Gibbsian properties

Ermolaev, Victor Nikolaevich

Gibbsian properties:

examples of failures and an application

Victor Ermolaev

Cover design: Alexander Vasilyev
hamster3d.com

Publisher: Wöhrmann print service
wps.nl

RIJKSUNIVERSITEIT GRONINGEN

Gibbsian properties:

examples of failures and an application

Proefschrift

ter verkrijging van het doctoraat in de
Wiskunde en Natuurwetenschappen
aan de Rijksuniversiteit Groningen
op gezag van de
Rector Magnificus, dr. E. Sterken,
in het openbaar te verdedigen op
vrijdag 9 september 2011
om 16.15 uur

door

Victor Nikolaevich Ermolaev

geboren op 23 mei 1984
te Volzhsky, Rusland

Promotores:	Prof. dr. C. Külske
	Prof. dr. A.C.D. van Enter
	Prof. dr. E. Verbitskiy
Beoordelingscommissie:	Prof. dr. R. Fernández
	Prof. dr. W. Th. F. den Hollander
	Prof. dr. F. Redig

ISBN: 978-90-367-5024-0

моим родителям

Acknowledgements

The title page claims that the author of the present thesis is myself, Victor Ermolaev, though this is true only partially. Thus, the second page — not to be taken as an offense — I would like to devote fully and completely to people without whom this thesis would not have been possible.

I am heartily thankful to my supervisor, professor Christof Külske, who decided to give me a chance in the field of Probability Theory, stoutly bore my endless questions about Gibbs measures, and took time explaining things in great detail. In the first place my Gibbs-measures ignorance fell on his shoulders. Another brilliant man — personally and professionally — whose encouragement, guidance, and support (especially at the final level) enabled me to develop a better understanding of the subject is professor Aernout van Enter. I owe my deepest gratitude to him for giving me an intuition of Gibbs problems and making me develop my own. He has made available his support not only in the mathematical sense but in linguistic sense, too. He supported me in my endless war with English articles (which every Russian person wages, I reckon :), explained a lot about commas, as well as about a good style of scientific writing. Also I offer my regards to professor Evgeny Verbitskiy, whose technology background allowed me to take a good look at the practical side of the Gibbs theory. He always was very helpful and willing to explain, even if a question was asked and answered several times before. I thank him for a great number of examples of applications of theoretic knowledge in industry. Also I would like to thank professors Roberto Fernandez, Frank Redig, and Frank den Hollander for painstaking efforts reading this thesis, professor Christian Maes who spent time discussing and clearing up a lot for me; these people had a lot of influence on the present piece of work both via personal dialogue and via their articles and lecture notes. I would like to also mention the person, who actually got me interested in science — Alexander Batkhin, my master-thesis supervisor.

Of course, I would never have been able to carry any scientific activities, if nobody created a suitable environment. I am indebted a lot to the faculty secretaries: Ineke Schelhaas, Esmee Elshof, and Desiree Hansen. “What”, “where”, “how”, “could you” were the questions they always could give an answer for. I am also grateful to Annette Korringa, Janieta de Jong, Yvonne

van der Weerd, Anneke Toxopeus for taking care of the administrative needs of my study at the University of Groningen.

Being scared to forget anybody, I would like to show gratitude to all my colleagues, you all were always open for a discussion and ready to offer me a help at any time. My special thanks for support goes to our probability community: Giulio, who showed me how to party blindly and work properly and honestly, Wioletta, for educative discussions and explaining what and why is wrong with the world today, Alex, I always got something new to know after several words with you, Siamak, for offering a different point of view; people from Leiden — Luca, Julián , Renato, Florian, Alessio, Kiamars, and Cristian.

I thank the University of Groningen for letting me be curious for other people's money.

Конечно же, моя благодарность была бы непростительно неполна без упоминания моих русских друзей в Нидерландах. Огромное спасибо Жене за разговоры о голландской культуре, о путешествиях, о deadline-ах и обо всём на свете, Володе — за бесконечный поток фильмов о зомби и советы относительно хорошей музыки, Змараю и Лорианне, Оле и Кате за общество и время, проведенное в одной компании.

Ане — тебе пришлось выслушать и вытерпеть больше всех за время моего PhD. Спасибо за твою поддержку и ободряющие слова.

И последнему, но не в последнюю очередь, спасибо Александру Васильеву за отличную обложку.

Thank you all! Спасибо!

Groningen, Autumn 2011, V.N.E.

Contents

1

Introduction

One may think of the starting point of the theory of Gibbs measures as an (successful) attempt to answer the question about physical reality: "How can a ferromagnet or a gas or a liquid in thermal equilibrium be described in mathematical terms?". The main feature to be captured is the existence of different *phases* of a system, for instance positive and negative magnetizations for magnets, existence as a liquid, steam, or ice for water. However, this question could be extended to a whole class of physical systems by identifying the common features of a ferromagnet, a gas, and a liquid. The most straightforward and promising characteristic is that all of them contain an enormously large number of microscopic components taking values in the same state-space. Therefore the question may be transformed into another: "How can a system consisting of large number of identical interacting components in equilibrium be described in mathematical terms?". The first thing we do, to simplify, is replacing physical space by a suitable graph. As an example one may think of a lattice , which is actually physically realistic for crystals, but it can also be used it as a simplified model for continuous space, or a tree (e.g. a Bethe lattice) as an even cruder model, although it is often appropriate for models met in biological applications. Despite the relative simplification, the microscopic structure of such a system is still extremely complex, while its macroscopic characteristics are often not very complicated (temperature or pressure for a gas, magnetization for a ferromagnet, etc.) The idea of Maxwell, Boltzmann, and Gibbs rephrased in modern mathematical language was as follows: the microscopic complexity can be overcome by a statistical approach; the macroscopic determinism may be regarded as a consequence of an application of a suitable law of large numbers. So to say, no state of a system should be described by a fixed configuration of the system's components. This description should rather be given in terms of a family of random variables $\{\sigma_i\}$. These random variables are associated with the *sites* i of the graph chosen as a mathematical abstraction of a physical space. Consequently, the joint distribution μ of $\{\sigma_i\}$ determines the system's state. The finite-volume prescription is given by Gibbs' canonical ensemble prescription

$$\mu(d\sigma) = \frac{1}{Z} e^{-\beta H(\sigma)} d\sigma, \tag{1.1}$$

where the function $H(\sigma)$ refers to the energy of any configuration σ — the *Hamiltonian* — and β is an inverse temperature. $H(\sigma)$ is formed by the interactions of all microscopic components and (perhaps) an external force h. In the case of ferromagnet, h has a meaning of an external magnetic field. The finite-volume Gibbs measure $\mu(d\sigma)$ is always unique, therefore there exists only one phase for any finite-volume system.

The next step in achieving the goal of an adequate description of existence of several phases is to identify large-volume (physically) or infinite-volume (mathematically) behaviour. It is easy to see that the formula (1.1) is ill-defined for any infinite configuration. This is due to the fact that the energy of an event in infinite volume is also infinite. There are two approaches developed to overcome this difficulty. The first one is expressing energy of a configuration as a function of its size and straightforward taking limits as the volume grows. Under some suitable conditions such limit(s) exist(s). Existence of several limiting measures will correspond to a *phase transition*. The second approach is based on the very fruitful idea that instead of performing this limiting procedure, one may study directly the set of all possible limiting objects and their *conditional restrictions* on finite sub-volumes fixing the configuration outside of them. These restrictions are given by the finite-volume prescription now depending on the changing inside and the fixed outside of particular volumes in some suitable way (this issue will be discussed shortly). Each extremal limiting object corresponds to a phase of a system. The family of the finite-volume conditional distributions may be compatible with more than one limiting measure, then a phase transition occurs, this phenomenon has been widely studied in many articles and books.

The former discussion can be summarized in two ways: physically, *a Gibbs measure is a mathematical idealization of an equilibrium state of a physical system which consists of a very large number of interacting components*, mathematically *a Gibbs measure is the distribution of a stochastic process which, instead of being indexed by time, is parametrized by the spatial coordinates of the system and has the special feature of admitting prescribed versions of the conditional distributions outside finite regions*, Georgii [26].

To be more specific we consider a model living on a lattice. Let $\mathbb{L}$ be the whole lattice and Λ its finite-volume subset. Let σ_i be a random variable (a *spin*) sitting at site i of $\mathbb{L}$ and taking values from S, then the joint distribution of all spins σ_Λ in finite volume Λ takes values in $S^\Lambda =: \Omega_\Lambda$, the probability space of the whole system is therefore $S^{\mathbb{L}} = \Omega$. Within this mathematical framework the question whether a family of conditional probabilities $\{\gamma_\Lambda, \Lambda \Subset \mathbb{L}\}$, a *specification*, is Gibbs arises naturally. A beautiful theorem was proven by Sullivan [54] and later extended by Kozlov [35], stating that uniform non-nullness and quasilocality properties are necessary and sufficient conditions for a specification to be Gibbs. Uniform non-nullness refers to the fact that

all conditional probabilities in the family of any event are separated from zero by a positive constant. Quasilocality is a property that guarantees that the energy function is well-defined, differently this property may be expressed as follows:

$$\sup_{\omega,\eta,\tilde{\eta}\in\Omega} |\gamma_\Lambda(\sigma_\Lambda|\omega_{\Lambda_n}\eta) - \gamma_\Lambda(\sigma_\Lambda|\omega_{\Lambda_n}\tilde{\eta})| \xrightarrow{n\to\infty} 0, \tag{1.2}$$

for any ω. In mathematical terms, (1.2) corresponds to the fact that all intermediate configurations ω shield σ_Λ from the influence of the outside of ω; in physical terms, this statement has a more intuitive form: quasilocality ensures that any local experiment is controllable. Imagine a well-tuned guitar and an astronaut playing it in a space ship. No matter if a space ship is being prepared to start its journey from Earth to outer space or if a space ship has already started and reached the vacuum of the cosmos, the guitar will sound the same. The exterior of a space ship plays no role. The interior of a space ship filled with air (or close to air mixture of gases) creates a barrier preventing the influence of either vacuum or air outside of a space ship. Hence, playing a guitar is quasilocal experiment.

If the spin space is discrete, as it will mostly be in this thesis, quasilocality equals continuity (in the product topology).

While non-nullness of a specification is not a property to be ignored, it is not usually the main problem. The lack of quasilocality leads to more serious consequences, namely in this case the Boltzmann-Gibbs prescription is not any more well-defined. In heuristic terms and in the light of (1.2), non-quasilocality means that there is some mechanism transmitting the influence of "far-away" regions to any finite sub-volume and remaining active even in the case of no fluctuations in the spins in-between. For a measure to be non-Gibbs it is enough that there exists only one such configuration of the in-between spins. This very idea is the main tool of the Gibbs-non-Gibbs investigation.

Now let us explain why the question whether or not a given measure was a Gibbs measure first arose. This happened in the theory of equilibrium statistical mechanics, in the study of phase transitions. To study the behaviour of a system close to critical points or to prove the absence of phase transitions for a system, a *renormalization map* $\mathcal{R}$ was applied to the Hamiltonian of system's Gibbs measure μ, [27, 28]. Although such maps were studied before, these papers for the first time put the renormalization-group ideas and the Gibbs formalism together. The absence of phase transitions at high temperature and/or low density was known before, and the analysis in this region served as an illustration. The map $\mathcal{R}$ is expected to define a new Hamiltonian $H' := \mathcal{R}(H)$ and moreover induces a map K mapping the original measure μ to μ'. Having K defined, the existence of H' depends on whether the image measure μ' is Gibbs or not [11, 33]. If the image measure μ' is Gibbs, then the associated Hamiltonian H' is well-defined, otherwise — not. It was found that

the application of many maps may result in such "pathologies" [12, 19, 41]. When such a phenomenon appears it means that finite-volume conditional probabilities of the image system will acquire long-range dependencies (therefore lose its quasilocality property), at least for some non-removable configurations.

After having discovered this, another question attracted a lot of interest and attention: "How stable is the Gibbs property of a measure describing a system in equilibrium, when the last is subjected to a time evolution?". There is a long expanding list of references on this: starting from spin contraction [9, 44], general single-site transformations [38], continuing with stochastic spin evolutions for bounded and unbounded spins in [7, 12, 14, 36, 37, 39, 48] for different kinds of underlying graphs of a system, and going further. A good example of losing Gibbsianness under time evolution is given in [12]. Before explaining it, we need to discuss informally the celebrated Ising model, a paradigmatic statistical mechanical model, designed to study magnetic properties with spins taking values in $\{-1, +1\}$. We choose as time evolution the *spin-flip* (or *Glauber*) dynamics: each spin is flipped according to independent Poissonian clocks attached to a site. Rephrasing the proof in [12, see Theorem 6.3, fact 2], we start with the Ising model on lattice. Such a model at low temperature will typically be found in one of two phases (low-temperature extremal Gibbs measures) , which look similar to a *ground* state — state having the biggest probability to be observed — almost everywhere pluses or almost everywhere minuses. We start with a low-temperature Gibbs measure. We choose a site 0, which will be called the *origin*, let the time flow and flip a spin when an associated to it clock rings. After some time a configuration $\omega_{\Lambda\setminus 0}\sigma_{\Gamma\setminus\Lambda}$ will be observed, where Λ is a finite subset of Γ with the property that Γ is a lot larger than Λ. We suppose that the configuration $\omega_{\Lambda\setminus 0}$ is *atypical* (or, very improbable) and $\sigma_{\Gamma\setminus\Lambda}$ is typical for any of the possible starting phases. Though the configuration in $\Gamma \setminus \Lambda$ may be not typical for the phase we started with, the cost of creating such a configuration is proportional to the length of contour separating Γ from the rest of the lattice. On the other hand, the cost of inserting any atypical configurations in $\Lambda \setminus 0$ is proportional to the volume of Λ. Suppose $\Lambda \setminus 0$ is not too small, that it could be considered as a barrier shielding the origin from the influence of $\sigma_{\Gamma\setminus\Lambda}$. Nonetheless, the effectiveness of such a barrier is poor because the values of the spin at the origin at time 0 and later time are positively correlated. Thus, there exists a time window (possibly infinite) with values such that they are large enough for the dynamics to create an atypical configuration around the origin and not too big to keep the correlation between the values of spin at the origin at different times. Within this time interval $\sigma_{\Gamma\setminus\Lambda}$ determines the value of σ_0 and the quasilocality is lost.

The general picture is that for very general dynamics and very general ini-

tial measures the time-evolved measures are again Gibbsian, for a sufficiently small time-interval [10, 12, 37, 41, 43]. Long times however, even for simple dynamics offer the possibility for the emergence of non-Gibbsian measures. The discontinuities in the conditional probabilities which are responsible for the Gibbs-non-Gibbs transitions are produced by hidden phase transitions which pop up as a result of the conditioning procedure. Depending on the specific nature of the system there may be many mechanisms of such singularities [12, 36].

The present thesis attempts to investigate when the Gibbs property of an initial measure is preserved or lost if the initial Gibbs model is subjected to some transformation. This question is addressed for two classes of statistical mechanics models. The first class is mean-field models, where the underlying structure is a complete graph, so each component influences on all the others. The simplifications made for mean-field models allow to develop tractable results. The importance of these models is due to their ability to mimic the behaviour of large-dimensional lattices [37, 52], as well as their tractability. Mean-field models are objects of interest on their own in computer science as they catch well the behaviour of computer networks [2, 40], for example. We warn the reader that in the mean-field case, the notion of Gibbsianness will be slightly changed. This will be discussed later, and in full detail in Chapter 4. The second class concerns models on trees, where components interact in a local fashion. Tree models form a first step way from a mean-field setup towards a proper lattice model. The main achievement of this thesis is a rather complete description of the Gibbs-non-Gibbs regions in the space of temperature and time followed by explicitly presented equations governing the dynamics in the mean-field setup in vanishing external field.

In the end of the present thesis we will address a problem of classification coming from Information Theory and show how to approach it with the Gibbsian formalism.

1.1 Strategy

Following the route suggested in [12], the study of a failure of the Gibbs property under stochastic time evolution is connected with understanding of a "constrained" or "two-layers" model. This model reflects the influence of a dynamics. The effect of evolution on the initial Gibbs measure results in some transformations of the energy function of its finite-volume restrictions. The lack of phase transitions in "constrained" models immediately guarantees the Gibbs nature of the evolved system. On the other hand the existence of phase transitions may or may not imply the non-Gibbsian nature depending on the

setup, often an extra step to show the lack of the quasilocality property is needed.

For tree models, as for more general lattice models, quasilocality means that the influence of "far-away" regions on any finite regions is effectively stopped by the spins in-between, in other words, the interaction between spins has a local nature. Thus, the conditioning on "middle-distanced" spins is essentially the same as the conditioning on them and any "far-away" configuration. This implies that the finite-volume probabilities of the transformed measure conditioned on configurations different only on "far-away" regions should be almost equal. If not — the "middle-distanced" configuration is called *bad* and quasilocality is lost. For discrete spins, which are of interest in the present thesis, the property of quasilocality is equivalent to the continuity property of finite-volume conditional probabilities as a function of the conditioning (in the product topology).

The complete graph, which is used in mean-field models, discards the notion of "far-away" regions, because all spins communicate with each other equally, so the concept of quasilocality is meaningless in this setup. The presence of a phase transition in the transformed system still plays a role, however. Due to the graph structure, any configuration and all its permutations may be identified with a *real number*, hence conditioning on a configuration translates in conditioning on this real number. The main characteristic of conditioning becomes a *magnetization* (or, more generally, some form of an *empirical* average) of a system. A mean-field system is called non-Gibbsian if the single-site conditional probabilities depend in a discontinuous way on the magnetization of the conditioned spins. The last statement suggests an existence of a phase transition is a sufficient condition for non-Gibbsianness.

Hence, to deduce a lack of Gibbsianness for the transformed system one is left with the problem of investigating continuity properties of conditional probabilities for the "constrained" model.

1.2 Overview of Thesis

In this section we provide an informal guide to what is contained in the main body of the thesis and outline the main achievements of the present work.

In Chapter 2 we review the formal side of the theory of Gibbs measures, making rigorous statements and notions used in the introduction. Chapter 2 provides a common description for many models of statistical mechanics. A broad description of dynamics is given, and the question of how applying a dynamics can cause the loss of the Gibbsian property for the evolved (according to this dynamics) measure starting from an initial Gibbsian measure

is addressed. A general strategy of proving non-Gibbsianness is described in detail.

In Chapter 3 we look at tree models. In particular, we consider Cayley trees for simplicity. As explained in Chapter 3, an Ising model on a Cayley tree exhibits a phase transition, therefore several (among which there are three extremal homogeneous) Gibbs measures are admitted. We discuss when a time-evolved initial Gibbs measure loses its quasilocal property. All three measures will be shown losing the quasilocality property under time evolution after some time. Moreover, we keep track of *how many* configurations transmit an influence of "far-away" regions. Surprisingly, for one of the initial measures after some time *all* configurations become bad. This fact holds independently of the preference of the system to be in a certain state, but the preference value has to be smaller than a certain real number. The preference of the system to a state is expressed by an external magnetic field aligning the spins. At the end of Chapter 3 we point out possibilities to extend the results for Cayley trees to a more general class and explain why this has to be possible by identifying common tokens of Cayley and general trees. Chapter 3 has appeared as [13]. Chapter 3 positively answers on the question whether non-Gibbsianness becomes worse as time progresses in the Cayley tree setup.

Chapter 4 is devoted to the description of the time evolution of mean-field systems in thermal equilibrium subjected to arbitrary-temperature Glauber dynamics. The results of Chapter 4 have appeared as [15] and extend the work of Külske and Le Ny [36] contributing an analysis of low-temperature dynamics. The low-temperature dynamics corresponds to a *dependent* spin-flip dynamics. A detailed description of the decomposition of the time-temperature space into Gibbs and non-Gibbs regions is given. Each couple of coordinates in the time-temperature plane depending on its location will correspond to a Gibbs/non-Gibbs regime of the evolved system. In Section 4.2 of Chapter 4 we formulate the main result which covers the up-to-now accumulated knowledge of arbitrary-temperature dynamics for mean-field models.

Chapter 5 demonstrates an interplay between the Gibbsian formalism and applied problems. In particular, we show how the theory of Gibbs measures can be involved in solving classification problems. Such problems are of great importance, for example, in biological science and Information Theory. While solving classification problems or problems of denoising, one is usually given random samples and aims to (re-)construct a probability distribution which generated these samples. We build our analysis based on the simple observation that the nature of the distribution of interest is intrinsically Gibbsian. This notice allows us to develop an approach related to the potential of a Gibbs measure. We show that this way of treating problems can successfully solve problems of classification.

* * *

2

Generalities on Gibbs measures

In this chapter we collect notation and some known facts about Gibbs measures for a general class of models which will be used further on in this thesis. Among the things we discuss are scenarios when the Gibbsian character of a measure fails.

2.1 Preliminaries

Let S denote the *single-site space*. The space S is sometimes also called a *state space*. This choice of the space S is governed both by physics (what do we want to model?) and by a wish for simplifications to be made. We can for example think of $\{0,1\}$ for lattice-gas models, $\{-1,+1\}$ for a ferromagnet or an antiferromagnet. In Chapter 3 and Chapter 4 we will consider S to be the space of Ising spins $\{-1,+1\}$. Let this single-site space be equipped with an a priori measure α. Let $G=(\mathcal{V},\mathcal{E})$ be a generic graph having countably many vertices. Each random variable from a family $\sigma=\{\sigma_i\}_{i\in\mathcal{V}}$ is called a *spin* and takes values in the space S. We shall call a *region* a finite subset of $\mathcal{V}$. The finiteness property will be indicated by the symbol "$\Subset$", i.e. for region Λ, $\Lambda \Subset \mathcal{V}$. We set $\Lambda^c := \mathcal{V}\setminus\Lambda$ for any subset Λ of $\mathcal{V}$ and whenever $\Lambda=\{i\}$ we write i^c for $\mathcal{V}\setminus\{i\}$. Furthermore a *configuration space*, in which the family $\{\sigma_i\}_{i\in\mathcal{V}}$ takes values, is denoted by $\Omega := S^{\mathcal{V}}$, which is endowed with a product σ-algebra $\mathcal{A}$. Each element $\sigma\in\Omega$ is called a *configuration*, a finite-volume configuration σ_Λ is a projection of σ, $\sigma_\Lambda=\{\sigma_i\}_{i\in\Lambda}$. A configuration σ_Λ takes values in a projection space Ω_Λ of Ω. The set of probability measures on Ω is denoted by $\mathcal{P}(\Omega,\mathcal{A})$.

A real-valued function f defined on the configuration space Ω is *quasilocal* if

$$\lim_{\Lambda\to\Omega}\sup_{\sigma,\eta\in\Omega:\sigma_\Lambda=\eta_\Lambda}|f(\sigma)-f(\eta)|=0 \tag{2.1}$$

For discrete spins quasilocality implies that a function is continuous in the product topology. Stated in words, it means that a function of a finite-volume

"circle" σ_Λ provided that a configuration in a "ring" $\sigma_{\Lambda'\setminus\Lambda}$ ($\Lambda \subset \Lambda'$) is fixed and thick enough, hardly depends on the outside of Λ'.

By $\mathcal{C}(\Omega)$ we will mean the set of all continuous functions on Ω.

2.2 Finite-volume Gibbs measures

In statistical mechanics, systems in equilibrium are described via finite-volume densities γ_Λ, which are defined through the Boltzmann-Gibbs prescription $\gamma_\Lambda \propto e^{-H_\Lambda}$, where the function H_Λ — a Hamiltonian — is the "interaction energy" of the region Λ. To attach a mathematical sense to the notion of "energy" we give the following definitions.

Definition 2.2.1. *An* interaction *or* interaction potential *or* potential *is a family $\Phi = \{\phi_\Lambda : \Lambda \Subset \mathcal{V}\}$ of functions $\phi_\Lambda : \Omega \mapsto \mathbb{R}$, such that $\phi_\Lambda \in \mathcal{A}_\Lambda$, that is, ϕ_Λ depends only on the spins in the finite set Λ, for every $\Lambda \Subset \mathcal{V}$.*

We would like to define the Hamiltonian of a region as the sum of interaction terms over all possible subsets of that region, but the meaningful definition will require something more, this is the convergence of relevant series. This gives rise to the following definition:

Definition 2.2.2. *Let Φ be an interaction.*

- *The* Hamiltonian *for a region $\Lambda \Subset \mathcal{V}$ with frozen external condition ω is the real-valued function defined by*

$$H_\Lambda^\Phi(\sigma_\Lambda|\omega_{\Lambda^c}) = \sum_{A \Subset \mathcal{V} : A \cap \Lambda \neq \emptyset} \phi_A(\sigma_\Lambda\, \omega) \tag{2.2}$$

  for $\sigma, \omega \in \Omega$ such that the sum exists.

- Φ *is* summable at $\omega \in \Omega$ *if $H_\Lambda^\Phi(\sigma_\Lambda|\omega_{\Lambda^c})$ exists for all $\Lambda \Subset \mathcal{V}$ and $\sigma_\Lambda \in \mathcal{A}_\Lambda$*

Definition 2.2.3. *The* Boltzmann weights *for interaction Φ are the functions defined for all $\Lambda \Subset \mathcal{V}$ and all boundary conditions ω at which Φ is summable by*

$$\vartheta_\Lambda^\Phi(\sigma_\Lambda|\omega_{\Lambda^c}) = \frac{e^{-\beta H_\Lambda^\Phi(\sigma_\Lambda|\omega_{\Lambda^c})}}{Z_\Lambda^\Phi(\omega)}, \tag{2.3}$$

where $Z_\Lambda^\Phi(\omega)$ is called the partition function.

Given boundary conditions ω and an a priori measure α we define a finite-volume Boltzmann-Gibbs measure $\gamma^{\Phi}_{\Lambda,\omega}$ as follows

$$\begin{aligned}\gamma^{\Phi}_{\Lambda,\omega} = \gamma^{\Phi}_{\Lambda}(d\sigma_\Lambda|\omega) &= \vartheta^{\Phi}_{\Lambda}(\sigma_\Lambda|\omega_{\Lambda^c}) \prod_{i\in\Lambda} \alpha(d\sigma_i) \\ &= \frac{e^{-\beta H^{\Phi}_{\Lambda}(\sigma_\Lambda|\omega_{\Lambda^c})}}{Z^{\Phi}_{\Lambda}(\omega)} \prod_{i\in\Lambda} \alpha(d\sigma_i) \\ &= \frac{e^{-\beta H^{\Phi}_{\Lambda}(\sigma_\Lambda|\omega_{\Lambda^c})}}{Z^{\Phi}_{\Lambda}(\omega)} \alpha^{\otimes}(d\sigma_\Lambda)\end{aligned} \tag{2.4}$$

2.3 Infinite-volume Gibbs measures

As stated before, the Boltzmann-Gibbs distribution does not admit a direct extension to infinite systems. However, when dealing with infinite systems we can still look at finite subsystems provided the "outside" is held fixed. Indeed, for any finite $\Lambda \Subset \mathcal{V}$ the Boltzmann-Gibbs distribution, satisfying some requirements, might be viewed as a map mapping each environment to the corresponding equilibrium distribution of subsystem Λ. Before exploring when a family of finite-volume Boltzmann-Gibbs measures indexed with boundary conditions could be interpreted as a family of conditional probabilities coming from an infinite-volume Gibbs measure, we cover two relevant notions.

A *probability kernel* from the probability space $(\Omega, \mathcal{A})$ to itself is a function $\pi(\cdot|\cdot)$ having two slots such that: (1) $\pi(\cdot|\omega)$ is a probability measure on $(\Omega, \mathcal{A})$ for each $\omega \in \mathcal{A}$, (2) $\pi(\sigma|\cdot)$ is $\mathcal{A}$-measurable for each $A \in \mathcal{A}$.

Remark 2.3.1. *Generally, probability kernels can be defined between two different spaces. We simplified this more general definition here to avoid unnecessary (for the present thesis) generality. For more details we refer the reader to e.g. [26, Chapter 1] or [19, Chapter 3].*

Definition 2.3.2. *A* specification *on* $(\Omega, \mathcal{A})$ *is a family* $\Pi = \{\pi_\Lambda,\ \Lambda \Subset \mathcal{V}\}$ *of probability kernels from* $(\Omega, \mathcal{A})$ *to itself and each* π_Λ *in the family is*

(i) proper, $\pi_\Lambda(A) = \mathbf{1}_A$, *if* A *is measurable w.r.t.* $\mathcal{A}_{\Lambda^c}$,

(ii) consistent, $(\pi_\Delta \pi_\Lambda)(A|\omega) = \int_\Omega \pi_\Delta(d\omega'|\omega)\pi_\Lambda(A|\omega') = \pi_\Delta(A|\omega)$, *if* $\Lambda \subset \Delta$

The above defined specification is called *Gibbsian* if it is quasilocal and uniformly non-null. A specification is said to be *uniformly non-null* if each of its members with fixed second slot to $\omega \in \mathcal{A}_{\Lambda^c}$ weighs any event $A \in \mathcal{A}_\Lambda$ with at least some positive value. Quasilocality requires that for any quasilocal function f the expectation $\pi_\Lambda(f|\omega)$ is quasilocal as a function of ω,

for all $\Lambda \Subset \mathcal{V}$. Although we used a general form of the probability kernels in Definition 2.3.2, we are guaranteed that a family $\Gamma = \{\gamma^{\Phi}_{\Lambda,\omega}, \Lambda \Subset \mathcal{V}\}$ (where the $\gamma^{\Phi}_{\Lambda,\omega}$ are as defined in (2.4)) is indeed a Gibbsian specification, [26, see Proposition 2.24 (b)].

Definition 2.3.3. *A measure μ is called is a* Gibbs measure *with interaction Φ and a priori measure α, if for all finite $\Lambda \Subset \mathcal{V}$ and continuous test functions f the following equation is satisfied*

$$\int f(\sigma_\Lambda)\mu(d\sigma) = \int\int f(\sigma_\Lambda)\gamma^{\Phi}_{\Lambda,\omega}(d\sigma_\Lambda)\mu(d\omega), \tag{2.5}$$

where $\Gamma = \{\gamma^{\Phi}_{\Lambda,\omega} : \Lambda \Subset \mathcal{V}\}$ is a Gibbsian specification.

The equations (2.5) *in shorthand are written*

$$\mu\,\gamma_\Lambda = \mu \tag{2.6}$$

and are called DLR-equations.

The measure μ is called consistent *with the specification Γ.*

The equations (2.5) mean that a measure μ is consistent with a quasilocal and uniformly non-null specification and that $\gamma^{\Phi}_{\Lambda,\omega}(d\sigma_\Lambda)$ is a *version* or a *realization* of conditional probability $\mu_\Lambda(d\sigma_\Lambda|\sigma_{\Lambda^c} = \omega) =: \mu^{\omega}_{\Lambda}(d\sigma_\Lambda)$.

Let $\mathcal{G}(\Gamma)$ be the set of Gibbs measures consistent with Γ. Clearly, $\mathcal{G}(\Gamma) \subset \mathcal{P}(\Omega, \mathcal{A})$. If $|\mathcal{G}(\Gamma)| > 1$, then the statistical mechanics system on $G = (\mathcal{V}, \mathcal{E})$ is said to have a phase transition. If the inverse temperature β is infinite or sufficiently high then the Boltzmann weights for any configuration in a fixed $\Lambda \Subset \mathcal{V}$ are either equal to 1 (the corresponding infinite-volume Gibbs measure is just a product measure) or close to 1 (the corresponding infinite-volume Gibbs measure is a small perturbation of a product measure). Both cases imply that $|\mathcal{G}(\Gamma)| = 1$, this means that the *high-temperature* regime is part of the *uniqueness* regime.

Hitherto we included the thermal constant (inverse temperature) β separately, from now on we reserve the right to absorb it into the Hamiltonian.

As was pointed out, consistency of a measure μ with a quasilocal and uniformly non-null specification is a necessary condition for Gibbsianness. We would like to identify *sufficient* conditions for a measure to be Gibbs. This question was answered in a simple and informative way by [29] for Markovian fields. That argument was later generalized by Sullivan [54] and Kozlov [35]. The Kozlov theorem has a constructive character, it answers the Gibbsianness question by reconstructing a (lattice gas or vacuum) potential from a given specification of general form.

Theorem 2.3.4. *A specification is Gibbs if and only if it is uniformly non-null and quasilocal.*

2.4 Block transformations, loss of Gibbsianness, and proving tools

Many examples of non-Gibbsianness refer to transformations of initially Gibbs measures according to some rule. One important category of these transformations is block transformations. For a moment consider two different probability spaces $\Omega = (S^{\mathcal{V}}, \mathcal{A})$ and $\Omega' = (S'^{\mathcal{V}'}, \mathcal{A}')$. A block transformation is a rule (which may be either deterministic or stochastic) such that for every $i' \in \mathcal{V}'$ there exists a *block* $B_{i'} \Subset \mathcal{V}$ such that the value of $\sigma'_{i'} (\in S')$ is a function on $\mathcal{A}_{B_{i'}}$. Spins in $\sigma_{B_{i'}}$ are called "original" spins, a configuration σ' generated according to this rule consists of "block" spins. Hereafter we discuss transformations on the same space $\Omega = \Omega'$.

The stochastic dynamics studied in Chapter 3 of the present thesis is a special case of a block transformation with a *single-site* block. In the mean-field setup of Chapter 4, where their stochastic evolution is treated, blocks will overlap[1].

2.4.1 Bad points for transformed measures

Mathematically, a block transformation is expressed by a probability (or *transition*) kernel $K(\sigma, \sigma')$ from $(\Omega, \mathcal{A})$ to itself. Such a map defines a probability distribution $\mu'(\sigma')$ of block spins from any given probability distribution $\mu(\sigma)$ of original (or internal) spins, i.e.

$$\mu'(\sigma') = (\mu K)(\sigma') = \sum_{\sigma} \mu(\sigma) K(\sigma, \sigma') \tag{2.7}$$

In other words, the dynamics is defined as a map *from measures to measures*. On the other hand, such a map may be expressed in terms of Hamiltonians H and H' for the finite-volume versions of μ and μ', respectively, and a renormalization map $\mathcal{R}$, i.e.

$$H'(\sigma') = (\mathcal{R}H)(\sigma) = -\ln \left(\sum_{\sigma} \exp\{-H(\sigma) + \ln K(\sigma, \sigma')\} \right) \tag{2.8}$$

This formula is invalid in infinite volume, because in this case both H and H' are ill-defined. Both K and $\mathcal{R}$ have advantages and disadvantages: T is linear, but acts on a space of large dimensions, $\mathcal{R}$ involves logarithms, but it is

[1] This fact causes no problem in the sense that if a starting measure was a Gibbs measure, the time-evolved measure is non-null, because all transformed configurations are reached with a positive probability. This is not always the case for deterministic rules. For an example of a problematic deterministic block-spin transformation, see e.g. [46]

always single-valued (if well-defined), [11, see Theorem 3.4(First Fundamental theorem), pp. 971].

To examine whether the Gibbsianness for the transformed measure is preserved, it is worthwhile to look at Theorem 2.3.4. This theorem points out intrinsic properties of Gibbs measures and provides a rather simple symptomatology of non-Gibbsianness: lack of uniform non-nullness and/or lack of quasilocality.

In the case when densities of a specification have the form of a Boltzmann-Gibbs weights, the absence of the non-nullness property means that infinite energies are allowed for finite regions. In the sequel of this thesis we exclude models dealing with infinite energies, e.g. systems of unbounded spins or hard-core-interaction systems.

Having excluded the possibility of infinite energies for finite regions, we are left with one property to investigate — quasilocality. In many instances, image measures may fail to be quasilocal and, consequently, to be Gibbsian. The non-Gibbsianness arises from the fact that internal spins σ may undergo a phase transition for a fixed configuration of block spins σ'_{spec}. Moreover, different phases of internal spins may be preselected by an appropriate choice of block-spins. In this way, the information can be broadcast from far-away block spins to a chosen block spin (in the presence of translation-invariance, to the block-spins near the origin) via an intermediate region of internal spins. Importantly, this transmission is possible *even when* the block spins in the intermediate region are *fixed*. Hence, the transformed measure μ' do not obey a quasilocality condition and, therefore, fail to be Gibbs. In this case, the renormalization map $\mathcal{R}$ between Hamiltonians is ill-defined, because there exists *no* reasonable interaction for μ'.

Puzzlingly, an existence of a single σ'_{spec} (which happens with zero probability) can cause the failure of Gibbsianess for the image measure. Indeed, this fact alone is not sufficient, but common non-zero probability weight of those block-spins configurations which agree with σ'_{spec} in a large volume and differ outside easily triggers loss of Gibbsianness for the image measure μ'.

From the mathematical point of view, the transformed measure μ' is not quasilocal if it is consistent with *no* quasilocal specification. To prove this it is enough to find a single, non-removable, point of discontinuity (in the product topology) for a single μ'_Λ for a single (quasi)local functions f, [11, 19]. Essentially non-quasilocality means that finite-volume realizations $\mu'_\Lambda(f|\cdot)$ of infinite-volume measure μ' as functions of the conditionings behave as *no* quasilocal functions, see (2.1). The relevant definitions read as follows:

Definition 2.4.1. *The measure μ' is not quasilocal at $\bar{\eta} \in \Omega$ if there exists $\Lambda_0 \Subset \mathcal{V}$ and f local (given that the single-site space is finite it suffices to look*

for f local, with support Λ_0) such that no realization of $\mu'_{\Lambda_0}(f|\cdot)$ is quasilocal at $\bar{\eta}$.

In other words, any realization of $\mu'_{\Lambda_0}(f|\cdot)$ must exhibit an essential discontinuity at $\bar{\eta}$; one that survives zero-measure modifications. (Remember that conditional probabilities are only defined up to measure-zero sets)

Definition 2.4.2. *For a local function f as above, $\mu'_{\Lambda_0}(f|\cdot)$ is μ'-essentially discontinuous at $\bar{\eta}$, if there exists an $\varepsilon > 0$ such that*

$$\limsup_{\Lambda\uparrow\infty} \sup_{\substack{\xi^1,\xi^2 \\ \Lambda'\supset\Lambda \\ |\Lambda'|<\infty}} |\mu'_{\Lambda_0}(f|\bar{\eta}_{\Lambda\setminus\Lambda_0}\xi^1_{\Lambda'\setminus\Lambda}) - \mu'_{\Lambda_0}(f|\bar{\eta}_{\Lambda\setminus\Lambda_0}\xi^2_{\Lambda'\setminus\Lambda})| > \varepsilon \tag{2.9}$$

If $\mu'_{\Lambda_0}(f|\cdot)$ is μ'-essentially discontinuous at $\bar{\eta}$, informally it means that there exists an $\varepsilon > 0$ such that for every $\Lambda \Subset \mathcal{V}$ there exists $\Lambda' \supset \Lambda$ and configurations ξ^1, ξ^2, such that

$$\left|\mu'_{\Lambda_0}(f|\bar{\eta}_{\Lambda\setminus\Lambda_0}\xi^1_{\Lambda'\setminus\Lambda}\eta) - \mu'_{\Lambda_0}(f|\bar{\eta}_{\Lambda\setminus\Lambda_0}\xi^2_{\Lambda'\setminus\Lambda}\eta)\right| > \varepsilon \tag{2.10}$$

for $\eta \in A$, where $A \in \mathcal{A}_{(\Lambda')^c}$ is of positive μ'-measure.

