Multifractal Analysis of Dimensions and Entropies

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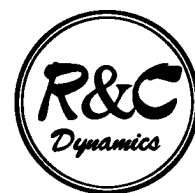
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MULTIFRACTAL ANALYSIS OF DIMENSIONS AND ENTROPIES

The theory of dynamical systems has undergone a dramatical revolution in the 20th century. The beauty and power of the theory of dynamical systems is that it links together different areas of mathematics and physics.

In the last 30 years a great deal of attention was dedicated to a statistical description of strange attractors. This led to the development of notions of various dimensions and entropies, which can be associated to the attractor, dynamical system or invariant measure.

In this paper we review these notions and discuss relations between those, among which the most prominent is the so-called *multifractal formalism*.

In the study of dynamical systems, and especially chaotic systems, sets with extremely complicated geometric structure occur quite naturally. Often these sets are regarded as *fractals*, meaning that these sets have some sort of self-similar structure. However, only few of these sets are truly self-similar. The following situation is more typical: there exists not one, but an infinite number of different so-called scalings which are intrinsic for the set. Such sets are called *multifractals*.

Various quantitative characteristics have been introduced for the global description of attractors and other invariant sets with complicated geometry. Examples of such quantities include various dimensions and entropies. At the same time, many of these global characteristics admit local or pointwise versions. This means that it is possible to define a dimension or an entropy at any given point. These quantities determine certain scalings at these points. What happens in the case of multifractals is the following. The range of possible values of pointwise dimensions (or entropies) is an uncountable bounded set. Often this set is an interval. One can decompose a set into subsets of points where the pointwise quantity (dimension or entropy) is constant. We can also say that each of these sets has a particular type of scaling. Multifractal analysis studies such decompositions, and describes its geometry in terms of so-called multifractal spectra. One of the main aims of the multifractal formalism is to relate these multifractal spectra to the globally defined quantities like dimensions and entropies. The different scalings can also be interpreted as inhomogeneities in the geometry (in the case of dimensions) or the dynamics (in the case of entropies). In this respect, we can say that the multifractal analysis is concerned with such inhomogeneities, and provides their characterization.

Historically the multifractal formalism for dimensions has been studied first; quite recently, a more general setup was introduced, developing a concept of multifractality for a wide set of dynamic characteristics including entropy and Lyapunov exponents. In this paper we will mainly concentrate on the analysis of entropies. In the case of dimensions, we recall existing results and compare them with our results [76, 73, 74, 75, 77, 79] for entropies. A striking difference, which we will emphasize, is

that despite obvious similarities in formal definitions and concepts, rigorous results for the multifractal entropy spectra (see definition below) can be proven under substantially weaker hypothesis.

1. Basic concepts of Dynamical Systems

Dynamics. In this paper we will consider the following 3 types of dynamical systems:

- (measurable dynamics) $f\colon X \to X$ is a measure-preserving transformation of a Lebesgue space $(X, \mathfrak{B}, \mu)$.

- (topological dynamics) $f\colon X \to X$ is a continuous transformation of a compact metric space (X, d).

- (differentiable dynamics) $f\colon M \to M$ is a C^k-diffeomorphism, $k \geqslant 1$, of a smooth manifold M.

In the sequel we will often combine the above settings. For example, the first and the second, or the first and the third.

REMARK 1. It is interesting to mention the relation between different types of the dynamical systems considered above. Firstly, any measure-preserving transformation can be represented in an equivalent (i. e., measure-theoretically isomorphic) way as a continuous transformation of a compact metric space preserving a Borel probability measure [19]. Secondly, any measure-preserving transformation can also be represented as a diffeomorphism of a two-dimensional torus preserving some Borel measure [38]. It is not known if in the latter case, the invariant measure can be made absolutely continuous with respect to the Lebesgue measure.

Attractors. Consider a transformation $f\colon X \to X$, where X is a compact metric space or a smooth manifold, and where accordingly f is a continuous transformation or a diffeomorphism. The *limit set* $\omega(x)$ of $x \in X$ is

$$\omega(x) = \big\{y \in X \colon \exists n_i \to \infty \quad \text{such that} \quad \lim_{i\to\infty} f^{n_i}(x) \to y\big\}.$$

Suppose now that we are given some measure m on X. We will call m the *reference measure*. We do not assume m to be invariant. In the case f is a diffeomorphism of some open domain in $\mathbb{R}^n$, or f is a diffeomorphism of a smooth Riemannian manifold, the Lebesgue measure is typically chosen as a reference measure. We say that a closed set $K \subseteq X$ is an attractor if the set

$$B(K) = \{x \in X \colon \omega(x) \subseteq K\},$$

called the basin of K, has positive m-measure, and there is no strictly smaller closed set $K' \subseteq K$ such that $B(K')$ coincides with $B(K)$ up to a set of measure 0. Attractors can be stationary, periodic, quasi-periodic (tori), or "strange". It is conjectured (see [49]) that a generic diffeomorphism has a finite number of attractors $K_1, \dots, K_N$ which attract almost every point, i. e., $m(X \setminus \cup_j B(K_j)) = 0$. However it is possible that there are no attractors at all (e. g., f is the identity), or there may coexist infinitely many attractors [45].

We have to mention that many other definitions of attractors can be found in the literature, see e. g. [43] for a discussion. The above definition is an attempt to formalize a "naive" understanding of an attractor as a set of points to which a large set of points evolve under iterates of a dynamical system. Another useful property of the above definition is the so-called *indecompossability* property: an attractor can not contain any strictly smaller attractor.

EXAMPLE 1 (SOLENOID). This attractor appeared in the seminal paper [69] by S. Smale in 1967. Consider a solid torus $\mathbf{T}^2 = S^1 \times D = \{\varphi \in \mathbb{R} \mod 1\} \times \{z \in \mathbb{C} \colon |z| \leqslant 1\}$ as a subset of $\mathbb{R}^3$. We define a diffeomorphism f on $\mathbb{R}^3$, mapping the torus $\mathbf{T}^2$ into itself, by the following formula:

$$f(\varphi, z) = (2\varphi, \lambda z + \varepsilon e^{2\pi i\varphi}),$$

where $\lambda, \varepsilon > 0$ are such that $\lambda < \varepsilon$ and $\lambda + \varepsilon < 1$.

The torus $\mathbf{T}^2$ is mapped by f into a solid tube which is wrapped twice inside $\mathbf{T}^2$. Note that the solid torus is expanded in the φ-direction, and is contracted in the z-directions. This, namely, continuous and complementary bundles of contracting and expanding directions, is typical for a *hyperbolic* dynamical system.

The solenoid is defined as the infinite intersection

$$K = \bigcap_{n \geqslant 0} f^n(\mathbf{T}^2).$$

Almost every point with respect to the Lebesgue measure on $\mathbf{T}^2$ belongs to the basin of attraction of K, and K is an attractor in the above sense. However there exists a Lebesgue measure zero set of points x such that $\omega(x) \subsetneqq K$.

We would like to mention that a similar construction, as an example of a specific topological space, has also appeared in the PhD thesis of D. van Dantzig defended in Groningen in 1931.
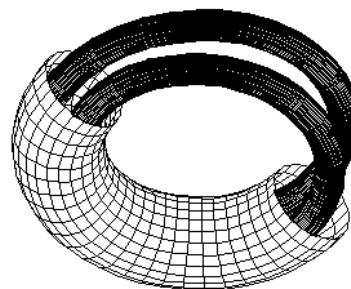
Fig. 1. Solenoid in $\mathbb{R}^3$

A formal definition of hyperbolicity is due to Smale [69], and it clearly applies to the solenoid.

Definition 1. For a given diffeomorphism f of a smooth Riemannian manifold M, a compact set $\Lambda \subseteq M$, which is invariant, i.e. $f(\Lambda) = \Lambda$, is called **hyperbolic** if there exists a continuous decomposition

$$T_x M = E_x^s \bigoplus E_x^u,$$

such that

$$(D_x f) E_x^s = E_{f(x)}^s, \qquad (D_x f) E_x^u = E_{f(x)}^u,$$

and for some constants $C > 0$, $\lambda \in (0, 1)$ (independent of x) the following holds

$$||(D_x f^n) u|| \leqslant C\lambda^n ||u||, \qquad ||(D_x f^{-n}) v|| \leqslant C\lambda^n ||v|| \qquad (1.1)$$

for every $x \in M$, $n \geqslant 0$ and all $u \in E_x^s$ and $v \in E_x^u$.

