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# The many faces of $OSp(1|32)$

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**Abstract.** We show that the complete superalgebra of symmetries, including central charges, that underlies F-theories, M-theories and type II string theories in dimensions 12, 11 and 10 of various signatures correspond to rewriting the same $OSp(1|32)$ algebra in different covariant ways. One only has to distinguish the complex and the unique real algebra. We develop a common framework to discuss all signature theories by starting from the complex form of $OSp(1|32)$. Theories are distinguished by the choice of basis for this algebra. We formulate dimensional reductions and dualities as changes of basis of the algebra. A second ingredient is the choice of a real form corresponding to a specific signature. The existence of the real form of the algebra selects preferred spacetime signatures. In particular, we show how the real $d = 10$ IIA and IIB superalgebras for various signatures are related by generalized T-duality transformations that not only involve spacelike but also timelike directions. A third essential ingredient is that the translation generator in one theory plays the role of a central charge operator in the other theory. The identification of the translation generator in these algebras leads to the star algebras of Hull, which are characterized by the fact that the positive-definite energy operator is not part of the translation generators. We apply our results to discuss different T-dual pictures of the D-instanton solution of Euclidean IIB supergravity.

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## 1. Introduction

In some recent applications of string theory an important role has been played by the D-instanton, as introduced in [1, 2]. The brane solution corresponding to such a D-instanton was considered in [3]. There, the instanton was presented as a brane solution with transverse directions only of a 10-dimensional Euclidean theory containing the metric, the dilaton and the Ramond–Ramond scalar. It was suggested that this was part of a Euclidean IIB supergravity theory in 10 dimensions. At first sight, the definition of such a Euclidean IIB supergravity theory seems problematic, since one cannot extend the self-duality constraint of the real IIB Ramond–Ramond 5-form field strength from a Minkowski spacetime to a spacetime of signature $(10, 0)$‖. This paper addresses the issue of in what sense the D-instanton can be represented as a brane solution of a Euclidean supersymmetric theory.

Euclidean supersymmetric theories have been considered in lower dimensions [4–6]. In some cases, the Euclidean theory could be obtained only by complexifying the fields, and considering holomorphic actions. This procedure also seems appropriate for the IIB Euclidean

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‖ See the imaginary factor in (A.2) for $d = 10$, $t = 0$.

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theory mentioned above, as the self-duality of the 5-form can be maintained for complex fields only†.

The D-instanton is a solution of the Euclidean action [3]

$$S_{\mathrm{E}} = \int \mathrm{d}^{10}x\, \sqrt{g}\, \big[R - \tfrac{1}{2}(\partial\phi)^2 + \tfrac{1}{2}\mathrm{e}^{2\phi}(\partial\ell)^2\big], \tag{1.1}$$

where $\phi$ is the dilaton and $\ell$ is the Ramond–Ramond scalar. In the solution, the metric is flat, $g_{\mu\nu} = \delta_{\mu\nu}$, and

$$\pm\ell = \mathrm{e}^{-\phi} = H^{-1}, \tag{1.2}$$

where $H$ is a general harmonic function over the 10-dimensional flat Euclidean space. If we want to connect the D-instanton with other branes by T-duality, we have to reduce it to a spacetime of signature $(9, 0)$, thus yielding a D-instanton in nine Euclidean dimensions. The only standard D-brane that also gives rise, upon reduction, to a D-instanton in nine Euclidean dimensions is the D0-brane of the type IIA string. This D0-brane is a solution of the Minkowskian action

$$S_{\mathrm{M}} = \int \mathrm{d}^{10}x\, \sqrt{-g}\, \big[-R - \tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{4}\mathrm{e}^{3\phi/2}F_{\mu\nu}F^{\mu\nu}\big], \tag{1.3}$$

given by

$$\begin{aligned} &\mathrm{d}s^2 = -H^{-7/8}\,\mathrm{d}t^2 + H^{1/8}\,\mathrm{d}x_{(9)}^2, \\ &\mathrm{e}^{\phi} = H^{3/4}, \qquad A_t = \pm H^{-1}, \end{aligned} \tag{1.4}$$

where $H$ is a harmonic function over the nine-dimensional Euclidean transverse space. The D-instanton and the D0-brane are mapped onto each other under the type II T-duality rules [7] adapted to the Euclidean case for the D-instanton, as follows (we use the Einstein frame) [8]:

$$\begin{array}{ccl} \text{IIB}, (10,0) & \leftrightarrow & \text{IIA}, (9,1) \\ \hline g_{\mu\nu} & \leftrightarrow & \mathrm{e}^{\phi/8}g_{\mu\nu}(-g_{\mathrm{tt}})^{1/4}, \\ g_{xx} & \leftrightarrow & \mathrm{e}^{-7\phi/8}(-g_{\mathrm{tt}})^{-3/4}, \\ \ell & \leftrightarrow & A_t, \\ \mathrm{e}^{\phi} & \leftrightarrow & \mathrm{e}^{3\phi/4}(-g_{\mathrm{tt}})^{-1/2}, \end{array} \tag{1.5}$$

where $x$ in the IIB case and $t$ in the IIA case are the directions in which the duality is performed, and $\mu, \nu$ are the remaining directions.

To obtain the above duality transformation, we had to compactify one theory over a spacelike direction and the other over a timelike direction‡. This has to be contrasted with the usual duality transformations, where both theories are compactified over a spacelike direction. To distinguish these dualities, we will refer to the usual type of duality as a space/space duality, and the above one as a space/time duality. The special feature of a space/time duality is that it connects theories with different signatures of spacetime. 10- and 11-dimensional theories with various signatures, connected by different space/time dualities have been studied in [12]. For further work on these theories, see [13]. The purpose of this paper is to reconsider the results of [12] from a superalgebra point of view§. All the

† We thank Peter van Nieuwenhuizen for a discussion on this point.

‡ The case in which both theories are compactified over a timelike direction has been studied in [9, 10]. Relations between Euclidean and Minkowski supersymmetry by reducing over a different number of space versus time directions played an important role in [11].

§ Some remarks on the underlying (complex) algebraic structure were already made in [12].

theories that we consider have 32 supersymmetries†. Including all the central charges, the superalgebra must then be (a contraction of) $OSp(1|32)$, which is the *unique* maximal simple superalgebra with 32 fermionic generators, already mentioned in [17] as the natural candidate for a geometrical interpretation of 11-dimensional supergravity. We will show in what sense all the theories occurring in [12] are indeed related by this common algebraic structure. In the $OSp(1|32)$ algebra, the bosonic generators that occur on the right-hand side of the $\{Q, Q\}$ anticommutator have non-trivial commutation relations, both between themselves and with the supersymmetry generators. In the Poincaré theories which we consider, the algebra is, in fact, the contraction of the $OSp(1|32)$ algebra such that all of these commutators vanish. One might also consider anti-de Sitter-type theories where the uncontracted $OSp(1|32)$ algebra exists. The connection to anti-de Sitter has been made in [18], by identifying the 2-index generator on the right-hand side of $\{Q, Q\}$ with the Lorentz generator $M_{\mu\nu}$, such that $[P_\mu, P_\nu] = M_{\mu\nu}$.

The maximal dimension in which the generators of $OSp(1|32)$ can be considered as representations of a Lorentz algebra is 12. In the decomposition of the generators of $OSp(1|32)$ into representations of the 12-dimensional Lorentz algebra there is no vector, i.e. there is no room for translations. Translations are essential to define a spacetime, and thus a supersymmetric action. Only after reduction to 11 dimensions do translations appear in the algebra, and we can start to consider actions. We will see that in various methods of reduction (first over spacelike or timelike directions), the translations get identified with different generators of the 11-dimensional algebra. To identify the symmetry algebras of the different theories occurring in [12], generators that play the role of translations in one theory, become central charges in others. The appearance of different twisted algebras in [12] can be understood from this point of view, and leads in our approach to identical complete superalgebras once the central charges have been included.

As mentioned above, sometimes it is necessary to work with complex fields. This reflects itself in the necessity to use the complex form of $OSp(1|32)$. Only for certain signatures can one use the real form of the algebra and work with real fields in the corresponding supergravity theory. We remind the reader that the complex form of an algebra with generators $T_A$ means that the algebra vector space consists of all elements $\epsilon^A T_A$ with $\epsilon^A$ complex numbers. The real form is a subspace of this vector space defined by a reality condition on all the $\epsilon^A$, e.g. that they are all real. In general, a complex form of a (super-)algebra can have different real forms. Consider for instance the three generators of $SU(2)$ with commutation relations

$$[T_i, T_j] = \varepsilon_{ijk} T_k. \tag{1.6}$$

The real form defined by $a^i T_i$ with all $a^i$ real is the real form of $SU(2)$, while $\mathrm{i}b^1 T_1 + \mathrm{i}b^2 T_2 + b^3 T_3$ with all $b^i$ real defines $SU(1, 1)$. Conveniently rewriting the above expression as $b^i S_i$, such that the factors of i disappear, the commutation relations are

$$[S_1, S_2] = -S_3, \qquad [S_2, S_3] = S_1, \qquad [S_3, S_1] = S_2, \tag{1.7}$$

showing a sign difference in the commutation relations. Note that this sign is not relevant in the complex form, as $b^i S_i$ with all $b_i$ complex is the same complex algebra as that mentioned in the $T$-basis.

An important property is that $OSp(1|32)$ *has only one real form.* This can be seen by looking at the list of real forms of superalgebras established in [19] and conveniently summarized in a table in [20, 21]. In order that this real form can be used for a particular

† For earlier work on superalgebras with 32 supersymmetries, see [14]. Superalgebras with more than 32 supersymmetries, containing both IIA and IIB algebras, have been considered in, for example, [15, 16].

dimension and signature, the parameters should form a real 32-dimensional representation of the corresponding Lorentz algebra. We can thus determine the relevant cases by considering for which signatures the smallest spinor has a real dimension of at most 32. Spinors for arbitrary spacetime signatures have been studied in [22] and for a convenient table, one can consider table 2 in [21]. This shows that the highest dimension is again 12, but only for signatures (10, 2), (6, 6) or (2, 10). In 11 dimensions the allowed signatures are (10, 1), (9, 2), (6, 5) or the $s \leftrightarrow t$ interchanged ones (which are equivalent using redefinitions). In 10 dimensions and lower, all signatures are possible, sometimes with irreducible spinors and otherwise as type II theories (see table 1 in section 3). The complex form of the algebra is independent of the signature of spacetime. Indeed, the complex form of the Lorentz algebra for any signature is the same, similar as we saw above in that the complex form of $SU(2)$ is the same as that of $SU(1,1)$.

We will show that manipulations such as reductions and dualities can be understood as different reparametrizations of the same $OSp(1|32)$ algebra, in its complex or real form. *Thus all different possibilities of M-type or type II theories can be viewed as different faces of the same $OSp(1|32)$ superalgebra*, hence explaining the title of this paper.

In section 2, we will consider the complex form of $OSp(1|32)$, written as a symmetry algebra in 10, 11 and 12 dimensions. It is convenient to start with the complex form, as this is independent of the signature. We will discuss the formulation of the algebras in these dimensions, their dimensional reductions and the (complex) T-dualities relating them. Section 3 considers the restriction of these complex algebras to real algebras, for which the signature becomes important. In parallel to the previous section we will first discuss the formulation of the real algebras in the different dimensions and next their dimensional reductions and the (real) T-dualities relating them. The final result is summarized in table 2. In section 4 we discuss the relation between the superalgebra and the corresponding supergravity action. In particular, we will identify the translations between the bosonic charges, and show that different choices of the translation generator (interchanging it with central charges) lead to the different $\star$-algebras of [12]. The different theories thus obtained are summarized in table 3 which is similar to that given in [12]. Finally, in the conclusions we will come back to the issue of the D-instanton and its T-dual formulations, as discussed in the beginning of this introduction.

