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The Standard Model prediction for ε'/ε

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Abstract

We present a detailed analysis of ε'/ε within the Standard Model, taking into account the strong enhancement through final-state interactions identified by Pallante and Pich in Phys. Rev. Lett. 84 (2000) 2568 and Nucl. Phys. B 592 (2000) 294. The relevant hadronic matrix elements are fixed at leading order in the $1/N_C$ expansion, through a matching procedure between the effective short-distance Lagrangian and its corresponding low-energy description in Chiral Perturbation Theory. All large logarithms are summed up, both at short and long distances. Two different numerical analyses are performed, using either the experimental or the theoretical value for ε, with compatible results. We obtain $\mathrm{Re}(\varepsilon'/\varepsilon) = (1.7 \pm 0.9) \times 10^{-3}$. The error is dominated by the uncertainty in the value of the strange quark mass and the estimated corrections from unknown $1/N_C$-suppressed local contributions. A better estimate of the strange quark mass would reduce the uncertainty to about 30%. The Standard Model prediction agrees with the present experimental world average $\mathrm{Re}(\varepsilon'/\varepsilon) = (1.93 \pm 0.24) \times 10^{-3}$.

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1. Introduction

The CP-violating ratio ε'/ε constitutes a fundamental test for our understanding of flavour-changing phenomena within the Standard Model framework. It represents a great source of inspiration for physics research and has motivated in recent years a very interesting scientific controversy, both on the experimental and theoretical sides.

The experimental status [3,4] has been clarified recently. The CERN NA48 Collaboration [5] has announced a preliminary value

$$\mathrm{Re}(\varepsilon'/\varepsilon) = (1.40 \pm 0.43) \times 10^{-3}. \tag{1}$$

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A larger result was obtained by the Fermilab KTeV Collaboration [6],

$$\mathrm{Re}(\varepsilon'/\varepsilon) = (2.80 \pm 0.41) \times 10^{-3}. \tag{2}$$

The present world average [5],

$$\mathrm{Re}(\varepsilon'/\varepsilon) = (1.93 \pm 0.24) \times 10^{-3}, \tag{3}$$

provides clear evidence for a non-zero value and, therefore, direct CP violation phenomena.

The theoretical status is more involved and not very satisfactory. There is no universal agreement on the ε'/ε value predicted by the Standard Model, since different groups, using different models or approximations, obtain different results [7–16]. Nevertheless, it has been often claimed that the Standard Model predicts a too small value of ε'/ε, failing to reproduce its experimental world average by at least a factor of two. This claim has generated a very intense theoretical activity, searching for new sources of CP violation beyond the Standard Model framework [17].

It has been pointed out [1] that the theoretical short-distance evaluations of ε'/ε had overlooked the important role of final-state interactions (FSI) in $K \to \pi\pi$ decays. Although it has been known for more than a decade that the rescattering of the two final pions induces a large correction to the isospin-zero decay amplitude, this effect was not taken properly into account in the theoretical predictions. [1] From the measured π–π phase shifts one can easily infer that FSI generate a strong enhancement of the predicted ε'/ε value, by roughly the needed factor of two [1,2]. A detailed analysis of the corrections induced by FSI has been already given in Ref. [2], where the low-energy (infrared) physics involved has been investigated and the size of the FSI enhancement and the associated uncertainties have been quantified.

In this paper, we present a complete reevaluation of ε'/ε within the Standard Model. We will show that with our present understanding of the different inputs, it is possible to pin down the prediction of this important parameter with a theoretical accuracy of about 50%. In order to achieve this goal, one needs to identify the most important corrections and find appropriate expansion parameters to perform a perturbative approach with well-defined power counting.

The large-N_C expansion [18,19], with N_C the number of QCD colours, turns out to be a very useful tool to organize the calculation. It is a unique non-perturbative approach, with a clear meaning within the usual perturbative expansion in powers of the QCD coupling. At leading (non-trivial) order in $1/N_C$ it is possible to compute all needed ingredients and, what is even more important, the matching between short- and long-distance physics can be done exactly. Moreover, FSI are zero at leading order in $1/N_C$; this allows a clear separation of these corrections, avoiding any possible ambiguity or double-counting.

Since $N_C = 3$ in the real world, the natural size to be expected for the $1/N_C$-suppressed contributions is 30%. Actually, there is a quite compelling phenomenological evidence that

[1] Some pion rescattering corrections have been included in Refs. [9–11]. Although computed in a model-dependent way, those effects push their ε'/ε predictions to the correct 10^{-3} range, explaining the numerical discrepancies with the estimates done in Refs. [7,8,16] where FSI are totally ignored.

those corrections are usually smaller. For this to be true, however, one needs to make sure that the $1/N_C$ expansion does not involve large logarithms [20]; i.e., one should expand in powers of $1/N_C$ and not in powers of $\frac{1}{N_C}\ln(M/m)$, with $M \gg m$ two widely separated scales. Large logarithms are in fact the main source of complications in low-energy flavour-changing processes, because the electroweak scale M_W where the short-distance quark transition takes place is much larger than the long-distance hadronic scale.

The large short-distance logarithms can be summed up with the use of the Operator Product Expansion (OPE) [21] and the renormalization group [22]. The proper way to proceed makes use of modern Effective Field Theory techniques [23]. One starts above the electroweak scale where the flavour-changing process, in terms of quarks, leptons and gauge bosons, can be analyzed within the usual gauge-coupling perturbative expansion in a rather straightforward way. The renormalization group is used to evolve down in energy from the scale M_Z, where the top quark and the Z and $W^{\pm}$ bosons are integrated out. That means that one changes to a different Effective Theory where those heavy particles are no longer explicit degrees of freedom. The new Lagrangian contains a tower of operators constructed with the light fields only, which scale as powers of $1/M_Z$. The information on the heavy fields is hidden in the (Wilson) coefficients of those operators, which are fixed by "matching" the high- and low-energy theories at the point $\mu = M_Z$. One follows the evolution further to lower energies, using the Effective Theory renormalization group equations, until a new particle threshold is encountered. Then, the whole procedure of integrating the new heavy scale and matching to another Effective Field Theory starts again. In this way, one proceeds down to scales $\mu < m_c$.

In this picture, the physics is described by a chain of different Effective Field Theories, with different particle content, which match each other at the corresponding boundary (heavy threshold). This procedure permits to perform an explicit summation of large logarithms $t \equiv \ln(M/m)$, where M and m refer to any scales appearing in the evolution. One gets finally an effective $\Delta S = 1$ Lagrangian, defined in the three-flavour theory [24–27],

$$\mathcal{L}_{\text{eff}}^{\Delta S=1} = -\frac{G_F}{\sqrt{2}} V_{ud} V_{us}^* \sum_i C_i(\mu) Q_i(\mu), \tag{4}$$

which is a sum of local four-fermion operators Q_i, constructed with the light degrees of freedom,

$$\begin{aligned}
Q_1 &= (\bar{s}_\alpha u_\beta)_{\text{V-A}} (\bar{u}_\beta d_\alpha)_{\text{V-A}}, \\
Q_2 &= (\bar{s}u)_{\text{V-A}} (\bar{u}d)_{\text{V-A}}, \\
Q_{3,5} &= (\bar{s}d)_{\text{V-A}} \sum_q (\bar{q}q)_{\text{V}\mp\text{A}}, \\
Q_{4,6} &= (\bar{s}_\alpha d_\beta)_{\text{V-A}} \sum_q (\bar{q}_\beta q_\alpha)_{\text{V}\mp\text{A}}, \\
Q_{7,9} &= \frac{3}{2} (\bar{s}d)_{\text{V-A}} \sum_q e_q (\bar{q}q)_{\text{V}\pm\text{A}},
\end{aligned}$$

$$Q_{8,10} = \frac{3}{2}(\bar{s}_\alpha d_\beta)_{V-A} \sum_q e_q (\bar{q}_\beta q_\alpha)_{V\pm A}, \tag{5}$$

modulated by Wilson coefficients $C_i(\mu)$ which are functions of the heavy masses. Here α, β denote colour indices and e_q are the quark charges ($e_u = 2/3$, $e_d = e_s = -1/3$). Colour indices for the colour singlet operators are omitted. The labels $(V \pm A)$ refer to the Dirac structures $\gamma_\mu(1 \pm \gamma_5)$.

We have explicitly factored out the Fermi coupling G_F and the Cabibbo–Kobayashi–Maskawa (CKM) matrix elements V_{ij} containing the usual Cabibbo suppression of K decays. The unitarity of the CKM matrix allows to write the Wilson coefficients in the form

$$C_i(\mu) = z_i(\mu) + \tau\ y_i(\mu), \tag{6}$$

where $\tau = -V_{td}V_{ts}^*/V_{ud}V_{us}^*$. The CP-violating decay amplitudes are proportional to the y_i components.

The overall renormalization scale μ separates the short- ($M > \mu$) and long- ($m < \mu$) distance contributions, which are contained in $C_i(\mu)$ and Q_i, respectively. The physical amplitudes are independent of μ; thus, the explicit scale (and scheme) dependence of the Wilson coefficients should cancel exactly with the corresponding dependence of the Q_i matrix elements between on-shell states.

Our knowledge of $\Delta S = 1$ transitions has improved qualitatively in recent years, thanks to the completion of the next-to-leading logarithmic-order calculation of the Wilson coefficients [28,29]. All gluonic corrections of $\mathcal{O}(\alpha_s^n t^n)$ and $\mathcal{O}(\alpha_s^{n+1} t^n)$ are already known. Moreover the full m_t/M_W dependence (to first order in α_s and α) has been taken into account at the electroweak scale. We will fully use this information up to scales $\mu \sim \mathcal{O}(1\ \text{GeV})$, without making any unnecessary expansion in powers of $1/N_C$.

In order to predict physical amplitudes one is still confronted with the calculation of hadronic matrix elements of quark operators. This is a very difficult problem, which so far remains unsolved. As indicated in Fig. 1, below the resonance region one can use global symmetry considerations to define another Effective Field Theory in terms of the QCD Goldstone bosons (π, K, η). The Chiral Perturbation Theory (χPT) formulation of the Standard Model [30–34] is an ideal framework to describe the pseudoscalar-octet dynamics, through a perturbative expansion in powers of momenta and light quark masses over the chiral symmetry breaking scale ($\Lambda_\chi \sim 1$ GeV). Chiral symmetry fixes the allowed χPT operators, at a given order in momenta. The only remaining problem is then the calculation of the corresponding chiral couplings from the effective short-distance Lagrangian; this requires to perform the matching between the two Effective Field Theories.

It is here where the $1/N_C$ expansion proves to be useful. At leading order in $1/N_C$, the matching between the 3-flavour quark theory and χPT can be done exactly. We will determine the needed chiral couplings in the large-N_C limit, in a quite straightforward way. The scale and scheme dependences of the short-distance Wilson coefficients are of course completely removed in the matching process, at leading order in $1/N_C$. Any remaining

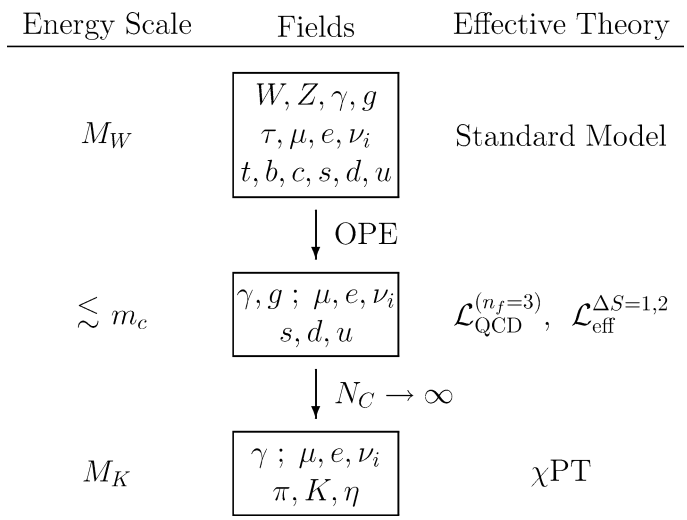

Fig. 1. Evolution from M_W to M_K.

dependences are higher-order in the $1/N_C$ expansion and, thus, numerically suppressed; they are included in our estimated theoretical uncertainty.

There is still an important source of large logarithms that needs to be identified and kept under control. The FSI of the pseudo-Goldstone pions generate large infrared logarithms, involving the light pion mass, which are next-to-leading in $1/N_C$. These chiral logarithms can be computed within the effective χPT framework. Moreover, as shown in Refs. [1,2] they can be exponentiated to all orders in the momentum expansion. Since this is a $1/N_C$ suppressed (but numerically large) effect, it generates an important correction, not included in the previous leading-order determination of chiral couplings.

The paper is organized as follows. The usual isospin formalism for $K \to \pi\pi$ decays and the relevant formulae for ε'/ε are collected in Section 2. Section 3 presents the low-energy χPT description. The matching between the short- and long-distance effective theories is performed in Section 4, at leading order in $1/N_C$. Section 5 summarizes the large-N_C predictions for the different isospin amplitudes. The one-loop chiral corrections are discussed in Section 6. Section 7 incorporates higher-order corrections induced by FSI, within the chiral framework. The Standard Model prediction for ε'/ε is worked out in Section 8, where two different numerical analyses are presented. The first one incorporates the experimental value of ε, while in the second one its theoretical prediction is used instead. Both analyses give compatible results. Our conclusions are finally given in Section 9. We have collected in several appendices the analytical results from the one-loop chiral calculation of the different $K \to \pi\pi$ amplitudes.