Other well-known examples of attractors include the *Logistic* and *Hénon* attractors. However, these attractors are not hyperbolic.

Statistical description of attractors. Suppose $K \subseteq X$ is a compact attractor for $f \colon X \to X$ and some reference measure m. We say that a Borel probability measure μ on K is a *Sinai–Ruelle–Bowen* (SRB) measure, (sometimes also called a *natural* or a *physical* measure), if there exists a set $B_0(K) \subseteq B(K)$ of full measure, i.e., $m(B(K) \setminus B_0(K)) = 0$, such that for every continuous function $\varphi \colon X \to \mathbb{R}$ and each $x \in B_0(K)$ one has

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi\big(f^i(x)\big) = \int \varphi \, d\mu.$$

According to our definition, if an SRB measure exists, then it is unique. Nowadays the problem of existence of an SRB measure for various dynamical systems is an important mathematical question.

For hyperbolic attractors existence of the SRB measures was established by Ya. Sinai [67], D. Ruelle [58], R. Bowen and D. Ruelle [12]. Examples of attractors without SRB measures are known as well. A simple example of such sort was suggested by R. Bowen, and is now known as an *eye attractor*.
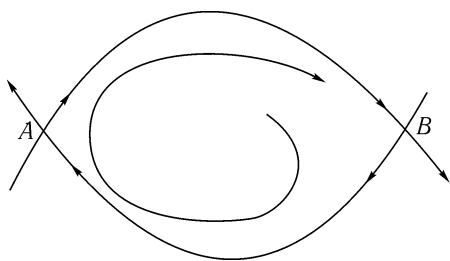
Fig. 2. Bowen's example of an eye attractor

The attractor K consists of two saddle points A, B and two separatrices, connecting them, see Fig. 2. The eigenvalues at the saddle points are assumed to be such that the loop K is attracting. The basin of attraction $B(K)$ is a one-sided neighborhood of the loop. Any trajectory starting in the basin $B(K)$ oscillates between A and B. This oscillation has the following property: for any function φ such that $\varphi(A) \neq \varphi(B)$, one has

$$\varliminf_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1}\varphi(f^i(x)) \neq \varlimsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1}\varphi(f^i(x)).$$

for every $x \in B(K)$. One can actually explicitly identify the lower and upper limits above in terms of eigenvalues at the two saddle points. This example was studied in great details by F. Takens in [72]. It also serves as a source for other abnormal examples, see [4]. Bowen's example has co-dimension 2 and is not persistent under small perturbations. It is also possible to construct examples of attractors without SRB measures in co-dimension 1, see [17]. It is an interesting question if such examples are possible in co-dimension 0, i.e., persistent in some sense.

For partially hyperbolic systems the question of existence of SRB measures has been studied by Ya. B. Pesin and Ya. G. Sinai in [54]. Recently, this investigation has been continued by J. Alves, C. Bonatti, and M. Viana in [2, 9] Also, the existence of SRB measures has been recently established by M. Benidicks and L. S. Young [7] for a large set of parameter values in the Hénon family.

Gibssian approach. In his classical paper Ya. G. Sinai [67] introduced the notion of Gibbs measures (states), inspired by the corresponding concepts in Statistical Physics, into Dynamical Systems. This approach proved to be extremely useful. It turns out that for hyperbolic dynamical systems the SRB measures are Gibbs measures as well. In the sequel we will see that Gibbs measures for certain dynamical systems allow for a complete multifractal analysis. A successful multifractal analysis of Gibbs measures is possible mainly due to the fact that one has a good control over the local structure of such measures. Another useful property of the Gibbs states is that they admit a global description as well; e.g. one can use the Variational Principles and the Large Deviations Principles, proved for a large class of dynamical systems and Gibbs states.

2. Dimensions and attractors

Very soon after the first examples of strange attractors emerged around 1970 it was realized that all these examples have a similar topological structure — a product of a smooth manifold with a Cantor set, see e.g. the collection of papers [68]. It also became apparent that new ways for describing the structure of attractors were needed. In this respect, the notion of Hausdorff dimension, introduced in the first quarter of 20-th century, proved to be of a great interest. The Hausdorff dimension is important for the description of sets, like these strange attractors. Within the further development of the dimension theory of dynamical systems some other dimension-like characteristics were introduced. In this section we recall some of them.

Hausdorff dimension. Consider a metric space (X, d) and some subset $Z \subseteq X$. We say that an at most countable collection of open sets $\mathcal{U} = (U_i)$ is a δ-cover of Z if $\mathrm{diam}(U_i) < \delta$ for all i, and $Z \subseteq \cup_i U_i$. For $s \geqslant 0$ define an s-dimensional Hausdorff measure of Z as follows

$$\mathcal{H}^s(Z) = \lim_{\delta\to 0}\mathcal{H}^s_\delta(Z), \quad \text{where} \quad \mathcal{H}^s_\delta(Z) = \inf_{\mathcal{U}=(U_i)}\sum_i \big(\mathrm{diam}(U_i)\big)^s,$$

and the infimum is taken over all δ-covers $\mathcal{U} = (U_i)$. It is not very difficult to show that there exists a value, $\dim_H(Z) = \hat{s}$, such that

$$\mathscr{H}^s(Z) = \begin{cases} +\infty, & \text{if } s < \dim_H(Z), \\ 0, & \text{if } s > \dim_H(Z). \end{cases} \tag{2.1}$$

The number $\dim_H(Z)$ is called the *Hausdorff dimension* of Z. The following properties of the Hausdorff dimension justify the use of the word dimension:

1) $\dim_H(\varnothing) = 0$, $\dim_H(\mathbb{R}^n) = n$;

2) if $Z_1 \subseteq Z_2$, then $\dim_H(Z_1) \leqslant \dim_H(Z_2)$.

3) if $Z_1, \ldots, Z_k \subseteq X$, then

$$\dim_H\left(\bigcup_{i=1}^k Z_i\right) = \max_{i=1,\ldots,k} \dim_H(Z_i).$$

Note however that not all these properties (e. g. 3) will be enjoyed by some other "dimensions" defined below.

Hausdorff dimension of a measure. The Hausdorff dimension of an attractor K, being a static characteristic, does not take into account the dynamics. For a given dynamical system the SRB measure (if it exists) contains more information about the dynamics than the attractor as an invariant set. It turns out that one can associate various dimensions to measures as well. One of these notions — the Hausdorff dimension of a measure — proved to be extremely useful for the purposes of the statistical description of attractors.

Suppose we are given a probability measure μ with compact support. The Hausdorff dimension of the measure μ is defined as

$$\dim_H(\mu) = \inf\{\dim_H(A): \ \mu(A) = 1\}.$$

Clearly, $\dim_H(\mu) \leqslant \dim_H(\text{supp}\mu)$.

Now we give an example of a compact attractor which supports a lot of invariant measures, and these measures can be distinguished using the notion of the Hausdorff dimension of a measure. We will also use this example when we discuss the multifractal formalism.

EXAMPLE 2 (SKEW TENT MAP). Consider the unit interval $I = [0, 1]$, and for any $\lambda \in (0, 1)$ define $f_\lambda \colon I \to I$ as follows

$$f_\lambda(x) = \begin{cases} \dfrac{x}{\lambda}, & x \in [0, \lambda] \\ \dfrac{1 - x}{1 - \lambda}, & x \in (\lambda, 1]. \end{cases}$$

There is a natural partition $\xi_\lambda = \{I_0, I_1\} = \{[0, \lambda], (\lambda, 1]\}$ associated to the map f_λ. For any $(j_0, \ldots, j_n) \in \{0, 1\}^n$ define a cylinder $I_{j_0, \ldots, j_n}$ by

$$I_{j_0, \ldots, j_n} = I_{j_0} \cap f_\lambda^{-1} I_{j_1} \cap \ldots \cap f_\lambda^{-n} I_{j_n}.$$

Now for any $p \in (0, 1)$ there exist a unique f_λ-invariant measure μ_p such that

$$\mu_p(I_{j_0, \ldots, j_n}) = p^m (1 - p)^{n+1-m},$$

where $m = \#\{k: j_k = 0\}$ — the number of zeros in the symbolic representation of the cylinder $I_{j_0, \ldots, j_n}$. In fact, μ_p is the projection of the Bernoulli (or the product) measure with probabilities $(p, 1 - p)$, defined on the set of all infinite sequences of 0's and 1's, onto the unit interval. The support of the

measure μ_p is the whole interval $[0, 1]$. The Hausdorff dimension of the measure μ_p is given by the following formula [22]:

$$\dim_H(\mu_p) = \frac{p \log p + (1 - p) \log(1 - p)}{p \log \lambda + (1 - p) \log(1 - \lambda)}.$$

The equality $\dim_H(\mu_p) = 1$ is possible if and only if $p = \lambda$. In this case μ_p is the Lebesgue measure on I.

