# The Continuous Spin Random Field Model

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# THE CONTINUOUS SPIN RANDOM FIELD MODEL: FERROMAGNETIC ORDERING IN $d \geq 3$*

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We investigate the Gibbs-measures of ferromagnetically coupled continuous spins in double-well potentials subjected to a random field (our specific example being the $\phi^4$ theory), showing ferromagnetic ordering in $d \geq 3$ dimensions for weak disorder and large energy barriers. We map the random continuous spin distributions to distributions for an Ising-spin system by means of a single-site coarse-graining method described by local transition kernels. We derive a contour-representation for them with notably *positive* contour activities and prove their Gibbsianness. This representation is shown to allow for application of the discrete-spin renormalization group developed by Bricmont/Kupiainen implying the result in $d \geq 3$.

*Keywords*: Disordered systems, contour models, cluster expansions, renormalization group, random field model.

## 1. Introduction

The study of phase transitions in continuous spin lattice models has a long history. An important prototypical example of a random model in this class is the continuous spin random field model, where ferromagnetically coupled real valued spins fluctuate in randomly modulated local double-well potentials.

In the present paper we study this model for weak disorder in dimensions $d \geq 3$ proving ferromagnetic ordering. Our aim is more generally to describe an expansion method mapping multiple-well continuous spin models to discrete spin models with exponentially decaying interactions by means of a single-site coarse-graining. Then we make use of information about the latter ones. This transformation can be regarded as an example of a useful (and moreover non-pathological) single-site "renormalization group" transformation. While it is already interesting in a translation-invariant situation, it is particularly useful for non-translational invariant systems since it allows to "factorize" the degrees of freedom provided by the fluctuations of the spins around their local minima.

It is ten years now since the existence of ferromagnetic ordering for small disorder at small temperatures was proved for the ferromagnetic random field Ising-model (with spins $\sigma_x$ taking values in $\{-1, 1\}$) by Bricmont–Kupiainen [5], answering a question that had been open for long in the theoretical physics community. The

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"converse", namely the a.s. uniqueness of the Gibbs-measure in $d = 2$ was proved later by Aizenman and Wehr [1]. For an overview on the random field model from the perspective of theoretical physics, see e.g. [19]. Given the popularity of *continuous* spin models it is however certainly desirable to have a transparent method that is able to treat the additional degrees of freedom present in such a model.

Bricmont and Kupiainen introduced in [5] the conceptually beautiful method of the renormalization group (RG) to the rigorous analysis of the low temperature behavior of a disordered system, that turned out to be very powerful in this situation although there is no scale-invariance in the problem. The heuristic idea is: map the initial spin-system onto a coarse-grained one that appears to be at lower temperature and smaller disorder. Then iterate this transformation. This idea has to be implemented in a suitable representation of contours (that are the natural variables at low temperatures). (For a pedagogical presentation of such a RG in application to the proof of stability of solid-on-solid interfaces in disordered media, see also [8, 17].) An alternative treatment of disordered lattice systems with finite local spin-space was sketched by Zahradník [22], however also using some iterated coarse graining.

It is also clear that in the more difficult situation of continuous spins, spatial renormalization will be needed. However, continuous spins being more "flexible" than Ising spins make it difficult to cut the analysis in local pieces. It is then to be expected that the difficulties to control the *locality* of a suitably defined renormalization group transformation acting directly on continuous spins in a rigorous way would blow up tremendously compared with the discrete spin case of [5]. (The amount of technical work needed in their proof is already not small!) For an example of a rigorous construction of an RG-group for a continuous spin-lattice system, see [2, 3] for the ordered Heisenberg–Ferromagnet. (This might give some idea of the complexities of such a method.)

Indeed, despite the conceptual beauty, technical difficulties have kept the number of rigorous applications of the RG to low-temperature disordered lattice spin systems limited. Moreover, usually a lot of technical work has to be repeated when extending such a method to a more complex situation, while it would be desirable to make use of older results in a more transparent way.

We will therefore describe a different and more effective way to the continuous spin problem: (1) Construct a single-site "RG"-transformation that maps the continuous model to a discrete one. Obtain bounds on the first in terms of the latter one. In our specific $\phi^4$ double-well situation this transformation is just a suitable stochastic mapping to the sign-field. (2) Apply the RG group to the discrete model. As we will show, the discrete (Ising-) model in our case has a representation as a contour model whose form is invariant under the discrete-spin RG that was constructed in [5]. So we need not repeat the RG analysis for this part but can apply their results, avoiding work that has already been done.

In the last few years there has been an ongoing discussion about the phenomenon of RG pathologies. It was first observed by Griffith, Pearce, Israel (and extended in various ways by van Enter, Fernandez, Sokal [13]) that even very "innocent" transformations like taking marginals on a sub-lattice of the original lattice can map a Gibbs-measure of a lattice spin system to an image measure that need not be a Gibbs-measure for any absolutely summable Hamiltonian. (See [13] for a clear presentation and more information about what pathologies can and cannot occur, see also the references given therein.) On the other hand, as a reaction to this, there has been the "Gibbsian restoration program" initiated by the late Dobrushin [11] whose aim it is to exhibit sets of "bad configurations" of measure zero (w.r.t. the renormalized measure) outside of which a "renormalized" Hamiltonian with nicely decaying interactions can be defined. This program has been carried out in [7] for a special case (again using RG based on [5]).

