrt{-\bar{\gamma}} \eta^{\alpha\beta} \partial^i \Phi_{\alpha\beta} = 0. \tag{3.17}$$

These equations of motion simplify if we choose the static gauge

$$x^\alpha = \sigma^{\bar{\alpha}}. \tag{3.18}$$

In this gauge, we have that $\bar{\gamma}_{\bar{\alpha}\bar{\beta}} = \eta_{\alpha\beta}$.

The equation of motion of $\Phi_{\alpha\beta}(x)$ itself is easily obtained by requiring that it is of second order in spatial derivatives and invariant under the acceleration-extended stringy Galilei symmetries (3.15) and (3.16). Since the variation of $\Phi_{\alpha\beta}(x)$, see equation (3.16), contains an arbitrary function of the longitudinal coordinates and is linear in the transverse coordinates, it follows that the unique second-order differential operator satisfying the above requirement is the Laplacian $\Delta \equiv \delta^{ij}\partial_i\partial_j$. Requiring that the source term is provided by the mass density function $\rho(x)$, which transforms as a scalar with respect to (3.15), this leads to the following Poisson equation:

$$\triangle \Phi_{\alpha\beta}(x) = V_{D-2} G \rho(x) \eta_{\alpha\beta}. \tag{3.19}$$

This finishes our first approach where we only gauge the transverse translations. In this approach, we have presented both the equations of motion for the transverse coordinates $\{x^i\}$ of a string, see equation (3.17), as well as the bulk equations of motion for the gravitational potential $\Phi_{\alpha\beta}$; see equation (3.19).

We now proceed with the second gauging procedure in which we gauge the full deformed stringy Galilei algebra. This algebra consists of longitudinal translations, transverse

[18] Note that $\bar{\gamma}_{\bar{\alpha}\bar{\beta}}$ corresponds to a factor $-(\dot{x}^0)^2$ in the particle action.

translations, longitudinal Lorentz transformations, 'boost' transformations, transverse rotations and two distinct extension transformations. The explicit commutation relations of the generators corresponding to these symmetries are given in appendix B. As a first step one associates a gauge field with each of these symmetries:

$$\begin{aligned} \tau_\mu{}^a &: \text{longitudinal translations,} \\ e_\mu{}^{a'} &: \text{transverse translations,} \\ \omega_\mu{}^{ab} &: \text{longitudinal Lorentz transformations,} \\ \omega_\mu{}^{a'a} &: \text{'boost' transformation,} \\ \omega_\mu{}^{a'b'} &: \text{transverse rotations,} \\ m_\mu{}^a, m_\mu{}^{ab} &: \text{extension transformations.} \end{aligned} \tag{3.20}$$

At the same time, the constant parameters describing the transformations are promoted to arbitrary functions of the spacetime coordinates $\{x^\mu\}$:

$$\begin{aligned} \tau^a(x^\mu) &: \text{longitudinal translations,} \\ \zeta^{a'}(x^\mu) &: \text{transverse translations,} \\ \lambda^{ab}(x^\mu) &: \text{longitudinal Lorentz transformations,} \\ \lambda^{a'a}(x^\mu) &: \text{'boost' transformations,} \\ \lambda^{a'b'}(x^\mu) &: \text{transverse rotations,} \\ \sigma^a(x^\mu), \sigma^{ab}(x^\mu) &: \text{extension transformations.} \end{aligned} \tag{3.21}$$

The explicit gauge transformations of the gauge fields, together with the expressions for the gauge-invariant curvatures and the Bianchi identities that they satisfy, can be found in appendix B. Besides the gauge transformations all gauge fields transform under general coordinate transformations with parameters $\xi^\mu(x^\mu) = (\xi^\alpha(x^\mu), \xi^i(x^\mu))$.

Like in the particle case, we would like to express the Γ-connection in terms of the previous gauge fields. In order to do that we first impose a set of so-called conventional constraints on the curvatures of the gauge fields:

$$R_{\mu\nu}{}^a(H) = R_{\mu\nu}{}^{a'}(P) = R_{\mu\nu}{}^a(Z) = 0. \tag{3.22}$$

These constraints are required to convert the local H_a and $P_{a'}$ transformations into general coordinate transformations through the identity (2.18). Besides this, the constraints (3.22) also imply that the gauge fields $\omega_\mu{}^{a'b'}$, $\omega_\mu{}^{a'a}$ and $\omega_\mu{}^{ab}$ become dependent:

$$\omega_\mu{}^{a'b'} = \partial_{[\mu}e_{\nu]}{}^{a'}e^{\nu b'} - \partial_{[\mu}e_{\nu]}{}^{b'}e^{\nu a'} + e_\mu{}^{c'}\partial_{[\nu}e_{\rho]}{}^{c'}e^{\nu a'}e^{\rho b'} - \tau_\mu{}^a e^{\rho[a'}\omega_\rho{}^{b']a}, \tag{3.23}$$

$$\begin{aligned} \omega_\mu{}^{a'a} = {} & 2\tau_\mu{}^b\big(\tau^{\nu b}e^{\rho a'}[\partial_{[\nu}m_{\rho]}{}^a - \omega_{[\nu}{}^{ac}m_{\rho]}{}^c] - e^{\nu a'}m_\nu{}^{ab}\big) \\ & +2e_\mu{}^{b'}\tau^{\rho a}e^{\nu(b'}\partial_{[\nu}e_{\rho]}{}^{a')} + e_\mu{}^{b'}e^{\nu b'}e^{\rho a'}[\partial_{[\nu}m_{\rho]}{}^a - \omega_{[\nu}{}^{ab}m_{\rho]}{}^b], \end{aligned} \tag{3.24}$$

$$\omega_\mu{}^{ab} = \partial_{[\mu}\tau_{\nu]}{}^a\tau^{\nu b} - \partial_{[\mu}\tau_{\nu]}{}^b\tau^{\nu a} + \tau^{\nu a}\tau^{\rho b}\tau_\mu{}^c\partial_{[\nu}\tau_{\rho]}{}^c. \tag{3.25}$$

The solution for $\omega_\mu{}^{ab}$ is familiar from the Poincaré theory and reflects the fact that the foliation space is given by a two-dimensional Minkowski spacetime. The same constraints have a third effect, namely that they lead to constraints on the curl of the gauge field $\tau_\mu{}^a$. More precisely, the conventional constraint $R_{\mu\nu}{}^a(H) = 0$ cannot only be used to solve for the spin connection $\omega_\mu{}^{ab}$, see equation (3.25). Substituting this solution back into the constraint also implies that the following projections of $\partial_{[\mu}\tau_{\nu]}{}^a$ vanish:

$$\mathrm{e}^{\mu a'}\tau^{\nu(a}\partial_{[\mu}\tau_{\nu]}{}^{b)} = 0, \quad \mathrm{e}^\mu{}_{a'}\mathrm{e}^\nu{}_{b'}\partial_{[\mu}\tau_{\nu]}{}^a = 0. \tag{3.26}$$

It is instructive to verify how the other two spin connections are solved for. First, the conventional constraints $R_{\mu\nu}{}^{a'}(P) = 0$ can not only be used to solve for the spin connection

$\omega_\mu{}^{a'b'}$, see equation (3.23), but also for the following projections of the spin connection field $\omega_\mu{}^{a'a}$:

$$\mathrm{e}^{\mu(a'}\omega_\mu{}^{b')b} = 2\tau^{\nu b}\mathrm{e}^{\mu(a'}\partial_{[\mu}e_{\nu]}{}^{b')}, \quad \omega_\rho{}^{a'[a}\tau^{b]\rho} = -\tau^{\mu a}\tau^{\nu b}\partial_{[\mu}e_{\nu]}{}^{a'}. \tag{3.27}$$

Making different contractions of the third conventional constraint $R_{\mu\nu}{}^a(Z) = 0$, one can solve for two more projections of the same spin connection field:

$$\tau^{\mu b}\omega_\mu{}^{a'a} = 2\tau^{\mu b}\mathrm{e}^{\nu a'}\left(\partial_{[\mu}m_{\nu]}{}^a - \omega_{[\mu}{}^{ac}m_{\nu]}{}^c\right) - 2\mathrm{e}^{\mu a'}m_\mu{}^{ab}, \tag{3.28}$$

$$\mathrm{e}^{\mu[a'}\omega_\mu{}^{b']a} = \mathrm{e}^{\mu a'}\mathrm{e}^{\nu b'}\left(\partial_{[\mu}m_{\nu]}{}^a - \omega_{[\mu}{}^{ab}m_{\nu]}{}^b\right). \tag{3.29}$$

Combining the solutions (3.27), (3.28) and (3.29) for the different projections and using the decomposition

$$\omega_\mu{}^{a'a} = \tau_\mu{}^b\tau^{\nu b}\omega_\nu{}^{a'a} + e_\mu{}^{b'}\mathrm{e}^{\nu(b'}\omega_\nu{}^{a')a} + e_\mu{}^{b'}\mathrm{e}^{\nu[b'}\omega_\nu{}^{a']a}, \tag{3.30}$$

one can solve for the spin connection field $\omega_\mu{}^{a'a}$, see (3.24). Finally, it turns out that beyond the contractions already considered, there is one more contraction of the conventional constraint $R_{\mu\nu}{}^a(Z) = 0$. It leads to the following constraint on the gauge field $m_\mu{}^{ab}$:

$$\tau^{\mu[c}m_\mu{}^{d]a} = \tau^{\mu c}\tau^{\nu d}\left(\partial_{[\mu}m_{\nu]}{}^a - \omega_{[\mu}{}^{ab}m_{\nu]}{}^b\right). \tag{3.31}$$

This constraint relates the longitudinal projection of $D_{[\mu}m_{\nu]}{}^a$ to a certain projection of the gauge field $m_\mu{}^{ab}$, but does not allow one to solve $m_\mu{}^{ab}$ completely; the other projections remain unspecified. We will return to the meaning of the constraint (3.31) after equation (3.45).

