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Symmetry-preserving regularization of wall-bounded turbulent flows

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Abstract.
   The incompressible Navier-Stokes equations constitute an excellent mathematical modelization of turbulence. Unfortunately, attempts at performing direct simulations are limited to relatively low-Reynolds numbers because of the almost numberless small scales produced by the non-linear convective term. Alternatively, a dynamically less complex formulation is proposed here. Namely, regularizations of the Navier-Stokes equations that preserve the symmetry and conservation properties exactly. To do so, both convective and diffusive term are altered in the same vein. In this way, the convective production of small scales is effectively restrained whereas the modified diffusive term introduces an hyper-viscosity effect and consequently enhances the destruction of small scales. In practice, the only additional ingredient is a self-adjoint linear filter whose local filter length is determined from the requirement that vortex-stretching must stop at the smallest grid scale. To do so, a new criterion based on the invariants of the local strain tensor is proposed here. Altogether, the proposed method constitutes a parameter-free turbulence model.

1. Introduction

The incompressible Navier-Stokes (NS) equations form an excellent mathematical model for turbulent flows. In primitive variables the equations are

$$\partial_t u + \mathcal{C}(u, u) = \mathcal{D}u - \nabla p ; \qquad \nabla \cdot u = 0 \tag{1}$$

where u denotes the velocity field, p represents the pressure, the non-linear convective term is defined by $\mathcal{C}(u, v) = (u \cdot \nabla) v$, and the diffusive term reads $\mathcal{D}u = (1/Re)\Delta u$, where Re is the Reynolds number.

   Preserving the symmetries of the continuous differential operators when discretizing them has been shown to be a very suitable approach for direct numerical simulation (DNS) (see the work by Verstappen & Veldman (2003), for instance). Doing so, certain fundamental properties such as the inviscid invariants - kinetic energy, enstrophy (in 2D) and helicity (in 3D) - are exactly preserved in a discrete sense. However, direct simulations at high Reynolds numbers are not feasible because the convective term produces far too many relevant scales of motion. Therefore, a dynamically less complex mathematical formulation is needed. In the quest for

such a formulation, we consider regularizations (Guermond *et al.*, 2004; Geurts & Holm, 2003; Guermond & Prudhomme, 2005) of the nonlinearity. The first outstanding approach in this direction goes back to Leray (1934). The Navier-Stokes-α model also forms an example of regularization modeling (see the works by Geurts & Holm (2003) and Guermond *et al.* (2003), for instance). The regularization methods basically alter the convective terms to reduce the production of small scales of motion. In doing so, Verstappen (2008) proposed to preserve exactly the symmetry and conservation properties of the convective terms. This requirement yielded a family of *symmetry-preserving regularization* models: a novel class of regularizations that restrains the convective production of smaller and smaller scales of motion in an unconditionally stable manner, meaning that the velocity cannot blow up in the energy-norm (in 2D also: enstrophy-norm). In our previous works, we restrict ourselves to the $\mathcal{C}_4$ approximation: the convective term in the NS equations (1) is then replaced by the following $\mathcal{O}(\epsilon^4)$-accurate smooth approximation $\mathcal{C}_4(u,v)$ given by

$$\mathcal{C}_4(u,v) = \mathcal{C}(\overline{u},\overline{v}) + \overline{\mathcal{C}(\overline{u},v')} + \overline{\mathcal{C}(u',\overline{v})}, \tag{2}$$

where the prime indicates the residual of the filter, *e.g.* $u' = u - \overline{u}$, which can be explicitly evaluated, and $\overline{(\cdot)}$ represents a symmetric linear filter with filter length ϵ. Therefore, the governing equations result to

$$\partial_t u_\epsilon + \mathcal{C}_4(u_\epsilon, u_\epsilon) = \mathcal{D}u_\epsilon - \nabla p_\epsilon; \qquad \nabla \cdot u_\epsilon = 0, \tag{3}$$

where the variable names are changed from u and p to u_ϵ and p_ϵ, respectively, to stress that the solution of (3) differs from that of (1). Note that the $\mathcal{C}_4$ approximation is also a skew-symmetric operator like the original convective operator. Hence, the same inviscid invariants than the original NS equations are preserved for the new set of partial differential equations (3). The $\mathcal{C}_4$ regularization method has already been successfully applied to several configurations (Verstappen, 2008; Trias *et al.*, 2010). However, two main drawbacks have been observed: (i) due to the energy conservation, the model solution tends to display an additional hump in the tail of the spectrum and (ii) for very coarse meshes the damping factor can eventually take very small values.