Since we will be dealing with contour representations of finite volume measures that provide uniform bounds on the initial spin system we do not have to worry about non-Gibbsianness vs. Gibbsianness to get our results. Nevertheless, to put our work in perspective with the mentioned discussion, we will in fact construct a uniformly convergent "renormalized Hamiltonian" for the measure on the sign-field, for *all* configurations. In other words, there are no pathologies in our single-site coarse graining and the situation is as nice and simple as it can be.

Let us introduce our model and state our main results. We are interested in the analysis of the Gibbs measures on the state space $\Omega = \mathbb{R}^{\mathbb{Z}^d}$ of the continuous spin model given by the Hamiltonians in finite volume $\Lambda$

$$\begin{aligned} E_\Lambda^{\tilde m_{\partial\Lambda},\eta_\Lambda}(m_\Lambda) = \frac{q}{2}\sum_{\substack{\{x,y\}\subset\Lambda\\ d(x,y)=1}}(m_x-m_y)^2 + \frac{q}{2}\sum_{\substack{x\in\Lambda;y\in\partial\Lambda\\ d(x,y)=1}}(m_x-\tilde m_y)^2 \\ + \sum_{x\in\Lambda}V(m_x) - \sum_{x\in\Lambda}\eta_x m_x \end{aligned} \tag{1.1}$$

for a configuration $m_\Lambda \in \Omega_\Lambda = \mathbb{R}^\Lambda$ with boundary condition $\tilde m_{\partial\Lambda}$. Here we write $\partial\Lambda = \{x \in \Lambda^c; \exists\, y \in \Lambda : d(x,y) = 1\}$ for the outer boundary of a set $\Lambda$ where $d(x,y) = \|x-y\|_1$ is the 1-norm on $\mathbb{R}^d$. $q \geq 0$ will be small. Given its history and its popularity we will consider mainly the example of the well-known double-well $\phi^4$-theory. As we will see during the course of the proof, there is however nothing special about this choice. We use the normalization where the minimizers are $\pm m^*$, the curvature in the minima is 1, and the value of the potential in the minima is zero and write

$$V(m_x) = \frac{(m_x^2-(m^*)^2)^2}{8m^{*2}}\,, \tag{1.2}$$

where the parameter $m^* \geq 0$ will be large. We consider i.i.d. random fields $(\eta_x)_{x\in\mathbb{Z}^d}$ that satisfy

(i) $\eta_x$ and $-\eta_x$ have the same distribution,
(ii) $\mathbb{P}[\eta_x \geq t] \leq e^{-\frac{t^2}{2\sigma^2}}$,
(iii) $|\eta_x| \leq \delta$,

where $\sigma^2 \geq 0$ is sufficiently small. The assumption (iii) of having uniform bounds is not essential for the problem of stability of the phases but made to avoid uninteresting problems with our transformation and keep things as transparent as possible.

The finite volume Gibbs-measures $\mu_\Lambda^{\tilde m_{\partial\Lambda},\eta_\Lambda}$ are then defined as usual through the expectations

$$\mu_\Lambda^{\tilde m_{\partial\Lambda},\eta_\Lambda}(f)=\frac{1}{Z_\Lambda^{\tilde m_{\partial\Lambda},\eta_\Lambda}}\int_{\mathbb{R}^\Lambda}dm_\Lambda f(m_\Lambda,\tilde m_{\Lambda^c})e^{-E_\Lambda^{\tilde m_{\partial\Lambda},\eta_\Lambda}(m_\Lambda)} \tag{1.3}$$

for any bounded continuous $f$ on $\Omega$ with the partition function

$$Z_\Lambda^{\tilde m_{\partial\Lambda},\eta_\Lambda}=\int_{\mathbb{R}^\Lambda}dm_\Lambda e^{-E_\Lambda^{\tilde m_{\partial\Lambda},\eta_\Lambda}(m_\Lambda)}\,. \tag{1.4}$$

We look in particular at the measures with boundary condition $\tilde m_x=+m^*$ (for all $x\in\mathbb{Z}^d$) in the positive minimum of the potential, for which we write $\mu_\Lambda^{+m^*,\eta_\Lambda}$.

To prove the existence of a phase transition we will show that, for a suitable range of parameters, with large probability w.r.t. the disorder, the Gibbs-expectation of finding the field left to the positive well is very small. Indeed, we have as the main result:

**Theorem 1.** *Let $d\geq 3$ and assume the conditions* (i), (ii), (iii) *with $\sigma^2$ small enough. Then, for any* (*arbitrarily small*) $\gamma>0$, *there exist $q_0>0$* (*small enough*), $\delta_0,\delta_1>0$ (*small enough*), $\tau_0$ (*large enough*) *such that, whenever $\delta\leq\delta_0$, $q(m^*)^2\geq\tau_0$ and $q(m^*)^{\frac{2}{3}}\leq\delta_1$ we have that*

$$\mathbb{P}\left[\limsup_{N\uparrow\infty}\mu_{\Lambda_N}^{+m^*,\eta_{\Lambda_N}}\left[m_{x_0}\leq\frac{m^*}{2}\right]\geq\gamma\right]\leq e^{-\frac{\text{const}}{\sigma^2}} \tag{1.5}$$

*for an increasing sequence of cubes $\Lambda_N$.*

**Remark.** Note that the quantity $q(m^*)^2$ gives the order of magnitude of the minimal energetic contribution of a nearest neighbor pair of spins with opposite signs to the Hamiltonian (1.1); it will play the role of a (low temperature) Peierls constant. Smallness of $q$ (to be compared with the curvature unity in the minima of the potential) is needed to ensure a fast decay of correlations of the thermal fluctuations around the minimizer in a given domain. The stronger condition on the smallness, $q\leq\text{const}(m^*)^{-\frac{2}{3}}$, however is needed in our approach to ensure the positivity and smallness of certain anharmonic corrections.