Definition 2.4.3. *$\mu'_{\Lambda_0}(f|\cdot)$ is strongly discontinuous at $\bar{\eta}$, if and only if there exists an $\varepsilon > 0$ such that*

$$\limsup_{\Lambda\uparrow\infty} \sup_{\substack{\xi^1,\xi^2 \\ \Lambda'\supset\Lambda \\ |\Lambda'|<\infty}} \inf_{\substack{\eta^1,\eta^2 \\ \Lambda''\supset\Lambda' \\ |\Lambda''|<\infty}} |\mu'_{\Lambda_0}(f|\bar{\eta}_{\Lambda\setminus\Lambda_0}\xi^1_{\Lambda'\setminus\Lambda}\eta^1_{\Lambda''\setminus\Lambda'}) - \mu'_{\Lambda_0}(f|\bar{\eta}_{\Lambda\setminus\Lambda_0}\xi^2_{\Lambda'\setminus\Lambda}\eta^2_{\Lambda''\setminus\Lambda'})| > \varepsilon \tag{2.11}$$

Remark 2.4.4. *Intuitively the difference is that whereas for μ'-essential discontinuity one needs to estimate a difference on two measurable sets of positive measure, for a strong discontinuity one needs an estimate of a difference on open sets; however, because of the impossibility of conditioning on individual configurations, we get the somewhat unwieldy definitions above.*

Such $\bar{\eta} \in \Omega$ are called *bad* configurations.

In practice, the lack of quasilocality has been detected by proving (2.10) for functions of the form $f(\sigma_\Lambda) = \sigma_\Lambda$. Furthermore, only single-site regions need to be checked due to the Theorem 2.3.4. If the graph G is translation invariant, then, non-quasilocality proofs typically refer to (2.10) for $\Lambda = \{\text{“origin”}\}$ and $f(\sigma_{\{\text{“origin”}\}}) = \sigma_{\{\text{“origin”}\}}$. In what follows we substitute $\sigma_{\{\text{“origin”}\}}$ by σ_0.

2.4.2 Two-layered models

A useful tool to study whether Gibbsianness is preserved or lost for a model on a graph $G = (\mathcal{V}, \mathcal{E})$ under a transformation is *joint* or *two-layered* model living on $G \cup G$. A joint model is obtained by coupling the initial model to a transformed model through transition kernel $K(\sigma_\Lambda, \eta_\Lambda)$, where Λ is a finite volume. Thinking of site-wise independent spin-flips, $K(\sigma_\Lambda, \eta_\Lambda) = \prod_{i\in\Lambda} k(\sigma_i, \eta_i)$, then the two-layered system produced in this way is given by prescribing a formal Hamiltonian

$$H_\Lambda(\sigma_\Lambda, \eta_\Lambda) = H_\Lambda(\sigma_\Lambda \sigma_{\Lambda^c}) - \sum_{i\in\Lambda} \ln k(\sigma_i, \eta_i), \tag{2.12}$$

where $H_\Lambda(\sigma_\Lambda \sigma_{\Lambda^c})$ is the Hamiltonian of the original model. As explained, in Chapter 1 this prescription works only in a finite volume. Naturally, two-layered model results in the original model, when the η-spins are integrated out.

If two-layered infinite-volume measure exhibits a phase transition when $\bar{\eta}_{\Lambda_0}$—a configuration suspected to be bad for the transformed measure—is fixed, this suggests a possibility to vary initial configurations σ^1 and σ^2 in a such way as to create a discontinuous behaviour for the transformed measure and to make (2.10) with a choice $\xi^1 = \sigma^1$ and $\xi^2 = \sigma^2$ hold. Nevertheless, in some situations the configuration $\eta_\Lambda, \Lambda_0 \subset \Lambda$ created during evolution may stop the influence of survived σ-spins. This discussion connects the presence of hidden phase transitions and the possibility to select phases by choosing correct boundary conditions via a choice σ^1 and σ^2 for two-layered models together with loss of quasilocality for the transformed measure. This approach to study the evolved measure via the marginal of a two-layers Gibbs measure was introduced in [12], and has been applied repeatedly since. We will be more specific in the following chapters identifying the properties of two-layered models for a tree in Chapter 3 and in the mean-field context in Chapter 4.

2.5 Ising spins and spin-flip evolution

In Chapter 4 and Chapter 3 we consider S to be the space $\{-1, +1\}$. These variables are called Ising spins and correspond to physical spins in ferromagnets pointing downwards and upwards. This simplification was introduced by Ising and his Ph.D. advisor Lenz in [32]. The relevant definitions involving the single-site space formally stay the same, but have to be thought in connection with the present S.

We will consider single-site spin-flip or, equivalently, Glauber time evolution. We attach a Poissonian clock to each site i of $\mathcal{V}$ with mean-time $\frac{1}{c(i,\sigma)}$ for site i of staying in the "+"- or "−"-state, when the current configuration

is σ, for every $\sigma \in \Omega$. The dynamics of the whole system is governed by a collection of spin-flip rates $\{c(i,\sigma)\}$, $i \in \mathcal{V}$, $\sigma \in \Omega$ satisfying the following conditions:

(i) $c_i : \sigma \mapsto c(i,\sigma)$ is a local function of σ. In our setup c_i alters just the spin at site i;

(ii) c_i's are translation-invariant for all i;

(iii) $c(i,\sigma) > 0$ for all $i \in \mathcal{V}$ and $\sigma \in \Omega$.

Condition (i) allows to treat only rates of flipping from "+" to "-" and vice versa — $\{c(\pm,\sigma)\}$, which are dependent of other sites for each $\sigma \in \Omega$. This means that we have a finite-temperature dynamics, an infinite-temperature dynamics lacks such a dependence.

Differently to usual notation we shall refer to s as time variable, when t will mean a *fixed* moment of time.

We consider a site-wise independent spin-flip dynamics. Hence, the spin-flip process of σ_L is nothing but a product of $|\Lambda|$ independent Markov chains. Consequently, given the rates $\{c(i,\sigma)\}$ the single site linear generator acting on local functions f mapping $\{-1,+1\}$ to itself reads

$$(L_i f)(\sigma_i) = c(i,\sigma)(f(-\sigma_i) - f(\sigma_i)), \tag{2.13}$$

then the linear generator L for spins in Ω is defined for local functions $\bar{F} : \Omega \mapsto \mathbb{R}$ by

$$(L\bar{F})(\sigma) = \sum_{i\in\mathcal{V}} c(i,\sigma)\left[\bar{F}(\sigma^i) - \bar{F}(\sigma)\right], \tag{2.14}$$

where σ^i is a "flipped" configuration, obtained from σ by flipping a spin at site i:

$$\left(\sigma^i\right)_j = \begin{cases} -\left(\sigma^i\right)_i, & j = i \\ \left(\sigma^i\right)_j, & j \neq i \end{cases} \tag{2.15}$$

Spin-flip evolution transforms an intial configuration σ at $s = 0$ to a time-evolved configuration η at time $s = t$.

It is proved that the closure of L on $\mathcal{C}(\Omega)$ is the generator of a unique Feller process $\{\sigma_s : s \geq 0\}$, [45, see Theorem 3.9]. We denote by $S(s) = e^{sL}$ the corresponding semigroup governing the time evolution. The semigroup defines a continuous-time transformation of a probability measure $\kappa \in \mathcal{P}(\Omega,\mathcal{A})$ via the following rule:

$$\mathbb{E}_\kappa(S(s)f)(\sigma) = \mathbb{E}_{\kappa S(s)} f(\sigma) \tag{2.16}$$

The former suggests that $\kappa S(t)$ is the distribution of the configuration η at time $s = t$ if at time $s = 0$ the initial distribution of σ was κ.

* * *

3

Tree models

In this Chapter (which is based on [13]) we turn to a much more specific class of graphs than the completely general $G = (\mathcal{V}, \mathcal{E})$ as in Chapter 2. These graphs are the Cayley trees $\mathfrak{CT}$. Cayley trees form a special case of a tree possessing some regularity properties which will be used to prove the theorems of this chapter. These properties are constant number of "adjacent" vertices for each site, and it is known that with at least three neighbours per site, co-existence of several extreme homogeneous Gibbs measures — plus, minus, and intermediate — holds in some parameter region of initial field and temperature. The first property simplifies calculations, the second one allows a *possibility* of a non-Gibbsian behaviour of an initial Gibbs measure subjected to an infinite-temperature time evolution. We show that this possibility is used under some conditions on the *time* during which the model evolves.

At high initial temperatures, or for sufficiently short times, standard methods can be used to prove Gibbsianness, also in the setup of this chapter. Thus the interesting case is to find out what happens for low initial temperatures. As usual low-temperature dynamics are beyond reach so far. For simplicity we will consider infinite-temperature dynamics, but high-temperature evolutions are expected to behave qualitatively similarly.

In contrast to what happens on regular lattices such as $\mathbb{Z}^d$, the Gibbsian properties of evolved Gibbs measures for models on trees turn out to depend on which of the different Gibbs measures (plus or minus, versus intermediate) one considers. In all cases there are two transition times: for the intermediate measure after the first transition time it becomes non-Gibbsian in the familiar sense that some, but not all, configurations are " bad" (that is, they are points of discontinuity), while it turns out that after a certain later time the evolved intermediate Gibbs measure becomes "totally bad"; thereafter it has the surprising property that *all* spin configurations are discontinuity points.

This last property is something which will not happen for plus and minus states. For those measures, although after a first transition time they also become non-Gibbsian, after the second transition time they become Gibbsian again.

Although the proofs are provided for the Cayley tree, we will indicate why these results should be expected to hold more generally. Results are presented

both in zero and non-zero external fields.

Analysis done in this chapter illustrates (again) how different models on trees are as compared to models on regular (amenable) lattices.

3.1 Preliminaries

3.1.1 Cayley tree

As before let the state-space S be $\{-1, +1\}$. We ask from the graph $G = (\mathcal{V}, \mathcal{E})$ to satisfy the following three properties.

(i) *Local finiteness.* For each $i \in \mathcal{V}$ the set

$$\partial i = \{j \in \mathcal{V} : \text{a bond } \{i, j\} \in \mathcal{E}\} \tag{3.1}$$

of all neighbours of i is finite.

(ii) *Connectedness.* For any two sites $i, j \in \mathcal{V}$ there is a sequence $i = i_0, i_1, \ldots i_n = j$ in $\mathcal{V}$ such that bonds $\{i_{k_1}, i_k\} \in \mathcal{E}$ for all $1 \leq k \leq n$. Such a sequence is called a path from i to j.

(iii) *Tree property.* For all $i, j \in \mathcal{V}$, there is only one path from i to j.

When moreover for an integer $d \geq 1$ the cardinality of the set ∂i is $d + 1$ for all $i \in \mathcal{V}$, the graph G is called a *Cayley tree* or *Bethe lattice* of degree d and denoted by $\mathfrak{CT}(d)$. Such a tree on a plane could be viewed as in Figure 3.1.

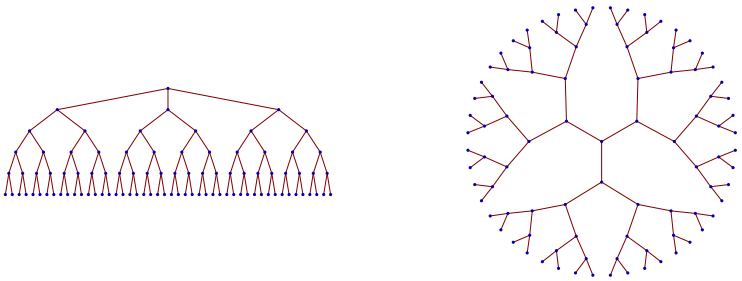


Figure 3.1: Two embeddings of $\mathfrak{CT}(3)$ into the plane

In this Chapter we shall be concerned with the Gibbs properties of a Gibbs measure living on a Cayley tree when evolved in time according to an independent spin-flip dynamics.

3.1.2 Ising model on Cayley trees

Let $\mathfrak{CT}(d)$ be a Cayley tree for some $d \geq 1$, that is the unique connected tree with $|\partial i| = d + 1$ for all $i \in \mathfrak{CT}(d)$. Let $\Omega = \{-1, +1\}^{\mathfrak{CT}(d)}$, endowed with the product topology. Elements in Ω are denoted by σ. A configuration σ assigns to each vertex $i \in \mathfrak{CT}(d)$ a spin value $\sigma_i = \pm 1$. Denote by $\mathbb{S}$ the set of all finite subtrees of $\mathfrak{CT}(d)$. For $\Lambda \in \mathbb{S}$ and $\sigma \in \Omega$ we denote by σ_Λ the restriction of σ to Λ, while Ω_Λ denotes the set of all such restrictions. Let $\Lambda \in \mathfrak{CT}(d)$ be any set, finite or infinite. We denote by $\mathcal{E}_\Lambda$ its set of edges and by $\mathcal{V}_\Lambda$ its set of vertices.

Let now $\Lambda \in \mathbb{S}$, hence finite. We will consider the nearest-neighbour Ising model on the tree. The finite-volume Gibbs measure on any finite subtree Λ for an Ising model in an inhomogeneous external field, given by fields h_i at sites i, boundary condition ω, at inverse temperature β, is defined by the following Boltzmann-Gibbs distribution

$$\gamma_\Lambda^\omega(\sigma_\Lambda) = \frac{1}{Z_\Lambda^\omega(\beta, \{h_i\}_{i\in\mathcal{V}_\Lambda})} \exp\left\{ \beta \sum_{\{i,j\}\in\mathcal{E}_\Lambda} \sigma_i\sigma_j + \sum_{i\in\mathcal{V}_\Lambda} h_i\sigma_i + \sum_{\substack{\{i,j\} \\ i\in\mathcal{V}_\Lambda, j\in\mathcal{V}_{\Lambda^c}}} \sigma_i\omega_i \right\} \tag{3.2}$$

Infinite-volume Gibbs measures are defined by having their conditional probabilities of finite-volume configurations, conditioned on the configurations outside the volume, of this Gibbsian form, see e.g. [11, 26]. In equation form we require that for all volumes Λ and configurations σ_Λ μ satisfies

$$\mu(\sigma_\Lambda) = \int \gamma_\Lambda^\omega(\sigma_\Lambda)\mu(d\omega) \tag{3.3}$$

This equation is the DLR-equation as explained in (2.6).

The infinite-volume Gibbs measures are parametrized by the external magnetic fields (in most of what follows we will consider a homogeneous field h_0), and by the inverse temperature $\beta \geq 0$. This will lead us to consider finite-volume Gibbs measures with this same homogeneous field plus a possibly different boundary field. We put $\beta(1) = \infty$ and, for $d > 1$,

$$\begin{aligned} \beta(d) &= \operatorname{arccoth} d = \frac{1}{2} \ln \frac{d+1}{d-1} \\ h(\beta, d) &= \left[d \operatorname{arctanh} \left(\frac{dw - 1}{d\bar{w} - 1} \right)^{\frac{1}{2}} - \operatorname{arctanh} \left(\frac{d - \bar{w}}{d - w} \right)^{\frac{1}{2}} \right] \mathbb{I}_{\beta > \beta(d)}, \end{aligned} \tag{3.4}$$

where $w = \tanh \beta = \bar{w}^{-1}$.

It is known [26, see Chapter 12], that if $\beta > \beta(d)$ and $|h_0| \leq h(\beta, d)$, then the system exhibits a phase transition. Throughout this chapter we will

assume $|h_0| < h(\beta, d)$, $\beta > \beta(d)$, and $d > 1$, whenever the opposite is not indicated. This condition ensures the existence of three homogeneous phases μ^-, $\mu^\sharp$, μ^+ ($-, +$, and $\sharp$ upper-scripts correspond to minus, plus, and free boundary conditions) of the model at time $t = 0$.

These phases are extremal in the set of invariant infinite-volume Gibbs measures; μ^+ and μ^- are also extremal in the set of all infinite-volume Gibbs measures, whereas $\mu^\sharp$ becomes non-extremal in this set below a certain temperature strictly smaller than the phase transition temperature [5, 31]; however, this second transition will not concern us here.

Let Λ_n be the Cayley tree with n generations and $\Lambda_{n-1} = \Lambda_n \setminus \partial\Lambda_n$ the (sub-) Cayley tree with $n-1$ generations, where $\partial\Lambda_n$ stands for the inner boundary of Λ_n. It is a known result for the Ising model on trees that the marginal on Λ_{n-1} of the finite-volume Gibbs measure on Λ_n is a finite-volume Gibbs measure on Λ_{n-1}, with a possibly different external magnetic field at the boundary. See Appendix A for how this works out in marginalizing infinite-volume Gibbs measures by using boundary laws.

Marginalizing on Λ_{n-1}, that is to a tree of one generation less, leaves us with a finite-volume Gibbs measure on Λ_{n-1}, parametrized by the following external fields

$$\begin{aligned} &i \in \partial\Lambda_{n-1}, \ h_i = h_0 + d\varphi(h_n), \\ &i \in \Lambda_{n-2}, \quad h_i = h_0 \end{aligned} \tag{3.5}$$

where $\varphi(x) = \operatorname{atanh}(\tanh\beta \tanh x)$.

Thus, summarising, taking the marginal of an Ising model Gibbs measure on a tree with n generations with homogeneous boundary field h_n results in an Ising model on an $(n-1)$-generation tree with a homogeneous boundary field h_{n-1}. The map from h_n to h_{n-1}, (3.5), has three fixed points $h^+, h^\sharp$ and h^-. (Equivalently, one could consider the map from the magnetisation at generation n to the magnetisation at generation $n-1$, which again has the corresponding three fixed points $m^+, m^\sharp$ and m^-.) Whereas h^+ and h^- are stable, $h^\sharp$ is an unstable fixed point which implies that weak positive boundary conditions will result in a plus state, once one is far enough from the boundary. In other words, the phase transition is *robust* [50].

These three fixed points determine the three homogeneous extremal invariant infinite-tree Gibbs measures mentioned above.

3.2 Main questions

With slight abuse of notation we define a set $\mathcal{G}(\beta, h_0)$ as a set of all Gibbs measures of the Ising model with an inverse temperature β and external field h_0. Let $\mathcal{P}_I(\Omega, \mathcal{A})$ denote the set of all $\mu \in \mathcal{P}(\Omega, \mathcal{A})$ which are invariant under

all the graph automorphisms (translations, rotations, reflections etc). Let $\mu \in \mathcal{G}_I(\beta, h_0)$, where $\mathcal{G}_I(\beta, h_0) = \mathcal{G}(\beta, h_0) \bigcap \mathcal{P}_I(\Omega, \mathcal{A})$.

We aim to study here the time-dependence of the Gibbsian property of the tree Gibbs measure $\mu^\star$, for $\star \in \{+, -, \sharp\}$, under an infinite-temperature Glauber dynamics (or, equivalently, under independent spin-flip dynamics). As defined in Section 2.5 of Chapter 2, this dynamics is the stochastic evolution $S(t)$ which is obtained by having independent spin flips at each vertex at a certain given rate. In other words, we want to investigate whether or not $\mu^\star S(t) =: \mu'_t$ is a Gibbs measure at a given time $t > 0$.

By assumption the initial measure μ is a Gibbs measure. This immediately guarantees the non-nullness of the measure $\mu^\star S(t)$ for all t (including $t = 0$). It will thus suffice to study the question whether the transformed measure is quasilocal or not. This study will generally refer to a check-up of correctness of (2.10) when an appropriate choice of ξ^1 and ξ^2 is done. We will slightly modify that inequality to match our needs and the graph structure.

To study the question whether the transformed measure μ'_t stays Gibbs we consider the joint two-layered distribution ν on (σ, η), where the initial spins σ are distributed according to μ, and the evolved spins η according to μ'_t. It can be viewed as a Gibbs measure on $S^{\mathfrak{K}}$ with $\mathfrak{K} = \mathfrak{CT}(d) \cup \mathfrak{CT}(d)$ consisting of two "layers" of $\mathfrak{CT}(d)$. Formally, the Hamiltonian of ν^t is

$$H_{\nu_t}(\sigma, \eta) = H_\mu(\sigma) - \ln K_t(\sigma, \eta), \tag{3.6}$$

where $K_t(\sigma, \eta)$ is the transition kernel of the dynamics. We consider independent spin-flip dynamics, so

$$\ln K_t(\sigma, \eta) = \sum_{i \in \mathfrak{CT}(d)} \frac{1}{2} \ln \frac{1 + e^{-t}}{1 - e^{-t}} \sigma_i \eta_i \tag{3.7}$$

Let us denote

$$h^t = \frac{1}{2} \ln \frac{1 + e^{-t}}{1 - e^{-t}} \tag{3.8}$$

Remark 3.2.1. *Here we will find for $\mu^\sharp S(t)$, by making the choices $\xi^1 = +1$, $\xi^2 = -1$, that in any open neighborhood of $\bar{\eta}$ two positive-measure sets exist, on which the limits differ, however, in contrast to amenable graphs, these sets are not open (which allows different behaviour between different evolved Gibbs measures $\mu^\sharp$ and μ^+ as regards their Gibbsianness, something which is excluded on amenable graphs such as $\mathbb{Z}^d$). In other words we will show a μ'-essential, although non-strong, discontinuity.*

As explained in the Appendix A we have the representation of the conditional probabilities of the time-evolved measure μ'_t of the form

$$\mu'_t(\eta_0 | \eta_{\Delta \setminus 0}) = \int \mu[\eta_{\Delta \setminus 0}](d\sigma_0) K_t(\sigma_0, \eta_0) \tag{3.9}$$

with the perturbed η-dependent measure on spin configurations $\mu[\eta_{\Lambda\setminus 0}](d\sigma) \equiv \mu[\eta_{\Lambda\setminus 0}, \eta_0 = 0](d\sigma)$ whose finite-volume marginals look like

$$\mu[\eta_{\Delta'}](\sigma_{\Delta'}) = C \exp\left\{ \beta \sum_{(i,j)\in\Lambda'} \sigma_i\sigma_j + \sum_{i\in\Delta'\setminus\partial\Delta'} h_i\sigma_i + \sum_{i\in\partial\Delta'} \tilde{h}_i\sigma_i \right\}, \qquad (3.10)$$

where

$$\begin{aligned} h_i &= h_0 + \eta_i h^t, \\ \tilde{h}_i &= h_0 + \eta_i h^t + h^\star \end{aligned} \qquad (3.11)$$

where the external fields at the boundaries are given in terms of $h^\star$. This value represents the fixed point of the recursion relation with homogeneous field h_0, (3.5), and is bijectively related with the starting measure $\mu^\star$. More generally, such a representation is always valid if the initial measure is a Markov chain on the tree. Markov chains can be described by boundary laws, and conditional probabilities of infinite-temperature time evolutions, are, for finite-volume conditionings, described by boundary laws obeying recursions which are local perturbations of those of the initial measure, see Appendix A and [26].

In what follows we choose $\xi^1 = (+)$ and $\xi^2 = (-)$. With this notation, for non-Gibbsianness it is enough to prove that, at $\bar{\eta}$, there exists an $\varepsilon > 0$ such that, for all Δ there exists $\Delta' \supset \Delta$ such that

$$\left|\mu[\bar{\eta}_{\Delta\setminus 0}, \xi^1_{\Delta'\setminus\Delta}](\sigma_0) - \mu[\bar{\eta}_{\Delta\setminus 0}, \xi^2_{\Delta'\setminus\Delta}](\sigma_0)\right| > \varepsilon \qquad (3.12)$$

3.3 Model in zero field

3.3.1 Finite-volume marginals and η-dependent fields

To prove the non-Gibbsianness of μ'_t, we will have to consider the phase transition behaviour of the Gibbs measures on the first layer in various external fields. These external fields are determined by the various conditionings, as well as by the choice of the initial Gibbs measure.

Let k, m be integers with $k < m$, let us denote $\Delta' = \Lambda_m$ and $\Delta = \Lambda_k$. Consider first the case $h_0 = 0$. Marginalizing on Λ_m leaves us with a finite-volume Gibbs measure on Λ_m denoted by $\nu^{h^\star}_{\Lambda_m}$ and parametrized by the following external fields

$$\begin{aligned} &i \in \partial\Lambda_m,\ h_i = \eta_i h^t + d\varphi(h^\star), \\ &i \in \Lambda_{m-1},\ h_i = \eta_i h^t \end{aligned} \qquad (3.13)$$

In order to apply the (marginalisation) procedure to the η-dependent finite-volume Gibbs measure $\nu_{\Lambda_m}^{h^\star}$ on Λ_m we need to identify the role played by η. It can be shown that taking the marginal on Λ_{m-1} of the finite-volume Gibbs measure on Λ_m (summing out the spin $\sigma \in \partial\Lambda_m$) gives us a finite-volume Gibbs measure on Λ_{m-1} with an external field at the boundary equal to

$$h_i = \eta_i h^t + \sum_{l\sim i} \varphi(\eta_l h^t) \tag{3.14}$$

Here the sum is over the nearest neighbours $l \in \Lambda_m$.

The equation (3.14) tells us how the configurations $\eta_{\partial\Lambda_m}$ will affect the field acting on $i \in \partial\Lambda_{m-1}$ after having taken a one-generation marginal.

The configuration $\eta_{\Lambda_m\setminus\Lambda_k}$ will govern the value of the fields at $\partial\Lambda_k$, when the marginal on Λ_k is taken. Let us see how:

- $\eta_{\Lambda_m\setminus\Lambda_k} = +$

$$\begin{aligned} &i \in \partial\Lambda_m,\ h_i^{(0)} = h^t + d\varphi(h^\star), \\ &\text{after summing out the m-th generation we have} \\ &i \in \partial\Lambda_{m-1},\ h_i^{(1)} = h^t + d\varphi(h_i^{(0)}), \\ &i \in \partial\Lambda_j, k < j < m-1,\ h_i^{(j)} = h^t + d\varphi(h_i^{(j-1)}) \end{aligned} \tag{3.15}$$

- $\eta_{\Lambda_m\setminus\Lambda_k} = -$

$$\begin{aligned} &i \in \partial\Lambda_m,\ h_i^{(0)} = -h^t + d\varphi(h^\star), \\ &\text{after summing out the m-th generation we have} \\ &i \in \partial\Lambda_{m-1},\ h_i^{(1)} = -h^t + d\varphi(h_i^{(0)}), \\ &i \in \partial\Lambda_j, k < j < m-1,\ h_i^{(j)} = -h^t + d\varphi(h_i^{(j-1)}) \end{aligned} \tag{3.16}$$

Note that the above-chosen η-conditioning on the annulus makes the recursion homogeneous. Choosing m big enough guarantees that the recursions (3.15), (3.16) approach their time-dependent fixed points; we denote them respectively by $H_t^\pm$, $H_t^\sharp$ and $h_t^\pm$, $h_t^\sharp$, see Figure(3.2).

Assume that we start at time $t = 0$ with the measure $\mu^\sharp$, then $h^\star = h^\sharp = 0$. It ensures that the recursions (3.15), (3.16) will approach, respectively, $H_t^+ > 0$ and $h_t^- = -H_t^+ < 0$. H_t^+ represents the biggest stable fixed point for the $\eta = +$ recursion (3.15), and h_t^- the smallest stable fixed point for the $\eta = -$ recursion (3.16). The fact that both recursions have as a starting point the unstable fixed point $h^\sharp = 0$ guarantees that the plus conditioning will drag the field towards H_t^+ and the minus one towards h_t^-. This will not be the case for μ^+ and μ^- as we will see later.

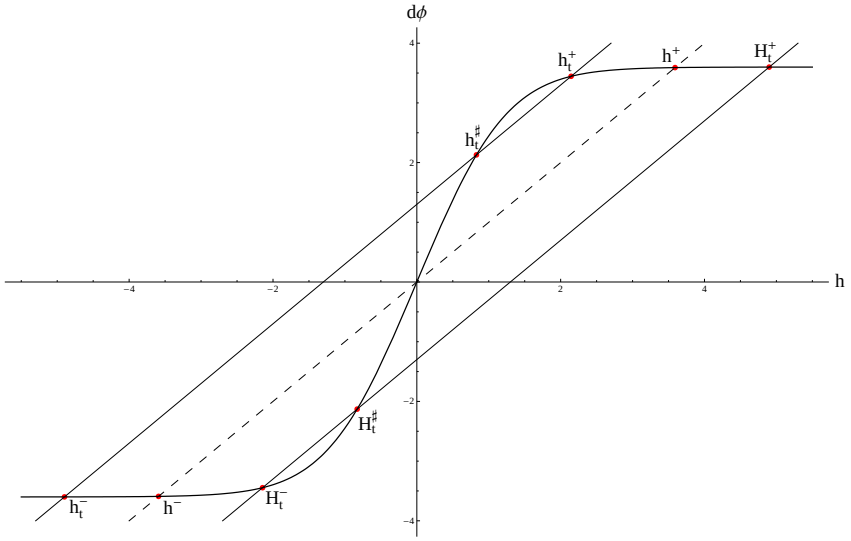


Figure 3.2: Fixed points[1]

The $(\eta_{\Lambda_m \setminus \Lambda_k} = \pm)$-dependent marginals on Λ_k, of the measure on Λ_m, are finite-volume Gibbs measures parametrized by the following fields: for the case $(\eta_{\Lambda_m \setminus \Lambda_k} = +)$

$$\begin{aligned} &i \in \partial\Lambda_k,\ h_i^{+,(0)} = \eta_i h^t + d\varphi(H_t^+), \\ &i \in \Lambda_{k-1},\ h_i^{(0)} = \eta_i h^t \end{aligned} \tag{3.17}$$

and in the case $(\eta_{\Lambda_m \setminus \Lambda_k} = -)$

$$\begin{aligned} &i \in \partial\Lambda_k,\ h_i^{-,(0)} = \eta_i h^t + d\varphi(h_t^-), \\ &i \in \Lambda_{k-1},\ h_i^{(0)} = \eta_i h^t \end{aligned} \tag{3.18}$$

Remark 3.3.1. *Notice that only the fields at $\partial\Lambda_k$ depend on $\eta_{\Lambda_m \setminus \Lambda_k}$ and not the ones acting on the interior. We emphasize that the broadcasting is absorbed by the boundary and has no direct influence on the interior.*

Now we investigate how the recursion relation $h_i^{(j)} = \eta_i h^t + \sum_{l \sim i} \varphi(h_l^{(j-1)})$, obtaining by summing out generations in Λ_k, will depend on the fixed configurations $\eta_{\Lambda_m \setminus \Lambda_k} = \pm$, namely on the fields H_t^+, h_t^- acting on the generation $\partial\Lambda_{k+1}$. We emphasize that the annulus configurations determine the starting point of the recursion. We will also show how the aforementioned recursion relation can be bounded from below if we are coming from $\eta_{\Lambda_m \setminus \Lambda_k} = +$, and from above for $\eta_{\Lambda_m \setminus \Lambda_k} = -$. Furthermore these bounds will turn out to be uniform with respect to η_{Λ_k} and with respect to the number of iterations j.

[1] "Longum est iter per praecepta, breve et efficax per exempla", Seneca.

Lemma 3.3.2. *Given the recursion relation* $h_i^{(j)} = \eta_i h^t + \sum_{l\sim i} \varphi(h_l^{(j-1)})$ *we have :* $h_i^{(j)} \geq h_t^+ > 0$*, for all* i *and* j*, if* $h_i^{(0)} = H_t^+$*; and* $h_i^{(j)} \leq H_t^- = -h_t^+$*, for all* i *and* j*, if* $h_i^{(0)} = h_t^-$*. Here* h_t^+ *is the fixed point for the homogeneous recursion* $h^{(j)} = -h^t + d\varphi(h^{(j-1)})$ *with* $h^{(0)} = H_t^+$*.*

Proof: Fixed points of the discussed recursion relation are given in the picture (3.2). The proof follows by induction. Take first the case $h_i^{(0)} = H_t^+$. Naturally $H_t^+ > h_t^+$, so $h_i^{(0)} > h_t^+$ for all i. If we now assume $h_i^{(j)} > h_t^+$ for all i, then $h_i^{(j+1)} = \eta_i h^t + \sum_{l\sim i} \varphi(h_l^{(j)}) > -h^t + d\varphi(h_t^+) = h_t^+$. The case $h_i^{(0)} = h_t^-$ follows by symmetry; the corresponding recursion relation will be bounded from above by H_t^-.

□

3.3.2 Results: total badness of the evolved $\mu^\sharp$

Let t_2 be defined by

$$h^{t_2} = h(\beta, d) \tag{3.19}$$

Theorem 3.3.3. *If* σ *is distributed according to* $\mu^\sharp$*, then after time* t_2 *all configurations* η *are bad configurations (points of essential discontinuity) for the transformed measure* $\mu^\sharp S(t)$*.*

Remark 3.3.4. *The main idea is as follows: If the plus configuration is bad (and by symmetry the same is true for the minus configuration), then all configurations* $\bar{\eta}$ *will be bad. This is because if minus boundary conditions give a minus magnetisation for the conditioned* σ*-spin at the origin, and plus boundary conditions a positive one, the same holds for all* $\bar{\eta}$ *(due to FKG e.g.). So take* $\bar{\eta}$ *to be plus. Choosing* ξ *to be plus in a large enough annulus* $\Lambda' \setminus \Lambda$ *and integrating the outside with* $\mu^\sharp$ *will lead to an effective plus boundary condition at* Λ*. The reason is that the positive magnetisation* m^+*is an attractive fixed point for the recursive relation, and any positively magnetised field in* Λ' *will lead into its domain of attraction. The same is true for the negative magnetisation. As there are different magnetisations with plus and minus boundary conditions, even in the presence of a weak plus field (the field is plus due the* $\bar{\eta}$ *being plus), the choice of plus or minus in the annulus influences the expected magnetisation at the origin, however big* Λ *is.*

Proof. The definition of t_2, (3.19), will assure that we are in the phase-transition regime for the transformed system (for $t \geq t_2$). Making use of Lemma 3.3.2, the value of ε we are after, in order to prove the essential discontinuity, is given by $\varepsilon = 2\tanh(h_t^+)$. This value corresponds to taking, for the measure coming from $\eta_{\Lambda_m \setminus \Lambda_k} = +$, the smallest positive field along all

the $k-1$ iterations, namely h_t^+. The field at the origin is given by $h^{(k)} = \eta_0 h^t + (d+1)\phi(h^{(k-1)})$ and could be roughly bounded from below

$$h^{(k)} = \eta_0 h^t + (d+1)\phi(h^{(k-1)}) \geq -h^t + d\phi(h_t^+) = h_t^+$$

Thus the corresponding single-site measure is given by $\nu_+(\sigma_0) = \frac{e^{h_t^+\sigma_0}}{e^{h_t^+}+e^{-h_t^+}}$, so

$$\mu[\bar{\eta}_{\Lambda_k}(+)_{\Lambda_m\setminus\Lambda_k}](\sigma_0) \geq \tanh(h_t^+)$$

Analogously for the measure coming from $\eta_{\Lambda_m\setminus\Lambda_k} = -$, we take the biggest negative value along all the $k-1$ iterations, that is $H_t^- = -h_t^+$, therefore $\nu_-(\sigma_0) = \frac{e^{-h_t^+\sigma_0}}{e^{h_t^+}+e^{-h_t^+}}$ and

$$\mu[\bar{\eta}_{\Lambda_k}(-)_{\Lambda_m\setminus\Lambda_k}](\sigma_0) \leq \tanh(-h_t^+)$$

For $\varepsilon = 2\tanh(h_t^+)$ the inequality (3.12) holds. Let us notice that ε is chosen uniformly with respect to η, thanks to the uniform bounds appearing in Lemma 3.3.2. This ensures the μ'-essential discontinuity in any point.

□

3.3.3 Results: difference between different phases

As mentioned before, the previous argument does not hold for μ^+ and μ^-. We treat here only the μ^+ case, the μ^- case is completely symmetrical. So, in case we start with the plus measure, even conditioning on a minus configuration in the annulus, due to the plus influence from the boundary will lead to a measure on Λ_k that looks like the plus measure in a negative field.

Lemma 3.3.5. *Given the starting measure μ^+, the fields acting on $\partial\Lambda_m$ for the marginal measure on Λ_m, which are given by $h_i^{(0)} = \eta_i h^t + d\phi(h^+)$, $i \in \partial\Lambda_m$, satisfy the following inequality*

$$\eta_i h^t + d\phi_\beta(h^+) > h_t^\sharp(d,\beta) \tag{3.20}$$

for all $d > 1$, $\beta > \beta(d)$ and for all $t \in [t_2, \infty)$.

Proof: Let t_2 be as in (3.19). It suffices to show that $d\varphi_\beta(h^+(d,\beta)) > h_t^\sharp(d,\beta) + h^t$ in the aforementioned region of parameters. First of all we note that the expression on the right-hand side is zero in the limit $t \uparrow \infty$, and it is a decreasing function of t. So in order to prove the lemma it is enough to show

$$d\varphi_\beta(h^+(d,\beta)) > h_{t_2}^\sharp(d,\beta) + h^{t_2} \tag{3.21}$$

Using that $h^\sharp_{t_2}(d,\beta)$ is a fixed point for the $(-)$ recursion at $t=t_2$, we arrive at

$$d\varphi_\beta(h^+(d,\beta)) > d\phi_\beta(h^\sharp_{t_2}(d,\beta)), \tag{3.22}$$

Note that $h^\sharp_{t_2}(d,\beta) = h_c(d,\beta) > 0$, where $h_c(d,\beta)$ is a tangent point to $d\varphi(x)$ such that $d\varphi'(h_c(d,\beta)) = 1$. We show that $h^+ > h_c(d,\beta)$. In fact we know that $d\varphi(h^+) - h^+ = 0$. Using the mean-value theorem together with the fact that $d\varphi(0) = 0$, we write $d\varphi'(\zeta)h^+ - h^+ = 0$. It implies that ζ is such that $d\phi'(\zeta) = 1$. Using then the fact that $d\varphi'$ is a decreasing function it follows that the domain of ζ, namely $(0,h^+)$ has to contain $h_c(d,\beta)$; so $h^+ > h_c(d,\beta)$. Using then the monotonicity of the functions φ_β the claim is proved.