## 2. Complex symmetry algebras

### *2.1. Algebras in 12 to 10 dimensions*

We consider algebras with 32 supersymmetries, not taking into account their reality properties. This implies that $d = 12$ is the highest dimension. In this subsection, we present the algebras in 12 to 10 dimensions, leaving the relations between them to the next two subsections. We start by writing the $OSp(1|32)$ algebra in a 12-dimensional covariant way. This means that the anticommutator of the supersymmetries is the most general expression in accordance with the symmetries, i.e. it reflects the decomposition of the symmetric product of two 32-dimensional spinor representations of $Sp(32)$ in $SO(12)$ representations†:

$$(32 \times 32)_S = 66 + 462^+. \tag{2.1}$$

The 32 spinor components are a chiral spinor in 12 dimensions, i.e. $\hat{\Gamma}_* \hat{Q} = \hat{Q}$. For the notation of spinors, gamma matrices and duals of tensors, we refer to appendix A. Note that for the

† In this section we discuss the complex algebras, and thus the notation $SO(12)$ should be understood as the corresponding algebra over the complex field.

complex case, the signature of spacetime is not relevant. However, we have to choose a metric for raising and lowering indices, etc. We will use the notation such that this metric is the identity, which implies that we can use the general formula as if we are in the Euclidean case $t = 0$. The anticommutator of two supersymmetries is†

$$\left\{\hat{Q}, \hat{Q}\right\} = \tfrac{1}{2}\hat{\mathcal{P}}^{+}\hat{\Gamma}^{\hat{M}\hat{N}}\hat{Z}_{\hat{M}\hat{N}} + \frac{1}{6!}\hat{\mathcal{P}}^{+}\hat{\Gamma}^{\hat{M}_1\cdots\hat{M}_6}\hat{Z}^{+}_{\hat{M}_1\cdots\hat{M}_6}, \tag{2.2}$$

where the gamma matrices are $64 \times 64$ matrices, but due to the chiral projection operators $\hat{\mathcal{P}}^{+}$, their relevant part is $32 \times 32$. The 6-index generator is self-dual, i.e.

$$\hat{Z}^{+}_{\hat{M}_1\cdots\hat{M}_6} = \tilde{\hat{Z}}^{+}_{\hat{M}_1\cdots\hat{M}_6}, \tag{2.3}$$

as a consequence of the chirality condition of the spinors. We do not consider the commutation relations between the bosonic generators. They are non-zero for $OSp(1|32)$, but zero for its contraction which, in $d = 10$ and 11, is the super-Poincaré algebra. These additional commutation relations do not play a role in the subsequent discussions.

In 11 dimensions, the reduction of the adjoint representation of $Sp(32)$ goes as

$$(32 \times 32)_S = 11 + 55 + 462. \tag{2.4}$$

The relevant spinors are in an irreducible spinor representation of $SO(11)$. The anticommutator is

$$\{Q, Q\} = \tilde{\Gamma}^{\tilde{M}}\tilde{Z}_{\tilde{M}} + \tfrac{1}{2}\tilde{\Gamma}^{\tilde{M}\tilde{N}}\tilde{Z}_{\tilde{M}\tilde{N}} + \frac{1}{5!}\tilde{\Gamma}^{\tilde{M}_1\cdots\tilde{M}_5}\tilde{Z}_{\tilde{M}_1\cdots\tilde{M}_5}, \tag{2.5}$$

where the gamma matrices are $32 \times 32$. Their relation with the 12-dimensional ones is given in appendix B.

In 10 dimensions the smallest irreducible spinors are the 16-dimensional chiral spinors. The 32 fermionic generators can be contained in two chiral spinors of opposite chirality (IIA) or of the same chirality (IIB). The 528 bosonic generators will be represented, respectively, as

$$\begin{aligned} &\text{IIA:} \quad 1 + 10 + 10 + 45 + 210 + 126^{+} + 126^{-}, \\ &\text{IIB:} \quad 10 + 10 + 10 + 120 + 126^{+} + 126^{+} + 126^{+}. \end{aligned} \tag{2.6}$$

For the IIA theory, the spinors satisfy $Q^{\pm} = \pm\Gamma_{*}Q^{\pm}$. The IIA anticommutators are

$$\begin{aligned} \left\{Q^{\pm}, Q^{\pm}\right\} &= \mathcal{P}^{\pm}\Gamma^{M}Z^{\pm}_{M} + \frac{1}{5!}\mathcal{P}^{\pm}\Gamma^{M_1\cdots M_5}Z^{\pm}_{M_1\cdots M_5}, \\ \left\{Q^{\pm}, Q^{\mp}\right\} &= \pm\mathcal{P}^{\pm}Z + \tfrac{1}{2}\mathcal{P}^{\pm}\Gamma^{MN}Z_{MN} \pm \frac{1}{4!}\mathcal{P}^{\pm}\Gamma^{M_1\cdots M_4}Z_{M_1\cdots M_4}. \end{aligned} \tag{2.7}$$

In this case $Z^{\pm}_{M_1\cdots M_5}$ is (anti-)self-dual. For later purposes, we note that this algebra has the $\mathbb{Z}_2$ automorphism which is denoted as $(-)^{\mathrm{F_L}}$, acting as

$$\begin{aligned} &Q^{\pm} \xrightarrow{(-)^{\mathrm{F_L}}} \pm Q^{\pm}, \\ &\left(Z, Z_{MN}, Z_{MNPQ}\right) \xrightarrow{(-)^{\mathrm{F_L}}} -\left(Z, Z_{MN}, Z_{MNPQ}\right), \end{aligned} \tag{2.8}$$

the other bosonic generators remaining invariant.

† Observe the omission of the charge conjugation matrix in our notation, see (A.8).

For the IIB theory we introduce a doublet of left-handed supersymmetries: $Q^i = \Gamma_* Q^i$ with $i = 1, 2$, and the anticommutators are

$$\left\{Q^i, Q^j\right\} = \mathcal{P}^+\Gamma^M Y_M^{ij} + \frac{1}{3!}\mathcal{P}^+\Gamma^{MNP}\varepsilon^{ij} Y_{MNP} + \mathcal{P}^+\frac{1}{5!}\Gamma^{M_1\cdots M_5} Y^{+ij}_{M_1\cdots M_5}, \quad (2.9)$$

where $Y_M^{ij}$ and $Y^{ij}_{M_1\cdots M_5}$ are the most general symmetric matrices in $(ij)$, which can be expanded in the $2 \times 2$ Pauli matrices as follows:

$$\begin{aligned} Y_M^{ij} &= \delta^{ij} Y_M^{(0)} + \tau_1^{ij} Y_M^{(1)} + \tau_3^{ij} Y_M^{(3)}, \\ Y^{+ij}_{M_1\cdots M_5} &= \left(\delta^{ij} Y^{+(0)}_{M_1\cdots M_5} + \tau_1^{ij} Y^{+(1)}_{M_1\cdots M_5} + \tau_3^{ij} Y^{+(3)}_{M_1\cdots M_5}\right). \end{aligned} \quad (2.10)$$

This IIB algebra has $\mathbb{Z}_2$ automorphisms, i.e. $Q^i \to M^i{}_j Q^j$, where $M$ is a matrix that squares to $\mathbb{1}$. In particular, the cases $M = \tau_3$ and $M = \tau_1$ are usually denoted by $(-)^{\mathrm{F_L}}$ and $\Omega$, which act as

$$\begin{aligned} &Q^i \xrightarrow{(-)^{\mathrm{F_L}}} (\tau_3)^i{}_j Q^j, \\ &\left(Y_M^{(1)}, Y_{MNP}, Y^{+(1)}_{M_1\cdots M_5}\right) \xrightarrow{(-)^{\mathrm{F_L}}} -\left(Y_M^{(1)}, Y_{MNP}, Y^{+(1)}_{M_1\cdots M_5}\right), \end{aligned} \quad (2.11)$$

$$\begin{aligned} &Q^i \xrightarrow{\Omega} (\tau_1)^i{}_j Q^j, \\ &\left(Y_M^{(3)}, Y_{MNP}, Y^{+(3)}_{M_1\cdots M_5}\right) \xrightarrow{\Omega} -\left(Y_M^{(3)}, Y_{MNP}, Y^{+(3)}_{M_1\cdots M_5}\right). \end{aligned} \quad (2.12)$$

Finally, there is also the $S$ duality map

$$\begin{aligned} &Q^i \xrightarrow{S} \left(\mathrm{e}^{\frac{1}{4}\mathrm{i}\pi\tau_2}\right)^i{}_j Q^j, \\ &\left(Y_M^{(1)}, Y^{(1)}_{M_1\cdots M_5}\right) \xrightarrow{S} \left(Y_M^{(3)}, Y^{(3)}_{M_1\cdots M_5}\right), \\ &\left(Y_M^{(3)}, Y^{(3)}_{M_1\cdots M_5}\right) \xrightarrow{S} -\left(Y_M^{(1)}, Y^{(1)}_{M_1\cdots M_5}\right), \end{aligned} \quad (2.13)$$

which has $S^8 = 1$.

### 2.2. *Dimensional reduction*

The algebras of the previous subsection can be related by dimensional reduction, except for the IIB algebra which is related to the IIA by T-duality, to be discussed in the next subsection. The explicit formulae that we will give below, depend on the representation of the Clifford algebra. We refer to appendix B for our choice of representation.

The $d = 12$ chiral spinor, $\hat{Q}$, is in this representation decomposed as

$$\hat{Q} = \begin{pmatrix} Q^+ \\ Q^- \end{pmatrix}. \quad (2.14)$$

The chirality in 12 dimensions, with $\hat{\Gamma}_* = \Gamma_* \otimes \sigma_3$, implies that the two components satisfy $Q^\pm = \pm\Gamma_* Q^\pm$. These components are the supersymmetry generators of the IIA algebra in 10 dimensions. The 11-dimensional spinor $Q$ is obtained as the sum of $Q^+$ and $Q^-$:

$$Q = Q^+ + Q^-. \quad (2.15)$$

It is easiest to reduce first from 12 to 10 dimensions. Then the IIA algebra is obtained with

$$\begin{aligned} Z^{\pm}_{M} &= \mathrm{i}\hat{Z}_{M\,12} \mp \hat{Z}_{M\,11}, \\ Z_{M_1\cdots M_5} &= 2\mathrm{i}\hat{Z}_{M_1\cdots M_5\,12}, \\ Z &= \mathrm{i}\hat{Z}_{11\,12}, \\ Z_{MN} &= \hat{Z}_{MN}, \\ Z_{M_1\cdots M_4} &= 2\mathrm{i}\hat{Z}_{M_1\cdots M_4\,11\,12}. \end{aligned} \tag{2.16}$$

After combining the anticommutators of $Q^{\pm}$ to obtain the anticommutation relations for $Q$ as defined in (2.15), we obtain the following reduction rules from 12 to 11 dimensions:

$$\begin{aligned} \tilde{Z}_{\tilde{M}} &= \mathrm{i}\hat{Z}_{\tilde{M}\,12}, \\ \tilde{Z}_{\tilde{M}\tilde{N}} &= \hat{Z}_{\tilde{M}\tilde{N}}, \\ \tilde{Z}_{\tilde{M}_1\cdots\tilde{M}_5} &= 2\mathrm{i}\hat{Z}_{\tilde{M}_1\cdots\tilde{M}_5\,12}. \end{aligned} \tag{2.17}$$