Let us now define the transition kernel $T_x\left(\cdot\,\middle|\,\cdot\right)$ from $\mathbb{R}$ to $\{-1,1\}$ we use and explain why we do it. Put, for a continuous spin $m_x\in\mathbb{R}$, and an Ising spin $\sigma_x\in\{-1,1\}$

$$T_x(\sigma_x|m_x):=\frac{1}{2}(1+\sigma_x\tanh(am^*m_x))\,, \tag{1.6}$$

where $a\geq 1$, close to 1, will have to be chosen later to our convenience. In other words, the probability that a continuous spin $m_x$ gets mapped to its sign is given by $\frac{1}{2}(1+\tanh(am^*|m_x|))$ which converges to one for large $m^*$. The above kernel defines a joint probability distribution $\mu_\Lambda^{\tilde m_{\partial\Lambda},\eta_\Lambda}(dm_\Lambda)T(d\sigma_\Lambda|m_\Lambda)$ on $\mathbb{R}^\Lambda \times \{-1,1\}^\Lambda$ whose non-normalized density is given by

$$e^{-E_\Lambda^{\tilde m_{\partial\Lambda},\eta_\Lambda}(m_\Lambda)} \prod_{x\in\Lambda} T_x(\sigma_x \big| m_x)\,. \tag{1.7}$$

Its marginal on the Ising-spins $\sigma_\Lambda$

$$\Big(T\left(\mu_\Lambda^{\tilde m_{\partial\Lambda},\eta_\Lambda}\right)\Big)(d\sigma_\Lambda) := \int_{\mathbb{R}^\Lambda} \mu_\Lambda^{\tilde m_{\partial\Lambda},\eta_\Lambda}(dm_\Lambda)T(d\sigma_\Lambda \big| m_\Lambda) \tag{1.8}$$

will be the main object of our study.

To prove the existence of a phase transition stated in Theorem 1 we will have to deal only with finite volume contour representations of (1.8), as given in Proposition 5.1. Nevertheless, it is perhaps most instructive to present the following infinite volume result in the Hamiltonian formulation to explain the nature of the transformation.

**Theorem 2.** *Assume the hypothesis of Theorem* 1 *and let* $\eta$ *be any fixed realization of the disorder. Suppose that* $\mu^\eta$ *is a continuous spin Gibbs-measure obtained as a weak limit of* $\mu_\Lambda^{\tilde m_{\partial\Lambda},\eta_\Lambda}$ *along a sequence of cubes* $\Lambda$ *for some boundary condition* $\tilde m \in \{-m^*, m^*\}^{\mathbb{Z}^d}$. *Then, for a suitable choice of the parameter* $a \geq 1$ (*close to* 1) *in the kernel* $T$ *the following is true.*

*The measure* $T(\mu^\eta)$ *on* $\{-1,1\}^{\mathbb{Z}^d}$ *is a Gibbs measure for the absolutely summable Ising–Hamiltonian*

$$\begin{aligned} H^\eta_{\text{Ising}}(\sigma) = &-\frac{a^2(m^*)^2}{2}\sum_{x,y}(a-q\Delta_{\mathbb{Z}^d})^{-1}_{x,y}\sigma_x\sigma_y \\ &- am^*\sum_{x,y}(a-q\Delta_{\mathbb{Z}^d})^{-1}_{x,y}\eta_x\sigma_y - \sum_{C:|C|\geq 2}\Phi_C(\sigma_C;\eta_C)\,, \end{aligned} \tag{1.9}$$

*where* $\Delta_{\mathbb{Z}^d}$ *is the lattice Laplacian in the infinite volume, i.e.* $\Delta_{\mathbb{Z}^d;x,y} = 1$ *iff* $x,y \in V$ *are nearest neighbors,* $\Delta_{\mathbb{Z}^d;x,y} = -2d$ *iff* $x = y$ *and* $\Delta_{\mathbb{Z}^d;x,y} = 0$ *else.*

*The many-body potentials are symmetric under joint flips of spins and random-fields,* $\Phi_C(\sigma_C,\eta_C) = \Phi_C(-\sigma_C,-\eta_C)$, *and translation-invariant under joint lattice-shifts. They obey the uniform bound*

$$|\Phi_C(\sigma_C,\eta_C)| \leq e^{-\tilde\gamma|C|} \tag{1.10}$$

*with a positive constant* $\tilde\gamma$.

**Remark 1.** As in Theorem 1, $\tilde\gamma$ can be made arbitrarily small by choosing $q_0, \delta_0, \delta_1$ small and $\tau_0$ large. More information about estimates on the value of $\gamma$ and $\tilde\gamma$ can in principle be deduced from the proofs.