□

Theorem 3.3.6. *If σ is distributed according to μ^+, then after time t_2 all configurations η are good configurations for the transformed measure $\mu^+S(t)$.*

Proof. Based on Lemma 3.3.5, choosing Λ_m big enough we make sure that the recursion relation coming from the fixed "+"-annulus $\Lambda_m \setminus \Lambda_k$ will approach its fixed value H_t^+, so do we for the fixed "−"-annulus to approach its fixed value h_t^+. Then the magnetic fields for the finite-volume Gibbs measure on Λ_k are respectively given by

$$\begin{aligned} &i \in \partial\Lambda_k,\ h_i^{+,(0)} = \eta_i h^t + d\varphi(H_t^+), \\ &i \in \Lambda_{k-1},\ h_i^{(0)} = \eta_i h^t \end{aligned} \tag{3.23}$$

and

$$\begin{aligned} &i \in \partial\Lambda_k,\ h_i^{-,(0)} = \eta_i h^t + d\varphi(h_t^+), \\ &i \in \Lambda_{k-1},\ h_i^{(0)} = \eta_i h^t \end{aligned} \tag{3.24}$$

Define $T^j = \max_i(h_i^{+,j} - h_i^{-,j})$. This maximum is always positive, as an inductive argument shows. We are about to prove that $\exists\delta \in (0,1)$ such that $T^j \geq (1-\delta)T^{j+1}$; this is equivalent to say that $\lim_{j\uparrow\infty} T^j = 0$.

$$\begin{aligned} T^{j+1} &= \max_i(h_i^{+,(j+1)} - h_i^{-,(j+1)}) = \max_i \left[\frac{1}{d}\sum_{l\sim i}\left(d\varphi(h_l^{+,(j)}) - d\varphi(h_l^{-,(j)})\right)\right] \\ &= \max_i \left[\sum_{l\sim i}\frac{d\varphi'(c_l)}{d}\left(h_l^{+,(j)} - h_l^{-,(j)}\right)\right] \leq (1-\delta)\max_i\left[\frac{1}{d}\sum_{l\sim i}\left(h_l^{+,(j)} - h_l^{-,(j)}\right)\right] \\ &\leq (1-\delta)\max_i\max_{l\sim i}\left((h_l^{+,(j)} - h_l^{-,(j)})\sum_{l\sim i}\frac{1}{d}\right) = (1-\delta)\max_i\max_{l\sim i}\left(h_l^{+,(j)} - h_l^{-,(j)}\right) \\ &= (1-\delta)T^j \end{aligned}$$

We used the mean-value theorem together with the fact that $d\varphi'(x) < 1$ for $x > h_c(d, \beta)$.

□

For σ distributed according to $\mu^\sharp$, we will show the existence of an intermediate time interval, where some, but not all, configurations are bad for μ'. Theorem 3.3.11 will express this. We will show that the all plus and all minus configurations are good for $\mu^\sharp S(t)$ at all times in $(0, t_2)$. Moreover we will impose a condition on the field h^t (therefore on t itself), such that it guarantees the existence of at least one bad configuration for $\mu^\sharp S(t)$.

We will find a t_1, which is larger than the minimal value of time for which this condition is satisfied. This value t_1 will turn out to be strictly less than t_2. This will guarantee that t_1 is small enough so that the transformed measure, conditioned on an all plus or all minus η will not exibit a phase transition.

Remark 3.3.7. *Note that this implies that at the same time t_2 the intermediate state has a transition to a totally non-Gibbsian regime, where all spin configurations are discontinuity points, whereas the plus and minus state have a transition to a Gibbsian regime, without discontinuity points.*

Lemma 3.3.8. *If σ is distributed according to $\mu^\sharp$ then for all $t \in (0, t_2)$ the $\eta = +$ and $\eta = -$ configurations are good configurations for the transformed measure $\mu^\sharp S(t)$.*

Proof. As was shown before, the recursions (3.15), (3.16) (related to the annuli) give us respectively H_t^+ and h_t^-. Let first η be the plus configuration. In this case $h_i^{+,(j)} = H_t^+$ for all i and j. In other words the field will stick to the fixed point value along the iterations. Using an inductive argument we show that $h_i^{-,(j)} = h^{-,(j)}$; that's to say that it does not depend on i. Based on that, it is straightforward to get a monotonicity property for $h^{-,(j)}$, namely that $h^{-,(j+1)} > h^{-,(j)}$ for all j. Indeed $h^{-,(j+1)} = h^t + d\phi(h^{-,(j)}) > h^{-,(j)}$. The last inequality follows from the fact that $d\phi(x) > x - h^t$ for all $x \in [h_t^-, H_t^+)$, due to the chosen range of t. Recalling that for $t \in (0, t_2)$ the recursion relation $h^{-,(j+1)} = h^t + d\phi(h^{-,(j)})$ has only one fixed point, namely H_t^+, the lemma is proven for $\eta = +$. The $\eta = -$ case follows by symmetry.

□

Remark 3.3.9. *The chosen range of times enables the existence of a unique fixed point for each of the recursions* (3.15), (3.16), *independently of $h^\star$. This means that the fields we obtain at $\partial\Lambda_k$ depend on the annuli, but they do not depend on the exterior Λ_m^c. For this reason Lemma 3.3.8 applies to σ's distributed according to μ^+ and μ^- too.*

For the sake of clarity, let us recall that h^+ indicates the positive stable fixed point for the recursion (3.5) with $h_0 = 0$.

Lemma 3.3.10. *Let t_1 be given by*

$$h^{t_1} = h^+ \tag{3.25}$$

then $t_1 \in (0, t_2)$

Proof. Recalling equation (3.8), the fact that t_1 lies in the interval $(0, t_2)$ is guaranteed by the truth of the inequality $h(d, \beta) < d\varphi(h^+)$, for $\beta > \beta(d)$ and $d > 1$. Indeed

$$\begin{aligned} h(d, \beta) &< d\text{atanh}\left(w\left(\frac{d-\bar{w}}{d-w}\right)^{\frac{1}{2}}\right) \\ &= d\text{atanh}\left(w\tanh(h_c)\right) = d\varphi(h_c) \end{aligned} \tag{3.26}$$

Knowing that $h_c < h^+$, the monotonicity of the function φ concludes the proof.

□

Define the "alternating" configuration η^A to be $\eta_i^A = (-1)^n$ for $i \in \partial\Lambda_n$ and $n \in \mathbb{N}$, i.e. all vertices at each generation have the same sign different from the sign of the previous and the next generations. Naturally the configuration for which $-\eta_i^A = (-1)^n$ is also an "alternating" one. Let us call $h_i^{\pm,(j)}$ the field at the vertex $i \in \partial\Lambda_{k-j}$ after $(j+1)$ applications of the recursion formula (3.14), starting respectively at H_t^+ or h_t^-. The particular structure of the "alternating" configuration makes the fields homogeneous at each generation; i.e., $h_i^{\pm,(j)} = h^{\pm,(j)}$, for all $i \in \partial\Lambda_{k-j}$.

Theorem 3.3.11. *If σ is distributed according to $\mu^\sharp$, and t_1 is given by* (3.25)*, then for all $t \in [t_1, t_2)$ some, but not all, configurations η are bad for the transformed measure $\mu^\sharp S(t)$.*

Proof. Making use of Lemma 3.3.8, Lemma 3.3.10, to prove the theorem it is enough to find a particular configuration η that will be bad for all $t \in [t_1, t_2)$. The "alternating" configurations will be shown to be bad for all $t \geq t_1$, in other words they transmit the influence of the annulus to the origin, no matter how "distant" the annulus and the origin are. As remarked before, $h_i^{\pm,(j)}$ associated to the η^A configurations depend only on j, and we call the corresponding values $h^{\pm,(j)}$. Without loss of generality let us assume $\eta_i^A = +$, for $i \in \partial\Lambda_k$. By an inductive argument, based on the hypothesis $t \in [t_1, t_2)$ (which in terms of fields means $h^t \leq h^+$), and on the particular structure of the configuration η^A, we show that $h^{+,(j)} \geq h^+$ and $h^{-,(j)} \leq 0$, for all j even, namely for those j which relate to generations at which η^A is set to be $+$, and that $h^{-,(j)} \leq -h^+$ and $h^{+,(j)} \geq 0$ for j odd. This will imply $h^{+,(j)} - h^{-,(j)} \geq h^+$ for all j. Consider the case j even.
For $j = 0$ we have:

$$h^{+,(0)} = H_t^+ \geq h^+, \qquad h^{-,(0)} = h^t + d\varphi(h_t^-) \leq 0$$

Both inequalities hold, because H_t^+ is a decreasing function of t whose lower bound is given by h^+.

Assuming the statement is true for j, let us see that it holds for $j+2$. We focus first on $h^{+,(j+2)}$.

$$h^{+,(j+2)} = h^t + d\varphi(h^{+,(j+1)}) = h^t + d\varphi(-h^t + d\varphi(h^{+,(j)})), \tag{3.27}$$

where the second equality is justified by the particular structure of the alternating configuration. Using the assumption $h^{+,(j)} \geq h^+$ and the monotonicity of ϕ we arrive at

$$h^{+,(j+2)} \geq h^t + d\varphi(-h^t + d\varphi(h^+)) \tag{3.28}$$

The fact that $0 \leq -h^t + h^+ \leq h^+$ ensures that $d\varphi(-h^t + h^+) \geq -h^t + h^+$. This concludes the proof for $h^{+,(j)}$.
For $h^{-,(j+2)}$ we have:

$$h^{-,(j+2)} = h^t + d\varphi(h^{-,(j+1)}) = h^t + d\varphi(-h^t + d\phi(h^{-,(j)})) \tag{3.29}$$

Using always the assumption $h^{-,(j)} \leq 0$, the monotonicity of φ, and the assumption $h^t \leq h^+$, which guarantees $h^t \leq d\phi(h^t)$, we obtain

$$h^{-,(j+2)} \leq h^t + d\varphi(-h^t) \leq 0 \tag{3.30}$$

The case j odd is analogous.

□

Remark 3.3.12. *The above result also applies to the evolved plus and minus measures. Indeed the alternating configuration displays a strong discontinuity here, whereas the above analysis shows that for large times all configurations display a $\mu^\sharp S(t)$-essential but non-strong discontinuity. Whether the t_1 used above is optimal in any sense is not known. We conjecture that it may be for the intermediate state, but not for the plus or minus states.*

3.4 Non-zero initial field: shifted view

Recall that $|h_0| < h(d,\beta)$, $\beta > \beta(d)$ and $d > 1$; these conditions guarantee existence of three homogeneous phases for the original measure; we denote them, though not fully consistent with the notation we have been using so far, $\mu^+_{h_0}$, $\mu^-_{h_0}$, and $\mu^\sharp_{h_0}$, to emphasize their dependence on h_0. We show that the previous results found for $h_0 = 0$ will also apply to the case $h_0 \neq 0$ but for different time values. Let $t_+(h_0), t_-(h_0)$ be given by the following equations:

$$\begin{aligned} h_0 + h^{t_+} &= h(d,\beta), \\ h_0 - h^{t_-} &= -h(d,\beta) \end{aligned} \tag{3.31}$$

Call

$$\begin{aligned} t_2(h_0) &= \min\{t_+(h_0), t_-(h_0)\}, \\ t_3(h_0) &= \max\{t_+(h_0), t_-(h_0)\} \end{aligned} \tag{3.32}$$

Depending on the sign of the initial field, $t_+(h_0)$ might be either bigger or smaller then $t_-(h_0)$, as follows from (3.8). Nevertheless the definitions of $t_2(h_0)$, and $t_3(h_0)$ will always assure $t_2(h_0) < t_3(h_0)$ (e.g. for $h_0 < 0$ the order is $t_2(h_0) = t_+ < t_- = t_3(h_0)$).

The time $t_2(h_0)$ indicates the time value for which the dynamic field h^t, taken in the opposite direction to h_0, will first reach a value which guarantees phase transition for the conditioned transformed measure. The time $t_3(h_0)$ refers to the analogous value, but for h^t taken with the same sign to h_0.

Suppose w.l.o.g. that $h_0 < 0$. Note that for h_0 negative the magnetization corresponding to $\mu^\sharp_{h_0}$ is positive, [26, see Chapter 12]. For $t > t_3(h_0)$ there exist three fixed points for the $(-)$-recursion $h^{(k+1)} = h_0 - h^t + d\phi(h^{(k)})$, namely two stable ones $h^-_t(h_0)$, $h^+_t(h_0)$, and an unstable $h^\sharp_t(h_0)$. The existence of several fixed points makes the convergence to them be dependent on the starting point. In particular the recursion will take us to $h^+_t(h_0)$ if and only if the starting point, $h^{(k=0)}$, lies to the right of the unstable one, that is when $h^{(k=0)} > h^\sharp_t(h_0)$; it will take us to $h^-_t(h_0)$ if and only if $h^{(k=0)} < h^\sharp_t(h_0)$, and will stick to $h^\sharp_t(h_0)$ if and only if $h^{(k=0)} = h^\sharp_t(h_0)$.

Given that $t_3(h_0) > t_2(h_0)$, the assumption $t > t_3(h_0)$ ensures the existence of three fixed points also for the $(+)$-recursion $h^{(k+1)} = h_0 + h^t + d\phi(h^{(k)})$; they are denoted by $H^\pm_t(h_0)$, and $H^\sharp_t(h_0)$.

Assume that we start at time $t = 0$ with the measure $\mu^\sharp_{h_0}$, then the starting point for the $(\pm)$-recursions is $h^\star = h^\sharp(h_0) > 0$. However, for the chosen range of time, $t > t_3(h_0)$, it can be shown that $h^\sharp(h_0)$ will always lie to the right of $H^\sharp_t(h_0)$ and always to the left of $h^\sharp_t(h_0)$. So the theorem reads:

Theorem 3.4.1. *If σ is distributed according to $\mu^\sharp_{h_0}$, then after time $t_3(h_0)$ all configurations η are bad configurations for the transformed measure $\mu^\sharp_{h_0} S(t)$.*

□

Analogously to the analysis for $h_0 = 0$, the former result will not hold for σ distributed according to $\mu^\pm_{h_0}$.

Two other results, obtained in the previous section, have equivalents for non-zero external field.

Lemma 3.4.2. *If σ is distributed according to $\mu^\sharp_{h_0}$, then for all $t \in (0, t_2(h_0))$ the $\eta = +$ and $\eta = -$ configurations are good configurations for the transformed measure $\mu^\sharp_{h_0} S(t)$.*

□

Theorem 3.4.3. *If σ is distributed according to $\mu^{\pm}_{h_0}$, then after time $t_3(h_0)$ all configurations η are good configurations for the transformed measure $\mu^{\pm}_{h_0}S(t)$.*

□

Remark 3.4.4. *It is worth remarking that the strict inequality $t_2(h_0) < t_3(h_0)$, always holding for $h_0 \neq 0$, implies the non-emptiness of the interval of times $[t_2(h_0), t_3(h_0))$. A similar result to the one given in Theorem 3.3.11 holds in the case $h_0 \neq 0$, namely that for $t \in [t_2(h_0), t_3(h_0))$ some, but not all, configurations are bad. In fact, it can be shown, for example in case $h_0 < 0$, that the time $t_2(h_0)$ corresponds to the time for which the plus configuration becomes bad, while for all times $t < t_3(h_0)$ the minus configuration will remain good. In case $h_0 > 0$, as symmetry may suggest, the time $t_2(h_0)$ will be the threshold for the minus configuration to become bad, while the plus configuration will be good till $t = t_3(h_0)$.*

Encouraged by the many analogies between the $h_0 = 0$ case and the $h_0 \neq 0$ case, one might ask what one can say about the $(h_0 \neq 0)$-equivalent of the time t_1, (3.25). Pursuing the former, let us define the values of times $\hat{t}_+$, $\hat{t}_-$ by the following equalities

$$\begin{aligned} h^{\hat{t}_+} &= h_0 + d\phi(h^+(h_0)) - h^\sharp(h_0), \\ -h^{\hat{t}_-} &= h_0 + d\phi(h^-(h_0)) - h^\sharp(h_0) \end{aligned} \tag{3.33}$$

and define further

$$t_1(h_0) = \max\left\{\hat{t}_+, \hat{t}_-\right\} \tag{3.34}$$

The Figure 3.3 helps to understand the role played by the different times so far defined.

It can be shown that $t_1(h_0) < t_3(h_0)$ for $|h_0| < h(d, \beta)$. Nonetheless the relation between time $t_1(h_0)$ and $t_2(h_0)$ is not trivial as we will show. The next lemma formalizes that for all time $t \geq t_1(h_0)$ the "alternating" configurations are bad for σ distributed according to $\mu^\sharp_{h_0}$.

Lemma 3.4.5. *If σ is distributed according to $\mu^\sharp_{h_0}$, and $t_1(h_0)$ is given by (3.34), then for all $t > t_1(h_0)$ "alternating" configurations are bad for the transformed measure $\mu^\sharp_{h_0}S(t)$.*

Proof. The proof follows the same route taken in the proof of Theorem(3.3.11) with some modifications on the bounds. Nontheless we reckon it is instructive to sketch the main points at least for $h_0 < 0$. For $t > t_1(h_0)$ an inductive argument leads to the following bounds:

$$\text{for even } j, \qquad h^{+,(j)} \geq h^+(h_0) \text{ and } h^{-,(j)} \leq h^\sharp(h_0),$$

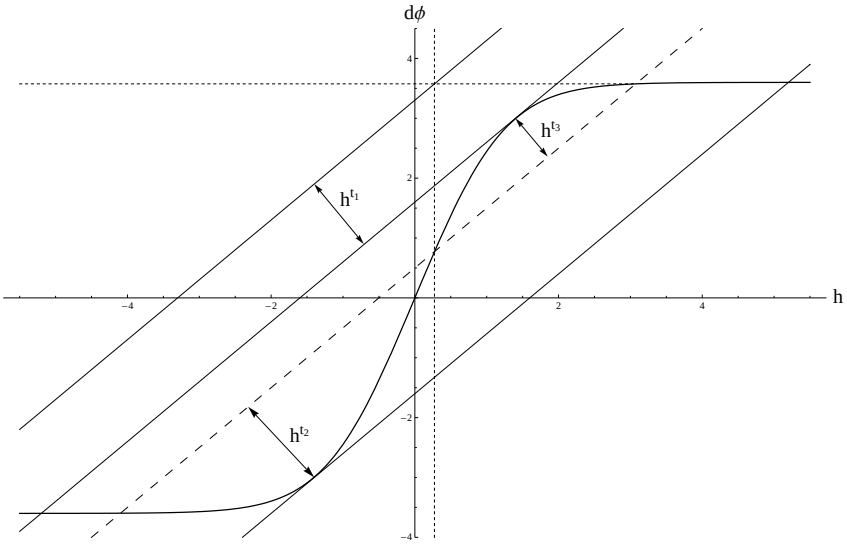


Figure 3.3: Times t_1, t_2, and t_3

$$\text{for odd } j, \qquad h^{+,(j)} \geq -h^\sharp(h_0) \text{ and } h^{-,(j)} \leq -h^+(h_0),$$

therefore $h^{+,j} - h^{-,j} \geq h^+(h_0) - h^\sharp(h_0)$ for all j.

□

The previous lemma together with Remark 3.4.4 shows that if σ is distributed according to $\mu^\sharp_{h_0}$, then for all $t \in [t_1(h_0), t_3(h_0))$ some, but not all, configurations are bad. There are then two different time intervals where some, but not all, configurations are bad. We will not leave the reader wondering how these two intervals relate. We will show the existence of a critical value h_0^c such that for $|h_0| > h_0^c$ we have $[t_1(h_0), t_3(h_0)) \subset [t_2(h_0), t_3(h_0))$, for $|h_0| < h_0^c$ the inclusion is reversed, namely $[t_1(h_0), t_3(h_0)) \supset [t_2(h_0), t_3(h_0))$, and for $|h_0| = h_0^c$ the two intervals coincide.

Remark 3.4.6. *In the small-field regime $|h_0| < h_0^c$ we have that the "alternating" configuration becomes bad before the all plus and the all minus configurations. In that case, the dominant effect is that the alternating character of the conditioning provides some cancellations, just as in the zero-field case.*

In the other regimes we can just say what follows from $t_1(h_0) < t_3(h_0)$, i.e. that the "alternating" configurations become bad before the homogeneous configuration with all η's aligned with h_0, that is $\eta = sign(h_0)$. The impossibility to state something more in the other regimes is due to the fact that $t_1(h_0)$ is not a "sharp" threshold for the "alternating" configurations to become bad. However, in this case having a "bad" configuration, one may need to counteract the effect of the field, thus in a positive external field, the minus configuration becomes bad at an earlier time than the alternating one.

To explore the latter inclusions we need to compare the values $t_1(h_0)$ and $t_2(h_0)$, or equivalently $h^{t_1(h_0)}$ and $h^{t_2(h_0)}$. Consider the difference between the fields

$$f(h_0) := h^{t_1(h_0)} - h^{t_2(h_0)} \tag{3.35}$$

Based on the definitions of the times, (3.34), (3.32) it turns out that the function f is even. So we might focus on its behaviour only for negative values of the initial field h_0. For such values of the field the function has the following form

$$f(h_0) = h^+(h_0) - h^\sharp(h_0) + h_0 - h(d,\beta) \tag{3.36}$$

First of all the limit values of f in the interval $(-h(d,\beta), 0)$ are given by

$$\lim_{h_0 \downarrow -h(d,\beta)} f(h_0) = -2h(d,\beta),$$

$$\lim_{h_0 \uparrow 0} f(h_0) = h^+ - h(d,\beta)$$

Note that the second limit value is positive, as has been explained in the proof of Lemma 3.3.5, while the first one is negative by the definition of $h(d,\beta)$, and by (3.4). Taking now the derivative of f with respect to h_0 we obtain

$$f'(h_0) = (h^+(h_0))' - (h^\sharp(h_0))' + 1 \tag{3.37}$$

Using the only thing we know about $h^+(h_0)$, $h^\sharp(h_0)$, namely that they are fixed points for the recursion $h^{k+1} = h_0 + d\phi(h^k)$, the following equalities turn out to hold

$$(h^+(h_0))' = \frac{1}{1 - d\phi'(h^+(h_0))},$$

$$(h^\sharp(h_0))' = \frac{1}{1 - d\phi'(h^\sharp(h_0))}$$

Because $h^\sharp(h_0) < h_c(d,\beta)$ and $h^+(h_0) > h_c(d,\beta)$, the monotonicity of $d\phi'$ assures that $f'(h_0) > 0$ for all $h_0 \in (-h(d,\beta), 0)$. Therefore the existence and uniqueness of h_0^c is guaranteed by an application of the intermediate-value theorem. We point out that the function f is not differentiable in $h_0 = 0$. Indeed, being f an even function and $\lim_{h_0 \uparrow 0} f'(h_0) > 0$ clarify the discontinuity.

We would like to remark that the case $h_0 = 0$ might be obtained from the previous analysis by taking the limit $h_0 \uparrow 0$. Indeed $\lim_{h_0 \uparrow 0} t_1(h_0) = t_1$, $\lim_{h_0 \uparrow 0} t_2(h_0) = \lim_{h_0 \uparrow 0} t_3(h_0) = t_2$.

3.5 Conclusions and probable extensions

This chapter shows that the Gibbs-non-Gibbs transition on trees has a number of different aspects, as compared to that on regular lattices. In particular, that different evolved Gibbs measures can have different Gibbsian properties. For the evolved intermediate state there are two transitions, one from being Gibbsian to being "standard non-Gibbsian" (having some, but not all configurations bad) and a second transition to a "totally non-Gibbsian" regime where *all* configurations are bad. Both these properties do not occur in the more familiar lattice and mean-field situations.

For the plus and minus measure there are also two transitions, namely one after which the evolved measure becomes non-Gibbsian, and some, but not all, configurations become discontinuity points and a second one after which the measure becomes Gibbsian again; this is the behaviour which on the lattice occurs for an initial Gibbs measure in an external field.

High-temperature dynamics should behave in a similar way as infinite-temperature dynamics, but although the proofs probably will be messier, qualitatively nothing new is expected.

As mentioned in the beginning of the chapter, a Cayley tree is a special case of trees. Although the regularity property of Cayley trees was intensively used in our proofs, we reckon these results to be true for general trees. The instability of the fixed point $h^\sharp$ for example corresponds to the phase transition being robust, which is true in general for Ising models on trees [50]. Also, the property of plus boundary conditions in a not too strong minus field inducing a positively magnetised state, which was used in the proof that the plus configuration was good for the plus state holds quite generally. The choice of bad configuration in the intermediate regime may be somewhat tree-dependent. Moreover, it seems problematical to identify a unique measure $\mu^\sharp$ in a field (on random Galton-Watson trees for example).

* * *

4

Mean-field models

As has been seen in Chapter 3, even the physically simple transformation of heating produces non-Gibbsian behaviour. It would even be more interesting to say something about *cooling dynamics*. More generally one would like to study a Gibbs measure μ_0 for an initial Hamiltonian H which is subjected to a Glauber dynamics for another Hamiltonian $\bar{H}$, which gives rise to a trajectory μ_t where t denotes time. Glauber dynamics at low temperatures describes fast cooling or "quenching". The question is to understand the behaviour of μ_t, and in particular for which times it will be Gibbs. Since this is as yet too difficult on the lattice, we present our results for mean-field models.

While we give a formal definition of a mean-field model in the latter part, informally mean-field models can be viewed as the "zeroth-order" approximation to lattice systems. Physically, the idea of mean-field theory has a source in the idea that one replaces all interactions with a "mean field". Quite often, mean-field theory provides a convenient launching point to studying higher-order approximations (model on trees, e.g.) In general, dimensionality plays a strong role in determining whether a mean-field approach will work for any particular problem. In mean-field theory, many interactions are replaced by one effective interaction. Then it naturally follows that if the field or particle participates in many interactions in the original system, a mean-field model will be more accurate for such a system. This is true in cases of high dimensionality, or when the Hamiltonian includes long-range forces. The Ginzburg criterion is the formal expression of how fluctuations render mean-field theory a poor approximation, depending upon the number of spatial dimensions in the system of interest [1]. Investigations for mean-field models tend to reproduce the lattice results in many situations [37, 52] but often lead to an explicit knowledge of the parameter regions where Gibbsianness and non-Gibbsianness occur.

As we discussed in Section 2.4, the Gibbsian property strongly depends on continuity properties of single-site conditional probabilities. To understand discontinuous behaviour of conditional probabilities for the time-evolved model at fixed time t one needs to look at the model resulting from the initial measure at time $s = 0$ under application of the dynamics in the space-time region for times s between 0 and t. The hidden phase transitions responsible

for the non-Gibbsian behaviour occur if there is a sensitive dependence of the model at time $s = 0$ obtained from constraining the space-time measure to certain configurations at time $s = t$. If a small variation of such a constraining configuration leads to a jump in the constrained initial measure it will (generically) be a bad configuration for the conditional probabilities of the system at time t. *Small variation* means in the lattice case a perturbation in an annulus far away from the origin. *Small variation* means in the mean-field case a small variation of the magnetization as a real number. In the independent spin-flip lattice example of [11] the chessboard configuration was a bad one, correspondingly in the independent spin-flip mean-field case of [36] the configurations with neutral magnetization equal to zero were bad ones for large enough times. Moreover, configurations with non-zero magnetization also appeared as points of discontinuity for the limiting conditional probabilities, in a particular bounded region of the parameter space of initial temperature and time. This phenomenon was called biased non-Gibbsianness in [36]. The complete analysis for the mean-field independent spin-flip situation was possible since the constrained system on the first layer could be understood on the level of the magnetization. The relevant quantities could be computed in terms of the rate-function for a standard quenched disordered model, namely the Curie-Weiss random-field Ising model with possibly non-symmetric random-field distribution of the quenched disorder.

To deal with the dependent-dynamics case a different route has to be taken since the dependence of the initial system on the conditioning is more intricate. As we will see, we will need to invoke the path large deviation principle for the dynamics with temperature β'^{-1} on the level of magnetizations. We will then have to minimize a cost functional of paths of magnetizations which is composed of the rate function along the path and an initial "punishment" term, which depends both on the initial Hamiltonian H and the dynamical Hamiltonian $\bar{H}$, evaluated at the unknown initial point of the trajectory. The solution of the problem gives a surprising connection between path properties of the corresponding (integrable) dynamical system and Gibbs properties of a model of statistical mechanics. Solutions of the corresponding dynamical system will correspond to *most probable ways* for the system to evolve from an (unknown) initial state to a present. A phase transition in the "constrained" model will be shown to be connected with existence of several most probable histories of a current state of the system. We represent the aforementioned phase transition graphically in the form of the most probable histories for magnetization of the system in a space-time diagram. Such a diagram allows us to visualize all most probable curves for all types of conditioning. Moreover *forbidden regions* pop up, the history-curves do not enter these regions. These regions grow with time. In the case of independent spin-flip dynamics $\beta' = 0$ starting configurations (positively magnetized) with magnetizations located

within an ε-neighbourhood of a certain positive value are only allowed. This effect corresponds to a *memory-losing*. As a result we will provide a full description of the regions of Gibbsian and non-Gibbsian behaviour as a function of time, initial temperature, and dynamical temperature. As a special case the previous results for infinite-temperature dynamics are reproduced. Furthermore we observe a new mechanism for the appearance of bad configurations in the region of cooling from low temperatures with even lower temperatures. These are related to periodic motion in the dynamical system. The corresponding periodic curves are found numerically from the differential equations governing both independent and dependent spin-flip dynamics.

4.1 Probabilistic analysis

4.1.1 Preliminaries

This section is devoted to a Gibbsian description of mean-field models and one type of dynamics defined for them. We transfer the relevant definitions of Gibbsianness and non-Gibbsianness from the lattice setup to the mean-field setup. A very broad and a review type description of the connection between these systems could be found in Le Ny [42].

Curie-Weiss Ising model

We first set the graph $G = (\mathcal{V}, \mathcal{E})$ of Chapter 2 to be a complete graph. Let spin-variables sitting at vertices in $\mathcal{V}$ take values in $S = \{-1, +1\}$, as before. Let Λ be a sub-volume of $\mathcal{V}$. The graph structure suggests the isomorphism $\Lambda \simeq [1, |\Lambda|]$, where $|\cdot|$ is the cardinality of the set, and the interval $[1, |\Lambda|]$ contains only natural numbers. Hereafter we consider that the cardinality of Λ is N and identify it with the interval $[1, N]$. We shall write $\sigma_{[1,N]}$ for σ_Λ, the spin configuration $\sigma_{[1,N]}$ takes values in S^N. The a priori measure $\alpha(d\sigma_1)$ is chosen to be equidistribution. The finite-volume Gibbs measure at inverse temperature β and zero external field on $\sigma_{[1,N]}$ is given by

$$\mu_{\beta,N}(d\sigma_{[1,N]}) = \frac{\exp\left\{\frac{\beta}{2N}\sum_{\{i,j\}\in\mathcal{E}}\sigma_i\sigma_j\right\}}{Z_{\beta,N}}\alpha^{\otimes}(d\sigma_{[1,N]}),$$

where the normalization factor $Z_{\beta,N}$ is the standard partition function. The completeness of the graph allows a replacement of many-spin interactions by

an effective interaction in the following way.

$$\begin{aligned}\mu_{\beta,N}(d\sigma_{[1,N]}) &= \frac{\exp\left\{\frac{\beta}{2N}\left(\sum_{i\in\mathcal{V}}\sigma_i\right)^2\right\}}{Z_{\beta,N}}\alpha^{\otimes}(d\sigma_{[1,N]}) \\ &= \frac{\exp\left\{-N\Phi\left(\frac{1}{N}\sum_{i\in\mathcal{V}}\sigma_i\right)\right\}}{Z_{\beta,N}}\alpha^{\otimes}(d\sigma_{[1,N]}),\end{aligned} \tag{4.1}$$

where the function $\Phi(x) = -\frac{\beta x^2}{2}$ is called a *mean-field interaction*, while the Hamiltonian of the system is $H = N\Phi(\frac{1}{N}\sum_{i\in\mathcal{V}}\sigma_i)$. We shall refer to this form of the finite-volume prescription as the mean-field finite-volume Gibbs measure in the sequel.

An infinite-volume Gibbs measure μ is obtained by taking limits when $N \to \infty$ in (4.1). A sequence $\{\mu_{\beta,N}\}$ has a weak limit μ according to the results of [16].

Generally, a *mean-field model* is defined as a sequence $\{\mu_N\}_{N\in\mathbb{N}}$ of probability measures, such that each element μ_N of this sequence is invariant under permutations of $\sigma_{[1,N]}$. Moreover, this sequence is required to have a weak limit as $N \to \infty$. As clear from the previous discussion, all these requirements are satisfied for μ_N of form (4.1). Such a model is called *Curie-Weiss Ising model* in a vanishing external field. Having in mind that we will always work with this model, we shall refer to it also as a mean-field model.

Gibbsianness for mean-field models

We aim to transfer the notion of non-Gibbsianness to mean-field models. We will make use (again) of the underlying graph structure. Let us remind the reader what the tokens intrinsically identifying a Gibbs nature of a measure are. An infinite-volume measure μ is Gibbs if and only if its finite-volume restrictions are uniformly non-null w.r.t. conditioning on exterior and quasi-local as functions of the conditioning. This definition does not exploit the graph's structure and refers to a general graph. We note that this definition cannot be modified anyhow in the case when spins of a statistical mechanics model interact in a local fashion. This was the case of trees (see Chapter 3) and the case for general lattices $\mathbb{Z}^d$.

There are two crucial differences between the lattice and the mean-field case:

(i) each site is a neighbour of any other. This fact implies that there are *no* "far-away" regions and an effect of conditioning on "outside" of any finite region is immediately transmitted to the concerned region;

(ii) in the lattice setup the continuity properties of a family of finite-volume conditioned restrictions are studied. The exact state where this family

comes from was identified by a boundary law. In the mean-field situation there is necessity of actual taking limits as volumes grow, not only considering the infinite-volume measure.

This requires to adjust the idea of conditioning. We start from noticing that the random variables σ_i, $i \in [1, N]$ are *exchangeable*. This means that their joint distribution is independent of the order in which σ_i's are observed, for instance for σ_1 and σ_2 vectors (σ_1, σ_2) and (σ_2, σ_1) have the same distribution. Exchangeability comes from the fact that all spin-variables are i.i.d. Thus the *number* of pluses and minuses plays a role. The product of a priori measures $\alpha^{\otimes}(\cdot)$ for large N behaves as follows [22](or, in English and in greater generality, [30])

$$\alpha^{\otimes}(d\sigma_{[1,N]}) = \int_{-1}^{1} \theta^{t_N}(1-\theta)^{N-t_N} dG(\theta),$$

where t_N is the number of pluses. It further holds that G is the distribution function of the limiting frequency $\underline{m}(\sigma) = \lim_{N\uparrow\infty} \underline{m}_1^N = \lim_{N\uparrow\infty} \frac{1}{N}\sum_{i=1}^N \sigma_i$. The empirical measure $\underline{m}$ in the language of statistics is *sufficient* for the unknown parameter θ. Some estimates on the speed of the convergence of $\underline{m}_1^N$ to $\underline{m}$ could be found in [8].

The exchangeability of $\sigma_{[1,N]}$ implies that the conditionings which are *permutations* of one another will give rise to the same result, effectively the empirical measure only matters. Generally, the empirical measure L of a configuration $\sigma_{[1,N]}$ is defined as $L(\sigma_{[1,N]})(\cdot) = \frac{1}{N}\sum_{i=1}^N \delta_{\sigma_i}(\cdot)$, as in e.g. [48, see Chapter 4]. For binary single-site state space S, a *magnetization* or *empirical mean* $\underline{m}$ is treated.

We stress that the non-nullness is always granted for mean-field models as follows from the finite-volume representation of μ.

When a system's size is finite and equal to N, the "rest of the world" for a single spin is a configuration of size $N-1$. By convention we will refer to a single spin as σ_1 and to the "rest of the world" as $\sigma_{[2,N]}$. A magnetization value $\underline{m}_2^N$ for a system of a configuration of size $N-1$ belongs to $\{-1, -1+\frac{2}{N-1}, \ldots, 1-\frac{2}{N-1}, 1\}$.

Definition 4.1.1. *A* single-site finite-volume conditional probability $\gamma_{\beta,1,N}(d\sigma_1|\underline{m}_2^N)$ *is defined as follows:*

$$\gamma_{\beta,1,N}(d\sigma_1|\underline{m}_2^N) := \mu_{\beta,N}(d\sigma_1|\sigma_{[2,N]}), \tag{4.2}$$

where $\sigma_{[2,N]}$ *is any spin-configuration such that* $\underline{m}_2^N = \frac{1}{N-1}\sum_{j=2}^N \sigma_j$.

The infinite-volume conditioning is obtained by taking a formal limit when the N grows. A discrete-valued magnetization $\underline{m}_2^N \in \{-1, -1+\frac{2}{N-1}, \ldots, 1-$

$\frac{2}{N-1}, 1\}$ converges to a real-valued magnetization $\underline{m} \in [-1,1]$ under such a limiting procedure.

$$\lim_{N\uparrow\infty} \gamma_{\beta,1,N}(d\sigma_1|\underline{m}_2^N) =: \gamma_{\beta,1}(d\sigma_1|\underline{m}) \tag{4.3}$$

The former definition reflects the main idea of mean-field theory of replacing many-bodies interaction with an average interaction. Equivalently, the whole mean-field system could be seen as a system containing only two spins — σ_1 and σ_{aver}, where $\sigma_1 \in S$ and σ_{aver} lives in a continuous space $[-1,1]$.

For a lattice or a tree a configuration is said to be bad if it broadcasts the information of "far-away" regions. Oppositely, a configuration is said to be good if it stops the influence of far regions. In the mean-field setup this reads:

Definition 4.1.2. *A point $\widehat{m} \in [-1,1]$ is said to be* good *for a mean-field model if and only if:*

1. *There exists a neighborhood of $\underline{m}_2^N$ such that, for all α in this neighborhood the following holds. For all sequences $\alpha_N \in \{-1, -1+\frac{2}{N-1}, \dots, 1-\frac{2}{N-1}, 1\}$ with the property $\lim_{N\uparrow\infty} \alpha_N = \alpha$ the limit*

$$\gamma_{\beta,1}(\sigma_1|\alpha) = \lim_{N\uparrow\infty} \gamma_{\beta,1,N}(\sigma_1|\alpha_N) \tag{4.4}$$

   exists and is independent of the choice of the sequence α_N.

2. *The function $\alpha \mapsto \gamma_{\beta,1}(\sigma_1|\alpha)$ is continuous at $\alpha = \widehat{m}$.*

A point $\hat{m}$ is *bad*, if it is not good. Here we are in the position to give a rigorous definition of Gibbsianness for a mean-field model.