2. $K \to \pi\pi$ amplitudes

We adopt the usual isospin decomposition:

$$A\big[K^0 \to \pi^+\pi^-\big] \equiv \mathcal{A}_0 + \frac{1}{\sqrt{2}}\,\mathcal{A}_2,$$
$$A\big[K^0 \to \pi^0\pi^0\big] \equiv \mathcal{A}_0 - \sqrt{2}\,\mathcal{A}_2. \tag{7}$$

The complete amplitudes $\mathcal{A}_I \equiv A_I \exp\{i\delta_0^I\}$ include the strong phase shifts δ_0^I. The S-wave π–π scattering generates a large phase-shift difference between the $I = 0$ and $I = 2$ partial waves [35]:

$$\big(\delta_0^0 - \delta_0^2\big)\big(M_K^2\big) = 45^\circ \pm 6^\circ. \tag{8}$$

There is a corresponding dispersive FSI effect in the moduli of the isospin amplitudes, because the real and imaginary parts are related by analyticity and unitarity. The presence of such a large phase-shift difference clearly signals an important FSI contribution to A_I.

In terms of the $K \to \pi\pi$ isospin amplitudes,

$$\frac{\varepsilon'}{\varepsilon} = e^{i\Phi}\,\frac{\omega}{\sqrt{2}|\varepsilon|}\left[\frac{\mathrm{Im}(A_2)}{\mathrm{Re}(A_2)} - \frac{\mathrm{Im}(A_0)}{\mathrm{Re}(A_0)}\right]. \tag{9}$$

Owing to the well-known "$\Delta I = 1/2$ rule", ε'/ε is suppressed by the ratio

$$\omega = \mathrm{Re}(A_2)/\,\mathrm{Re}(A_0) \approx 1/22. \tag{10}$$

The phases of ε' and ε turn out to be nearly equal:

$$\Phi \approx \delta_0^2 - \delta_0^0 + \frac{\pi}{4} \approx 0. \tag{11}$$

The CP-conserving amplitudes $\mathrm{Re}(A_I)$, their ratio ω and $|\varepsilon|$ are usually set to their experimentally determined values. A theoretical calculation is then only needed for $\mathrm{Im}(A_I)$.

Using the short-distance Lagrangian (4), the CP-violating ratio ε'/ε can be written as [7]

$$\frac{\varepsilon'}{\varepsilon} = \mathrm{Im}\big(V_{ts}^* V_{td}\big)e^{i\Phi}\,\frac{G_F}{2|\varepsilon|}\,\frac{\omega}{|\,\mathrm{Re}(A_0)|}\left[P^{(0)}(1-\Omega_{\mathrm{IB}}) - \frac{1}{\omega}P^{(2)}\right], \tag{12}$$

where the quantities

$$P^{(I)} = \sum_i y_i(\mu)\langle(\pi\pi)_I|Q_i|K\rangle \tag{13}$$

contain the contributions from hadronic matrix elements with isospin I and

$$\Omega_{\mathrm{IB}} = \frac{1}{\omega}\,\frac{\mathrm{Im}(A_2)_{\mathrm{IB}}}{\mathrm{Im}(A_0)} \tag{14}$$

parameterizes isospin breaking corrections. The factor $1/\omega$ enhances the relative weight of the $I = 2$ contributions.

The hadronic matrix elements $\langle(\pi\pi)_I|Q_i|K\rangle$ are usually parameterized in terms of the so-called bag parameters B_i, which measure them in units of their vacuum insertion

approximation values. In the Standard Model, $P^{(0)}$ and $P^{(2)}$ turn out to be dominated by the contributions from the QCD penguin operator Q_6 and the electroweak penguin operator Q_8, respectively [9]. Thus, to a very good approximation, ε'/ε can be written (up to global factors) as [7]

$$\frac{\varepsilon'}{\varepsilon} \sim \left[B_6^{(1/2)}(1-\Omega_{\rm IB}) - 0.4B_8^{(3/2)}\right]. \tag{15}$$

The isospin-breaking correction coming from π^0–η mixing was originally estimated to be $\Omega_{\rm IB}^{\pi^0\eta} = 0.25$ [36,37]. Together with the usual ansatz $B_i \sim 1$, this produces a large numerical cancellation in Eq. (15) leading to low values of ε'/ε around 7×10^{-4}. A recent improved calculation of π^0–η mixing at $\mathcal{O}(p^4)$ in χPT has found the result [38]

$$\Omega_{\rm IB}^{\pi^0\eta} = 0.16 \pm 0.03. \tag{16}$$

This smaller number, slightly increases the naive estimate of ε'/ε.

3. Chiral Perturbation Theory description

In the limit m_u, m_d, $m_s \to 0$, the QCD Lagrangian for light quarks has a $SU(3)_L \otimes SU(3)_R$ symmetry, which is spontaneously broken to $SU(3)_V$. The lightest particles of the hadronic spectrum, the pseudoscalar octet (π, K, η), can be identified with the corresponding Goldstone bosons. Their low-energy interactions can be analyzed within χPT [30–34], which is an expansion in terms of momenta and meson (quark) masses. The Goldstone fields are parameterized as

$$\boldsymbol{\Phi} = \begin{pmatrix} \sqrt{\frac{1}{2}}\,\pi^0 + \sqrt{\frac{1}{6}}\,\eta & \pi^+ & K^+ \\ \pi^- & -\sqrt{\frac{1}{2}}\,\pi^0 + \sqrt{\frac{1}{6}}\,\eta & K^0 \\ K^- & \overline{K}^0 & -\sqrt{\frac{2}{3}}\,\eta \end{pmatrix}, \tag{17}$$

and appear in the Lagrangian via the exponential representation $U = \exp(\sqrt{2}\,i\boldsymbol{\Phi}/f)$, with $f \sim f_\pi = 92.4$ MeV the pion decay constant at lowest order. Under a chiral transformation $g \equiv (g_L, g_R) \in SU(3)_L \otimes SU(3)_R$, the matrix U changes as $U \to g_R U g_L^\dagger$.

The effect of strangeness-changing non-leptonic weak interactions with $\Delta S = 1$ is incorporated [39] in the low-energy chiral theory as a perturbation to the strong effective Lagrangian. At lowest order, the most general effective bosonic Lagrangian, with the same $SU(3)_L \otimes SU(3)_R$ transformation properties and quantum numbers as the short-distance Lagrangian (4), contains three terms:

$$\mathcal{L}_2^{\Delta S=1} = -\frac{G_F}{\sqrt{2}} V_{ud} V_{us}^* f^4 \left\{ g_8 \left[\langle \lambda L_\mu L^\mu \rangle + e^2 f^2 g_{\rm ew} \langle \lambda U^\dagger Q U \rangle\right] + g_{27}\left(L_{\mu 23} L_{11}^\mu + \frac{2}{3} L_{\mu 21} L_{13}^\mu \right) \right\}, \tag{18}$$

where the matrix $L_\mu = -iU^\dagger D_\mu U$ represents the octet of $V-A$ currents, at lowest order in derivatives, $\mathcal{Q} = \mathrm{diag}(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3})$ is the quark charge matrix, $\lambda \equiv (\lambda^6 - i\lambda^7)/2$ projects onto the $\bar{s} \to \bar{d}$ transition [$\lambda_{ij} = \delta_{i3}\delta_{j2}$] and $\langle \mathrm{A} \rangle$ denotes the flavour trace of A.

The chiral couplings g_8 and g_{27} measure the strength of the two parts of the effective Lagrangian (4) transforming as $(8_L, 1_R)$ and $(27_L, 1_R)$, respectively, under chiral rotations. Chiral symmetry forces the lowest-order Lagrangian to contain at least two derivatives (Goldstone bosons are free particles at zero momenta). In the presence of electroweak interactions, however, the explicit breaking of chiral symmetry generated by the quark charge matrix $\mathcal{Q}$ induces the $\mathcal{O}(p^0)$ operator $\langle \lambda U^\dagger \mathcal{Q} U \rangle$ [40,41], transforming as $(8_L, 8_R)$ under the chiral group. In the usual chiral counting $e^2 \sim \mathcal{O}(p^2)$ and, therefore, the g_{ew} term appears at the same order in the derivative expansion than g_8 and g_{27}. One additional term [42] proportional to the quark mass matrix, which transforms as $(8_L, 1_R)$, has not been written in the lowest-order Lagrangian (18), since it does not contribute [2] to physical $K \to \pi\pi$ matrix elements [43–45].

The tree-level $K \to \pi\pi$ amplitudes generated by the $\mathcal{O}(p^2)$ χPT Lagrangian (18) are:

$$\begin{aligned}
\mathcal{A}_0 &= -\frac{G_F}{\sqrt{2}} V_{ud} V_{us}^* \sqrt{2}\, f \left\{ \left(g_8 + \frac{1}{9} g_{27} \right) \left(M_K^2 - M_\pi^2 \right) - \frac{2}{3} f^2 e^2 g_8 g_{\mathrm{ew}} \right\}, \\
\mathcal{A}_2 &= -\frac{G_F}{\sqrt{2}} V_{ud} V_{us}^* \frac{2}{9} f \left\{ 5 g_{27} \left(M_K^2 - M_\pi^2 \right) - 3 f^2 e^2 g_8 g_{\mathrm{ew}} \right\}.
\end{aligned} \tag{19}$$

The strong phase shifts are zero at lowest order. Taking the measured phase shifts into account, the moduli of g_8 and g_{27} can be extracted from the CP-conserving $K \to 2\pi$ decay rates. A lowest-order phenomenological analysis [46], neglecting [3] the tiny electroweak corrections proportional to $e^2 g_{\mathrm{ew}}$, gives:

$$|g_8| \simeq 5.1, \qquad |g_{27}| \simeq 0.29. \tag{20}$$

The huge difference between these two couplings shows the well-known enhancement of octet $|\Delta I| = 1/2$ transitions.

The isospin amplitudes $\mathcal{A}_I$ have been computed up to next-to-leading order in the chiral expansion [44,45,47–50]. Decomposing the isoscalar amplitudes in their octet and 27-plet components as $\mathcal{A}_0 = \mathcal{A}_0^{(8)} + \mathcal{A}_0^{(27)}$, the results of those calculations can be written in the form:

$$\begin{aligned}
\mathcal{A}_0^{(8)} = -\frac{G_F}{\sqrt{2}} V_{ud} V_{us}^* \sqrt{2}\, f_\pi g_8 \Big\{ & \left(M_K^2 - M_\pi^2 \right) \left[1 + \Delta_L \mathcal{A}_0^{(8)} + \Delta_C \mathcal{A}_0^{(8)} \right] \\
& - \frac{2}{3} e^2 f_\pi^2 \left[g_{\mathrm{ew}} \left(1 + \Delta_L \mathcal{A}_0^{(\mathrm{ew})} \right) + \Delta_C \mathcal{A}_0^{(\mathrm{ew})} \right] \Big\}
\end{aligned} \tag{21}$$

[2] The contributions of this term to $K \to \pi\pi$ amplitudes vanish at $\mathcal{O}(p^2)$, while at $\mathcal{O}(p^4)$ they can be reabsorbed through a redefinition of the local $\mathcal{O}(p^4)$ $\Delta S = 1$ chiral couplings [43–45].

[3] A general analysis of isospin breaking and electromagnetic corrections to $K \to \pi\pi$ transitions is presented in Refs. [47–49].

for the octet isoscalar amplitude,

$$\mathcal{A}_0^{(27)} = -\frac{G_F}{\sqrt{2}} V_{ud} V_{us}^* \frac{\sqrt{2}}{9} f_\pi g_{27} \big(M_K^2 - M_\pi^2\big)\big[1 + \Delta_L \mathcal{A}_0^{(27)} + \Delta_C \mathcal{A}_0^{(27)}\big] \tag{22}$$

for the 27-plet isoscalar amplitude and

$$\mathcal{A}_2 = -\frac{G_F}{\sqrt{2}} V_{ud} V_{us}^* \frac{2}{9} f_\pi \Big\{ 5 g_{27} \big(M_K^2 - M_\pi^2\big)\big[1 + \Delta_L \mathcal{A}_2^{(27)} + \Delta_C \mathcal{A}_2^{(27)}\big] - 3\, e^2\, f_\pi^2 g_8 \big[g_{\rm ew}\big(1 + \Delta_L \mathcal{A}_2^{(\rm ew)}\big) + \Delta_C \mathcal{A}_2^{(\rm ew)}\big]\Big\} \tag{23}$$

for the $I = 2$ amplitude. The electroweak penguin contributions have been also included. These formulae contain chiral loop corrections $\Delta_L \mathcal{A}_I^{(R)}$, coming from the lowest-order Lagrangian (18) and its strong counterpart. Loop corrections are always subleading in the $1/N_C$ expansion, so that they do not enter the large-N_C matching procedure outlined in the introduction. One-loop corrections to $K \to \pi\pi$ have been extensively analyzed in Ref. [2], with the aim of identifying and resum FSI effects. Those effects, subleading in $1/N_C$ but numerically relevant, will be taken into account in Sections 6 and 7.

