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Note on the power divergence in lattice calculations of $\Delta I = 1/2$ $K \to \pi\pi$ amplitudes at $M_K = M_\pi$

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In this Brief Report, we clarify a point concerning the power divergence in lattice calculations of $\Delta I = 1/2$ $K \to \pi\pi$ decay amplitudes. There have been worries that this divergence might show up in the Minkowski amplitudes at $M_K = M_\pi$ with all the mesons at rest. Here we demonstrate, via an explicit calculation in leading-order chiral perturbation theory, that the power divergence is absent at the above kinematic point, as predicted by *CPS* symmetry.

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The subtraction of a power divergence, which arises via the mixing of dimension-six four-fermion operators with those of lower dimension, has been one of the central issues in lattice calculations of $\Delta I = 1/2$ $K \to \pi\pi$ amplitudes. This power divergence is, of course, unphysical, and can be related to a shift of the vacuum due to the inclusion of the weak interaction in chiral perturbation theory (χPT) [1–4]. It results in the so-called tadpole operators, which contribute to the processes $K^0 \to |0\rangle$ and $\overline{K^0} \to |0\rangle$, in χPT with weak interactions.

As argued in Ref. [5], this power divergence should be absent for $K \to \pi\pi$ amplitudes when $m_s = m_d = m_u$ ($m_{u,d,s}$ are the masses of u, d, and s quarks), due to the exact *CPS* symmetry [1] of the four-fermion operators that mediate $K \to \pi\pi$ decays. In Ref. [6], it was argued that the power divergence indeed does disappear in *Euclidean space* at $M_K = M_\pi$. However, a naive calculation in Minkowski space suggests that this power divergence might still be present at $M_K = M_\pi$ when all mesons are at rest. The issue is relevant, as it has been proposed that this unphysical kinematic point can be used to extract the low-energy constants relevant for $\Delta I = 1/2$ $K \to \pi\pi$ to order p^4 in χPT [7,8].[1]

In this Brief Report, we show, via an explicit calculation in χPT, that also in *Minkowski* space the power divergence is not present in $\Delta I = 1/2$ $K \to \pi\pi$ amplitudes at $M_K = M_\pi$, with all mesons at rest. Since it has already been argued in Ref. [9] that the $\Delta I = 1/2$ $K \to \pi\pi$ amplitudes in partially quenched χPT at the kinematic point $M_K = M_\pi$ suffer from problems related to the lack of unitarity [6,10,11], we concentrate here on full QCD. Our conclusions on the power divergence will, however, not change in the (partially) quenched case.

To simplify the discussion, we only consider weak operators in the (8,1) irreducible representation (irrep) of $SU(3)_L \times SU(3)_R$. The weak mass operator in this irrep at $\mathcal{O}(p^2)$ in the chiral expansion is

$$\mathcal{O}_2^{(8,1)} = \alpha_2 \{ 2 B_0 \mathrm{Tr}[\lambda_6 (\mathcal{M}^\dagger \Sigma + \Sigma^\dagger \mathcal{M})] \}, \tag{1}$$

where α_2 is the (power-divergent) low-energy constant associated with this operator, $B_0 = -\langle 0 | \overline{u}u + \overline{d}d | 0 \rangle / f^2$ (in the chiral limit), λ_6 is a Gell-Mann matrix, $\mathcal{M}$ is the quark-mass matrix, and Σ is the standard nonlinear Goldstone field.

We first observe that *CPS* symmetry implies that the parity-odd part of this operator is proportional to $m_s - m_d$. In fact,

$$\mathcal{O}_2^{(8,1)} = \alpha_2 \{ B_0 (m_s + m_d) \mathrm{Tr}[\lambda_6 (\Sigma + \Sigma^\dagger)] + i B_0 (m_s - m_d) \mathrm{Tr}[\lambda_7 (\Sigma - \Sigma^\dagger)] \}. \tag{2}$$

Therefore, at $m_s = m_d$ the parity-odd part of the operator vanishes, and thus its $K \to \pi\pi$ matrix element should vanish as well for $M_K = M_\pi$. This was confirmed by an explicit calculation in Euclidean space (as reported in Ref. [6]), and should be true in Minkowski space as well.