Definition 4.1.3. *A mean-field model at inverse temperature β is called* Gibbs *if and only if it has no bad points.*

The Definition 4.1.1 could be extended to any finite volume and, moreover, finite-volume conditional probabilities may be expressed in therms of single-site conditional probabilities. This representation is proven in [48, see Chapter 4, Proposition 4.2.2] for a general mean-field interaction. From the present standpoint the most important statement of the aforementioned proposition is

$$\gamma_{\beta,n}(d\sigma_n|\underline{m}) := \lim_{N\to\infty} \mu_{\beta,N}(d\sigma_n|\sigma_{[n+1,N]}) = \prod_{i=1}^{n} \gamma_{\beta,1}(d\sigma_i|\underline{m}), \tag{4.5}$$

where $\underline{m} = \lim_{N\to\infty} \frac{1}{N-n} \sum \sigma_i$

4.1.2 Spin-flip transform of mean-field model

In the present setup the finite-volume measure (4.1) may be rewritten in such a way that it will explicitly contain the effective parameter $\underline{m}_1^N$, namely

$$\mu_{\beta,N}(d\sigma_{[1,N]}) = \frac{\exp\left\{N\frac{\beta(\underline{m}_1^N)^2}{2}\right\}}{Z_{\beta,N}}\alpha^{\otimes}(d\sigma_{[1,N]}), \tag{4.6}$$

where $\underline{m}_1^N$ is an empirical mean of the configuration $\sigma_{[1,N]}$. This representation is equivalent to (4.1) and both will be treated as a definition of the finite-volume Gibbs measure for Curie-Weiss Ising model in a vanishing external field.

Given an initial Gibbs mean-field model with measures $\mu_{\beta,N}$ (4.6), our aim is to investigate the Gibbs properties of the transformed model under site-wise independent spin-flip evolution. Hence, rewriting (2.7), we get

$$\mu'_{\beta,\beta',t,N}(d\eta_{[1,N]}) = \sum_{\sigma_{[1,N]}} \tilde{\mu}_{\beta,\beta',t,N}(d\sigma_{[1,N]}) = \sum_{\sigma_{[1,N]}} \mu_{\beta,N}(d\sigma_{[1,N]}) \prod_{i=1}^{N} k(\sigma_i,\eta_i) \tag{4.7}$$

The corresponding joint model is given by $\tilde{\mu}_{\beta,\beta',t,N}(d\sigma_{[1,N]})$.

The study of the Gibbs properties is based on the investigation of continuity properties of the single-site conditional distributions $\gamma'_{\beta,\beta',t,1}$ of the corresponding transformed model. As before,

$$\gamma'_{\beta,\beta',t,1}(d\eta_1|\underline{m}') := \lim_{N\to\infty} \mu'_{\beta,\beta',t,N}\left(d\eta_1 \middle| \frac{1}{N-1}\sum_{i=2}^{N}\eta_i\right)$$

Fix a transformed configuration η, equivalently, fix the corresponding magnetization $\underline{m}'$. The completeness of the graph relaxes requirements for non-Gibbsian behaviour of the transformed measure. An existence of a phase transition for the two-layered model with *fixed* second layer becomes a sufficient condition for non-Gibbsianness. Suppose that a finite-volume joint model with the fixed particular $\underline{m}'$ admits several infinite-volume measures corresponding to different magnetizations (say) $\underline{m}_1^*(\underline{m}')$ and $\underline{m}_2^*(\underline{m}')$. The distributions $\gamma'_{\beta,\beta',t,1}(\cdot|\underline{m}')$ turn out to be functions of the magnetization $\underline{m}^*$. Therefore varying the conditioning in a neighbourhood of $\underline{m}'$, the single-site conditional distributions will obtain discontinuities of a jump type. We are left with a problem of identifying such a magnetization(s) $\underline{m}^*$ of the initial spins σ, so the model started at σ and evolved ends up in configuration η, equivalently, with magnetization $\underline{m}'$. The study of evolution of the system from the state $\underline{m}^*$ to the state $\underline{m}'$ involves large deviations theory for trajectories of stochastic processes. Such an approach allows to keep track, in addition, of a most probable evolution of the system.

4.1.3 Another view on spin-flip evolution

The spin-flip evolution on the level of individual spins was explained in Section 2.5. To be more specific, let $c(+, \underline{m}_2^N)$ be a single-site rate for σ_1 to flip from plus- to minus-state, when the magnetization of the rest of the system (consisting $N-1$ components) is $\underline{m}_2^N$. Analogously, $c(-, \underline{m}_2^N)$ is defined. We are going to apply a *temperature-dependent* spin-flip dynamics. This is reflected in the fact that the spin-flip rates depend implicitly (for now) on the temperature β' of the dynamics. Moreover, their values depend on the initial inverse temperature β and time. The time-dependence of the rates occurs via the time-dependence of the magnetization $\underline{m}$. The explicit formula for the rates will be given later. The dynamics is called *constrained* or *interacting* if the inverse temperature β' of the dynamics is non-zero. Unless the opposite is stated a dynamics is considered with both β and β' not equal to zero.

We require the dynamical measure $\mu_{\beta'}$ to be *time-reversible*. Equivalently, this requirement for the single-site spin-flip rates may be re-expressed as follows:

$$\frac{c(-, \underline{m})}{c(+, \underline{m})} = \exp\left\{2\beta' \underline{m}\right\}, \tag{4.8}$$

where $c(\pm, \underline{m})$ are the rates depending on the magnetization of infinite-volume. See Appendix B.1 for an explanation.
In other words,

$$c(\sigma_1, \underline{m}) = R(\underline{m}) \exp\left\{-\sigma_1 \beta' \underline{m}\right\}, \tag{4.9}$$

with a function $R(\underline{m})$ giving a time-rescaling.

Adjusting the form of the linear generator L of the configurations spin-flip dynamics from the Section 2.5, we define a generator $L_{\beta',N}$ for local functions $\bar{F} : \Omega_{[1,N]} \mapsto \mathbb{R}$

$$L_{\beta',N}\bar{F}(\sigma_{[1,N]}) = \sum_{i=1}^{N} c\left(\sigma_i, \frac{1}{N-1}\sum_{j:j\neq i}\sigma_j\right)\left(\bar{F}(\sigma^i_{[1,N]}) - \bar{F}(\sigma_{[1,N]})\right), \tag{4.10}$$

where $\sigma^i_{[1,N]}$ is the configuration that is flipped at the site i.

In the light of (4.6) it is wise to trace how spin-flip dynamics acts on the magnetization. We remark that by permutation invariance the continuous time process, induced on the empirical average, is a Markov chain. Namely, suppose that $F : \left\{-1, -1+\frac{2}{N}, \ldots, 1-\frac{2}{N}, 1\right\} \to \mathbb{R}$ is a function on the possible magnetization values at size N, then the linear generator $\hat{L}_{\beta',N}$ acting

on such functions F defined as $(\hat{L}_{\beta',N}F)\circ \underline{m}_1^N = L_{\beta',N}(\bar{F}\circ\sigma_{[1,N]})$ with

$$\begin{aligned}\hat{L}_{\beta',N}F(\underline{m}_1^N) = \\ N\frac{1+\underline{m}_1^N}{2}c\left(+,l(N)\left(\underline{m}_1^N-\frac{1}{N}\right)\right)\left(F\left(\underline{m}_1^N-\frac{2}{N}\right)-F\left(\underline{m}_1^N\right)\right) \\ +N\frac{1-\underline{m}_1^N}{2}c\left(-,l(N)\left(\underline{m}_1^N+\frac{1}{N}\right)\right)\left(F\left(\underline{m}_1^N+\frac{2}{N}\right)-F\left(\underline{m}_1^N\right)\right)\end{aligned} \tag{4.11}$$

where $l(N)=\frac{N}{N-1}$. Clearly, $l(N)$ may be omitted when N is sufficiently large.

The asymptotic rates in (4.11) with $N\to\infty$, for the magnetization are

$$c_\pm(\underline{m}) = \frac{1\pm\underline{m}}{2}c(\pm,\underline{m}) = R(\underline{m})\frac{1\pm\underline{m}}{2}\exp\{\mp\beta'\underline{m}\} \tag{4.12}$$

We treat the prefactor N in (4.11) as of a scaling for test functions F.

Summarizingly, we would like to study a stochastic spin-flip evolution of infinite spin-configurations σ. The dynamics of the process $\{\sigma_s, s\geq 0\}$ is described by a semi-group $S(s)=e^{sL_{\beta',N}}$ on the level of spins. The exchangeability allows to study equivalent real-valued process $\{m_s := \underline{m}(\sigma_s), s\geq 0\}$. The corresponding semi-group of the dynamics is $S_m(s)=e^{s\hat{L}_{\beta',N}}$.

4.1.4 Deterministic behaviour

As a warm-up we ask ourselves how the typical paths for the infinite-temperature starting measure look like for large N, when the former measure is subjected to the unconstrained dynamics. To answer this question we look how a magnetization of the system behaves as a function of time. Mathematically, we consider a process $\{m_s, s\geq 0\}$ taking values in $[-1,1]$ and depending on time s started at the initial value m_0. For fixed N we write

$$\frac{d}{ds}\mathbb{E}_N^{m_0}m_s = \mathbb{E}_N^{m_0}\frac{d}{ds}m_s = \mathbb{E}_N^{m_0}\frac{d}{ds}(S_m(s)m_0) = \mathbb{E}_N^{m_0}\hat{L}_{\beta',N}m_s \tag{4.13}$$

Interchanging expectation and derivative causes no problem because N is finite. Choosing as a test function in (4.11) a simple identity, we get

$$\begin{aligned}\frac{d}{ds}\mathbb{E}_N^{m_0}m_s = \mathbb{E}_N^{m_0}\hat{L}_{\beta',N}m_s = \\ \mathbb{E}_N^{m_0}\left[(1-m_s)c\left(-,l(N)(m_s+\frac{1}{N})\right)-(1+m_s)c\left(-,l(N)(m_s-\frac{1}{N})\right)\right]\end{aligned} \tag{4.14}$$

Taking the limit $N\to\infty$ at both sides, the stochastic process for the magnetization m_s concentrates on a certain deterministic path $m(s)$, see e.g. [23,

Chapter 2]. The expectations on both sides may be omitted, because $m(s)$ is not random anymore. As a shortcut we write m every time the function $m(s)$ is meant. Its time derivative is denoted by a dot, i.e. $\dot{m}$. Choosing F to be an identity function, we get a deterministic differential equation.

$$\begin{aligned}\frac{d}{ds}m &= (1-m)c(-,m) - (1+m)c(+,m) \\ &= (1-m)R(m)e^{\beta' m} - (1+m)R(m)e^{-\beta' m} \\ &= 2R(m)\left(\sinh(\beta' m) - m\cosh(\beta' m)\right)\end{aligned} \tag{4.15}$$

The total rate is

$$c(m) := c_+(m) + c_-(m)$$

Employing the expressions for the asymptotic rates (4.12), the total rate is given by

$$\begin{aligned}c(m) &= R(m)\left(\cosh(\beta' m) - m\sinh(\beta' m)\right) \\ &= R(m)\cosh(\beta' m)\left(1 - m\tanh(\beta' m)\right)\end{aligned} \tag{4.16}$$

To simplify the problem, we choose the time rescaling to be

$$R(m) = \frac{1}{\cosh(\beta' m) - m\sinh(\beta' m)}, \tag{4.17}$$

making the total rate constant. Taking into account the chosen time rescaling (4.17), the deterministic time evolution results in the ODE

$$\dot{m} = 2\,\frac{\sinh(\beta' m) - m\cosh(\beta' m)}{\cosh(\beta' m) - m\sinh(\beta' m)} \tag{4.18}$$

In the case $\beta' = 0$ the equation reduces to the linear equation $\dot{m} = -2m\,ds$ which describes the relaxation of the magnetization to zero under the unconstrained infinite-temperature dynamics.

4.1.5 Large deviations for stochastic processes

Consider a finite-volume configuration σ_N. Let, as usual, s be a time variable. The induced Markov process $\{m_s, s \geq 0\}$ on a finite volume with N spins performs jumps upwards or downwards of size $\frac{2}{N}$. We remind that m_s concentrates on a deterministic path $m(s)$ when N is large. Let $p_{\beta'}(m) = \frac{c_-(m(s))}{c_-(m(s))+c_+(m(s))}$ be the temperature- and $m(s)$-*dependent* probability for a magnetization to go down for large N. Clearly, the asymptotic probability to go up is then given by $1 - p_{\beta'}(m)$. Whenever it is clear from the context, we, as before, write m instead of $m(s)$ and, additionally, $p_{\beta'}$ for $p_{\beta'}(m)$.

Let $z_{\sigma_{[1,N]}}(s) \in \left\{-1, -1+\frac{2}{N}, \ldots, 1-\frac{2}{N}, 1\right\}$ be the path of the magnetization for the Markov chain (with the generator as in (4.11)) evolved at inverse temperature β' with the initial condition to be distributed according to the Curie-Weiss measure $\mu_{\beta,N}$. Denote by $\mathbb{P}_{\beta,\beta',N}$ the law of the paths $\left\{z_{\sigma_{[1,N]}}(s)\right\}_{s\in[0;t]}$.

Theorem 4.1.4. *Suppose* $\left\{z_{\sigma_{[1,N]}}(0)\right\}$ *satisfies a large deviation principle (LDP) with rate function* I_0 *and rate* N*, then the measure* $\mathbb{P}_{\beta,\beta',N}$ *satisfies a large deviation principle in* $L^2[-1,1]$ *with rate* N *and rate function*

$$I(m) = I_0(m(0)) + \int_0^t \mathcal{L}_{\beta'}(m(s), \dot{m}(s))ds \tag{4.19}$$

with Lagrange density $\mathcal{L}_{\beta'}(m(s), \dot{m}(s))$ *given by*

$$\mathcal{L}_{\beta'}(m, \dot{m}) = \frac{\dot{m}}{2} \ln\left(\frac{\dot{m} + \sqrt{16p_{\beta'}(1-p_{\beta'}) + \dot{m}^2}}{4p_{\beta'}}\right) - \frac{1}{2}\sqrt{16p_{\beta'}(1-p_{\beta'}) + \dot{m}^2} + 1, \tag{4.20}$$

where $p_{\beta'} = p_{\beta'}(m)$.

Proof of Theorem. There are two ingredients needed to prove the statement of the theorem. The first ingredient we need is the static large deviation principle for the magnetization in the initial measure, the Curie-Weiss measure with inverse temperature β. Secondly, the formalism of Feng and Kurtz [17] allows us to obtain the form of Lagrangian density, then the correctness of the present statement is a mere application of [17, Theorem 13.7].

We start with the proposition about static LDP. It reads as follows.

Proposition 4.1.5. *The distribution of magnetization* $\underline{m}_1^N = \frac{1}{N}\sum_{i=1}^N \sigma_i$ *w.r.t. the Curie-Weiss measure at inverse temperature* β *obeys a large deviation principle with rate* N *and rate function* I_0*:*

$$I_0(\underline{m}) = \Phi(\underline{m}_1^N) + I(\underline{m}_1^N), \tag{4.21}$$

where

$$I(\underline{m}_1^N) = \frac{1+\underline{m}_1^N}{2}\log(1+\underline{m}_1^N) + \frac{1-\underline{m}_1^N}{2}\log(1-\underline{m}_1^N) \tag{4.22}$$

is the rate function for the symmetric Bernoulli distribution and $\Phi(\underline{m}_1^N) = -\frac{\beta(\underline{m}_1^N)^2}{2}$ *is a mean-field interaction.*

Proof of Proposition. Proposition 4.1.5 follows from Varadhan's Lemma. Suppose a probability distribution satisfies an LDP principle with a known rate function and rate N and suppose we consider the probability distribution with density $Ce^{-N\Phi(x)}$ relative to the first density. Under suitable conditions of the function $\Phi(x)$ (boundedness and continuity will suffice) this probability distribution will satisfy an LDP with the same rate N and rate function obtained by adding $\Phi(x)$ to the first rate function and subtracting a constant. Let k be the number of pluses in a configuration $\sigma_{[1,N]}$ with mean $\underline{m}_1^N$. The number of pluses may be expressed as $\frac{1+\underline{m}_1^N}{2}N$ Applying the apparatus of [55] to our framework, the heuristics described before, slightly more formally, reads:

$$-\frac{1}{N}\ln\alpha^{\otimes}(d\sigma_{[1,N]}) = -\frac{1}{N}\ln\binom{N}{\frac{1+\underline{m}_1^N}{2}N}2^{-N} \to I(\underline{m}_1^N),$$

where $\sigma_{[1,N]}$ is any configuration such that $\sum_{i=1}^N \sigma_i = \underline{m}_1^N$ and

$$-\frac{1}{N}\ln\int Ce^{-N\Phi(\underline{m}_1^N)}\alpha^{\otimes}(d\sigma_{[1,N]}) \to \\ \inf\left(-\Phi(\underline{m}_1^N) - I(\underline{m}_1^N)\right) = \sup\left(\Phi(\underline{m}_1^N) + I(\underline{m}_1^N)\right)$$

□

When a magnetization path is considered, it is clear that $\underline{m}_1^N$ is nothing but $z_{\sigma_{[1,N]}}(0)$ and for large N the value $\underline{m}_1^N$ concentrates on $m(0)$.

The Lagrangian density is computed employing the notion of *non-linear generator* introduced in [17]. The non-linear generator $\hat{\mathfrak{H}}_{\beta',N}$ acts on test functions $F : \left\{-1, -1+\frac{2}{N}, \ldots, 1-\frac{2}{N}, 1\right\} \to \mathbb{R}$ and is defined as

$$\left(\hat{\mathfrak{H}}_{\beta',N}F\right)(x) = \lim_{N\to\infty}\frac{1}{N}e^{-NF(x)}\left(\hat{L}_{\beta',N}e^{NF}\right)(x) \tag{4.23}$$

The integrand in (4.19) is given by the Legendre transform of $\left(\hat{\mathfrak{H}}_{\beta',N}F\right)(m)$ with F being an identity function.

As explained in Appendix B.2, computing the right-hand side for a finite N and taking limits with respect to the volume size yields

$$\mathcal{H}(m, F'(m)) := \left(\hat{\mathfrak{H}}_{\beta',N}F\right)(m) = c_-(m)(e^{2F'(m)} - 1) + c_+(m)(e^{-2F'(m)} - 1) \tag{4.24}$$

Due to the chosen rescaling $R(m)$ (4.17) involved into the definition of the rates we deduce that $c_- = p_{\beta'}(m)$ and $c_+ = 1 - p_{\beta'}(m)$. Thence our non-linear generator reads

$$\mathcal{H}(m, F'(m)) = p_{\beta'}(m)(e^{2F'(m)} - 1) + (1 - p_{\beta'}(m))(e^{-2F'(m)} - 1) \tag{4.25}$$

Performing a Legendre transform with $F(m) = m$

$$\mathcal{L}_{\beta'}(m, \dot{m}) = \sup_{\lambda} \left(\lambda \dot{m} - \mathcal{H}(m, \lambda)\right),$$

we arrive at the Lagrangian density which governs the path large deviations as stated in [17, Theorem 13.7 or, in particular situations, Examples 1.5 or 1.12]. This concludes our treatment of the proof.

□

4.1.6 Minimal cost problem

The joint model connects a layer of initial spins σ and a layer of the evolved (during time $s = t$) spins η. We would like to identify whether the two-layer model exhibits a phase transition for initial spins when the configuration of the second layer is fixed. Analogously, we say that the second layer has a certain magnetization $\underline{m}'$. Conditioning on the second layer induces a conditional distribution on the magnetization values $\underline{m}_0$ at time $s = 0$. This is reflected in the following corollary.

Corollary 4.1.6. *The conditional distribution of the initial magnetization $\underline{m}_0$ taken according to the law of the paths $\mathbb{P}_{\beta',\beta,N}$, conditioned to end in the final condition $\underline{m}'$ at time $s = t$, satisfies a large deviation principle with rate N and rate function given by*

$$E_{\underline{m}'}(\underline{m}_0, \beta, \beta') = \Phi(\underline{m}_0) + I(\underline{m}_0) + \inf_{\substack{m(s):(0)=\underline{m}_0, \\ \varphi(t)=\underline{m}'}} \int_0^t \mathcal{L}_{\beta'}(\varphi, \dot{\varphi}) ds - Const(\underline{m}') \tag{4.26}$$

This rate function allows to compute the large-N asymptotics of the probability to find the system in a final magnetization $\underline{m}'$ at time $s = t$ by computing the value of the rate function in the minimizing path to $\underline{m}'$.

The existence of a phase transition for the joint model in our setup corresponds to the fact that a large deviations functional (4.20) weighs several functions $\varphi_i(s), i \in \mathbb{N}$ constrained to take value $\underline{m}'$ at time t, with an equal minimally possible mass. Thence, we may talk of a cost of any function $\varphi(s)$ defined on $[0, t]$ and valued in $[-1, 1]$ such that $\varphi(t) = \underline{m}'$.

Definition 4.1.7. *Let a function $\varphi(s)$ be such that $\varphi(t) = \underline{m}'$. The functional $E_{\underline{m}'}(\underline{m}_0, \beta, \beta')$ defined in Corollary 4.1.6 is called a* cost *of evolving from an unknown configuration σ, such that its magnetization is $\varphi(0)$, to the fixed configuration η.*

Therefore, the existence of a phase transition of the joint-model is well captured by the fact whether the solution of the cost-minimizing problem is unique or not.

We denote by $m(s;t,\underline{m}')$ any path started at unknown state $\underline{m}_0$ at the time $s=0$ and ended up at $\underline{m}'$ at the time $s=t$. The magnetization $m(s;t,\underline{m}')$ viewed as a function of time with a constraint on the final point is called a *history*. If the cost functional reaches its minimum at $m(s;t,\underline{m}')$, then such a function is denoted by $m^*(s;t,\underline{m}')$ and called *most probable history* (or a *most probable curve*). The value of magnetization $m^*(0;t,\underline{m}')$ given by the value a most probable curve at zero time is called *most probable starting point*.

4.1.7 Gibbsianness for transformed measures and limiting conditional distributions

In this section we prove the form of the single-site conditional distributions for the transformed measure. The nature of the transformed measure — Gibbsian or non-Gibbsian— strongly depends on the behaviour of these distributions. To see the problem in full detail we first transfer the relevant definitions of Gibbs measure on the transformed finite-volume measure $\mu'_{\beta,\beta',t,N}$ defined in (4.7).

The definition of a good point reads the same as Definition 4.1.2 replacing $\gamma_{\beta,1}$ with $\gamma_{\beta,\beta',t,1}$.

Definition 4.1.8. *Let β,β',t be given. A point $\widehat{m}\in[-1,1]$ is said to be* good *for a mean-field model if and only if:*

1. *There exists a neighborhood of $\underline{m}_2^N$ such that, for all α in this neighborhood the following holds. For all sequences $\alpha_N\in\{-1,-1+\frac{2}{N-1},\dots,1-\frac{2}{N-1},1\}$ with the property $\lim_{N\uparrow\infty}\alpha_N=\alpha$ the limit*

$$\gamma_{\beta,\beta',t,1}(\eta_1|\alpha)=\lim_{N\uparrow\infty}\gamma_{\beta,\beta',t,1,N}(\eta_1|\alpha_N) \tag{4.27}$$

   exists and is independent of the choice of the sequence α_N.

2. *The function $\alpha\mapsto\gamma_{\beta,\beta',t,1}(\eta_1|\alpha)$ is continuous at $\alpha=\underline{m}_2^N$.*

The functions $\gamma_{\beta,\beta',t,1}(\eta_1|\cdot)$ are called *limiting conditional distributions* of a spin value η_1. It is conditional because the spins outside a single-site volume are frozen to have a particular magnetization and limiting since the volume we condition on is infinite.

Definition 4.1.9. *The time-evolved mean-field model $\{\mu'_{\beta,\beta',t,N}\}$ with parameters β,β',t is called* Gibbs *if and only if it has no bad points.*

We would like to establish the connection between the notion of a solution for the minimization problem (4.26) and the form of limiting conditional distribution. This requires more knowledge for the single-site evolution. We rewrite the form of the single-site linear generator (2.13) according to our needs for fixed β, β', t, and m', that is

$$(L_i(s;t,\underline{m}')f)(\sigma_i) = c(\sigma_i, m^*(s;t,\underline{m}'))(f(-\sigma_i) - f(\sigma_i)), \tag{4.28}$$

with rates which are obtained by substitution of the optimal path for the constrained problem for the empirical magnetization (4.26) into the single-site flip rates. For shortcut we write

$$\begin{aligned} &\text{rate of flipping from “ + ” } c_s(+) := c(+, m(s;t,\underline{m}')), \\ &\text{rate of flipping from “ − ” } c_s(-) := c(-, m(s;t,\underline{m}')), \end{aligned} \tag{4.29}$$

We first provide the following theorem.

Theorem 4.1.10. ***Single-site transition probability.*** *Fix β, β', t, m'. Consider a Markov jump process on $\{-1,1\}$ which is defined by the time-dependent generator (4.28) for these values. Then a probability $k_s(\sigma_i, \eta_i; t, \underline{m}')$ for a single spin to evolve from an initial value $\sigma_i \in \{-1,1\}$ at time $s = 0$ to $\eta_i \in \{-1,1\}$ at time $s \le t$ is given by the expression*

$$\begin{aligned} k_s(+,+;t,\underline{m}') = 1 - k_s(+,-;t,\underline{m}') = k_s(-,-;t,\underline{m}') = 1 - k_s(-,+;t,\underline{m}') \\ = e^{-\int_0^s (c_u(-)+c_u(+))du} \times \left[\int_0^s c_u(-) e^{-\int_0^u (c_v(-)+c_v(+))dv} du + 1 \right] \end{aligned} \tag{4.30}$$

Proof. We omit the proof of this fact here and refer to Appendix B.4. □

In the second place we define a configuration compatible with a minimizing solution $m^*(s;t,\underline{m}')$

Definition 4.1.11. *A configuration $\eta_{[1,N]}$ is called consistent with minimizer $m^*(s;t,\underline{m}')$ at time $s = s_0$, if*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \eta_i = m^*(s_0;t,\underline{m}')$$

We are now ready to give our formula for the limiting conditional distributions of the model started at the inverse temperature β and evolved with β' during the time t .

Theorem 4.1.12. *Fix β, β', t, m'. Suppose the constrained variational problem (4.26) for paths φ, taken over the paths with fixed right endpoint $\varphi(t) = \underline{m}'$, has a unique minimizing path $s \mapsto m^*(s; t, \underline{m}')$.*

Then the limiting probability kernels of the time-evolved measure $\mu_{\beta,\beta',t;N}$ have a well-defined infinite-volume limit $\gamma_{\beta,\beta',t}(\cdot|\underline{m}')$ in the sense of Definition 4.1.8 of the following form

$$\gamma_{\beta,\beta',t}(\eta_1|\underline{m}') = \frac{\sum_{\sigma_1=\pm 1} e^{\sigma_1 \beta\, m^*(0;t,\underline{m}')} k_t(\sigma_1, \eta_1; t, \underline{m}')}{\sum_{\sigma_1,\tilde{\eta}_1=\pm 1} e^{\sigma_1 \beta\, m^*(0;t,\underline{m}')} k_t(\sigma_1, \tilde{\eta}_1; t, \underline{m}')}, \tag{4.31}$$

Here $k_t(\sigma_1, \eta_1; t, \underline{m}')$ as in Theorem 4.1.10.

Proof. We start with initial finite-volume configuration $\sigma_{[1,N]}$. With a little abuse of notation we denote by $\sigma_i(s)$ the value of a spin at site i at time s. Spins η_i of the transformed configuration could be viewed as values of functions $\sigma_i(s)$ started at the value $\sigma_i(0) =: \sigma_i$ for corresponding sites.

Take a sequence $\alpha_N \in \{-1, -1 + \frac{2}{N-1}, \ldots, 1 - \frac{2}{N-1}, 1\}$ with the property $\lim_{N\uparrow\infty} \alpha_N = \alpha$. We denote by $\underline{m}_2^N(s) = \frac{1}{N-1}\sum_{i=2}^N \sigma_1(s)$ the empirical magnetization of the spins of site 2 to N at time s of the evolution. To prove that the promised form for the limiting conditional probabilities is correct we must show that

$$\lim_{N\uparrow\infty} \frac{\mu'_{\beta,\beta',N}(\sigma_1(t) = +|\underline{m}_2^N(t) = \alpha_N)}{\mu'_{\beta,\beta',N}(\sigma_1(t) = -|\underline{m}_2^N(t) = \alpha_N)} = \frac{\gamma_{\beta,\beta',t,1}(\eta_1 = +|\alpha)}{\gamma_{\beta,\beta',t,1}(\eta_1 = -|\alpha)}, \tag{4.32}$$

where the terms of the right-hand side are given by (4.31).

We drop the subscripts of temperatures and time for the conditional probabilities and the transformed measure during the proof of the theorem. The subscript explicitly identifying finiteness of both objects is kept.

$$\begin{aligned}
&\frac{\gamma_{1,N}(\eta_1 = +|\alpha_N)}{\gamma_{1,N}(\eta_1 = -|\alpha_N)} = \frac{\mu'_N(\eta_1 = +|\underline{m}_2^N(t) = \alpha_N)}{\mu'_N(\eta_1 = -|\underline{m}_2^N(t) = \alpha_N)} = \\
&\frac{\int \mathbb{P}_N(dz_{\sigma_{[1,N]}}|\underline{m}_2^N(t) = \alpha_N)\mu'_N(\eta_1 = +|z_{\sigma_{[1,N]}})}{\int \mathbb{P}_N(dz_{\sigma_{[1,N]}}|\underline{m}_2^N(t) = \alpha_N)\mu'_N(\eta_1 = -|z_{\sigma_{[1,N]}})} = \\
&\frac{\int \mathbb{P}_N(dz_{\sigma_{[1,N]}}|\underline{m}_2^N(t) = \alpha_N) \sum_{\tilde{\sigma}=\pm 1} \tilde{\mu}_N(\eta_1 = +|\sigma_1 = \tilde{\sigma}_1, z_{\sigma_{[1,N]}})\mu_N(\sigma_1 = \tilde{\sigma}_1|z_{\sigma_{[1,N]}})}{\int \mathbb{P}_N(dz_{\sigma_{[1,N]}}|\underline{m}_2^N(t) = \alpha_N) \sum_{\tilde{\sigma}=\pm 1} \tilde{\mu}_N(\eta_1 = +|\sigma_1 = \tilde{\sigma}_1, z_{\sigma_{[1,N]}})\mu_N(\sigma_1 = \tilde{\sigma}_1|z_{\sigma_{[1,N]}})}
\end{aligned} \tag{4.33}$$

Let's compute $\mu_N(\sigma_1 = \tilde{\sigma}_1 | z_{\sigma_{[1,N]}})$.

$$\mu_N(\sigma_1 = + | z_{\sigma_{[1,N]}}) = \sum_{\sigma_{[2,N]}} \frac{e^{\frac{\beta}{2N}(\sum_{i=2}^n \sigma_i + 1)^2}}{e^{\frac{\beta}{2N}(\sum_{i=2}^n \sigma_i + 1)^2} + e^{\frac{\beta}{2N}(\sum_{i=2}^n \sigma_i - 1)^2}} \Bigg|_{z_{\sigma_{[1,N]}}} =$$

$$\sum_{\sigma_{[2,N]}} \frac{1}{1 + \exp\left\{\frac{\beta}{2N}\left((\sum_{i=2}^n \sigma_i - 1)^2 - (\sum_{i=2}^n \sigma_i + 1)^2\right)\right\}} \Bigg|_{z_{\sigma_{[1,N]}}} =$$

$$\sum_{\sigma_{[2,N]}} \frac{1}{1 + \exp\left\{-2\beta \frac{1}{N} \sum_{i=2}^n \sigma_i\right\}} \Bigg|_{z_{\sigma_{[1,N]}}} = \sum_{\sigma_{[2,N]}} \frac{\exp\left\{\beta\left(z_{\sigma_{[1,N]}}(0) - \frac{1}{N}\right)\right\}}{Z} \tag{4.34}$$

Generalizing the last statement to any value of σ_1, we write:

$$\mu_N(\sigma_1 = \tilde{\sigma}_1 | z_{\sigma_{[1,N]}}) = \sum_{\sigma_{[2,N]}} \frac{\exp\left\{\tilde{\sigma}_1 \beta\left(z_{\sigma_{[1,N]}}(0) - \frac{\tilde{\sigma}_1}{N}\right)\right\}}{Z} \tag{4.35}$$

While taking an infinite-volume limit, we make use of the assumption on the uniqueness of the solution of the constrained path large deviation principle made in the statement of the theorem. Under our assumption the distribution $\mathbb{P}_{\beta,\beta',N}(z_{\sigma_N}(s) | \underline{m}_2^N(t) = \alpha_N)$ concentrates exponentially fast on the unique deterministic trajectory $m^* : s \mapsto m^*(s; t, \underline{m}')$ as N tends to infinity. This collapses the outer expected value and simplifies the formula a lot. Next we have that whenever $z_{\sigma_{[1,N]}} \to m^*(s; t, \underline{m}')$, we are guaranteed that at each moment of time $s \in [0, t]$ there exists a unique (up to permutations) configuration $\eta_{[1,N]}$, that is consistent with $m^*(s; m', t)$, meaning that summation over $\sigma_{[2,N]}$ in (4.35) counts just the number of permutations of the configurations consistent with $m^*(0; t, \underline{m}')$ at time $s = 0$. Finally, we have that the single-site Markov chain describing the time-evolution of the spin at site 1, *conditional* on the path of the empirical mean of the other $N - 1$ spins and its initial value at time $s = 0$, converges to the Markov chain with deterministic but time-dependent generator (4.28). The corresponding transition probabilities converge to the limiting expression from the theorem and we have

$$\lim_{N \uparrow \infty} \tilde{\mu}_N(\eta_1 | \sigma_1 = \tilde{\sigma}_1, z_{\sigma_{[1,N]}}) = k_t(\tilde{\sigma}_1, \eta_1; t, \underline{m}') \tag{4.36}$$

Therefore combining these three ingredients ensued from the assumption we

made we get

$$\frac{\int \mathbb{P}_N(dz_{\sigma_{[1,N]}}|\underline{m}_2^N(t)=\alpha_N)\sum_{\tilde{\sigma}=\pm1}\tilde{\mu}_N(\eta_1=+|\sigma_1=\tilde{\sigma}_1,z_{\sigma_{[1,N]}})\mu_N(\sigma_1=\tilde{\sigma}_1|z_{\sigma_{[1,N]}})}{\int \mathbb{P}_N(dz_{\sigma_{[1,N]}}|\underline{m}_2^N(t)=\alpha_N)\sum_{\tilde{\sigma}=\pm1}\tilde{\mu}_N(\eta_1=-|\sigma_1=\tilde{\sigma}_1,z_{\sigma_{[1,N]}})\mu_N(\sigma_1=\tilde{\sigma}_1|z_{\sigma_{[1,N]}})} \tag{4.37}$$

$$\xrightarrow{N\to\infty} \frac{\sharp\left\{\sigma_{[2,N]}:\lim_{N\to\infty}\frac{1}{N}\sum_{i=2}^N\sigma_i=m^*(0;m',t)\right\}\sum_{\tilde{\sigma}=\pm1}k_t(\tilde{\sigma}_1,+;t,\underline{m}')e^{\tilde{\sigma}_1\beta m^*(0;t,\underline{m}')}}{\sharp\left\{\sigma_{[2,N]}:\lim_{N\to\infty}\frac{1}{N}\sum_{i=2}^N\sigma_i=m^*(0;m',t)\right\}\sum_{\tilde{\sigma}=\pm1}k_t(\tilde{\sigma}_1,-;t,\underline{m}')e^{\tilde{\sigma}_1\beta m^*(0;t,\underline{m}')}} \tag{4.38}$$

Canceling counting constants and adding normalizing constants to numerator and denominator completes the proof.

□

4.2 Main result

Hitherto, we gave the notion of Gibbsianness for the transformed model and established the link between the single-site conditional probabilities and a solution of the variational problem (4.26). The latter singles out the set of relevant parameters determining the regime of the system. These parameters are an initial temperature β^{-1}, a temperature of the dynamics β'^{-1}, a value $\widehat{m}$ tested to be a continuity point, and a time of evolution t. The importance of their combinations is reflected in our main result of this chapter.

Theorem 4.2.1. *Consider the time-evolved Curie-Weiss model with initial and dynamical temperatures β^{-1}, β'^{-1}. Then the following holds.*

1. *Initial high temperature, any temperature of the dynamics.*
   If $\beta^{-1} \geq 1$ then the time-evolved model is Gibbs for all $t \geq 0$.

2. *Heating from an initial low-temperature, with a either high-temperature or a low-temperature dynamics.*
   For any β' there exists a value $\beta_{SB}^{-1}(\beta') < \beta'^{-1}$ (which is explicitly computable, see below) such that the following is true. Assume that $0 < \beta^{-1} < \min\{\beta'^{-1}, 1\}$.

   a) If $\beta_{SB}^{-1}(\beta') \leq \beta^{-1}$ then

   - *for all $0 \leq t \leq t_{nGS}(\beta,\beta') := \frac{\ln\frac{\beta'-\beta}{1-\beta}}{4(1-\beta')}$ the time-evolved model is Gibbs.*

- *for all* $t > t_{nGS}(\beta, \beta')$ *the model is not Gibbs and the time-evolved conditional probabilities are discontinuous at* $\widehat{m} = 0$ *and continuous at any* $\widehat{m} \neq 0$.

b) If $0 < \beta^{-1} < \beta_{SB}^{-1}(\beta')$ *there exist sharp values* $0 < t_0(\beta, \beta') < t_1(\beta, \beta') < \infty$ *such that*

- *for all* $0 \leq t \leq t_0(\beta, \beta')$ *the time-evolved model is Gibbs,*
- *for all* $t_0(\beta, \beta') < t < t_1(\beta, \beta')$ *there exists* $\widehat{m}_c = \widehat{m}_c(\beta, \beta'; t) \in (0, 1)$ *such that the limiting conditional probabilities are discontinuous at the points* $\pm\widehat{m}_c$, *and continuous otherwise,*
- *for all* $t > t_1(\beta, \beta')$ *the limiting conditional probabilities are discontinuous at* $\widehat{m} = 0$ *and continuous at any* $\widehat{m} \neq 0$.

3. *Cooling from initial low temperature.*
For $\beta'^{-1} < \beta^{-1} < 1$ *there exists a time-threshold* $t_{per}(\beta, \beta')$ *such that,*

- *for all* $0 \leq t \leq t_{per}(\beta, \beta')$ *the time-evolved model is Gibbs.*
- *for all* $t > t_{per}(\beta, \beta')$ *the model is not Gibbs and the time-evolved conditional probabilities are discontinuous at non-zero configurations* $\widehat{m}_c$ *(and continuous at* $\widehat{m} = 0$*).*

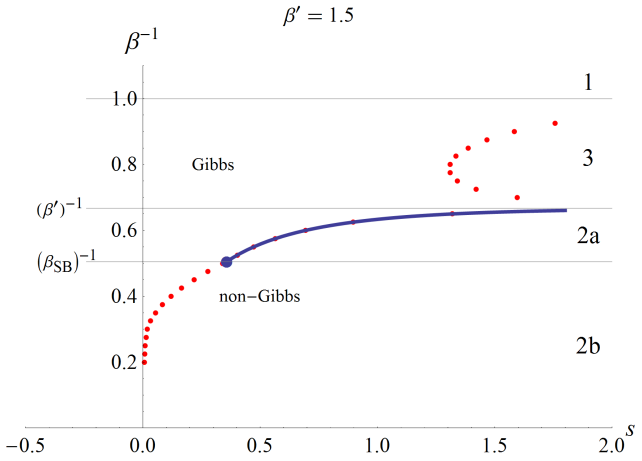


Figure 4.1: Division between Gibbs and non-Gibbs area for low-temperature dynamics, the thick curve is obtained by computation, the dots are given by numerics

Note that for high-temperature dynamics $\beta'^{-1} > 1$ the Region 3 of initial temperatures in Figure 4.1 is empty. Part 2 of the theorem generalizes the structure which we already know from the independent spin-flip dynamics $\beta' = 0$ (see [36]) which is contained as a special case. This means that a

symmetric (w.r.t. starting measure) bad point $m_0 = 0$ will appear after a sharp transition time if the initial temperature is not too low (see Subregion 2a). For lower temperatures (in Subregion 2b) symmetry-breaking in the set of bad configurations for the time-evolved measure appears in an intermediate time-interval: at the beginning of this interval a symmetric pair of bad configuration appears which merges at the end of the time interval.