In this context, here we propose to alter diffusion term in the same vein than convection. In Section 2, this new regularization method is presented and discussed. Firstly, a family of fourth-order accurate regularizations of convective term is obtained. Then, the modification of linear diffusive operator follows when trying to restore (approximately) the Galilean invariance of the regularized equations. The modified diffusive term introduces an hyper-viscosity effect and consequently enhances the destruction of small scales. Then, the only additional ingredient is a self-adjoint linear filter whose local filter length is determined from the requirement proposed by Trias *et al.* (2010), *i.e.* the vortex-stretching mechanism must be stopped at the smallest grid scale. This is addressed in Section 3 where a new and more accurate criterion to bound the vortex-stretching term contribution in physical space is presented. It is based on the invariants of the local strain tensor and, from a theoretical point-of-view, it can be shown that the method is suitable to be applied to wall-bounded flows.

2. Spectrally-consistent regularization of NS equations

2.1. Symmetries and conservation properties
For convenience, we introduce first the following notation:

$$d(u,v) = (u, \mathcal{D}v) \qquad \text{and} \qquad c(u,v,w) = (\mathcal{C}(u,v), w), \tag{4}$$

where the innerproduct of functions is defined in the usual way: $(a, b) = \int_\Omega a \cdot b \, d\Omega$. The bilinear operator $d(u, v)$ satisfies the following properties

$$d(u, v) = d(v, u) \qquad \text{and} \qquad d(u, u) < 0, \tag{5}$$

whereas the trilinear form $c(u, v, w)$ satisfies two fundamental symmetry properties

$$c(u, v, w) = -c(u, w, v) \qquad \text{if} \quad \nabla \cdot u = 0, \tag{6}$$
$$c(u, v, \Delta v) = c(\Delta v, v, u) \qquad \text{in 2D}, \tag{7}$$

provided that the contribution of boundaries vanishes. These properties are extensively used to prove the conservation properties of the inviscid invariants of original NS equations. Namely, the skew-symmetry (6) ensures the conservation of kinetic energy, $1/2(u, u)$, and helicity, (u, ω) where $\omega = \nabla \times u$ is the vorticity. The enstrophy, (ω, ω), also forms an inviscid in the case of 2D flows. Actually, the stronger form of enstrophy invariance given by (7) also holds for NS equations. For details the reader is referred to Guermond *et al.* (2004), for instance.

2.2. $\{\mathcal{CD}\}_4$ regularization modeling

Regularization aims to modify the convective term in such a way that a dynamically less complex mathematical formulation results. Let us assume that we have a self-adjoint linear filter $\overline{(\cdot)} : u \longrightarrow \overline{u}$ with the requirements that it filters out high frequency components and it commutes with differential operators. Now, for convenience, let us define the following function

$$\varphi_i(u) = \begin{cases} u, & \text{if } i = 0 \\ \overline{u}, & \text{if } i = 1 \end{cases} \tag{8}$$

Then, a family of modified (regularized) non-linear operators can be easily constructed

$$\tilde{\mathcal{C}}(u, v) = \sum_{i,j,k=0}^{1} a_{ijk} \tilde{\mathcal{C}}_{ijk}(u, v), \tag{9}$$

where $\tilde{\mathcal{C}}_{ijk}(u, v) = \varphi_k(\mathcal{C}(\varphi_i(u), \varphi_j(v)))$. Hence, regularization $\tilde{\mathcal{C}}(u, v)$ results into a linear combination of (up to) eight terms. Among all the possible combinations we find the regularization proposed by Leray (1934), $a_{100} = 1$ (with the rest of $a_{ijk} = 0$). Firstly, the equality $\sum_{i,j,k=0}^{1} a_{ijk} = 1$ must be satisfied to guarantee that $\tilde{\mathcal{C}}(u, v) \approx \mathcal{C}(u, v) + \mathcal{O}(\epsilon^n)$ with $n \geq 2$. Then, several restriction may be applied to the coefficients a_{ijk}. Namely,