The obvious relation between the Hausdorff dimension of a measure and its support motivates the following definition.

Definition 2. Given a compact set K, we say that a measure μ with $\mathrm{supp}(\mu) \subseteq K$ is a measure of maximal dimension if $\dim_H(\mu) = \dim_H(K)$.

Of course, the problem of finding the measure of maximal dimension for a given compact K does not make too much sense in general. Usually one would like to solve a more specific problem: given a transformation f and an invariant set K, what is the maximal Hausdorff dimension of an invariant measure with a support in K? This is a difficult question about which not much is known. We refer to a survey paper [26] and later work [27, 39].

Pointwise or local dimensions. Consider a probability measure μ on a metric space (X, d). Define the *lower and upper pointwise dimensions*

$$\underline{d}_\mu(x) = \varliminf_{\varepsilon \to 0} \frac{\log \mu(\mathfrak{B}(x, \varepsilon))}{\log \varepsilon}, \quad \text{and} \quad \overline{d}_\mu(x) = \varlimsup_{\varepsilon \to 0} \frac{\log \mu(\mathfrak{B}(x, \varepsilon))}{\log \varepsilon}.$$

If $\underline{d}_\mu(x) = \overline{d}_\mu(x)$ for some point $x \in X$, then we say that a local dimension at the point x exists and denote it by

$$d_\mu(x) = \underline{d}_\mu(x) = \overline{d}_\mu(x).$$

A measure μ is called *exact dimensional* if there exists a constant d such that $d_\mu(x) = \underline{d}_\mu(x) = \overline{d}_\mu(x) = d$ for $\mu - \text{a. e. } x \in X$. There are several known examples of invariant measures which are not exact dimensional, see [16, 40]. Recently it was shown [28] that a typical Borel probability measure on a compact metric space is not exact dimensional. In [36], a generic set of circle diffeomorphisms, admitting a unique invariant measure, was studied. It was shown that for a such diffeomorphism, the corresponding unique invariant measure is not exact dimensional as well. L. Barreira, Ya. B. Pesin and J. Schmeling recently proved [6] the so-called Eckmann–Ruelle conjecture. This conjecture states that for any $C^{1+\alpha}$, $\alpha > 0$, diffeomorphism f of a smooth Riemannian manifold, every hyperbolic measure is exact dimensional. We recall that the measure is called hyperbolic, if it is ergodic and the Lyapunov exponents of f are non-zero μ-almost everywhere.

Now let us come back to the problem of computing the Hausdorff dimension of a given set Z. This is usually a very difficult task. With the exception of a few simple examples (mainly Cantor sets) explicit calculations are not possible. For sets more general than Cantor sets the most powerful and widely used method is based on the so-called *Frostmann lemma*. This approach gives an estimate of the Hausdorff dimension of a set in terms of the pointwise dimensions of some measure concentrated on that set. We recall one result of such nature, proved by L.-S. Young [81].

Theorem 1. *Consider $Z \subseteq X$, and let μ be a finite Borel measure such that $\mu(Z) > 0$.*

1) *If $\underline{d}_\mu(x) \geqslant d$ for μ-a. e. $x \in Z$ then $\dim_H(Z) \geqslant d$.*

2) *If $\overline{d}_\mu(x) \leqslant d$ for all $x \in Z$ then $\dim_H(Z) \leqslant d$.*

Generalized dimensions. One could also associate an infinite number of different dimensions to a given measure. There are two known approaches: one named after Rényi and another suggested by Hentschel–Procaccia.

Rényi dimensions. Suppose μ is a compactly supported measure in $\mathbb{R}^n$. For a given $\varepsilon > 0$ consider a partition of $\mathbb{R}^n$ into boxes of size ε:

$$\mathscr{P}_\varepsilon = \left\{ P_{\mathbf{i}} = \prod_{k=1}^{n} [i_k\varepsilon, (i_k+1)\varepsilon) \colon \ \mathbf{i} = (i_1, \ldots, i_n) \in \mathbb{Z}^n \right\}.$$

Following [34], see also [52], for $q > 1$ define the lower and upper Rényi dimensions of μ as

$$\underline{R}_q(\mu) = \varliminf_{\varepsilon \to 0} \frac{1}{q-1} \frac{\log \sum_{\mathbf{i}} \mu(P_{\mathbf{i}})^q}{\log \varepsilon},$$

$$\overline{R}_q(\mu) = \varlimsup_{\varepsilon \to 0} \frac{1}{q-1} \frac{\log \sum_{\mathbf{i}} \mu(P_{\mathbf{i}})^q}{\log \varepsilon}.$$

The Rényi dimensions show the scaling (with respect to ε) of the $L_{q-1}(\mathbb{R}^n, \mu)$ norm of the following function

$$\xi(x) = \mu(P_{\mathbf{i}}(x)),$$

where $x \in \mathbb{R}^n$ and $P_{\mathbf{i}}(x)$ is the unique cell of the partition $\mathscr{P}$ containing x. The function $\xi(x)$ can be interpreted in the following way: $\xi(x)$ is the probability of the event that a μ-random point y lies in the same cell of our partition $\mathscr{P}_\varepsilon$ as x. Obviously,

$$\sum_{\mathbf{i}} \mu(P_{\mathbf{i}})^q = \int \left(\int \mathbb{I}(x, y \ \ \text{are in the same cell}) \ \ d\mu(y) \right)^{q-1} d\mu(x),$$

where $\mathbb{I} \colon \mathbb{R}^n \times \mathbb{R}^n \to \{0, 1\}$ is the indicator function of the event.

For $q = 1$ we need a different definition. The following definition is suggested by a continuity argument:

$$\underline{R}_1(\mu) = \varliminf_{\varepsilon \to 0} \frac{\sum_{\mathbf{i}} \mu(P_{\mathbf{i}}) \log \mu(P_{\mathbf{i}})}{\log \varepsilon},$$

$$\overline{R}_1(\mu) = \varlimsup_{\varepsilon \to 0} \frac{\sum_{\mathbf{i}} \mu(P_{\mathbf{i}}) \log \mu(P_{\mathbf{i}})}{\log \varepsilon}.$$

The Rényi dimensions of order 1 determine how the entropy of the partition $\mathscr{P}_\varepsilon$ scales with ε.

Hentschel−Procaccia dimensions. The one-parameter family of dimensions suggested in [34] basically uses the same motivation as the Rényi dimensions introduced above. However cubes of size ε are substituted by balls of radius ε, which makes it possible to use these dimensions for spaces more general than $\mathbb{R}^n$. The Hentschel−Procaccia dimension of order q, $q > 1$, is defined as follows ([34, 52]):

$$\underline{HP}_q(\mu) = \varliminf_{\varepsilon \to 0} \frac{1}{1-q} \frac{\log \int \mu(\mathscr{B}(x, \varepsilon))^{q-1} d\mu}{\log \varepsilon},$$

$$\overline{HP}_q(\mu) = \varlimsup_{\varepsilon \to 0} \frac{1}{1-q} \frac{\log \int \mu(\mathscr{B}(x, \varepsilon))^{q-1} d\mu}{\log \varepsilon},$$

$$(2.2)$$

and for $q = 1$

$$\underline{HP}_1(\mu) = \varliminf_{\varepsilon \to 0} \frac{\int \log \mu(\mathscr{B}(x, \varepsilon)) \, d\mu}{\log \varepsilon},$$

$$\overline{HP}_1(\mu) = \varlimsup_{\varepsilon \to 0} \frac{\int \log \mu(\mathscr{B}(x, \varepsilon)) \, d\mu}{\log \varepsilon}.$$

$$(2.3)$$

Basic properties of generalized dimensions. The generalized dimensions defined above are non-increasing functions of the parameter q. One can easily establish that

$$\underline{HP}_q(\mu) \leqslant \underline{R}_q(\mu) \quad \text{and} \quad \overline{HP}_q(\mu) \leqslant \overline{R}_q(\mu)$$

for all $q \geqslant 1$. It was shown in [52] that for all $q > 1$

$$\underline{R}_q(\mu) = \underline{HP}_q(\mu) \quad \text{and} \quad \overline{R}_q(\mu) = \overline{HP}_q(\mu).$$

From now on we will understand $D_\mu(q)$ as one of the generalized dimensions of μ defined above. We prefer the choice of the lower Hentschel–Procaccia dimension, i.e., $D_q(\mu) = \underline{HP}_q(\mu)$. The motivation for this choice is as follows. The Hentschel–Procaccia dimensions can be defined for all metric spaces, while the Rényi dimensions are defined only for $\mathbb{R}^n$. Secondly we would like the generalized dimension $D_q(\mu)$ to be as small as possible, and the lower Hentschel–Procaccia dimension is the minimal among the four generalized dimensions defined above.