**Remark 2.** By imposing the smallness of $\delta$ we exclude pathologies due to exceptional realizations of the disorder variable $\eta$ ("Griffiths singularities") in the transformation $T$. (We stress that this does not simplify the physical problem of the study of the low-temperature phases which is related to the study of the formation of large contours.) Starting from the joint distribution (1.7) it is natural to consider the distribution of continuous spins *conditional* on the Ising spins; here the Ising spins $\sigma_x$ will play the role of a second sort of external fields. Then, as it was explained in [7], possible pathologies in the transformation $T$ would be analogous to Griffiths-singularities created by pathological Ising configurations. In this sense, Theorem 2 states that there are neither Griffiths singularities of the first type (w.r.t. $\eta$) nor the second type (w.r.t $\sigma$). The treatment of unbounded random fields would necessitate the analysis of so-called "bad regions" in space (where the realizations of the random fields are anamolously large). This should be possible but would however obscure the nature of the transformation $T$.

Let us now motivate the form of $T_x$ and comment on the structure of the Hamiltonian. Introducing quadratic potentials, centered at $\pm m^*$,

$$Q^{\sigma_x}(m_x) := \frac{a}{2}(m_x - \sigma_x m^*)^2 + b \tag{1.11}$$

with $b > 0$ (close to zero) to be chosen later, we can rewrite the transition kernel in the form

$$T_x(\sigma_x|m_x) = \frac{e^{-Q^{\omega_x}(m_x)}}{\sum_{\bar\omega_x=\pm1} e^{-Q^{\bar\omega_x}(m_x)}}\,. \tag{1.12}$$

The crucial point is that the joint density (1.7) contains a product over $x$ over the quantities

$$e^{-V(m_x)}T_x(\sigma_x|m_x) = e^{-Q^{\sigma_x}(m_x)}(1 + w(m_x)) \tag{1.13}$$

where, using (1.12), we can write the remainder in the form

$$(1 + w(m_x)) := \frac{e^{-V(m_x)}}{\sum_{\bar\sigma_x=\pm1} e^{-Q^{\bar\sigma_x}(m_x)}}\,. \tag{1.14}$$

Now, if the initial potential $V(m_x)$ is sufficiently Gaussian around its minima and the quadratic potential $Q^{\sigma_x}$ is suitably chosen, $w(m_x)$ should be *small* in some sense. If $w(m_x)$ were even zero, we would be left with $\sigma_\Lambda$-dependent Gaussian integrals that can be readily carried out. They lead to the first two terms in the Ising–Hamiltonian (1.9), containing only pair-interactions. This can be understood by a formal computation. The modification of the measure for "small" $w(m_x)$ then gives rise indeed to exponentially decaying many-body interactions, as one could naively hope for.

Expanding $\prod_{x\in\Lambda}(1 + w(m_x))$ then leads in principle to an expansion around a Gaussian field.[a] However, one problem with this direct treatment is that resulting contour activities will in general be nonnegative only if $w(m_x) \geq 0$ for all $m_x$. But note that the latter can only be true for the narrow class of potentials such that

[a]The author is grateful to M. Zahradník for pointing out the idea to decompose $e^{-V(m_x)}$ into a sum of two Gaussians and a remainder term that should be expanded. However, contrary to [23] we write the remainder in a multiplicative form which allows for the transition kernel interpretation.

$V(m_x) \le \text{Const } m_x^2$ for large $|m_x|$. Thus, $w(m_x)$ will have to become negative for some $m_x$ e.g. for $V$ compact support or in the $\phi^4$-theory. While it is not necessary to have positive contour activities for some applications (see [4, 23]) it is crucial for the random model: A RG, as devised in [5], needs non-negative contour weights.[b] We are able to solve this problem and define positive effective anharmonic weights by a suitable resummation and careful choice of the parameters $a, b$ of the quadratic potential $Q^{\sigma_x}$; these will be kept fixed. This choice is the only point of the proof that has to be adapted to the specific form of the initial potential $V$. Later the positivity of weights will also be used for the control of the original measure in terms of the Ising-measure (see Proposition 5.2).

In Sec. 2 it is shown how non-negative effective anharmonic weights obeying suitable Peierls bounds can be defined. Section 3 finishes the control of the anharmonicity around the Ising model arising from the purely Gaussian theory (i.e. $w(m_x) \equiv 0$) in terms of a uniformly convergent expansion. Section 4 treats the simple but instructive case of the Ising field without the presence of anharmonicity, showing the emergence of (generalized) Peierls bounds on Ising contours. In Sec. 5 we obtain our final contour model for the full theory and prove Theorems 1 and 2. The Appendix collects some facts about Gaussian random fields and random walk expansions we employ.

## 2. Anharmonic Contours with Positive Weights

We will explain in this section how (preliminary) "anharmonic contours" with "anharmonic weights" that are non-negative and obey a Peierls estimate can be constructed. We start with a combinatorial Lemma 2.1 and a suitable organization of the order of Gaussian integrations appearing to derive algebraically the representation of Lemma 2.3. We will make no specific assumptions about the potential at this point that should however be thought to be symmetric "deep" double-well. Our later treatment is valid once we have the properties of "positivity" and "uniform Peierls condition of anharmonic weights" that are introduced in (2.19) and (2.20). These are then verified for the $\phi^4$-theory in an isolated part of the proof that can be adapted to specific cases of interest.

We will have to deal with the interplay of three different fields: continuous spins $m_x$ (to be integrated out), Ising spins $\sigma_x$ and (fixed) random fields $\eta_x$, subjected to various boundary conditions in various volumes. In some sense, the general theme of the expansions to come is: keep track of the locality of the interaction of these fields in the right way. For the sake of clarity we found it more appropriate in this context to keep a notation that indicates the dependence on these quantities in an explicit way in favor of a more space-saving one.