### *2.3. T-duality*

The connection between the IIA and IIB algebras is obtained by T-duality in a particular direction. Since we work with the complex form of the algebras as if we are in the Euclidean case, we denote this direction by $s$ (spacelike). We make the following identifications between the supersymmetry generators:

$$Q^{+} = Q^{1}, \qquad Q^{-} = \Gamma^{s} Q^{2}. \tag{2.18}$$

Splitting the 10-dimensional index $M$ in $(\mu, s)$, with $\mu = 1, \ldots, 9$, this leads to the following identifications between the bosonic generators:

$$\begin{aligned} Z^{\pm}_{\mu} &= Y^{(0)}_{\mu} \pm Y^{(3)}_{\mu}, \\ Z^{\pm}_{s} &= \pm Y^{(0)}_{s} + Y^{(3)}_{s}, \\ Z^{\pm}_{\mu_1\cdots\mu_5} &= Y^{+(0)}_{\mu_1\cdots\mu_5} \pm Y^{+(3)}_{\mu_1\cdots\mu_5}, \\ Z &= -Y^{(1)}_{s}, \\ Z_{\mu\nu} &= -Y_{\mu\nu s}, \\ Z_{\mu s} &= -Y^{(1)}_{\mu}, \\ Z_{\mu_1\cdots\mu_4} &= -2Y^{+(1)}_{\mu_1\cdots\mu_4 s}, \\ Z_{\mu\nu\rho s} &= -Y_{\mu\nu\rho}. \end{aligned} \tag{2.19}$$

For later purposes, to compare with the real case discussed in subsection 3.3, we explain how to derive the first two lines of (2.19). We start by writing out the relevant terms of the IIA algebra and split the index $M$ into $(\mu, s)$:

$$\{Q^{\pm}, Q^{\pm}\} = \mathcal{P}^{\pm}\Gamma^{\mu} Z^{\pm}_{\mu} + \mathcal{P}^{\pm}\Gamma^{s} Z^{\pm}_{s} + \cdots. \tag{2.20}$$

We next substitute the identifications (2.18) into the left-hand side of this equation. We thus obtain

$$\begin{aligned} \{Q^{1}, Q^{1}\} &= \mathcal{P}^{+}\Gamma^{\mu} Z^{+}_{\mu} + \mathcal{P}^{+}\Gamma^{s} Z^{+}_{s} + \cdots, \\ \Gamma^{s}\{Q^{2}, Q^{2}\}(\Gamma^{s})^{T} &= \mathcal{P}^{-}\Gamma^{\mu} Z^{-}_{\mu} + \mathcal{P}^{-}\Gamma^{s} Z^{-}_{s} + \cdots. \end{aligned} \tag{2.21}$$

In the second line we bring the $\Gamma^s$ matrices over to the right-hand side by using the identity $(\Gamma^s)^2 = 1$. Using some simple gamma matrix identities like $C^{-1}(\Gamma^s)^T = -\Gamma^s C^{-1}$ (note again the omitted $C^{-1}$ in our notation for the anticommutation relations) and $\Gamma^s\Gamma^\mu\Gamma^s = -\Gamma^\mu$, one finds that the above algebra coincides with the relevant terms in the IIB algebra provided the identifications are made given in the first two lines of (2.19).

## 3. Real symmetry algebras

Up to now, we did not consider the Hermiticity properties of the generators. For the real forms of the algebras, we can impose Hermiticity conditions on the realizations of the generators. For the fermionic generators, Hermiticity conditions are the Majorana conditions, whose consistency depends on the signature of spacetime as mentioned in the introduction and discussed in appendix A. The situation is summarized in table 1.

**Table 1.** The table summarizes the possible reality conditions for spinors in dimensions $d = 10, 11, 12$ for different signatures $(s, t)$ in which the minimal spinor is at most 32 dimensional. M denotes Majorana spinors, SM is a shorthand for symplectic Majorana spinors (given our convention for charge conjugation, otherwise this is equivalent to M), MW indicates the possibility of Majorana–Weyl spinors, while SMW indicates the possibility of symplectic Majorana–Weyl spinors. We only give the signatures with $s \geqslant t$, as those with $s < t$ are equivalent up to interchange of $s$ with $t$. The last row indicates, for each signature, whether in $d = 10$ a real form for type IIA (A), type IIB (B) or both (A/B) exists.

| $d$ | $(s,t)$ | | | | | |
|---|---|---|---|---|---|---|
| 12 | (10, 2) | | | | (6, 6) | |
| | MW | | | | MW | |
| 11 | (10, 1) | (9, 2) | | | (6, 5) | |
| | M | M | | | M | |
| 10 | (10, 0) | (9, 1) | (8, 2) | (7, 3) | (6, 4) | (5, 5) |
| | SM | MW | M | SMW | SM | MW |
| | A | A/B | A | B | A | A/B |

### *3.1. Algebras in dimensions 12 to 10*

To consider the Hermiticity properties of the generators, it is convenient to replace complex conjugation by the operation of charge conjugation†. This operator is chosen such that Majorana spinors $\lambda$ satisfy $\lambda^C = \lambda$. For more details, see appendix A. The definition of the $C$ operation involves an arbitrary phase factor $\alpha$ (or a matrix, see (A.9)) which may depend on the spacetime dimension and signature. This phase factor is equivalent to a possible redefinition of all spinors by $\alpha^{1/2}$. For bosonic generators, the $C$ operation is Hermitian conjugation together with multiplication with a factor of $\alpha^{-2}\beta$ (see (A.13)). The parameter $\beta$ is a sign that depends on the convention of whether one maintains ($\beta = 1$) or interchanges ($\beta = -1$) the order of fermions. Finally, for $\Gamma$-matrices the important properties are

$$\Gamma_a^C = (-)^{t+1}\Gamma_a, \qquad \Gamma_*^C = (-)^{d/2+t}\Gamma_*, \tag{3.1}$$

where $a$ denotes a Lorentz index in an arbitrary dimension.

† As can be seen from appendix A, the charge conjugation operation on spinors squares to one in the signatures that appear in table 1, except for (10, 0), (7, 3) and (6, 4). For (10, 0) and (6, 4) one could define another charge conjugation matrix, multiplying the one we defined in appendix A by $\Gamma_*$, such that the operation with respect to the newly defined charge conjugation matrix also squares to one. The charge conjugation matrix which we use in this paper leads in the latter signatures to a symplectic Majorana condition as discussed in the text.

To connect gamma matrices in different spacetime signatures, we use for gamma matrices in timelike directions

$$\Gamma^a = -\mathrm{i}\overset{0}{\Gamma}{}^a, \tag{3.2}$$

where $\overset{0}{\Gamma}$ denotes the gamma matrix in Euclidean space, which was used in the complex form of the algebras. We will not change the form of the commutators of the previous section, by redefining simultaneously the generators, e.g.

$$\overset{0}{\Gamma}{}^{13}\overset{0}{Z}_{13} = \Gamma^{13} Z_{13} \qquad \text{with} \quad Z_{13} = \mathrm{i}\overset{0}{Z}_{13}, \tag{3.3}$$

with 1 being a timelike direction and 3 a spacelike direction.

Considering the consistency of the anticommutators of the previous section with the charge conjugation, we find for $d = 12$

$$d = 12: \quad \hat{Z} = \hat{Z}^{\otimes} \tag{3.4}$$

for all generators. Here we have introduced a $\otimes$-operation for the bosonic generators that basically is a complex conjugation that hides prefactors involving $\alpha$ and $\beta$ (see equation (A.13)). For $d = 12$ the $\otimes$-operation involves a factor $\alpha_{10,2}^{-2}\beta$ or $\alpha_{6,6}^{-2}\beta$. The factor $\alpha$ is in this case a number with modulus one, that can be chosen for each spacetime signature.

Similarly, for $d = 11$ and $t = 1$ we find the reality conditions

$$d = 11, \quad t = 1, 5: \qquad \tilde{Z} = \tilde{Z}^{\otimes}, \tag{3.5}$$

for all bosonic generators $\tilde{Z}$ with one, two or five indices. For two time directions in $d = 11$, other signs occur:

$$d = 11, \quad t = 2: \qquad \begin{aligned} \tilde{Z}_{\tilde{M}} &= -\tilde{Z}^{\otimes}_{\tilde{M}}, \\ \tilde{Z}_{\tilde{M}\tilde{N}} &= \tilde{Z}^{\otimes}_{\tilde{M}\tilde{N}}, \\ \tilde{Z}_{\tilde{M}_1\cdots\tilde{M}_5} &= -\tilde{Z}^{\otimes}_{\tilde{M}_1\cdots\tilde{M}_5}. \end{aligned} \tag{3.6}$$

For 10 dimensions the discussion becomes more involved as there are different types of spinors. Consider first the signatures (9, 1) and (5, 5), where they are Majorana–Weyl spinors

$$\begin{aligned} &\text{IIA, (9, 1) or (5, 5)}: \qquad (Q^+)^C = Q^+, \qquad (Q^-)^C = Q^-, \\ &\text{IIB, (9, 1) or (5, 5)}: \qquad (Q^i)^C = Q^i. \end{aligned} \tag{3.7}$$

This is possible because $\Gamma_*$ is invariant under $C$. We could have inserted arbitrary phase factors in the right-hand sides of these equations. However, these amount to a redefinition of the phase factor $\alpha$ in the definition of the charge conjugation of spinors. This is true for any Majorana condition. For IIA we have two independent Majorana conditions, and correspondingly we could have two independent phase factors hidden in the $C$ operation. In other words, the matrix $\alpha$ is in this case

$$\alpha_{9,1}^{(A)} = \begin{pmatrix} \alpha_{9,1}^{+} & 0 \\ 0 & \alpha_{9,1}^{-} \end{pmatrix}. \tag{3.8}$$

For IIB the Majorana condition can mix the two chiral spinors, and $\alpha$ can thus be a matrix $\alpha^i{}_j$, that has to satisfy (A.10). For clarity, we give below the explicit form of the IIB reality condition given in the second line of (3.7):

$$(Q^i)^C \equiv (\alpha^{-1})^i{}_j\, \Gamma_t C^{-1} Q^{j*} = Q^i. \tag{3.9}$$

One obtains for all bosonic generators in the superalgebra the reality conditions

$$\begin{aligned} &\text{IIA, (9, 1) or (5, 5):} \qquad Z = Z^{\otimes},\\ &\text{IIB, (9, 1) or (5, 5):} \qquad Y = Y^{\otimes}. \end{aligned} \tag{3.10}$$

In the first equation it is understood that, in the case that one uses different phase factors $\alpha^{\pm}$ for the reality conditions on $Q^{\pm}$ the $\otimes$-operation reads

$$\begin{aligned} &\left(Z^{\pm}\right)^{\otimes} = \beta\,(\alpha^{\pm})^{-2}(Z^{\pm})^{*},\\ &Z^{\otimes} = \beta\,(\alpha^{+}\alpha^{-})^{-1}Z^{*}, \end{aligned} \tag{3.11}$$

where the second line is for the Ramond–Ramond operators. In the second equation of (3.10), the implicit $\alpha$ factors now act as a matrix, thus

$$Y^{ij} = Y^{\otimes ij} = \beta(\alpha^{-1})^{i}{}_{k}\,Y^{*k\ell}\,(\alpha^{-1})^{j}{}_{\ell}, \qquad \varepsilon^{ij}Y_{MNP} = \beta(\alpha^{-1})^{i}{}_{k}\,\varepsilon^{k\ell}Y^{*}_{MNP}\,(\alpha^{-1})^{j}{}_{\ell}. \tag{3.12}$$