At this point, the symmetries of the theory are the general coordinate transformations, the longitudinal Lorentz transformations, 'boost' transformations, transverse rotations and extension transformations, all with parameters that are arbitrary functions of the spacetime coordinates. The gauge fields $\tau_\mu{}^a$ of longitudinal translations and $e_\mu{}^{a'}$ of transverse translations are identified as the (singular) longitudinal and transverse vielbeins. One may also introduce their inverses (with respect to the longitudinal and transverse subspaces) $\tau^\mu{}_a$ and $\mathrm{e}^\mu{}_{a'}$:

$$e_\mu{}^{a'}\mathrm{e}^\mu{}_{b'} = \delta^{a'}_{b'}, \quad e_\mu{}^{a'}\mathrm{e}^\nu{}_{a'} = \delta^\nu_\mu - \tau_\mu{}^a\tau^\nu{}_a, \quad \tau^\mu{}_a\tau_\mu{}^b = \delta^b_a,$$
$$\tau^\mu{}_a e_\mu{}^{a'} = 0, \quad \tau_\mu{}^a\mathrm{e}^\mu{}_{a'} = 0. \tag{3.32}$$

The spatial and temporal vielbeins are related to the spatial metric $h^{\mu\nu}$ with 'inverse' $h_{\mu\nu}$, and the temporal metric $\tau_{\mu\nu}$ with 'inverse' $\tau^{\mu\nu}$, as follows:

$$\tau_{\mu\nu} = \tau_\mu{}^a\tau_\nu{}^b\,\eta_{ab}, \qquad \tau^{\mu\nu} = \tau^\mu{}_a\tau^\nu{}_b\,\eta^{ab},$$
$$h_{\mu\nu} = e_\mu{}^{a'}e_\nu{}^{b'}\,\delta_{a'b'}, \qquad h^{\mu\nu} = \mathrm{e}^\mu{}_{a'}\mathrm{e}^\nu{}_{b'}\,\delta^{a'b'}. \tag{3.33}$$

These tensors satisfy the Newton–Cartan metric conditions

$$h^{\mu\nu}h_{\nu\rho} + \tau^{\mu\nu}\tau_{\nu\rho} = \delta^\mu_\rho, \quad \tau^{\mu\nu}\tau_{\mu\nu} = 2,$$
$$h^{\mu\nu}\tau_{\nu\rho} = h_{\mu\nu}\tau^{\nu\rho} = 0. \tag{3.34}$$

We note that for the point particle one would have $\tau^{\mu\nu}\tau_{\mu\nu} = 1$ instead of $\tau^{\mu\nu}\tau_{\mu\nu} = 2$.

A Γ-connection can be introduced by imposing the following vielbein postulates:

$$\partial_\mu e_\nu{}^{a'} - \omega_\mu{}^{a'b'}e_\nu{}^{b'} - \omega_\mu{}^{a'a}\tau_\nu{}^a - \Gamma^\lambda_{\nu\mu}e_\lambda{}^{a'} = 0,$$
$$\partial_\mu\tau_\nu{}^a - \omega_\mu{}^{ab}\tau_\nu{}^b - \Gamma^\rho_{\nu\mu}\tau_\rho{}^a = 0. \tag{3.35}$$

These vielbein postulates allow one to solve for Γ uniquely. The torsion $\Gamma^\rho_{[\nu\mu]}$ vanishes because of the constraints $R(P) = R(H) = 0$, and with this the vielbein postulates give the solution

$$\Gamma^\rho_{\nu\mu} = \tau^\rho{}_a\left(\partial_{(\mu}\tau_{\nu)}{}^a - \omega_{(\mu}{}^{ab}\tau_{\nu)}{}^b\right) + e^\rho{}_{a'}\left(\partial_{(\mu}e_{\nu)}{}^{a'} - \omega_{(\mu}{}^{a'b'}e_{\nu)}{}^{b'} - \omega_{(\mu}{}^{a'a}\tau_{\nu)}{}^a\right) \tag{3.36}$$

in terms of the dependent spin connections $\omega_\mu{}^{a'b'}$, $\omega_\mu{}^{a'a}$ and $\omega_\mu{}^{ab}$. If one plugs in the explicit solutions of these spin connections, one obtains

$$\Gamma^\rho_{\mu\nu} = \tfrac{1}{2}\tau^{\rho\sigma}(\partial_\nu\tau_{\sigma\mu} + \partial_\mu\tau_{\sigma\nu} - \partial_\sigma\tau_{\mu\nu}) + \tfrac{1}{2}h^{\rho\sigma}(\partial_\nu h_{\sigma\mu} + \partial_\mu h_{\sigma\nu} - \partial_\sigma h_{\mu\nu}) + h^{\rho\sigma}K_{\sigma(\mu}{}^a\,\tau_{\nu)}{}^a, \tag{3.37}$$

where $K_{\mu\nu}{}^a = -K_{\nu\mu}{}^a$ is given by the covariant curl of $m_\mu{}^a$:

$$K_{\mu\nu}{}^a = 2D_{[\mu}m_{\nu]}{}^a. \tag{3.38}$$

An important observation is that $m_\mu{}^{ab}$ does not appear in (3.37). The origin of this absence is the fact that expression (3.36) is invariant under the shift transformations

$$\omega_\mu{}^{a'a} \to \omega_\mu{}^{a'a} + \tau_\mu{}^b X^{a'}_{ab}, \tag{3.39}$$

where $X^{a'}_{ab} = X^{a'}_{[ab]}$ is an arbitrary shift parameter. The field $m_\mu{}^{ab}$ appears in the form $X^{a'}_{ab} = \mathrm{e}^\lambda{}_{a'}m_\lambda{}^{ab}$ in the solution of $\omega_\mu{}^{a'a}$, and as such $m_\mu{}^{ab}$ will drop out of the connection (3.36), and thus out of (3.37).

The Riemann tensor can be obtained, using the vielbein postulates, from the curvatures of the spin connection fields:

$$R^\mu{}_{\nu\rho\sigma}(\Gamma) = -\tau^\mu{}_a R_{\rho\sigma}{}^{ab}(M)\tau_\nu{}^b - e^\mu{}_{a'}R_{\rho\sigma}{}^{a'b'}(M'')e_{\nu b'} - e^\mu{}_{a'}R_{\rho\sigma}{}^{a'a}(M')\tau_{\nu a}. \tag{3.40}$$

Note that this Riemann tensor has no dependence on the gauge field $m_\mu{}^{ab}$.

At this stage, the independent fields are given by $\{\tau_\mu{}^a, e_\mu{}^{a'}, m_\mu{}^a\}$, whereas we saw that $m_\mu{}^{ab}$ was partially solved for through equation (3.31) and does not enter the dynamics[19]. The dynamics of a Newton–Cartan string is now described by the following Lagrangian[20]:

$$L = -\frac{T}{2}\sqrt{-\det(\tau)}\,\tau^{\bar\alpha\bar\beta}\partial_{\bar\alpha}x^\mu\partial_{\bar\beta}x^\nu\left(h_{\mu\nu} - 2m_\mu{}^a\tau_\nu{}^a\right), \tag{3.41}$$

where the induced world-sheet metric $\tau_{\bar\alpha\bar\beta}$ is given by

$$\tau_{\bar\alpha\bar\beta} \equiv \partial_{\bar\alpha}x^\mu\partial_{\bar\beta}x^\nu\tau_{\mu\nu}. \tag{3.42}$$

Equation (3.41) is the stringy generalization of the particle action (2.29). The first term in equation (3.41) can be seen as the covariantization of the Lagrangian of (3.10) with the Newton–Cartan metrics $h_{\mu\nu}$ and $\tau_{\mu\nu}$, where the induced world-sheet metric (3.42) is the covariantization of (3.5) with $\tau_{\mu\nu}$. Analogously to the point particle, the Lagrangian (3.41) is quasi-invariant under the gauged deformed stringy Galilei algebra. Under Z_a-transformations $\delta m_\mu{}^a = \partial_\mu\sigma^a$, the Lagrangian (3.41) transforms as a total derivative, while the other transformations leave the Lagrangian invariant. In particular, this applies to the Z_{ab}-transformations which are given by

$$\delta m_\mu{}^a = -\sigma^{ab}\tau_\mu{}^b \quad \text{or} \quad \tau^{\mu[a}\delta m_\mu{}^{b]} = \sigma^{ab}. \tag{3.43}$$

The latter way of writing shows that the projection $\tau^{\mu[a}m_\mu{}^{b]}$ of the gauge field $m_\mu{}^a$ can be gauged away. The $m_{(\mu}{}^a\tau_{\nu)}{}^a$ term in the Lagrangian (3.41) is needed in order to render the action invariant under boost transformations which transform both the spatial metric $h_{\mu\nu}$ and the extension gauge field $m_\mu{}^a$ as follows:

$$\delta h_{\mu\nu} = 2\lambda^{a'a}e_{(\mu}{}^{a'}\tau_{\nu)}{}^a, \quad \delta m_\mu{}^a = \lambda^{a'a}e_\mu{}^{a'}. \tag{3.44}$$

Like in the particle case, the presence of the extension gauge field $m_\mu{}^a$ represents an ambiguity when trying to solve the Γ-connection in terms of the (singular) metrics (3.33) of Newton–Cartan spacetime. Namely, the metric compatibility conditions on $h^{\mu\nu}$ and $\tau_{\mu\nu}$

[19] An analogous results holds for the dynamics of the non-relativistic string, see equation (32) of [18].