At next-to-leading order in the chiral expansion, i.e., $\mathcal{O}(G_F p^4)$ and $\mathcal{O}(G_F e^2 p^2)$, the complete Lagrangian which mediates non-leptonic weak interactions with $\Delta S = 1$ can be written as follows [44,45,47–51]:

$$\mathcal{L}_4^{\Delta S=1} = -\frac{G_F}{\sqrt{2}} V_{ud} V_{us}^* f^2 \left(g_8 \sum_i E_i O_i^8 + g_{27} \sum_i D_i O_i^{27} + g_8 e^2 f^2 \sum_i Z_i O_i^{\rm EW} \right). \tag{24}$$

For the octet and 27-plet weak operators O_i^8 and O_i^{27} the basis constructed in Ref. [45] has been adopted. [4] For the electroweak operators $O_i^{\rm EW}$ we use the basis [5] of Ref. [47]. We refer to those references for the explicit form of the operators.

The $\mathcal{O}(p^4)$ and $\mathcal{O}(e^2 p^2)$ tree-level contributions to the $K \to \pi\pi$ amplitudes are easily computed with the Lagrangian (24) and its strong counterpart. The complete expressions can also be obtained from Refs. [45] and [47]:

$$\begin{aligned} \Delta_C \mathcal{A}_0^{(8)} &= \tilde{\Delta}_C + \frac{2M_K^2}{f_\pi^2}(E_{10} - 2E_{13} + E_{15}) \\ &\quad + \frac{2M_\pi^2}{f_\pi^2}(-2E_1 - 4E_2 - 2E_3 + 2E_{10} + E_{11} + 4E_{13}), \end{aligned} \tag{25}$$

$$\begin{aligned} \Delta_C \mathcal{A}_0^{(27)} &= \tilde{\Delta}_C + \frac{M_K^2}{f_\pi^2}(D_4 - D_5 - 9D_6 + 4D_7) \\ &\quad + \frac{2M_\pi^2}{f_\pi^2}(-6D_1 - 2D_2 + 2D_4 + 6D_6 + D_7), \end{aligned} \tag{26}$$

[4] For the octet operators one can use either the basis of Ref. [45] or the basis of Ref. [50]. For completeness we provide the transformation rules between the two bases in Appendix A.

[5] Our operators $O_i^{\rm EW}$ are denoted with Q_i in Ref. [47] and their coupling G_8 is related to our g_8 via the identity $G_8 = -(G_F/\sqrt{2}) V_{ud} V_{us}^* g_8$.

$$\Delta_C\mathcal{A}_0^{(\mathrm{ew})} = g_{\mathrm{ew}}\tilde{\Delta}_C^{(\mathrm{ew})} + \frac{2M_K^2}{f_\pi^2}(Z_1+2Z_2) + \frac{M_\pi^2}{f_\pi^2}(4Z_1+2Z_2-Z_6) - \frac{M_K^2-M_\pi^2}{6f_\pi^2}(8Z_3-24Z_4+9Z_5+6Z_7-3Z_8-3Z_9 - 2Z_{10}+2Z_{11}+2Z_{12}), \tag{27}$$

$$\Delta_C\mathcal{A}_2^{(27)} = \tilde{\Delta}_C + \frac{M_K^2}{f_\pi^2}(D_4-D_5+4D_7) + \frac{2M_\pi^2}{f_\pi^2}(-2D_2+2D_4+D_7), \tag{28}$$

$$\Delta_C\mathcal{A}_2^{(\mathrm{ew})} = g_{\mathrm{ew}}\tilde{\Delta}_C^{(\mathrm{ew})} + \frac{M_K^2}{f_\pi^2}(2Z_1+4Z_2-Z_6) + \frac{2M_\pi^2}{f_\pi^2}(2Z_1+Z_2) + \frac{M_K^2-M_\pi^2}{3f_\pi^2}(-4Z_3+12Z_4-3Z_8-3Z_9-2Z_{10}+2Z_{11}+2Z_{12}), \tag{29}$$

where

$$\tilde{\Delta}_C = -\frac{4L_5}{f_\pi^2}\big(M_K^2+3M_\pi^2\big) - \frac{16L_4}{f_\pi^2}\big(2M_K^2+M_\pi^2\big),$$
$$\tilde{\Delta}_C^{(\mathrm{ew})} = -\frac{4L_5}{f_\pi^2}\big(M_K^2+5M_\pi^2\big) - \frac{24L_4}{f_\pi^2}\big(2M_K^2+M_\pi^2\big). \tag{30}$$

There are seven $(8_L,1_R)$ operators O_i^8 $(i=1,2,3,10,11,13,15)$, six $(27_L,1_R)$ operators O_i^{27} $(i=1,2,4,5,6,7)$ and twelve electroweak operators O_i^{EW} $(i=1,\ldots,12)$ contributing to $K\to\pi\pi$ matrix elements [45,47]. The practical limitation of a systematic χPT evaluation of the $K\to\pi\pi$ isospin amplitudes is in the fact that the counterterms which appear at next-to-leading order are not fully known and their determination would require the experimental knowledge of a large set of weak $\Delta S=1$ processes.

In addition, there are contributions involving the lowest-order $\Delta S=1$ Lagrangian (18) combined with the $\mathcal{O}(p^4)$ strong chiral operators with couplings L_i, introduced in Ref. [31]. In previous analyses [44,45,48,49] these corrections, shown in Eqs. (30), were factorized as global factors in front of the corresponding amplitudes: $1+\tilde{\Delta}_C \doteq f^4/(f_\pi^3 f_K)\approx 0.65$, $1+\tilde{\Delta}_C^{(\mathrm{ew})} \doteq f^6/(f_\pi^5 f_K)\approx 0.58$. A factor $f^3/(f_\pi^2 f_K)$ arises from wave-function renormalization, while the remaining powers of f/f_π are needed to rewrite in terms of the physical pion decay constant the explicit dependences of the tree-level amplitudes (19) on the chiral Lagrangian coupling f. This procedure induces a sizeable suppression which is finally compensated by large and positive corrections from the $\mathcal{O}(p^4)$ weak counterterms. We prefer to keep all $\mathcal{O}(p^4)$ local contributions together and perform a consistent large-N_C calculation of their global size.

4. Large-N_C matching

In the large-N_C limit the T-product of two colour-singlet quark currents factorizes:

$$\langle J\cdot J\rangle = \langle J\rangle\langle J\rangle\left\{1+\mathcal{O}\left(\frac{1}{N_c}\right)\right\}. \tag{31}$$

In other words, colour exchanges between the two currents J are $1/N_C$ suppressed and in this limit the factorization of four-quark operators is exact. Since quark currents have well-known realizations in χPT [31–33], the hadronization of the weak operators Q_i can then be done in a quite straightforward way. Thus, at large-N_C the matching between the short-distance Lagrangian (4) and its long-distance χPT realization can be explicitly performed.

The chiral couplings of the lowest-order Lagrangian (18) have the following large-N_C values:

$$\begin{aligned} g_8^\infty &= -\frac{2}{5}C_1(\mu) + \frac{3}{5}C_2(\mu) + C_4(\mu) - 16L_5\left(\frac{\langle\bar{q}q\rangle^{(2)}(\mu)}{f^3}\right)^2 C_6(\mu), \\ g_{27}^\infty &= \frac{3}{5}\big[C_1(\mu) + C_2(\mu)\big], \\ \big(g_8 e^2 g_{\rm ew}\big)^\infty &= -3\left(\frac{\langle\bar{q}q\rangle^{(2)}(\mu)}{f^3}\right)^2 C_8(\mu). \end{aligned} \tag{32}$$

Together with the $\mathcal{O}(p^2)$ amplitudes in Eqs. (19), these results are equivalent to the standard large-N_C evaluation of the usual bag parameters B_i. In particular, for ε'/ε, where only the imaginary part of the g_i couplings matter [i.e., $\mathrm{Im}(C_i)$], Eqs. (32) amount to $B_8^{(3/2)} \approx B_6^{(1/2)} = 1$. Therefore, up to minor variations on some input parameters, the corresponding ε'/ε prediction, obtained at lowest order in both the $1/N_C$ and χPT expansions, reproduces the published results of the Munich [7] and Rome [8] groups.

The large-N_C limit has been only applied to the matching between the 3-flavour quark theory and χPT, as indicated in Fig. 1. The evolution from the electroweak scale down to $\mu < m_c$ has to be done without any unnecessary expansion in powers of $1/N_C$; otherwise, one would miss large corrections of the form $\frac{1}{N_C}\ln(M/m)$, with $M \gg m$ two widely separated scales [20]. Thus, the Wilson coefficients contain the full μ dependence.

The operators Q_i $(i \neq 6, 8)$ factorize into products of left- and right-handed vector currents, which are renormalization-invariant quantities. The matrix element of each single current represents a physical observable which can be directly measured; its χPT realization just provides a low-energy expansion in powers of masses and momenta. Thus, the large-N_C factorization of these operators does not generate any scale dependence. Since the anomalous dimensions of Q_i $(i \neq 6, 8)$ vanish when $N_C \to \infty$ [20], a very important ingredient is lost in this limit [52]. To achieve a reliable expansion in powers of $1/N_C$, one needs to go to the next order where this physics is captured [52,53]. This is the reason why the study of the $\Delta I = 1/2$ rule has proved to be so difficult. Fortunately, these operators are numerically suppressed in the ε'/ε prediction.

The only anomalous dimensions which survive when $N_C \to \infty$ are the ones corresponding to Q_6 and Q_8 [20,37]. One can then expect that the matrix elements of these two operators are well approximated by this limit [6] [52–54]. These operators factorize into

[6] Some insight on these matrix elements can be obtained from the two-point functions $\Psi_{ii}(q^2) \equiv i\int d^4x e^{iqx}\langle T(Q_i(x)Q_i(0)^\dagger)\rangle$, since their absorptive parts correspond to an inclusive sum of hadronic matrix elements squared. The known $\mathcal{O}(\alpha_s)$ results [52–54] show that the large-N_C limit provides an excellent approximation to Ψ_{66}, but an incorrect description of Ψ_{22}.

colour-singlet scalar and pseudoscalar currents, which are μ dependent. Since the products $m_q\bar{q}(1,\gamma_5)q$, are physical observables, the scalar and pseudoscalar currents depend on μ like the inverse of a quark mass. Conversely, the Wilson coefficients of the operators Q_6 and Q_8 scale with μ like the square of a quark mass in the large-N_C limit.

The χPT evaluation of the scalar and pseudoscalar currents provides, of course, the right μ dependence, since only physical observables can be realized in the low-energy theory. What one actually finds is the chiral realization of the renormalization-invariant products $m_q\bar{q}(1,\gamma_5)q$. This generates the factors [$m_q \equiv m_u = m_d$]

$$\begin{aligned}\frac{\langle\bar{q}q\rangle^{(2)}(\mu)}{f^3} &\equiv -\frac{B_0}{f} = -\frac{B_0}{f_\pi}\frac{f_\pi}{f}\\ &= -\frac{M_\pi^2}{2m_q(\mu)f_\pi}\left[1+\frac{4L_5}{f_\pi^2}M_\pi^2+4\frac{2M_K^2+M_\pi^2}{f_\pi^2}(3L_4-4L_6)\right.\\ &\qquad\left.-8\frac{M_\pi^2}{f_\pi^2}(2L_8-L_5)-3v_\pi-v_K-\frac{1}{3}v_\eta\right]\\ &= -\frac{M_K^2}{(m_s+m_q)(\mu)f_\pi}\left[1+\frac{4L_5}{f_\pi^2}M_\pi^2+4\frac{2M_K^2+M_\pi^2}{f_\pi^2}(3L_4-4L_6)\right.\\ &\qquad\left.-8\frac{M_K^2}{f_\pi^2}(2L_8-L_5)-2v_\pi-v_K-\frac{2}{3}v_\eta\right],\end{aligned} \tag{33}$$

in Eqs. (32), which exactly cancel the μ dependence of $C_{6,8}(\mu)$ at large N_C [20,37,51–54]. It remains a dependence at next-to-leading order. The parameter B_0 is a low-energy coupling of the $\mathcal{O}(p^2)$ strong chiral Lagrangian, which accounts for the vacuum quark condensate at lowest order in the momentum expansion. The one-loop corrections v_P ($P = \pi, K, \eta$), defined in Appendix B, are identically zero in the limit $N_C \to \infty$.

While the real part of g_8 gets its main contribution from C_2, $\mathrm{Im}(g_8)$ and $\mathrm{Im}(g_8 g_{\mathrm{ew}})$ are governed by C_6 and C_8, respectively. Thus, the analyses of the CP-conserving and CP-violating amplitudes are very different. There are large $1/N_C$ corrections to $\mathrm{Re}(g_i)$ [52–54], which are needed to understand the observed enhancement of the $(8_L, 1_R)$ coupling. However, the large-N_C limit can be expected to give a good estimate of $\mathrm{Im}(g_i)$.