Next, we consider the signature $(8, 2)$. The Majorana condition can only be imposed on non-chiral spinors. Indeed, the $C$ operation changes a chiral spinor to an antichiral one as $\Gamma_{*}^{C} = -\Gamma_{*}$. Therefore, we cannot impose Majorana conditions on chiral spinors, and thus type IIB does not have a real form. For type IIA, the Majorana spinor is $Q^{+} + Q^{-}$, or explicitly we have

$$\text{IIA, (8, 2):} \qquad (Q^{+})^{C} = Q^{-}, \qquad (Q^{-})^{C} = Q^{+}. \tag{3.13}$$

Considering here the consistency of the algebra with the $C$ operation, we find

$$\begin{aligned} \text{IIA, (8, 2):} \qquad &(Z^{+}_{M})^{\otimes} = -Z^{-}_{M}, \qquad (Z^{+}_{M_1\cdots M_5})^{\otimes} = -Z^{-}_{M_1\cdots M_5},\\ &Z^{\otimes} = -Z, \qquad Z^{\otimes}_{MN} = Z_{MN}, \qquad Z^{\otimes}_{MNPQ} = -Z_{MNPQ}. \end{aligned} \tag{3.14}$$

For the signatures (10, 0) and (6, 4) the discussion is similar to the (8, 2) case. Again, a real form of type IIB does not exist. For type IIA there are only signs to be changed with respect to the previous case. Indeed, as mentioned in the first footnote in section 1, by another definition of charge conjugation the (10, 0) and (6, 4) cases could be made identical to the (8, 2) case. However, we do not change our definition of the charge conjugation, which implies that now the charge conjugation operator squares to $-1$. Therefore, a consistent Majorana condition looks rather as a symplectic Majorana condition:

$$\text{IIA, (10, 0) or (6, 4):} \qquad (Q^{+})^{C} = Q^{-}, \qquad (Q^{-})^{C} = -Q^{+}. \tag{3.15}$$

The consistency of the algebra with the $C$ operation now gives other signs for the Ramond–Ramond generators,

$$\begin{aligned} \text{IIA, (10, 0) or (6, 4):} \qquad &(Z^{+}_{M})^{\otimes} = -Z^{-}_{M}, \qquad (Z^{+}_{M_1\cdots M_5})^{\otimes} = -Z^{-}_{M_1\cdots M_5},\\ &Z^{\otimes} = Z, \qquad Z^{\otimes}_{MN} = -Z_{MN},\\ &Z^{\otimes}_{MNPQ} = Z_{MNPQ}. \end{aligned} \tag{3.16}$$

The remaining case is (7, 3). Now, the charge conjugation leaves $\Gamma_{*}$ invariant, and squares to $-1$. Therefore, in type IIA we cannot impose Majorana conditions, or, in other words, there is no real form for IIA. For type IIB there is the symplectic Majorana–Weyl condition

$$\text{IIB, (7, 3):} \qquad (Q^{i})^{C} = \varepsilon^{i}{}_{j}\,Q^{j}. \tag{3.17}$$

The reality conditions of the bosonic generators are

$$\begin{aligned}
\text{IIB}, (7,3): \quad & (Y_M^{(0)})^{\otimes} = Y_M^{(0)}, && (Y^{+(0)}_{M_1\cdots M_5})^{\otimes} = Y^{+(0)}_{M_1\cdots M_5},\\
& (Y_M^{(1)})^{\otimes} = -Y_M^{(1)}, && (Y^{+(1)}_{M_1\cdots M_5})^{\otimes} = -Y^{+(1)}_{M_1\cdots M_5},\\
& (Y_M^{(3)})^{\otimes} = -Y_M^{(3)}, && (Y^{+(3)}_{M_1\cdots M_5})^{\otimes} = -Y^{+(3)}_{M_1\cdots M_5},\\
& (Y_{MNP})^{\otimes} = Y_{MNP}.
\end{aligned} \tag{3.18}$$

Finally, we consider the automorphisms mentioned in subsection 2.1. For type IIA, the automorphism $(-)^{\mathrm{F_L}}$, equation (2.8), is only preserved for the signatures (9, 1) and (5, 5). In these cases a projection to the even part under this automorphism leads to the $N = 1$ algebra, i.e. $OSp(1|16)$. Also for type IIB, the automorphisms $(-)^{\mathrm{F_L}}$ and $\Omega$, see (2.11) and (2.12), are only preserved for these signatures. The type IIA and IIB automorphisms for the (9, 1) signature have been discussed in [23]. The $S$ duality is preserved in all real forms of IIB, i.e. also for (7, 3).

### *3.2. Dimensional reduction*

We start with the 12-dimensional Majorana condition for (10, 2) signature,

$$\hat{Q} = \frac{1}{\alpha_{10,2}}\hat{B}^{-1}\hat{Q}^*, \qquad \hat{B}^{-1} = \hat{\Gamma}_{t_1}\hat{\Gamma}_{t_2}\hat{C}^{-1}, \tag{3.19}$$

where we have indicated the signature for the phase factor explicitly. We now reduce this to 10 dimensions. There are different ways to do so, related to the identification of the 12-dimensional timelike directions $t_1$ and $t_2$ with either timelike directions in the 10 uncompactified dimensions or the two compactified dimensions. Neglecting the order of the two (which is another sign factor† in $\alpha$), there are four different choices. If the compactification directions are 11 and 12, and we use the representation of 12-dimensional gamma matrices as in (B.4), inserting factors of i for timelike directions, we obtain 10-dimensional Majorana conditions. Comparing these with those in the previous subsection, we find the following relations between the phase factors in 10 and 12 dimensions:

$$\begin{aligned}
t_1 &= 11, \quad t_2 = 12 \quad (\text{via } (10,1)): && \alpha_{10,0} = -\mathrm{i}\alpha_{10,2},\\
t_1 &= 1, \quad t_2 = 11 \quad (\text{via } (9,2)): && \alpha^{\pm}_{9,1} = \pm\mathrm{i}\alpha_{10,2},\\
t_1 &= 1, \quad t_2 = 12 \quad (\text{via } (10,1)): && \alpha^{\pm}_{9,1} = -\alpha_{10,2},\\
t_1 &= 1, \quad t_2 = 2 \quad (\text{via } (9,2)): && \alpha_{8,2} = \alpha_{10,2}.
\end{aligned} \tag{3.20}$$

Observe that the results for signature (9, 1), which are obviously type IIA theories, depend on whether the first compactification direction (direction 12) is spacelike, and the second one (direction 11) timelike, as it is in the second line, or the reverse as in the third line.

The above rules apply as well to the reduction of the (6, 6) spinors. In that case the expressions for $\alpha$ are obtained from the above ones by replacing $\alpha_{s,t}$ by $\alpha_{s-4,t+4}$, e.g. the first line in (3.20) gives $\alpha_{6,4} = -\mathrm{i}\alpha_{6,6}$.

We can combine the 10-dimensional Majorana spinors to 11-dimensional ones. The 11-dimensional spinor is given by (2.15). In the above scheme, the (10, 1) theory where '1' is the

† A sign change of $\alpha$ amounts to a redefinition of the fermionic generators with i and of the bosonic generators with a sign.

time direction can be obtained from the third line in (3.20). One can check that this leads to the Majorana condition of (10, 1) with

$$\alpha_{10,1} = \alpha^{\pm}_{9,1} = -\alpha_{10,2}. \tag{3.21}$$

In principle, one could also start from the first line in (3.20).

For the (9, 2) theory, we have standard the timelike directions as '1' and '2'. Thus this can be obtained from the fourth line of (3.20). Combining the corresponding chiral spinors leads to a (9, 2) Majorana spinor with

$$\alpha_{9,2} = \alpha_{8,2} = \alpha_{10,2}. \tag{3.22}$$

To obtain the dimensional reduction of the bosonic generators, we start from those in the complex case, but we have to take into account extra factors of i due to (3.3). For instance, when we reduce from (10, 2) to (10, 1), we start from (2.17), which is a formula for the $\overset{0}{Z}$, i.e. this formula refers to the Euclidean case. Then taking into account that the 12th direction is a time direction, the factors of i cancel for the relation between the $Z$ generators:

$$(10,2) \to (10,1): \qquad \begin{aligned} \tilde{Z}_{\tilde{M}} &= \hat{Z}_{\tilde{M}\,12}, \\ \tilde{Z}_{\tilde{M}\tilde{N}} &= \hat{Z}_{\tilde{M}\tilde{N}}, \\ \tilde{Z}_{\tilde{M}_1\cdots\tilde{M}_5} &= 2\hat{Z}_{\tilde{M}_1\cdots\tilde{M}_5\,12}. \end{aligned} \tag{3.23}$$

On the other hand, when we reduce from (10, 2) to (9, 2) the 12th direction is spacelike, so there are no extra factors of i when we replace $\overset{0}{Z}$ by $Z$ in (2.17):

$$(10,2) \to (9,2): \qquad \begin{aligned} \tilde{Z}_{\tilde{M}} &= \mathrm{i}\hat{Z}_{\tilde{M}\,12}, \\ \tilde{Z}_{\tilde{M}\tilde{N}} &= \hat{Z}_{\tilde{M}\tilde{N}}, \\ \tilde{Z}_{\tilde{M}_1\cdots\tilde{M}_5} &= 2\mathrm{i}\hat{Z}_{\tilde{M}_1\cdots\tilde{M}_5\,12}. \end{aligned} \tag{3.24}$$

One can check the consistency of these rules with the reality properties of the previous subsection, see (3.4)–(3.6). One has to keep in mind that the ⊗-operation involves factors of $\alpha$, see (A.13). In this case $\alpha^2_{10,2} = \alpha^2_{10,1} = \alpha^2_{9,2}$, so that this does not lead to a change of sign in the reality property.