$$a_{ijk} = a_{ikj} \qquad \text{and} \qquad a_{ijk} = a_{jik}, \tag{10}$$

where the latter ensures that the skew-symmetry property (6) is exactly preserved whereas the former is needed to guarantee that the form of vorticity transport equation is not altered. They impose four additional restrictions to the coefficients a_{ijk} and lead to a family of second-order accurate regularization models. Among them, we find the second-order approximation proposed by Verstappen (2008),

$$\mathcal{C}_2(u, v) = \tilde{\mathcal{C}}_{111}(u, u) = \overline{\mathcal{C}(\overline{u}, \overline{u})}. \tag{11}$$

It must be noticed that the following restriction

$$a_{ijk} = a_{kji}, \tag{12}$$

needed to preserve the strong form of the enstrophy invariance (7) follows automatically from (10). Then, if we want to cancel second-order terms three additional conditions need to be imposed:

$$\sum_{j,k=0}^{1} a_{1jk} = 0 \qquad \sum_{i,k=0}^{1} a_{i1k} = 0 \qquad \sum_{i,j=0}^{1} a_{ij1} = 0. \tag{13}$$

Finally, without the loss of generality we can restrict ourselves to solutions where $a_{000} = 0$. This leads to a family of fourth-order accurate regularization methods

$$\mathcal{C}_4^{\gamma}(u,v) = \mathcal{C}_4^0(u,v) + \gamma \mathcal{E}_4(u,v), \tag{14}$$

where

$$\mathcal{C}_4^0(u,v) = \frac{1}{2}\left(\tilde{\mathcal{C}}_{001} + \tilde{\mathcal{C}}_{010} + \tilde{\mathcal{C}}_{100} - \tilde{\mathcal{C}}_{111}\right)(u,v), \tag{15}$$

$$\mathcal{E}_4(u,v) = \left(\tilde{\mathcal{C}}_{011} + \tilde{\mathcal{C}}_{101} + \tilde{\mathcal{C}}_{110}\right)(u,v) - \frac{1}{2}\left(\tilde{\mathcal{C}}_{001} + \tilde{\mathcal{C}}_{010} + \tilde{\mathcal{C}}_{100} + 3\tilde{\mathcal{C}}_{111}\right)(u,v). \tag{16}$$

Note that $\mathcal{C}_4^0(u,v) = \mathcal{C}(u,v) + \mathcal{O}(\epsilon^4)$ whereas $\mathcal{E}_4(u,v) = \mathcal{O}(\epsilon^4)$. Even more important,

$$\mathcal{C}_4^{\gamma}(u,v) = \mathcal{C}(u,v) + (\gamma + 1)\mathcal{O}(\epsilon^4) + \mathcal{O}(\epsilon^6). \tag{17}$$

Therefore, $\mathcal{C}_4^{\gamma}$ is fourth-order accurate except for $\gamma = -1$ that becomes sixth-order. Actually for $\gamma = 1$ and $\gamma = -1$, $\mathcal{C}_4^{\gamma}$ becomes the $\mathcal{C}_4$ and $\mathcal{C}_6$ approximations proposed by Verstappen (2008),

$$\mathcal{C}_4(u,v) = \mathcal{C}_4^{\gamma=1}(u,v) = \mathcal{C}(\overline{u},\overline{v}) + \overline{\mathcal{C}(\overline{u},v')} + \overline{\mathcal{C}(u',\overline{v})}, \tag{18}$$

$$\mathcal{C}_6(u,v) = \mathcal{C}_4^{\gamma=-1}(u,v) = \mathcal{C}(\overline{u},\overline{v}) + \mathcal{C}(\overline{u},v') + \mathcal{C}(u',\overline{v}) + \overline{\mathcal{C}(u',v')}, \tag{19}$$

respectively. Notice that the $\mathcal{C}_4^{\gamma}$ regularization can also be viewed as a linear combination of $\mathcal{C}_4$ and $\mathcal{C}_6$

$$\mathcal{C}_4^{\gamma}(u,v) = \frac{1}{2}((\mathcal{C}_4 + \mathcal{C}_6) + \gamma(\mathcal{C}_4 - \mathcal{C}_6))(u,v). \tag{20}$$