Contrary to the other Q_i operators, the leading-order contribution of Q_6 involves the coupling L_5 of the $\mathcal{O}(p^4)$ strong chiral Lagrangian. The large-N_C value of this chiral coupling can be estimated from the ratio of the kaon and pion decay constants:

$$L_5^\infty = \frac{f_\pi^2}{4(M_K^2-M_\pi^2)}\left(\frac{f_K}{f_\pi}-1\right) = 2.1\times 10^{-3}. \tag{34}$$

The Q_6 contribution dominates the numerical value of $\mathrm{Im}(g_8^\infty)$. In the large-N_C limit, the combined effect of all other operators only amounts to a 5% correction.

The $\mathcal{O}(p^4)$ corrections introduce dependences on three additional strong chiral couplings. At large N_C,

$$L_4^\infty = L_6^\infty = 0. \tag{35}$$

To determine L_8, we impose the stronger requirement of lowest-meson dominance [55,56] and assume that the scalar form factors vanish at infinite momentum transfer. This implies the relation [57]

$$(2L_8 - L_5)^\infty = 0, \tag{36}$$

which is well satisfied by the phenomenological determinations of those constants [31,58].

The operators Q_3 and Q_5 start to contribute at $\mathcal{O}(p^4)$, while the electroweak penguin operators Q_7, Q_9 and Q_{10} give their first contributions at $\mathcal{O}(e^2p^2)$. The large-N_C matching at the next-to-leading chiral order fixes the couplings E_i, D_i and Z_i of the long-distance chiral Lagrangian (24). We only quote the values of those couplings contributing to $K \to \pi\pi$ amplitudes.

For the $\mathcal{O}(p^4)$ couplings, one gets:

$$\begin{aligned}
(g_8E_1)^\infty &= -48X_{19}\left(\frac{\langle\bar{q}q\rangle^{(2)}(\mu)}{f^2}\right)^2 C_6(\mu),\\
(g_8E_2)^\infty &= -32X_{20}\left(\frac{\langle\bar{q}q\rangle^{(2)}(\mu)}{f^2}\right)^2 C_6(\mu),\\
(g_8E_3)^\infty &= -16X_{31}\left(\frac{\langle\bar{q}q\rangle^{(2)}(\mu)}{f^2}\right)^2 C_6(\mu),\\
(g_8E_{10})^\infty &= 2\,L_5\left[-\frac{2}{5}C_1(\mu) + \frac{3}{5}C_2(\mu) + C_4(\mu)\right]\\
&\quad - 8(2X_{14} + 2X_{15} + X_{38})\left(\frac{\langle\bar{q}q\rangle^{(2)}(\mu)}{f^2}\right)^2 C_6(\mu),\\
(g_8E_{11})^\infty &= 4L_5\left[-\frac{2}{5}C_1(\mu) + \frac{3}{5}C_2(\mu) + C_4(\mu)\right]\\
&\quad - 16(X_{15} + 2X_{17} - X_{38})\left(\frac{\langle\bar{q}q\rangle^{(2)}(\mu)}{f^2}\right)^2 C_6(\mu),\\
(g_8E_{13})^\infty &= 8(X_{15} - 4X_{16})\left(\frac{\langle\bar{q}q\rangle^{(2)}(\mu)}{f^2}\right)^2 C_6(\mu),\\
(g_8E_{15})^\infty &= 8(-X_{34} + X_{38})\left(\frac{\langle\bar{q}q\rangle^{(2)}(\mu)}{f^2}\right)^2 C_6(\mu),\\
(g_{27}D_4)^\infty &= 4\,L_5 g_{27}^\infty.
\end{aligned} \tag{37}$$

All other $(27_L, 1_R)$ couplings contributing to $K \to \pi\pi$ (D_1, D_2, D_5, D_6 and D_7) are zero at large-N_C.

The $\mathcal{O}(p^4)$ contributions from the operator Q_6 have been computed using the $\mathcal{O}(p^6)$ Lagrangian of Ref. [59]; the couplings X_i refer to the list of $\mathcal{O}(p^6)$ $SU(3)$ operators given in that reference. These couplings however are unknown, so in practice the Q_6 contribution is missing in Eqs. (37). The remaining terms are in agreement with the results obtained in Ref. [50].

The non-zero $\mathcal{O}(e^2p^2)$ couplings relevant for $K\to\pi\pi$ are:

$$\begin{aligned}
\big(g_8e^2Z_1\big)^\infty &= -24\left(\frac{\langle\bar{q}q\rangle^{(2)}(\mu)}{f^3}\right)^2 L_8C_8(\mu),\\
\big(g_8e^2Z_5\big)^\infty &= C_{10}(\mu),\\
\big(g_8e^2Z_6\big)^\infty &= -12\left(\frac{\langle\bar{q}q\rangle^{(2)}(\mu)}{f^3}\right)^2 L_5C_8(\mu),\\
\big(g_8e^2Z_8\big)^\infty &= \frac{3}{2}\big[C_9(\mu)+C_{10}(\mu)\big],\\
\big(g_8e^2Z_9\big)^\infty &= -\frac{3}{2}C_7(\mu).
\end{aligned}\tag{38}$$

5. Isospin amplitudes at leading order in $1/N_C$

Combining the results of the previous sections, one gets the predicted $K\to\pi\pi$ amplitudes at leading order in $1/N_C$. The different contributions to the isospin amplitudes take the following form:

$$\begin{aligned}
g_8^\infty\big[1+\Delta_C\mathcal{A}_0^{(8)}\big]^\infty = \Bigg\{&-\frac{2}{5}C_1(\mu)+\frac{3}{5}C_2(\mu)+C_4(\mu)\\
&-16L_5C_6(\mu)\left[\frac{M_K^2}{(m_s+m_q)(\mu)f_\pi}\right]^2\Bigg\}f_0^{K\pi}\big(M_\pi^2\big),
\end{aligned}\tag{39}$$

$$g_{27}^\infty\big[1+\Delta_C\mathcal{A}_0^{(27)}\big]^\infty = g_{27}^\infty\big[1+\Delta_C\mathcal{A}_2^{(27)}\big]^\infty = \frac{3}{5}\big[C_1(\mu)+C_2(\mu)\big]f_0^{K\pi}\big(M_\pi^2\big),\tag{40}$$

$$\begin{aligned}
e^2g_8^\infty\big[g_{\rm ew}+\Delta_C\mathcal{A}_0^{(\rm ew)}\big]^\infty = &-3C_8(\mu)\left[\frac{M_K^2}{(m_s+m_q)(\mu)f_\pi}\right]^2\left[1+\frac{4L_5}{f_\pi^2}M_K^2\right]\\
&-\frac{3}{4}[C_7-C_9+C_{10}](\mu)\frac{M_K^2-M_\pi^2}{f_\pi^2}f_0^{K\pi}\big(M_\pi^2\big),
\end{aligned}\tag{41}$$

$$\begin{aligned}
e^2g_8^\infty\big[g_{\rm ew}+\Delta_C\mathcal{A}_2^{(\rm ew)}\big]^\infty = &-3C_8(\mu)\left[\frac{M_K^2}{(m_s+m_q)(\mu)f_\pi}\right]^2\left[1+\frac{4L_5}{f_\pi^2}M_\pi^2\right]\\
&+\frac{3}{2}[C_7-C_9-C_{10}](\mu)\frac{M_K^2-M_\pi^2}{f_\pi^2}f_0^{K\pi}\big(M_\pi^2\big).
\end{aligned}\tag{42}$$

For the operators Q_i $(i\neq 6,8)$, which are products of colour-singlet vector and axial-vector currents, these are exact large-N_C results to all orders in the chiral expansion, as can be easily seen factorizing the operators at the quark level. The χPT framework discussed before reproduces these results in a perturbative way, through the momentum expansion of the $K\pi$ scalar form factor at $\mathcal{O}(p^4)$:

$$f_0^{K\pi}\big(M_\pi^2\big)\equiv f_+^{K\pi}\big(M_\pi^2\big)+\frac{M_\pi^2}{M_K^2-M_\pi^2}f_-^{K\pi}\big(M_\pi^2\big)=1+\frac{4L_5}{f_\pi^2}M_\pi^2+\cdots.\tag{43}$$

Table 1
Numerical values of the weak chiral couplings in the large-N_C limit

$g_8^\infty[1+\Delta_C\mathcal{A}_0^{(8)}]^\infty$	$(1.3\pm0.2\pm0.1)+\tau\left(1.12\pm0.08^{+0.49}_{-0.30}\right)$
$g_{27}^\infty[1+\Delta_C\mathcal{A}_0^{(27)}]^\infty$	$(0.47\pm0.01\pm0.0)$
$e^2g_8^\infty[g_{\rm ew}+\Delta_C\mathcal{A}_0^{(\rm ew)}]^\infty$	$-\left(0.085\pm0.085^{+0.035}_{-0.023}\right)-\tau\left(2.33\pm0.07^{+0.83}_{-0.52}\right)$
$e^2g_8^\infty[g_{\rm ew}+\Delta_C\mathcal{A}_2^{(\rm ew)}]^\infty$	$-\left(0.07\pm0.07^{+0.03}_{-0.02}\right)-\tau\left(1.34\pm0.03^{+0.68}_{-0.42}\right)$

The form factors $f_\pm^{K\pi}(t)$ are defined through the matrix element of the vector current,

$$\langle\pi|\bar{s}\gamma^\mu q|K\rangle = C_{K\pi}\left\{(P_K+k_\pi)^\mu f_+^{K\pi}(t)+(P_K-k_\pi)^\mu f_-^{K\pi}(t)\right\}$$
$$(q=u,d), \tag{44}$$

where $t\equiv(P_K-k_\pi)^2$, $C_{K^0\pi^0}=-C_{K^+\pi^0}=1/\sqrt{2}$ and $C_{K^0\pi^-}=C_{K^+\pi^+}=-1$.

The wave-function renormalization corrections $\tilde{\Delta}_C$ [Eq. (30)] have been cancelled by weak $\mathcal{O}(p^4)$ contributions, as it should since we are dealing with conserved currents. Once the $\mathcal{O}(p^2)$ results are written in terms of the physical pion decay constant f_π, higher-order chiral contributions only introduce the small correction factor $f_0^{K\pi}(M_\pi^2)\approx1.02$.

The hadronic matrix elements of the operators Q_6 and Q_8 factorize into products of scalar and pseudoscalar currents, which cannot be directly measured. The χPT predictions are then needed to determine those hadronic currents. The electroweak penguin matrix elements are known to $\mathcal{O}(p^4)$. Again, one observes that the contributions from local weak terms (Z_1 and Z_6) cancel the negative contribution from $\tilde{\Delta}_C^{(\rm ew)}$ and reverse the sign of the $\mathcal{O}(p^4)$ correction. The contribution of the penguin operator Q_6 is only known at $\mathcal{O}(p^2)$. For Q_6 we cannot just include the $\tilde{\Delta}_C$ correction, because the corresponding weak $\mathcal{O}(p^4)$ counterterms are unknown and large cancellations can be expected. In Eq. (39) we have taken a global correction factor $f_0^{K\pi}(M_\pi^2)$ for the octet amplitude. This is a reasonable assumption, [7] since nearly all known pieces have this common correction. Only $\langle Q_8\rangle_0$ gets a different (and larger) correction.

The scalar and pseudoscalar currents introduce a quadratic dependence on quark masses in the contributions from the operators Q_6 and Q_8. At present, the most reliable determinations of the light quark masses give $m_s(1\ \text{GeV})=(150\pm25)$ MeV [60–65] and $(m_u+m_d)(1\ \text{GeV})=(12.8\pm2.5)$ MeV [66], at the scale $\mu=1$ GeV. We then take:

$$(m_s+m_q)(1\ \text{GeV})=(156\pm25)\ \text{MeV}. \tag{45}$$

Table 1 shows the resulting numerical predictions for the weak chiral couplings. The central values have been obtained at $\mu=1$ GeV. The first errors indicate the sensitivity to changes of the short-distance renormalization scale in the range $M_\rho<\mu<m_c$ and to the choice of γ_5 scheme in the next-to-leading order calculation of the Wilson coefficients. The second uncertainties correspond to the input values of the quark masses.

[7] In fact, the factor $f_0^{K\pi}(M_\pi^2)$ already appears in the lowest-order Q_6 contribution to g_8, through the $\mathcal{O}(p^4)$ correction in Eq. (33).

For historical reasons, the values of the short-distance Wilson coefficients are usually given in terms of $\Lambda_{\rm QCD}$ (in the three or four flavour theory). Nowadays, that α_s is experimentally known with rather good accuracy, it is unnecessary to introduce this additional auxiliary parameter which only complicates the final expressions. Since the most important α_s corrections appear at the lowest scale $\mu \sim \mathcal{O}(1\ \text{GeV})$, we have fixed the strong coupling at the τ mass, where it is known [67] with about a few percent level of accuracy:

$$\alpha_s(M_\tau) = 0.345 \pm 0.020. \tag{46}$$

The high-energy matching scale is chosen to be intermediate between the W-boson and the top quark mass scale. We have performed the matching directly at the Z-boson mass scale where α_s is best known [68],

$$\alpha_s(M_Z) = 0.119 \pm 0.002. \tag{47}$$

The measured values (46) and (47) are in perfect agreement, if one performs [69] a four-loop evolution of α_s between M_Z and M_τ, with the appropriate matching conditions at the different thresholds [70]. The values of α_s at the other needed scales can be deduced from (46). The numerical uncertainties associated with the present error on α_s have been included in our results, but they are negligible in comparison with the uncertainties from other sources.