The reduction from 12 to 10 dimensions can be done in four different ways. With time directions 11 and 12 we obtain

$$(10,2) \to (10,0): \qquad \begin{aligned} Z^{\pm}_{M} &= \hat{Z}_{M\,12} \pm \mathrm{i}\hat{Z}_{M\,11}, \\ Z_{M_1\cdots M_5} &= 2\hat{Z}_{M_1\cdots M_5\,12}, \\ Z &= -\mathrm{i}\hat{Z}_{11\,12}, \\ Z_{MN} &= \hat{Z}_{MN}, \\ Z_{M_1\cdots M_4} &= -2\mathrm{i}\hat{Z}_{M_1\cdots M_4\,11\,12}. \end{aligned} \tag{3.25}$$

With time directions 1 and 11 we obtain

$$(10,2) \to (9,1): \qquad \begin{aligned} Z^{\pm}_{M} &= \mathrm{i}(\hat{Z}_{M\,12} \pm \hat{Z}_{M\,11}), \\ Z_{M_1\cdots M_5} &= 2\mathrm{i}\hat{Z}_{M_1\cdots M_5\,12}, \\ Z &= \hat{Z}_{11\,12}, \\ Z_{MN} &= \hat{Z}_{MN}, \\ Z_{M_1\cdots M_4} &= 2\hat{Z}_{M_1\cdots M_4\,11\,12}. \end{aligned} \tag{3.26}$$

Note that to check the reality properties in this case, one has to take into account that for the Ramond–Ramond generators, the ⊗-operation involves $\alpha^{+}_{9,1}\alpha^{-}_{9,1}$, see (3.11). For time directions 1 and 12 we obtain

$$(10,2) \to (9,1): \qquad \begin{aligned} Z^{\pm}_{M} &= \hat{Z}_{M\,12} \mp \hat{Z}_{M\,11}, \\ Z_{M_1\cdots M_5} &= 2\hat{Z}_{M_1\cdots M_5\,12}, \\ Z &= \hat{Z}_{11\,12}, \\ Z_{MN} &= \hat{Z}_{MN}, \\ Z_{M_1\cdots M_4} &= 2\hat{Z}_{M_1\cdots M_4\,11\,12}. \end{aligned} \tag{3.27}$$

The two (9, 1) algebras we obtain from (3.26) and (3.27) are related to each other via a simple redefinition. To exhibit this redefinition, it is convenient to denote the (9, 1) generators obtained from (3.26) with $Q$ and $Z$, while those obtained from (3.27) with $\tilde{Q}$ and $\tilde{Z}$. Using this notation we find

$$\begin{aligned} &Q^{+} = \mathrm{i}^{1/2}\tilde{Q}^{-}, && Q^{-} = (-\mathrm{i})^{1/2}\tilde{Q}^{+}, \\ &-\mathrm{i}Z^{+}_{M} = \tilde{Z}^{-}_{M}, && \mathrm{i}Z^{-}_{M} = \tilde{Z}^{+}_{M}, \\ &-\mathrm{i}Z^{+}_{M_1\cdots M_5} = \tilde{Z}^{-}_{M_1\cdots M_5}, && \mathrm{i}Z^{-}_{M_1\cdots M_5} = \tilde{Z}^{+}_{M_1\cdots M_5}, \\ &Z = -\tilde{Z}, && Z_{MN} = \tilde{Z}_{MN}, \\ &Z_{M_1\cdots M_4} = -\tilde{Z}_{M_1\cdots M_4}. \end{aligned} \tag{3.28}$$

This redefinition is consistent with the 12-dimensional origin of the generators given in (3.26) and (3.27) provided we interchange the 11 and 12 directions in the latter equation.

Finally, for time directions 1 and 2 there is no change with respect to the rules of the complex algebra:

$$(10,2) \to (8,2): \qquad \begin{aligned} Z^{\pm}_{M} &= \mathrm{i}\hat{Z}_{M\,12} \mp \hat{Z}_{M\,11}, \\ Z_{M_1\cdots M_5} &= 2\mathrm{i}\hat{Z}_{M_1\cdots M_5\,12}, \\ Z &= \mathrm{i}\hat{Z}_{11\,12}, \\ Z_{MN} &= \hat{Z}_{MN}, \\ Z_{M_1\cdots M_4} &= 2\mathrm{i}\hat{Z}_{M_1\cdots M_4\,11\,12}. \end{aligned} \tag{3.29}$$

### *3.3. T-duality*

We first discuss the conventional *space/space* dualities. These dualities do not change the signature of spacetime and only exist for the signatures (9, 1) and (5, 5) where both a IIA and a IIB algebra can be defined (see also table 1):

$$\mathrm{IIA}_{(9,1)} \xleftrightarrow{\mathrm{T_{ss}}} \mathrm{IIB}_{(9,1)}, \qquad \mathrm{IIA}_{(5,5)} \xleftrightarrow{\mathrm{T_{ss}}} \mathrm{IIB}_{(5,5)}. \tag{3.30}$$

The identification between the supersymmetry generators is as in the complex case, see equation (2.18) and one obtains the same T-duality rules as in the complex case, see equation (2.19). Let us check that these T-duality rules are consistent with the reality properties of the different generators. Starting with the reality conditions on the IIA supersymmetry generators as given in the first line of equation (3.7) and applying the T-duality rule (2.18) one finds the reality condition (3.9) on the IIB supergenerators with the matrix $\alpha_{9,1}^{(B)}$

$$\alpha_{9,1}^{(B)} = \alpha_{9,1}^{(A)} = \begin{pmatrix} \alpha_{9,1}^{+} & 0 \\ 0 & \alpha_{9,1}^{-} \end{pmatrix}. \tag{3.31}$$

Applying (3.12), this leads to the following reality conditions on the bosonic generators of the IIB (9, 1) algebra:

$$\begin{aligned} &\big(Y_M^{(0)} \pm Y_M^{(3)}\big)^* = \beta(\alpha_{9,1}^{\pm})^2\big(Y_M^{(0)} \pm Y_M^{(3)}\big), \\ &\big(Y_M^{(1)}\big)^* = \beta\alpha_{9,1}^{+}\alpha_{9,1}^{-}Y_M^{(1)}, \end{aligned} \tag{3.32}$$

$$\begin{aligned} &\big(Y_{M_1\cdots M_5}^{(0)} \pm Y_{M_1\cdots M_5}^{(3)}\big)^* = \beta(\alpha_{9,1}^{\pm})^2\big(Y_{M_1\cdots M_5}^{(0)} \pm Y_{M_1\cdots M_5}^{(3)}\big), \\ &Y_{MNP}^* = \beta\alpha_{9,1}^{+}\alpha_{9,1}^{-}Y_{MNP}. \end{aligned} \tag{3.33}$$

One may verify that the above reality conditions on the bosonic IIB generators and the corresponding conditions (3.11) on the bosonic IIA generators are consistent with the T-duality rules (2.19).

Also *time/time* dualities give a relation between the IIA and IIB (9, 1) algebras. In this case the duality is performed in a timelike direction $t$. To apply the rules of subsection 2.3, we write the gamma matrix $\Gamma^s$ as $\Gamma^s = \mathrm{i}\Gamma^t$ such that $(\Gamma^s)^2 = (\mathrm{i}\Gamma^t)^2 = 1$. We thus make the following identifications between the supersymmetry generators:

$$Q^+ = Q^1, \qquad Q^- = \mathrm{i}\Gamma^t Q^2, \tag{3.34}$$

The calculation for the spacelike case can thus be copied, replacing everywhere $\Gamma^s$ by $\mathrm{i}\Gamma^t$ and $s$-like components of the bosonic charges by $-\mathrm{i}$ times the $t$-like components as explained in equation (3.3). We thus we find the following result:

$$\text{IIA, (9, 1)} \overset{\mathrm{T_{tt}}}{\longleftrightarrow} \text{IIB, (9, 1)}: \qquad \begin{aligned} Z_\mu^{\pm} &= Y_\mu^{(0)} \pm Y_\mu^{(3)}, \\ Z_t^{\pm} &= \big(\pm Y_t^{(0)} + Y_t^{(3)}\big), \\ Z_{\mu_1\cdots\mu_5}^{\pm} &= Y_{\mu_1\cdots\mu_5}^{+(0)} \pm Y_{\mu_1\cdots\mu_5}^{+(3)}, \\ Z &= \mathrm{i}Y_t^{(1)}, \\ Z_{\mu\nu} &= \mathrm{i}Y_{\mu\nu t}, \\ Z_{\mu t} &= \mathrm{i}Y_\mu^{(1)}, \\ Z_{\mu_1\cdots\mu_4} &= 2\mathrm{i}Y_{\mu_1\cdots\mu_4 t}^{+(1)}, \\ Z_{\mu\nu\rho t} &= \mathrm{i}Y_{\mu\nu\rho}. \end{aligned} \tag{3.35}$$

Again, these T-duality rules should be consistent with the reality conditions of the generators discussed before. The reality condition of the IIA supersymmetry generators is determined by (3.7), in terms of the matrix $\alpha_{9,1}^{(A)}$. Calculating then $Q^{i*}$, we find

$$Q^{1*} = B\alpha_{9,1}^{+}Q^1, \qquad Q^{2*} = -B\alpha_{9,1}^{-}Q^2, \tag{3.36}$$

which are the IIB reality conditions of (3.7) when we take

$$\alpha_{9,1}^{(B)} = \begin{pmatrix} \alpha_{9,1}^{+} & 0 \\ 0 & -\alpha_{9,1}^{-} \end{pmatrix}. \tag{3.37}$$

Given this $\alpha^{(B)}$ one can check that the time/time duality rules (3.35) are consistent with the reality conditions of the real IIA (9, 1) and IIB (9, 1) algebras.

To explain how the *space/time* T-duality between the real algebras works, we first consider the identification between the IIA (10, 0) and the IIB (9, 1) algebras. Note that we are dealing with a space/time T-duality, i.e. in the IIA (10, 0) algebra the duality is performed in a spacelike direction $s$, whereas in the IIB (9, 1) algebra we perform T-duality in the single timelike direction $t$. Again, we can copy the results for subsection 2.3, this time we only have to insert factors of i on the IIA side as explained for the time/time duality above. We thus find

$$\text{IIA}, (10,0) \overset{\mathrm{T_{st}}}{\longleftrightarrow} \text{IIB}, (9,1): \qquad \begin{aligned} Z^{\pm}_{\mu} &= Y^{(0)}_{\mu} \pm Y^{(3)}_{\mu}, \\ Z^{\pm}_{s} &= -\mathrm{i}\big(\pm Y^{(0)}_{t} + Y^{(3)}_{t}\big), \\ Z^{\pm}_{\mu_1\cdots\mu_5} &= Y^{+(0)}_{\mu_1\cdots\mu_5} \pm Y^{+(3)}_{\mu_1\cdots\mu_5}, \\ Z &= \mathrm{i}Y^{(1)}_{t}, \\ Z_{\mu\nu} &= \mathrm{i}Y_{\mu\nu t}, \\ Z_{\mu s} &= -Y^{(1)}_{\mu}, \\ Z_{\mu_1\cdots\mu_4} &= 2\mathrm{i}Y^{+(1)}_{\mu_1\cdots\mu_4 t}, \\ Z_{\mu\nu\rho s} &= -Y_{\mu\nu\rho}. \end{aligned} \tag{3.38}$$

The reality condition of the (10, 0) supersymmetry generators is, according to equation (3.15), given by

$$\frac{-1}{\alpha_{10,0}} C^{-1}(Q^{+})^{*} = Q^{-}, \qquad \frac{-1}{\alpha_{10,0}} C^{-1}(Q^{-})^{*} = -Q^{+}, \tag{3.39}$$

where $\alpha_{10,0}$ is an arbitrary phase factor. The reality conditions of the (10, 0) bosonic generators are given in (3.16). Applying the T-duality relation (3.34) on the IIA supersymmetry generators we find the reality condition (3.9) on the IIB supersymmetry generators with the matrix $\alpha^{(B)}_{9,1}$ given by

$$\alpha^{(B)}_{9,1} = -\mathrm{i}\,\alpha_{10,0}\,\tau_1. \tag{3.40}$$

Using this expression we can write out equation (3.12) in components. This leads to the following reality conditions on the bosonic generators of the IIB algebra:

$$\begin{aligned} Y^{(0)*}_{M} &= -\beta\alpha^2_{10,0}\,Y^{(0)}_{M}, & Y^{(0)*}_{M_1\cdots M_5} &= -\beta\alpha^2_{10,0}\,Y^{(0)}_{M_1\cdots M_5}, \\ Y^{(1)*}_{M} &= -\beta\alpha^2_{10,0}\,Y^{(1)}_{M}, & Y^{(1)*}_{M_1\cdots M_5} &= -\beta\alpha^2_{10,0}\,Y^{(1)}_{M_1\cdots M_5}, \\ Y^{(3)*}_{M} &= \beta\alpha^2_{10,0}\,Y^{(3)}_{M}, & Y^{(3)*}_{M_1\cdots M_5} &= \beta\alpha^2_{10,0}\,Y^{(3)}_{M_1\cdots M_5}, \\ Y^{*}_{MNP} &= \beta\alpha^2_{10,0}\,Y_{MNP}. \end{aligned} \tag{3.41}$$

It is now straightforward to verify that the reality conditions (3.16) and (3.41) are consistent with the T-duality rules (3.38) between the real IIA (10, 0) and IIB (9, 1) superalgebras.