The approximations $\mathcal{C}_4^{\gamma}$ maintains all the invariant transformations of the NS equations, except, in general, the Galilean transformation. These transformations are listed in (Pope, 2000), for instance. To restore the Galilean invariance we need to replace the time-derivative, $\partial_t u_\epsilon$, be the following forth-order approximation:

$$(\partial_t)_4^{\gamma} u_\epsilon = \partial_t(u_\epsilon - 1/2(1+\gamma)u_\epsilon'') = \mathcal{G}_4^{\gamma}(\partial_t u_\epsilon), \tag{21}$$

where $\mathcal{G}_4^{\gamma}(\phi) = \phi - 1/2(1+\gamma)\phi''$ In this case, the new set of PDEs reads

$$(\partial_t)_4^{\gamma} u_\epsilon + \mathcal{C}_4^{\gamma}(u_\epsilon, u_\epsilon) = \mathcal{D}u_\epsilon - \nabla p_\epsilon. \tag{22}$$

Therefore, Galilean invariance can be restored by simply setting $\gamma = -1$. However, it can be shown than such approach have several drawbacks. Another possibility relies on modifying appropriately other terms, *i.e.* viscous dissipation. The energy equation for (22) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}(|u_\epsilon|^2 - |u_\epsilon'|^2) = (u_\epsilon, \mathcal{D}u_\epsilon) < 0, \tag{23}$$

provided that the filter is self-adjoint and $|u|^2 = (u, u)$. Therefore, modification of time-derivative term (21) constitutes a dissipation model. Recalling that $(\mathcal{G}_4^\gamma)^{-1}(\phi) \approx 2\phi - \mathcal{G}_4^\gamma(\phi) + \mathcal{O}(\epsilon^6)$, we can obtain an energetically almost equivalent set of equations by modifying the viscous diffusive term

$$\partial_t u_\epsilon + \mathcal{C}_4^\gamma(u_\epsilon, u_\epsilon) = \mathcal{D}_4^\gamma u_\epsilon - \nabla p_\epsilon, \tag{24}$$

where the linear operator $\mathcal{D}_4^\gamma u$ is given by

$$\mathcal{D}_4^\gamma u = \mathcal{D}u + 1/2(1 + \gamma)(\mathcal{D}u')'. \tag{25}$$

In this way, we are reinforcing the dissipation at the smallest grid scales. At this point, there are two parameters that need to be fixed; namely, the constant γ and the local filter length, ϵ. The former will determine the exact form of the regularization model whereas the latter will define the convolution kernel of the linear filter. By setting $\gamma = 1$, $\mathcal{C}_4^\gamma$ becomes the fourth-order accurate $\mathcal{C}_4$ regularization originally proposed by Verstappen (2008). In the authors' opinion keeping the same form for the regularized non-linear convective term provides a good starting point. Thus, for the remainder of the paper $\gamma = 1$ and therefore the proposed regularization model results

$$\partial_t u_\epsilon + \mathcal{C}_4(u_\epsilon, u_\epsilon) = \mathcal{D}_4 u_\epsilon - \nabla p_\epsilon \tag{26}$$

where $\mathcal{D}_4(u_\epsilon) = \mathcal{D}u + (\mathcal{D}u')'$. Hereafter, this regularization model will be denoted by $\{\mathcal{CD}\}_4$.

3. Restraining the production of small scales of motion

3.1. Interscale interactions
To study the interscale interactions in more detail, we continue in the spectral space. The spectral representation of the convective term in the NS equations is given by

$$\mathcal{C}(u, u)_k = i\Pi(k) \sum_{p+q=k} \hat{u}_p q \hat{u}_q, \tag{27}$$

where $\Pi(k) = I - kk^T/|k|^2$ denotes the projector onto divergence-free velocity fields in the spectral space. Taking the Fourier transform of (26), we obtain the evolution of each Fourier-mode $\hat{u}_k(t)$ of $u_\epsilon(t)$ for the $\{\mathcal{CD}\}_4$ approximation[1]