The dominance of Q_6 and Q_8 in the CP-odd amplitudes (the ones proportional to the CKM factor τ) is apparent in Table 1, where those pieces show a very strong dependence on quark masses (second error bars). In comparison, the short-distance uncertainties are much smaller. The opposite behaviour is observed in the CP-conserving couplings $\text{Re}(g_8)$ and $\text{Re}(g_{27})$, which are dominated by Q_1 and Q_2. The 27-plet coupling, which does not get any penguin contribution, satisfies $\text{Im}(g_{27}) = 0$ for all practical purposes.

Taking $\Omega_{\rm IB}^{\pi^0\eta} = 0.16$, $\text{Im}(V_{ts}^* V_{td}) = 1.2 \times 10^{-4}$ and the central values in Table 1 for the CP-odd amplitudes, one gets the large-N_C prediction $\text{Re}(\varepsilon'/\varepsilon) \approx 0.8 \times 10^{-3}$. Although numerically suppressed, the operators Q_1, Q_2 and Q_4, which are not well approximated by the large-N_C limit, provide also small corrections to $\text{Im}(A_0)$. In Refs. [7,28] the measured CP-conserving rates are used to estimate those contributions. This amounts to multiply the corrections from these operators by a factor $\xi_0 \approx 4.9$, to compensate for the underestimated coupling $\text{Re}(g_8)$. Adopting this prescription, one gets $\text{Re}(\varepsilon'/\varepsilon) \approx 0.5 \times 10^{-3}$, in agreement with the findings of Refs. [7,8].

6. Chiral loop corrections

The previous tree-level amplitudes do not contain any strong phases δ_0^I. Those phases originate in the final rescattering of the two pions and, therefore, are generated by chiral loops which are of higher order in the $1/N_C$ expansion. Analyticity and unitarity require the presence of a corresponding dispersive FSI effect in the moduli of the isospin amplitudes. Since the strong phases are quite large, specially in the isospin-zero case,

one should expect large higher-order unitarity corrections. Intuitively, the behaviour of the $I = 0$ and $I = 2$ S-wave phase shifts as a function of the total invariant mass of the two pions suggests a large enhancement of the $I = 0$ amplitude and a small suppression of the $I = 2$ amplitude.

The size of the FSI effect can be estimated at one loop in χPT. The dominant one-loop correction to the octet amplitude comes indeed from the elastic soft rescattering of the two pions in the final state. The existing one-loop analyses [44,45] show that pion loop diagrams provide an important enhancement of the A_0 amplitude by about 40%, implying a sizeable reduction of the phenomenologically fitted value of $|g_8|$ in Eq. (20).

The complete formulae for the one-loop corrections $\Delta_L \mathcal{A}_I^{(R)}$ are compiled in the appendices. The usual one-loop function $B(M_1^2, M_2^2, p^2)$ is defined in Appendix B, while appendix Appendix C contains explicit results for the different isospin amplitudes. The contributions proportional to $B(M_P^2, M_P^2, M_K^2)$, with $P = \pi, K, \eta$, arise from intermediate $\pi\pi$, $K\overline{K}$ and $\eta\eta$ states. At $s \equiv (p_{\pi_1} + p_{\pi_2})^2 = M_K^2$, the only possible absorptive contribution comes from the elastic $\pi\pi$ rescattering:

$$\Delta_L \mathcal{A}_0^{(R)} = -\frac{1}{2}\big(2M_K^2 - M_\pi^2\big) B\big(M_\pi^2, M_\pi^2, M_K^2\big) + \cdots, \tag{48}$$

$$\Delta_L \mathcal{A}_2^{(R)} = \frac{1}{2}\big(M_K^2 - 2M_\pi^2\big) B\big(M_\pi^2, M_\pi^2, M_K^2\big) + \cdots, \tag{49}$$

where

$$B\big(M_\pi^2, M_\pi^2, M_K^2\big) = \frac{1}{(4\pi f_\pi)^2}\left\{\sigma_\pi\left[\ln\left(\frac{1+\sigma_\pi}{1-\sigma_\pi}\right) - i\pi\right] - \ln\left(\frac{\nu^2}{M_\pi^2}\right) - 1\right\}, \tag{50}$$

with ν the chiral loop renormalization scale and

$$\sigma_\pi \equiv \sqrt{1 - \frac{4M_\pi^2}{M_K^2}}. \tag{51}$$

Thus, all isoscalar amplitudes get the same absorptive contribution, as it should, since they have identical strong phase shifts. The same is true for the two amplitudes with $I = 2$. The one-loop absorptive contributions reproduce the leading χPT values of the strong rescattering phases δ_0^I, with $I = 0, 2$:

$$\tan\delta_0^{0;2}\big(M_K^2\big) = \frac{1}{32\pi f_\pi^2}\sigma_\pi\big(2M_K^2 - M_\pi^2; 2M_\pi^2 - M_K^2\big). \tag{52}$$

The numerical values of $\delta_0^{0;2}(M_K^2)$ predicted by χPT at leading order, $\delta_0^0(M_K^2) = 25^\circ$ and $\delta_0^2(M_K^2) = -12^\circ$, are significantly lower than their experimental values, implying that higher-order rescattering contributions are numerically relevant. The phase-shift difference, $\delta_0^0 - \delta_0^2 = 37^\circ$, is slightly less sensitive to higher-order chiral corrections [35].

The 2π intermediate state induces a large one-loop correction to the $I = 0$ amplitudes. At $\nu = M_\rho$, the 2π contribution to the isoscalar amplitudes is $\Delta_L \mathcal{A}_0^{(R)}|_{\pi\pi} = 0.43 + 0.46\,i$, while $\Delta_L \mathcal{A}_2^{(R)}|_{\pi\pi} = -(0.19 + 0.20\,i)$; i.e., the expected enhancement (suppression) of the $I = 0$ ($I = 2$) amplitudes. The contributions from other one-loop diagrams, not related to FSI, are different for the different amplitudes $\mathcal{A}_I^{(R)}$.

Let us write our isospin amplitudes in the form

$$\mathcal{A}_I^{(R)} = \mathcal{A}_I^{(R)\infty} \times \mathcal{C}_I^{(R)}, \tag{53}$$

where $\mathcal{A}_I^{(R)\infty}$ are the large-N_C results obtained in the previous section. The correction factors $\mathcal{C}_I^{(R)}$ contain the chiral loop contributions we are interested in. At the one-loop level, they take the following numerical values:

$$\begin{aligned}
\mathcal{C}_0^{(8)} &\approx 1 + \Delta_L \mathcal{A}_0^{(8)} = 1.27 \pm 0.05 + 0.46\, i, \\
\mathcal{C}_0^{(27)} &\approx 1 + \Delta_L \mathcal{A}_0^{(27)} = 2.0 \pm 0.7 + 0.46\, i, \\
\mathcal{C}_0^{(\mathrm{ew})} &\approx 1 + \Delta_L \mathcal{A}_0^{(\mathrm{ew})} = 1.27 \pm 0.05 + 0.46\, i, \\
\mathcal{C}_2^{(27)} &\approx 1 + \Delta_L \mathcal{A}_2^{(27)} = 0.96 \pm 0.05 - 0.20\, i, \\
\mathcal{C}_2^{(\mathrm{ew})} &\approx 1 + \Delta_L \mathcal{A}_2^{(\mathrm{ew})} = 0.50 \pm 0.24 - 0.20\, i.
\end{aligned} \tag{54}$$

The central values have been evaluated at the chiral renormalization scale $\nu = M_\rho$. To estimate the corresponding uncertainties we have allowed the scale ν to change between 0.6 and 1 GeV. The scale dependence is only present in the dispersive contributions and should cancel with the corresponding ν dependence of the local counterterms. However, this dependence is next-to-leading in $1/N_C$ and, therefore, is not included in our large-N_C estimate of the $\mathcal{O}(p^4)$ and $\mathcal{O}(e^2p^2)$ chiral couplings. The ν dependence of the chiral loops would be cancelled by the unknown $1/N_C$-suppressed corrections $\Delta_C \mathcal{A}_I^{(R)}(\nu) - \Delta_C \mathcal{A}_I^{(R)\infty}$, that we are neglecting in the factors $\mathcal{C}_I^{(R)}$. The numerical sensitivity of our results to the scale ν gives then a good estimate of those missing contributions.

The absorptive contribution induces a large one-loop correction to the $I = 0$ amplitudes. The dispersive correction to $\Delta_L \mathcal{A}_0^{(27)}$ is even larger, but it has a smaller phenomenological impact because the isoscalar $K \to \pi\pi$ amplitude is dominated by its octet component; this 27-plet correction has a strong dependence on ν and, therefore, a rather large uncertainty. Although the one-loop correction to the $I = 2$ $(27_L, 1_R)$ amplitude is rather moderate, the electroweak $I = 2$ amplitude gets a large dispersive correction with negative sign. This induces a corresponding suppression of $|\mathcal{A}_2^{(\mathrm{ew})}|$ by about 46%.

The numerical corrections to the 27-plet amplitudes do not have much phenomenological interest for CP-violating observables, because $\mathrm{Im}(g_{27}) = 0$. Remember that the CP-conserving amplitudes $\mathrm{Re}(A_I)$ are set to their experimentally determined values. What is relevant for the ε'/ε prediction is the 35% enhancement of the isoscalar octet amplitude $\mathrm{Im}[A_0^{(8)}]$ and the 46% reduction of $\mathrm{Im}[A_2^{(\mathrm{ew})}]$. Just looking to the simplified formula (15), one realizes immediately the obvious impact of these one-loop chiral corrections, which destroy the accidental lowest-order cancellation between the $I = 0$ and $I = 2$ contributions, generating a sizeable enhancement of ε'/ε.

A complete $\mathcal{O}(p^4)$ calculation [38,47] of the isospin-breaking parameter Ω_{IB} is not yet available. The value 0.16 quoted in Eq. (16) only accounts for the contribution from π^0–η mixing [38] and should be corrected by the effect of chiral loops. Since $|\mathcal{C}_2^{(27)}| \approx 0.98 \pm 0.05$, one does not expect any large correction of $\mathrm{Im}(A_2)_{\mathrm{IB}}$, while we know that $\mathrm{Im}[A_0^{(8)}]$ gets enhanced by a factor 1.35. Taking this into account, one gets the corrected

value

$$\Omega_{\rm IB} \approx \Omega_{\rm IB}^{\pi^0\eta} \left| \frac{\mathcal{C}_2^{(27)}}{\mathcal{C}_0^{(8)}} \right| = 0.12 \pm 0.05, \tag{55}$$

where the quoted error is an educated theoretical guess. This value agrees with the result $\Omega_{\rm IB} = 0.08 \pm 0.05 \pm 0.01$, obtained in Ref. [71] by using three different models [9,50,53, 55,72,73] to estimate the relevant $\mathcal{O}(p^4)$ chiral couplings.

The one-loop corrections increase the large-N_C estimate from $\varepsilon'/\varepsilon \approx 0.8 \times 10^{-3}$ to [8] $\varepsilon'/\varepsilon \approx 1.8 \times 10^{-3}$. The contributions to $\mathrm{Im}(A_0)$ from the operators $Q_{1,2,4}$ can be corrected phenomenologically, as advocated in Ref. [28]; this requires now a smaller factor $\xi_0 \approx 3.5$, which results in [9] $\varepsilon'/\varepsilon \approx 1.5 \times 10^{-3}$.

7. Final state interactions at higher orders

Given the large size of the one-loop contributions, one should worry about higher-order chiral corrections. The fact that the one-loop calculation still underestimates the observed δ_0^0 phase shift indicates that a further enhancement could be expected at higher orders.

The large one-loop FSI correction to the isoscalar amplitudes is generated by large infrared chiral logarithms involving the light pion mass [2]. These logarithms are universal, i.e., their contribution depends exclusively on the quantum numbers of the two pions in the final state [2]. As a result, they give the same correction to all isoscalar amplitudes. Identical logarithmic contributions appear in the scalar pion form factor [31], where they completely dominate the $\mathcal{O}(p^4)$ χPT correction.