The T-duality between the real IIA (10, 0) and IIB (9, 1) algebras can be extended to the following chain of space/time T-dualities:

$$\text{IIA}_{(10,0)} \overset{\mathrm{T_{st}}}{\longleftrightarrow} \text{IIB}_{(9,1)} \overset{\mathrm{T_{st}}}{\longleftrightarrow} \text{IIA}_{(8,2)} \overset{\mathrm{T_{st}}}{\longleftrightarrow} \text{IIB}_{(7,3)} \overset{\mathrm{T_{st}}}{\longleftrightarrow} \text{IIA}_{(6,4)} \overset{\mathrm{T_{st}}}{\longleftrightarrow} \text{IIB}_{(5,5)}. \tag{3.42}$$

Consider, for instance, the IIA (8, 2) supergenerators. They satisfy the following reality conditions:

$$\frac{1}{\alpha_{8,2}}\Gamma_1\Gamma_2 C^{-1}(Q^+)^* = Q^-, \qquad \frac{1}{\alpha_{8,2}}\Gamma_1\Gamma_2 C^{-1}(Q^-)^* = Q^+, \tag{3.43}$$

where $\alpha_{8,2}$ is an arbitrary phase factor and 1 and 2 are the two timelike directions. Performing a T-duality in one of the timelike directions, say 1, we should obtain the IIB (9, 1) algebra with the single time direction in the 2 direction. Performing the T-duality

$$Q^+ = Q^1, \qquad Q^- = \mathrm{i}\Gamma^1 Q^2, \tag{3.44}$$

we indeed obtain the correct reality conditions for the IIB (9, 1) supergenerators, see (3.9), with the matrix $\alpha^{(B)}_{9,1}$ given by

$$\alpha^{(B)}_{9,1} = -\mathrm{i}\,\alpha_{8,2}\,\tau_1. \tag{3.45}$$

Another possibility is to T-dualize the IIA (8, 2) algebra in one of the eight spacelike directions, say 3. This duality proceeds similar to the T-duality of the IIA (10, 0) algebra discussed before and leads us to the IIB (7, 3) algebra with the three timelike directions in the 1, 2 and 3 directions. Performing the T-duality

$$Q^+ = Q^1, \qquad Q^- = \mathrm{i}\Gamma^3 Q^2, \tag{3.46}$$

we indeed obtain the reality conditions of the IIB (7, 3) algebra, see (3.17), with

$$\alpha_{7,3} = -\mathrm{i}\,\alpha_{8,2}. \tag{3.47}$$

Finally, one may also T-dualize in two different ways the IIA (6, 4) superalgebra. The reality conditions on the supergenerators for this case are given by

$$\frac{-1}{\alpha_{6,4}}\Gamma_1\Gamma_2\Gamma_3\Gamma_4 C^{-1}(Q^+)^* = Q^-, \qquad \frac{-1}{\alpha_{6,4}}\Gamma_1\Gamma_2\Gamma_3\Gamma_4 C^{-1}(Q^-)^* = -Q^+, \tag{3.48}$$

where 1, 2, 3, 4 refer to the four timelike directions. The first possibility is to T-dualize in one of the six spacelike directions, say 5. This leads us to the IIB (5, 5) algebra in exactly the same way as the T-duality of the IIA (10, 0) algebra in a spacelike direction gave us the IIB (9, 1) algebra so we will not discuss this case further. The second possibility is to T-dualize one of the four timelike directions, say 1. This leads us to the IIB (7, 3) algebra with the three timelike directions in the directions 2, 3 and 4. Indeed, applying the T-duality rule

$$Q^+ = Q^1, \qquad Q^- = \mathrm{i}\Gamma^1 Q^2, \tag{3.49}$$

we find the proper reality condition on the IIB (7, 3) supergenerators with

$$\alpha_{7,3} = -\mathrm{i}\,\alpha_{6,4}. \tag{3.50}$$

This concludes our discussion of the real T-dualities. We have summarized the situation in table 2.

## 4. The $*$-algebras

In this section we will indicate the connection between the algebras and spacetime for the different theories. In this way, we will find a place for the $*$-algebras of [12].

Up to here, all the generators that occurred on the right-hand side of the $\{Q, Q\}$ anticommutators have been treated on an equal footing. Now, we will identify two special operators.

**Table 2.** The realizations in $d = 12$, 11 and 10 of the real algebra $OSp(1|32)$ for different signatures. The diagonal one-sided arrows indicate dimensional reductions. The both-sided arrows refer to the space/space ($\mathrm{T_{ss}}$), time/time ($\mathrm{T_{tt}}$) and space/time ($\mathrm{T_{st}}$) dualities discussed in the text.

$$
\begin{tikzcd}
 & & \mathrm{F}_{10,2} \arrow[dl] \arrow[dr] & & & & & & \mathrm{F}_{6,6} \arrow[dl] \\
 & \mathrm{M}_{10,1} \arrow[dl] \arrow[dr] & & \mathrm{M}_{9,2} \arrow[dl] \arrow[dr] & & & & \mathrm{M}_{6,5} \arrow[dl] \arrow[dr] & \\
\mathrm{IIA}_{10,0} \arrow[drr, leftrightarrow, "\mathrm{T_{st}}"] & & \mathrm{IIA}_{9,1} \arrow[d, leftrightarrow, "\mathrm{T_{ss}}"', "\mathrm{T_{tt}}"] & & \mathrm{IIA}_{8,2} \arrow[dll, leftrightarrow, "\mathrm{T_{st}}"] \arrow[dr, leftrightarrow, "\mathrm{T_{st}}"] & & \mathrm{IIA}_{6,4} \arrow[drr, leftrightarrow, "\mathrm{T_{st}}"] & & \mathrm{IIA}_{5,5} \arrow[d, leftrightarrow, "\mathrm{T_{ss}}"', "\mathrm{T_{tt}}"] \\
 & & \mathrm{IIB}_{9,1} & & & \mathrm{IIB}_{7,3} \arrow[ur, leftrightarrow, "\mathrm{T_{st}}"] & & & \mathrm{IIB}_{5,5}
\end{tikzcd}
$$

*The spacetime translations $P_\mu$.* To associate the algebra with a field theory, one has to define spacetime points $x^\mu$. The spacetime is defined as the space generated from a basic point by these translation operators, as in a coset approach. In subsection 4.1 we will show how different choices of this translation generator leads to the different $\star$-algebras of [12]. The duality relations between these $\star$-algebras is discussed in subsection 4.2.

*The energy operator $E$.* As we will show below this is the unique operator that is positive. It will be treated in subsection 4.3.

### *4.1. The translation operator*

We first discuss the translation operator. It has to be a vector in spacetime, and we will further only require that it appears in a non-singular way† in the supersymmetry anticommutator. This requirement is imposed in order that all supersymmetries act on the spacetime.

There exists no vector operator in the 12-dimensional algebra, and thus there is no translation operator in this case. Indeed, for F theory there is no associated field theory. In 11 dimensions there is just one vector in the algebra (2.5), and we thus identify

$$\tilde{P}_{\tilde{M}} \equiv \tilde{Z}_{\tilde{M}}. \tag{4.1}$$

The situation becomes more interesting in 10 dimensions, where there are several vectors. In the IIA theory (2.7) there are two vectors, $Z^+_M$ and $Z^-_M$. Each separately is not suitable as a translation generator as, for example, $Z^+_M$ does not appear in the anticommutator of the $Q^-$ supersymmetries. Instead, we may consider any linear combination of these two as a possible translation generator:

$$P_M = p_+ Z^+_M + p_- Z^-_M. \tag{4.2}$$

Different values of $p_+$ and $p_-$ can be related by redefinitions of the supersymmetries. Redefining $Q^\pm$ by a factor $q^\pm$, the $p_\pm$ are redefined to $p_\pm q^2_\pm$.

When going from complex to real algebras, there is a further restriction, namely we want a real‡ translation operator. For the (9, 1) signature the reality condition on $P_M$ and $Z^\pm_M$ implies that $p_\pm$ are real. On the other hand, the reality conditions on the supersymmetries imply that $q_\pm$ are real, for fixed reality conditions on the supersymmetries, i.e. a fixed matrix $\alpha^{(A)}_{9,1}$. We now consider the basis where $\alpha^{(A)}_{9,1} = \mathbb{1}$. Then, up to redefinitions and an overall sign, there are two possible choices for the translation generator:

$$\begin{aligned} (9,1):\qquad & \mathrm{IIA}{:}P_M \equiv Z^+_M + Z^-_M \\ & \mathrm{IIA}^*{:}P_M \equiv Z^+_M - Z^-_M. \end{aligned} \tag{4.3}$$

In Lagrangian theories, the spacetime is defined, and thus the translation operators are also defined. In the IIA theories, they appear in the algebra as the operator in the first line of (4.3). On the other hand, in the so-called IIA* theories of [12], translations appear as in the second line of (4.3). The complete algebras of the two theories only differ in the identification of the translation operator.

In the case of the (10, 0) or (8, 2) IIA algebras, the reality condition of $Z^\pm_M$ in (3.16) or (3.14) implies $p_- = -p^*_+$, and the reality conditions on the supersymmetries allow

† To be precise, we require that the translation vector part of the matrix $\mathcal{Z}^{ik}$ as defined in (4.12) is a non-singular matrix.

‡ We will use $P^\otimes = P$ as a reality condition, as the difference with $P^* = P$ are just phase factors.

redefinitions with $q_- = q_+^*$. These are sufficient to transform the translation operator to that with $p_+ = p_- = \mathrm{i}$. In these cases there is thus only one translation operator,

$$\text{IIA:} \quad (10,0) \text{ or } (8,2): \qquad P_M \equiv \mathrm{i}(Z_M^+ + Z_M^-). \tag{4.4}$$

Of course, we could also have chosen $p_+ = 1$, in which case

$$\text{IIA}': \quad (10,0) \text{ or } (8,2): \qquad P_M \equiv Z_M^+ - Z_M^-. \tag{4.5}$$

This choice, which we label by IIA′, is thus equivalent to the choice labelled IIA, by a redefinition of the generators. This redefinition is similar to the dualities treated at the end of subsection 2.1. The obtained results for the IIA algebras can, of course, be generalized from $(t, s)$ to $(t+4, s-4)$ signatures.

The possibilities for choosing the translation operator in the IIB algebras follow a similar pattern. For (9, 1) (or (5, 5)) signature, there are two inequivalent choices of the translation operator: in the basis where $\alpha = \mathbb{1}$ these can be

$$\begin{aligned} (9,1): \qquad & \text{IIB:} \quad P_M = Y_M^{(0)}, \\ & \begin{cases} \text{IIB}^*: & P_M = Y_M^{(3)}, \\ \text{IIB}': & P_M = Y_M^{(1)}. \end{cases} \end{aligned} \tag{4.6}$$

The two choices IIB* and IIB′, following the notation of [12], are equivalent up to a redefinition, which is the $S$-duality transformation given in (2.13).