$$\left(\frac{d}{dt} + \frac{h_4(\widehat{G}_k)}{Re} |k|^2 \right) \hat{u}_k + i\Pi(k) \sum_{p+q=k} f_4\left(\widehat{G}_k, \widehat{G}_p, \widehat{G}_q \right) \hat{u}_p q \hat{v}_q = F_k, \tag{28}$$

where $\widehat{G}_k$ denotes the k-th Fourier-mode of the kernel of the convolution filter, *i.e.*, $\overline{\overline{u}}_k = \widehat{G}_k \hat{u}_k$. The mode $\hat{u}_k$ interacts only with those modes whose wavevectors p and q form a triangle with the vector k. Thus, compared with (27), every triad interaction is multiplied by

$$f_4\left(\widehat{G}_k, \widehat{G}_p, \widehat{G}_q \right) = \widehat{G}_k \widehat{G}_p + \widehat{G}_k \widehat{G}_q + \widehat{G}_p \widehat{G}_q - 2\widehat{G}_k \widehat{G}_p \widehat{G}_q, \tag{29}$$

where $0 < f_4 \leq 1$. On the other hand, the k-th Fourier mode of the diffusive term is multiplied by

$$h_4(\widehat{G}_k) = 1 + (1 - \widehat{G}_k)^2 \tag{30}$$

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[1] Hereafter, for simplicity, the subindex ϵ is dropped.

where $h_4 \geq 1$. Moreover, since for a generic symmetric convolution filter (see Carati *et al.* (1999), for instance), $\widehat{G}_k = 1 - \alpha^2 |k|^2 + \mathcal{O}(\alpha^4)$ with $\alpha^2 = \epsilon^2/24$, the functions f_4 and h_4 can be approximated by $f_4 \approx 1 - \alpha^4(|k|^2|p|^2 + |k|^2|q|^2 + |p|^2|q|^2)$ and $h_4 \approx 1 + \alpha^4|k|^4$, respectively. Therefore, the interactions between large scales of motion ($\epsilon|k| < 1$) approximate the NS dynamics up to $\mathcal{O}(\epsilon^4)$. Hence, the triadic interactions between large scales are only slightly altered. All interactions involving longer wavevectors (smaller scales of motion) are reduced. The amount by which the interactions between the wavevector-triple (k, p, q) are lessened depends on the length of the legs of the triangle $k = p + q$. For example, all triadic interactions for which at least two legs are (much) longer than $1/\epsilon$ are (strongly) attenuated; whereas, interactions for which at least two legs are (much) shorter than $1/\epsilon$ are reduced to a small degree only.

3.2. Stopping the vortex-stretching mechanism
Taking the curl of Eq.(26) leads to

$$\partial_t \omega + \mathcal{C}_4(u, \omega) = \mathcal{C}_4(\omega, u) + \mathcal{D}_4 \omega. \tag{31}$$

This equation resembles the vorticity equation that results from the NS equations: the only difference is that $\mathcal{C}$ and $\mathcal{D}$ are replaced by their regularizations $\mathcal{C}_4$ and $\mathcal{D}_4$, respectively. If it happens that the vortex stretching term $\mathcal{C}_4(\omega, u)$ in Eq.(31) is so strong that the dissipative term $\mathcal{D}_4 \omega$ cannot prevent the intensification of vorticity, smaller vortical structures are produced. Left-multiplying the vorticity transport Eq.(31) by ω, we can obtain the evolution of $|\omega|^2$. In this way, the vortex-stretching and dissipation term contributions to $\partial_t |\omega|^2$ result

$$\omega \cdot \mathcal{C}_4(\omega, u) \quad \text{and} \quad \omega \cdot \mathcal{D}_4 \omega, \tag{32}$$

respectively. In order to prevent local intensification of vorticity, dissipation must dominate the vortex-stretching term contribution at the smallest grid scale, $k_c = \pi/h$. In spectral space, this requirement leads to the following inequality

$$\frac{1}{2} \frac{\left(\hat{\omega}_{k_c} \cdot \mathcal{C}_4(\omega, u)^*_{k_c} + \mathcal{C}_4(\omega, u)_{k_c} \cdot \hat{\omega}^*_{k_c}\right)}{\hat{\omega}_{k_c} \cdot \hat{\omega}^*_{k_c}} \leq \frac{h_4(\hat{g}_k)}{Re} k_c^2, \tag{33}$$

where the vortex-stretching term, $\mathcal{C}_4(\omega, u)_{k_c}$, is given by

$$\mathcal{C}_4(\omega, u)_{k_c} = \sum_{p+q=k_c} f_4\left(\widehat{G}_{k_c}, \widehat{G}_p, \widehat{G}_q\right) \hat{\omega}_p iq \hat{u}_q. \tag{34}$$