Using analyticity and unitarity constraints [74], these logarithms can be exponentiated to all orders in the chiral expansion [1,2]. The result can be written as:

$$\mathcal{C}_I^{(R)} \equiv \mathcal{C}_I^{(R)}\big(M_K^2\big) = \Omega_I\big(M_K^2, s_0\big)\mathcal{C}_I^{(R)}(s_0). \tag{56}$$

The Omnès [74–76] exponential [10]

$$\Omega_I(s, s_0) \equiv e^{i\delta_0^I(s)}\Re_I(s, s_0) = \exp\left\{ \frac{(s - s_0)}{\pi} \int \frac{dz}{(z - s_0)} \frac{\delta_0^I(z)}{(z - s - i\epsilon)} \right\} \tag{57}$$

provides an evolution of $\mathcal{C}_I^{(R)}(s)$ from an arbitrary low-energy point s_0 to $s \equiv (p_{\pi_1} + p_{\pi_2})^2 = M_K^2$. The physical amplitudes are of course independent of the subtraction point s_0. Intuitively, what the Omnès solution does is to correct a local weak $K \to \pi\pi$ transition with an infinite chain of pion-loop bubbles, incorporating the strong $\pi\pi \to \pi\pi$ rescattering to all orders in χPT. The Omnès exponential only sums a particular type of higher-order Feynman diagrams, related to FSI. Therefore, Eq. (56) does not provide the complete

[8] This number is obtained taking the experimental value for ε and $\mathrm{Im}(V_{ts}^* V_{td}) = 1.2 \times 10^{-4}$. Using instead the theoretical prediction for ε, one would get $\varepsilon'/\varepsilon \approx 2.2 \times 10^{-3}$. See Section 8 for more details on this second kind of numerical analysis.

[9] Using the theoretical value of ε, one finds $\varepsilon'/\varepsilon \approx 1.8 \times 10^{-3}$.

[10] Equivalent expressions with an arbitrary number of subtractions for the dispersive integral can be written [2].

result. Nevertheless, it allows us to perform a reliable estimate of higher-order effects because it does sum the most important corrections.[11] Moreover, the Omnès exponential enforces the decay amplitudes to have the right physical phases.

The Omnès resummation of chiral logarithms is uniquely determined up to a polynomial (in s) ambiguity [2,74,77], which has been solved with the large-N_C amplitude $\mathcal{A}_I^{(R)\infty}$. The exponential only sums the elastic rescattering of the final two pions, which is responsible for the phase shift. Since the kaon mass is smaller than the inelastic threshold, the virtual loop corrections from other intermediate states ($K \to K\pi, K\eta, \eta\eta, K\overline{K} \to \pi\pi$) can be safely estimated at the one loop level; they are included in $\mathcal{C}_I^{(R)}(s_0)$.

Taking the chiral prediction for $\delta_0^I(z)$ and expanding $\Omega_I(M_K^2, s_0)$ to $\mathcal{O}(p^2)$,

$$\Omega_I\big(M_K^2, s_0\big) \approx 1 + \frac{(M_K^2 - s_0)}{\pi} \int \frac{dz}{(z - s_0)} \frac{\delta_0^I(z)}{(z - M_K^2 - i\epsilon)} \equiv 1 + \delta\Omega_I\big(M_K^2, s_0\big), \tag{58}$$

one should reproduce the one-loop χPT result. This determines the factor $\mathcal{C}_I^{(R)}(s_0)$ to $\mathcal{O}(p^4)$ in the chiral expansion:

$$\mathcal{C}_I^{(R)}(s_0) = \mathcal{C}_I^{(R)}\big[1 - \delta\Omega_I\big(M_K^2, s_0\big)\big] \approx 1 + \Delta_L \mathcal{A}_I^{(R)} - \delta\Omega_I\big(M_K^2, s_0\big). \tag{59}$$

It remains a local ambiguity at higher orders [2,74,77].

Eq. (56) allows us to improve the one-loop calculation, by taking s_0 low enough that the χPT corrections to $\mathcal{C}_I^{(R)}(s_0)$ are moderate and exponentiating the large logarithms with the Omnès factor. Moreover, using the experimental phase shifts in the dispersive integral one achieves an all-order resummation of FSI effects. The numerical accuracy of this exponentiation has been successfully tested [2] through an analysis of the scalar pion form factor, which has identical FSI than A_0.

At $s_0 = 0$, the dispersive parts of the experimentally determined Omnès exponentials are [2]:

$$\Re_0\big(M_K^2, 0\big) = 1.55 \pm 0.10, \qquad \Re_2\big(M_K^2, 0\big) = 0.92 \pm 0.03. \tag{60}$$

The quoted errors take into account uncertainties in the experimental phase-shifts data and additional inelastic contributions above the first inelastic threshold. These numbers fit very well with the findings of the chiral one-loop calculation discussed in the previous section. The corrections induced by FSI in the moduli of the decay amplitudes $\mathcal{A}_I$ generate an enhancement of the $\Delta I = 1/2$ to $\Delta I = 3/2$ ratio [1],

$$\Re_0\big(M_K^2, 0\big)\big/\Re_2\big(M_K^2, 0\big) = 1.68 \pm 0.12. \tag{61}$$

This factor multiplies the enhancement already found at short distances.

[11] A more elaborated dispersive framework including "crossed-channel" contributions has been recently discussed in Ref. [77]. The available non-perturbative information, needed to fix the corresponding subtraction constants, does not allow an accurate calculation of those additional effects. Nevertheless, using the present knowledge on πK scattering phase shifts [78], this dispersive analysis [77] corroborates that higher-order πK rescattering corrections are indeed negligible, as expected.

At $\mathcal{O}(p^4)$, the previous numbers should be corrected with the factors $\mathcal{C}_I^{(R)}(s_0)$, which incorporate additional one-loop contributions not related to FSI. These factors compensate the obvious s_0 dependence of the Omnès exponentials, up to $\mathcal{O}(p^6)$ corrections. To estimate the remaining sensitivity to this parameter, we have changed the subtraction point between $s_0 = 0$ and $s_0 = 3M_\pi^2$ and have included the resulting fluctuations in the final uncertainties. The detailed numerical analysis is given in Appendix D. At $\nu = M_\rho$, we get the following values for the resummed loop corrections:

$$\begin{aligned}
\left|\mathcal{C}_0^{(8)}\right| &= \Re_0\left(M_K^2, s_0\right)\mathcal{C}_0^{(8)}(s_0) = 1.31 \pm 0.06,\\
\left|\mathcal{C}_0^{(27)}\right| &= \Re_0\left(M_K^2, s_0\right)\mathcal{C}_0^{(27)}(s_0) = 2.4 \pm 0.1,\\
\left|\mathcal{C}_0^{(\mathrm{ew})}\right| &= \Re_0\left(M_K^2, s_0\right)\mathcal{C}_0^{(\mathrm{ew})}(s_0) = 1.31 \pm 0.07,\\
\left|\mathcal{C}_2^{(27)}\right| &= \Re_2\left(M_K^2, s_0\right)\mathcal{C}_2^{(27)}(s_0) = 1.05 \pm 0.05,\\
\left|\mathcal{C}_2^{(\mathrm{ew})}\right| &= \Re_2\left(M_K^2, s_0\right)\mathcal{C}_2^{(\mathrm{ew})}(s_0) = 0.62 \pm 0.05.
\end{aligned} \tag{62}$$

These results agree within errors with the one-loop chiral calculation of the moduli of the isospin amplitudes, indicating a good convergence of the chiral expansion.

To derive the Omnès representation, one makes use of Time-Reversal invariance, so that it can be strictly applied only to CP-conserving amplitudes. Nevertheless, the procedure can be directly extended to the CP-violating components relevant for the estimate of ε'/ε. Working to first order in the Fermi coupling, the CP-odd phase is fully contained in the ratio of CKM matrix elements τ which appears in the short-distance Wilson coefficients and, therefore, in $\mathcal{A}_I^{(R)\infty}$. Decomposing the isospin amplitudes as $\mathcal{A}_I^{(R)} = \mathcal{A}_I^{(R)CP} + \tau\mathcal{A}_I^{(R)C\not{P}}$, the Omnès solution can be derived separately for the two amplitudes $\mathcal{A}_I^{(R)CP}$ and $\mathcal{A}_I^{(R)C\not{P}}$ which respect Time-Reversal invariance.

8. Numerical analysis

The CP-violating ratio ε'/ε is proportional to the CKM factor $\mathrm{Im}(V_{ts}^* V_{td})$. The standard unitarity triangle analyses [79] have estimated this parameter to be in the range

$$\mathrm{Im}\left(V_{ts}^* V_{td}\right) = (1.2 \pm 0.2) \times 10^{-4}. \tag{63}$$

This determination is obtained combining the present information on various flavour-changing processes; mainly, ε, B_0–$\overline{B}_0$ mixing and the ratio $\Gamma(b \to u)/\Gamma(b \to c)$. The final number is sensitive to the input values adopted for several non-perturbative hadronic parameters and, thus, there are large theoretical uncertainties [80] which are not easy to quantify.

Since the Standard Electroweak Model has a unique source of CP violation, the same combination of CKM factors appears in the theoretical prediction for ε, which is proportional to the K^0–$\overline{K}^0$ matrix element of the $\Delta S = 2$ operator:

$$\left\langle \overline{K}^0 \right| (\bar{s}_L \gamma_\mu d_L)\left(\bar{s}_L \gamma^\mu d_L\right) \left| K^0 \right\rangle \equiv \frac{4}{3} f_K^2 M_K^2 B_K(\mu). \tag{64}$$

The factor $B_K(\mu)$ parameterizes this hadronic matrix element in vacuum insertion units. The corresponding Wilson coefficient $C_{\Delta S=2}(\mu)$ is known at the next-to-leading logarithmic order [27,81]. Taking appropriate values for the different inputs one finds:

$$|\varepsilon| = \frac{4}{3}\widehat{B}_K \,\mathrm{Im}\big(V_{ts}^* V_{td}\big)(18.9 - 14.4\bar{\rho}), \tag{65}$$

with $\bar{\rho}$ one of the two CKM parameters, in the Wolfenstein [82] parameterization, which characterize the upper vertex of the unitarity triangle. The standard analyses [79] favour the range $\bar{\rho} = 0.2 \pm 0.1$, implying

$$|\varepsilon| = \frac{4}{3}\widehat{B}_K \,\mathrm{Im}\big(V_{ts}^* V_{td}\big)\,(16.0 \pm 1.4)\,, \tag{66}$$

where $\widehat{B}_K = C_{\Delta S=2}(\mu) B_K(\mu)$ is the scale-invariant bag parameter. In the large-N_C limit, $\widehat{B}_K = B_K(\mu) = 3/4$.

The numerical values of both $\mathrm{Im}(V_{ts}^* V_{td})$ and $\bar{\rho}$ depend on hadronic inputs. However, ε is rather insensitive to the precise value of $\bar{\rho}$; it changes by less than 10% when $\bar{\rho}$ is varied within the previously quoted range.

Thus, we can make two different numerical analyses of ε'/ε:

1. The usual one, taking the experimental value of ε and adopting the range (63) for the relevant CKM factor.
2. Using instead the theoretical prediction of ε in Eq. (66), the ratio ε'/ε does not depend on $\mathrm{Im}(V_{ts}^* V_{td})$ [10]. The sensitivity of this CKM factor to different hadronic inputs is then reduced to the explicit remaining dependence on $\widehat{B}_K$.

The second type of analysis is more suitable to a systematic $1/N_C$ approach. The theoretical prediction for ε'/ε depends on ratios of hadronic matrix elements, i.e., $B_i/\widehat{B}_K$. It is known [80] that $\widehat{B}_K$ has sizeable large-N_C [53,83,84] and chiral [85] corrections, which are of opposite sign and could then cancel to some extent. Thus, one can expect the limit $N_C \to \infty$ to provide a good starting point to analyze the relevant ratios $B_6^{(1/2)}/\widehat{B}_K$ and $B_8^{(3/2)}/\widehat{B}_K$.

We have performed the two types of numerical analysis, obtaining consistent results. This allows us to estimate better the theoretical uncertainties, since the two analyses have different sensitivity to hadronic inputs. The contributions to $\mathrm{Im}(A_0)$ from the operators $Q_{1,2,4}$ have been estimated, following the strategy adopted in Ref. [28]; i.e., we have corrected them with the factor $\xi_0 \approx 3.5$.

As a first estimate, we can perform the calculation of ε'/ε to $\mathcal{O}(p^4)$ in χPT, without making any Omnès resummation of higher-order corrections. Once the large one-loop corrections are taken into account, all important ingredients are already caught. We find, for the two different types of analysis:

$$\mathrm{Re}(\varepsilon'/\varepsilon) = 1.5 \times 10^{-3} \frac{\mathrm{Im}(V_{ts}^* V_{td})}{1.2 \times 10^{-4}} = 1.8 \times 10^{-3}. \tag{67}$$

To quantify the uncertainties, we need to consider higher-order effects. Performing the Omnès resummation, as indicated in Eq. (56), one finds:

$$\mathrm{Re}\big(\varepsilon'/\varepsilon\big) = 1.4 \times 10^{-3} \frac{\mathrm{Im}(V_{ts}^* V_{td})}{1.2 \times 10^{-4}} = 1.6 \times 10^{-3}. \tag{68}$$

These numbers are quite close to the one-loop results (67), which indicates that the error induced by the chiral loop calculation is not large.

From the previous numbers, we derive:

$$\mathrm{Re}(\varepsilon'/\varepsilon) = \big(1.7 \pm 0.2^{+0.8}_{-0.5} \pm 0.5\big) \times 10^{-3}. \tag{69}$$

The first error indicates the sensitivity to the short-distance renormalization scale, which we have taken in the range $M_\rho < \mu < m_c$. The uncertainty coming from varying the strange quark mass in the interval $(m_s + m_q)(1\ \mathrm{GeV}) = 156 \pm 25\ \mathrm{MeV}$ [60–66] is indicated by the second error. We have added a 30% uncertainty from unknown next-to-leading in $1/N_C$ local contributions (third error).