It is remarkable that the picture we observe in Region 2 is similar to the independent spin-flip case. This is even true for low temperatures $\beta'^{-1} < 1$ of the dynamics. As we will see, we can moreover compute the symmetry-breaking inverse temperature β_{SB} in terms of β' as the largest solution of the following cubic equation

$$4\beta_{\mathrm{SB}}^3 + 12\beta_{\mathrm{SB}}\beta' - 6\beta_{\mathrm{SB}}^2(1+\beta') - \beta'(3+3\beta'-\beta'^2) = 0 \tag{4.39}$$

In the independent spin-flip case $\beta' = 0$ we get exactly $\beta^{-1} = \frac{2}{3}$, which was already found in [36]. We will also give an explicit expression of the critical time in Subregion 2a, for all β'.

In Region 3 of cooling from an already low initial temperature we observe an entirely new mechanism for the production of non-Gibbsian points. These are related to periodic orbits of the flow of the β'-dependent vector field which is created by the Euler-Lagrange equations for the variational problem (4.26).

4.3 Deterministic analysis

The key feature that allows to treat our problem of initially stochastic nature from the deterministic point of view is the concentration property of possible evolution ways on an optimal path for large N. The optimality of a path means that it delivers a minimum to the functional (4.26). The connection established by the Theorem 4.1.12 shows that the knowledge of the optimal path is sufficient to judge whether the transformed model retains its Gibbs nature or not. This distinction depends on whether the prescribed end-state there was only one possibility for the system to start from or several. Many options in this case we consider as rather a defect, because they imply the lack of quasilocality for the transformed system. While looking for the most probable starting state of the system which is conditioned to be in a certain state, we have to look for the *whole* evolution path. This approach does not only answer the question of the most probable starting state, but develops a strong insight into hidden mechanisms of the evolution of the system.

4.3.1 Variational problem

Fix β, β', t, m'. We look at the constrained variational problem (4.26) taken over the paths φ with $\varphi(t) = \underline{m}'$ with the aim to find (the) minimizing path(s) $s \mapsto m^*(s; m', t)$. With a little abuse of notation we shall further write this problem in the form:

$$E_{m'}(m_0, \beta, \beta') = \Phi(m_0) + I(m_0) + \inf_{\substack{\varphi:\varphi(0)=m_0 \\ \varphi(t)=m'}} \int_0^t \mathcal{L}_{\beta'}(\varphi, \dot{\varphi}) ds - \mathrm{Const}(m'),$$

where the function $\varphi(s)$ may viewed as the infinite-volume magnetization functionally dependent on time, when the system is observed. As a remainder, we say that Φ is the original mean-field interaction and I is a large deviations function of the initial measure.

To bring the functional into a more usual form for calculus of variations, we drop the irrelevant constant, embed the constants depending on the starting value into infimum and further under the integral sign. Let us abbreviate $U(\varphi(s)) = \Phi(\varphi(s)) + I(\varphi(s))$, whenever it is clear from the context we shall write $U(\varphi)$ and for $U(\varphi)$ computed at time $s = r$ we shall write $U(\varphi(r))$, then the updated form of the functional reads:

$$E_{m'}(m_0, \beta, \beta') = \inf_{\substack{m:m(0)=m_0 \\ m(t)=m'}} \int_0^t \left\{ \mathcal{L}_{\beta'}(\varphi, \dot{\varphi}) - \frac{dU(\varphi)}{d\varphi}\dot{\varphi} + \frac{U(\varphi(t))}{t} \right\} ds \quad (4.40)$$

We may formulate the deterministic problem in the following way: *find an extremal $\varphi(s)$ fixed at time $s = t$ to take value m' and free at time $s = 0$ delivering minimum to the cost functional* (4.40).

We shall equivalently refer to a time $s = 0$ as the *left end* and to a time $s = t$ as the *right end*. The end is *fixed* if an extremal $\varphi(s)$ has to attain a prescribed value on it and *open* in the opposite case. Thus, the problem of finding extrema of (4.40) is the problem of calculus of variations with an open left end.

It is known in the calculus of variations [25, see Chapter 3, Section 14] that a necessary condition for an extremum is given by the corresponding Euler-Lagrange equation and an additional transversality condition for the free left end of the form

$$\begin{array}{lcll} \frac{d}{ds}\mathcal{L}_{\dot{\varphi}}(\phi(s), \dot{\varphi}(s)) - \mathcal{L}_{\varphi}(\varphi(s), \dot{\varphi}(s)) & = & 0 \text{ for all } s \in [0, t] & \\ \mathcal{L}_{\dot{\varphi}}(\varphi(s), \dot{\varphi}(s)) - \Phi_{\varphi}(\varphi(s)) - I_{\varphi}(\varphi(s))|_{s=0} & = & 0 & (4.41) \\ \varphi(t) & = & m' & \end{array}$$

Here we have dropped the subscript β' for the function $L(\phi, \dot{\phi})$ and written subscripts to denote partial derivatives.

To stress that these equations describe the behaviour of histories $m(s)$ which were defined previously, we shall explicitly write $m(s)$ in the place of $\varphi(s)$.

For a general probability distribution $p_{\beta'}(m)$ the first equation of (4.41) reads:

$$\ddot{m} = 8(2p_{\beta'}(m) - 1)\frac{dp_{\beta'}(m)}{dm} \tag{4.42}$$

The above equation does not contain explicitly the time variable s, thus it possesses a first integral. This integral is the preserved total energy of the system whose existence is a consequence of non-dissipativity of the system.

$$\dot{m}^2 + 16p_{\beta'}(m)(1 - p_{\beta'}(m)) = \mathcal{E} \tag{4.43}$$

This equation is integrable in quadratures.

$$\begin{aligned} t &= m_0 \pm \int_0^m (s)\frac{d\zeta}{\sqrt{\mathcal{E} - 16p_{\beta'}(\zeta)(1 - p_{\beta'}(\zeta))}} \\ \mathcal{E} &= \dot{m}_0 + 16p_{\beta'}(m_0)(1 - p_{\beta'}(m_0)), \end{aligned} \tag{4.44}$$

where m_0 and $\dot{m}_0$ are the unknown initial conditions.

In the situation of the present chapter, we are led to choose the distribution $p_{\beta'}(m)$ in the following to match the rates in the generator $\hat{L}_{\beta',N}$:

$$p_{\beta'}(m) = \frac{c_-(m)}{c_-(m) + c_+(m)} = \frac{e^{2\beta' m}(1 - m)}{e^{2\beta' m}(1 - m) + (1 + m)}, \tag{4.45}$$

where $m = m(s)$.

Substituting the form of $p_{\beta'}(m)$ and $\mathcal{L}(m, \dot{m})$ in the equations (4.41) after computations we get:

$$\begin{aligned} \ddot{m} &= 16e^{2\beta' m}\frac{\Big((1+m) - e^{2\beta' m}(1-m)\Big)\Big(1 + \big(m^2 - 1\big)\beta'\Big)}{\Big((1+m) + e^{2\beta' m}(1-m)\Big)^3} \\ \dot{m}\Big|_{s=0} &= g(m)\Big|_{s=0} \\ m(t) &= m' \end{aligned} \tag{4.46}$$

with the function

$$g(m) = 2\frac{(1 + m)e^{-2\beta m} - (1 - m)e^{2m(\beta - \beta')}}{(1 + m)e^{-2\beta' m} + (1 - m)} \tag{4.47}$$

We call the function $g(m)$ which gives the relationship between initial point and initial slope of the solution curve the *curve of the "allowed" initial configurations (ACC)*. We note that it is independent from the final value m'.

4.3.2 Typical paths for non-interacting time-evolution

Let us start with a discussion of the independent time-evolution — when $\beta' = 0$.
(i) For $\beta' = 0, \beta = 0$, the system becomes

$$\begin{aligned} \ddot{m}(s) &= 4m(s) \\ \dot{m}(s)\Big|_{s=0} &= 2m(s)\Big|_{s=0} \\ m(t) &= m' \end{aligned} \tag{4.48}$$

and the solution becomes $m(s) = m'e^{2(s-t)}$. This describes how a curve which is conditioned to end in m' away from zero is built up from the initial condition $m'e^{-2t}$ close to zero.
(ii) For independent dynamics $\beta' = 0$ and initial inverse temperature $\beta \neq 0$ the simplified system is

$$\begin{aligned} \ddot{m}(s) &= 4m(s) \\ \dot{m}(s)\Big|_{s=0} &= e^{-2\beta m(s)}(1+m(s)) - e^{2\beta m(s)}(1-m(s))\Big|_{s=0} \\ m(t) &= m' \end{aligned} \tag{4.49}$$

In this case the general solution is a linear combination of the $e^{\pm 2s}$. Looking at the right-end condition one gets

$$m(s) = (m' - C_2 e^{2t})e^{2(t-s)} + C_2 e^{2s}, \tag{4.50}$$

where C_2 is a constant and must be determined by the left-end condition. This can be done numerically.

It is possible to match the current approach with the one of [36] by plugging the solution curves with an initial condition $m(0) = m_0$ which are given by

$$m(s) = \frac{m_0 e^{2t} - m'}{e^{2t} - e^{-2t}} e^{-2s} + \frac{m' - m_0 e^{-2t}}{e^{2t} - e^{-2t}} e^{2s}, s \in [0, t] \tag{4.51}$$

into the rate function and carrying out the time integral explicitly. This gives

$$\begin{aligned} &E_{m'}(m_0, \beta, 0) = H(m_0) + I(m_0) \\ &+ \frac{1}{4}\Bigg(4t + \ln\left[\frac{1-m'^2}{1-m_0^2}\right] + 2m' \ln\left[\frac{R - C_1 e^{-2t} + C_2 e^{2t}}{1-m'}\right] \\ &- 2m_0 \ln\left[\frac{R - C_1 + C_2}{1-m_0}\right] + \ln\left[\frac{1-R-2C_1 m' e^{-2t}}{1+R-2C_1 m' e^{-2t}} \cdot \frac{1+R-2C_1 m_0}{1-R-2C_1 m_0}\right]\Bigg), \\ &\text{where } R = \sqrt{1-4C_1 C_2},\ C_1 = \frac{m_0 e^{2t} - m'}{e^{2t} - e^{-2t}},\ C_2 = \frac{m' - m_0 e^{-2t}}{e^{2t} - e^{-2t}} \end{aligned} \tag{4.52}$$

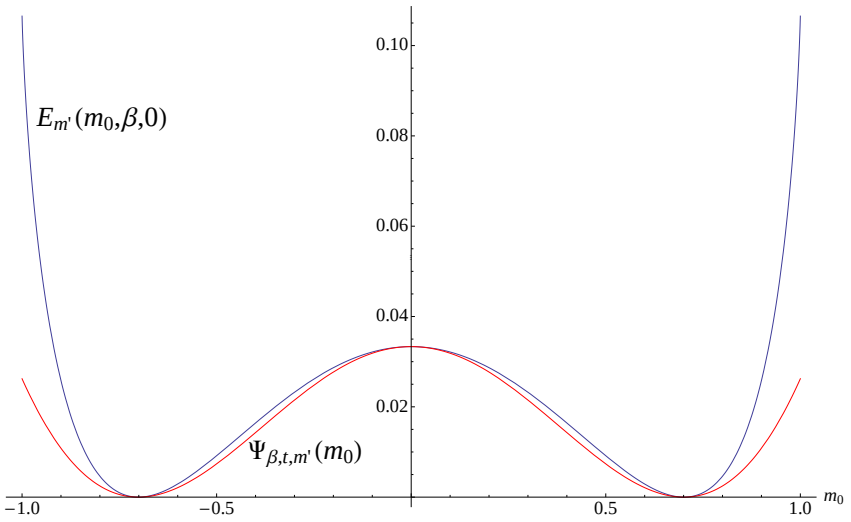


Figure 4.2: Cost functional $E_{m'}(m_0, \beta, 0)$ and known function $\Psi_{\beta,t,m'}(m_0)$ for $\beta' = 0$, and $\beta^{-1} \approx 1.744, t \approx 0.251$

In the approach of [36] a related function called $\Psi_{\beta,t,m'}(m_0)$ was obtained by Hubbard-Stratonovitch transformation, whose minimizers with a given conditioning (t, m') correspond to the most probable initial conditions. This provides an opportunity to check if the results of the present analysis done via path large deviations coincide with the approach employing the function $\Psi_{\beta,t,m'}(m_0)$.

It is known that the functions $\Psi_{\beta,t,m'}(m_0)$ (4.3.2) and $E_{m'}(m_0, \beta, 0)$ have the same set of extrema (see [48] in a more general context). In Figure 4.2 is the plot of these functions (after normalization to have zero as a minimum) for the same set of parameters $(\beta, \beta' = 0, m', t)$ which shows that the minima appear in fact at the same value.

The form (4.51) of the curves delivering minimum to the cost functional induces fast relaxation and fast concentration properties for the magnetization of the system being transformed[1]. In other words, the evolution time could be split into three stages: the magnetization 1) relaxes in a short time to a value close to zero, 2) stays close to zero for a long time, and 3) at time $s = t$ quickly — just before time $s = t$ — approaches the prescribed value m'. A simple proof of this fact is given in Appendix B.3.

Further we turn to the case of interacting dynamics $\beta' \neq 0$. In this case trajectories can only be obtained numerically. Before we go on, let us discuss

[1]Proposed by R.Fernandez at Nature-Nurture workshop, 12-13.01.2009, University of Groningen

in more detail the geometrical properties of the vector field and the allowed-configurations curve.

4.3.3 Geometric interpretation of Euler-Lagrange vector-field and curve of allowed initial configurations.

Since the Euler-Lagrange density $\mathcal{L}(m(s), \dot{m}(s))$ (4.20) does not contain an explicit dependence on the time s, the generalized energy given by the Legendre transformation of (4.20) is the system first integral of motion

$$\mathcal{L}(m(s), \dot{m}(s)) - \dot{m}(s)\mathcal{L}_{\dot{m}}(m(s), \dot{m}(s)) = \mathcal{E} \tag{4.53}$$

This can be rewritten as

$$\frac{e^{4\beta' m}(1-m)^2\dot{m}^2 + (1+m)^2\dot{m}^2 + 2e^{2\beta' m}\left(1-m^2\right)\left(8+\dot{m}^2\right)}{\left(1+e^{2\beta' m}(1-m)+m\right)^2} = \mathcal{E} \tag{4.54}$$

and explicitly solved for the velocity

$$\dot{m} = \pm\sqrt{\mathcal{E} + \frac{16e^{2\beta' m}(m^2-1)}{\left(1-e^{2\beta' m}(m-1)+m\right)^2}} \tag{4.55}$$

Looking at the integral curves in phase space we get some geometric intuition.

First let us understand what it means to have several equiprobable initial states (for the system) which could be led by the evolution to the same final state in therms of our differential equations. Phase diagrams for different values of β' help us.

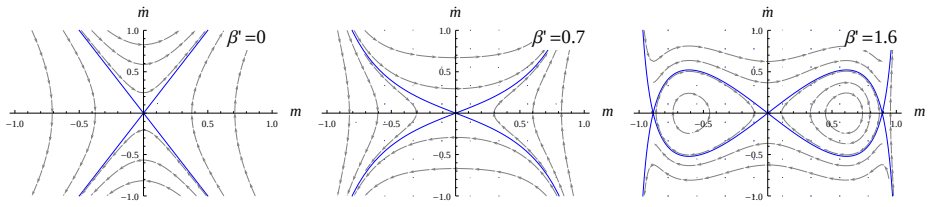


Figure 4.3: Phase portrait for several values of β'

Two initial states $m_{0,1}$, $m_{0,2}$ (*necessarily* lying on the ACC) are equiprobable for the final prescribed state m' if the corresponding points $(m_{0,1}, g(m_{0,1}))$, $(m_{0,2}, g(m_{0,2}))$ in the phase space are transferred by the phase flows at time $s = t$ to points having equal projections on the m-axis (see Figure 4.3). This corresponds to the fact that the solution-functions started at different magnetizations collapse in m' after time t with different speeds (slopes).

We would like to identify areas on the phase portrait where a possibility to start with two different points and after some time to end up with the transferred points having equal projections on m-axis is excluded. The only requirement we have for the initial points is that they have to lay on the graph of a *function*. Nonetheless, this requirement suffices to find "safe" areas (see Figure 4.4). Plot a graph of a function f crossing only "safe" regions, take any

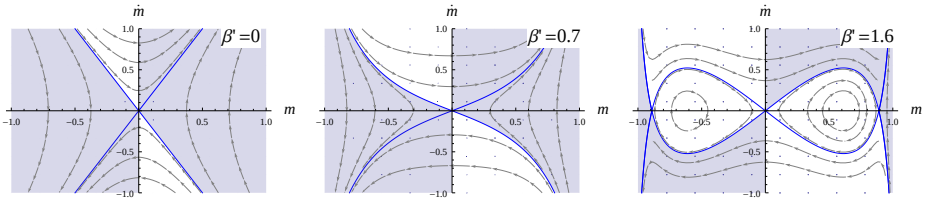


Figure 4.4: "Safe" (filled) vs. not "safe" regions

two magnetization values (w.l.g) $m_{0,1} < m_{0,2}$, then the corresponding points lying on the graph of f will be $(m_{0,1}, f(m_{0,1}))$, $(m_{0,2}, f(m_{0,2}))$. Safety of the filled regions is a combination of three facts: 1) a driving force is alway bigger for a point with a greater m-coordinate in absolute value, 2) the driving force is a smooth function of m, and 3) for (w.l.g) $m > 0$ the phase flow keeps the same direction of a drift for both of the starting points. Thus, the phase flow leaves no possibility for the point with the projection $m_{0,1}$ to m-axis to speed up and catch up with the another one and for the point with the projection $m_{0,2}$ to slow down and let the first one to reach it. On the other hand, the empty areas in Figure 4.4 suggest the very possibility excluded in the "safe" areas. Areas with a periodic motion are not "safe" because of the nature of the motion itself.

Let us go back to the notion of the ACC (4.47) on which all possible "allowed" starting conditions lie. The curve of allowed initial configurations for different combinations of initial and dynamical temperatures crosses both "safe" and not "safe" regions. In figure Figure 4.5 there are several ACC's drawn which correspond to different values of β, but the same value of the dynamical inverse temperature $\beta' = \frac{3}{2}$, which is relatively low. The production of discontinuities of the limiting conditional probabilities will be related to the time-evolution of the curve of allowed initial configurations under the Euler-Lagrange vector field, as we will describe now.

Let us first give a definition of a bad quadruple of initial temperature, dynamical temperature, time, and final magnetization in terms of dynamical-systems quantities. We start by defining candidate quadruples making use of the Euler-Lagrange flow in the following way.

Definition 4.3.1. *The quadruple* $(\beta, \beta', t, m_{pb})$ *is called* pre-bad *iff there ex-*

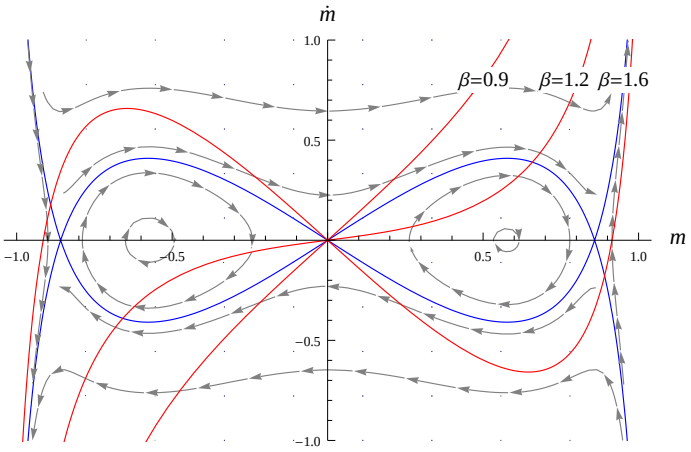


Figure 4.5: Phase portrait with level curves and ACC, $\beta' = \frac{3}{2}$

ists a pair $m_{0,1} \neq m_{0,2}$ *of initial magnetizations s.t. the solution of the initial value problem of the Euler-Lagrange equations started in the corresponding points* $(m_{0,1}, g(m_{0,1}))$ *and* $(m_{0,2}, g(m_{0,2}))$ *on the allowed-configurations curve for* β, β' *has the same magnetization value* m_{pb} *at time* t*, that is*

$$m(t; m_{0,1}, g(m_{0,1})) = m(t; m_{0,2}, g(m_{0,2})) = m_{pb}$$

While this first definition refers only to the existence of overhangs of the time-evolved allowed-configurations curve, the next definition involves also the value of the cost (4.40), which makes it much more restrictive.

Definition 4.3.2. *The pre-bad quadruple* $(\beta, \beta', t, m_{bad})$ *is called* bad *if and only if the two different paths started at the corresponding* $m_{0,1} \neq m_{0,2}$ *are both minimizers for the cost, i.e.*

$$E_{m_{bad}}(m_{0,1}, \beta, \beta') = E_{m_{bad}}(m_{0,2}, \beta, \beta') = \inf_m E_{m_{bad}}(m, \beta, \beta') \tag{4.56}$$

We will exploit both definitions both to gain geometric insight as well as numerical results. The important connection to non-Gibbsian behaviour of the time-evolved measure lies in the fact that m_{bad} of a bad quadruple will (generically) be a bad configuration for $\gamma_{\beta,\beta',t}(\cdot|m)$. Indeed, to see this, let us go back to the explicit expression of the limiting conditional probabilities, given by

$$\gamma_{\beta,\beta',t}(\eta_1|m') = \frac{\sum_{\sigma_1=\pm 1} e^{\sigma_1 \beta m^*(0;m',t)} k_t(\sigma_1, \eta_1; m', t)}{\sum_{\sigma_1,\tilde{\eta}_1=\pm 1} e^{\sigma_1 \beta m^*(0;m',t)} k_t(\sigma_1, \tilde{\eta}_1; m', t)} \tag{4.57}$$

Note that the function $m^*(0; m', t)$ is not well defined for $m' = m_{\text{bad}}$ itself since at time t there are two minimizing paths available, one from $m_{0,1}$ to

$m_{\rm bad}$ and one from $m_{0,2}$ to $m_{\rm bad}$. Varying however *around* $m_{\rm bad}$ the paths will become unique and we might select the minimizing paths (and hence their initial points) by approaching the bad configuration from the right or left, obtaining (say) $\lim_{m'\downarrow m_{\rm bad}} m^*(0;m',t) = m_{0,1}$ and $\lim_{m'\uparrow m_{\rm bad}} m^*(0;m',t) = m_{0,2}$. Note that we also expect that (generically) $\lim_{m'\downarrow m_{\rm bad}} k_t(\sigma_1,\tilde\eta_1;m',t) \neq \lim_{m'\uparrow m_{\rm bad}} k_t(\sigma_1,\tilde\eta_1;m',t)$. This follows since the k_t are probabilities for two different single-particle Markov chains, one depending on the path starting from $(m_{0,1},g(m_{0,1}))$, the other one on the path starting from $(m_{0,2},g(m_{0,2}))$. We note that, knowing the paths entering the k_t's, an explicit formula for k_t in terms of time-integrals can be written, and so, given (numerical) knowledge of the minimizing path, the $\gamma_{\beta,\beta',t}(\eta_1|m')$ can be obtained by simple integrations. Unless these two discontinuities compensate each other (which is generically not happening and which can be quickly checked by numerics) we will have that $\lim_{m'\downarrow m_{\rm bad}} \gamma_{\beta,\beta',t}(\eta_1|m') \neq \lim_{m'\uparrow m_{\rm bad}} \gamma_{\beta,\beta',t}(\eta_1|m')$. Consequently the model will be non-Gibbs at the time t.

Conversely, if $(\beta,\beta',t,m_{\rm pb})$ is not *bad*, then $m' \mapsto \gamma_{\beta,\beta',t}(\eta_1|m')$ is a continuity point. This follows since in that case all m'-dependent terms in (4.31) deform in a continuous way. So the absence of bad points (and a fortiori the absence of pre-bad points) implies Gibbsianness at (β,β',t).

4.3.4 Time-evolved allowed initial configurations

We just saw that non-Gibbsianness is produced by multiple histories which means in other words the production of overhangs in the time-evolved curve of allowed initial configurations. To get an intuition for this let us discuss the Regions 2) and 3) of the Theorem 4.2.1 in more detail. Let us begin with the phase-space picture for the non-interacting dynamics $\beta' = 0$. We are starting with the Region 2a) of non-symmetry-breaking non-Gibbsianness i.e. $\frac{2}{3} = \beta_{\rm SB}^{-1}(\beta'=0) \leq \beta^{-1} < \min\{\beta'^{-1},1\} = 1$.

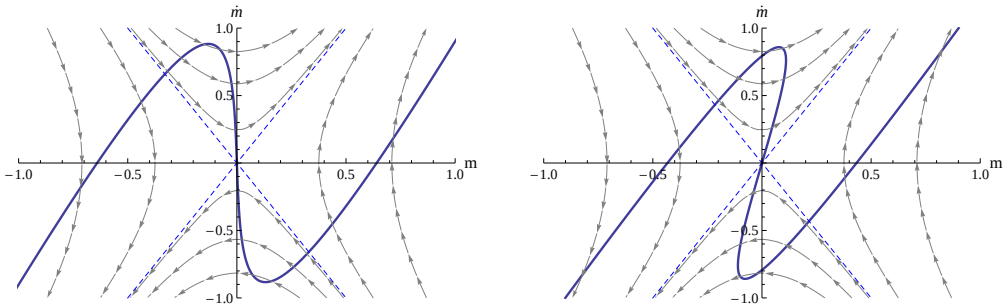


Figure 4.6: Non-symmetry-breaking mechanism, $\beta' = 0, \beta^{-1} = 0.8$

The time-evolved allowed-configurations curve for $t = t_{\mathrm{nGS}}(\beta, \beta' = 0)$ is shown at the left plot of Figure 4.6 where it acquires a vertical slope at zero. The right plot shows the time-evolved allowed-configurations curve for $t > t_{\mathrm{nGS}}(\beta, \beta' = 0)$ where it has two symmetric overhangs. In particular $(\beta, \beta' = 0, t, m' = 0)$ is pre-bad. It is also bad, since the preimages of the upper and lower time-evolved allowed-configurations curve which intersect the vertical axis have paths with the same cost, by the symmetry of the model. Note that $(\beta, \beta' = 0, t, m')$ is pre-bad for a whole interval of values of m', but (as the study of the cost shows and as it was proved in [36]) there are no other bad points. We note that $m' = 0$ is easily checked to be indeed a bad configuration (discontinuity point) of $\gamma_{\beta,\beta'=0,t}(\cdot|m')$ since there are no cancellations of discontinuities in this case, as we will explain now. Indeed, $k_t(\sigma_1, \tilde{\eta}_1; m', t)$ does not depend on the trajectory of the empirical magnetization and is given by the independent spin-flip at the site 1 between plus and minus with rate 1,

$$\gamma_{\beta,\beta'=0,t}(\eta_1|m') = \frac{\sum_{\sigma_1=\pm 1} e^{\sigma_1 \beta m^*(0;m',t)} k_t(\sigma_1, \eta_1)}{\sum_{\sigma_1,\tilde{\eta}_1=\pm 1} e^{\sigma_1 \beta m^*(0;m',t)} k_t(\sigma_1, \tilde{\eta}_1)} \tag{4.58}$$

where $k_t(+,+) = \frac{1}{2}(1 + e^{-2t})$, and $k_t(+,+) = k_t(-,-) = 1 - k_t(+,-) = 1 - k_t(-,+)$. So, a discontinuity under variation of m' is entering the formula only through $m^*(0; m', t)$, and hence $m' \mapsto \gamma_{\beta,\beta'=0,t}(\eta_1|m')$ is discontinuous if and only if $m \mapsto m^*(0; m', t)$ is discontinuous.

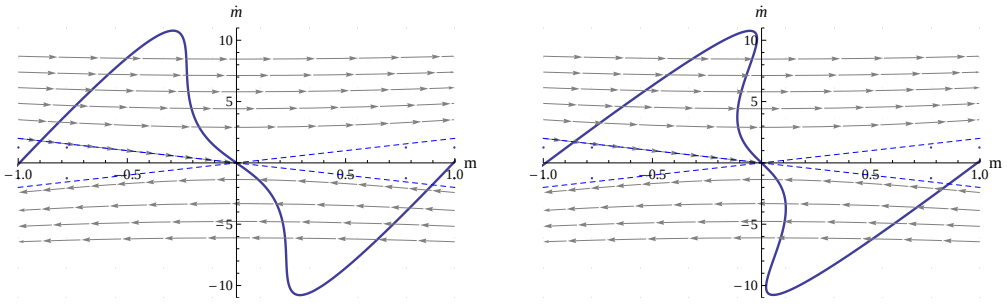


Figure 4.7: Symmetry-breaking mechanism, $\beta' = 0, \beta^{-1} = 0.4$

Let us now look at region 2b) of symmetry-breaking non-Gibbsianness i.e. $\beta^{-1} < \beta_{\mathrm{SB}}^{-1}(\beta' = 0)$

The left plot of Figure 4.7 shows the time-evolved allowed-configurations curve at $t = t_0(\beta, \beta' = 0)$ where it acquires a vertical slope away from zero. The right plot shows the time-evolved allowed-configurations curve for $t_0(\beta, \beta') < t < t_1(\beta, \beta')$ where it has two symmetric overhangs away from zero. This means that $(\beta, \beta' = 0, t, m')$ is pre-bad for a whole range of values of final magnetizations m'. Due to the lack of symmetry it is not clear to identify in the picture which of the $(\beta, \beta' = 0, t, m')$'s will be bad. It turns

out that it is precisely one such value $(\beta, \beta' = 0, t, m_c)$, and this can be found looking numerically at the cost.

Perturbations of these pictures stay true for ${\beta'}^{-1} > 1$, where they describe the only mechanism of non-Gibbsianness. Perturbations of these pictures *also* stay true for ${\beta'}^{-1} < 1$, but then there is also the Region 3 of the main theorem which describes the cooling from an initial low temperature. We choose $\frac{2}{3} = {\beta'}^{-1} < \beta^{-1} = 0.85 < 1$. Then the vector field has periodic orbits which are intersected by the allowed-configurations curve, and the time-evolution will create overhangs and smear out the allowed-configurations curve over time.

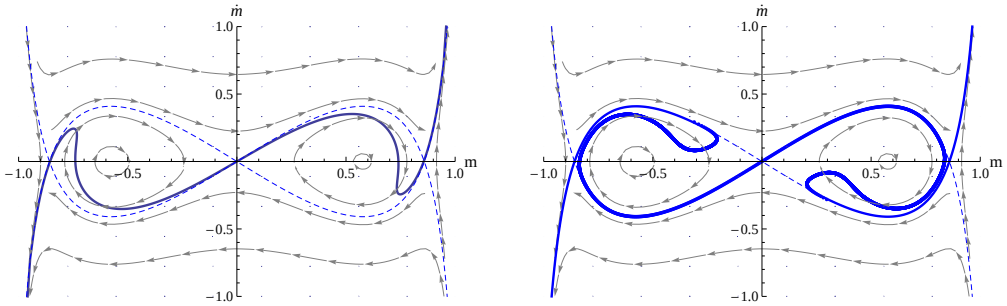


Figure 4.8: Non-Gibbsianness by periodicity, ${\beta'}^{-1} = \frac{2}{3}, \beta^{-1} = 0.85$

The left plot of Figure 4.8 shows the time-evolved allowed-configurations curve at $t = t_{\text{per}}(\beta, \beta')$ where it acquires a vertical slope away from zero inside the area of periodic motion.

The right plot shows the time-evolved allowed-configurations curve for a time $t > t_0(\beta, \beta')$ where it has overhangs. Again, from the interval of pre-bad points, the bad point has to be selected by looking at the cost. When time gets larger more overhangs are created and the trajectory is smeared out. The corresponding potential function $m \mapsto E_{m'}(m, \beta, \beta')$ will acquire more and more local extrema as t increases. Then, by fine-tuning of the m' while keeping the β, β', t fixed, equality of the depths of the two lowest minima can be achieved. Since the number of available minima is increasing with t we conjecture that there will be also an increasing number of bad m's which becomes dense as t increases. To prove this conjecture however, more investigation is needed.

4.3.5 Emergence of bad points as a function of time

The notion of a bad point can be viewed from two different standpoints. A pre-bad point in the time-space diagram is a point where two (or more) histories collide. If the costs computed along these paths are equal, then a

pre-bad point is a bad point. In the phase space this means that the phase flow transported two (or more) points originally lying on the curve of allowed initial configurations to the same space-position within equal time but with different speeds. Two (or more) points have the same space-position if their projections to the m-axis are equal, as seen in Figures (4.6), (4.7), and (4.8). How can we identify analytically the first time t where time-evolved initial points from the curve of allowed initial configurations will obtain the same projection to the m-axis? As intuition suggests one has to look when the transported curve of allowed configurations acquires a vertical slope for the first time. This discussion brings us to the following computation.

Writing $v = \dot{m}$ for the velocity, let us consider the flow $m(t; m_0, v_0)$, $v(t; m_0, v_0)$ of our system under the Euler-Lagrange equations,

$$\begin{aligned} \dot{m} &= v \\ \dot{v} &= f_{\beta'}(m) \end{aligned} \tag{4.59}$$

We take the curve of allowed initial configurations to be transported by the flow $v_0 = g_{\beta,\beta'}(m_0)$ where we write in short $f = f_{\beta'}$ and $g = g_{\beta,\beta'}$. We are then interested in the projections to the m-axis of the time-evolved curves in phase space, that is the curves $m_0 \mapsto m(t; m_0, g(m_0))$, as they evolve with t. Restricted to suitable neighbourhoods this curve becomes a function, and we view it as a potential function with state variable m_0 and parameter t (keeping also β, β' as fixed parameters.)

Doing so we see that the derivatives of the flow with respect to the initial conditions obey at the threshold time t that

$$\begin{aligned} 0 = F_{\beta',\beta}(t, m_0) &:= \frac{dm(t; m_0, g(m_0))}{dm_0} = \frac{\partial m(t; m_0, v_0)}{\partial m_0} + \frac{\partial m(t; m_0, v_0)}{\partial v_0} g'(m_0) \\ 0 &= \frac{d^2 m(t; m_0, g(m_0))}{(dm_0)^2} \end{aligned} \tag{4.60}$$

The first equation means that in the (m, v) plane the time-evolved curve will obtain a vertical slope which is clear by the interpretation of the variable m_0 as a parametrization of the curve of allowed initial configurations.

Moreover we have that the second derivative will also vanish, since a minimum and a maximum of $m_0 \mapsto m(s; m_0, g(m_0))$ collide for $s \downarrow t$, in a *fold bifurcation*.

4.3.6 The threshold time for non-symmetry-breaking non-Gibbsianness for dependent dynamics

We can use these equations to obtain quantitative information about the threshold time for non-symmetry-breaking non-Gibbsianness also for depend-

ent dynamics. For this it suffices to look at the dynamics locally around the origin $(m, \dot{m}) = (0, 0)$ in phase space which is a stationary point for the dynamics independently of β'.

Linearizing $f_{\beta'}$ we get

$$\begin{pmatrix} \dot{m} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 4(1-\beta')^2 & 0 \end{pmatrix} \begin{pmatrix} m \\ v \end{pmatrix} \tag{4.61}$$

Nonetheless the linearizing procedure provides corrections of third order. The eigenvalues of the matrix are $\lambda_{1,2} = \pm 2(1-\beta')$, these eigenvalues are real and have different signs, so $(m, \dot{m}) = (0, f_{\beta'}(0)) = (0, 0)$ is a saddle point. This ensures that the nature of solutions close to $(0; 0)$ stays the same whatever β' is taken.

Let us now discuss the phase flow around the origin $(0, 0)$. At this point non-Gibbsianness without symmetry-breaking occurs, by the following argument. Suppose a symmetric pair of initial conditions $(m_0, v(m_0))$ and $(-m_0, v(-m_0)) = (-m_0, -v(m_0))$ is given which has the same time-evolved magnetization 0 at time t. This corresponds to the fact that the transported curve will have overhangs at the points $(0, v_1(m))$ and $(0, -v_1(m))$. If we look at the phase portraits of the dynamics as a function of time we see that for times larger than but very close to the first time where this occurs the speed $v_1(m)$ will be very close to 0. It converges to 0 when t approaches the transition time for Gibbsianness. Indeed, the whole path was evolving in an arbitrarily small neighborhood of the origin and hence it suffices to look at the linearized dynamics. We also note that there is no need to look at the cost functional in this case, due to the symmetry of the paths. As time becomes larger than the transition-time (as in the right picture of Figure 4.6) the intersection points of the time-evolved curve with the vertical axis will move away from zero and so it would not be sufficient to use the linearization of the dynamics to compute the relation between bad magnetization values and time.

Clearly the general solution of the linearized system is

$$m(s) = C_1 e^{-2(1-\beta')s} + C_2 e^{2(1-\beta')s} \tag{4.62}$$

Putting the initial condition to be (m_0, v_0) the phase flow becomes

$$\begin{aligned} m(s; m_0, v_0) &= \frac{2(1-\beta')m_0 - v_0}{4(1-\beta')} e^{-2(1-\beta')s} + \frac{2(1-\beta')m_0 + v_0}{4(1-\beta')} e^{2(1-\beta')s} \\ v(s; m_0, v_0) &= \frac{v_0 - 2(1-\beta')m_0}{2} e^{-2(1-\beta')s} + \frac{v_0 + 2(1-\beta')m_0}{2} e^{2(1-\beta')s} \end{aligned} \tag{4.63}$$

Computing the function $F_{\beta',\beta}(t, m_0)$ (4.60) for this phase flow and setting it to zero having in mind that $v_0 = g(m_0)$, we solve it w.r.t. time t and get

$$t = \frac{1}{4(1-\beta')} \ln \frac{g'(m_0) - 2(1-\beta')}{g'(m_0) + 2(1-\beta')} \tag{4.64}$$

Putting $m_0 = 0$ we obtain from this for the transition time

$$t = \frac{1}{4(1-\beta')} \ln \frac{\beta' - \beta}{1-\beta} \tag{4.65}$$

By setting $\beta' = 0$ for the independent evolution in the last expression, the result $t = \frac{1}{4}\ln(1-\beta^{-1})$ given in [36] is reproduced. We note that the transition time given by formula (4.65) is positive only in the case when $\beta > 1$. This confirms the intuition obtained via "safe" regions. For $\beta \leq 1$ the curve of allowed configurations (w.l.g. $m > 0$) lies either higher than any branch of separatrix in a "safe" area or coincides with it providing the invariance of ACC under the phase flow.