Note that $f_4\left(\widehat{G}_{k_c}, \widehat{G}_p, \widehat{G}_q\right)$ depends on the filter length ϵ and, in general, on the wavevectors p and $q = k_c - p$. This makes very difficult to control the damping effect because f_4 cannot be taken out of the summation in (34). To avoid this, filters should be constructed from the requirement that the damping effect of all the triadic interactions at the smallest scale must be virtually independent of the interacting pairs, *i.e.*

$$f_4(\widehat{G}_{k_c}, \widehat{G}_p, \widehat{G}_q) \approx f_4(\widehat{G}_{k_c}). \tag{35}$$

This is a crucial property to control the subtle balance between convection and diffusion in order to stop the vortex-stretching mechanism. This point was addressed in detail by Trias & Verstappen (2011). Then, the overall damping effect at the smallest grid scale, $H_4(\widehat{G}_{k_c})$, follows straightforwardly

$$H_4(\widehat{G}_{k_c}) = \frac{f_4(\widehat{G}_{k_c})}{h_4(\widehat{G}_{k_c})} = \frac{(2/Re)k_c^2 \hat{\omega}_{k_c} \cdot \hat{\omega}^*_{k_c}}{\hat{\omega}_{k_c} \cdot \mathcal{C}(\omega, u)^*_{k_c} + \mathcal{C}(\omega, u)_{k_c} \cdot \hat{\omega}^*_{k_c}}, \tag{36}$$

with the condition that $0 < H_4(\widehat{G}_{k_c}) \leq 1$. Notice that $h_4(\widehat{G}_{k_c}) = 2 - f_4(\widehat{G}_{k_c})$ and therefore the damping function $f_4(\widehat{G}_{k_c})$ reads

$$f_4(\widehat{G}_{k_c}) = \frac{2H_4(\widehat{G}_{k_c})}{1 + H_4(\widehat{G}_{k_c})}. \tag{37}$$

3.3. From spectral to physical space

In the previous subsection we applied our analysis on a spectral space. However, the method needs to be applied on a physical domain in $\mathbb{R}^3$. To that end, here we propose to express the overall damping effect, $H_4(\widehat{G}_{k_c})$, as a function of the invariants of the local strain tensor, $S(u) = 1/2(\nabla u + \nabla u^T)$. Recalling that the velocity field, u, is solenoidal $(\nabla \cdot u = 0)$; $tr(S) = 0$ and the characteristic equation of S reads

$$\lambda^3 + Q\lambda + R = 0, \tag{38}$$

where $R = -1/3 tr(S^3) = -det(S) = -\lambda_1\lambda_2\lambda_3$ and $Q = -1/2 tr(S^2) = -1/2(\lambda_1^2 + \lambda_2^2 + \lambda_3^2)$ are the invariants of S, respectively. We order the eigenvalues of S by $\lambda_1 \leq \lambda_2 \leq \lambda_3$. Let us now consider an arbitrary part of the flow domain Ω with periodic boundary conditions. The innerproduct is defined in the usual way: $(a, b) = \int_\Omega a \cdot b d\Omega$. Then, taking the L^2 innerproduct of (1) with $-\Delta u$ leads to the enstrophy equation

$$\frac{1}{2}\frac{d}{dt}|\omega|^2 = (\omega, \mathcal{C}(\omega, u)) - \frac{1}{Re}(\nabla\omega, \nabla\omega), \tag{39}$$

where $|\omega|^2 = (\omega, \omega)$ and the convective term contribution $(\mathcal{C}(u, \omega), \omega) = 0$ vanishes because of the skew-symmetry (6) of the convective operator. Using the results obtained by Chae (2005) and following the same arguments than in (Verstappen, 2009), it can be shown that the vortex-stretching term can be expressed in terms of the invariant R of $S(u)$