[20] Note that the stringy Newton–Cartan theory does not only contain the metric $h_{\mu\nu}$ but also the extension gauge field $m_\mu{}^a$, see equation (3.20).

give the solution (3.37), but $K_{\mu\nu}{}^{a} = -K_{\nu\mu}{}^{a}$ is an ambiguity which is not fixed by the metric compatibility conditions. It is the specific solution (3.36) of the vielbein postulates which fixes this ambiguity to be (3.38). A new feature of the string case is that the ambiguity $K_{\mu\nu}{}^{a}$ has its own ambiguity. In other words, there is an ambiguity in the ambiguity! To show how this works we first note that from equation (3.37) it follows that the longitudinal projection of (3.38) does not contribute to the connection because it is multiplied by $h^{\rho\sigma}$. This is equivalent to saying that expression (3.37) is invariant under the shift transformations[21]

$$K_{\mu\nu}{}^{a} \rightarrow K_{\mu\nu}{}^{a} + \tau_{[\mu}{}^{c}\tau_{\nu]}{}^{b}Y^{a}_{bc} \tag{3.45}$$

for arbitrary parameters Y^{a}_{bc}. We will now argue that this ambiguity in $K_{\mu\nu}{}^{a}$ is related to the second extension gauge field $m_{\mu}{}^{ab}$, which in contrast to $m_{\mu}{}^{a}$ does *not* enter the Lagrangian (3.41). We have seen that the absence of $m_{\mu}{}^{ab}$ in the dynamics follows from the shift symmetry (3.39), which prevents the field $m_{\mu}{}^{ab}$ to enter the Γ-connection. We now come back to the role of the constraint (3.31) Using equation (3.38), we see that this constraint relates a certain projection of $m_{\mu}{}^{ab}$ to the longitudinal projection of the ambiguity $K_{\mu\nu}{}^{a}$. This longitudinal projection of the ambiguity is precisely the part that drops out of the expression for Γ corresponding to the shift invariance of (3.37) under (3.45). Therefore, the constraint (3.31) implies that a certain projection of the extension gauge field $m_{\mu}{}^{ab}$ can be regarded as an ‘ambiguity in the ambiguity’.

Summarizing, we conclude that the extension gauge field $m_{\mu}{}^{a}$, like in the particle case, corresponds to an ambiguity in the Γ-connection. This gauge field occurs in the string action (3.41). A new feature, absent in the particle case, is that there is a second extension gauge field $m_{\mu}{}^{ab}$ which corresponds to an ambiguity in the ambiguity. This extension gauge field does not occur in the string action (3.41).

Having clarified the role of the extension gauge fields, we now vary the Lagrangian (3.41) which gives, after a long calculation[22] similar to the one leading to (2.31),

$$\tau^{\bar{\alpha}\bar{\beta}}\left(\nabla_{\bar{\alpha}}\partial_{\bar{\beta}}x^{\rho} + \partial_{\bar{\alpha}}x^{\mu}\partial_{\bar{\beta}}x^{\nu}\Gamma^{\rho}_{\mu\nu}\right) = 0, \tag{3.46}$$

where the Γ-connection is given by (3.36). This geodesic equation can be seen as the covariantization of (3.17), and in the particle case reduces to (2.31) as one would expect. The equations describing the dynamics of stringy Newton–Cartan spacetime are given by

$$R_{\mu\nu}(\Gamma) = V_{D-2}G\rho\tau_{\mu\nu}, \tag{3.47}$$

just as for the point particle. The Ricci tensor however now is given in terms of the Γ-connection (3.36).

To make contact with a Galilean observer, we impose the additional kinematical constraints

$$R_{\mu\nu}{}^{ab}(M) = R_{\mu\nu}{}^{a'b'}(M'') = 0. \tag{3.48}$$

Here, M'' refers to the generators of spatial rotations, whereas M refers to the generator of a longitudinal rotation which was absent for the particle. It should be stressed that one is not forced to impose these curvature constraints, and one could stay more general and try to solve the resulting theory of gravity for a curved longitudinal and transverse space. In particular, in adding a cosmological constant in the next section, we will impose a different constraint for the longitudinal space. The first constraint of (3.48) allows one to gauge fix $\omega_{\mu}{}^{ab} = 0$, expressing the flatness of the longitudinal space. This solves the constraints (3.26) and allows one to go to the so-called adapted coordinates, in which $\tau_{\mu}{}^{a}$ is given by

$$\tau_{\mu}{}^{a} = \delta_{\mu}{}^{a}. \tag{3.49}$$

[21] An analogous result was obtained in [18].
[22] Some details are given in appendix C.

In terms of these adapted coordinates, the (longitudinal and transverse) vielbeins and their inverses are given by

$$\tau_\mu{}^a = \big(\delta^a_\alpha, 0\big), \qquad e_\mu{}^{a'} = \big(-e_k{}^{a'}\tau^k{}_a, e_i{}^{a'}\big),$$
$$\tau^\mu{}_a = \big(\delta^\alpha_a, \tau^i{}_a\big), \qquad e^\mu{}_{a'} = \big(0, e^i{}_{a'}\big), \tag{3.50}$$

in terms of the independent components $\tau^i{}_a$ and the transverse vielbeins $e_i{}^{a'}$ together with their inverse $e^i{}_{a'}$. Note that in adapted coordinates the transverse vielbein is non-singular in the transverse space, i.e.

$$e_i{}^{a'}e^j{}_{a'} = \delta^j_i, \quad e_i{}^{a'}e^i{}_{b'} = \delta^{a'}_{b'}. \tag{3.51}$$

The second kinematical constraint of (3.48) expresses the choice of flat transverse directions. It implies, using equation (3.40), that $R^i{}_{jkl}(\Gamma) = 0$ and allows us to choose a flat Cartesian coordinate system in the transverse space such that

$$e_i{}^{a'} = \delta_i{}^{a'}, \quad e^i{}_{a'} = \delta^i{}_{a'}. \tag{3.52}$$

As such the constraints (3.48) can be regarded as metric ansätze in which one is looking for solutions of the metrics describing both a flat transverse space and a flat foliation space. All metric components can now be expressed in terms of the only nontrivial components $\tau^i{}_a$:

$$\tau_\mu{}^a = \big(\delta^a_\alpha, 0\big), \qquad e_\mu{}^{a'} = \big(-\tau^{a'}{}_a, \delta_i{}^{a'}\big),$$
$$\tau^\mu{}_a = \big(\delta^\alpha_a, \tau^i{}_a\big), \qquad e^\mu{}_{a'} = \big(0, \delta^i{}_{a'}\big), \tag{3.53}$$

where we no longer distinguish between (longitudinal, transverse) curved indices (α, i) and (longitudinal, transverse) flat indices (a, a').

Plugging the conventional constraints (3.22) and the kinematical constraints (3.48) into the Bianchi identities (B.6), we find that

$$R_{\alpha\beta}(\Gamma) = -\delta^a_{(\alpha}\delta^b_{\beta)}e^\rho{}_{a'}\tau^\sigma{}_b R_{\rho\sigma}{}^{a'a}(M') \tag{3.54}$$

are the only nonzero components of the Ricci tensor. Furthermore, the remaining nonzero curvatures $R(M')$ and $R(Z)$ are constrained by the following algebraic identities:

$$R_{[\lambda\mu}{}^{a'a}(M')\tau_{\nu]}{}^a = R_{[\lambda\mu}{}^{a'a}(M')e_{\nu]}{}^{a'} - R_{[\lambda\mu}{}^{ab}(Z)\tau_{\nu]}{}^b = 0. \tag{3.55}$$

The kinematical constraint $R_{\mu\nu}{}^{a'b'}(M'') = 0$ also allows one to gauge fix $\omega_\mu{}^{a'b'} = 0$. We will now show that in this gauge

$$\Gamma^i_{\alpha j} = 0, \quad \Gamma^i_{\alpha\beta} = \partial^i\Phi_{\alpha\beta}, \tag{3.56}$$

where the latter equation defines the gravitational potential $\Phi_{\alpha\beta}$.

We first show that $\Gamma^i_{\alpha j} = 0$. Using expressions (3.53), equation (3.36) and the fact that $\omega_j{}^{ab} = \omega_\mu{}^{a'b'} = 0$, we find that $\Gamma^i_{\alpha j}$ is given by

$$\Gamma^i_{aj} = \tfrac{1}{2}\big(-\partial_j\tau^i{}_a - \omega_j{}^{ia}\big). \tag{3.57}$$

Next, using expressions (3.23)–(3.25), we find that

$$\omega_j{}^{ia} = -\partial_{[i}m_{j]a} - \partial^{(i}\tau^{j)a}, \tag{3.58}$$

where we have used that $\omega_i{}^{ab} = 0$. Furthermore, the gauge-fixing condition $\omega_k{}^{ij} = 0$ is already satisfied but the gauge-fixing condition $\omega_\alpha{}^{a'b'} = 0$ leads to the constraint

$$\omega_a{}^{ij} = -\partial_{[i}m_{j]a} - \partial^{[i}\tau^{j]a} = 0. \tag{3.59}$$

This constraint equation implies that m_{ia} can be written as

$$m_{ia} = -\tau^i{}_a - \partial_i m_a, \tag{3.60}$$

where m_a are the transverse spatial gradient components of m_{ia}. Substituting the expression for ${\omega_j}^{ia}$ into that of Γ^i_{aj}, the result becomes proportional to the right-hand side of the constraint equation (3.59), and hence, we find $\Gamma^i_{aj} = 0$.

We next show that $\Gamma^i_{\alpha\beta}$ can be written as $\partial^i\Phi_{\alpha\beta}$ defining a gravitational potential $\Phi_{\alpha\beta}$. Using (3.36), we derive the following expression[23]:

$$\Gamma^i_{ab} = -\partial_{(a}{\tau^i}_{b)} - \omega_{(a}{}^i{}_{b)}, \tag{3.61}$$

where we have used that ${\omega_\alpha}^{ab} = {\omega_\alpha}^{ij} = 0$. Following equations (3.23)–(3.25), we find that ${\omega_a}^{ib}$ is given by

$$ {\omega_a}^{ib} = \partial_a m_{ib} - \partial_i m_{ab} + {\tau^k}_a \partial_{[k} m_{i]b} + \tfrac{1}{2}{\tau^k}_a\big(\partial_i {\tau^k}_b\big) + \tfrac{1}{2}{\tau^k}_a\partial_k{\tau^i}_b + 2{m_i}^{ab}. \tag{3.62}$$

Substituting this expression for ${\omega_a}^{ib}$ back into that of Γ^i_{ab} and using (3.60), we indeed find that $\Gamma^i_{ab} = \partial^i\Phi_{ab}$ with

$$\Phi_{\alpha\beta}(x) = m_{(\alpha\beta)}(x) - \tfrac{1}{2}\delta_{ij}{\tau^i}_\alpha(x){\tau^j}_\beta(x) + \partial_{(\alpha}m_{\beta)}(x), \tag{3.63}$$

where $m_{(\alpha\beta)} = m_{(\alpha}{}^a\delta^a_{\beta)}$. This is the stringy generalization of equation (2.37).