9. Discussion

The infrared effect of chiral loops generates an important enhancement of the isoscalar $K \to \pi\pi$ amplitude. This effect gets amplified in the prediction of ε'/ε, because at lowest order (in both $1/N_C$ and the chiral expansion) there is an accidental numerical cancellation between the $I = 0$ and $I = 2$ contributions. Since the chiral loop corrections destroy this cancellation, the final result for ε'/ε is dominated by the isoscalar amplitude. Thus, the Standard Model prediction for ε'/ε is finally governed by the matrix element of the gluonic penguin operator Q_6.

There are three major ingredients in our theoretical analysis:

1. A short-distance calculation at the electroweak scale and its renormalization-group evolution to the three-flavour theory ($\mu \lesssim m_c$), which sums the leading ultraviolet logarithms.
2. The matching to the χPT description.
3. Chiral loop corrections, which induce large infrared logarithms related to FSI.

The first step is already known at the next-to-leading logarithmic order [28,29]. The short-distance results are then very reliable.

We have tried to achieve an acceptable control of the large infrared chiral corrections, which are fully known at the one-loop level. A complete two-loop χPT calculation is not yet available. Nevertheless, since the leading one-loop corrections are generated by the FSI of the two pions, we can use the Omnès resummation to get an idea about the size to be expected for the unknown higher-order contributions. The Omnès exponential only sums a particular type of higher-order Feynman diagrams, related to FSI. Although it does not give the complete result, it allows us to estimate the theoretical uncertainty in a very reliable way, because it does sum the most important corrections. The one-loop results and the Omnès calculation agree within errors, indicating a good convergence of the chiral expansion.

The most critical step is the matching between the short- and long-distance descriptions. We have performed this matching at leading order in the $1/N_C$ expansion, where the result

is exactly known to $\mathcal{O}(p^4)$ and $\mathcal{O}(e^2p^2)$ in χPT [$\mathcal{O}(p^2)$ for Q_6]. This can be expected to provide a good approximation to the matrix elements of the leading Q_6 and Q_8 operators. Since all ultraviolet and infrared logarithms have been resummed, our educated guess for the theoretical uncertainty associated with $1/N_C$ corrections is $\sim 30\%$.

As our final result we quote:

$$\mathrm{Re}(\varepsilon'/\varepsilon) = (1.7 \pm 0.9) \times 10^{-3}. \tag{70}$$

A better determination of the strange quark mass would allow to reduce the uncertainty to the 30% level. In order to get a more accurate prediction, it would be necessary to have a good analysis of next-to-leading $1/N_C$ corrections. This is a very difficult task, but progress in this direction can be expected in the next few years [9,12,53,84,86,87].

Note added

After this paper was submitted for publication, new experimental results have been announced both by NA48 [88] and KTeV [89]:

$$\mathrm{Re}(\varepsilon'/\varepsilon) = \begin{cases} (1.53 \pm 0.26) \times 10^{-3} & \text{[NA48]}, \\ (2.07 \pm 0.28) \times 10^{-3} & \text{[KTeV]}. \end{cases}$$

The new world average,

$$\mathrm{Re}(\varepsilon'/\varepsilon) = (1.72 \pm 0.18) \times 10^{-3},$$

is in excellent agreement with the Standard Model prediction in Eq. (70).

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Appendix A. Octet basis transformation rules

Following the same notation as the original references, one can change from the octet basis $\sum_i E_i O_i^8$ of Ref. [45] to the one of Ref. [50], $\sum_i N_i W_i^8$, using either the following identities for the operators,

$$\begin{aligned} W_5^8 &= O_{10}^8, & W_{10}^8 &= O_1^8, \\ W_6^8 &= \frac{1}{2} O_{12}^8, & W_{11}^8 &= O_2^8, \\ W_7^8 &= O_{13}^8, & W_{12}^8 &= O_3^8, \end{aligned}$$

$$W_8^8 = O_{10}^8 + O_{11}^8 - \frac{1}{2}\big(O_{12}^8 + O_{13}^8\big), \qquad W_{13}^8 = O_4^8,$$
$$W_9^8 = O_{15}^8, \qquad W_{36}^8 = O_1^8 - O_3^8 + O_5^8, \tag{A.1}$$

or the coefficient relations

$$N_5 = E_{10} - E_{11}, \qquad N_{10} = E_1 - E_5,$$
$$N_6 = E_{11} + 2E_{12}, \qquad N_{11} = E_2,$$
$$N_7 = \frac{1}{2}E_{11} + E_{13}, \qquad N_{12} = E_3 + E_5,$$
$$N_8 = E_{11}, \qquad N_{13} = E_4,$$
$$N_9 = E_{15}, \qquad N_{36} = E_5. \tag{A.2}$$

Appendix B. One-loop functions

The one-loop function $B(M_1^2, M_2^2, p^2)$ is defined by the (dimensionally regularized) basic scalar integral with two bosonic propagators:

$$i\int \frac{d^Dq}{(2\pi)^D}\,\frac{1}{(M_1^2 - q^2)[M_2^2 - (p-q)^2]} = f_\pi^2 B\big(M_1^2, M_2^2, p^2\big) + 2\Lambda\big(\mu^2\big), \tag{B.1}$$

where

$$\Lambda\big(\mu^2\big) \equiv \frac{1}{16\pi^2}\left\{\frac{\mu^{D-4}}{D-4} + \frac{1}{2}\big[\gamma_E - \ln(4\pi) - 1\big]\right\}. \tag{B.2}$$

It can be expressed in terms of the function $\bar{J}_{12}(p^2)$ [31]:

$$f_\pi^2 B\big(M_1^2, M_2^2, p^2\big) = -\bar{J}_{12}\big(p^2\big) + \frac{1}{16\pi^2}\left(\ln\frac{M_2^2}{\nu^2} + \frac{M_1^2}{M_1^2 - M_2^2}\ln\frac{M_1^2}{M_2^2}\right), \tag{B.3}$$

with

$$\bar{J}_{12}\big(p^2\big) = \frac{1}{16\pi^2}\left\{1 - \frac{1}{2}\left(1 + \frac{M_1^2}{p^2} - \frac{M_2^2}{p^2}\right)\ln\frac{M_1^2}{M_2^2} + \frac{M_1^2}{M_1^2 - M_2^2}\ln\frac{M_1^2}{M_2^2} - \frac{1}{2}\frac{\lambda}{p^2}\ln\frac{(p^2+\lambda)^2 - (M_1^2 - M_2^2)^2}{(p^2-\lambda)^2 - (M_1^2 - M_2^2)^2}\right\}, \tag{B.4}$$

where

$$\lambda^2 \equiv \lambda^2\big(p^2, M_1^2, M_2^2\big) = \big[p^2 - (M_1 + M_2)^2\big]\big[p^2 - (M_1 - M_2)^2\big]. \tag{B.5}$$

For $M_1 = M_2 \equiv M$ one gets

$$f_\pi^2 B\big(M^2, M^2, p^2\big) = -\bar{J}\big(p^2\big) + \frac{1}{16\pi^2}\left(\ln\frac{M^2}{\nu^2} + 1\right), \tag{B.6}$$

where $\bar{J}(p^2)$ is given by

$$\bar{J}\big(p^2\big) = \frac{1}{(4\pi)^2}\left\{2 - \sigma\ln\left(\frac{\sigma+1}{\sigma-1}\right)\right\}, \quad \sigma \equiv \sqrt{1 - \frac{4M^2}{p^2}}. \tag{B.7}$$

The one-loop amplitudes contain an additional logarithmic dependence on the chiral renormalization scale ν, through the factors ($P = \pi, K, \eta$):

$$\nu_P = \frac{M_P^2}{32\pi^2 f_\pi^2} \ln \frac{M_P^2}{\nu^2}. \tag{B.8}$$

Appendix C. One-loop amplitudes

The one-loop $K \to \pi\pi$ amplitudes have been computed in Refs. [44,45], in the absence of electroweak corrections. The electromagnetic contributions have been recently calculated in Refs. [47–49]. The results take the form:

$$\begin{aligned}
\Delta_L \mathcal{A}_0^{(8)} = \Bigg\{ & -\frac{1}{2}\big(2M_K^2 - M_\pi^2\big) B\big(M_\pi^2, M_\pi^2, M_K^2\big) - \frac{1}{18} M_\pi^2 B\big(M_\eta^2, M_\eta^2, M_K^2\big) \\
& + \frac{1}{4}\frac{M_K^2}{M_\pi^2}\big(M_K^2 - 4M_\pi^2\big) B\big(M_K^2, M_\pi^2, M_\pi^2\big) + \frac{1}{12}\frac{M_K^4}{M_\pi^2} B\big(M_K^2, M_\eta^2, M_\pi^2\big) \\
& + \frac{M_K^4}{4(M_K^2 - M_\pi^2)M_\pi^2}\Bigg[\left(2 + 15\frac{M_\pi^2}{M_K^2} - 21\frac{M_\pi^4}{M_K^4}\right)\nu_\pi \\
& + 2\frac{M_\pi^2}{M_K^2}\left(3 + \frac{M_\pi^2}{M_K^2}\right)\nu_K + \left(-2 + 3\frac{M_\pi^2}{M_K^2} - 5\frac{M_\pi^4}{M_K^4}\right)\nu_\eta\Bigg]\Bigg\},
\end{aligned} \tag{C.1}$$

$$\begin{aligned}
\Delta_L \mathcal{A}_0^{(\mathrm{ew})} = \Bigg\{ & -\frac{1}{2}\big(2M_K^2 - M_\pi^2\big) B\big(M_\pi^2, M_\pi^2, M_K^2\big) + \frac{3M_K^2}{8} B\big(M_K^2, M_K^2, M_K^2\big) \\
& + \frac{1}{4}\frac{M_K^2}{M_\pi^2}\big(M_K^2 - 4M_\pi^2\big) B\big(M_K^2, M_\pi^2, M_\pi^2\big) + \frac{M_K^4}{8M_\pi^2} B\big(M_K^2, M_\eta^2, M_\pi^2\big) \\
& + \frac{1}{4}\left(17 + 2\frac{M_K^2}{M_\pi^2}\right)\nu_\pi + \frac{1}{4}\left(4 + \frac{M_K^2}{M_\pi^2}\right)\nu_K + \frac{3}{4}\left(1 - \frac{M_K^2}{M_\pi^2}\right)\nu_\eta\Bigg\},
\end{aligned} \tag{C.2}$$

for the octet isoscalar amplitude,

$$\begin{aligned}
\Delta_L \mathcal{A}_0^{(27)} = \Bigg\{ & -\frac{1}{2}\big(2M_K^2 - M_\pi^2\big) B\big(M_\pi^2, M_\pi^2, M_K^2\big) + \frac{1}{2} M_\pi^2 B\big(M_\eta^2, M_\eta^2, M_K^2\big) \\
& + \frac{1}{4}\frac{M_K^2}{M_\pi^2}\big(M_K^2 - 4M_\pi^2\big) B\big(M_K^2, M_\pi^2, M_\pi^2\big) - \frac{1}{3}\frac{M_K^4}{M_\pi^2} B\big(M_K^2, M_\eta^2, M_\pi^2\big) \\
& + \frac{M_K^4}{4(M_K^2 - M_\pi^2)M_\pi^2}\Bigg[\left(2 + 15\frac{M_\pi^2}{M_K^2} - 21\frac{M_\pi^4}{M_K^4}\right)\nu_\pi \\
& - \left(10 + 4\frac{M_\pi^2}{M_K^2} - 22\frac{M_\pi^4}{M_K^4}\right)\nu_K + \left(8 - 27\frac{M_\pi^2}{M_K^2} + 15\frac{M_\pi^4}{M_K^4}\right)\nu_\eta\Bigg]\Bigg\},
\end{aligned} \tag{C.3}$$

for the 27-plet isoscalar amplitude and

$$\begin{aligned}\Delta_L\mathcal{A}_2^{(27)} =& \left\{\frac{1}{2}\left(M_K^2-2M_\pi^2\right)B\left(M_\pi^2,M_\pi^2,M_K^2\right)\right.\\ &+\frac{5}{8}\frac{M_K^2}{M_\pi^2}\left(M_K^2-\frac{8}{5}M_\pi^2\right)B\left(M_K^2,M_\pi^2,M_\pi^2\right)\\ &+\frac{1}{24}\frac{M_K^4}{M_\pi^2}B\left(M_K^2,M_\eta^2,M_\pi^2\right)\\ &+\frac{M_K^4}{4(M_K^2-M_\pi^2)M_\pi^2}\left[\left(5-18\frac{M_\pi^2}{M_K^2}+21\frac{M_\pi^4}{M_K^4}\right)\nu_\pi\right.\\ &\left.\left.-\left(4-2\frac{M_\pi^2}{M_K^2}+2\frac{M_\pi^4}{M_K^4}\right)\nu_K-\left(1+3\frac{M_\pi^4}{M_K^4}\right)\nu_\eta\right]\right\},\end{aligned} \tag{C.4}$$

$$\begin{aligned}\Delta_L\mathcal{A}_2^{\mathrm{ew}} =& \left\{\frac{1}{2}\left(M_K^2-2M_\pi^2\right)B\left(M_\pi^2,M_\pi^2,M_K^2\right)\right.\\ &+\frac{5}{8}\frac{M_K^2}{M_\pi^2}\left(M_K^2-\frac{8}{5}M_\pi^2\right)B\left(M_K^2,M_\pi^2,M_\pi^2\right)+\frac{M_K^4}{8M_\pi^2}B\left(M_K^2,M_\eta^2,M_\pi^2\right)\\ &\left.+\frac{5}{4}\left(\frac{M_K^2}{M_\pi^2}+\frac{11}{5}\right)\nu_\pi-\frac{1}{2}\left(\frac{M_K^2}{M_\pi^2}-5\right)\nu_K-\frac{3}{4}\left(\frac{M_K^2}{M_\pi^2}-1\right)\nu_\eta\right\},\end{aligned} \tag{C.5}$$

for the $I=2$ amplitude.