Now, since we are interested here in a contour-representation of the image measure $T\left(\mu_\Lambda^{\tilde{m}_{\partial\Lambda},\eta_\Lambda}\right)$ under the stochastic transformation (1.6), let us look at the

[b] Vaguely speaking, the method keeps lower bounds on the energies of all configurations, but also upper bounds on the energies of some configurations (that are candidates for the true groundstates). This can be seen nicely in the groundstate-analysis of the models treated in [8]. To do an analogue of this for finite temperatures, non-negative (probabilistic) contour weights are necessary in this framework.

non-normalized weights on Ising-spins given by

$$Z_\Lambda^{\tilde{m}_{\partial\Lambda},\eta_\Lambda}(\sigma_\Lambda) := \int_{\mathbb{R}^\Lambda} dm_\Lambda e^{-E_\Lambda^{\tilde{m}_{\partial\Lambda},\eta_\Lambda}(m_\Lambda)} \prod_{x\in\Lambda} T_x(\sigma_x|m_x) \tag{2.1}$$

so that we get the desired Ising-probabilities dividing by $Z_\Lambda^{\tilde{m}_{\partial\Lambda},\eta_\Lambda} = \sum_{\sigma_\Lambda\in\{-1,1\}^\Lambda} Z_\Lambda^{\tilde{m}_{\partial\Lambda},\eta_\Lambda}(\sigma_\Lambda)$.

To describe our expansions conveniently let us define the following quadratic continuous-spin Hamiltonians, that are made to collect the quadratic terms that arise from the use of (1.13) to the above integral. We write, for finite volume $V\subset\mathbb{Z}^d$,

$$\begin{aligned} H_V^{\tilde{m}_{\partial V},\eta_V,\sigma_V}(m_V) = \frac{q}{2}\sum_{\substack{\{x,y\}\subset V\\ d(x,y)=1}}(m_x-m_y)^2 + \frac{q}{2}\sum_{\substack{x\in V;y\in\partial V\\ d(x,y)=1}}(m_x-\tilde{m}_y)^2 \\ + \frac{a}{2}\sum_{x\in V}(m_x-m^*\sigma_x)^2 - \sum_{x\in V}\eta_x m_x\,. \end{aligned} \tag{2.2}$$

Here and throughout the paper we shall write $\partial G$ for the outer boundary *inside* $\Lambda$, i.e. $\partial G = \{x\in\Lambda\cap G^c; d(x,G)=1\}$. The notion "nearest neighbor" is always meant in the usual sense of the 1-norm. The fixed Ising-spin $\sigma_V\in\{-1,1\}^V$ thus signifies the choice of the well at each site. From the point of view of the continuous fields it is just another parameter.

With this definition we can write the non-normalized Ising-weights (2.1) in the form

$$Z_\Lambda^{\tilde{m}_{\partial\Lambda},\eta_\Lambda}(\sigma_\Lambda) = e^{-b|\Lambda|}\int_{\mathbb{R}^\Lambda} dm_\Lambda e^{-H_\Lambda^{\tilde{m}_{\partial\Lambda},\eta_\Lambda,\sigma_\Lambda}(m_\Lambda)} \prod_{x\in\Lambda}(1+w(m_x))\,. \tag{2.3}$$

If the $w(m_x)$ were identically zero, we would be left with purely Gaussian integrals over Ising-spin dependent quadratic expressions. This Gaussian integration can be carried out and yields

$$\int_{\mathbb{R}^\Lambda} dm_\Lambda e^{-H_\Lambda^{\tilde{m}_{\partial\Lambda},\eta_\Lambda,\sigma_\Lambda}(m_\Lambda)} = C_\Lambda \times e^{-\inf_{m_\Lambda\in\mathbb{R}^\Lambda} H_\Lambda^{\tilde{m}_{\partial\Lambda},\eta_\Lambda,\sigma_\Lambda}(m_\Lambda)} \tag{2.4}$$

with a constant $C_\Lambda$ that does not depend on $\sigma_\Lambda$ (and $\eta_\Lambda$). The latter fact is clear since $\sigma_\Lambda$ (and $\eta_\Lambda$) only couple as linear terms ("magnetic fields") to $m_\Lambda$ while they do not influence the quadratic terms. Note the pleasant fact that no spacial decomposition of the Gaussian integral is needed here and no complicated boundary terms arise.

Now the minimum of the continuous-spin Hamiltonian in the expression on the r.h.s. of (2.4) provides weights for an effective random field Ising model for the spins $\sigma_\Lambda$; its (infinite volume) Hamiltonian is given by the first two terms in (1.9). The treatment of this model is much simpler than that of the full model; all this will be postponed to Sec. 4. There it is discussed in detail how this model can be transformed into a disordered contour model by a mixed low- and high-temperature expansion. However, since this model provides the main part of the final contour model that is responsible for the ferromagnetic phase transition some readers might want to take a look to Sec. 4 to understand the form of our final contour-representation in a simpler situation.

Our present aim is to show how the anharmonic perturbation induced by the $w$-terms can be treated as a positive-weight perturbation of the purely Gaussian model.

Let $U = U^+ \cup (-U^+) \subset \mathbb{R}$, where $U^+$ is a suitable "small" neighborhood of the positive minimizer of the potential $m^*$ that will be determined later and that will depend on the specific form of the potential. The first key step to define non-negative activities is to use the following combinatorial identity on the set $\mathcal{U} = \{x \in \Lambda; m_x \in U\}$.