Generalized dimensions of order $q < 1$. Without loss of generality we can assume that μ is positive on open sets; otherwise, we restrict μ to its support. Formally, we can use the same formulae, for example (2.2), to define generalized dimensions for $q < 1$. It is, however, possible that for some q_0 the limits in (2.2) are equal to $+\infty$. Due to the monotonicity for all $q < q_0$ the situation will then be the same. In this case, we are left with a choice: either to say that the generalized dimensions of all orders q, $q \leqslant q_0$, do not exist, or to accept the possibility of infinite dimensions. We tend to accept the latter. It is, however, a matter of taste. In all concrete cases, the generalized dimensions will be finite for all $q \in \mathbb{R}$. However examples where D_q becomes infinite, have been studied in the physics literature as well. For a simple example of this see [15]. There are a few other known examples of singular behavior of generalized dimensions D_q with to respect to q, see [15, 46].

Multifractal formalism. The multifractal formalism, first suggested on the heuristic level in [78] and [32, 31], describes the relation between various dimension characteristics of measures. Let us describe this formalism in the form presently accepted in the mathematical literature [52]. Let μ be a Borel probability measure on some separable metric space X. Consider the following decomposition of X in level sets of the function assigning to a point x the pointwise dimensions of μ at x:

$$X = \bigcup_{\alpha \in \mathbb{R}_+} K_\alpha \bigcup K_{no} =$$

$$:= \bigcup_{\alpha \in \mathbb{R}_+} \left\{ x \in X \colon \underline{d}_\mu(x) = \overline{d}_\mu(x) = \alpha \right\} \bigcup \left\{ x \in X \colon \underline{d}_\mu(x) \neq \overline{d}_\mu(x) \right\}.$$

The multifractal dimension spectrum $\mathscr{D}_\mu(\cdot)$, by definition, is the function which assigns to each α the Hausdorff dimension of K_α

$$\mathscr{D}_\mu(\alpha) = \dim_H(K_\alpha).$$

The *domain*, $\mathrm{dom}_{\mathscr{D}}(\mu)$, of the dimension spectrum of μ is the set of all α's with a non-trivial set K_α:

$$\mathrm{dom}_{\mathscr{D}}(\mu) = \left\{ \alpha \geqslant 0 \colon \ K_\alpha \neq \varnothing \right\}.$$

Definition 3. We say that the multifractal formalism is *valid* for the dimension spectrum of μ if

i) there exist $\underline{\alpha}, \overline{\alpha} \geqslant 0$ such that

$$(\underline{\alpha}, \overline{\alpha}) \subseteq \mathrm{dom}_{\mathscr{D}}(\mu) \subseteq [\underline{\alpha}, \overline{\alpha}];$$

ii) for any $\alpha \in (\underline{\alpha}, \overline{\alpha})$ the following is true

$$\mathcal{D}_\mu(\alpha) = \inf_{q \in \mathbb{R}} \big(q\alpha + T(q)\big),$$

where $T(q) = (1 - q)D_\mu(q)$;

iii) the dimension spectrum $\mathcal{D}_\mu(\alpha)$ is a smooth function of α on $(\underline{\alpha}, \overline{\alpha})$;

Let us discuss the meaning of this definition. We require that the domain of the dimension spectrum is a finite interval. It can be open or closed from the right or left. Secondly we require some smoothness of the dimension spectrum. Known examples vary from a real-analytic to a continuous behavior. However for the moment we will understand "smooth" as being at least continuously differentiable. The second condition in the definition says that the dimension spectrum $\mathcal{D}_\mu(\alpha)$ is the *Legendre transform* of $T(q) = (1 - q)D_\mu(q)$. Let us discuss the notion of the Legendre transform in greater details.

Legendre transform. A classical definition of the *Legendre–Fenchel transform* [57] is as follows. Suppose f is a function defined on some interval I which maybe finite or infinite. Its Legendre–Fenchel transform is the function f^* given by

$$f^*(y) = \sup_{x \in I}\big(xy - f(x)\big) = -\inf_{x \in I}(f(x) - xy). \tag{2.4}$$

The Legendre–Fenchel transform $f^*(y)$ is a convex function on its domain $I^* = \{y \colon f^*(y) < +\infty\}$. Moreover, on the class of strictly convex functions the Legendre–Fenchel transform is invertible

$$f(x) = \sup_{y}\big(xy - f^*(y)\big),$$

and is also an involution, i. e., $f^{**} = f$. A pair of functions (f, g) is said to be a Legenre–Fenchel pair if $g = f^*$ and $f = g^*$. Hence, (f, f^*) is such a pair, if f is strictly convex.

In the Definition 3, $\mathcal{D}_\mu(\alpha)$ and $T(q)$ are required to satisfy the equality

$$\mathcal{D}_\mu(\alpha) = \inf_{q \in \mathbb{R}}\big(q\alpha + T(q)\big). \tag{2.5}$$

Formally, $\mathcal{D}_\mu(\alpha)$ is not a Legendre–Fenchel transform of $T(q)$, but the function $F(\alpha) = -\mathcal{D}_\mu(-\alpha)$ satisfies $F(\alpha) = T^*(\alpha)$, where $T^*(\alpha)$ is given by (2.4).

In order to avoid the unnecessary complications, it is an accepted practice in the multifractal literature to use a different definition of the Legendre transform:

$$f^*(y) = \inf_{x}\big(xy + f(y)\big). \tag{2.6}$$

According to such definition, (2.5) can be rewritten as

$$\mathcal{D}_\mu(\alpha) = T^*(\alpha).$$

Everywhere in the sequel, we use (2.6) as the definition of the Legendre transform. Observe also that according to (2.6), transform of a convex functions is a concave function.

Validity of multifractal formalism. It is rather obvious that such a strong statement can not be true in a great generality. Let us start with a simple example where the multifractal formalism is valid. In fact, this example motivated the above definition of multifractal formalism.

EXAMPLE 3 (MULTIFRACTAL ANALYSIS OF THE SKEW TENT MAPS). Let f be a skew tent map with $\lambda = 1/2$ and μ_p for $p \in (0,1)$ be as in Example 2. For simplicity we assume $p > 1/2$. It is not very difficult to see that the pointwise dimensions of μ_p can take any values between $-\log(p)/\log 2$ and $-\log(1-p)/\log 2$. Using a natural partition $\xi = \{I_0, I_1\} = \{[0, 1/2], (1/2, 1]\}$, one can show by a direct computation that the generalized dimensions are

$$D_\mu(q) = -\frac{1}{q-1}\frac{\log(p^q + (1-p)^q)}{\log 2}, \quad q \neq 1,$$

$$D_\mu(1) = -\frac{p\log p + (1-p)\log(1-p)}{\log 2}.$$

The family of generalized dimensions $D_\mu(q)$ depends real-analytically on q.

For every $x \in [0,1]$ there exists a unique infinite sequence $\mathbf{j} = (j_0, j_1, \ldots)$ of zeros and ones such that

$$x = \bigcap_{n \geqslant 0} I_{j_0, \ldots, j_n}.$$

Such sequence $\mathbf{j}$ is called the symbolic representation of x. Define (if the limit exists)

$$l(x) = \lim_{n \to \infty} \frac{1}{n}\#\{k : \ 1 \leqslant k \leqslant n, \ j_k = 0\},$$

i.e., $l(x)$ is the asymptotic frequency of 0's in the symbolic representation of x. One readily checks that $d_{\mu_p}(x)$ exists if and only if $l(x)$ exists, and

$$d_{\mu_p}(x) = -\frac{l(x)\log p + (1-l(x))\log(1-p)}{\log 2}. \tag{2.7}$$

Using some large deviation results from probability theory, one can show [23] that indeed

$$\mathscr{D}_{\mu_p}(\alpha) = \inf_{q \in \mathbb{R}}\big(q\alpha + T(q)\big),$$

where $T(q) = (1-q)D_{\mu_p}(q)$. Hence, the multifractal formalism for μ_p is valid.