Using the expressions for the components of the Γ-connection calculated above, we may now verify that the Newton–Cartan geodesic equation (3.46) and the Poisson equation (4.32) corresponding to the second gauging procedure reduce to equations (3.17) and (3.19) derived in the first gauging procedure. After gauge fixing the Newton–Cartan symmetries to the acceleration-extended Galilei symmetries as described above, the Lagrangian (3.41) reduces to the Lagrangian associated with the action (3.14), with the potential $\Phi_{\alpha\beta}$ given by (3.63) and $\bar{\gamma}_{\bar{\alpha}\bar{\beta}} = \tau_{\bar{\alpha}\bar{\beta}}$:[24]

$$L = -\frac{T}{2}\sqrt{-\det(\tau)}\,\tau^{\bar{\alpha}\bar{\beta}}\big(\partial_{\bar{\alpha}}x^i\partial_{\bar{\beta}}x^j\delta_{ij} + \partial_{\bar{\alpha}}x^\alpha\partial_{\bar{\beta}}x^\beta\big[{\tau^i}_\alpha{\tau^j}_\beta\delta_{ij} - 2m_{(\alpha\beta)} - 2\partial_{(\alpha}m_{\beta)}\big]\big). \tag{3.64}$$

The longitudinal components $R_{\alpha\beta}(\Gamma)$ of the Ricci tensor become

$$R_{\alpha\beta}(\Gamma) = -\delta^a_{(\alpha}\delta^b_{\beta)}{\mathrm{e}^\rho}_{a'}{\tau^\sigma}_b{R_{\rho\sigma}}^{a'a}(M') = \delta^{ij}\partial_i\partial_j\Phi_{\alpha\beta}, \tag{3.65}$$

such that indeed (4.32) gives the stringy Poisson equation (3.19). This finishes our discussion of the string moving in a flat Minkowski spacetime. In the next section, we will consider the addition of a cosmological constant.

4. Adding a cosmological constant

In order to study applications of the AdS/CFT correspondence based on the symmetry algebra corresponding to a non-relativistic string, it is necessary to include a (negative) cosmological constant Λ. To explain how this can be done, we will discuss in section 4.1 the particle case. In section 4.2, we will show how to go from particles to strings.

4.1. The particle case

Adding a negative cosmological constant in the relativistic case means that the Poincaré algebra gets replaced by an AdS algebra corresponding to a particle moving in an AdS background. It is well known that one cannot obtain general relativity with a (negative) cosmological constant by gauging the AdS algebra in the same way that one can obtain general relativity by gauging the Poincaré algebra [23]. The (technical) reason for this is that one cannot find a set of (so-called conventional) curvature constraints whose effect is to convert the translation

[23] Remember that we no longer distinguish between flat indices a and curved indices α.
[24] After the gauge fixing, one has $\tau_{\bar{\alpha}\bar{\beta}} = \partial_{\bar{\alpha}}x^\alpha\partial_{\bar{\beta}}x^\beta\eta_{\alpha\beta}$.

transformations into general coordinate transformations and, at the same time, to make certain gauge fields to be dependent on others. The same is true for the non-relativistic limit of the AdS algebra which is the Newton–Hooke algebra [15, 16]. Therefore, we cannot apply the same gauging procedure to the Newton–Hooke algebra that we used for the Bargmann algebra in section 2. It turns out that we do not need to apply a full gauging procedure to the Newton–Hooke algebra. When taking the non-relativistic limit of a particle moving in an AdS background, which is a Λ-deformation of the Minkowski background, one ends up with the action of a non-relativistic particle moving in a harmonic oscillator potential. This is a particular case of the non-relativistic particle action for a Galilean observer with *zero* cosmological constant but with a particular nonzero value of the potential $\Phi(x)$. In view of this, it is convenient to write the potential $\Phi(x)$ as the sum of a purely gravitational potential $\phi(x)$ and an effective background potential $\phi_\Lambda(x)$ describing the harmonic oscillator due to the cosmological constant:

$$\Phi(x) = \phi(x) + \phi_\Lambda(x). \tag{4.1}$$

Note that equation (4.1) points out a conceptual difference between the relativistic and the non-relativistic notion of a cosmological constant, which will also be true for the string. Namely, according to (4.1), one is always able to redefine the potential $\phi(x)$ in order to absorb the cosmological constant into $\Phi(x)$. But in the relativistic case such a redefinition of Λ into the metric $g_{\mu\nu}(x)$ is not possible. The non-relativistic particle action in the presence of a cosmological constant is invariant under the Newton–Hooke symmetries which is a Λ-deformation of the Galilei symmetries that we considered in section 2. A particularly useful feature of the Newton–Hooke symmetries is that the Λ-deformed symmetries can all be viewed as particular time-dependent transverse translations. This means that, when gauging the Galilei symmetries like we did in section 2, the Newton–Hooke symmetries are automatically included. The consequence of this is that although we cannot perform the second gauging procedure of section 2, i.e. gauge the full Newton–Hooke algebra, it is straightforward to apply the first gauging procedure, i.e. gauge the transverse translation leading to arbitrary accelerations between different frames, as is appropriate for a Galilean observer. Independent of whether we are starting from the Galilei or Newton–Hooke symmetries, when we gauge the transverse translations, we end up with precisely the same answer which we already derived in section 2, but with a different interpretation of the potential $\Phi(x)$. The difference is seen when we turn off gravity. Without a cosmological constant, turning off gravity means setting $\Phi(x) = \phi(x) = 0$ and there is no background potential, i.e. $\phi_\Lambda(x) = 0$. However, when $\Lambda \neq 0$, turning off gravity means a different thing since now we want to end up with a nonzero background potential $\phi_\Lambda(x) \neq 0$. According to equation (4.1), it means setting $\Phi(x) = \phi_\Lambda(x)$ or $\phi(x) = 0$. One can view this as a different gauge condition and that is the reason why, in the presence of a nonzero cosmological constant, the symmetries that relate inertial frames are given by the Newton–Hooke symmetries instead of the Galilei symmetries. For a Galilean observer, however, we end up with precisely the same geodesic equation and bulk equation of motion that we derived in the absence of a cosmological constant in the previous section.

Before showing how the Newton–Hooke symmetries arise as the transformations that relate inertial frames, it is instructive to first re-derive the Galilei symmetries starting from a Galilean observer. Consider the acceleration-extended Galilei symmetries given in equations (2.11) and (2.12). Without a cosmological constant, turning off gravity means setting $\Phi(x) = 0$. Given the transformation rule (2.12) of the background potential $\Phi(x)$, this implies the following restriction on the transverse translations:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\dot{\xi}^i}{\dot{t}}\right) = 0, \tag{4.2}$$

where we have ignored the standard ambiguity in the potential represented by the function $g(t)$ in equation (2.12). This restriction implies that $\dot{\xi}^i = v^i \dot{t}$ or $\xi^i(t) = v^i t + \zeta^i$. This brings us back to the Galilei transformations given in equation (2.7).

We now turn to the case of a nonzero cosmological constant Λ. It turns out that when taking the non-relativistic limit as is described in section 2 of a particle moving in an (A)dS background[25], one ends up with a particle moving in an effective background potential $\phi_\Lambda = -\frac{1}{2}\Lambda x^i x^i$ describing a harmonic oscillator [15]:

$$S = \frac{m}{2}\int \left(\frac{\dot{x}^i \dot{x}^j \delta_{ij}}{\dot{t}} + \dot{t}\,\Lambda x^i x^j \delta_{ij}\right) \mathrm{d}\tau. \tag{4.3}$$

We take the convention in which $\Lambda > 0$ describes a dS space, whereas $\Lambda < 0$ gives an AdS space. In the following, we will consider the AdS case only. The action (4.3) is nothing else than the action (2.10), with $\Phi(x)$ being the harmonic oscillator potential:

$$\Phi(x) = \phi_\Lambda(x) = -\tfrac{1}{2}\Lambda x^i x^i. \tag{4.4}$$

Viewed as a gauge condition, and using the transformation rule (2.12), this equation is invariant under transverse translations that satisfy the following constraint:

$$\frac{1}{\dot{t}}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\dot{\xi}^i}{\dot{t}}\right) = \Lambda \xi^i. \tag{4.5}$$

Here, we have again ignored the ambiguity in the potential represented by the function $g(t)$ in equation (2.12). For $\Lambda < 0$, i.e. AdS space, the restriction (4.5) on ξ^i is solved by[26]

$$\xi^i(t) = v^i R \sin\left(\frac{t}{R}\right) + \zeta^i \cos\left(\frac{t}{R}\right), \tag{4.6}$$

where

$$R^2 \equiv -\frac{1}{\Lambda}. \tag{4.7}$$

Note that for $\Lambda \to 0$ or $R \to \infty$ this expression reduces to the Galilei result $\xi^i(t) = v^i t + \zeta^i$.