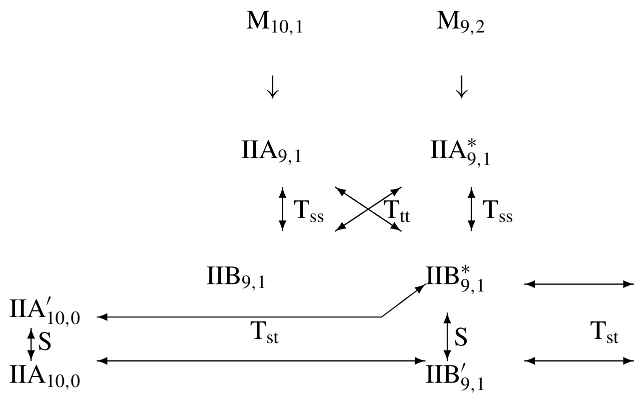
We stress that, in IIB as well as in IIA theories, the adapted terminology refers to a basis in which $\alpha = \mathbb{1}$. If $\alpha$ is a different matrix, then one first has to redefine the supersymmetries with $\alpha^{1/2}$:

$$Q'^i = \left(\alpha^{1/2}\right)^i{}_j Q^j. \tag{4.7}$$

The newly defined generators $Q'^i$ satisfy reality conditions with $\alpha' = \mathbb{1}$. The bosonic operators appearing in the anticommutator

$$\left\{Q^i, Q^j\right\} = \mathcal{Y}^{ij} \tag{4.8}$$

**Table 3.** Detail of table 2, indicating various theories distinguished by the choice of the translation operator. Our notation is explained in the caption of table 2. For a discussion of this table, see subsection 4.2.



are in the new basis given by

$$\mathcal{Y}'^{ij} = \left(\alpha^{1/2}\right)^i{}_k \mathcal{Y}^{k\ell} \left(\alpha^{1/2}\right)^j{}_\ell, \tag{4.9}$$

and these are to be compared with (4.3)–(4.6).

For the (7, 3) signature, there is only one inequivalent choice, similar to the (10, 0) or (8, 2) case in IIA:

$$(7,3): \quad P_M = Y^{(0)}_M. \tag{4.10}$$

Another way to express this fact is to say that there are three S-dual versions.

### 4.2. T-dualities

The theories distinguished in the previous section are represented in table 3, which is in fact, a close look at part of table 2. These theories are connected by various dualities, which we now discuss.

Let us start with the reduction from 11 dimensions, where the translation operator is always (4.1), or in 12-dimensional language $\hat{Z}_{M\,12}$. The reduction to any 10-dimensional theory always leads to a translation generator of the form $Z^+_M + Z^-_M$. However, the $\alpha$ matrix is not always $\mathbb{1}$. Indeed, from (3.20), we see that reducing via the (10, 1) theory gives $\alpha^{(A)}_{9,1}$ proportional to the unit matrix, but reducing via (9, 2) gives instead a matrix proportional to $\tau_3$. Therefore, in the first case the translation operator is the one for the IIA theory, while in the second case, we have to apply the correction (4.9), after which the translation operator is $\mathrm{i}(Z^+_M - Z^-_M)$, thus that of IIA$^*$. This explains the two vertical arrows at the top of table 3.

For T dualities in 10 dimensions, the translation generators of one algebra are split into one component in the direction of the duality direction and nine components orthogonal to that direction. They are mapped to components of different Lorentz-covariant vectors or tensors in the dual algebra. To identify dual algebras, we consider only how the nine translations orthogonal to the duality direction transform.

Under space/space dualities, the operators are related by (2.19), and the $\alpha$ matrices remain in standard form, following (3.31). It follows that under space/space duality IIA is mapped to IIB and IIA$^*$ to IIB$^*$. This is indicated by the two vertical double-sided arrows in the middle of table 3. Under time/time dualities, the bosonic operators are related in the same way as in the previous case (according to (3.35)). However, a unit $\alpha^{(A)}$ matrix is, according to (3.37), mapped to a $\tau_3$ matrix. Therefore, $Y^{(0)}$ and $Y^{(3)}$ are interchanged by the transformation (4.9), resulting in the $\mathrm{T_{tt}}$ rules indicated in the table. Finally, under space/time dualities, the maps of the generators in (3.38) are the same, but the $\alpha^{(B)}_{9,1}$ matrix is proportional to $\tau_1$ according to (3.40). When we start from IIA in (10, 0), the translation generator is proportional to the unit matrix in $(ij)$ space, which is, by the mapping (4.9), transformed to $\tau^1$. Therefore, it is mapped to the IIB$'$ translation generator. On the other hand, in the IIA$'$ theory, the translation generator is proportional to $\tau_3$, which is invariant under the redefinitions. Thus IIA$'$ is mapped to IIB$^*$. These two space/time duality relations are indicated by the lower two double-sided arrows in table 3.

### 4.3. The energy operator

We will now clarify the uniqueness of the energy operator†. A first technical issue is the positivity of states generated by supersymmetry creation operators. We remark that $\{Q^i, Q^{j\,*}\}$

† Of course, one can redefine the energy operator by adding some other charges and still obtain a positive-definite operator. However, then the amount of mixture of the other charges is limited, and our energy operator still has to appear in the combination.

cannot be positive in the case of $\beta = -1$, as its Hermitian conjugate is $\beta\{Q^{j*}, Q^i\} = \beta\{Q^i, Q^{j*}\}$, and is thus not even real for $\beta = -1$. This indicates that positivity for fermion bilinears is not a trivial issue. We can define the positivity for our purposes by taking for a square root of $\beta$

$$\sqrt{\beta}\left\{Q^i, Q^{j*}\right\} = \sqrt{\beta}\left\{Q^i, Q^{kC}\right\}\alpha^j{}_k B^T \geqslant 0. \tag{4.11}$$

Adopting the notation

$$\left\{Q^i, Q^{kC}\right\} = \mathcal{Z}^{ik}\mathcal{C}^{-1}, \tag{4.12}$$

where we have now written the charge conjugation explicitly, we obtain the positivity condition

$$\sqrt{\beta}\mathcal{Z}^{ik}\alpha^j{}_k\Gamma^t\cdots\Gamma^1 \geqslant 0, \tag{4.13}$$

where the timelike gamma matrices appear. Splitting $\mathcal{Z}$ in $\Gamma^{(\Lambda)}Z_\Lambda$ where $\Gamma^{(\Lambda)}$ are all the basic elements of the Clifford algebra, we find that the operator connected to $\Gamma^{1\cdots t}$ contains the one that always has to be non-zero and positive. We further have to make a projection in the $(ij)$ space. Let us show this by an example: the IIB algebra in (9, 1) spacetime. For a Minkowskian signature equation (4.13) contains only the timelike $\Gamma$ matrix, say $\Gamma^0$. The only operators that we thus have to consider in (2.9) are $Y_0^{ij}$. The condition (4.13) for states with all other central charges zero reads

$$\sqrt{\beta}Y_0^{ik}\alpha^j{}_k > 0. \tag{4.14}$$

For instance, for $\beta = 1$ and $\alpha = \mathbb{1}$, the operator $Y_0^{(0)}$ in (2.10) is thus the one that has to be positive, and is identified as the energy operator. Without imposing $\beta = 1$ and $\alpha = \mathbb{1}$, the reality condition on $Y_0^{ij}$ is (3.12), and one can check the consistency that the positive combination (4.14) is then indeed also real due to $\alpha\alpha^* = \mathbb{1}$ and $(\sqrt{\beta})^*\beta = \sqrt{\beta}$.

Another illustrative example is provided by the case that the $\alpha$ matrix is given by (3.40). This case appears in the duality between the IIB (9, 1) and IIA (10, 0) algebra. We now find from (4.14) that

$$-\mathrm{i}\sqrt{\beta}Y_0^{(1)}\alpha_{10,0} \geqslant 0. \tag{4.15}$$

One checks again from (3.41) that this energy operator is real. Under space/time duality this operator is mapped onto the operator $-\sqrt{\beta}Z\alpha_{10,0}$ of the Euclidean IIA superalgebra. We thus conclude that the energy operator of the Euclidean IIA algebra is proportional to the scalar $Z$ generator. This is consistent with the general rule (4.13) in the case of a Euclidean space. We conclude from this example that the energy operator does not always occur as a component of the translation generator.

We found that the energy operator is a bosonic operator that appears in the anticommutator of two supersymmetries with the product of all timelike $\Gamma$ matrices $\Gamma^{1\cdots t}$. An immediate consequence of this is that it can only be one of the generators of translations if the space is Minkowskian. Indeed, the translations appear by definition with one $\Gamma$ matrix in this anticommutator. In dimension (10, 1), this is the usual situation. The time component of the translations is the energy operator. However, this is thus not the case in the (9, 2) theory, where the energy is part of the two-index central charge operator.

In non-Minkowskian signatures, energy is not part of the translations. In Euclidean spacetime, it is part of a scalar central charge. If there are two or more timelike directions, it is part of a tensor central charge.

Finally, for the (9, 1) theories, we can identify the energy operator as being the timelike component of the vector operator that is identified as the translation in the IIA and IIB theories.

Thus, the timelike component of the translation operators in IIA$^*$, IIB$^*$ or IIB$'$ algebras is *not* the positive-definite operator. In the corresponding field theories, the terms with a timelike derivative are often called 'kinetic-energy terms'. As this time coordinate corresponds to the timelike component of the translation generator, this is not what corresponds to the positive-definite 'energy' operator that we defined here. Therefore, the terms that appear with derivatives $\partial_t$ in the action, are not positive definite. This is indeed the characteristic feature of the IIA$^*$, IIB$^*$ or IIB$'$ Lagrangian theories.

## 5. Conclusions

In this paper we have studied how the symmetries of F-theories, M-theories and type II string theories exhibit many faces of the $OSp(1|32)$ algebra, rewriting it covariantly in dimensions 12, 11 and 10 of different signatures. One only has to distinguish the complex algebra and the unique real algebra. We have shown explicitly how, both in the complex and in the real case, the $d = 10$, 11 and 12 algebras are related by dimensional reduction. Furthermore, we have shown how the (complex and real) $d = 10$ IIA and IIB superalgebras are related by T-duality (see table 2). The identification of the translation generator in these algebras leads to the $\star$-algebras of [12]. This leads to a more detailed set of T-dualities whose structure is indicated in table 3.

From a physical point of view one might prefer the Minkowskian algebras above the other signatures since *a priori* it is not clear how to make sense out of non-Minkowskian signatures. On the other hand, we have seen that the different signatures are related to each other by a generalized space/time T-duality map. One of the lessons we have learned from the modern developments on dualities in string theory is that what looks strange in one picture can make perfect sense in the dual picture. Many-time physics could make sense within the context of an underlying string theory. This was the philosophy adapted in [12].

There are several ways in which string or M-theory hints at the relevance of non-Minkowskian signatures. For instance, they occur naturally in a classification of supermembranes [24]. Also, it has been suggested that string theory has a 12-dimensional F-theory origin with signature $(10, 2)$, i.e. two times [25]. For recent work on two-time physics, see [26] and references therein. Also for embeddings of branes in flat spaces one needs two time directions [27]. Two times also occur when one considers $N = 2$ superstrings leading to a $d = 4$ target spacetime of $(2, 2)$ signature [28]. It would be interesting to see whether the superalgebraic approach of this work could be applied to this case. Space/time T-duality might then relate physics in $(2, 2)$ dimensions to the Minkowskian $(3, 1)$ signature.