EXAMPLE 4 (NON-CONCAVE SPECTRUM OF LOCAL DIMENSION). If we consider mixtures of two measures μ_{p_1}, μ_{p_2} with $p_1 \neq p_2$ from the above example, like

$$\mu = t\mu_{p_1} + (1-t)\mu_{p_2}, \quad t \in (0,1),$$

then for every x such that $d_{\mu_1}(x)$, $d_{\mu_2}(x)$ exist, the pointwise dimension $d_\mu(x)$ exists as well and
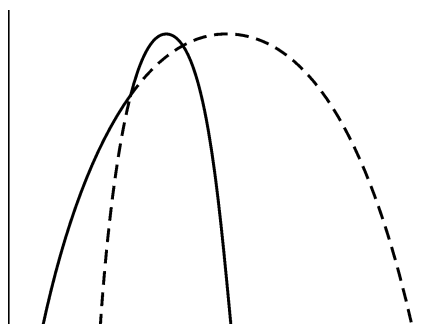
$$d_\mu(x) = \min\{d_{\mu_1}(x), d_{\mu_2}(x)\}.$$

Hence, from (2.7) we conclude that $d_\mu(x)$ exists if and only if $l(x)$ exists. Finally, using the the spectra $\mathscr{D}_{\mu_1}(\alpha)$ and $\mathscr{D}_{\mu_2}(\alpha)$, we obtain the graph of $\mathscr{D}_\mu(\alpha)$, see Fig. 3. Therefore the dimension spectrum of μ is not a concave function, and hence it cannot be a Legendre transform of any function. Thus the multifractal formalism for such measures is not valid.

Fig. 3. Example of a non-concave spectrum $\mathscr{D}_\mu(\alpha)$ (solid line) and the corresponding concave spectra $\mathscr{D}_{\mu_i}(\alpha)$ (dashed lines).

It is also possible to construct a measure for which the domain of the dimension spectrum is a union of two intervals, hence violating the first condition of our definition. For this it is sufficient

to consider a measure which is a mixture of two measures with disjoint domains of their dimension spectra.

An important class of measures is the class of the so-called *diametrically regular* or *doubling* measures. A measure μ is said to satisfy the doubling condition if there exists $A > 1$ such that for every sufficiently small $\varepsilon > 0$ and all x one has

$$\mu(\mathfrak{B}(x, 2\varepsilon)) \leqslant A\mu(\mathfrak{B}(x, \varepsilon)). \tag{2.8}$$

L. Olsen [46] showed that for *any* measure μ, satisfying the doubling condition (2.8), the dimension spectrum $\mathscr{D}_\mu(\alpha)$ is bounded above by the Legendre transform of $T(q) = (1 - q)D_\mu(q)$.

Dimension spectrum for Gibbs measures, invariant under hyperbolic surface diffeomorphisms. The above example 3 of a successful multifractal analysis of an invariant measure for one-dimensional expanding maps, raises a question: can the same be done in higher dimensions? The following result, due to D. Simpelaere [65, 66] and Ya. B. Pesin & H. Weiss [50, 51], shows that this is indeed the case, provided the dimension of the manifold is 2, and the dynamical system is hyperbolic.

Theorem 2. *Let $f\colon M \to M$ be a diffeomorphism of a compact smooth Riemannian surface M and Λ be a closed hyperbolic set for f. Let $\varphi\colon \Lambda \to \mathbb{R}$ be a Hölder continuous function and let μ be the corresponding Gibbs measure. Then the multifractal formalism is valid for the dimension spectrum of μ. Moreover, the dimension spectrum $\mathscr{D}_\mu(\alpha)$ is real-analytic on $(\underline{\alpha}, \overline{\alpha})$.*

There are no results of such sort for diffeomorphisms on manifolds of higher dimensions.

One has to mention that successful multifractal analysis in higher dimensions is possible if one assumes that the dynamics is given by an *expanding conformal* map [50, 51] , i.e., a map $f\colon M \to M$ with

$$D_x f = a(x)I(x),$$

where $a(x)\colon M \to \mathbb{R}$ is a Hölder continuous function, such that $a(x) \geqslant \gamma > 1$ for some γ and all x, and $I(x)$ is an isometry at every x. However, these assumptions, though multi-dimensional by nature, effectively reduce the problem to the one-dimensional setting considered above.

3. Entropy in Dynamical Systems

The notion of topological entropy for continuous transformations of compact metric spaces was introduced by R. Adler, A. G. Konheim and M. H. McAndrew in [1], later R. Bowen gave an equivalent definition using a different approach. One of the fundamental results, the so-called *Variational Principle*, establishes the equality between the topological entropy and the supremum of measure-theoretic (or Kolomogorov–Sinai) entropies over all invariant measures. The Variational Principle was first proved by E. I. Dinaburg [20] in a finite dimensional setting, and later generalized by W. L. Goodwyn [30] to any compact metric space; M. Misiurewicz [44] gave a short and very elegant proof.

It became apparent that the topological entropy is in fact a dimension-like characteristic of a dynamical system. R. Bowen [11] gave a definition of topological entropy of any set, not necessarily invariant or compact, along the lines of a standard definition of Hausdorff dimension, see (2.1). Later Ya. B. Pesin and B. S. Pitskel' [53] generalized this definition even further, allowing now the notion of topological pressure to be extended to a non-compact setting.

We recall another equivalent definition of topological entropy which can be found in [52]. Now we come to the formal definitions.

Topological entropy. Once again, let (X, d) be a compact metric space, and let $f\colon X \to X$ be a continuous transformation. For any $n \in \mathbb{N}$ we define a new metric d_n on X as follows:

$$d_n(x, y) = \max\{d(f^k(x), f^k(y))\colon\ k = 0, \ldots, n-1\},$$

and for every $\varepsilon > 0$ we denote by $\mathfrak{B}_n(x, \varepsilon)$ an open ball of radius ε in the metric d_n around x, i.e.,

$$\mathfrak{B}_n(x, \varepsilon) = \{y \in X : \ d_n(x, y) < \varepsilon\}.$$

Suppose we are given some set $Z \subseteq X$. Fix $\varepsilon > 0$. We say that an at most countable collection of balls $\Gamma = \{\mathfrak{B}_{n_i}(x_i, \varepsilon)\}_i$ covers Z if

$$Z \subseteq \bigcup_i \mathfrak{B}_{n_i}(x_i, \varepsilon).$$

For $\Gamma = \{\mathfrak{B}_{n_i}(x_i, \varepsilon)\}_i$ put $n(\Gamma) = \min_i n_i$. Let $s \geqslant 0$ and define

$$M(Z, s, N, \varepsilon) = \inf_{\substack{\Gamma \text{ covers } Z \\ n(\Gamma) \geqslant N}} \sum_i e^{-sn_i}.$$

The quantity $M(Z, s, N, \varepsilon)$ does not decrease with N, hence the following limit exists

$$M(Z, s, \varepsilon) = \lim_{N \to \infty} M(Z, s, N, \varepsilon) = \sup_{N > 0} M(Z, s, N, \varepsilon).$$

It easy to show that there exists a critical value of the parameter s, which we will denote by $h_{\text{top}}(f, Z, \varepsilon)$, where $M(Z, s, \varepsilon)$ jumps from $+\infty$ to 0, i.e.,

$$M(Z, s, \varepsilon) = \begin{cases} +\infty, & s < h_{\text{top}}(f, Z, \varepsilon), \\ 0, & s > h_{\text{top}}(f, Z, \varepsilon). \end{cases}$$

As usual, there are no restrictions on the value $M(Z, s, \varepsilon)$ for $s = h_{\text{top}}(f, Z, \varepsilon)$. It can be infinite, zero, or positive and finite.

Local (pointwise) entropy, the Brin−Katok formula. Consider a compact metric space (X, d). Let $f : X \to X$ be a continuous map and μ an invariant non-atomic Borel measure. Without loss of generality we may assume that μ is positive on open sets. In this case we define the lower (upper) local (pointwise) entropies as follows:

$$\underline{h}_\mu(f, x) = \lim_{\varepsilon \to 0} \varliminf_{n \to \infty} -\frac{1}{n} \log \mu(\mathfrak{B}_n(x, \varepsilon)), \tag{3.1}$$

$$\overline{h}_\mu(f, x) = \lim_{\varepsilon \to 0} \varlimsup_{n \to \infty} -\frac{1}{n} \log \mu(\mathfrak{B}_n(x, \varepsilon)). \tag{3.2}$$

Note that the limits in ε exist due to monotonicity.