At leading order in the chiral expansion, $\mathcal{O}_2^{(8,1)}$ contributes to the $K \to \pi\pi$ amplitudes via the diagrams in Fig. 1, where the gray circles represent the weak-mass operator, and the

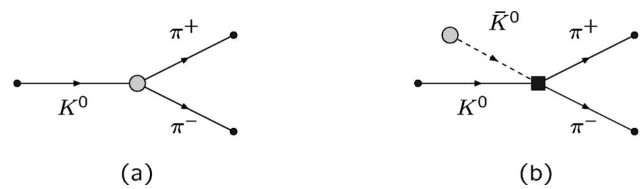

FIG. 1. Diagrams involving the weak mass operator at the lowest order in the chiral expansion for the $\Delta I = 1/2$ $K \to \pi\pi$ amplitudes. The gray circles represent the operator $\mathcal{O}_2^{(8,1)}$, and the square is the $K^0 \overline{K^0} \to \pi^+ \pi^-$ vertex from the lowest-order strong chiral Lagrangian. The dashed line in (b) indicates that the $\overline{K^0}$ could be off-shell, while all the other mesons are always on-shell.

[1] It follows from our analysis that the low-energy constants α_2 and $e^r_{1,2,5}$ should not appear in Eq. (31) of Ref. [8].

square is from the leading-order strong chiral Lagrangian. In Fig. 1(b), there is a pole associated with the $\bar{K}^0$ propagator (the dashed line in Fig. 1), which takes the form

$$\frac{i}{(M_K-2M_\pi)^2-M_K^2+i\epsilon}, \tag{3}$$

when all the other three on-shell particles are at rest. For fixed $M_K \neq M_\pi$, one may take $\epsilon \to 0$ at any stage of the calculation, since the denominator of Eq. (3) does not vanish in that case. However, for $M_K = M_\pi$, the $i\epsilon$ prescription is needed in order to define the propagator, and should be taken to zero *only at the end of the calculation*. In that case, one finds

$$\begin{aligned}\langle \pi^+\pi^-|\mathcal{O}_2^{(8,1)}|K^0\rangle_{M_K=M_\pi} &= \lim_{\epsilon\to 0}\Bigg\{ \lim_{M_K\to M_\pi} \alpha_2(M_K^2-M_\pi^2) \\ &\times \left[\frac{8(3M_\pi(M_K-2M_\pi)-i\epsilon)}{3f^3(4iM_\pi(M_K-M_\pi)+\epsilon)}\right]\Bigg\}=0, \end{aligned} \tag{4}$$

which indicates that there is no need to perform the subtraction of a power divergence.

Let us discuss this claim in more detail. We begin by noting that it was shown long ago [1–3] that if the weak mass term $\int d^4x \mathcal{O}_2^{(8,1)}$ is treated as a perturbation to the strong chiral Lagrangian, it does not have any observable effect. However, here we consider the unphysical situation of an energy nonconserving matrix element of $\mathcal{O}_2^{(8,1)}$ (corresponding to the insertion of this operator at a fixed time, see below), and the above consideration does not apply.

The factor $(M_K^2-M_\pi^2)$ on the right-hand side of Eq. (4) originates from the *CPS* symmetry of the operator [c.f. Eq. (2)], while the quantity in the square brackets is determined by the kinematics. This latter quantity indeed diverges in the limit $M_K \to M_\pi$ (and $\epsilon \to 0$). That this is exactly what one expects to happen because the $\bar{K}^0$ propagator in Fig. 1(b) goes on-shell without being amputated. In fact, Fig. 1(b) also represents the process of K^0-$\bar{K}^0$ scattering into π^+-π^-, but in that case in order to obtain a finite amplitude, the LSZ reduction formula tells us to amputate the $\bar{K}^0$ external leg, before putting it on-shell. Since in our case this leg is not amputated, the diagram is divergent in the on-shell limit. In the case in which K^0, π^+, and π^- are all at rest, this "$\bar{K}^0$ on-shell" point coincides with the limit $M_K \to M_\pi$, and *CPS* symmetry prevents the divergence from happening: the amplitude actually vanishes at $M_K = M_\pi$.

However, one may consider the following more general situation. Consider for instance kinematics with K^0 at rest but π^+ and π^- carrying spatial momenta $\vec{p}$ and $-\vec{p}$, respectively. In that case, the $\bar{K}^0$ on-shell point is at $M_K = E_\pi = \sqrt{M_\pi^2+|\vec{p}|^2}$, and Fig. 1(b) is proportional to $\vec{p}^{\,2}/\epsilon$ at this point. The extra factor $(M_K^2-M_\pi^2)$ clearly does not help in this case, and the divergence occurs of course for exactly the same reason as described above.