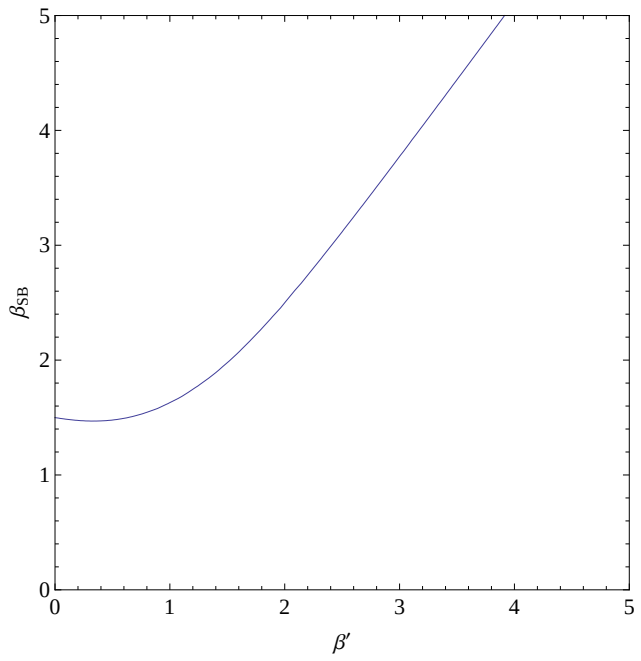


Figure 4.9: The symmetry-breaking inverse temperature β_{SB} as a function of β'

To identify for which temperature-values the phenomenon of non-Gibbsianness without symmetry-breaking ends, let us look when the function (4.64) starts having several minima. In order to do this we compute the second

derivative of (4.64) with respect to m_0 in $m_0 = 0$ and put it equal to zero. The computations are made easier due to the second equation in (4.60). This results in the equation

$$4\beta^3 + 12\beta\beta' - 6\beta^2(1+\beta') - \beta'(3+3\beta'-\beta'^2) = 0 \tag{4.66}$$

In the independent-dynamics case $\beta' = 0$ we get exactly $\beta = \frac{3}{2}$, which was already found in the paper [36]. The algebraic curve (4.66) is plotted in Figure 4.9.

4.3.7 Cooling and non-Gibbsianness by periodic orbits

Let us specialize to the case of a low-temperature dynamics $\beta' > 1$. In that case the phase space decomposes into the areas of periodic and non-periodic dynamics. The separatrix is given by (4.55) with $C = 4$.

$$f_{\pm}(m) = \pm 2\frac{(1+m) - e^{2\beta' m}(1-m)}{(1+m) + e^{2\beta' m}(1-m)} \tag{4.67}$$

Note that the curve $f_+(m)$ coincides with the curve of the "allowed" configurations (4.47) when $\beta' = \beta$. This means that it will be stable under the phase flow in that case. In particular the time-evolved curve will not acquire overhangs which corresponds to the fact that the time-evolved measure will be invariant under the dynamics and the model Gibbs.

Note also that the negative branch of the separatrix coincides with the right-hand side of the ODE describing the unconstrained typical evolution (4.18) and so the intersection point with the m-axis is given given by the biggest solution of the ordinary mean-field equation $m = \tanh(\beta' m)$. Let us first concentrate of the existence of pre-bad points, that is different initial points of the allowed-configurations curve leading to the same projection to the m-axis after time t.

Now multiple overhangs are created if the allowed curve of initial configurations intersects the periodic motion area, as seen in Figure 4.8. Indeed, this part of the curve will perform periodic motion and while doing so it will acquire more and more overhangs, filling out the part of the periodic motion area which is bounded by its extremal value of the integral of motion over time. It is now interesting to note for which temperatures this phenomenon can happen and this is the content of the following theorem.

Theorem 4.3.3 (Non-Gibbsianness by periodicity)**.** *Suppose $\beta' > 1$ and let m_1^* and m_2^* be the biggest solutions of the mean-field equations for β' and β. Then the following is true.*

1. *if $1 < \beta < \beta'$ (or equivalently $0 < m_2^* < m_1^*$) holds then*

- *The curve of allowed initial configurations for β, β' has non-zero intersection with the (open) periodic motion area in phase phase for β'.*
- *Consequently there exists a threshold time $t_{per}(\beta, \beta')$ such that for all $t > t_{per}(\beta, \beta')$ there exists pre-bad (β, β', t, m')s.*

2. *if $1 < \beta < \beta'$ fails, there is either no periodic motion areas, or the curve of allowed-configurations has no intersection with them.*

Proof. Denote $f = f_-$ (here we take the branch which bounds the periodic motion area from above), and the curve of the "allowed" configurations by $g(m)$ so that we have

$$\begin{aligned} f(m) &= -2\frac{(1+m) - e^{2m\beta'}(1-m)}{(1+m) + e^{2m\beta'}(1-x)}, \\ g(m) &= 2e^{2\beta' m}\frac{(1+m) - e^{2m(\beta-\beta')}(1-m)}{(1+m) + e^{2m\beta'}(1-m)} \end{aligned} \tag{4.68}$$

Previously it was mentioned that periodic motion arises only in the case $\beta' > 1$, and so we will consider this along the proof, also w.l.g. we say that $m > 0$. Let us show what the condition $1 < \beta < \beta'$ means and its equivalence to $0 < m_2^* < m_1^*$. First, we put $f(m) = 0$ to determine the right border of the periodic motion area, and we get that it's given by the equation

$$(1+m) - e^{2\beta' m}(1-m) = 0,$$

which is equivalent to the mean-field equation for β'. Let its biggest solution be given by m_1^*. Second, consider $f(m) = g(m)$ to determine their intersection point. This is simply

$$(1+m) - e^{2\beta m}(1-m) = 0,$$

which is again the same mean-field equation, but for β, where m_2^* has the same meaning as before.

The allowed-configurations curve comes into the region of periodic motion and stays there when the following condition is satisfied

$$-f'(x)\Big|_{m=0} < g'(m)\Big|_{m=0} < f'(m)\Big|_{m=0},$$

which turns out to be just equivalent to

$$-(2\beta' - 2) < 2 - 4\beta + 2\beta' < 2\beta' - 2 \tag{4.69}$$

or $1 < \beta < \beta'$. One can get an intuitive understanding of this mechanism from Figure 4.10. □

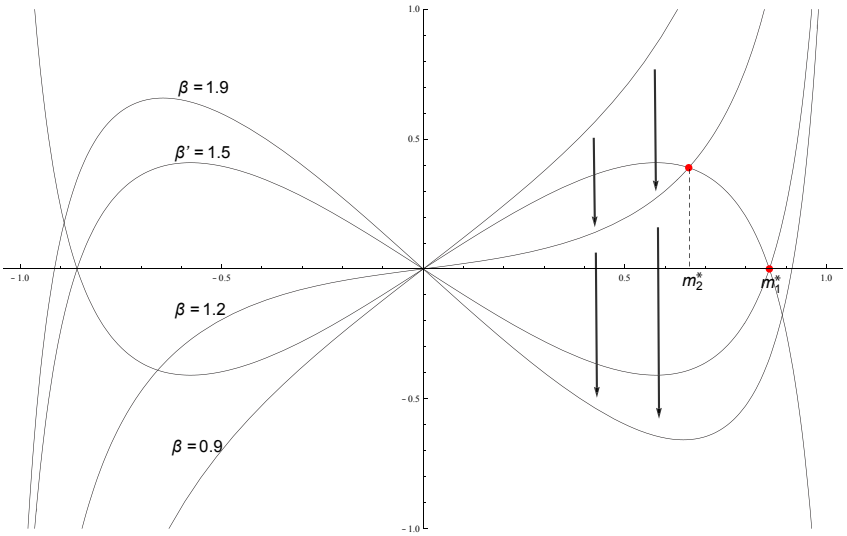


Figure 4.10: Allowed-configurations curve for different β keeping β' constant

4.4 Numerical results

Since the variational problem with fixed endpoint (4.46) cannot be solved in closed form unless the dynamics is independent, let us now describe some of the key features which are seen in a numerical study.

4.4.1 General approach

We start with the describing the numerical procedure we used to discover pre-bad points. These pre-bad points later are examined as to whether they are bad by computing corresponding costs. We look at the variational problem of finding an extremal constrained to take value m' at time $s = t$ from another standpoint. We apply a modification of the shooting method [53, see Section 7.3], when the variational problem is solved for a couple of initial conditions which later on are examined for the collision with each other at any point. The difference with the shooting method is that originally a final value of a solution of the variational problem has to be prescribed.

Fix the initial and dynamical inverse temperatures β and β'. We want to identify all initial magnetizations leading to the same (unknown) pre-bad value of magnetization at (unknown) time $\widehat{t}$. Since the model is symmetric, we may consider only positive initial conditions. As before, denote as $m(s)$ a solution of the Euler-Lagrange (EL, shortly) equation.

1. Select M, a fine enough discretization of $(0, 1)$. Also choose a partition $\mathbb{T}$ of time interval $[0, t]$, $\mathbb{T}_i = (t_i, t_{i+1})$

2. For each element $m_0 \in M$ the corresponding $\dot{m}_0$ could be computed from *ACC*. This defines two initial conditions for the Euler-Lagrange differential equation.

$$\begin{aligned} m(0) &= m_0 \\ \left.\frac{d}{ds}\right|_{s=0} m(s) &= \dot{m}_0 \end{aligned} \tag{4.70}$$

3. Solve the EL-equation for each couple $(m_0, \dot{m}_0)$ on the time interval $[0, t]$.

4. Set $i = 0$, and T to be the corresponding time-interval, $T = \mathbb{T}_i$

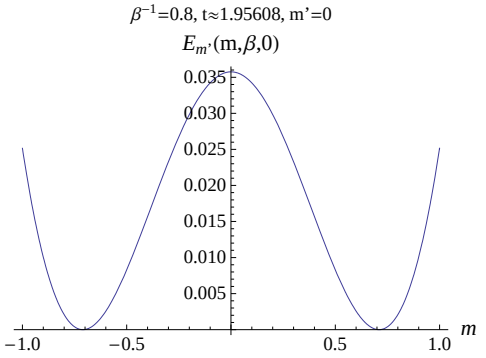
5. For each couple of any two intersecting solutions $m_1(s)$ and $m_2(s)$ on T at $(\widetilde{t}, \widetilde{m})$ look around for intersection points in ε-neighbourhood. Find all curves intersecting at least one of $m_1(s), m_2(s)$ within this neighbourhood. Call the union of these curves C_ε.

6. For each curve $m_\approx(s) \in C_\varepsilon$ (w.l.g. let $m_\approx(s)$ and $m_1(s)$ intersect at $(\widetilde{t}_\approx, \widetilde{m}_\approx)$) perturb the initial condition corresponding to $m_\approx(s)$ until the distance between $(\widetilde{t}_\approx, \widetilde{m}_\approx)$ and $(\widetilde{t}, \widetilde{m})$ is sufficiently small. In the degenerate case $m_\approx(s)$ will converge to one of $m_1(s), m_2(s)$. In other case we have found that if started at any of $m_1(0), m_2(0), m_\approx(0)$ with β', β^{-1} fixed paths will go trough $(\widetilde{t}, \widetilde{m})$ which is, therefore, a pre-bad point.

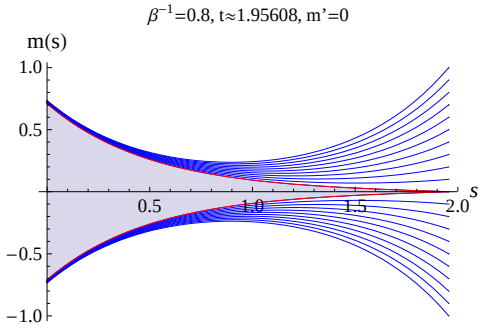
7. If $\mathbb{T}_{i+1}$ does not exceed $[0, t]$, increase i by a unit and go to (5) with the new $T = \mathbb{T}_i$, otherwise algorithm stops here.

4.5 Typical paths, bad configurations, multiple histories, forbidden regions

We remind the reader that for given conditioning (β', β, t, m') a solution of (4.46) with this set of parameters is called a *history curve*. Let us first discuss such curves for the example of independent dynamics. Figure 4.11 shows on the right such history curves conditioned to end at time t at m', for different values of m'. There is a jump in the optimal trajectory when we change $m' = 0+$ to $m' = 0-$. The associated cost functional at $m' = 0$, depicted on the left, has two symmetric minima, and their minimizers are the two possible initial magnetization values. This is an example of a multiple history scenario. We call the regions showing on the right plot which cannot be visited by any integral curve *forbidden regions*.

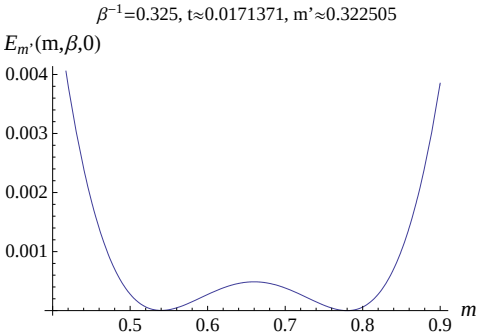


Cost functional

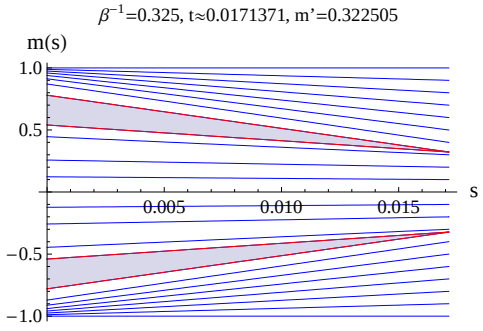


Corresponding history curves

Figure 4.11: Symmetric forbidden regions



Cost functional



Corresponding history curves

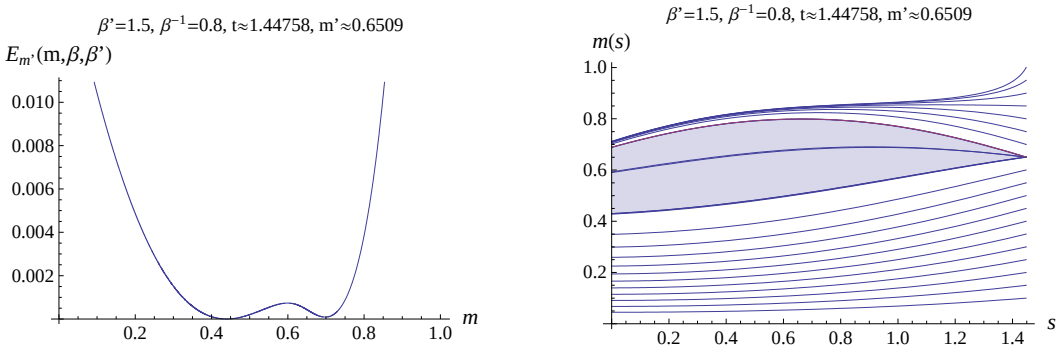
Figure 4.12: Non-symmetric forbidden region

Figure 4.12 shows on the right history curves for the independent dynamics with a low initial temperature smaller than $\frac{2}{3}$ where symmetry-breaking in the set of bad configurations takes place. We see on the right two discontinuity points m' and correspondingly two components of forbidden regions for the trajectory. The cost functional corresponding to the positive one of them is depicted on the right. Deformations of these pictures describe the phenomena for all temperatures of the dynamics, as long as the initial temperature is lower.

Finally, Figure 4.13 displays history curves and cost functional at the critical conditioning for an example of cooling dynamics.

Next, let us fix β, β' and describe the possible change of the set of bad configurations as a function of the time. Again we look at the independent dynamics first.

The top line of Figure 4.14 has an initial temperature in which non-Gibbsian behaviour without symmetry-breaking takes place. In the Figure 4.14(b) we see the bad configurations m' as a function of the time s which



Cost functional

Corresponding history curves

Figure 4.13: Forbidden region for $\beta' = \frac{3}{2}$

were found numerically depicted by dots. Since $m' = 0$ appears at a threshold time and stays to be the only bad configuration from that on, the graph of bad configurations is just a straight line starting at the threshold time. In the Figure 4.14(a) we see the corresponding initial points of the history curves which are conditioned to end at m'.

The lower line of Figure 4.14 has an initial temperature for which non-Gibbsian behaviour with symmetry-breaking takes place, in an intermediate time-interval. The right plot shows the corresponding non-negative branch of bad configurations m'. (By the symmetry of the model, taking the negative of these one obtains the full set of bad configurations.) The left plot shows the corresponding initial points of the history curves which are conditioned to end at the non-negative bad configurations m' on the right.

Finally, the Figure 4.15 displays the time-evolution of bad configurations and their initial points for a low-temperature dynamics. The lowest line corresponds to heating from very low initial temperature and shows non-Gibbsianness with symmetry-breaking at an intermediate time-interval. The middle line corresponds to heating from an intermediate lower temperature and shows non-Gibbsianness without symmetry-breaking. These two mechanisms are known from high-temperature dynamics. Figures 4.15(a) and 4.15(b) correspond to cooling and shows data from the region of periodic orbits.

Applying numerical integration of the Euler-Lagrange equations from initial conditions chosen on the allowed-configurations curve, check for intersecting trajectories and numerical computation of the cost function we can get (numerical approximations to) the array of bad quadruples $(\beta, \beta', t, m_{\mathrm{pb}})$, augmented by the possible initial points. With this procedure we re-derived the Gibbs-non-Gibbs phase diagram for $\beta' = 0$ (which was obtained earlier in[36]). Based on it we can draw the Gibbs-non-Gibbs phase-diagram at any dynamical temperature β'. An example for this was presented in the section

where the main result of the present chapter was stated in the Figure 4.1 for a fixed relatively low dynamical temperature.

4.6 Final remarks

The paper [15] which this chapter is based on is to our knowledge the first one where Gibbs properties of a model subjected to a low-temperature dynamics are investigated. Shortly after, the paper of van Enter, A.C.D. et al. [14] appeared where the lattice case was treated. In that paper a large-deviation approach was proposed to understand dynamical transitions in the Gibbs properties for lattice systems, too. While there is a beautiful formalism available for path large deviations of empirical measures of lattice systems on an abstract level, explicit results are very hard and given only for an infinite-temperature dynamics, which underlines also the use of our work, and the necessity of future research.

There are several possible ways for extension of our work. We first mention theoretical issues. As previously conjectured, the set of bad configurations expands, this requires more numerical experiments to be done. Moreover, a similar effect has been seen in Chapter 3. The investigated model was considered in a vanishing field, while it is not always the case and it will be interesting to see which effects persist (if at all) for the model in a field. We expect a shift in the phase diagram leaving less possibilities for non-Gibbsian behaviour. A more challenging generalization and first to think of is to run the same analysis for mean-field Potts model.

Viewing the problem from the standpoint of applications, the questions and methods used should have interest also in models of population dynamics. In such models a population of N individuals, each individual carrying genes from a finite alphabet of possible types, performs a stochastic dynamics which can be described on the level of empirical distributions. Starting the dynamics from a known initial measure corresponds to an a-priori belief (prior distribution) over the distribution of types. Conditioning to a final configuration m' at time $s = t$ corresponds to measuring the distribution of types. The occurrence of multiple histories leading to the same m' (which is responsible for non-Gibbsianness in the spin-model) has the interesting interpretation of a non-unique best estimator for the path explaining the present mix of genes.

* * *

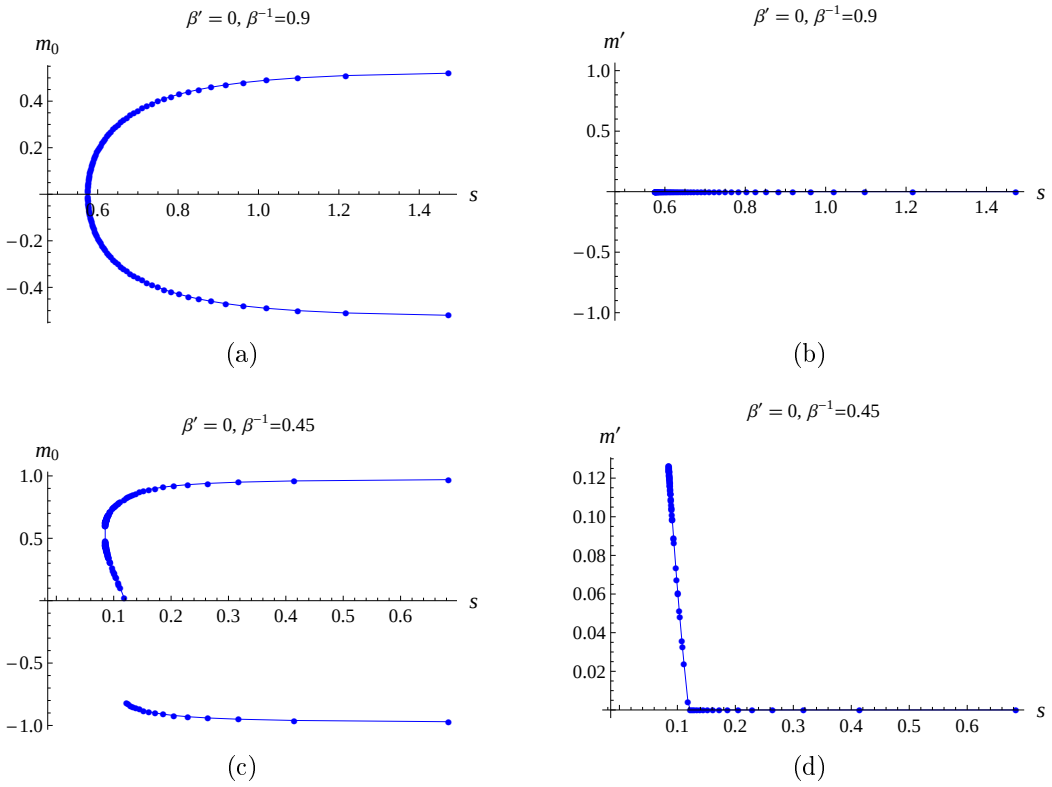


Figure 4.14: Initial points of trajectories (left) and bad configurations as function of time (right), $\beta' = 0$

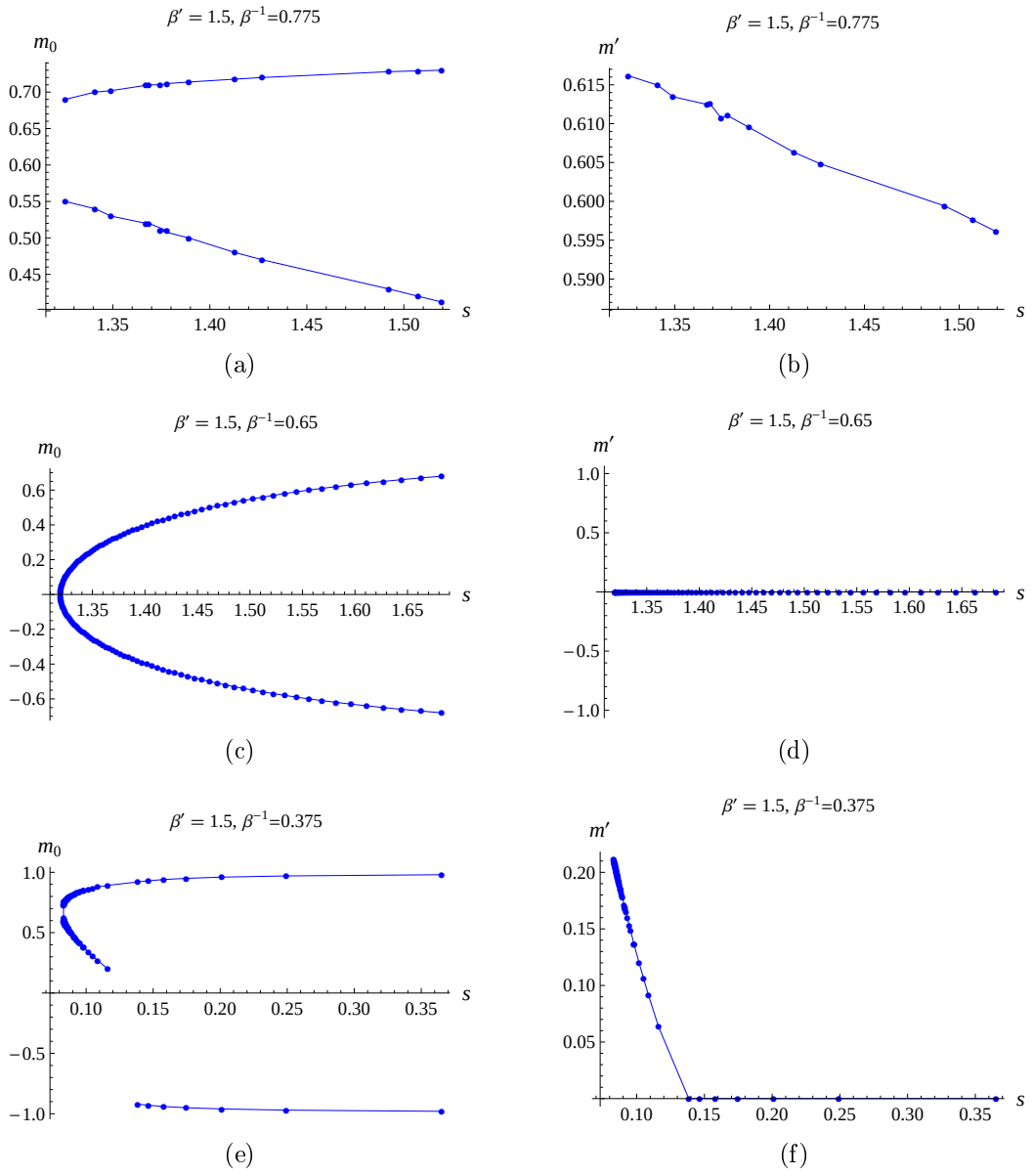


Figure 4.15: Initial points of trajectories (left) and bad configurations as function of time (right),low-temperature dynamics — $\beta' = \frac{3}{2}$

5

Some applications of the Gibbsian formalism

This chapter is devoted to an application of the Gibbsian formalism to some problems of Information Theory. We address a specific type of problems and overview approaches of Information Theory to this kind of problems and explain why the Gibbsian formalism might be useful there. As an illustration we run a series of experiments where we compare the approach based on the Gibbs approach and the more traditional one from Information theory.

5.1 Historical remarks

Markov chains of variable length memory first appeared in information theory in Rissannen's paper in 1983 [51]. That article addressed a problem of compressing a string of symbols. His idea was to model such a string as a realization of a stochastic chain with variable-length memory, that is when the length of the memory needed to predict the next symbol is not fixed, but is a deterministic function of the past symbols. Allowing a memory to be of variable length captures long-range dependence, but also to track "pasts" only to relevant depth. Consider a simple example, let $\{X_t\}$ be a stochastic chain defined on a binary alphabet $\{0, 1\}$ obeying the following rules: $\mathbb{P}(X_0 = 0|X_{-1} = 0) = \frac{1}{2}$ and $\mathbb{P}(X_0 = 0|X_{-1} = 1, X_{-2} = 0) = 1$. Clearly, it is not a Markov chain, though it may be viewed as a Markov chain with memory 2. With a bigger dependence range such an analogy is inefficient for practical purposes. The algorithm "Context" developed in [51] is able to reconstruct conditional distributions of the same nature as in our example from a random sample. As an output the algorithm provides a tree, encoding all information on the estimated conditional distributions. Each leaf (a node without children) is uniquely associated with a shortest memory one has to look at in order to have a "correct" conditional distribution for a given past and with a corresponding conditional distribution.

This discussion gives the basic description of, as we will abbreviate them, VLMC (also known as g-measures, random Markov chains) and describes an

efficient way to store such an object. VLMC's extensively employ the idea that conditional distributions of a stochastic chain are functions of the past values of that chain. Moreover, the length into the past which influences the conditional distribution of the variable at present itself depends on the past configuration. In 1990 Kalikow showed that a chain with complete connections (or, equivalently, infinite memory) under certain continuity assumptions on transition kernels can be decomposed as a countable mixture of Markov transition laws [34], more on this equivalence could be found in Fernández et al. [21, Chap. 7]

Rissannen's paper drew a lot of attention and became a foundation for later work both in theoretical science, e.g. [6, 20], and in applied research. A good example of applied research is the study of protein families in [4]. The idea of VLMC was applied to identify significant patterns in a set of related protein sequences. The developed method based on the "Context" singles out significant patterns of variable length surprisingly well without assuming any preliminary biological information. That implementation is publicly available [3]. We will be using it during our comparison analysis. Another interesting example occurs in linguistic studies. In [24] the application of VLMC was motivated by the linguistic challenge of retrieving rhythmic features from written texts (a set of daily newspapers). As a result an illustration compatible with the long standing conjecture that Brazilian Portuguese and European Portuguese belong to different rhythmic classes was provided.

More recently, a strong interest in two-sided models appeared. The source of this interest lies in Information Theory, more precisely in problems related to universal denoising. That is rather than considering distributions conditioned on the past, conditional distributions with two-sided conditioning — on past and future — are considered. In practice, a one-sided approach can relatively easily be extended to a two-sided point of view in two ways — either utilizing the knowledge of conditional distributions estimated with the one-sided approach or via a direct generalization, although mathematically both approaches are equivalent [20]. Practical differences come from the finiteness of any sample. We will review both of the approaches.

It is important to stress that dealing with two-(one-)sided conditional probabilities is rather common for the theory of Gibbs measures. Traditionally, in the Gibbsian formalism information about conditional distributions is encoded with a potential — a family of functions describing the interactions between random variables. Therefore, the information about a stochastic chain encoded in the aforementioned tree could be written in terms of a potential. A maximal memory needed to "correctly" describe a conditional distribution may be associated to a *range* of a potential. The idea of Rissanen and its further developments so successfully used in Information Theory are Gibbs in nature. The question we address in this chapter is whether a direct applica-

tion of Gibbsian formalism might improve the solution of some Information Theory problems. In other words, we aim to reconstruct a potential from a sample of random process and see whether it shows any superiority w.r.t. the one-sided approach while both are applied some problems of Information Theory (not yet denoising, however).

5.2 Theory

5.2.1 General set-up

We add to the general notation of Chapter 2 some new definitions. Denote by $\mathcal{G} = \{\Lambda : |\Lambda| < \infty\}$ the set of all finite subsets of G. We shall refer to the single-site space S also as an *alphabet*. Realizations of configurations $\sigma \in \Omega$ will be denoted by lower-case Latin letters b, s, i.e. the realization of σ_Λ is s_Λ. We denote by $\langle S^\Lambda \rangle$ the set of all possible realizations of a configuration σ_Λ and define

$$S^* = \bigcup_{\substack{|\Lambda| \geq 1 \\ \Lambda \subset \mathcal{V}}} \langle S^\Lambda \rangle$$

Definition 5.2.1. *Let μ be a probability measure on $(\Omega, \mathcal{A})$, then μ is called a* variable neighbourhood random field *if for any $\Lambda \in \mathcal{G}$ and for μ-almost all ω_{Λ^c} there exists a function $l : S^* \mapsto \langle S^\Lambda \rangle$, $\Lambda \in \mathcal{G}$, such that*

$$\mu(\sigma_\Lambda | \omega_{\Lambda^c} = b_{\Lambda^c}) = \mu(\sigma_\Lambda | \omega_{l(b_{\Lambda^c})} = b_{l(b_{\Lambda^c})}) \tag{5.1}$$

We will be concerned with reconstructing the original distribution μ when a realization $b \in S^*$ is known.

As a special case we treat G being a one-dimensional lattice, i.e. integer sites of the real line, $G = \mathbb{Z}$. The general notation keeps its meaning, but has to be thought in connection with the chosen graph.

Adopting the general notation for the one-dimensional case we write $\sigma_m^n = \sigma_\Lambda$ for configurations and $b_m^n = b_\Lambda$ for their realizations, where $\Lambda = (m, m+1, \ldots, n-1, n)$. The state-space of $\sigma_{-\infty}^{+\infty}$ is $\Omega = S^{\mathbb{Z}}$. Realizations of length j belong to the set $\langle S^j \rangle$, the set of all realizations on $\mathbb{Z}$ is $S^* = \bigcup_{j=1}^{\infty} \langle S^j \rangle$. Let $|\cdot|$ be the length function of a string, i.e. $|b| = j$ for $b \in \langle S^j \rangle$. We shall refer to one-dimensional realizations also as *strings*.

We say that a sequence $s_{-j}^{-1} \in \langle S^j \rangle$ is a *suffix* of a sequence $b_{-k}^{-1} \in \langle S^j \rangle$ if $j \leq k$ and $s_{-i} = b_{-i}$ for all $i = 1, \ldots, j$. We denote it as $s_{-j}^{-1} \preceq b_{-k}^{-1}$. If $j < k$, then it is said that s is a proper suffix of b and written as $s \prec b$.

5.2.2 Overview: One-sided modelling

Due to the one-dimensionality of the set-up, we shall call a string $b_{-j}^{-1} \in \langle S^j \rangle$ for any j a *past* for a random variable σ_0. Moreover, the same term "past" will refer to a family of random variables $\sigma_{-j}^{-1} \in S^j$.

The relevant definition of variable-neighbourhood random field (5.2.1) in the case of one-sided models transforms into the following.

Definition 5.2.2. *Let μ be a probability measure on $(\Omega, \mathcal{A})$, then μ is called a* variable length Markov chain *if there exists a function $l : S^* \mapsto \langle S^j \rangle$, j is finite, such that*

$$\mu(\sigma_0 | \sigma_{-\infty}^{-1} = b_{-\infty}^{-1}) = \mu(\sigma_0 | \sigma_{-l(b_{-\infty}^{-1})}^{-1} = b_{-l(b_{-\infty}^{-1})}^{-1}) \tag{5.2}$$

Definition 5.2.3. *A finite subset $\mathcal{T}$ of S^* is called a* tree *if it satisfies a* suffix property*, that is for no $b_{-k}^{-1} \in \mathcal{T}$ we have that $b_{-k+j}^{-1} \in \mathcal{T}$ for $j = 1, \ldots, k-1$.*

Here we explain the reference to $\mathcal{T}$ as a tree. The relation "$\prec$" induces a natural order for any sequence b_{-k}^{-1} in the following manner: $b_{-1} \prec b_{-2}^{-1} \prec b_{-3}^{-1} \cdots \prec b_{-k}^{-1}$. Therefore, given a set $\mathcal{T}$, satisfying the suffix property, all its elements might be represented as leaves of a certain tree rooted at the empty string $\mathbf{e}$. Nodes of such a tree in generation l are marked with suffixes s_{-l}^{-1} of $b_{-k}^{-1} \in \mathcal{T}$, $l < k$. Each node has possibly as many offspring as the cardinality of the set S which differ from the ancestor only by one symbol from S.

Definition 5.2.4. *A* probabilistic context tree *over S is an ordered pair $(\mathcal{T}, \mathbf{p})$ such that*

(i) Irreducibility. No string in $\mathcal{T}$ may be replaced by its suffix without violating the suffix property.

(i) $\mathbf{p} = \{\mathbf{p}(\cdot|b) : b \in \mathcal{T}\}$ is a family of transition probabilities over S.

Definition 5.2.5. *A realization $b_{-j}^{i-1} \in \langle S^{i+j} \rangle$ is a* one-sided context *for $\sigma_i, i \in \mathbb{Z}$ if $\mu(\sigma_{-j}^{i-1} = b_{-j}^{i-1}) > 0$ and if b_{-j}^{i-1} is a suffix for a semi-infinite string $s_{-\infty}^{i-1}$ such that*

$$\mu(\sigma_i = a | \sigma_{-\infty}^{i-1} = b_{-\infty}^{i-1}) = \mathbf{p}(a | b_{-j}^{i-1}), \text{ for all } a \in S, \tag{5.3}$$

and no suffix of b_{-j}^{i-1} satisfies this equality.

Definition 5.2.6. *The conditional distribution μ of a single σ_i is* compatible *with the probabilistic context tree $(\mathcal{T}, \mathbf{p})$ if the following conditions are satisfied*

(i) $b \in \mathcal{T}$ if and only if b is a context for σ_i;

(ii) for any $b \in \mathcal{T}$ and any $a \in S$, $\mathbf{p}(a|b) = \mu(\sigma_i = a | \sigma_{i-1-|b|}^{i-1} = b)$.

In the light of the last definition, the set $\mathcal{T}$ will also be referred to as a *uni-directional contexts set.*

5.2.3 Overview: Two-sided modelling

Let each of $\mathcal{T}_1$ and $\mathcal{T}_2$ be a finite subset of S^*. Given a couple of strings $(b_{-k}^{-1}, b_1^l) \in \mathcal{T}_1 \times \mathcal{T}_2$ we keep calling b_{-k}^{-1} a *past* for a random variable σ_0 and a string b_1^l a *future* for σ_0, for any integer l and k.

The relevant definition of variable-neighbourhood random field (5.2.1) in the case of two-sided models transforms into the following.

Definition 5.2.7. *Let μ be a probability measure on $(\Omega, \mathcal{A})$, then μ is called a* variable length Markov field *if there exists a function $l : S^* \mapsto \langle S^j \rangle$, j is finite, such that*

$$\mu(\sigma_0 | \sigma_{-\infty}^{-1} = b_{-\infty}^{-1}, \sigma_1^{+\infty} = b_1^{+\infty}) = \mu(\sigma_0 | \sigma_{-l(b_{-\infty}^{-1})}^{-1} = b_{-l(b_{-\infty}^{-1})}^{-1}, \sigma_1^{l(b_1^{+\infty})} = b_1^{l(b_1^{+\infty})}) \tag{5.4}$$

Definition 5.2.8. *A product $\mathcal{T}_1 \times \mathcal{T}_2$ satisfies a* suffix *property if for no couple of strings (b_{-k}^{-1}, b_1^l) we have that $(b_{-k'}^{-1}, b_1^{l'})$ belongs to $\mathcal{T}_1 \times \mathcal{T}_2$ for $k' \le k, l' \le l, k' + l' < k + l$.*

As before, given $\mathcal{T}_1 \times \mathcal{T}_2$ satisfying the suffix property, let $\mathbf{p}(\cdot | b, c)$ be a family of transition probabilities over S with conditioning $(b, c) \in \mathcal{T}_1 \times \mathcal{T}_2$.