We say that the local entropy exists at x if

$$\underline{h}_\mu(f, x) = \overline{h}_\mu(f, x). \tag{3.3}$$

In this case the common value will be denoted by $h_\mu(f, x)$.

The following well-known result establishes the existence of the local entropies.

Theorem 3 (Brin−Katok formula, [13]). *Let $f : X \to X$ be a continuous map on a compact metric space (X, d) preserving a non-atomic Borel measure μ, then*

1) *for μ-a. e. $x \in X$ the local entropy exists, i. e.,*

$$h_\mu(f, x) = \underline{h}_\mu(f, x) = \overline{h}_\mu(f, x);$$

2) *$h_\mu(f, x)$ is an f-invariant function of x, and*

$$\int h_\mu(f, x) \, d\mu = h_\mu(f),$$

where $h_\mu(f)$ is the measure-theoretic entropy of f.

REMARK 2. An invariant measure μ is called ergodic if every invariant set has measure zero or one. If μ is ergodic, then every invariant function is constant almost surely. Hence if μ is ergodic, then $h_\mu(f, x) = h_\mu(f)$ for μ-a. e. $x \in X$.

Generalized entropies. Similar to the case of generalized dimensions, see Section 2, two families of generalized entropies were proposed. First we consider the Rényi entropies.

Rényi entropies. A formal definition of the Rényi entropy of order q [21, 75] goes along the lines of the standard definition of the measure-theoretic entropy, except that Shannon's information function:

$$I(\mathbf{p}) = -\sum_i p_i \log p_i,$$

defined on a probability vectors $\mathbf{p} = (p_1, \ldots, p_n)$, $p_i \geqslant 0$, $\sum_i p_i = 1$, is replaced by a more general Rényi's information of order q, $q \neq 1$,

$$I_q(\mathbf{p}) = \frac{1}{1-q} \log \sum_i p_i^q.$$

Note that for $\mathbf{p}$ given, $I_q(\mathbf{p})$, with $I_1(\mathbf{p}) = I(\mathbf{p})$, depends continuously on q. The Rényi entropies of the measure-preserving dynamical systems are *measure-theoretic invariants*, i. e., two systems, isomorphic in the measure-theoretic sense, must have the same Rényi entropies of all orders. For ergodic dynamical systems these invariants do not contain any new information compared to the standard Kolmogorov–Sinai entropy, see [75, 80].

Theorem 4 ([75, 80]). *For an ergodic dynamical system (X, μ, f), the Rényi entropies are given by the following formula*

$$h_\mu(f, q) = \begin{cases} +\infty, & q < 1, \\ h_\mu(f), & q \geqslant 1. \end{cases} \tag{3.4}$$

However if $X = \mathbb{R}^n$, consideration of only the cubic partitions (like $\mathscr{P}_\varepsilon$ above) might provide additional information about the dynamics. Nevertheless, if one wishes the Rényi entropies to be measure-theoretic invariants, then one has to consider *all* possible partitions, arriving to (3.4), see [75, 80].

For non-ergodic dynamical systems the answer is slightly different [74], but nevertheless, the Rényi entropies are not very informative from a purely measure-theoretic point of view.

Correlation entropies. Another family of generalized entropies was proposed by F. Takens [71] as an entropy analogue of the Hentschel–Procaccia spectrum of generalized dimensions.

Let (X, d) be a compact metric space, and $f : X \to X$ a continuous transformation preserving a Borel probability measure μ. Without loss of generality we may assume that μ is positive on open sets. For each $q \in \mathbb{R}$, $q \neq 1$, we define the *lower and upper correlation entropies of order q* as follows

$$\underline{H}_\mu(f, q) = \lim_{\varepsilon \to 0} \varliminf_{n \to \infty} -\frac{1}{(q-1)n} \log \int \mu(\mathfrak{B}_n(x, \varepsilon))^{q-1} \, d\mu,$$

$$\overline{H}_\mu(f, q) = \lim_{\varepsilon \to 0} \varlimsup_{n \to \infty} -\frac{1}{(q-1)n} \log \int \mu(\mathfrak{B}_n(x, \varepsilon))^{q-1} \, d\mu,$$

and for $q = 1$:

$$\underline{H}_\mu(f, 1) = \lim_{\varepsilon \to 0} \varliminf_{n \to \infty} -\frac{1}{n} \int \log \mu(\mathfrak{B}_n(x, \varepsilon)) \, d\mu,$$

$$\overline{H}_\mu(f, 1) = \lim_{\varepsilon \to 0} \varlimsup_{n \to \infty} -\frac{1}{n} \int \log \mu(\mathfrak{B}_n(x, \varepsilon)) \, d\mu.$$

If for some $q \in \mathbb{R}$ one has $\underline{H}_\mu(f, q) = \overline{H}_\mu(f, q)$, we say that the correlation entropy $H_\mu(f, q)$ exists and let

$$H_\mu(f, q) = \underline{H}_\mu(f, q) = \overline{H}_\mu(f, q).$$

Let us recall some elementary properties of correlation entropies (see [75, 76]):

1) for every $q_1 < q_2$ one has

$$\underline{H}_\mu(f, q_1) \geqslant \underline{H}_\mu(f, q_2) \geqslant 0, \quad \overline{H}_\mu(f, q_1) \geqslant \overline{H}_\mu(f, q_2) \geqslant 0;$$

2) $\underline{H}_\mu(f, 1) = \overline{H}_\mu(f, 1) = h_\mu(f)$, where $h_\mu(f)$ is the measure-theoretic entropy;

3) for $q > 1$ one has

$$\underline{H}_\mu(f, q) \leqslant \overline{H}_\mu(f, q) \leqslant h_\mu(f);$$

4) for $q \in [0, 1]$ one has

$$h_\mu(f) \leqslant \underline{H}_\mu(f, q) \leqslant \overline{H}_\mu(f, q) \leqslant h_{\text{top}}(f),$$

where $h_{\text{top}}(f)$ is the topological entropy of f;

5) $\underline{H}_\mu(f, q)$ and $\overline{H}_\mu(f, q)$ depend continuously on q for $q \in [0, 1)$ and $q \in (1, \infty)$.

Multifractal entropy spectrum. Similar to the case of dimensions, consider the following decomposition of X:

$$X = \bigcup_{\alpha \in \mathbb{R}_+} K_\alpha \bigcup K_{no} =$$

$$: = \bigcup_{\alpha \in \mathbb{R}_+} \left\{ x \in X \colon \underline{h}_\mu(f, x) = \overline{h}_\mu(f, x) = \alpha \right\} \bigcup \left\{ x \in X \colon \underline{h}_\mu(f, x) \neq \overline{h}_\mu(f, x) \right\}.$$

A multifractal entropy spectrum $\mathcal{E}_\mu(\cdot)$ assigns to each α the topological entropy of K_α

$$\mathcal{E}_\mu(\alpha) = h_{\text{top}}(f, K_\alpha).$$

The domain of the entropy spectrum $\text{dom}_\mathcal{E}(\mu)$, similar to the dimension case, is

$$\text{dom}_\mathcal{E}(\mu) = \left\{ \alpha \colon K_\alpha \neq \varnothing \right\}.$$

Multifractal formalism for the entropy spectrum. The multifractal formalism for local entropies was first suggested on a heuristic level in [48, 29], a rigorous definition and the first results were obtained by L. Barreira, Ya. B. Pesin and J. Schmeling in [5].

As it was in the case of the dimension spectrum, we say that a multifractal formalism is valid for the entropy spectrum $\mathcal{E}_\mu(\alpha)$ if

i) there exist $\underline{\alpha}, \overline{\alpha} \geqslant 0$ such that

$$(\underline{\alpha}, \overline{\alpha}) \subseteq \text{dom}_\mathcal{E}(\mu) \subseteq [\underline{\alpha}, \overline{\alpha}];$$

ii) for any $\alpha \in (\underline{\alpha}, \overline{\alpha})$ the following is true

$$\mathcal{E}_\mu(\alpha) = \inf_{q \in \mathbb{R}} \left(q\alpha + T(q) \right),$$

where $T(q) = (1 - q)H_\mu(q)$;

iii) the entropy spectrum $\mathcal{E}_\mu(\alpha)$ is a smooth function of α on $(\underline{\alpha}, \overline{\alpha})$.