**Lemma 2.1.** *Let $\Lambda \subset \mathbb{Z}^d$ be finite and connected. For any set $\mathcal{U} \subset \Lambda$ we can write the polynomial $\prod_{x\in\Lambda}(1+w_x)$ in the $|\Lambda|$ variables $(w_x)_{x\in\Lambda}$ in the form*:

$$\prod_{x\in\Lambda}(1+w_x) = 1 + \sum_{G:\emptyset\neq G\subset\Lambda} \prod_{\substack{G_i \\ \text{conn.cp of } G}} \prod_{x\in\partial G_i} 1_{x\in\mathcal{U}} \left[ \prod_{x\in G_i} (1_{x\notin\mathcal{U}} + w_x) - \prod_{x\in G_i} 1_{x\notin\mathcal{U}} \right]. \tag{2.5}$$

The proof is given at the end of this section. Application of Lemma 2.1 gives us the expansion

$$\begin{aligned} e^{b|\Lambda|} Z_\Lambda^{\tilde m_{\partial\Lambda},\eta_\Lambda}(\sigma_\Lambda) &= \int_{\mathbb{R}^\Lambda} dm_\Lambda e^{-H_\Lambda^{\tilde m_{\partial\Lambda},\eta_\Lambda,\sigma_\Lambda}(m_\Lambda)} + \sum_{G:\emptyset\neq G\subset\Lambda} \int_{\mathbb{R}^\Lambda} dm_\Lambda e^{-H_\Lambda^{\tilde m_{\partial\Lambda},\eta_\Lambda,\sigma_\Lambda}(m_\Lambda)} \\ &\quad \times \prod_{\substack{G_i \\ \text{conn.cp of } G}} \prod_{x\in\partial G_i} 1_{m_x\in U} \left[ \prod_{x\in G_i} (1_{m_x\notin U} + w(m_x)) - \prod_{x\in G_i} 1_{m_x\notin U} \right]. \end{aligned} \tag{2.6}$$

Note that the expression under the integral factorizes over connected components of $\overline{G} := G \cup \partial G$.

To introduce the anharmonic (preliminary) weights we need a little preparation. To avoid unnecessary complications in the expansions it is important to organize the Gaussian integral in the following conceptually simple but useful way: We decompose the nonnormalized Gaussian expectation over the terms in the last line into an *outer* integral over $m_{\partial G}$ and a "conditional integral" over $m_{\Lambda\setminus\partial G}$ *given* $m_{\partial G}$. The latter integral factorizes of course over connected components of $\Lambda\setminus\partial G$; in particular the integrals over $\Lambda\setminus\overline{G}$ and $G$ become conditionally independent. W.r.t. this decomposition they appear in a symmetric way.

To write down the explicit formulae we need to introduce:

**Some notation.** The $V \times V$-matrix $\Delta_V$ is the lattice Laplacian with Dirichlet boundary conditions on $V \subset \Lambda$, i.e. $\Delta_{V;x,y} = 1$ iff $x, y \in V$ are nearest neighbors, $\Delta_{V;x,y} = -2d$ iff $x = y \in V$ and $\Delta_{V;x,y} = 0$ else. $\Pi_V$ is the projection operator onto $\Omega_V$ (in short: onto $V$), i.e. $\Pi_{V;x,y} = 1_{x=y\in V}$. We also use the redundant but intuitive notations $m_\Lambda|_V \equiv \Pi_V m_\Lambda \equiv m_V$ for the same thing. $1_V$ is the vector in

$\mathbb{R}^\Lambda$ given by $1_{V;x} = 1_{x\in V}$. For disjoint $V_1, V_2 \subset \Lambda$ we write $\partial_{V_1,V_2}$ for the matrix with entries $\partial_{V_1,V_2;x,y} = 1$ iff $x \in V_1$, $y \in V_2$ are nearest neighbors and $\partial_{V_1,V_2;x,y} = 0$ else. We write $R_V := (c - \Delta_V)^{-1}$ for the corresponding resolvent in the volume $V$. Here and later we put $c = \frac{a}{q}$.

For the sake of clarity we keep (at least for now) the dependence of all quantities on continuous spin-boundary conditions, random fields, Ising-spins, as superscripts. Then we have:

**Lemma 2.2.** *For any subset $G\subset\Lambda$ the random quadratic Hamiltonians* (2.2) *have the decomposition*

$$H_\Lambda^{\tilde m_{\partial\Lambda},\eta_\Lambda,\sigma_\Lambda}(m_\Lambda) = \Delta H_{\partial G,\Lambda}^{\tilde m_{\partial\Lambda},\eta_\Lambda,\sigma_\Lambda}(m_{\partial G}) + \Delta H_{\Lambda\setminus\partial G}^{\tilde m_{\partial\Lambda},m_{\partial G},\eta_{\Lambda\setminus\partial G},\sigma_{\Lambda\setminus\partial G}}(m_{\Lambda\setminus\partial G}) + \inf_{m'_\Lambda} H_\Lambda^{\tilde m_{\partial\Lambda},\eta_\Lambda,\sigma_\Lambda}(m'_\Lambda)\,. \tag{2.7}$$