Definition 5.2.9. *A pair of strings $(b_{-k}^{-1}, b_1^l) \in \mathcal{T}_1 \times \mathcal{T}_2$ for some finite integers k, l such that $k \ge 1, l \ge 1$, such that b_{-k}^{-1} is a suffix of a semi-infinite realization $s_{-\infty}^{-1}$ and b_1^l is a suffix of a semi-infinite realization $s_1^{+\infty}$ is called a* bidirectional context, *if for any $a \in S$*

$$\mu(\sigma_0 = a | \sigma_{-\infty}^{-1} = s_{-\infty}^{-1}, \sigma_1^{+\infty} = s_1^{+\infty}) = \mathbf{p}(a | b_{-k}^{-1}, b_1^t) \tag{5.5}$$

The set $\mathcal{T}_1 \times \mathcal{T}_2$ is, therefore, called a *bi-directional contexts set.*

Two-sided modelling as an extension of one-sided modelling

We now concentrate on how studying of the set of bi-directional contexts could be reduced to studying of the sets of uni-directional contexts and how the corresponding distributions $\mathbf{p}(a | b_{-k}^{-1}, b_1^t)$ and $\mathbf{p}(a | b_{-k}^{-1})$ and $\mathbf{p}(a | b_1^t)$ could be related. In the process we repeat arguments of Yu and Verdú [58] pointing out the disadvantages of this approach.

1) Backward-Forward product [58]. A conditional distribution of σ_0 with given past and future is proportional to the product of two one-sided conditional distributions with a given future and past, respectively. The disadvantage of the Backward-Forward product lies in its ability to describe a rather small set of two-sided distributions.

2) Generalized Markov scheme [58]. Yu and Verdú [58] make an assumption that σ is a $2m+1$-Markov field. The Markovian nature then implies that there exists an integer m (in case of a finite sub-graph of $\mathbb{Z}$, $m < \frac{1}{2}n$, where n is the number of vertices in the sub-graph) such that for any given σ_{j-m}^{j+m}, the families $\sigma_{-\infty}^{j-m-1}$ and $\sigma_{j+m+1}^{+\infty}$ are conditionally independent given σ_{j-m}^{j+m}, $\forall j$. Under this assumption the two-sided conditional distribution is shown to be proportional to an expression involving only uni-directional conditional probabilities as follows

$$\begin{aligned}\mu(\sigma_j = a|\sigma_{-\infty}^{j-1} = b_{-\infty}^{j-1}, \sigma_{j+1}^{+\infty} = b_{j+1}^{\infty}) \propto \\ \frac{\mu(\sigma_{j-m}^{j+m} = b_{j-m}^{j-1} a b_{j+1}^{j+m} | \sigma_{-\infty}^{j-m-1} = b_{-\infty}^{j-m-1})}{\mu(\sigma_{j-m}^{j+m} = b_{j-m}^{j-1} a b_{j+1}^{j+m})} \times \\ \mu(\sigma_{j-m}^{j+m} = b_{j-m}^{j-1} a b_{j+1}^{j+m} | \sigma_{j+m+1}^{+\infty} = b_{j+m+1}^{+\infty})\end{aligned} \tag{5.6}$$

The distributions conditioned either on a past or a future involved in (5.6) at the right-hand side are well described by uni-directional probabilistic context trees. The unconditioned distribution at the right-hand side is a measure of a cylinder and can also be computed using uni-directional conditional distributions.

This approach is inaccurate if variables in the configurations σ have a relatively short-range dependence, therefore either, for big m — σ_{j-m}^{j+m} starts being independent of $\sigma_{-\infty}^{j-m-1}$ and $\sigma_{j+m+1}^{+\infty}$, — or, for small m — long memory is not captured,— the aforementioned approach fails.

3) Forward and Backward One-sided Generalized Markov schemes [58]. The family $\sigma_{-\infty}^{+\infty}$ is assumed to be Markovian of k-th order. The reversed family $\sigma_{+\infty}^{-\infty}$ also possesses the k-th order Markovian nature. This knowledge allows one to observe that

$$\begin{aligned}\mu(\sigma_j = a|\sigma_{-\infty}^{j-1} = b_{-\infty}^{j-1}, \sigma_{j+1}^{+\infty} = b_{j+1}^{+\infty}) = \\ \mu(\sigma_j = a|\sigma_{j-k}^{j-1} = b_{j-k}^{j-1}, \sigma_{j+1}^{j+k} = b_{j+1}^{j+k})\end{aligned} \tag{5.7}$$

The equation (5.7) allows us to write the following expressions for the two-sided model:

$$\begin{aligned}\mu(\sigma_j = a|\sigma_{-\infty}^{j-1} = b_{-\infty}^{j-1}, \sigma_{j+1}^{+\infty} = b_{j+1}^{+\infty}) \propto \\ \mu(\sigma_j = a|\sigma_{j-k}^{j-1} = b_{j-k}^{j-1}) \times \mu(\sigma_{j+1} = b_{j+1}|\sigma_{j-k+1}^{j} = b_{j-k+1}^{j}) \times \\ \prod_{t=j+2}^{j+k} \mu(\sigma_t = b_t|\sigma_{t-k}^{t-1} = b_{t-k}^{j-1} a b_{j+1}^{t-1})\end{aligned} \tag{5.8}$$

$$\begin{aligned}\mu(\sigma_j = a|\sigma_{-\infty}^{j-1} = b_{-\infty}^{j-1}, \sigma_{j+1}^{+\infty} = b_{j+1}^{+\infty}) \propto \\ \mu(\sigma_j = a|\sigma_{j+1}^{j+k} = b_{j+1}^{j+k}) \times \mu(\sigma_{j-1} = b_{j-1}|\sigma_j^{j+k-1} = b_j^{j-k+1}) \times \\ \prod_{t=j-k}^{j-2} \mu(\sigma_t = b_t|\sigma_{t+1}^{t+k} = b_{t+1}^{j-1} a b_{j+1}^{t+k})\end{aligned} \tag{5.9}$$

The equation (5.8) is referred as forward one-sided generalized Markov scheme (or f-OGM) and the equation (5.9) — as b-OGM, in [58]. The previously discussed GM scheme is related with the f(b)-OGM scheme and with the correct choice of the order k they coincide up to normalizing constant. However, the GM-scheme combines the uni-directional probabilities from the two sides, whereas the f(b)-OGM scheme utilizes only one-sided conditional probabilities. Computationally the f(b)-OGM scheme is less complex.

4) The adaptive bi-directional model based on CTW [58]. The idea behind the CTW (context-tree weighting) method is slightly different than the idea of one-sided probability suffix trees. Instead of dealing with conditional probabilities $\mu(\sigma_j = a|\cdot)$, CTW employs distributions of block probabilities (i.e. joint probabilities) $\mu(\sigma_m^n), m < n$. The CTW method is well studied in [56, 57]. The definition of conditional probability together with the total probability law help to find a link between ideas of the CTW method and bi-directional conditional distributions.

$$\mu(\sigma_j = a|\sigma_{-\infty}^{j-1} = b_{-\infty}^{j-1}, \sigma_{j+1}^{+\infty} = b_{j+1}^{+\infty}) = \frac{\mu(\sigma_{-\infty}^{+\infty} = b_{-\infty}^{j-1} a b_{j+1}^{+\infty})}{\sum_{c \in A} \mu(\sigma_{-\infty}^{+\infty} = b_{-\infty}^{j-1} c b_{j+1}^{+\infty})} \tag{5.10}$$

In reality dealing with distributions of infinite blocks is an unreasonable aim, therefore a condition on the length of a block is put. This condition poses no problem in the case of an infinite string $s \in S^*$ where μ is reconstructed from, on the other hand , if $s \in S^k$ with k finite, a block $b_1^{j-1} a b_{j+1}^n$ might not exist for all $a \in A$ in $s \in S^k$. Let $2d+1$ be the maximum length of each block, then $\mu(\sigma_j = a|\sigma_{j-d}^{j-1} = b_{j-d}^{j-1}, \sigma_{j+1}^{j+d} = b_{j+1}^{j+d})$ is given as a sum of probabilities of all blocks of form σ_{j-d}^{j+d}, where the j-th symbol is perturbed. The corresponding probabilities could be found in CTW built from a given realization.

Often the realization is finite, therefore this method might lack some of the blocks, making the estimation imprecise.

Direct two-sided modelling

Another approach is to directly construct a bi-directional conditional distribution instead of estimating it from uni-directional distributions. Doing this

poses additional representation problems. The lack of natural order in the set of $\mathcal{T}_1 \times \mathcal{T}_2$ implies non-uniqueness of a bi-directional context for conditional distributions of μ. Moreover, representation of such contexts as leaves of a tree is only possible under additional requirements on a way a tree grows.

1) Bi-directional contexts trees of Yu and Verdú [58]. To preserve the desired tree structure for representation of context set, a simple rule was suggested in [58]. Recall that $|b_n^m|$ is the length function for a string, $\mathbf{e}$ is the empty string, $|\mathbf{e}| = 0$. Consider a bi-directional context (v, w) with past v and future w. Each of $(aw, v\mathbf{e})$, $(\mathbf{e}w, vb)$, and (aw, vb) for some $a, b \in A$ will be called a (left-, right-, left-right-) child-node (or *extension*) of (w, v). For each node (v, w) in the bi-directional tree rooted at $(\mathbf{e}, \mathbf{e})$ the following growing rule is applied:

- $|w| = |v|$, add three possible kinds of child-nodes with the forms $(aw, v\mathbf{e})$, $(\mathbf{e}w, vb)$, and (aw, vb) for some $a, b \in A$,
- $|w| < |v|$, add one possible kind of child-node with the form $(\mathbf{e}w, vb)$ for some $b \in A$,
- $|w| > |v|$, add one possible kind of child-node with the form $(aw, v\mathbf{e})$ for some $a \in A$.

This rule ensures that the constructed graph is connected and no offspring-node has several ancestor-nodes, therefore the tree structure is preserved and bi-directional contexts may be represented as leaves of such a tree.

An optimal choice of the context for $(b_{-\infty}^{j-1}, b_{j+1}^{+\infty})$ is a harder problem. Let $H(b_{-\infty}^{j-1}, b_{j+1}^{+\infty})$ be the set of all contexts for the given bi-directional conditioning $(b_{-\infty}^{j-1}, b_{j+1}^{+\infty})$ (clearly, the cardinality of this set in the uni-directional case, $b_{j+1}^{+\infty} = \mathbf{e}$, is 1). To resolve the ambiguity in finding a bi-directional context for $(b_{-\infty}^{j-1}, b_{j+1}^{+\infty})$, an exponential weighting scheme is used. This is motivated by noticing the similarity of the problem under discussion with that of estimating an unknown distribution P from a collection of distributions $(P_i)_{i=1}^k$ on a finite alphabet A. A natural method is to minimize the divergence between the mixture of $(P_i)_{i=1}^k$ and P. An optimization problem can be formulated as

$$\begin{aligned} &\text{minimize} && D(\sum_{i=1}^{k} \alpha_i P_i \| P) \\ &\text{with constraint} && \sum_{i=1}^{k} \alpha_i = 1,\ \alpha_i \geq 0,\ i = 1, \ldots, k \end{aligned} \tag{5.11}$$

Solving (5.11) (for the details we refer to [58]) allows to write the following estimator for the conditional distribution μ:

$$\hat{\mu}(\sigma_j = a|\sigma_{-\infty}^{j-1} = b_{-\infty}^{j-1}, \sigma_{j+1}^{+\infty} = b_{j+1}^{+\infty}) = \sum_{(w,v)\in H(b_{-\infty}^{j-1}, b_{j+1}^{+\infty})} \alpha_{w,v}(b_{-\infty}^{j-1}, b_{j+1}^{+\infty})\mu(\sigma_j = a|b_{j-|w|}^{j-1}, b_{j+1}^{j+|v|}),$$

where (5.12)

$$\alpha_{w,v}(b_{-\infty}^{j-1}, b_{j+1}^{+\infty}) = \frac{e^{\beta l(w,v)}}{\sum_{(x,y)\in H(b_{-\infty}^{j-1}, b_{j+1}^{+\infty})} e^{\beta l(x,y)}},$$

where β is some positive pre-specified constant.

2) Bi-directional contexts trees of Ordentlich et al.[49] In general, the rule (5.2.3) is not satisfied, so the representation of the contexts sets is not a tree anymore, but a graph. This is due to the fact that two different bi-directional context might have the same extension. It is suggested in Ordentlich et al. [49] to first construct a directed graph where nodes having no outgoing edges are associated with bi-directional contexts. Nodes having both incoming and outgoing edges are marked with suffixes of bi-directional contexts. An algorithm of pruning such a graph in order to delete all "doubling" nodes and get as an output a tree was proposed in Ordentlich et al. [49]. Having a tree with each node having necessarily exactly one ancestor automatically solves the problem of non-uniqueness of bi-directional context for a given conditioning.

3) Bi-directional contexts trees of Fernández et al.[18] The authors of [18] also pursue the idea of constructing a bi-directional contexts tree without a special rule obeyed during construction. In the set of two-sided contexts of $(b_{-\infty}^{-1}, b_1^{+\infty})$ "representative" contexts (satisfying a certain property [18]) are proven to be doubly-branching, that is past- and future- conditionings have equal length. A corresponding bi-directional tree is constructed as a product of two uni-directional context trees.

5.2.4 Gibbs potentials: Vacuum and telescoping formulas

As Definition 2.2.1 suggests the form of a potential is far from being unique. Non-uniqueness can be shown by a simple rearrangement of terms in Hamiltonian, nevertheless such potentials are *physically equivalent*. Potentials are equivalent if they assign the same amount of energy up to constants and boundary terms [11] to corresponding regions. Among physically equivalent interactions there are interactions satisfying the vacuum property. The relevant definition is as follows [19].

Definition 5.2.10. *An interaction* Φ *in* Ω *has* vacuum θ *if*

$$\phi_A(\omega) = 0, \text{ if } \omega_i = \theta_i \text{ for some } i \in A \tag{5.13}$$

for all $A \Subset \mathcal{V}$

The vacuum property was used to reconstruct the form of the potential in the course of proving the Kozlov theorem [35], from the specification Γ.

Theorem 5.2.11. *Let* $\Gamma = \{\gamma_\Lambda, \Lambda \Subset \mathcal{V}\}$ *be a specification with a certain potential such that* $\gamma_\Lambda(\sigma_\Lambda|\omega_{\Delta\setminus\Lambda}\theta_{\Delta^c}) > 0, \Lambda \subset \Delta$, *then the* θ-vacuum potential *for* Γ *verifies*

$$H_\Lambda(\sigma_\Lambda|\theta_{\Lambda^c}) = -\ln \frac{\gamma_\Lambda(\sigma_\Lambda|\theta_{\Lambda^c})}{\gamma_\Lambda(\theta_\Lambda|\theta_{\Lambda^c})} \tag{5.14}$$

for each $\Lambda \Subset \mathcal{V}$ *and each* $\sigma \in \Omega$.

For the Hamiltonian write

$$H_\Lambda(\sigma_\Lambda|\theta_{\Lambda^c}) = \sum_{A\subset\Lambda} v_A(\sigma_\Lambda\theta_{\Lambda^c}) \tag{5.15}$$

The last formula can be inverted with the Möbius transform and we get

$$v_A(\sigma\theta_{\Lambda^c}) = -\sum_{\substack{\Lambda\subset A\\ \Lambda\neq\emptyset}} (-1)^{|A\setminus\Lambda|} H_\Lambda(\sigma_\Lambda|\theta_{\Lambda^c}) \tag{5.16}$$

Such a potential satisfies the vacuum property. A possible problem with the vacuum potential is that it may not be absolutely convergent. In [35] Kozlov introduces another type of potential. Here we present a simplified derivations (in the spirit of [47]) and, for the occasion, we will call it a *telescoping* potential. Recall (5.14)

$$\exp\{-H_\Lambda(\sigma_\Lambda|\theta_{\Lambda^c})\} = \frac{\gamma_\Lambda(\sigma_\Lambda|\theta_{\Lambda^c})}{\gamma_\Lambda(\theta_\Lambda|\theta_{\Lambda^c})} \tag{5.17}$$

The last formula yields the following decomposition

$$\frac{\gamma_\Lambda(\sigma_\Lambda|\theta_{\Lambda^c})}{\gamma_\Lambda(\theta_\Lambda|\theta_{\Lambda^c})} = \prod_{k=1}^{n} \exp\{F_k(\sigma_{i_1}\ldots\sigma_{i_k})\}, \tag{5.18}$$

where we lexicographically ordered $n = |\Lambda|$ sites in Λ so that $i_1 < i_2 < \ldots < i_n$ and

$$F_k(\sigma_{i_1},\ldots,\sigma_{i_k}) = \ln \frac{\gamma_\Lambda(\sigma_{i_1}\ldots\sigma_{i_{k-1}}\sigma_{i_k}\theta\ldots\theta|\theta_{\Lambda^c})}{\gamma_\Lambda(\sigma_{i_1}\ldots\sigma_{i_{k-1}}\ \theta\ \ \theta\ldots\theta|\theta_{\Lambda^c})} \tag{5.19}$$

Each of $F_k(\sigma_{i_1},\dots,\sigma_{i_k})$ obeys a *bar displacement property*, i.e.

$$\ln\frac{\gamma_\Lambda(\sigma_{i_1}\dots\sigma_{i_{k-1}}\sigma_{i_k}\theta\dots\theta|\theta_{\Lambda^c})}{\gamma_\Lambda(\sigma_{i_1}\dots\sigma_{i_{k-1}}\ \theta\ \ \theta\dots\theta|\theta_{\Lambda^c})}=\ln\frac{\gamma_\Lambda(\sigma_{i_1}\dots\sigma_{i_{k-1}}\sigma_{i_k}|\theta_{\Lambda^c})}{\gamma_\Lambda(\sigma_{i_1}\dots\sigma_{i_{k-1}}\ \theta\ \ |\theta_{\Lambda^c})} \tag{5.20}$$

Note the following property of $F_k(\sigma_{i_1},\dots,\sigma_{i_k})$

$$\begin{aligned}-F_k(\sigma_{i_1},\dots,\sigma_{i_k})&=-\ln\frac{\gamma_\Lambda(\sigma_{i_1}\dots\sigma_{i_{k-1}}\ \theta\ \ |\theta_{\Lambda^c})}{\gamma_\Lambda(\sigma_{i_1}\dots\sigma_{i_{k-1}}\sigma_{i_k}|\theta_{\Lambda^c})}\\&=-\ln\frac{\gamma_\Lambda(\sigma_{i_1}\dots\sigma_{i_{k-1}}\ \theta\ \ |\theta_{\Lambda^c})}{\gamma_\Lambda(\sigma_{i_1}\dots\sigma_{i_{k-1}}\sigma_{i_k}|\theta_{\Lambda^c})}\frac{\gamma_\Lambda(\theta\sigma_{i_2}\dots\sigma_{i_{k-1}}\sigma_{i_k}|\theta_{\Lambda^c})}{\gamma_\Lambda(\theta\sigma_{i_2}\dots\sigma_{i_{k-1}}\ \theta\ \ |\theta_{\Lambda^c})}-\ln\frac{\gamma_\Lambda(\theta\sigma_{i_2}\dots\sigma_{i_{k-1}}\ \theta\ \ |\theta_{\Lambda^c})}{\gamma_\Lambda(\theta\sigma_{i_2}\dots\sigma_{i_{k-1}}\sigma_{i_k}|\theta_{\Lambda^c})}\\&=-\ln\frac{\gamma_\Lambda(\sigma_{i_1}\dots\sigma_{i_{k-1}}\ \theta\ \ |\theta_{\Lambda^c})}{\gamma_\Lambda(\sigma_{i_1}\dots\sigma_{i_{k-1}}\sigma_{i_k}|\theta_{\Lambda^c})}\frac{\gamma_\Lambda(\theta\sigma_{i_2}\dots\sigma_{i_{k-1}}\sigma_{i_k}|\theta_{\Lambda^c})}{\gamma_\Lambda(\theta\sigma_{i_2}\dots\sigma_{i_{k-1}}\ \theta\ \ |\theta_{\Lambda^c})}-F_{k-1}(\sigma_{i_2},\dots,\sigma_{i_k}),\end{aligned} \tag{5.21}$$

where the bar displacement property was used at the second equality sign. This observation allows to telescope functions F_k into

$$F_k(\sigma_{i_1},\dots,\sigma_{i_k})=-\sum_{m=0}^{k}U_m(\sigma_{\{i_m,\dots i_k\}}|\theta_{\{i_m,\dots i_k\}^c}), \tag{5.22}$$

where

$$\begin{aligned}&U_m(\sigma_{\{i_m,\dots i_k\}}|\theta_{\{i_m,\dots i_k\}^c})=\\&\qquad=\ln\frac{\gamma_{\{i_m,\dots i_s\}}(\theta\sigma_{\{i_{m+1},\dots i_k\}}|\theta)\gamma_{\{i_m,\dots i_k\}}(\sigma_{\{i_m,\dots i_{k-1}\}}\theta|\theta)}{\gamma_{\{i_m,\dots i_k\}}(\sigma_{\{i_m,\dots i_k\}}|\theta)\gamma_{\{i_m,\dots i_k\}}(\theta\sigma_{\{i_{m+1},\dots i_{k-1}\}}\theta|\theta)}\end{aligned} \tag{5.23}$$

As a consequence, the Hamiltonian (5.14) is telescoped to

$$H_\Lambda(\sigma_\Lambda|\theta_{\Lambda^c})=\sum_{A\cap\Lambda\neq\emptyset}U_A(\sigma_\Lambda|\theta_{\Lambda^c}) \tag{5.24}$$

As before, to match our purposes we set the graph G to be a one-dimensional lattice $\mathbb{Z}$ and explicitly write the telescoping potential

$$U_{[i,j]}(\sigma_{[i,j]})=\ln\frac{\gamma_{[i,j]}(\theta\sigma_{(i,j]}|\theta_{[i,j]^c})\gamma_{[i,j]}(\sigma_{[i,j)}\theta|\theta_{[i,j]^c})}{\gamma_{[i,j]}(\sigma_{[i,j]}|\theta_{[i,j]^c})\gamma_{[i,j]}(\theta\sigma_{[(i,j)}\theta|\theta_{[i,j]^c})} \tag{5.25}$$

5.3 Applications

5.3.1 The general problem and our approach

Given two realizations $s_{\mathbb{P}}\in S^{l_1}$ and $s_{\mathbb{Q}}\in S^{l_2}$ with l_1,l_2 finite, of random fields corresponding to different Gibbs measures $\mathbb{P}$ and $\mathbb{Q}$, one is asked to identify

whether a third realization $b \in S^{l_3}$, l_3 finite, of unknown nature is more typical for $\mathbb{P}$ or for $\mathbb{Q}$ or for neither of them. In general, $l_1 \neq l_2 \neq l_3$. Moreover, we do not exclude the situation when some pieces of the concerned realization are more likely generated by $\mathbb{P}$ and some by $\mathbb{Q}$. For that purpose we build two measure-estimators $\hat{\mathbb{P}}$ and $\hat{\mathbb{Q}}$ and consider a map $\Upsilon : S^{l_3} \mapsto \{t_n\}_{n=1}^{l_3}$, that is for any symbol $b_n \in S$ part of b we treat a number

$$t_n = \ln \frac{\hat{\mathbb{P}}(b_n|b_1^{n-1}, b_{n+1}^{l_3})}{\hat{\mathbb{Q}}(b_n|b_1^{n-1}, b_{n+1}^{l_3})}. \tag{5.26}$$

In particular, the ability to distinguish pieces of different nature is expected to become apparent via sharp transitions between consecutive sub-sequences having mostly negative terms and consecutive sub-sequences having mostly positive terms. For practical reasons the sequence $\{x_n\}_{n=1}^{l_3}$ should also be viewed as a sequence of points $\{(n, x_n)\}_{n=1}^{l_3}$. If plotted, the graph of the sequence of points is expected to perform sudden jumps at certain values of $1 < n < l_3$ which are meaningfully interpreted as transitions between pieces in $b \in S^{l_3}$ typical for different measures.

5.3.2 Statistical procedure

Given a realization $s \in S^*$ of the random field $\sigma \in \Omega$, let $\#(\cdot) : S^j \mapsto \mathbb{N}$, for any j, be the function mapping a string of length j into the number of its occurrences in the realization s. In the case of uni-directional modelling the relevant estimator for $\mathbf{p}(a|b_{-j}^{-1})$ in (5.2.5) with $a \in S$ and j integer is

$$\hat{\mathbf{p}}(a|b_{-j}^{-1}) = \frac{\#(b_{-j}^{-1}a)}{\sum_{\overline{a}\in S} \#(b_{-j}^{-1}\overline{a})}$$

Analogously to uni-directional case, the estimator for $\mathbf{p}(a|b_{-k}^{-1}, b_1^t)$ in (5.2.9), $a \in S$, integer k and t, is

$$\hat{\mathbf{p}}(a|b_{-k}^{-1}, b_1^t) = \frac{\#(b_{-k}^{-1}ab_1^t)}{\sum_{\overline{a}\in S} \#(b_{-k}^{-1}\overline{a}b_1^t)}$$

The estimation of Gibbs distributions requires more computations that just simply collecting occurrence-statistics. This is heavily related to estimating the Hamiltonian associated to the measure and its corresponding terms, that is functions $U_{[i,j]}$ of the form (5.25). We suppose that a true potential has finite range, that is functions $U_{[i,j]}$ are take the following form

$$U_{[i,j]}^{\text{fin}}(\sigma_{[i,j]}) = \ln \frac{\gamma_{[i,j]}(\theta\sigma_{(i,j]})\gamma_{[i,j]}(\sigma_{[i,j)}\theta)}{\gamma_{[i,j]}(\sigma_{[i,j]})\gamma_{[i,j]}(\theta\sigma_{(i,j)}\theta)} \tag{5.27}$$

Given a vacuum $\mathsf{O} \in S$, we choose the estimator $\hat{U}_{[i,j]}$ for $U^{\text{fin}}_{[i,j]}$ to be

$$\hat{U}_{[i,j]}(s_{[i,j]}) = \ln \frac{(\varepsilon + \#(\mathsf{O}s_{(i,j]}))(\varepsilon + \#(s_{[i,j)}\mathsf{O}))}{(\varepsilon + \#(s_{[i,j]}))(\varepsilon + \#(\mathsf{O}s_{(i,j)}\mathsf{O}))}, \tag{5.28}$$

where ε is a small positive constant.

5.3.3 Algorithmic implication

We aim to build a Gibbs potential from a given sample. The algorithm estimating the Gibbs potential could be roughly divided in two parts: 1) collecting statistics and 2) approximating a weight of each sub-string of the sample via the formula (5.28). See Appendix C for pseudo-code. A distribution of b_0 given past and future b_p and b_f, respectively then is nothing but a normalized exponent of a negative sum over all sub-strings of a string $b_p\, b_0\, b_f$ containing b_0.

The advantage of our approach is that instead of a tree structure having many leaves at different levels, where each of them describes (generically) different distributions, we encode all the information just in potential. The height of the tree is directly associated to the range of potential, in our case the approximation to a potential of finite range will weigh either with low weight or with no weight strings longer than its range.

5.3.4 Examples

We present the output of two programs, where the first one utilizes a unidirectional approach and the second one uses the above described algorithm. The output set of points $\{(n, x_n)\}_{n=1}^{l_3}$ is drawn: gray dots are the points themselves, solid line is a moving average taken in a fixed neighbourhood of each of x_n. For complicated examples we enlarge plots to make jumps in $\{(n, x_n)\}_{n=1}^{l_3}$ more visible. Obviously the set of parameters for the smoothing procedure is fixed in every example. For each of the examples we allowed the maximal memory to be 4 in the one-sided case and 2 to the left and 2 to the right in the two-sided one.

1-Markovian chain

Take $S = \{0, 1\}$. We generate two samples of different Markov chains: 1) the first one jumps between different states with a small probability, 2) the second one prefers to switch states as often as possible. The sample to classify is a mixture of them containing several equally long pieces of each of them . The result of classification is drawn in Figure 5.1.

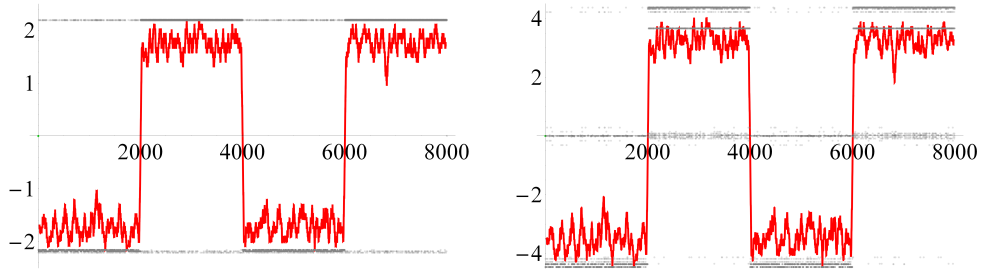


Figure 5.1: One-sided (left) vs two-sided (right) approach: Markov chains

Languages with common alphabet

Take $S = \{a, \ldots, z, \text{“ ”}\}$. Initial samples are: 1) an article from Wikipedia about the Netherlands in English, 2) the same article but in Dutch. The sample to classify is two articles in English and in Dutch about the city of Antwerpen copy-pasted one after another. The result is presented in Figure 5.2 and an enlarged version in Figure 5.3.

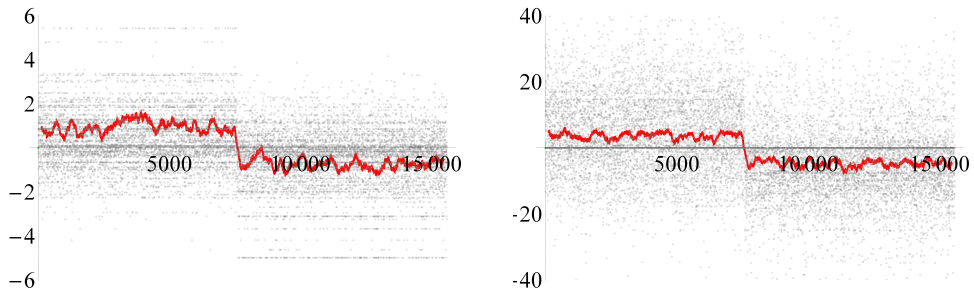


Figure 5.2: One-sided (left) vs two-sided (right) approach: Languages

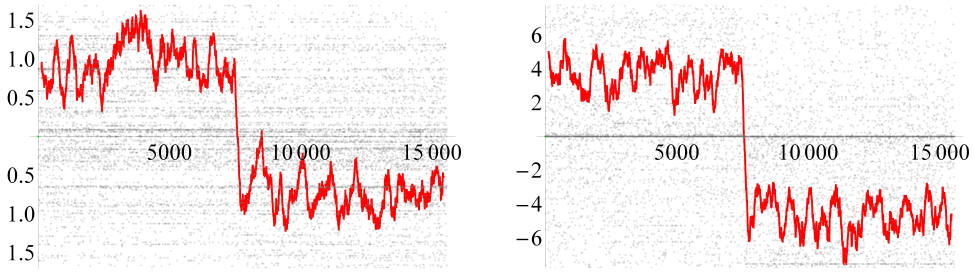


Figure 5.3: One-sided (left) vs two-sided (right): Languages, enlarged

Example: different authors

Take $S = \{a, \ldots, z, 0, \ldots, 9, \text{“ ”}, \text{punctuation marks}\}$. We extract only information according to a chosen set S from “Alice’s Adventures in Wonderland” by Charles Lutwidge Dodgson (commonly know as Lewis Caroll) and “Mansfield Park” by Jane Austen and take 4000 symbols of each of them as initial samples. The mixture is combined as follows: 1) 4000 symbols of “Alice’s Adventures in Wonderland” used to build an estimator, 2) 4000 symbols of “Mansfield Park” used to build an estimator, too, 3) 4000 symbols of “Alice’s Adventures in Wonderland” previously not used, 4) 4000 symbols of “Mansfield Park” previously not used either, 5) 4000 symbols of “Through the Looking-Glass, and What Alice Found There” by Lewis Caroll, and 6) 4000 symbols of “Pride and Prejudice” by Jane Austen.

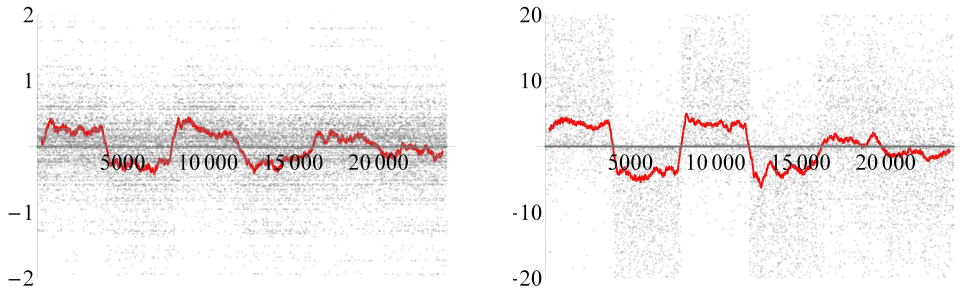


Figure 5.4: One-sided (left) vs two-sided (right): Literature

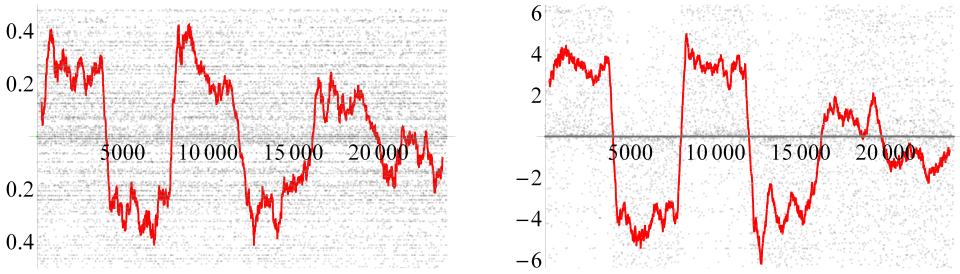


Figure 5.5: One-sided (left) vs two-sided (right): Literature, enlarged

5.3.5 Conclusions and remarks

The figures above show that the one-sided approach and the Gibbs two-sided approach work equally well for simple Markov chains. On the other hand

the Gibbs bi-directional approach shows a better ability to distinguish more realizations of different random field having more complex dependence. In the example with languages, the one-sided approach performed worse that the Gibbs two-sided approach showing a jump of a smaller height when a piece of text in Dutch switches to one in English. The third example showed most clearly the superiority of the Gibbs approach: the uni-directional approach was not able to distinguish between the different works of the same authors, i.e. "Pride and Prejudice' was not distinguished as a work of Jane Austen by the conditional distributions built upon "Mansfield Park". The two-sided model successfully managed to do this.

* * *

Appendices

A

Boundary laws, beyond homogeneity

The purpose of this appendix is to explain the relation between the notion of a boundary law as it is used in the book by Georgii [26] and the one-sided simple recursions which are used in Chapter 3 of the present thesis. The notion of a boundary law is necessary to describe all the extremal phases (or more generally, all Markov chains on trees).

To follow the notation used in Georgii [26], let us denote, for two connected sites $i \sim j$, by $Q_{ij}(\sigma_i, \sigma_j) = e^{\beta\sigma_i\sigma_j + g_i\sigma_i + g_j\sigma_j}$ the transition matrix of the random field Ising model on the tree with Hamiltonian $-\beta \sum_{\{i,j\}\in E} \sigma_i\sigma_j - \sum_i h_i\sigma_i$, where $g_i = h_i/(d+1)$, so the local field at each site has been symmetrically distributed among the edges to its neighbors.

Every extremal Gibbs measure μ for the random field Ising model on the Cayley tree is a Markov chain on the tree [26, see Theorem 12.12]. To define what it means to be a Markov chain on the tree, consider an oriented bond ij, draw this bond horizontally such that i lies to the left of j, and draw the tree embedded into the plane in such a way that there is no intersection between the tree and the axis crossing the oriented bond ij in a perpendicular way. A measure μ is a Markov chain on the tree if conditioning on the semi-infinite spin configurations extending from i to the left (the past) is the same as conditioning on the spin configuration at the site i alone, and this holds for all oriented bonds ij. Not all Markov chains are extremal Gibbs measures however, as the example of the free boundary condition Gibbs measure of the Ising model in zero field at sufficiently low temperatures shows. The meaning and importance of a boundary law lies in the following fact. A Markov chain on the tree always has a representation in terms of a boundary law $l_{ij}(a)$, $a = \pm$, that is for the finite-volume marginals it holds

$$\mu[h](\sigma_{\Lambda\cup\partial_+\Lambda}) = \frac{1}{Z_\Lambda(\beta,h)} \prod_{k\in\partial_+\Lambda} l_{kk_\Lambda}(\sigma_k) \prod_{\{ij\}\cap\Lambda\neq\emptyset} Q_{ij}(\sigma_i,\sigma_j) \tag{A.1}$$

where $\partial_+\Lambda$ denotes the outer boundary of Λ and k_Λ is the unique nearest neighbor of k in Λ. A boundary law is a function on oriented edges ij which

depends on the possible spin values. From its appearance in the last formula we see that, at any ij, it is defined only up to a multiplicative constant, not depending on the spin configuration a. Define therefore $q_{ij} = \frac{1}{2}\log\frac{l_{ij}(+)}{l_{ij}(-)}$ in the Ising case. This quantity has the character of a local field at the site i and contains the full information about the boundary law in the Ising case. More precisely q_{kk_Λ} has the meaning of a local field acting on the spin σ_k which has to be added to the Hamiltonian with free boundary conditions in the volume $\Lambda \cup \partial_+\Lambda$ if the site k is attached at the site k_Λ.

Assuming the validity of the last formula for the finite-volume marginals one arrives at a Q-dependent consistency (or recursion) relation that a boundary law has to satisfy. This recursion is formulated as (12.10) in Georgii; in the case of the Ising model with site-dependent fields it translates equivalently into the recursion

$$q_{ij} = \sum_{k\in\partial_+ i\setminus j} \frac{1}{2}\log\frac{e^{2q_{ki}+\beta+g_k+g_i} + e^{-\beta-g_k+g_i}}{e^{2q_{ki}-\beta+g_k-g_i} + e^{\beta-g_k-g_i}} \tag{A.2}$$

Conversely, a function q_{ij} on all oriented bonds which is consistent in the sense of (A.2) defines a Markov chain by formula (A.1) with the corresponding boundary law l_{ij}.

Note that (A.2) is a one-sided recursion which has no beginning and no end. It is interesting in the first step to look at homogeneous solutions, i.e. solutions not depending on the bond ij, but there may be also many other solutions, even in the case when the local magnetic field in the initial Hamiltonian is site-independent. In that case there can be non-homogeneous solutions when there are more than one fixed points for the homogeneous recursion. Indeed, to construct a non-homogeneous solution one picks a site j and looks to all oriented bonds ij pointing to it, and picks values of q_{ij} not at the fixed point. Then one defines a boundary law by preimages for q_b's for the oriented bonds b going up to ij. In order to make sure that there are such preimages under all orders of iterations, the value has to be chosen such that it lies between a stable and an unstable fixed point.