$$(\omega, \mathcal{C}(\omega, u)) = \int_\Omega \omega \cdot S\omega = -\frac{4}{3}\int_\Omega tr(S^3)d\Omega = 4\int_\Omega Rd\Omega, \tag{40}$$

and the $L^2(\Omega)$-norm of ω in terms of the invariant Q

$$|\omega|^2 = -4\int_\Omega Qd\Omega. \tag{41}$$

Then, the diffusive term can be bounded by

$$(\nabla\omega, \nabla\omega) = -(\omega, \Delta\omega) \leq -\lambda_\Delta(\omega, \omega), \tag{42}$$

where $\lambda_\Delta < 0$ is the largest (smallest in absolute value) non-zero eigenvalue of the Laplacian operator Δ on Ω. If we now consider that the domain Ω is a periodic box of volume h, then $\lambda_\Delta = -(\pi/h)^2$. In a numerical simulation h would be related with the local grid size. Then, to prevent a local intensification of vorticity, *i.e.* $|\omega|_t \leq 0$, the following inequality must be hold

$$H_4(\widehat{G}_{k_c})\frac{(\omega, S\omega)}{(\omega, \omega)} \leq -\frac{\lambda_\Delta}{Re}, \tag{43}$$

where, in this case, $k_c = \pi/h$. This inequality is the analog to Eq.(36) in physical space. Rayleigh's principle states that

$$\max_{\omega \neq 0}\frac{(\omega, S\omega)}{(\omega, \omega)} = \lambda_3, \tag{44}$$

and therefore gives a lower bound for the damping function, $H_4(\widehat{G}_{k_c}) \leq Re^{-1}(-\lambda_\Delta/\lambda_3)$. This was the approach consider in our previous work (Trias *et al.*, 2010). However, the maximum value is attained only if ω is aligned with the eigenvector corresponding to λ_3, and therefore the convective terms tends to be over-damped. This becomes especially relavant near the walls. In order to overcome this drawback here we propose to rewrite the inequality (43) in terms of the invariants Q and R. From Eqs. (40)-(43) we deduce

$$H_4(\widehat{G}_{k_c}) \leq \frac{\lambda_\Delta}{Re}\frac{Q}{R^+}, \tag{45}$$

where $R^+ = \max\{R, 0\}$ and the overall damping factor $0 < H_4 \leq 1$. Thus, a proper definition of the overall damping factor at the smallest grid scale is given by

$$H_4(\widehat{G}_{k_c}) = \min\left\{\frac{\lambda_\Delta}{Re}\frac{Q}{R^+}, 1\right\}. \tag{46}$$

Notice that the invariant Q is always negative whereas R can be either positive or negative. In terms of the Reynolds number, the quotient of R and Q scales like $R/Q \propto (Re^{3/2})/Re = Re^{1/2}$. Then, recalling that $\lambda_\Delta \propto h^{-2}$, it yields to $H_4(\widehat{G}_{k_c}) \propto h^{-2}Re^{-1}Q/R \propto h^{-2}Re^{-3/2}$. Therefore, we obtain $H_4(\widehat{G}_{k_c}) \to 1$ if $h \propto Re^{-3/4}$. This shows that the model switches off when h approaches to the smallest scale in a turbulent flow. Another interesting feature of the model is that it automatically switches off ($R \to 0$) for laminar flows (no vortex-stretching) and 2D flows ($\lambda_2 = 0 \to R = 0$). The near-wall behavior of the invariants is given by $R \propto y^3$ and $Q \propto y^0$, respectively, where y is the distance to the wall. Consequently, it results into a model that switches off in the wall.