*Here the "fluctuation-Hamiltonians" are given by*

$$\begin{aligned}\Delta H_{\partial G,\Lambda}^{\tilde m_{\partial\Lambda},\eta_\Lambda,\sigma_\Lambda}&(m_{\partial G})\\ &= \frac{1}{2}\Big\langle \Big(m_{\partial G} - m_\Lambda^{\tilde m_{\partial\Lambda},\eta_\Lambda,\sigma_\Lambda}\big|_{\partial G}\Big), (\Pi_{\partial G}(a - q\Delta_\Lambda)^{-1}\Pi_{\partial G})^{-1}\\ &\quad\times \Big(m_{\partial G} - m_\Lambda^{\tilde m_{\partial\Lambda},\eta_\Lambda,\sigma_\Lambda}\big|_{\partial G}\Big)\Big\rangle_{\partial G}\end{aligned} \tag{2.8}$$

*and the "conditional fluctuation-Hamiltonian"* (*i.e. conditional on $m_{\partial G}$*)

$$\begin{aligned}\Delta H_{\Lambda\setminus\partial G}^{\tilde m_{\partial\Lambda},m_{\partial G},\eta_{\Lambda\setminus\partial G},\sigma_{\Lambda\setminus\partial G}}&(m_{\Lambda\setminus\partial G})\\ &= \frac{1}{2}\Big\langle \Big(m_{\Lambda\setminus\partial G} - m_{\Lambda\setminus\partial G}^{\tilde m_{\partial\Lambda},m_{\partial G},\eta_{\Lambda\setminus\partial G},\sigma_{\Lambda\setminus\partial G}}\Big), (a - q\Delta_{\Lambda\setminus\partial G})\\ &\quad\times \Big(m_{\Lambda\setminus\partial G} - m_{\Lambda\setminus\partial G}^{\tilde m_{\partial\Lambda},m_{\partial G},\eta_{\Lambda\setminus\partial G},\sigma_{\Lambda\setminus\partial G}}\Big)\Big\rangle_{\Lambda\setminus G}\,.\end{aligned} \tag{2.9}$$

*As centerings are occuring: the "global minimizer"*

$$m_\Lambda^{\tilde m_{\partial\Lambda},\eta_\Lambda,\sigma_\Lambda} = R_\Lambda\left(cm^*\sigma_\Lambda + \frac{\eta_\Lambda}{q} + \partial_{\Lambda,\partial\Lambda}\tilde m_{\partial\Lambda}\right) \tag{2.10}$$

*and the "conditional minimizer"*

$$\begin{aligned}m_{\Lambda\setminus\partial G}^{\tilde m_{\partial\Lambda},m_{\partial G},\eta_{\Lambda\setminus\partial G},\sigma_{\Lambda\setminus\partial G}}&\\ = R_{\Lambda\setminus\partial G}&\left(cm^*\sigma_{\Lambda\setminus\partial G} + \frac{\eta_{\Lambda\setminus\partial G}}{q} + \partial_{\Lambda\setminus\partial G,\partial G}m_{\partial G} + \partial_{\Lambda\setminus\partial G,\partial\Lambda}\tilde m_{\partial\Lambda}\right).\end{aligned} \tag{2.11}$$

The proof is a consequence of Appendix Lemma A.1(iii) which is just a statement about symmetric positive definite matrices. Lemma 2.2 can be seen as an explicit expression of the compatibility property for the Gaussian local specifications defined through the Hamiltonian (2.7) in the volumes $\Lambda\setminus\partial G\subset\Lambda$. Indeed, the

Gaussian measure defined with the quadratic form (2.8) describes the distribution on $\Lambda$ projected onto $\partial G$. (Since we will use this formula later for subsets of $\Lambda$ it is convenient to make the $\Lambda$ explicit at this point, too.) The Gaussian measure on $\Lambda\backslash\partial G$ defined with (2.9) is the conditional measure given $m_{\partial G}$.

We would like to stress the following decoupling properties of the conditional expressions. Equation (2.11) for the conditional minimizer decouples over connected components $V_i$ of $\Lambda\backslash\partial G$ since the resolvent $R_{\Lambda\backslash\partial G}$ is just the direct sum of the $R_{V_i}$'s. So we have that

$$\begin{aligned} m_{\Lambda\backslash\partial G}^{\tilde m_{\partial\Lambda},m_{\partial G},\eta_{\Lambda\backslash\partial G},\sigma_{\Lambda\backslash\partial G}}\Big|_{V_i} &= R_{V_i}\left(cm^*\sigma_{V_i}+\frac{\eta_{V_i}}{q}+\partial_{V_i,\partial V_i}m_{\partial V_i}+\partial_{V_i,\partial\Lambda}\tilde m_{\partial\Lambda}\right)\\ &=: m_{V_i}^{\tilde m_{\partial\Lambda},m_{\partial V_i},\eta_{V_i},\sigma_{V_i}} \end{aligned} \tag{2.12}$$

is a function depending only on what is appearing as superscripts, namely random fields and Ising-spins *inside* $V_i$ and continuous-spin boundary condition on $\partial V_i$. (The dependence on the global boundary condition $\tilde m_{\partial\Lambda}$ is of course only through $\tilde m_x$ for $d(x,G_i)=1$. We don't make this explicit in the notation.)