Appendix D. Resummation of higher-order corrections

In this appendix we provide some details on the Omnès procedure for calculating the isospin amplitudes. The resummed loop corrections are contained in the factors $\mathcal{C}_I^{(R)}$, as defined in Eq. (56). At $\mathcal{O}(p^4)$ in the chiral expansion, these quantities should reproduce the one-loop χPT results in (54); this determines the factors $\mathcal{C}_I^{(R)}(s_0)$, with s_0 the subtraction point, up to higher-order local contributions:

$$\mathcal{C}_I^{(R)}(s_0)=\mathcal{C}_I^{(R)}\left[1-\delta\Omega_I\left(M_K^2,s_0\right)\right]\approx 1+\Delta_L\mathcal{A}_I^{(R)}-\delta\Omega_I\left(M_K^2,s_0\right). \tag{D.1}$$

Here, $\Delta_L\mathcal{A}_I^{(R)}$ is the one-loop χPT result and $\delta\Omega_I(M_K^2,s_0)$ is obtained by taking the chiral prediction for the phase shift $\delta_0^I(z)$ in $\Omega_I(M_K^2,s_0)$ and expanding $\Omega_I(M_K^2,s_0)$ to $\mathcal{O}(p^2)$,

$$\Omega_I\left(M_K^2,s_0\right)=1+\delta\Omega_I\left(M_K^2,s_0\right)+\mathcal{O}\left(p^4\right). \tag{D.2}$$

The explicit expressions for $\Delta_L\mathcal{A}_I^{(R)}$ are listed in Appendix C. The once-subtracted Omnès exponential

$$\Omega_I\left(M_K^2,s_0\right)=\exp\left\{\frac{(M_K^2-s_0)}{\pi}\int\limits_{4M_\pi^2}^{\bar{z}}\frac{dz}{(z-s_0)}\,\frac{\delta_0^I(z)}{(z-M_K^2-i\epsilon)}\right\} \tag{D.3}$$

contains the integral over the experimentally determined phase shift $\delta_0^I(z)$ with $I = 0$ or 2. The upper edge of the integral $\bar{z}$ should correspond to the first inelastic threshold in the given isospin channel. The corresponding expansion factor,

$$\delta\Omega_I\big(M_K^2, s_0\big) = \frac{(M_K^2 - s_0)}{\pi} \int\limits_{4M_\pi^2}^{\bar{z}} \frac{dz}{(z - s_0)} \frac{\delta_0^I(z)}{(z - M_K^2 - i\epsilon)}, \tag{D.4}$$

contains the same dispersive integral, but with the phase shift $\delta_0^I(z)$ determined at $\mathcal{O}(p^2)$ in χPT. The explicit expressions for $\delta\Omega_I(M_K^2, s_0)$ with $I = 0$ and 2 are as follows:

$$\begin{aligned}
\delta\Omega_0\big(M_K^2, s_0\big) = \frac{1}{32\pi^2 f_\pi^2} \Bigg\{ & \big(2M_K^2 - M_\pi^2\big)\sigma\big(M_K^2\big) \ln\left[\frac{\sigma(M_K^2) - \sigma(\bar{z})}{\sigma(M_K^2) + \sigma(\bar{z})}\right] \\
& - \big(2s_0 - M_\pi^2\big)\sigma(s_0) \ln\left[\frac{\sigma(s_0) - \sigma(\bar{z})}{\sigma(s_0) + \sigma(\bar{z})}\right] \\
& - 2\big(M_K^2 - s_0\big) \ln\left[\frac{1 - \sigma(\bar{z})}{1 + \sigma(\bar{z})}\right] \Bigg\}, \\
\delta\Omega_2\big(M_K^2, s_0\big) = \frac{1}{32\pi^2 f_\pi^2} \Bigg\{ & \big(2M_\pi^2 - M_K^2\big)\sigma\big(M_K^2\big) \ln\left[\frac{\sigma(M_K^2) - \sigma(\bar{z})}{\sigma(M_K^2) + \sigma(\bar{z})}\right] \\
& - \big(2M_\pi^2 - s_0\big)\sigma(s_0) \ln\left[\frac{\sigma(s_0) - \sigma(\bar{z})}{\sigma(s_0) + \sigma(\bar{z})}\right] \\
& + \big(M_K^2 - s_0\big) \ln\left[\frac{1 - \sigma(\bar{z})}{1 + \sigma(\bar{z})}\right] \Bigg\},
\end{aligned} \tag{D.5}$$

where for convenience we have defined $\sigma(s) \equiv \sqrt{1 - 4M_\pi^2/s}$.

In the following numerical analysis we have varied the subtraction point between $s_0 = 0$ and $s_0 = 3M_\pi^2$, together with the upper edge of the Omnès integral $\bar{z}$, to estimate the sensitivity of our predictions to these parameters. We have fixed the χPT renormalization scale at $\nu = M_\rho$. In Tables 2 and 3 the dispersive part of the Omnès factors and $\delta\Omega_I(M_K^2, s_0)$ are reported as functions of s_0, for $\bar{z} = 1\ \text{GeV}^2$ and $\bar{z} = 2\ \text{GeV}^2$, respectively. The corresponding moduli of the corrections $\mathcal{C}_I^{(R)}$, derived according to Eq. (62), are given in Tables 4 and 5. The residual tiny dependence of $|\mathcal{C}_I^{(R)}|$ on the subtraction point s_0 should be cancelled by missing $\mathcal{O}(p^6)$ contributions to $\mathcal{C}_I^{(R)}(s_0)$, since the local ambiguity of the

Table 2
The s_0 dependence of the once-subtracted Omnès parameters for $\bar{z} = 1\ \text{GeV}^2$

s_0	$\Re_0$	$\delta\Omega_0$	$\Re_2$	$\delta\Omega_2$
0	1.45	$0.32 + 0.46\,i$	0.94	$-0.16 - 0.20\,i$
M_π^2	1.40	$0.29 + 0.46\,i$	0.95	$-0.15 - 0.20\,i$
$2M_\pi^2$	1.33	$0.25 + 0.46\,i$	0.96	$-0.13 - 0.20\,i$
$3M_\pi^2$	1.26	$0.21 + 0.46\,i$	0.97	$-0.12 - 0.20\,i$

Table 3
The s_0 dependence of the once-subtracted Omnès parameters for $\bar{z} = 2$ GeV^2

s_0	$\Re_0$	$\delta\Omega_0$	$\Re_2$	$\delta\Omega_2$
0	1.58	$0.47 + 0.46\,i$	0.92	$-0.24 - 0.20\,i$
M_π^2	1.51	$0.43 + 0.46\,i$	0.93	$-0.22 - 0.20\,i$
$2M_\pi^2$	1.44	$0.39 + 0.46\,i$	0.94	$-0.20 - 0.20\,i$
$3M_\pi^2$	1.35	$0.33 + 0.46\,i$	0.95	$-0.17 - 0.20\,i$

Table 4
Resummed loop corrections with one subtraction and $\bar{z} = 1$ GeV^2

s_0	$\lvert\mathcal{C}_0^{(8)}\rvert$	$\lvert\mathcal{C}_0^{(27)}\rvert$	$\lvert\mathcal{C}_0^{(\mathrm{ew})}\rvert$	$\lvert\mathcal{C}_2^{(27)}\rvert$	$\lvert\mathcal{C}_2^{(\mathrm{ew})}\rvert$
0	1.37	2.47	1.38	1.06	0.62
M_π^2	1.36	2.42	1.37	1.05	0.61
$2M_\pi^2$	1.35	2.36	1.36	1.05	0.60
$3M_\pi^2$	1.33	2.28	1.34	1.04	0.59

Table 5
Resummed loop corrections with one subtraction and $\bar{z} = 2$ GeV^2

s_0	$\lvert\mathcal{C}_0^{(8)}\rvert$	$\lvert\mathcal{C}_0^{(27)}\rvert$	$\lvert\mathcal{C}_0^{(\mathrm{ew})}\rvert$	$\lvert\mathcal{C}_2^{(27)}\rvert$	$\lvert\mathcal{C}_2^{(\mathrm{ew})}\rvert$
0	1.26	2.45	1.27	1.10	0.68
M_π^2	1.27	2.41	1.27	1.10	0.67
$2M_\pi^2$	1.27	2.36	1.28	1.09	0.65
$3M_\pi^2$	1.26	2.28	1.27	1.08	0.64

Omnès procedure has been only solved to $\mathcal{O}(p^4)$ in the chiral expansion. From Tables 4 and 5 one can also verify that the once-subtracted result is sensitively dependent on $\bar{z}$.

As it was noticed in Ref. [2], the sensitivity to the higher energy region of the dispersive integral (i.e., the numerical dependence on the upper edge $\bar{z}$) is reduced by performing more subtractions. However, a better knowledge of the theory is required in this case. Indeed, the sensitivity to unknown higher-order corrections in the chiral expansion will increase with the number of subtractions, so that the resulting amplitudes can only be trusted at the lowest values of the subtraction point ($s_0 \sim 0$), where χPT corrections are moderate. We have checked these statements using the twice-subtracted Omnès

Table 6
The $\bar{z}$ dependence of the twice-subtracted Omnès parameters for $s_0 = 0$

$\bar{z}$ (GeV^2)	$\Re_0$	$\delta\Omega_0$	$\Re_2$	$\delta\Omega_2$
1	1.44	$0.40 + 0.46\,i$	0.86	$-0.20 - 0.20\,i$
2	1.46	$0.42 + 0.46\,i$	0.85	$-0.21 - 0.20i$

Table 7
Resummed loop corrections with two subtractions and $s_0 = 0$

$\bar{z}$ (GeV^2)	$\lvert\mathcal{C}_0^{(8)}\rvert$	$\lvert\mathcal{C}_0^{(27)}\rvert$	$\lvert\mathcal{C}_0^{(\mathrm{ew})}\rvert$	$\lvert\mathcal{C}_2^{(27)}\rvert$	$\lvert\mathcal{C}_2^{(\mathrm{ew})}\rvert$
1	1.25	2.34	1.26	1.00	0.60
2	1.23	2.34	1.24	1.00	0.61

exponential [2]:

$$\Omega_I\big(M_K^2, s_0\big) = \exp\left\{ \big(M_K^2 - s_0\big)\frac{g_I'(s_0)}{1 + g_I(s_0)} + \frac{(M_K^2 - s_0)^2}{\pi} \int_{4M_\pi^2}^{\bar{z}} \frac{dz}{(z - s_0)^2} \frac{\delta_0^I(z)}{(z - M_K^2 - i\epsilon)} \right\}, \tag{D.6}$$

where the functions $g_I(s)$ (and their first derivatives $g_I'(s)$) are the one-loop contributions (and their derivatives) to the isospin amplitudes due to the elastic $\pi\pi$ rescattering:

$$g_0(s) = -\frac{1}{2}\big(2s - M_\pi^2\big)B\big(M_\pi^2, M_\pi^2, s\big),$$
$$g_2(s) = \frac{1}{2}\big(s - 2M_\pi^2\big)B\big(M_\pi^2, M_\pi^2, s\big). \tag{D.7}$$

The expansion of $\Omega_I(M_K^2, s_0)$ at $\mathcal{O}(p^2)$ defines

$$\delta\Omega_I\big(M_K^2, s_0\big) = \big(M_K^2 - s_0\big)g_I'(s_0) + \frac{(M_K^2 - s_0)^2}{\pi} \int_{4M_\pi^2}^{\bar{z}} \frac{\mathrm{d}z}{(z - s_0)^2} \frac{\delta_I(z)}{(z - M_K^2 - i\epsilon)}, \tag{D.8}$$

where the phase shift $\delta_0^I(z)$ is taken at $\mathcal{O}(p^2)$ in χPT. The numerical results obtained at $s_0 = 0$, with the twice-subtracted Omnès procedure, are reported in Tables 6 and 7.

The final results for the moduli of the correction factors $\mathcal{C}_I^{(R)}$, quoted in Eq. (62), take into account the sensitivity to s_0 and $\bar{z}$ of the once-subtracted solution and the values obtained at $s_0 = 0$ with two subtractions.

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