Entropy spectrum for Gibbs measures. The first result on the validity of the multifractal formalism for the entropy spectrum in the case of Gibbs measures was obtained in [5]. The main interest of [5] was a multifractal analysis of the dimension spectrum, the result for the entropy spectrum was obtained as a corollary of the corresponding result for the dimension spectrum, and hence, is valid under the conditions of Theorem 2, i.e., in dimension 2.

In [77] we considered the entropy spectrum independent from the dimension spectrum and obtained the following result.

Theorem 5. *Let $f\colon M \to M$ be a diffeomorphism and Λ be a closed hyperbolic set for f. Let $\varphi\colon \Lambda \to \mathbb{R}$ be a Hölder continuous function and μ be the corresponding Gibbs state. Then the multifractal formalism is valid for the entropy spectrum $\mathcal{E}_\mu$.*

In fact this result can be proven under much less restrictive conditions on the transformation f. It can be shown that the same result is true for expansive homeomorphism with the specification property (see definition 4 below) and Gibbs measures. In [77] it was shown that $\mathcal{E}_\mu(\alpha)$ is continuously differentiable in α, $\alpha \in (\underline{\alpha}, \overline{\alpha})$; recall that the dimension spectrum of Gibbs measures invariant under 2-dimensional hyperbolic diffeomorphism is real-analytic.

As was done in the case of dimensions by considering mixtures of measures for which the multifractal formalism is valid, we can easily obtain new measures with non-concave entropy spectra, and hence the multifractal formalism can not be valid for such measures. Nevertheless it still makes sense to look at the Legendre transform of $T(q) = (1 - q)H_\mu(q)$, since under some mild conditions on the measure this Legendre transform gives an upper estimate for the spectrum of local entropies.

Theorem 6 ([76]). *Let f be a continuous transformation of a compact metric space (X, d) with finite topological entropy. Consider an invariant probability non-atomic Borel measure μ, which satisfies the following condition, called the weak entropy doubling condition: for every sufficiently small $\varepsilon > 0$ one has*

$$\lim_{n \to \infty} \frac{1}{n} \log \sup_x \frac{\mu(\mathfrak{B}_n(x, \varepsilon))}{\mu(\mathfrak{B}_n(x, \varepsilon/2))} = 0. \tag{3.5}$$

Then there exist $\underline{\alpha}, \overline{\alpha} \in \mathbb{R}$ such that

i) $K_\alpha = \varnothing$ for $\alpha \in\!\!\!/ \, [\underline{\alpha}, \overline{\alpha}]$;

ii) for $\alpha \in (\underline{\alpha}, \overline{\alpha})$ one has

$$\mathcal{E}(\alpha) = h_{top}(f, K_\alpha) \leqslant \inf_q (q\alpha + T(q)),$$

where $T(q) = (1 - q)H_\mu(f, q)$.

4. Expansiveness and specification

In this section we discuss in greater details the properties of dynamical systems satisfying expansiveness and specification. These notions are not as widely used in Dynamical Systems as for example the notion of hyperbolicity. In what follows we stress similarities and differences between systems with these properties, and the more traditional hyperbolic dynamical systems.

Specification. The notion of specification was introduced to dynamical systems by R. Bowen in [10]. We recall now the definition of specification.

Definition 4. We say that a homeomorphism $f\colon X \to X$ has the specification property (abbreviated to f satisfies specification) if for each $\varepsilon > 0$ there exists an integer $m = m(\varepsilon)$ such that the following holds:

if

a) $I_1, \ldots, I_k$ are intervals of integers, $I_j \subseteq [a, b]$ for some $a, b \in \mathbb{Z}$ and all j,

b) $\mathrm{dist}(I_i, I_j) \geqslant m(\varepsilon)$ for $i \neq j$,

then for arbitrary $x_1, \ldots, x_k \in X$ there exists a point $x \in X$ such that

1) $f^{b-a+m}(x) = x$,

2) $d(f^n(x), f^n(x_j)) < \varepsilon$ for $n \in I_j$.

The meaning of this definition is as follows: given $\varepsilon > 0$ and any finite number of pieces of orbits, sufficiently separated in time, one can find a periodic point, which ε-shadows the specified pieces of orbits.

Various modifications of the above definition appeared in the literature as well. Among the most important are the following:

1) [64, 19] Specification is restricted to 2 intervals, i.e., $k = 2$.

2) [60, 33] Assumption that the shadowing point x is periodic is dropped.

3) [42] The switching time $m = m_\varepsilon$ is no longer assumed uniform: it can depend on the lengths of intervals $I_1, \ldots, I_k$, but not on the points $x_1, \ldots, x_k$. Moreover $m_\varepsilon(\cdot)$ must be small compare to the $\max(|I_1|, \ldots, |I_k|)$, in the sense that

$$\lim_{\max(|I_1|, \ldots, |I_k|) \to \infty} \frac{m_\varepsilon(|I_1|, \ldots, |I_k|)}{\max(|I_1|, \ldots, |I_k|)} = 0.$$

It is well-known that Axiom A (and therefore Anosov) systems satisfy specification, e.g. see [37]. In [42] B. Marcus introduced the above variant 3 and showed that ergodic (not necessarily hyperbolic!) toral automorphisms satisfy this version of weak specification.

The specification property is rather strong. Transformations with specification are *topologically mixing*, i.e., for any two open sets U, V one has $T^{-n}U \cap V \neq \varnothing$ for all sufficiently large n. Moreover, for continuous transformations of compact metric spaces with specification, measures concentrated on periodic points are dense in the set of all invariant measures, see e.g. [19].

Surprisingly for continuous transformations on the interval the following holds.

Theorem 7 ([8]). *A continuous transformation of the interval satisfies specification if and only if it is topologically mixing.*

Topological mixing is not exceptional even for non-hyperbolic systems. For example, from the results by M. Jakobson [35] we know that in the logistic family $f_r(x) = rx(1 - x)$ topological mixing holds for a set of parameters of positive Lebesgue measure.

However the next result shows that Theorem 7 is not true for piecewise continuous transformations of the interval.

Theorem 8 ([14, 62]). *Consider a family of piecewise monotonic transformations $\{f_\beta\}$, $\beta > 1$, called β-shifts, and given by*

$$f_\beta(x) = \beta x \mod 1.$$

Then the set of parameters β for which f_β satisfies specification is dense, has Lebesgue measure 0 and Hausdorff dimension 1.

The so-called *Markov property* of piecewise continuous transformations of the interval is often used in literature, and is, for example, crucial for the existence of ergodic absolutely continuous invariant

measures in the case of expanding maps. The Markov property consists of the following. Consider a piecewise monotonic transformation f of the interval I and let $\{I_1, \ldots, I_M\}$ be the corresponding partition into the intervals of monotonicity. Then f is said to have a Markov property if for every $j = 1, \ldots, M$, the image $f(I_j)$ is a union of some I_k's.

The above result on the specification property of β-shifts should be compared to another well-known fact (see [14, 62]): for the family $\{f_\beta\}$ as above, the set of parameter values for which f_β has a *Markov property* is at most countable, and hence has measure and the Hausdorff dimension zero. Thus we can say that for a family of β-shifts, the specification property is more typical than the Markov property.

It is also interesting to mention that J. Schmeling formulated the following conjecture.

Conjecture. In the family of β-shifts $\{f_\beta\}$, $\beta > 1$, weak specification in the sense of Marcus (variant 3 above) is a generic property in the measure-theoretic sense, i. e., it holds for Lebesgue almost all $\beta > 1$.

Expansiveness. Another important property of a dynamical system which is closely related to the sensitive dependence on initial conditions is the so-called expansiveness.

Definition 5. An invertible transformation $f\colon X \to X$ of a metric space (X, d) is called *expansive* if there exists a constant $\gamma > 0$ such that for any $x, y \in X$, $x \neq y$, there exists $n \in \mathbb{Z}$ with

$$d(f^n(x), f^n(y)) \geqslant \gamma. \tag{4.1}$$

If $f\colon X \to X$ is not invertible, we say that f is *positively expansive* if there exists $\gamma > 0$ such that for any $x, y \in X$, $x \neq y$, one can find $n \in \mathbb{Z}_+$ satisfying (4.1).

This definition corresponds to the intuitive understanding of chaotic dynamics: different points, eventually, will be separated by the dynamics. A similar, but stronger, property can be formulated:

Definition 6. A map $f\colon X \to X$ on a metric space (X, d) is called (*locally*) *expanding* if there exists $\lambda > 1$ and $\varepsilon_0 > 0$ such that for all $x, y \in X$, with $d(x, y) < \varepsilon_0$, one has

$$d(f(x), f(y)) \geqslant \lambda d(x, y).$$

It is clear that expanding transformations are positively expansive. One could also easily construct examples of positively expansive, but not expanding transformations. Nevertheless, these notions are somewhat equivalent, as the next statement shows.