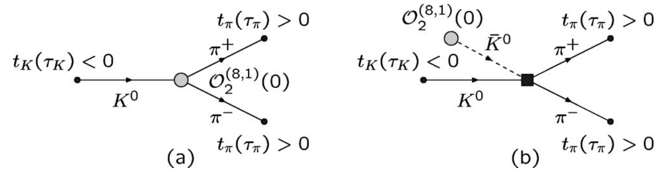

FIG. 2. Diagrams involving the weak mass operator at the lowest order in the chiral expansion for the correlator $\langle 0|\pi^+\pi^- Q^{(8,1)}\bar{K}^0|0\rangle$. The gray circles represent the weak mass operator $\mathcal{O}_2^{(8,1)}$, and the square is the $K^0\bar{K}^0 \to \pi^+\pi^-$ vertex from the lowest-order strong chiral Lagrangian. The dashed line in (b) means $\bar{K}^0$ could be off-shell, while all the other mesons are always on-shell. The weak operator is at the space-time origin. K^0 is created at t_K (τ_K), and the pions are annihilated at t_π (τ_π) in Minkowski (Euclidean) space.

We gain more insight by considering the amplitude in position space, as in Fig. 2(b). This diagram contains a factor $e^{-iM_K|t_w-t_s|}$ from the $\bar{K}^0$ propagator, where t_w is the location (in time) of the weak operator $\mathcal{O}_2^{(8,1)}$ (taken as $t_w=0$ in the diagram), and t_s is the location of the strong vertex. The LSZ prescription for this $\bar{K}^0$ line corresponds to taking a Fourier transform with respect to t_w, and putting the corresponding momentum on-shell. For this to work, the integral over t_w needs to be regulated by replacing $M_K \to M_K - i\epsilon$, and this is precisely what leads to the $i\epsilon$ prescription in Eq. (3). It follows that the divergence encountered here is regulated by considering the amplitude at finite t_w (by time-translation invariance we may choose $t_w=0$). This is of course what one does anyway in a lattice computation of this amplitude. It is therefore instructive to consider this amplitude in position space rather than momentum space [6], which is what we will do next.

Since we take all our mesons to have vanishing spatial momentum, we will consider the relevant correlators in the time-momentum representation, i.e., study the correlators as functions of three-momentum and time. In this setup, a free meson propagator with energy $E_{\vec{p}} = \sqrt{m^2+|\vec{p}|^2}$ (m is the mass and $\vec{p}$ is the three-momentum of the meson) is

$$\frac{e^{-iE_{\vec{p}}|t|}}{2E_{\vec{p}}} \ \text{(Minkowski)},$$

$$\frac{e^{-E_{\vec{p}}|\tau|}}{2E_{\vec{p}}} \ \text{(Euclidean)},$$

where t (τ) is the Minkowski (Euclidean) time. The time dependence of the Minkowski expression is of course in accordance with the $i\epsilon$ prescription of Eq. (3). We now consider the Minkowski correlator

$$C_2 = \langle 0|\pi_0^+(t_\pi)\pi_0^-(t_\pi)\mathcal{O}_2^{(8,1)}(0)K_0^0(t_K)|0\rangle$$

and its Euclidean counterpart

$$\mathcal{C}_2 = \langle 0|\pi^+_{\vec{0}}(\tau_\pi)\pi^-_{\vec{0}}(\tau_\pi)\mathcal{O}_2^{(8,1)}(0)K^0_{\vec{0}}(\tau_K)|0\rangle.$$

For simplicity, we choose to annihilate the two pions at the same time, and assume that $t_K(\tau_K)<0$ and $t_\pi(\tau_\pi)>0$. The weak operator is inserted at time $t_w=0$ $(\tau_w=0)$. All particles are at rest (as indicated by the subscripts $\vec{0}$). The relevant diagrams for the above correlators are shown in Fig. 2. In the following, we only present the result in Minkowski space, but stress that the calculation in Euclidean space is virtually identical, and leads to the same conclusion [6].

The contribution from Fig. 2(a) to the correlator C_2 is

$$C_{2(\mathrm{a})} = \frac{-8i\alpha_2}{3f^3}(M_K^2-M_\pi^2)\left[\frac{e^{-iM_K|t_K|}e^{-2iM_\pi t_\pi}}{(2M_K)(2M_\pi)(2M_\pi)}\right],$$

while Fig. 2(b) leads to

$$\begin{aligned}C_{2(\mathrm{b})} = {} & \frac{4i\alpha_2}{3f^3}(M_K^2-M_\pi^2)\frac{i}{(2M_K)(2M_K)(2M_\pi)(2M_\pi)}\\ & \times\int dt_s e^{-iM_K|t_s-t_K|}e^{-iM_K|t_s|}e^{-2iM_\pi|t_\pi-t_s|}\\ & \times\{M_K^2[1+\epsilon(t_s)\epsilon(t_s-t_K)]\\ & +M_KM_\pi[\epsilon(t_s)+\epsilon(t_s-t_K)]\epsilon(t_\pi-t_s)+2M_\pi^2\},\end{aligned} \tag{5}$$

where t_s is the time component of the space-time position of the strong chiral Lagrangian vertex in this diagram. The function $\epsilon(t)$ is defined as