The complete transformation rules are now obtained by combining the transformations (4.6) with the constant time translations and the spatial rotations:

$$\delta t = \zeta^0, \qquad \delta x^i = \lambda^i{}_j x^j + v^i R \sin\left(\frac{t}{R}\right) + \zeta^i \cos\left(\frac{t}{R}\right). \tag{4.8}$$

This defines the Newton–Hooke algebra whose nonzero commutators are given by [15] (see also [16])

$$\begin{aligned} [P_{a'}, H] &= R^{-2} G_{a'}, & [G_{a'}, H] &= -P_{a'}, \\ [M_{a'b'}, P_{c'}] &= -2\eta_{c'[a'} P_{b']}, & [M_{a'b'}, G_{c'}] &= -2\eta_{c'[a'} G_{b']}, \\ [M_{a'b'}, M_{c'd'}] &= 4\eta_{[a'[c'} M_{d']b']}. \end{aligned} \tag{4.9}$$

Here, $H, P_{a'}, G_{a'}$ and $M_{a'b'}$ are the generators of time translations, spatial translations, boosts and spatial rotations, with parameters $\zeta^0, \zeta^{a'}, v^{a'}$ and $\lambda^{a'b'}$, respectively. We note that the cosmological constant shows up in the $[P_{a'}, H]$ commutator, but not in the $[P_{a'}, P_{b'}]$ commutator. This is consistent with the fact that the transverse space is flat. We also observe that at this stage the Newton–Hooke algebra (4.9) does not contain a central extension like the Bargmann algebra, i.e. $[P_{a'}, G_{b'}] = 0$. Similar to the Galilei particle action (2.6), the Newton–Hooke

[25] For this, the cosmological constant Λ must be rescaled with a factor of ω^{-2}.

[26] For $\Lambda > 0$, i.e. dS space, one obtains a similar expression but with the sine and cosine replaced by their hyperbolic counterparts.

particle action (4.3) suggests a central extension: the corresponding Lagrangian is quasi-invariant under both boosts *and* translations, described by the parameter (4.6):

$$\delta L = \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{m\delta_{ij}x^i\dot{\xi}^j}{\dot{t}}\right) = \frac{\mathrm{d}}{\mathrm{d}\tau}\left(mx^i v^j\delta_{ij}\cos\left(\frac{t}{R}\right) - mx^i\zeta^j\delta_{ij}\sin\left(\frac{t}{R}\right)\right). \quad (4.10)$$

This is most easily seen by using the restriction (4.5) directly in the variation of the Lagrangian corresponding to the action (4.3). In the limit $R \to \infty$, i.e. $\Lambda \to 0$, the variation (4.10) reduces to the variation (2.9). Calculating the Noether charges Q_P and Q_G for the translations and the boosts, respectively, the Poisson brackets suggest the same central extension M as for the Galilei particle:

$$[P_{a'}, G_{b'}] = \delta_{a'b'}M. \quad (4.11)$$

Given the transformation rules (4.8), it is straightforward to calculate the commutators between the different transformations and to verify that they are indeed given by the Newton–Hooke algebra (4.9). As explained above, when viewed as the symmetries of the Newton–Hooke particle described by the action (4.3), one obtains a centrally extended Newton–Hooke algebra. The contraction $R \to \infty$ on this algebra reproduces the Bargmann algebra. This the non-relativistic analogue of the fact that the $R \to \infty$ contraction on the (A)dS algebra yields the Poincaré algebra.

To obtain the cosmological constant in the gauging procedure of the Bargmann algebra, we relate the expression for the potential (2.37) in terms of the gauge field components to the potential (4.1):

$$\Phi(x) = m_0(x) - \tfrac{1}{2}\delta_{ij}\tau^i(x)\tau^j(x) + \partial_0 m(x) = \phi(x) - \tfrac{1}{2}\Lambda x^i x^j\delta_{ij}. \quad (4.12)$$

The Poisson equation (2.15) can then be written as

$$\triangle\phi(x) = V_{D-2}G\rho(x) + (D-1)\Lambda, \quad (4.13)$$

where D is the dimension of spacetime.

4.2. The string case

We now wish to discuss the string case following the same philosophy as we used for the particle case in the previous subsection.

Like in the particle case, we write the potential $\Phi_{\alpha\beta}(x)$ as the sum of a purely gravitational potential and a background potential that represents the extra gravitational force represented by the nonzero cosmological constant Λ:

$$\Phi_{\alpha\beta}(x) = \phi_{\alpha\beta}(x) + \phi_{\alpha\beta,\Lambda}(x). \quad (4.14)$$

We first consider the case of a zero cosmological constant and show how the stringy Galilei symmetries are recovered after turning off gravity. According to equation (3.16) the condition $\Phi_{\alpha\beta}(x) = 0$ leads to the following restriction on the transverse translations:

$$\partial_{\bar{\alpha}}(\sqrt{-\bar{\gamma}}\,\bar{\gamma}^{\bar{\alpha}\bar{\beta}}\,\partial_{\bar{\beta}}\xi^i) = 0, \quad (4.15)$$

where we have ignored the standard ambiguity in $\Phi_{\alpha\beta}(x)$ represented by the arbitrary functions $g_\beta(x^\epsilon)$ in equation (3.16). This restriction is the stringy analogue of the restriction (4.2) that we found in the particle case. It is precisely the same restriction that one finds if one requires that the non-relativistic string action (3.10) is invariant under transverse translations. The solution of equation (4.15) is given by $\xi^i(x^\alpha) = \lambda^i{}_\beta x^\beta + \zeta^i$, which can be checked using expression (3.6) of $\bar\gamma^{\bar\alpha\bar\beta}$. This brings us back to the stringy Galilei symmetries given in equation (3.11).

We now consider a nonzero cosmological constant Λ. It turns out that when one considers the non-relativistic limit of a string moving in an AdS background, one ends up with an effective background potential given by [11]

$$\phi_{\alpha\beta,\Lambda} = \tfrac{1}{4}\Lambda x^i x^j \delta_{ij}\tau_{\alpha\beta}, \tag{4.16}$$

where $\tau_{\alpha\beta}$ is an AdS_2 metric. At the same time one should replace the flat foliation of spacetime by an AdS_2 foliation. This means that both in the definition of $\bar\gamma_{\bar\alpha\bar\beta}$ given in equation (3.5) and the action (3.14), one should replace the flat metric $\eta_{\alpha\beta}$ by the AdS_2 metric $\tau_{\alpha\beta}$. Setting also $\Phi_{\alpha\beta}(x) = \frac{1}{4}\Lambda x^i x^j \delta_{ij}\tau_{\alpha\beta}$ in equation (3.14), one obtains the action [11]

$$S = -\frac{T}{2}\int d^2\sigma\sqrt{-\bar\gamma}\,(\bar\gamma^{\bar\alpha\bar\beta}\partial_{\bar\alpha}x^i\partial_{\bar\beta}x^j\delta_{ij} + \Lambda x^i x^j\delta_{ij}), \tag{4.17}$$

with $\bar\gamma_{\bar\alpha\bar\beta}$ given by

$$\bar\gamma_{\bar\alpha\bar\beta} = \partial_{\bar\alpha}x^\alpha\partial_{\bar\beta}x^\beta\tau_{\alpha\beta}. \tag{4.18}$$

The replacement of $\eta_{\alpha\beta}$ by $\tau_{\alpha\beta}$ also applies to the transformation rule (3.16). This leads to the following modified restriction on the transverse translations:

$$\frac{1}{\sqrt{-\bar\gamma}}\,\partial_{\bar\alpha}(\sqrt{-\bar\gamma}\,\bar\gamma^{\bar\alpha\bar\beta}\,\partial_{\bar\beta}\xi^i) = -\Lambda\xi^i. \tag{4.19}$$

Note that we have again ignored the arbitrary functions $g_\beta(x^\epsilon)$ in equation (3.16). For $\Lambda < 0$, i.e. AdS space, the restriction (4.19) is solved for by the following expression for $\xi^i(x^\alpha)$:

$$\xi^i(x^\alpha) = \lambda^i{}_0\sqrt{z^2+R^2}\sin\left(\frac{t}{R}\right) + \lambda^i{}_1 z + \zeta^i\frac{\sqrt{z^2+R^2}}{R}\cos\left(\frac{t}{R}\right), \tag{4.20}$$

where we have written $x^\alpha = \{t, z\}$ and used that $\Lambda = -R^{-2}$. Note that for $R \to \infty$ this expression reduces to the stringy Galilei one given by $\xi^i(x^\alpha) = \lambda^i{}_\beta x^\beta + \zeta^i$.

The complete transformation rules are obtained by combining the transformation rules (4.20) with the spatial transverse rotations and the isometries of the AdS_2 space that act on $x^\alpha = \{t, z\}$. The form of the latter transformations in an explicit coordinate frame is given in appendix D, see equation (D.6), where a few useful properties of the AdS_2 foliation space have been collected. All these transformations together define the stringy Newton–Hooke algebra:

$$\begin{aligned}
&[H_a, H_b] = R^{-2}M_{ab}, && [M_{bc}, H_a] = -2\eta_{a[b}H_{c]},\\
&[M_{cd}, M_{ef}] = 4\eta_{[c[e}M_{f]d]}, &&\\
&[P_{a'}, H_a] = R^{-2}M_{a'a}, && [M_{c'd'}, M_{e'f'}] = 4\eta_{[c'[e'}M_{f']d']},\\
&[M_{b'c}, H_a] = \eta_{ac}P_{b'}, && [M_{b'c'}, P_{a'}] = -2\eta_{a'[b'}P_{c']},\\
&[M_{c'd}, M_{ef}] = 2\eta_{d[e}M_{|c'|f]}, && [M_{c'd'}, M_{e'f}] = -2\eta_{e'[c'}M_{d']f}.
\end{aligned} \tag{4.21}$$

Note that the generators $\{H_a, M_{ab}\}$ span an $\mathfrak{so}(2,1)$ algebra describing the isometries of the AdS_2 foliation. Using the transformation rules given above and in appendix D, one may calculate the different commutators and verify that the algebra defined by (4.21) is satisfied. Note how the cosmological constant ends up in the $[H_a, H_b]$ and $[P_{a'}, H_a]$ commutators, but not in the $[P_{a'}, P_{b'}]$ commutator. This is consistent with the fact that the transverse space is flat but that the two-dimensional longitudinal space is not flat. Like in the case of the point particle, the stringy Newton–Hooke algebra (4.21) allows for an extension [11]. This is motivated by the fact that the Lagrangian L corresponding to the string action (4.17) with the potential (4.16) transforms as a total derivative under the boosts and translations described by the parameters (4.20):

$$\delta L = \partial_{\bar\alpha}(-T\sqrt{-\bar\gamma}\,\bar\gamma^{\bar\alpha\bar\beta}x^i\partial_{\bar\beta}\xi_i). \tag{4.22}$$

This is most easily seen by using the restriction (4.19) directly in the variation of the Lagrangian corresponding to (4.17). For $R \to \infty$, the variation (4.22) reduces to the variation (3.13), and in the particle case, it reduces to the variation (4.10). The resulting extension suggested by the Poisson brackets is given by equation (B.3).