To see the meaning of the boundary law in a more intuitive or physical way let us make explicit the difference to the field which is already present in the original Hamiltonian. We look at the asymmetric quantity which is centered at the local field for the first spin, namely $f_{ij} = q_{ij} - g_i d$ and note that it satisfies the equation

$$f_{ij} = \sum_{k\in\partial_+ i\setminus j} \phi_\beta(f_{ki} + h_k) \tag{A.3}$$

with $\phi_\beta(t) = \frac{1}{2}\log\frac{\cosh(t+\beta)}{\cosh(t-\beta)}$. With this variable we have

$$\mu[h](\sigma_{\Lambda\cup\partial_+\Lambda}) = \frac{1}{Z_\Lambda(\beta,h)} e^{\sum_{\{ij\}\cap\Lambda\neq\emptyset}\beta\sigma_i\sigma_j + \sum_{i\in\Lambda\cup\partial_+\Lambda} h_i\sigma_i + \sum_{k\in\partial_+\Lambda} f_{kk_\Lambda}\sigma_k} \tag{A.4}$$

So the f_{ij} has the meaning of an additional boundary field at the site i acting on top of the local fields which are present already in the Hamiltonian, when one computes the finite-volume marginals in a volume with a boundary site i when i is attached via the site j to the inside of the volume.

Now, let us enter in more detail the discussion on the dependence of boundary laws on a variation of local fields entering in the Hamiltonian. Suppose that a boundary law $l[h]$, not necessarily homogeneous, is given for the (not necessarily but possibly homogeneous) Hamiltonian with a field h. Recall that, as we just explained, homogeneous fields h may have very well inhomogeneous boundary laws. Let us consider the system now in the presence of a local perturbation of the field $h + \Delta h$, possibly site-dependent, but bounded, i.e. $\sup_k |\Delta h_k| < \infty$. Any Gibbs measure $\mu[h]$ gives rise to a Gibbs measure $\mu[h + \Delta h]$ which is related by the formula involving the local perturbation of the Hamiltonian of the form

$$\mu[h + \Delta h](\phi(\tilde\sigma)) = \frac{\mu[h](\phi(\tilde\sigma)e^{\sum_i \Delta h_i\tilde\sigma_i})}{\mu[h](e^{\sum_i \Delta h_i\tilde\sigma_i})} \tag{A.5}$$

where it is understood that integration is over $\tilde\sigma$. If the original Gibbs measure is actually a Markov chain described by the boundary law $l_{ij} \equiv l_{ij}[h]$, the perturbed measure is described by the boundary law $l_{ij}[h + \Delta h]$ which is obtained by putting $l_{ij}[h + \Delta h] := l_{ij}[h]$ for oriented bonds ij in the outside of the region of the perturbation of the fields which are pointing towards the perturbation region. When passing with the recursion through the perturbation region of the local fields the l_{ij}'s obtain a dependence on the size of the perturbations. Then the forward iteration is used to obtain an assignment of l's to all oriented bonds.

Summarizing we have the following lemma.

Lemma A.0.1. *Suppose that h is an arbitrary external-field configuration, Δh is an arbitrary finite-volume perturbation of the external fields, and $\mu[h+\Delta h]$ is the measure which results from a local perturbation of a Markov chain $\mu[h]$ which is described by a boundary law $l[h]$.*

Then $\mu[h + \Delta h]$ behaves in a quasilocal way (i.e. all expected values $\mu[h + \Delta h](\phi)$ on local spin functions ϕ are quasilocal functions of Δh) if and only if the boundary laws $\Delta h \mapsto l_{ij}[h + \Delta h]$, depending on field perturbations Δh_k's for k in the past of the oriented bond ij, behave in a quasilocal way, and this holds for all oriented bonds ij.

Here a vertex k is said to be in the past of ij if the path from k to j passes through i. Quasilocality is meant in the same way as it has been introduced in the context of finite-volume variations of spins, i.e. we say that l depends quasilocally on a variation of fields iff

$$\lim_{\Lambda\uparrow\mathfrak{CT}(d)} \sup_{\Lambda':\Lambda'\supset\Lambda} \sup_{\Delta h|_\Lambda=\Delta h'|_\Lambda} |l[\Delta h|_{\Lambda'}] - l[\Delta h'|_{\Lambda'}]| = 0 \tag{A.6}$$

where the supremum is taken over perturbations $\Delta h|_{\Lambda'}, \Delta h'|_{\Lambda'}$ in the finite volume Λ' which look the same on Λ.

Proof: The proof follows from the representation of the finite-volume Gibbs measures $\mu[h+\Delta h]$ in terms of the boundary laws $l_{ij}[h+\Delta h]$.

□

We note again that there is a one-to-one correspondence between simple directed field recursions with d neighbors, as used in the paper, and boundary laws. So we obtain the following corollary, which is used extensively in the paper.

Corollary. Suppose that h is a homogeneous external field, Δh is an arbitrary finite-volume perturbation of external fields, and $\mu[h+\Delta h]$ is the measure which results from a local perturbation from either one of the homogeneous measures $\mu[h]$, corresponding to the plus, the minus or the unstable fixed points. Then the measures $\mu[h+\Delta h]$ behave in a non-quasilocal way on the field perturbations Δh if and only if the corresponding solutions of the one-sided simple recursions for the effective fields behave in a non-quasilocal way.

Some non-homogeneous Gibbs measures. The discussion just given has consequences also for those Gibbs measures $\mu = \mu_{(l^b),\Lambda}$ which are obtained by pasting boundary laws l^b for oriented bonds b of the form kk_Λ for some fixed subtree Λ, so that (A.1) is true for the particular volume Λ. Then extend the boundary laws to have a prescription in the whole volume. Then the parameter region for non-quasilocal behavior of the resulting measure will be the union of the parameter regions of non-Gibbsianness of the original measures taken over the b's.

Connection to Gibbs vs. non-Gibbs under time evolution. Since the Gibbs properties of time-evolved Ising measures in infinite-temperature evolution can be expressed via quasilocality properties of $\Delta h \mapsto \mu[h+\Delta h]$, for *finite-volume* Δh, we are left with the investigation of the locality properties of the boundary law iteration. A local variation of the image spins amounts to a local perturbation Δh of the local fields. Indeed, denoting the time-evolved measure by $\hat{\mu}_t(d\eta)$, starting from the measure $\mu(d\sigma)$, we have for finite $\Lambda \ni 0$

the formula

$$\begin{aligned}\hat{\mu}_t(\eta_0|\eta_{\Lambda\setminus 0}) &= \frac{\int \mu(d\sigma) P_t(\sigma_0,\eta_0) e^{h_t \sum_{i\in\Lambda\setminus 0}\eta_i\sigma_i}}{\int \mu(d\sigma) e^{h_t \sum_{i\in\Lambda\setminus 0}\eta_i\sigma_i}} \\ &=: \int \mu[\eta_{\Lambda\setminus 0}](d\sigma_0) P_t(\sigma_0,\eta_0)\end{aligned} \tag{A.7}$$

with a measure $\mu[\eta_{\Lambda\setminus 0}](d\sigma)$ of the form $\mu[h+\Delta h]$ with a perturbation in the finite volume $\Lambda\setminus 0$. Finite-volume marginals of this measure have a representation, according to Lemma A.0.1, of the form (A.1) with an η-dependent transition matrix $Q_{ij}[\eta](\sigma_i,\sigma_j) = e^{\frac{h_t\eta_i 1_{i\in\Lambda\setminus 0}}{d+1}+\frac{h_t\eta_j 1_{j\in\Lambda\setminus 0}}{d+1}} Q_{ij}(\sigma_i,\sigma_j)$ where $Q_{ij}(\sigma_i,\sigma_j)$ is the transition matrix for the initial measure μ, and an $\eta_{\Lambda\setminus 0}$-dependent boundary law $l_{ij}[\eta_{\Lambda\setminus 0}]$ which obeys the locally modified iterations for the boundary law described below (A.5). Hence, non-Gibbsianness of time-evolved measures is detected by non-quasilocality of the perturbed boundary laws $l_{ij}[\eta_{\Lambda\setminus 0}]$.

A consequence of these remarks is that a time-evolved measure resulting from an initial Gibbs measure which is constructed by pasting finitely many boundary laws l^b as described above, will be non-Gibbsian at a parameter regime which is the union of the non-Gibbsian parameter regimes of the time-evolved Markov chains corresponding to l^b, over b.

* * *

B

Computations for the mean-field case

In this part of Appendix we present explicitly the computations we skipped in the body of Chapter 4 of the present thesis.

B.1 Reversibility of measures

If the mean-field system with magnetization $\underline{m}_1^N$ is invariant and reversible for an evolution constructed from rates $c(\pm, \underline{m}_1^N)$, we want to show that the following is true in the infinite-N limit:

$$\frac{c(+, \underline{m})}{c(-, \underline{m})} = \exp\left(-2\beta' \underline{m}\right) \tag{B.1}$$

Consider an initial configuration $\sigma_{[1,N]}$ of size N and its transformation $\eta_{[1,N]}$. Recall that the measure is defined:

$$\mu_{\beta',N}(d\sigma) \propto \exp\left\{ \frac{\beta'}{2N} \left(\sum_{i=1}^{N} \sigma_i \right)^2 \right\}$$

for $\sigma_i \in S = \{-1, +1\}$.

A measure is called reversible if and only if

$$\mu_{\beta',N}(\sigma_{[1,N]}) K(\sigma, \eta) = \mu_{\beta',N}(\eta) K(\eta, \sigma),$$

where $K(\theta_0, \theta_t)$ is a transition probability to start with an initial configuration θ_0 and observe a configuration θ_t after time t.

Let the initial configuration and its transformation be different in just one spin for the first time, w.l.g. let $\sigma_1 = -\eta_1$ and the rest of configuration is preserved. The transition probabilities are expressed in terms of the flip rates, namely

$$\begin{aligned} K(\{+, \sigma_2, \ldots, \sigma_N\}, \{-, \sigma_2, \ldots, \sigma_N\}) &= \frac{c(+, \underline{m}_2^N)}{c(+, \underline{m}_2^N) + c(-, \underline{m}_2^N)} \\ K(\{-, \sigma_2, \ldots, \sigma_N\}, \{+, \sigma_2, \ldots, \sigma_N\}) &= \frac{c(-, \underline{m}_2^N)}{c(+, \underline{m}_2^N) + c(-, \underline{m}_2^N)} \end{aligned} \tag{B.2}$$

This brings us to the following expression

$$\mu_{\beta',N}(\{+,\sigma_2,\dots,\sigma_N\})\frac{c(+,\underline{m}_2^N)}{c(+,\underline{m}_2^N)+c(-,\underline{m}_2^N)} = \\ = \mu_{\beta',N}(\{-,\sigma_2,\dots,\sigma_N\})\frac{c(-,\underline{m}_2^N)}{c(+,\underline{m}_2^N)+c(-,\underline{m}_2^N)}$$

Then the fraction $\frac{c(+,\underline{m})}{c(-,\underline{m})}$ with the rates depending on the infinite-volume magnetization m is computed as follows:

$$\begin{aligned}\frac{c(+,\underline{m})}{c(-,\underline{m})} &= \lim_{N\to\infty}\frac{c(+,\underline{m}_2^N)}{c(-,\underline{m}_2^N)} = \lim_{N\to\infty}\frac{\mu_{\beta',N}(\{-,\sigma_2,\dots,\sigma_N\})}{\mu_{\beta',N}(\{+,\sigma_2,\dots,\sigma_N\})} \\ &= \lim_{N\to\infty}\exp\left(-\frac{2\beta'}{N}\sum_{i=2}^{N}\sigma_i\right) = \exp\left(-2\beta'\underline{m}\right)\end{aligned} \tag{B.3}$$

B.2 The form of a mean-field non-linear generator

Here we provide a simple computation of a non-linear generator for a mean-field model. We recall that a non-linear generator of a process is given by the following formula.

$$\left(\hat{\mathfrak{H}}_{\beta',N}F\right)(x) = \lim_{N\to\infty}\frac{1}{N}e^{-NF(x)}\left(\hat{L}_{\beta',N}e^{NF}\right)(x) \tag{B.4}$$

Recall also the linear generator of the concerned process:

$$\begin{aligned}&\hat{L}_{\beta',N}F(\underline{m}_1^N) = \\ &\quad N\frac{1+\underline{m}_1^N}{2}c\left(+,l(N)\left(\underline{m}_1^N-\frac{1}{N}\right)\right)\left(F\left(\underline{m}_1^N-\frac{2}{N}\right)-F\left(\underline{m}_1^N\right)\right) \\ &\quad +N\frac{1-\underline{m}_1^N}{2}c\left(-,l(N)\left(\underline{m}_1^N+\frac{1}{N}\right)\right)\left(F\left(\underline{m}_1^N+\frac{2}{N}\right)-F\left(\underline{m}_1^N\right)\right)\end{aligned}$$

For a shortcut we write

$$\begin{aligned}c_+ &= \frac{1+\underline{m}_1^N}{2}c\left(+,l(N)\left(\underline{m}_1^N-\frac{1}{N}\right)\right) \\ c_- &= \frac{1-\underline{m}_1^N}{2}c\left(-,l(N)\left(\underline{m}_1^N+\frac{1}{N}\right)\right)\end{aligned}$$

The computation for the right-hand side reads:

$$\begin{aligned}
&\frac{1}{N}e^{-NF(\underline{m}_1^N)}\left(\hat{L}_{\beta',N}e^{NF}\right)(\underline{m}_1^N) = \\
&\quad \frac{1}{N}e^{-NF(\underline{m}_1^N)}\left(Nc_-\left(e^{NF(\underline{m}_1^N+\frac{2}{N})}-e^{NF(\underline{m}_1^N)}\right)+Nc_+\left(e^{NF(\underline{m}_1^N-\frac{2}{N})}-e^{NF(\underline{m}_1^N)}\right)\right) = \\
&\quad c_-\left(\exp\left\{2\frac{F(\underline{m}_1^N+\frac{2}{N})-F(\underline{m}_1^N)}{\frac{2}{N}}\right\}-1\right)+c_+\left(\exp\left\{-2\frac{F(\underline{m}_1^N+\frac{2}{N})-F(\underline{m}_1^N)}{-\frac{2}{N}}\right\}-1\right) \\
&\quad \overrightarrow{N\to\infty}\ c_-\left(e^{2F'(m)}-1\right)+c_+\left(e^{-2F(m)}-1\right),
\end{aligned} \tag{B.5}$$

where we used the fact that in infinite-N limit $\underline{m}_1^N$ concentrates on a deterministic value m.

B.3 Relaxation and concentration property

In the case of non-interacting dynamics i.e. $\beta' = 0$, the form of the trajectories which are conditioned on ending at the value m' at time $s = t$ was obtained (4.51) and given as follows

$$m(s) = \frac{m_0e^{2t}-m'}{e^{2t}-e^{-2t}}e^{-2s}+\frac{m'-m_0e^{-2t}}{e^{2t}-e^{-2t}}e^{2s} \tag{B.6}$$

We start with the non-zero value m_0 at time $s = 0$, a simple computation $m(0)$ confirms this.

Mathematically, the quick relaxation of the magnetization for intermediate times reads $\forall\lambda \in (0,1)\lim_{t\to\infty} m(\lambda t) = 0$. Letting $s = \lambda t$ in (B.6) yields

$$\begin{aligned}
m(\lambda t) =& \frac{m_0e^{2t}-m'}{e^{2t}-e^{-2t}}e^{-2\lambda t}+\frac{m'-m_0e^{-2t}}{e^{2t}-e^{-2t}}e^{2\lambda t} \\
=& \frac{m_0\left(e^{2(1-\lambda)t}-e^{-2(1-\lambda)t}\right)+m'\left(e^{2\lambda t}-e^{-2\lambda t}\right)}{e^{2t}\left(1-e^{-4t}\right)} \\
=& e^{-2\lambda t}\frac{1-e^{-4(1-\lambda)t}}{1-e^{-4t}}m_0+e^{-2(1-\lambda)t}\frac{1-e^{-4\lambda t}}{1-e^{-4t}}m' \overset{t\to\infty}{\longrightarrow} 0
\end{aligned} \tag{B.7}$$

Trivially, at time $s = t$ the value of the magnetization is $m' > 0$. This can be seen either by direct computing $m(t)$ or by letting $\lambda = 1$ in (B.7).

B.4 The form of single-site transition probabilities

Recall the notations for the time-dependent spin-flip rates. For fixed values of β, β', t, and m' and a corresponding optimal trajectory $m^*(s;t,\underline{m}')$ in the sense of (4.26) $c_s(+) = c(+, m^*(s;t,\underline{m}'))$ is the rate to flip from "+" at time

$s = 0$ to "$-$" at time s. Analogously, $c_s(+)$ is defined. Here we treat the case of $k_s(+,+;t,\underline{m}')$.

$$\begin{aligned} & k_{s+ds}(+,+;t,\underline{m}') \\ & \qquad = k_s(+,+;t,\underline{m}')k_{ds}(+,+;t,\underline{m}') + k_s(+,-;t,\underline{m}')k_{ds}(-,+;t,\underline{m}') \\ & \qquad = k_s(+,+;t,\underline{m}')(1-c_s(+))ds + (1-k_s(+,+;t,\underline{m}'))c_s(-)ds \end{aligned} \tag{B.8}$$

implying the following equation

$$\frac{k_{s+ds}(+,+;t,\underline{m}') - k_s(+,+;t,\underline{m}')}{ds} = c_s(-) - (c_s(+)+c_s(-))\,k_s(+,+;t,\underline{m}') \tag{B.9}$$

Letting ds tend to zero, we obtain a differential equation for an unknown function $k_s(+,+;t,\underline{m}')$ with an initial condition $k_0(+,+;t,\underline{m}') = 1$. Solving this equation yields

$$\begin{aligned} & k_s(+,+;t,\underline{m}') \\ & \qquad = e^{-\int_0^s (c_u(-)+c_u(+))du} \times \left[\int_0^s c_u(-)e^{-\int_0^u (c_v(-)+c_v(+))dv}du + 1\right] \end{aligned} \tag{B.10}$$

The form (B.10) for transition probabilities in case of an independent spin-flip dynamics — $c_s(+) = c(-) = 1$ — repeats the well-known formula $k_s(+,+) = \frac{1}{2}(1+e^{-2s})$ of van Enter, A.C.D. et al. [12].

* * *

C

Pseudo-code for computing the potential-estimator

Given parameters M, δ, and O, where M is maximum of the memory of estimated Gibbs potential, δ is the constant identifying if the value of U computed for a certain string b should be ignored, and O is vacuum, and a realization s, we start.

```
Collect all words w into a set W;
Let N(w) = #(w occurs in s)

foreach( w ∈ W )
  u(w)=U([1, length(w)], w), use N(·)
  if u(w) > δ then Û(w)=u(w)
end foreach( w )
```

The constructed estimator $\hat{U}$ has to be used when the Hamiltonian of a certain string $b_p\, b_0\, b_f$ is computed.

* * *

Samenvatting

In de statistische mechanica bestaat het Gibbs formalisme uit het voorschrijven van voorwaardelijke kansen op locale gebeurtenissen, gegeven de rest van het systeem, in termen van wisselwerkingen. De toestanden met deze kansen beschrijven systemen die men observeert. Die systemen worden oneindig genomen in een dergelijke beschrijving, en bestaan uit inter-agerende componenten. Het verschijnsel van fase-overgangen in dergelijke systemen is gerelateerd aan het bestaan van verschillend globaal gedrag, consistent met bovengenoemde voorwaardelijke kansen. Meestal worden dergelijke wetten gemodelleerd in termen van voorwaardelijke verdelingen in eindige gebieden waarbij toestand het systeem in de rest van de ruimte volledig vastligt.

De Boltzmann-Gibbs aanpak voor deze voorwaardelijke verdelingen schrijft exponentiële gewichtsfactoren voor op configuraties in eindige gebieden die samenhangen met een energiefunctie (of potentiaal). De essentie van de Gibbs eigenschap van een systeem is de continuïteitseigenschap (in de product topologie) van de voorwaardelijke verdelingen met betrekking tot conditionering.

Diverse onderzoeken hebben aangetoond dat zelfs simpele transformaties kunnen leiden tot verlies van de Gibbs eigenschap. In het algemeen geldt dat de niet-Gibbs eigenschap van een systeem kan worden aangetoond door het vinden van een discontinuïteitpunt voor tenminste een voorwaardelijke kans.

Dit proefschrift beschrijft het onderzoek naar het behoud, of verlies, van Gibbs eigenschappen onder toepassing van transformaties op een Gibbs systeem. Het onderzoek is gericht op roostersystemen, op "mean-field" systemen en op systemen met een boomstructuur. Er wordt uitgegaan van stochastische, Bernoulli-verdeelde, variabelen, "spins", met waarden $+1$ of -1 op de knooppunten van een rooster. De onderzochte transformaties beschrijven de spin-flip dynamica waarbij de waarde van iedere spin met zekere kans omkeert.

In Hoofdstuk 3 worden twee typen discontinuïteiten voor de voorwaardelijke verdelingen besproken. De verschillen tussen deze twee soorten discontinuïteiten hangen samen met het bestaan van een metastabiele fase en een instabiele fase voor een initieel model met een boomstructuur, in een uitwendig magneetveld. Metastabiliteit van een fase houdt in de meeste gevallen in dat deze fase in de loop van de tijd convergeert naar een stabiele fase. Op een boom kan een metastabiele fase, die een hogere vrije energie heeft dan de

stabiele fase waarin de magnetisatie het veld volgt, invariant in de tijd zijn. In de metastabiele situatie is het mogelijk twee overgangsmechanismen van Gibbs naar niet-Gibbs gedrag te observeren. Voor een van de fasen wordt het volgende overgangsschema bewezen: Gibbs-systeem → "regulier" niet-Gibbs (er zijn discontinuïteiten in het gedrag van de voorwaardelijke kansen, maar niet alle punten zijn discontinuïteitspunten) → "totaal" niet-Gibbs (er zijn geen continuïteitspunten voor voorwaardelijke kansen). Andere fasen gedragen zich op een soortgelijke manier: Gibbs-systeem → "regulier" non-Gibbs → Gibbs systeem.

Vervolgens wordt aandacht besteed aan een "mean-field" model in een verwaarloosbaar klein magnetisch veld. In deze situatie zijn er twee bepalende grootheden, namelijk de begintemperatuur van het systeem en de temperatuur van de dynamica. De relatieve eenvoud van dit probleem maakt een opdeling mogelijk van het "tijd×initiële temperatuur" vlak bij een vaste dynamica temperatuur, in gebieden waarbij voorwaardelijke verdelingen wel of geen discontinuïteit vertonen.

Tot slot wordt ingegaan op een separaat onderwerp namelijk het verband tussen Markov ketens met variabele lengte (VLMC) en de Gibbs-grootheden. Het Gibbs-formalisme wordt toegepast op zogenaamde classificatie-problemen, waarbij de superioriteit ten opzichte van de VLMC aanpak wordt aangetoond.

Summary

In statistical mechanics the Gibbsian description of infinite systems consisting of interacting components involves prescribing local laws which are consistent with the observed behaviour of the system. The phenomenon of phase transitions is connected with the existence of several global behaviours consistent with these local laws. Usually such laws are modelled in terms of conditional distributions in finite regions when the rest is fixed. The Boltzmann-Gibbs approach for these conditional distributions prescribes exponential weights for configurations in finite regions associated with an energy function (or potential). The quintessence of the Gibbs property of the system is the continuity property of the conditional distributions w.r.t. conditioning. Many studies showed that even simple transformations of the system can lead to a loss of the Gibbs property. Generally, to show the non-Gibbs nature of a system it is enough to find a single discontinuity point for the local laws.

The present thesis further investigates under which conditions the transformed system preserves or loses its Gibbs property. This question is addressed for systems living on a complete graph or a tree. The former type of system is referred to as a mean-field system. In either case we consider Bernoulli random variables — spins — at nodes of the graph taking either $+1$ or -1 values. The transformations of interest are simple spin-flip dynamics altering value of each of the spins to an opposite one.

In Chapter 3, we discuss two types of discontinuities for the conditional distributions. This difference originates in existing of a metastable phase for an initial model on tree in a field. Metastability of a phase means that this phase converges to the one of stable ones when the field is increased (decreased). Moreover, this situation results in possibility to observe two different mechanisms for Gibbs-non-Gibbs transition. For one of the phases we prove the following transition scheme: Gibbsian system $\to$ "regular" non-Gibbsianness (there are discontinuities in behaviour of the conditional probabilities) $\to$ "total" non-Gibbsianness (there are *no continuity points* for conditional probabilities). Other phases behave in an expected way: Gibbsian system $\to$ "regular" non-Gibbsianness $\to$ Gibbsian system. This result underlines the importance of the size of a boundary of a finite region, this is to be contrasted with similar models living on lattices.

In Chapter 4 we draw attention to a mean-field model in vanishing external magnetic field. In this case there are two governing parameters: the initial temperature of the system and the temperature of the dynamics. The relative simplicity of the problem allows to decompose the "time×initial temperature" plane at fixed dynamical temperature into regions where conditional distributions have discontinuities and where they do not, see Figure 4.1.

Finally, the last Chapter 5 standing relatively apart from the main topic discusses connection between variable length Markov chains and Gibbs measures. We apply the Gibbs formalism for classification problems (where VLMC were of great use) and see the superiority of the Gibbsian approach.

Заключение

В статистической механике суть гиббсовского описания систем взаимодействующих частиц заключается в указании вероятностных законов распределения частиц на микроуровне так, что указанное распределение согласуется с наблюдаемым макросостоянием. Явление, при котором одному макросостоянию соответствует несколько таких вероятностных распределений, называется фазовым переходом. Исторически такие микрозаконы имеют форму условных распределений частиц внутри некого объема при условии, что состояние системы вне этого объема зафиксировано. Распределение Больцмана–Гиббса имеет следующую форму: каждому состоянию в конечной области ставится в соответствие его экспоненциальный вес. Квинтэссенция так называемого гиббсовского свойства — это непрерывность условных распределений относительно условий. Существует множество примеров, когда простые изменения в рассматриваемой системе ведут к потере гиббсовского свойства. Система называется *негиббсовской,* если найдется хотя бы одно условие, при котором нарушается непрерывность условных распределений.

Настоящая работа продолжает исследование по выяснению того, какие из преобразований ведут к потере гиббсовости. Рассматриваются системы случайных величин живущих на полном графе и на специальном виде деревьев. В обоих случаях в вершины графов помещены бернуллиевские случайные величины — спины ± 1, а в качестве преобразований системы мы рассматриваем изменения значений случайных величин на противоположные с течением времени. Такая модель называется моделью Изинга, а правило, по которому изменяется состояние системы — динамикой типа spin–flip.

В главе 3 обсуждаются два возможных типа разрывов в поведении условных распределений для модели Изинга на дереве. Разрывы разных типов возможны благодаря существованию метастабильного состояния в стартовой модели на дереве, помещенной в магнитное поле. Метастабильность состояния означает, что оно сходится к одному из стабильных состояний при изменении силы поля. Более того подобная ситуация предоставляет возможность наблюдать две принципиально разные схемы перехода системы к негиббсовскому режиму. Для метастабильной фазы

мы показываем верность следующего механизма: гиббсовская система $\to$ "обычное" негиббсовское состояние (условные распределения имеют разрывы) $\to$ "полностью" негиббсковское состояние (условные распределения *не имеют* точек непрерывности). Другие состояния ведут себя предсказуемо: гиббсовская система переходит в "обычное" негиббсовское состояние. Результат главы подчеркивает насколько важны граничные условия для условных распределений частиц внутри зафиксированной области.

В главе 4 рассматриваются системы на полном графе (или mean-field системы) вне поля. Проблема осложнена наличием двух параметров: начальной и динамической температурами системы. В вышеописанном случае температура во время преобразования системы предполагалась бесконечной, вследствие этого значения спинов изменялись независимо друг от друга. Простая структура полного графа позволяет явно указать области в плоскости "время $\times$ начальная температура" при фиксированной динамической температуре, где условные распределения трансформированной системы сохраняет и где теряют гиббсовское свойство, см. 4.1

Последняя глава посвящена вопросу несколько отличному от основной темы исследования и анализирует связи между марковскими цепями с памятью переменной длины и гиббсовкими мерами. Мы применяем алгоритм на основе гиббсовского представления условных вероятностей для проблем классификации (которые так же относительно успешно решались с применением марковских цепей) и показываем превосходство нашего алгоритма на примерах.

References

[1] J. Als-Nielsen and R.J. Birgeneau. Mean field theory, the Ginzburg criterion, and marginal dimensionality of phase transitions. *American Journal of Physics*, 45(6):554–560, 1977.

[2] R. Bakhshi, L. Cloth, W. Fokkink, and B.R. Haverkort. Mean-field analysis for the evaluation of gossip protocols. *ACM SIGMETRICS Performance Evaluation Review*, 36(3):31–39, 2008.

[3] G. Bejerano and G. Yona. Source for building PST. `http://bejerano.stanford.edu/resources/pst_v201.zip`, 2001. Online.

[4] G. Bejerano and G. Yona. Variations on probabilistic suffix trees: statistical modeling and prediction of protein families. *Bioinformatics*, 17(1): 23, 2001. ISSN 1367-4803.

[5] P.M. Bleher, J. Ruiz, and V.A. Zagrebnov. On the purity of the limiting Gibbs state for the Ising model on the Bethe lattice. *Journal of Statistical Physics*, 79(1):473–482, 1995.

[6] M. Bramson and S. Kalikow. Nonuniqueness in g-functions. *Israel Journal of Mathematics*, 84(1):153–160, 1993. ISSN 0021-2172.

[7] D. Dereudre and S. Rœlly. Propagation of Gibbsianness for infinite-dimensional gradient Brownian diffusions. *Journal of Statistical Physics*, 121(3):511–551, 2005.

[8] P. Diaconis and D Freedman. Finite exchange sequences. *The Annals of Probability*, 8(4):745–767, 1980.

[9] T.C. Dorlas and A.C.D. van Enter. Non-Gibbsian limit for large-block majority-spin transformations. *Journal of Statistical Physics*, 55(1):171–181, 1989.

[10] van Enter, A.C.D. and W.M. Ruszel. Gibbsianness versus Non-Gibbsianness of time-evolved planar rotor models. *Stochastic Processes and their Applications*, 119(6):1866–1888, 2009.

[11] van Enter, A.C.D., R. Fernández, and A.D. Sokal. Regularity properties and pathologies of position-space renormalization-group transformations: scope and limitations of Gibbsian theory. *Journal of Statistical Physics*, 72:879–1167, 1993.

[12] van Enter, A.C.D., R. Fernández, F. den Hollander, and F. Redig. Possible Loss and Recovery of Gibbsianness During the Stochastic Evolution of Gibbs Measures. *Communications in Mathematical Physics*, 226(1): 101–130, 2002.

[13] van Enter, A.C.D., V.N. Ermolaev, G. Iacobelli, and C. Külske. Gibbs-non-Gibbs properties for evolving Ising models on trees. *Arxiv preprint arXiv:1009.2952*, 2010.

[14] van Enter, A.C.D., R. Fernández, F. den Hollander, and F. Redig. A large-deviation view on dynamical Gibbs-non-Gibbs transitions. *Moscow Mathematical Journal*, 10:687–711, 2010.

[15] V.N. Ermolaev and C. Külske. Low-temperature dynamics of the Curie-Weis model: Periodic orbits, multiple histories and loss of Gibbsianness. *Journal of Statistical Physics*, 141:727–756, 2010.

[16] M. Fannes, H. Spohn, and A. Verbeure. Equilibrium states for mean field models. *Journal of Mathematical Physics*, 21:355, 1980.

[17] J. Feng and T.G. Kurtz. *Large deviations for stochastic processes*. Am. Math. Soc., 2006. ISBN 0821841459.

[18] F. Fernández, A. Viola, and M.J. Weinberger. Efficient Algorithms for Constructing Optimal Bi-directional Context Sets. In *2010 Data Compression Conference*, pages 179–188. IEEE, 2010.

[19] R. Fernández. *Gibbsianness and non-Gibbsianness in lattice random fields, Session LXXXIII, 2005, Mathematical Statistical Physics*. Elsevier, 2006.

[20] R. Fernández and G. Maillard. Chains with complete connections: General theory, uniqueness, loss of memory and mixing properties. *Journal of Statistical Physics*, 118(3):555–588, 2005. ISSN 0022-4715.

[21] R. Fernández, P. A. Ferrari, and A. Galves. Coupling, renewal, and perfect simulation of chains of infinite order. *Lecture Notes for the Vth Brazilian school of Probability, Ubatuba, August*, 92, 2001.

[22] de Finetti, B. Funzione caratteristica di un fenomeno aleatorio. *Mathematice e Naturale*, 4:251–299, 1931.

[23] M.I. Freidlin and A.D. Wentzell. *Random perturbations of dynamical systems*. Springer Verlag, 1998. ISBN 0387983627.

[24] A. Galves, C. Galves, N.L. Garcia, and F. Leonardi. Context tree selection and linguistic rhythm retrieval from written texts. *arXiv:0902.3619*, 2009.

[25] I.M. Gelfand and I.M. Fomin. *Calculus of Variations*. Dover Publications, 2000.

[26] H.-O. Georgii. *Gibbs measures and phase transitions*. Walter de Gruyter, Berlin, ISBN 0-89925-462-4, 1988.

[27] R.B. Griffiths and P.A. Pearce. Position-space renormalization-group transformations: Some proofs and some problems. *Physical Review Letters*, 41(14):917–920, 1978.

[28] R.B. Griffiths and P.A. Pearce. Mathematical properties of position-space renormalization-group transformations. *Journal of Statistical Physics*, 20 (5):499–545, 1979.

[29] G. Grimmett. A theorem about random fields. *Bulletin of the London Mathematical Society*, 5(1):81, 1973.

[30] E. Hewitt and L. Savage. Symmetric measures on Cartesian products. *Transaction of the American Mathematical Society*, 80:470–501, 1955.

[31] D. Ioffe. On the extremality of the disordered state for the Ising model on the Bethe lattice. *Letters in Mathematical Physics*, 37(2):137–143, 1996.

[32] E. Ising. Beitrag zur theorie des ferromagnetismus. *Zeitschrift für Physik A Hadrons and Nuclei*, 31(1):253–258, 1925.

[33] R.B. Israel. Banach algebras and Kadanoff transformations in random fields. *Random Fields*, vol. I, II:593–608, 1979.

[34] S. Kalikow. Random Markov processes and uniform martingales. *Israel Journal of Mathematics*, 71(1):33–54, 1990. ISSN 0021-2172.

[35] O.K. Kozlov. Gibbs description of a system of random variables. *Problemy Peredachi Informatsii*, 10(3):94–103, 1974.

[36] C. Külske and A. Le Ny. Spin-flip dynamics of the Curie-Weiss model: Loss of Gibbsianness with possibly broken symmetry. *Communications in Mathematical Physics*, 271(2):431–454, 2007.

[37] C. Külske and A.A. Opoku. Continuous spin mean-field models: Limiting kernels and Gibbs properties of local transforms. *Journal of Mathematical Physics*, 49:125215, 2008.

[38] C. Külske and A.A. Opoku. The posterior metric and the goodness of Gibbsianness for transforms of Gibbs measures. *Electronic Journal of Probability*, 13:1307–1344, 2008.

[39] C. Külske and F. Redig. Loss without recovery of Gibbsianness during diffusion of continuous spins. *Probability Theory and Related Fields*, 135 (3):428–456, 2006.

[40] J.Y. Le Boudec, D. McDonald, and J. Mundinger. A generic mean-field convergence result for systems of interacting objects. In *Quantitative Evaluation of Systems, 2007. QEST 2007. Fourth International Conference on the*, pages 3–18. IEEE, 2007.

[41] A. Le Ny. Introduction to (generalized) Gibbs measures. *Ensaios Matemáticos*, 15:1–126, 2008.

[42] A. Le Ny. Gibbsian description of mean-field models. In and out of equilibrium 2. *Progress in Probability*, 60:463–480, 2008.

[43] A. Le Ny and F. Redig. Short time conservation of Gibbsianness under local stochastic evolutions. *Journal of Statistical Physics*, 109(5):1073–1090, 2002.

[44] J.L. Lebowitz and C. Maes. The effect of an external field on an interface, entropic repulsion. *Journal of Statistical Physics*, 46(1):39–49, 1987.

[45] T.M. Liggett. *Interacting particle systems*. Springer Verlag, 2004.

[46] J. Lörinczi. Non-Gibbsianness of the reduced SOS-measure. *Stochastic Processes and their Applications*, 74(1):83–88, 1998.

[47] C. Maes, F. Redig, and A. van Moffaert. The restriction of the Ising model to a Layer. *Journal of Statistical Physics*, 96(1):69–107, 1999. ISSN 0022-4715.

[48] A. A. Opoku. *On Gibbs measures of transforms of lattice and mean-field systems*. PhD thesis, Rijksuniversiteit Groningen, 2009.

[49] E. Ordentlich, M.J. Weinberger, and T. Weissman. Multi-directional context sets with applications to universal denoising and compression. In *International Symposium on Information Theory, 2005. ISIT 2005. Proceedings*, pages 1270–1274, 2005.

[50] R. Pemantle and J.E. Steif. Robust phase transitions for Heisenberg and other models on general trees. *The Annals of Probability*, 27(2):876–912, 1999.

[51] J. Rissanen. A universal prior for integers and estimation by minimum description length. *The Annals of Statistics*, 11(2):416–431, 1983. ISSN 0090-5364.

[52] S.R. Salinas and W.F. Wreszinski. On the mean-field Ising model in a random external field. *Journal of Statistical Physics*, 41(1):299–313, 1985.

[53] J. Stoer and R. Bulirsch. *Introduction to numerical analysis*. Springer Verlag, 2002. ISBN 038795452X.

[54] W.G. Sullivan. Potentials for almost Markovian random fields. *Communications in Mathematical Physics*, 33(1):61–74, 1973.

[55] S.R.S. Varadhan. Large deviations. *The Annals of Probability*, 36(2): 397–419, 2008.

[56] T. Weissman, E. Ordentlich, G. Seroussi, S. Verdú, and M.J. Weinberger. Universal discrete denoising: Known channel. *IEEE Transactions on Information Theory*, 51(1):5–28, 2005.

[57] F.M.J. Willems, Y.M. Shtarkov, and T.J. Tjalkens. The context-tree weighting method: Basic properties. *IEEE Transactions on Information Theory*, 41(3):653–664, 1995.

[58] J. Yu and S. Verdú. Schemes for bidirectional modeling of discrete stationary sources. *Information Theory, IEEE Transactions on*, 52(11): 4789–4807, 2006. ISSN 0018-9448.