Theorem 9 ([56]). *Suppose (X, d) is a compact metric space, and $f\colon X \to X$ is a continuous transformation which is positively expansive. Then there exists a metric d' on X compatible with d (i. e., the topology generated on X by d and d' are the same) such that f is expanding on (X, d').*

REMARK 3. The above result states that there are actually no truly positively expansive, but not expanding transformations: one can always get expanding dynamics by considering a different metric. However the new metric d' might not be *equivalent* to the original metric d. This might cause some problems. For example, it is well known, that the topological pressure $P(f, \varphi)$ of a continuous transformation $f\colon X \to X$ and a continuous function $\varphi\colon X \to \mathbb{R}$, though defined using the metric, is independent of this metric. Nevertheless the metric plays an important role in the question of existence and uniqueness of Gibbs states. One usually restricts oneself to a certain class of potentials which is strictly smaller than the space of all continuous functions. A typical example is the class of all Hölder continuous functions. It is not clear that such a class of functions may not be preserved when the metric is changed in a compatible, but nonequivalent way.

Markov partitions are widely used in dynamical systems, see e. g. [37]. However results, establishing existence of finite Markov partitions, are mainly known for hyperbolic dynamical systems. In this respect the following result might be of some interest.

Theorem 10 ([18]). *Let $f\colon X \to X$ is a continuous transformation of a compact metric space. Suppose that f is positively expansive and open, i. e. open sets are mapped to open sets by f. Then there exists a finite Markov partition of arbitrarily small diameter.*

The overall picture for expansive homeomorphisms is slightly different. This is not very surprising: two directions in time make possible a more intricate geometric behavior. Nevertheless, results similar to Theorems 9, 10 were established for expansive systems as well. Before we proceed with further exposition, we would like to recall some definitions.

Definition 7. For a homeomorphism f of a compact metric space (X, d) we define *local stable* and *unstable sets* of $x \in X$ as follows: for $\varepsilon > 0$ put

$$W_f^{s,\varepsilon}(x,d) = \big\{y \in X\colon\ d(f^n(x), f^n(y)) \leqslant \varepsilon \quad \text{for all}\quad n \geqslant 0\big\},$$
$$W_f^{u,\varepsilon}(x,d) = \big\{y \in X\colon\ d(f^{-n}(x), f^{-n}(y)) \leqslant \varepsilon \quad \text{for all}\quad n \geqslant 0\big\}.$$

The following result of K. Sakai [61] shows that by changing metric we can achieve "usual" hyperbolic behavior on stable and unstable sets.

Theorem 11 ([61]). *Let f be an expansive homeomorphism of a compact metric space (X, d). Then there exists a compatible metric $\bar{d}$ on X, $\lambda \in (0, 1)$ and $\varepsilon > 0$ such that for each $x \in X$ and any $n \geqslant 0$ one has:*

$$\bar{d}(f^n(x), f^n(y)) \leqslant \lambda^n \bar{d}(x,y) \quad \text{for all}\ \ y \in W_f^{s,\varepsilon}(x, \bar{d}), \tag{4.2}$$
$$\bar{d}(f^{-n}(x), f^{-n}(y)) \leqslant \lambda^n \bar{d}(x,y) \quad \text{for all}\ \ y \in W_f^{u,\varepsilon}(x, \bar{d}). \tag{4.3}$$

Estimates (4.2), (4.3) are analogous to estimates (1.1) for hyperbolic diffeomorphisms. Nevertheless these estimates by themselves are not enough to guarantee the existence of Markov partitions. For this we have to require more: namely, the so-called *local product structure* defined below. Homeomorphisms satisfying (4.2), (4.3) and having a local product structure appeared in the literature under different names. For example, in [59] D. Ruelle calls such homeomorphisms together with the underlying space X a *Smale space*, in [41] R. Mañé calls them *hyperbolic homeomorphisms*, and in [3] such homeomorphisms are called *topological Anosov homeomorphisms* (TAH). All these notions were proved to be equivalent, see [47].

Definition 8 ([41]). A homeomorphism f of (X, d) is called *hyperbolic*, if there exist $\varepsilon > 0$, $\lambda \in (0, 1)$ such that

1) on $W_f^{s,\varepsilon}(x,d)$, $W_f^{u,\varepsilon}(x,d)$ the estimates (4.2), (4.3) are true;

2) (local product structure) for any $x, y \in X$ with $d(x,y) < \varepsilon$, one has

$$\#\Big(W_f^{s,\varepsilon}(x,d) \cap W_f^{u,\varepsilon}(y,d)\Big) = 1.$$

Hyperbolic homeomorphisms admit finite Markov partitions. Moreover, in complete analogy with Anosov systems, hyperbolic homeomorphisms are structurally stable. One has to say that there are examples of diffeomorphisms which are not Anosov, but are topological Anosov homeomorphisms, i. e., hyperbolic homeomorphisms [3].

Also, for a hyperbolic homeomorphism the topological pressure $P(f, \cdot)\colon C^\alpha(X) \to \mathbb{R}$, where $C^\alpha(X)$ is the set of Hölder continuous functions with Hölder constant α, is real-analytic [59]. The latter becomes important for possible applications to the multifractal analysis.

We have seen (Theorem 10) that for positively expansive maps the fact that the map is open was sufficient for the existence of a finite Markov partition. For expansive homeomorphisms one has to require more. J. Ombach in [47] showed that for expansive homeomorphisms the *pseudo-orbit tracing property* (POTP) is equivalent to the local product structure, i. e., the second condition in the Definition 8. We recall the definition of the POTP.

Definition 9. A sequence of points $\{x_n\}_{n\in\mathbb{Z}}$ is called a δ-pseudo orbit if

$$d(f(x_n), x_{n+1}) \leqslant \delta \quad \text{for all } n \in \mathbb{Z}.$$

We say that a homeomorphism f has a pseudo-orbit tracing property (also called shadowing) if for every $\varepsilon > 0$ there exists $\delta > 0$ such that any δ-orbit can be ε-shadowed by some orbit, that is, there exists $x \in X$ satisfying

$$d(x_n, f^n(x)) \leqslant \varepsilon \quad \text{for all } n \in \mathbb{Z}.$$

Combining the results by K. Sakai (Theorem 11) and J. Ombach [47] we obtain the following statement.

Theorem 12. *Let f be a homeomorphism of a compact metric space (X, d). Then the following conditions are equivalent*:

1) *f is expansive and has the pseudo-orbit tracing property*;

2) *f is a hyperbolic homeomorphism on $(X, \bar{d})$ for some compatible metric $\bar{d}$.*

Expansiveness and positive expansiveness are rather strong properties of dynamical systems. There exist no expansive homeomorphisms on 1-dimensional manifolds. If a 2-dimensional manifold M admits an expansive homeomorphism f, then M must be a two-torus $\mathbb{T}^2$ and f is conjugated to an Anosov diffeomorphism. Also, on some 3-dimensional manifolds a similar classification is possible.

5. Concluding remarks

In this paper we have considered two multifractal spectra: for local dimensions and entropies, and compared results on the validity of the multifractal formalism. We observed that the results for entropies are valid under milder assumptions on the underlying dynamics, when compared to the corresponding results for dimensions.

The main problem in establishing the results on the validity of the multifractal formalism for local dimensions for higher dimensional dynamical systems, lies in the following unpleasant, and not yet understood, phenomenon [24, 25, 55, 70]: There exist examples of families of hyperbolic dynamical systems in dimensions 3 and higher such that the dimension of the attractors depends in a discontinuous way on the parameters. Moreover, the set of parameters can be decomposed into two sets. The first one is a large set of parameters, for which the dimension of the corresponding attractor is given by the same formula (the so-called Bowen–Ruelle equation) as it was for two dimensional systems. We may say that for these parameters, the dimension of the attractor is equal to what we have expected. The second set is complementary to the first set, is much smaller, and for parameters from this set, the dimension of the attractor is strictly smaller than the number, predicted by the Bowen–Ruelle equation. We have to stress that the Bowen–Ruelle equation is fundamental in the present treatment of the dimensions of attractors and the multifractal analysis of invariant measures.

One can ask, however, about the validity of the multifractal formalism not for all systems, but for those corresponding to a set of "good" parameters. Some results in this direction were obtained in [63]. However at the present moment we are quite far from answering this question.

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