We now fit the cosmological constant into the gauging procedure for the string. One important difference with the point particle case is that the foliation space for the string becomes AdS_2, whereas for the particle this foliation space is trivially flat. To accomplish this AdS_2 foliation, we change the on-shell curvature constraint (3.48) for the foliation space, whereas for the transverse space we keep it unaltered:

$$R_{\mu\nu}{}^{ab}(M) = \Lambda\tau_{[\mu}{}^{a}\tau_{\nu]}{}^{b}, \quad R_{\mu\nu}{}^{a'b'}(M'') = 0. \tag{4.23}$$

This gives an AdS_2 space in the longitudinal direction and a flat transverse space. We then choose coordinates such that

$$\begin{aligned} \tau_\mu{}^a &= \left(\tau_\alpha{}^a, 0\right), \qquad e_\mu{}^{a'} = \left(-\tau^{a'}{}_a\tau_\alpha{}^a, \delta_i^{a'}\right), \\ \tau^\mu{}_a &= \left(\tau^\alpha{}_a, \tau^i{}_a\right), \quad e^\mu{}_{a'} = \left(0, \delta^i_{a'}\right), \end{aligned} \tag{4.24}$$

where now we are not able to choose $\tau_\alpha{}^a = \delta^a_\alpha$, as we did in (3.50). Using the coordinates chosen in appendix D, one can choose

$$\tau_\alpha{}^a = \left(\left(1+\frac{z^2}{R^2}\right)^{1/2}\delta^a_0, \left(1+\frac{z^2}{R^2}\right)^{-1/2}\delta^a_1\right), \tag{4.25}$$

$$\tau^\alpha{}_a = \left(\left(1+\frac{z^2}{R^2}\right)^{-1/2}\delta^0_a, \left(1+\frac{z^2}{R^2}\right)^{1/2}\delta^1_a\right). \tag{4.26}$$

In view of this, we should carefully distinguish between the curved longitudinal coordinates $\{\alpha\}$ and the flat longitudinal coordinates $\{a\}$. In contrast, from now on, we will not distinguish between flat and curved transverse coordinates $\{a'\}$ and $\{i\}$ because the transverse space is flat. With the coordinates (4.24), the constraints (4.23) allow for the gauge choice

$$\omega_\mu{}^{a'b'} = 0, \quad \omega_i{}^{ab} = 0. \tag{4.27}$$

The condition $\omega_i{}^{a'b'} = 0$ is trivially satisfied, but an explicit calculation reveals that

$$\omega_\alpha{}^{ij} = -\tau_\alpha{}^a\left(\partial^{[i}\tau^{j]}{}_a + \partial_{[i}m_{j]}{}^a\right) = -\tfrac{1}{2}\Gamma^i_{\alpha j} = 0, \tag{4.28}$$

so the gauge condition $\omega_\alpha{}^{ij} = 0$ sets the connection component $\Gamma^i_{\alpha j}$ to zero, as in the Galilei string case. From (4.28), we again arrive at (3.60). One should now be careful in distinguishing between $\tau^i{}_a$, which is nonzero in general, and $\tau_i{}^a$, which is zero for the coordinate choice (4.24). With the spin connections (4.27) and (4.28), one can show that the expression for the connection, i.e. equation (3.36), implies that again $\Gamma^i_{\alpha\beta} = \partial^i\Phi_{\alpha\beta}$, i.e. the Γ-connection can also for the AdS_2 foliation be written as the transverse gradient of a potential. The potential $\Phi_{\alpha\beta}$ is now given by

$$\Phi_{\alpha\beta} = m_a\omega_{(\alpha}{}^{ab}\tau_{\beta)}{}^b + \tau_{(\alpha}{}^a\partial_{\beta)}m_a + \tau_{(\alpha}{}^a m_{\beta)}{}^a - \tfrac{1}{2}\tau_{(\alpha}{}^a\tau_{\beta)}{}^b\tau^j{}_a\tau^j{}_b, \tag{4.29}$$

which should be compared to the potential for the flat foliation, see equation (3.63). To describe the splitting described in the beginning of this section with the background given by (4.16), we put the potential (4.29) equal to (4.14). That the set of gauge fields appearing on the right-hand side of (4.29) can give rise to an arbitrary symmetric, $\Phi_{\alpha\beta}$ can be seen by taking, e.g., the realization $m_a = \tau^i{}_a = 0$ (and, thus, through (3.60), $m_i{}^a = 0$) in the potential (4.29) and expressing the remaining longitudinal components $m_\alpha{}^a$ in terms of $\Phi_{\alpha\beta}$. The symmetric longitudinal projection of $m_\mu{}^a$ is then given by

$$\tau^{\alpha(a}m_\alpha{}^{b)} = \tau^{\alpha a}\tau^{\beta b}\Phi_{\alpha\beta}, \tag{4.30}$$

whereas the antisymmetric longitudinal projection of ${m_\mu}^a$, given by $\tau^{\alpha[a}{m_\alpha}^{b]}$, can be gauged away through a Z_{ab} transformation as is clear from equation (3.43). As such ${m_\mu}^a$ can be expressed in terms of $\Phi_{\alpha\beta}$. With $\{\Gamma^i_{\alpha\beta}, \Gamma^\epsilon_{\alpha\beta}\}$ being the only nonzero connection coefficients, the longitudinal components of the Ricci tensor become

$$
\begin{aligned}
R_{\alpha\beta}(\Gamma) &= \Delta\Phi_{\alpha\beta} + R_{\alpha\beta}(\mathrm{AdS}_2) \\
&= \Delta\phi_{\alpha\beta} + (D-1)\Lambda\tau_{\alpha\beta},
\end{aligned}
\tag{4.31}
$$

where we have used that $R_{\alpha\beta}(\mathrm{AdS}_2) = \Lambda\tau_{\alpha\beta}$. Therefore, the nonzero components of the Poisson equation (4.32) read as follows [13]:

$$
\Delta\phi_{\alpha\beta} = (V_{D-2}G\rho - (D-1)\Lambda)\tau_{\alpha\beta}, \tag{4.32}
$$

where D is the dimension of spacetime. Notice how the Laplacian on the left only contains information about the transverse space, whereas the geometry of the AdS_2 foliation is only on the right-hand side of (4.32). This concludes our discussion of the addition of the cosmological constant to the theory.

5. Conclusions and outlook

We have shown how the theory of Newton–Cartan can be extended from particles moving in a flat background to strings moving in a cosmological background. One way to obtain the desired equations corresponding to these extensions is to gauge the transverse translations. This necessitates the introduction of a new field, which is identified as the gravitational potential. The resulting equations of motion are the ones used by a Galilean observer. Alternatively, one can first gauge the full extended (stringy) Galilei algebra and, next, gauge fix some of the symmetries in order to obtain the symmetries that are appropriate to a Galilean observer. The (central) extensions of the algebras involved play a crucial role in this procedure. To obtain the (stringy) Newton–Cartan theory, conventional constraints are imposed to convert the spacetime translations into general coordinate transformations and to make the spin connections dependent fields. Furthermore, on-shell constraints are imposed on the curvature of the transverse space and, in the string case, on the curvature of the foliation space. The transverse space is chosen to be flat, whereas for the string the on-shell constraint on the longitudinal boost curvature can be chosen such that one obtains either a flat foliation (corresponding to the stringy Galilei group) or an AdS_2 foliation (corresponding to the stringy Newton–Hooke group). The first choice describes the non-relativistic limit of a string moving in a Minkowski background, whereas the second choice describes the non-relativistic limit of a string moving in an AdS_D background. The analysis can easily be extended to arbitrary branes, in which case one should use extended brane Galilei algebras [18].

It is interesting to compare our results with the literature on the application of Newton–Cartan theory in the non-relativistic limit of the AdS/CFT correspondence. This has been discussed in, e.g., [24, 25] where some subtleties of this application are discussed. In [13], it was noted that the non-relativistic limit on the CFT side of the correspondence should give the so-called Galilei conformal algebra [26, 27]. This Galilean conformal algebra[27] is the boundary realization of the stringy Newton–Hooke algebra in the bulk [29]. The dual gravity theory should then be a Newton–Cartan theory with an AdS_2 foliation describing strings, instead of the usual $\mathbb{R}$ foliation which describes particle Newton–Cartan theory. The gauging procedure outlined in this work provides the framework of developing such a theory from a gauge perspective.

[27] The spacetime Bargmann structure has been analyzed in [28].

It is known that the Newton–Cartan theory can be obtained from a dimensional reduction of general relativity along a null-Killing vector; see, e.g., [7, 30].[28] The central charge gauge field m_μ is related to the Kaluza–Klein vector corresponding to this null direction. It would be interesting to investigate if the stringy version of the Newton–Cartan theory presented in this paper can also be obtained by a null reduction from higher dimensions such that the deformation potentials ${m_\mu}^a$ and ${m_\mu}^{ab}$ obtain a similar Kaluza–Klein interpretation. This possibility should be related to the fact that the extended Newton–Hooke p-brane algebra in D dimensions is a subalgebra of the 'multitemporal' conformal algebra $SO(D+1, p+2)$ in one dimension higher [18].