$$\epsilon(t) = \begin{cases} +1, & t>0 \\ -1, & t<0. \end{cases} \tag{6}$$

In the above two equations, only the integral of t_s between 0 and t_π can result in a "vanishing denominator" when $M_K \to M_\pi$. Explicitly, it is

$$\begin{aligned}C_{2(\mathrm{b})}|_{0\to t_\pi} = {} & -iC_{2(\mathrm{a})}\left(\frac{M_K^2+M_\pi^2+M_KM_\pi}{2M_K}\right)\\ & \times\left\{\frac{1}{-2i(M_K-M_\pi)}[e^{-2i(M_K-M_\pi)t_\pi}-1]\right\}.\end{aligned} \tag{7}$$

When $M_K\to M_\pi$, the factor

$$\frac{1}{-2i(M_K-M_\pi)}[e^{-2i(M_K-M_\pi)t_\pi}-1]$$

is just t_π. Therefore, for finite t_π (or finite τ_π in Euclidean space), C_2 (or $\mathcal{C}_2$ in Euclidean space) vanishes at $M_K=M_\pi$ (with both $C_{2(\mathrm{a})}=0$ and $C_{2(\mathrm{b})}=0$ separately) due to the explicit factor of $M_K^2-M_\pi^2$, and there is no power divergence. This conclusion remains true to all orders in χPT.

To conclude, we would like to discuss in some more detail why the factor linear in t_π appears in Eq. (7), even though $C_{2(\mathrm{b})}$ vanishes for $M_K=M_\pi$ because of the explicit factor (m_s-m_d) in Eq. (2). Omitting this factor, our result contains a term linear in t_π for $M_K=M_\pi$. One would expect that if one takes t_π large after taking the limit $M_K\to M_\pi$, it would be necessary to unitarize $C_{2(\mathrm{b})}$.[2] Reinterpreting Fig. 2(b) as the lowest-order contribution in χPT (in the strong vertex) to $K^0\overline{K^0}\to\pi^+\pi^-$ scattering (as we did above), the term linear in t_π can be understood as follows. For $M_K=M_\pi$, there is full $SU(3)$ symmetry, and $|KK\rangle$ and $|\pi\pi\rangle$ s wave, $I=0$ states can be expressed in terms of the ($I=0$ components of the) irreducible states $|1\rangle$, $|8\rangle$, and $|27\rangle$ through the relations [9]

$$\begin{aligned}|1\rangle &= \frac{1}{\sqrt{2}}|KK\rangle+\frac{1}{2\sqrt{2}}|\eta\eta\rangle+\frac{1}{2}\sqrt{\frac{3}{2}}|\pi\pi\rangle,\\ |8\rangle &= \frac{1}{\sqrt{5}}|KK\rangle+\frac{1}{\sqrt{5}}|\eta\eta\rangle-\sqrt{\frac{3}{5}}|\pi\pi\rangle,\\ |27\rangle &= -\sqrt{\frac{3}{10}}|KK\rangle+\sqrt{\frac{27}{40}}|\eta\eta\rangle+\frac{1}{\sqrt{40}}|\pi\pi\rangle,\end{aligned} \tag{8}$$

where

$$\begin{aligned}|KK\rangle &= \frac{1}{\sqrt{2}}(|K^0\overline{K^0}\rangle+|K^+K^-\rangle),\\ |\pi\pi\rangle &= \frac{1}{\sqrt{3}}(|\pi^0\pi^0\rangle+\sqrt{2}|\pi^+\pi^-\rangle).\end{aligned} \tag{9}$$

From these relations, it follows that

$$\begin{aligned}\langle\pi\pi(t=t_\pi)|KK(t=0)\rangle = {} & \frac{\sqrt{3}}{4}\langle 1(t=t_\pi)|1(t=0)\rangle\\ & -\frac{\sqrt{3}}{5}\langle 8(t=t_\pi)|8(t=0)\rangle\\ & -\frac{\sqrt{3}}{20}\langle 27(t=t_\pi)|27(t=0)\rangle.\end{aligned}$$

To leading order in χPT this expression contains a term linear in t_π, the coefficient of which is the corresponding linear combination of finite-volume two-particle energy shifts, thus explaining how a term linear in t_π appears in $C_{2(\mathrm{b})}$. Note

[2]The on-shell divergence discussed earlier has nothing to do with this term linear in t_π, but, as explained above, with the oscillatory behavior of $e^{-iM_K|t_w-t_s|}$.

that the normalization of $C_{2(b)}$ has been chosen such that (in leading nonvanishing order) it is independent of the spatial volume. Higher orders in χPT will indeed unitarize our result.

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