For convenience, let us now define the ratio between λ_2 and λ_3, $\eta = \lambda_2/\lambda_3$. Note that $\lambda_1 \leq 0$ and $\lambda_3 \geq 0$, whereas the middle eigenvalue, λ_2, can be both positive or negative. Actually, the sign of the invariant $R = -\lambda_1\lambda_2\lambda_3$, λ_2 and η are the same. Then, recalling that the strain tensor is traceless ($tr(S) = 0$), *i.e.* $\lambda_1 + \lambda_2 + \lambda_3 = 0$, the first eigenvalue can also be written in terms of λ_3 and η: $\lambda_1 = -(1 + \eta)\lambda_3$. Then, the ratio Q/R results into $(-Q/R)^{-1} = \lambda_{QR} = (1+\eta)\eta/(\eta^2+\eta+1)\lambda_3$. Here, λ_{QR} can be viewed as the rate of amplification of vorticity at the smallest grid scale. Then, assuming that $|\eta| \ll 1$, $\lambda_{QR} \approx \eta\lambda_3 = \lambda_2$ and therefore it is consistent with the preferential vorticity alignment with the intermediate eigenvector (see the work by Galanti *et al.* (1997) and references therein).

4. Concluding remarks

Since DNS simulations are not feasible for real-world applications the $\{\mathcal{CD}\}_4$-regularization of the NS equations has been proposed as a simulation shortcut. The convective and diffusive operators in the NS equations (1) are replaced by the $\mathcal{O}(\epsilon^4)$-accurate smooth approximation given by Eq.(2) and Eq.(25) (with $\gamma = 1$), respectively. The symmetries and conservation properties of the original convective term are exactly preserved. Doing so, the production of smaller and smaller scales of motion is restrained in an unconditionally stable manner. In this way, the new set of equations is dynamically less complex than the original NS equations, and therefore more amenable to be numerically solved. The only additional ingredient is a self-adjoint linear filter whose local filter length is determined from the requirement that vortex-stretching must be stopped at the scale set by the grid. This can be easily satisfied in spectral space via Eq.(36) provided that discrete filter satisfies Eq.(35), *i.e.* the triadic interactions at the smallest scale are virtually independent of the interacting pairs. This was addressed in detail by Trias & Verstappen (2011). However, in physical space it becomes more cumbersome. To circumvent this, here a novel criterion based on the two invariants, R and Q, of the local strain tensor has been proposed. Doing so, the expected behavior of a turbulence model is achieved:

it switches off (*i.e.* $H_4 = 1$) for laminar flows (no vortex-stretching), 2D flows ($R = 0$) and near the walls. Hence, the proposed methods constitutes a parameter-free turbulence model suitable for complex geometries and flows.

In the present paper, the parameter γ of Eq.(24) was fixed to $\gamma = 1$. Doing so, the $\mathcal{C}_4^\gamma$ becomes the fourth-order accurate $\mathcal{C}_4$ regularization originally proposed by Verstappen (2008) and tested through application to a turbulent channel flow. Later, $\mathcal{C}_4$ has also been successfully tested for a turbulent differentially heated cavity by Trias *et al.* (2010) and the flow around a wall-mounted cube by Trias *et al.* (2009). Hence, we expect that $\{\mathcal{CD}\}_4$ will perform at least as well $\mathcal{C}_4$ itself. Numerical results evaluating the performance of the $\{\mathcal{CD}\}_4$ method for wall-bounded configurations will be presented during the conference. Nevertheless, γ can take values different than unity. This opens the possibility to explore new forms of $\{\mathcal{CD}\}_4^\gamma$ regularization. In such a case, $h_4(\widehat{G}_{k_c}) = 1 + \gamma(1 - f_4(\widehat{G}_{k_c}))$, and therefore Eq.(37) would be replaced by the more general formula

$$f_4^\gamma(\widehat{G}_{k_c}) = \frac{(1+\gamma)H_4(\widehat{G}_{k_c})}{1+\gamma H_4(\widehat{G}_{k_c})}. \tag{47}$$

Therefore, increasing the values of $\gamma > 1$ leads to lower values of f_4. In particular, taking $\gamma = 1/f_4(\widehat{G}_{k_c})$, $h_4(\widehat{G}_{k_c}) = 1/f_4(\widehat{G}_{k_c})$ and the overall damping becomes $H_4(\widehat{G}_{k_c}) = f_4(\widehat{G}_{k_c})/h_4(\widehat{G}_{kc}) = (f_4(\widehat{G}_{k_c}))^2$. The analysis of these $\{\mathcal{CD}\}_4^\gamma$ regularization models is part of our future research plans.

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