One way to obtain null directions is to start from a relativistic string coupled to a constant B-field with vanishing field strength and to T-dualize this string along its spatial world-sheet direction and perform the non-relativistic limit. The T-dual picture is a pp-wave which has a null direction [17]. One could now use this null direction for a Kaluza–Klein reduction along the lines of [30] and see whether one obtains the stringy Newton–Cartan theory constructed in this paper.

Finally, an interesting extension of the stringy Newton–Cartan theory would be to apply the gauging procedure as presented here to the supersymmetric extension of the stringy Galilei algebra [17]. We hope to return to these issues in the nearby future.

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Appendix A. Notation and conventions

Our notation and conventions are as follows. For the metric, the mostly plus convention is taken. A positive cosmological constant $\Lambda > 0$ describes a de Sitter space, whereas $\Lambda < 0$ describes an AdS space.

Flat target-space indices are given by $A = \{a, a'\}$, where $\{a\}$ is longitudinal and $\{a'\}$ is transverse, e.g.

$$\zeta^A = \{\zeta^a, \zeta^{a'}\}. \tag{A.1}$$

For a particle, we write $\{a = \underline{0}\}$ and $\{a' = 1, \ldots, D-1\}$, whereas for a string we write $\{a = \underline{0}, \underline{1}\}$ and $\{a' = 2 \ldots D-1\}$. Curved target-space indices are given by $\mu = \{\alpha, i\}$, where $\{\alpha\}$ is longitudinal and $\{i\}$ is transverse, e.g.

$$\xi^\mu = \{\xi^\alpha, \xi^i\}. \tag{A.2}$$

For a particle, we write $\{\alpha = 0\}$ and $\{i = 1, \ldots, D-1\}$, and for a string, we write $\{\alpha = 0, 1\}$ and $\{i = 2, \ldots, D-1\}$. Finally, we indicate world-sheet indices with $\{\bar{\alpha}, \bar{\beta}, \ldots\}$, and the world-sheet coordinates as $\{\sigma^{\bar{\alpha}}\}$.

[28] In [30], also a proposal for an action describing the NC bulk dynamics has been made. For AdS/CFT applications, this is a desirable feature.

Appendix B. The extended stringy Galilei algebra

We associate the following generators with the symmetries of the extended stringy Galilei algebra [12] :

$$\begin{aligned} H_a &: \text{ longitudinal translations,}\\ P_{a'} &: \text{ transverse translations,}\\ M_{ab} &: \text{ longitudinal Lorentz transformations,}\\ M_{a'a} &: \text{ `boost' transformation,}\\ M_{a'b'} &: \text{ transverse rotations,}\\ Z_a, Z_{ab} &: \text{ extended transformations,} \end{aligned} \tag{B.1}$$

with $Z_{ab} = -Z_{ba}$.

The nonzero commutators of the un-deformed stringy Galilei algebra read

$$\begin{aligned} &[M_{b'c}, H_a] = \eta_{ac} P_{b'}, && [M_{b'c'}, P_{a'}] = -2\eta_{a'[b'} P_{c']},\\ &[M_{c'd}, M_{ef}] = 2\eta_{d[e} M_{|c'|f]}, && [M_{c'd'}, M_{e'f}] = -2\eta_{e'[c'} M_{d']f},\\ &[M_{c'd'}, M_{e'f'}] = 4\eta_{[c'[e'} M_{f']d']} && [M_{bc}, H_a] = -2\eta_{a[b} H_{c]}, \end{aligned} \tag{B.2}$$

where $a = \underline{0}, \underline{1}$ are the two longitudinal foliating directions and $a' = 2, \ldots, D-1$ are the $D-2$ transverse directions. Note that the Lorentz algebra $\mathfrak{so}(1, 1)$ of the two-dimensional foliation space is Abelian, while for general p-branes, where the symmetries of the foliation space are generated by the algebra $\mathfrak{so}(1, p)$, this would not be the case. The extensions suggested by the Poisson brackets corresponding to the non-relativistic string action (3.10) are given by [18]

$$\begin{aligned} &[P_{a'}, M_{b'b}] = \eta_{a'b'} Z_b, && [M_{a'a}, M_{b'b}] = -\eta_{a'b'} Z_{ab},\\ &[H_a, Z_{bc}] = 2\eta_{a[b} Z_{c]}, && [Z_{ab}, M_{cd}] = 4\eta_{[a[c} Z_{d]b]},\\ &[Z_a, M_{bc}] = 2\eta_{a[b} Z_{c]}. \end{aligned} \tag{B.3}$$

The gauge transformations of the gauge fields (3.20) corresponding to the generators (B.1) of the deformed stringy Galilei algebra are given by

$$\begin{aligned} &\delta\tau_\mu{}^a = \partial_\mu \tau^a - \tau^b \omega_\mu{}^{ab} + \lambda^{ab} \tau_\mu{}^b,\\ &\delta e_\mu{}^{a'} = \partial_\mu \zeta^{a'} - \zeta^{b'} \omega_\mu{}^{a'b'} + \lambda^{a'b'} e_\mu{}^{b'} + \lambda^{a'a} \tau_\mu{}^a - \tau^a \omega_\mu{}^{a'a},\\ &\delta\omega_\mu{}^{ab} = \partial_\mu \lambda^{ab},\\ &\delta\omega_\mu{}^{a'a} = \partial_\mu \lambda^{a'a} - \lambda^{a'b} \omega_\mu{}^{ab} + \lambda^{ab} \omega_\mu{}^{a'b} + \lambda^{a'b'} \omega_\mu{}^{b'a} - \lambda^{b'a} \omega_\mu{}^{a'b'},\\ &\delta\omega_\mu{}^{a'b'} = \partial_\mu \lambda^{a'b'} + 2\lambda^{c'[a'} \omega_\mu{}^{b']c'},\\ &\delta m_\mu{}^a = \partial_\mu \sigma^a + \lambda^{a'a} e_\mu{}^{a'} - \zeta^{a'} \omega_\mu{}^{a'a} + \lambda^{ab} m_\mu{}^b - \sigma^b \omega_\mu{}^{ab} + \tau^b m_\mu{}^{ab} - \sigma^{ab} \tau_\mu{}^b,\\ &\delta m_\mu{}^{ab} = \partial_\mu \sigma^{ab} - \lambda^{a'a} \omega_\mu{}^{a'b} + \lambda^{a'b} \omega_\mu{}^{a'a} + \sigma^{c[a} \omega_\mu{}^{b]c} + \lambda^{c[a} m_\mu{}^{b]c}, \end{aligned} \tag{B.4}$$

where we have used the gauge parameters (2.18). The corresponding gauge-invariant curvatures are given by[29]

$$\begin{aligned} &R_{\mu\nu}{}^a(H) = 2D_{[\mu} \tau_{\nu]}{}^a,\\ &R_{\mu\nu}{}^{a'}(P) = 2\big(D_{[\mu} e_{\nu]}{}^{a'} - \omega_{[\mu}{}^{a'a} \tau_{\nu]}{}^a\big),\\ &R_{\mu\nu}{}^{ab}(M) = 2\,\partial_{[\mu} \omega_{\nu]}{}^{ab},\\ &R_{\mu\nu}{}^{a'a}(M') = 2\,D_{[\mu} \omega_{\nu]}{}^{a'a},\\ &R_{\mu\nu}{}^{a'b'}(M'') = 2\big(\partial_{[\mu} \omega_{\nu]}{}^{a'b'} - \omega_{[\mu}{}^{c'a'} \omega_{\nu]}{}^{b'c'}\big),\\ &R_{\mu\nu}{}^a(Z) = 2\big(D_{[\mu} m_{\nu]}{}^a + e_{[\mu}{}^{a'} \omega_{\nu]}{}^{a'a} - \tau_{[\mu}{}^b m_{\nu]}{}^{ab}\big),\\ &R_{\mu\nu}{}^{ab}(Z) = 2\big(D_{[\mu} m_{\nu]}{}^{ab} + \omega_{[\mu}{}^{a'a} \omega_{\nu]}{}^{a'b}\big), \end{aligned} \tag{B.5}$$

[29] For general p-branes, we would have $\delta\omega_\mu{}^{ab} = \partial_\mu \lambda^{ab} + 2\lambda^{c[a} \omega_\mu{}^{b]c}$ and $R_{\mu\nu}{}^{ab}(M) = 2(\partial_{[\mu} \omega_{\nu]}{}^{ab} - \omega_{[\mu}{}^{ca} \omega_{\nu]}{}^{bc})$.

where M, M' and M'' indicate the generators corresponding to longitudinal Lorentz transformations, 'boost' transformations and transverse rotations, respectively. The derivative D_μ is covariant with respect to these M, M' and M'' transformations.