Let us now return to the original motivation of this paper, i.e. the study of type IIB string theory in $(10, 0)$ Euclidean dimensions. From table 2 we see that the $(10, 0)$ signature only allows a real Euclidean type IIA theory. Therefore, we confirm that, in order to discuss the IIB case, one needs complexification, as has been the case for other Euclidean theories [6]. Note that, after complexification, the theory does not refer to any specific signature. The original D-instanton must be considered in the context of this complex IIB theory.

At a first stage one can consider the complex T-dual of the D-instanton. Under this map the D-instanton will be mapped to a T-dual two-block solution of a complex IIA supergravity theory. At a second stage, one can impose reality conditions on the complex fields of the IIA supergravity theory. These reality conditions not only specify the signature but also, since the signs of the kinetic terms in the action are now fixed, the choice of the translation generator. The reality conditions thus lead to a Minkowskian supergravity theory based on either the IIA$_{9,1}$- or the IIA$^\star_{9,1}$-algebra. Since the D-instanton solution itself is real, we expect that the dual solution can be embedded either in the IIA$_{9,1}$- or the IIA$^\star_{9,1}$-supergravity theory. The first embedding corresponds to the standard D0-brane solution of a $\mathrm{IIA}_{9,1}$-supergravity theory as discussed in the introduction. The second T-dual picture of the D-instanton is, in the terminology of [12], an E1-brane of a $\mathrm{IIA}^{\star}_{9,1}$-supergravity theory.

It would be interesting to verify the existence of these two different T-dual pictures of the D-instanton by constructing explicitly the corresponding supergravity theories and T-duality maps.

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## Appendix A. Conventions

The conventions of this paper can be found in [21]. In any even dimension $d$, with $t$ timelike directions and $s$ spacelike dimensions, we define a chiral projection as follows:

$$\Gamma_* = (-\mathrm{i})^{(s-t)/2}\frac{1}{d!}\varepsilon_{a_1\cdots a_d}\Gamma^{a_1\cdots a_d}, \qquad \mathcal{P}^{\pm} = \tfrac{1}{2}(\mathbb{1} \pm \Gamma_*). \tag{A.1}$$

The dual of a rank-$n$ antisymmetric tensor in $d = 2n$ dimensions is defined by

$$\tilde{F}_{a_1\ldots a_n} = (\mathrm{i})^{d/2+t}\frac{1}{n!}\varepsilon_{a_1\ldots a_d}F^{a_d\ldots a_{n+1}}, \tag{A.2}$$

such that

$$\begin{aligned}&\tilde{\tilde{F}}_{a_1\ldots a_n} = F_{a_1\ldots a_n}, \qquad \Gamma_{a_1\ldots a_n} = \Gamma_*\tilde{\Gamma}_{a_1\ldots a_n},\\ &\tilde{F}_{a_1\ldots a_n}\tilde{G}^{a_1\ldots a_n} = (-)^n F_{a_1\ldots a_n}G^{a_1\ldots a_n}.\end{aligned} \tag{A.3}$$

These are used to define (anti-)self-dual tensors

$$F^{\pm} = \tfrac{1}{2}(F \pm \tilde{F}). \tag{A.4}$$

For the formulae of gamma matrices in dimensions 10 to 12, we only need the cases where $\epsilon = \eta = 1$ in the terminology of [21]. For 10 and 12 dimensions, this is a choice, while for 11 dimensions this is necessary. Therefore, for the purpose of this paper, we can suffice with part of the general formalism. With these choices, the essential ingredients are the following. One has a charge conjugation matrix $\mathcal{C}$, which is unitary and antisymmetric, and satisfies the property

$$\Gamma_a^T = -\mathcal{C}\Gamma_a\mathcal{C}^{-1}. \tag{A.5}$$

Another unitary matrix that plays an important role in the definition of complex conjugation of spinors (see below) is

$$B = -\mathcal{C}\Gamma_1\cdots\Gamma_t, \qquad B^{-1} = (-)^{(t-1)(t-2)/2}\Gamma_1\cdots\Gamma_t\mathcal{C}^{-1}, \tag{A.6}$$

where $\Gamma_1, \ldots, \Gamma_t$ are the gamma matrices in the $t$ timelike directions. An important property is

$$B^* B = (-)^{(t-1)(t-2)/2}. \tag{A.7}$$

We do not write spinor indices in the main text, and neither the charge conjugation matrices which should always appear at the right-hand side of anticommutation relations. For instance, we write

$$\left\{Q_\alpha, Q_\beta\right\} = (\Gamma^\mu)_\alpha{}^\gamma(\mathcal{C}^{-1})_{\gamma\beta}P_\mu \quad\longrightarrow\quad \{Q, Q\} = \Gamma^\mu P_\mu. \tag{A.8}$$

Using this shorthand notation $\Gamma_{a_1\cdots a_r}$ is symmetric for $r = 1, 2 \bmod 4$, and antisymmetric for $r = 0, 3 \bmod 4$.

Complex conjugation can be conveniently performed by replacing it with 'charge conjugation'. This operation works on spinors as

$$\lambda^C = \frac{1}{\alpha}B^{-1}\lambda^*, \tag{A.9}$$

where $\alpha$ is an arbitrary phase factor. When there are several spinors $\lambda^i$, then $\alpha$ may even be a matrix $\alpha^i{}_j$, such that

$$\alpha^{*i}{}_j\alpha^j{}_k = \delta^i{}_k. \tag{A.10}$$

On matrices in spinor space, $M$, charge conjugation acts as

$$M^C = B^{-1}M^*B. \tag{A.11}$$

On spinors, the square of the $C$ operation is determined by (A.7), i.e. it is the identity for $t = 1, 2 \bmod 4$, and minus the identity for $t = 0, 3 \bmod 4$. A Majorana spinor is a spinor that is invariant under this charge conjugation, which is thus only possible† for $t = 1, 2 \bmod 4$.

The charge conjugation operator applied to anticommutation relations, amounts in practice to the following rules. In the left-hand side, replace the fermionic generators by their $C$ conjugate. In the right-hand side, add a factor $\beta\alpha^{-2}$ (if $\alpha$ is just a number), and apply the $C$ operation to the gamma matrices and generators (for which it is just Hermitian conjugation). The parameter $\beta$ is a sign that depends on the convention of whether complex conjugations maintain ($\beta = 1$) or interchange ($\beta = -1$) the order of fermions. One can then neglect the unwritten charge conjugation matrix. Thus, for example, applying it to (A.8), it gives

$$\left\{Q^C, Q^C\right\} = \alpha^{-1}\beta(\Gamma^\mu)^C P_\mu^*\alpha^{-1T}. \tag{A.12}$$

In this example we can commute $\alpha^{-1T}$ through $P_\mu$, but we write the expression in this way to indicate its form when $P_\mu$ is replaced by a non-trivial matrix in the space of several supersymmetries $Q^i$. Usually, $\alpha$ is just a number, and the factors thus combine to $\beta\alpha^{-2}$.

It is convenient to use a $\otimes$-notation for the bosonic generators that hides the prefactor. We thus define for all bosonic generators $Z$

$$Z^\otimes \equiv \beta\alpha^{-1}Z^*\alpha^{-1T}, \tag{A.13}$$

e.g. $P_\mu^\otimes = \beta\alpha^{-2}P_\mu^*$. In practice the factor $\beta$ is a choice which is independent of the spacetime dimension or signature. One could choose $\beta = 1$. On the other hand, the $\alpha$ factors vary for different spacetime dimensions and signatures, and are thus relevant.

† Note that for $t = 0 \bmod 4$, a Majorana spinor would be possible if we first replace $\mathcal{C}$ by $\mathrm{i}\mathcal{C}\Gamma_*$. This would amount to the other signs of parameters ($\epsilon = \eta = -1$) in [21]. However, in this paper we will always choose $\epsilon = \eta = 1$. With this choice, we cannot define Majorana spinors for $t = 0, 3 \bmod 4$, but we can define symplectic Majorana spinors (see table 1 in section 3).

## Appendix B. Gamma matrices in 10 to 12 dimensions

To distinguish the different gamma matrices in 10 to 12 Euclidean dimensions we use the following notation:

$$\begin{aligned} &d = 10: && \Gamma_M && M = 1, \ldots, 10, \\ &d = 11: && \tilde{\Gamma}_{\tilde{M}} && \tilde{M} = 1, \ldots, 10, 11, \\ &d = 12: && \hat{\Gamma}_{\hat{M}} && \hat{M} = 1, \ldots, 10, 11, 12. \end{aligned} \tag{B.1}$$

The gamma matrices for other spacetime signatures are obtained via the relations

$$\begin{aligned} \Gamma^t &= -\mathrm{i}\overset{0}{\Gamma}{}^t, \\ \Gamma^s &= \overset{0}{\Gamma}{}^s, \end{aligned} \tag{B.2}$$

for any timelike direction $t$ and spacelike direction $s$. Here $\overset{0}{\Gamma}$ are the gamma matrices corresponding to the Euclidean signature. It is thus sufficient to construct the Euclidean gamma matrices which, for simplicity, we denote in the following by $\Gamma$ instead of $\overset{0}{\Gamma}$.

Our starting point is the set of $32 \times 32$ matrices $\Gamma_M$ in 10 Euclidean dimensions which are the basic building blocks of our construction. Using the general definition (A.1) for $s = 10$, $t = 0$, we have

$$\Gamma_* = -\mathrm{i}\Gamma_1 \ldots \Gamma_{10} \equiv \Gamma_{11}. \tag{B.3}$$

The $\Gamma_M$ together with $\Gamma_{11}$ define the gamma matrices in 11 Euclidean dimensions.

To construct the gamma matrices in 12 Euclidean dimensions we define the $64 \times 64$ matrices

$$\begin{aligned} \hat{\Gamma}_M &= \Gamma_M \otimes \mathbb{1}_{32}, \\ \hat{\Gamma}_{11} &= \Gamma_{11} \otimes \sigma_1, \\ \hat{\Gamma}_{12} &= \Gamma_{11} \otimes \sigma_2, \\ \hat{\Gamma}_* &= -\hat{\Gamma}_1 \ldots \hat{\Gamma}_{12} = \Gamma_{11} \otimes \sigma_3. \end{aligned} \tag{B.4}$$

All charge conjugation matrices we use in this paper are unitary and antisymmetric, i.e.

$$\begin{aligned} &d = 10, 11: && C^\dagger C = \mathbb{1}_{32}, && C^T = -C, \\ &d = 12: && \hat{C}^\dagger \hat{C} = \mathbb{1}_{64}, && \hat{C}^T = -\hat{C}. \end{aligned} \tag{B.5}$$

In the notation of [21], we always use $C = C_+$ $(d = 10, 11)$ and $\hat{C} = \hat{C}_+$ $(d = 12)$, i.e. $\eta = \hat{\eta} = 1$. The charge conjugation matrix $C$ of 10 dimensions can also be used as charge conjugation in 11 dimensions. The $64 \times 64$ $d = 12$ charge conjugation matrix $\hat{C}$ is constructed as follows:

$$\hat{C} = C \otimes \sigma_1. \tag{B.6}$$

All these charge conjugation matrices satisfy the property (A.5).

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