Finally, the curvatures (B.5) satisfy the Bianchi identities

$$
\begin{aligned}
&D_{[\rho}R_{\mu\nu]}{}^{a}(H) = -R_{[\rho\mu}{}^{ab}(M)\tau_{\nu]}{}^{b},\\
&D_{[\rho}R_{\mu\nu]}{}^{a'}(P) = -R_{[\rho\mu}{}^{a'b'}(M'')e_{\nu]}{}^{b'} - R_{[\rho\mu}{}^{a'a}(M')\tau_{\nu]}{}^{a},\\
&D_{[\rho}R_{\mu\nu]}{}^{ab}(M) = 0,\\
&D_{[\rho}R_{\mu\nu]}{}^{a'a}(M') = -R_{[\rho\mu}{}^{ab}(M)\omega_{\nu]}{}^{a'b} - R_{[\rho\mu}{}^{a'b'}(M'')\omega_{\nu]}{}^{b'a},\\
&D_{[\rho}R_{\mu\nu]}{}^{a'b'}(M'') = 0,\\
&D_{[\rho}R_{\mu\nu]}{}^{a}(Z) = -R_{[\rho\mu}{}^{ab}(M)m_{\nu]}{}^{b} + R_{[\rho\mu}{}^{a'}(P)\omega_{\nu]}{}^{a'a} - R_{[\rho\mu}{}^{a'a}(M')e_{\nu]}{}^{a'},\\
&\qquad\qquad\qquad -R_{[\rho\mu}{}^{a}(H)m_{\nu]}{}^{ab} + R_{[\rho\mu}{}^{ab}(Z)\tau_{\nu]}{}^{b},\\
&D_{[\rho}R_{\mu\nu]}{}^{ab}(Z) = R_{[\rho\mu}{}^{c[a}(M)m_{\nu]}{}^{b]c} + R_{[\rho\mu}{}^{a'a}(M')\omega_{\nu]}{}^{a'b} - R_{[\rho\mu}{}^{a'b}(M')\omega_{\nu]}{}^{a'a}.
\end{aligned}
\tag{B.6}
$$

Appendix C. Newton–Cartan geodesic equations

Here, we give some details about the derivation of the geodesic equations (2.31) and (3.46). We start with the point particle case. For that purpose, we write the Lagrangian (2.28) as

$$
\begin{aligned}
L &= \frac{m}{2}N^{-1}\dot{x}^\mu\dot{x}^\nu(h_{\mu\nu} - 2m_\mu\tau_\nu)\\
&\equiv \frac{m}{2}N^{-1}\dot{x}^\mu\dot{x}^\nu H_{\mu\nu},
\end{aligned}
\tag{C.1}
$$

where we defined

$$H_{\mu\nu} \equiv h_{\mu\nu} - 2m_{(\mu}\tau_{\nu)}, \quad N \equiv \tau_\mu\dot{x}^\mu. \tag{C.2}$$

Varying the Lagrangian (C.1) with respect to $\{x^\lambda\}$ and using the metric compatibility condition $\partial_{[\mu}\tau_{\nu]} = 0$ gives

$$
\begin{aligned}
-Nm^{-1}\frac{\delta L}{\delta x^\lambda} &= \left(N^{-2}\dot{N}\tau_\lambda H_{\mu\nu} - \frac{1}{2}N^{-1}\tau_\lambda\partial_\rho H_{\mu\nu}\dot{x}^\rho - \frac{1}{2}\partial_\lambda H_{\mu\nu} + \partial_\nu H_{\mu\lambda}\right)\dot{x}^\mu\dot{x}^\nu\\
&\quad - N^{-1}\tau_\lambda H_{\mu\nu}\dot{x}^\mu\ddot{x}^\nu - N^{-1}\dot{N}H_{\mu\lambda}\dot{x}^\mu + H_{\mu\lambda}\ddot{x}^\mu = 0.
\end{aligned}
\tag{C.3}
$$

First, we contract this equation with $h^{\lambda\sigma}$. This gives

$$h^{\lambda\sigma}\left(\partial_\nu H_{\mu\lambda} - \tfrac{1}{2}\partial_\lambda H_{\mu\nu}\right)\dot{x}^\mu\dot{x}^\nu + h^{\lambda\sigma}H_{\mu\lambda}\ddot{x}^\mu - N^{-1}\dot{N}h^{\lambda\sigma}H_{\mu\lambda}\dot{x}^\mu = 0. \tag{C.4}$$

One can now use the Newton–Cartan metric relations (2.22), $\partial_{[\mu}\tau_{\nu]} = 0$ and

$$\dot{N} = \tau_\mu\ddot{x}^\mu + \partial_\mu\tau_\nu\dot{x}^\mu\dot{x}^\nu. \tag{C.5}$$

Some manipulation then shows that (C.4) gives the geodesic equation (2.31),

$$\ddot{x}^\mu + \Gamma^\mu_{\nu\rho}\dot{x}^\nu\dot{x}^\rho = \frac{\dot{N}}{N}\dot{x}^\mu, \tag{C.6}$$

with the connection given by (2.26). Second, one can contract (C.3) with τ^λ. The resulting expression contains, among others, terms proportional to $\ddot{x}^\mu$. If one uses the geodesic equation (C.6) to rewrite these in terms of $\dot{x}^\mu$, one can finally show that this τ^λ-contraction of (C.3) is trivially satisfied.

The calculation concerning the string Lagrangian (3.41) leading to the stringy geodesic equation (3.46) can be made in a similar way. We first write

$$H_{\mu\nu} = h_{\mu\nu} - 2m_{(\mu}{}^{a}\tau_{\nu)}{}^{a}, \tag{C.7}$$

such that (3.41) becomes

$$L = -\frac{T}{2}\sqrt{-\det(\tau)}\tau^{\bar{\alpha}\bar{\beta}}\partial_{\bar{\alpha}}x^{\mu}\partial_{\bar{\beta}}x^{\nu}H_{\mu\nu}. \tag{C.8}$$

We next use the relations

$$\begin{aligned}
&\delta\sqrt{-\det(\tau)} = \tfrac{1}{2}\sqrt{-\det(\tau)}\tau^{\bar{\alpha}\bar{\beta}}\delta\tau_{\bar{\alpha}\bar{\beta}},\\
&\delta\tau^{\bar{\alpha}\bar{\beta}} = -\tau^{\bar{\alpha}\bar{\gamma}}\tau^{\bar{\beta}\bar{\epsilon}}\delta\tau_{\bar{\gamma}\bar{\epsilon}},\\
&\delta\tau_{\bar{\alpha}\bar{\beta}} = 2\partial_{\bar{\alpha}}x^{\mu}\partial_{\bar{\beta}}\delta x^{\lambda}\tau_{\mu\lambda} + \partial_{\bar{\alpha}}x^{\mu}\partial_{\bar{\beta}}x^{\nu}\partial_{\lambda}\tau_{\mu\nu}\delta x^{\lambda},\\
&\partial_{\bar{\alpha}}\left(\sqrt{-\det(\tau)}\tau^{\bar{\alpha}\bar{\beta}}\partial_{\bar{\beta}}x^{\mu}\right) = \sqrt{-\det(\tau)}\tau^{\bar{\alpha}\bar{\beta}}\nabla_{\bar{\alpha}}\partial_{\bar{\beta}}x^{\mu},\\
&\partial_{\rho}\tau_{\mu\nu} + \partial_{\mu}\tau_{\rho\nu} - \partial_{\nu}\tau_{\rho\mu} = \Gamma^{\lambda}_{\mu\rho}\tau_{\lambda\nu},
\end{aligned} \tag{C.9}$$

where the last identity follows from the metric compatibility condition $\nabla_{\rho}\tau_{\mu\nu} = 0$. Varying (C.8) with respect to $\{x^{\lambda}\}$ now gives the geodesic equation (3.46),

$$\tau^{\bar{\alpha}\bar{\beta}}\left(\nabla_{\bar{\alpha}}\partial_{\bar{\beta}}x^{\rho} + \partial_{\bar{\alpha}}x^{\mu}\partial_{\bar{\beta}}x^{\nu}\Gamma^{\rho}_{\mu\nu}\right) = 0, \tag{C.10}$$

with the connection $\Gamma^{\rho}_{\mu\nu}$ given by (3.37). This connection is equivalent to the connection (3.36) given by the vielbein postulates.

Appendix D. Some properties of AdS$_2$

In terms of coordinates $x^{\alpha} = \{t, z\}$, we write the AdS$_2$ metric as $\tau_{\alpha\beta}$, and the corresponding line interval as

$$\mathrm{d}s^2 = -\left(1 + \frac{z^2}{R^2}\right)\mathrm{d}t^2 + \left(1 + \frac{z^2}{R^2}\right)^{-1}\mathrm{d}z^2, \tag{D.1}$$

where R is the radius of curvature. The nonzero Christoffel components in this coordinate system are given by

$$\Gamma^{z}_{tt} = z\left(\frac{z^2 + R^2}{R^4}\right), \quad \Gamma^{z}_{zz} = \frac{-z}{z^2 + R^2}, \quad \Gamma^{t}_{zt} = \frac{z}{z^2 + R^2}. \tag{D.2}$$

The three isometries of the AdS$_2$-space parametrized by $\{\zeta^0, \zeta^1, \lambda^{01}\}$ are described by the Killing vectors[30]

$$\begin{aligned}
k_{\{01\}} &= \frac{zR\cos\frac{t}{R}}{\sqrt{z^2 + R^2}}\partial_t + \sqrt{z^2 + R^2}\sin\frac{t}{R}\partial_z,\\
k_{\{02\}} &= -R\partial_t,\\
k_{\{12\}} &= \frac{Rz\sin\frac{t}{R}}{\sqrt{z^2 + R^2}}\partial_t - \sqrt{z^2 + R^2}\cos\frac{t}{R}\partial_z.
\end{aligned} \tag{D.3}$$

One can check that these vectors indeed form an $\mathfrak{so}(2, 1)$ algebra and that the components of the vectors (D.3) obey the Killing equation

$$\mathscr{L}_k\tau_{\alpha\beta} = 2\nabla_{(\alpha}k_{\beta)} = 0. \tag{D.4}$$

[30] Note that $k_{\{02\}}$ describes the fact that the AdS$_2$ metric is static. We could rescale the time coordinate t with R to obtain $k_{\{02\}} = -\partial_t$.

Acting with the Killing vectors (D.3) on the coordinates $x^\alpha = \{t, z\}$ induces the infinitesimal isometry transformations

$$\begin{aligned} \delta_H t &= \zeta^0 - \zeta^1 \frac{z}{\sqrt{z^2+R^2}} \sin\left(\frac{t}{R}\right), \\ \delta_H z &= \zeta^1 \frac{\sqrt{z^2+R^2}}{R} \cos\left(\frac{t}{R}\right), \\ \delta_M t &= \lambda^{01} \frac{zR\cos\left(\frac{t}{R}\right)}{\sqrt{z^2+R^2}}, \\ \delta_M z &= -\lambda^{01}\sqrt{z^2+R^2}\sin\left(\frac{t}{R}\right). \end{aligned} \tag{D.5}$$

Note that in the limit $R \to \infty$ these rules reduce to the stringy Galilei ones given by $\xi^\alpha(x^\alpha) = \lambda^\alpha{}_\beta x^\beta + \zeta^\alpha$, which are the isometries of a flat $M_{1,1}$ foliation space.

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