Also, the conditional fluctuation-Hamiltonian on $\Lambda\backslash\partial G$ decomposes into a sum over connected components of its support $\Lambda\backslash\partial G$:

$$\Delta H_{\Lambda\backslash\partial G}^{\tilde m_{\partial\Lambda},m_{\partial G},\eta_{\Lambda\backslash\partial G},\sigma_{\Lambda\backslash\partial G}}(m_{\Lambda\backslash\partial G})=\sum_i \Delta H_{V_i}^{\tilde m_{\partial\Lambda},m_{\partial V_i},\eta_{V_i},\sigma_{V_i}}(m_{V_i})$$

where

$$\begin{aligned} &\Delta H_{V_i}^{\tilde m_{\partial\Lambda},m_{\partial V_i},\eta_{V_i},\sigma_{V_i}}(m_{V_i})\\ &\quad=\frac12\Big\langle\left(m_{V_i}-m_{V_i}^{\tilde m_{\partial\Lambda},m_{\partial V_i},\eta_{V_i},\sigma_{V_i}}\right),(a-q\Delta_{V_i})\\ &\qquad\times\left(m_{V_i}-m_{V_i}^{\tilde m_{\partial\Lambda},m_{\partial V_i},\eta_{V_i},\sigma_{V_i}}\right)\Big\rangle_{V_i}. \end{aligned} \tag{2.13}$$

Putting together the connected components of $\Lambda\backslash\overline{G}$ we can thus write

$$\begin{aligned} &\Delta H_{\Lambda\backslash\partial\bar G}^{\tilde m_{\partial\Lambda},m_{\partial G},\eta_{\Lambda\backslash\partial G},\sigma_{\Lambda\backslash\partial G}}(m_{\Lambda\backslash\partial G})\\ &\quad=\Delta H_{\Lambda\backslash\partial\bar G}^{\tilde m_{\partial\Lambda},m_{\partial G},\eta_{\Lambda\backslash\overline{G}},\sigma_{\Lambda\backslash\overline{G}}}(m_{\Lambda\backslash\overline{G}})\\ &\qquad+\sum_{\substack{G_i\\ \text{conn.cp of } G}}\Delta H_{G_i}^{\tilde m_{\partial\Lambda},m_{\partial G_i},\eta_{G_i},\sigma_{G_i}}(m_{G_i}). \end{aligned} \tag{2.14}$$

So, the sum over $G$'s in (2.6) can be written as

$$\begin{aligned} &\sum_{G:\emptyset\neq G\subset\Lambda}\int_{\mathbb{R}^\Lambda}dm_\Lambda e^{-H_\Lambda^{\tilde m_{\partial\Lambda},\eta_\Lambda,\sigma_\Lambda}(m_\Lambda)}\\ &\qquad\times\prod_{\substack{G_i\\ \text{conn.cp of } G}}\prod_{x\in\partial G_i}1_{m_x\in U}\left[\prod_{x\in G_i}(1_{m_x\notin U}+w(m_x))-\prod_{x\in G_i}1_{m_x\notin U}\right] \end{aligned}$$

$$
\begin{aligned}
&= \sum_{G:\emptyset\neq G\subset\Lambda} e^{-\inf_{m'_\Lambda} H_\Lambda^{\tilde m_{\partial\Lambda},\eta_\Lambda,\sigma_\Lambda}(m'_\Lambda)} \int dm_{\partial G} e^{-\Delta H_{\partial G,\Lambda}^{\tilde m_{\partial\Lambda},\eta_\Lambda,\sigma_\Lambda}(m_{\partial G})} \\
&\quad\times \prod_{x\in\partial G} 1_{m_x\in U} \int dm_{\Lambda\setminus\overline{G}} e^{-\Delta H_{\Lambda\setminus\overline{G}}^{\tilde m_{\partial\Lambda},m_{\partial G},\eta_{\Lambda\setminus\overline{G}},\sigma_{\Lambda\setminus\overline{G}}}(m_{\Lambda\setminus\overline{G}})} \\
&\quad\times \prod_{\substack{G_i \\ \text{conn.cp of } G}} \int dm_{G_i} e^{-\Delta H_{G_i}^{\tilde m_{\partial\Lambda},m_{\partial G_i},\eta_{G_i},\sigma_{G_i}}(m_{G_i})} \\
&\quad\times \left[\prod_{x\in G_i}(1_{m_x\notin U}+w(m_x)) - \prod_{x\in G_i} 1_{m_x\notin U}\right]. \qquad (2.15)
\end{aligned}
$$

Now we note the pleasant fact that the Gaussian integral over $\Lambda\setminus\overline{G}$ is independent of all of the superindexed quantities (since they appear only in the shift of the quadratic form), so that it can be pulled out of the $m_{\partial G}$-integral. It gives

$$
\int dm_{\Lambda\setminus\overline{G}} e^{-\Delta H_{\Lambda\setminus\overline{G}}^{\tilde m_{\partial\Lambda},m_{\partial G},\eta_{\Lambda\setminus\overline{G}},\sigma_{\Lambda\setminus\overline{G}}}(m_{\Lambda\setminus\overline{G}})} = (2\pi)^{\frac{|\Lambda-\overline{G}|}{2}}(\det(a-q\Delta_{\Lambda\setminus\overline{G}}))^{-\frac{1}{2}}. \qquad (2.16)
$$

Let us look at the last two lines now. Conditional on $m_{\partial G}$ we define anharmonic activities by the formula

$$
\begin{aligned}
I_{G_i}^{\tilde m_{\partial\Lambda},m_{\partial G_i},\eta_{G_i},\sigma_{G_i}} &:= \int dm_{G_i} e^{-\Delta H_{G_i}^{\tilde m_{\partial\Lambda},m_{\partial G_i},\eta_{G_i},\sigma_{G_i}}(m_{G_i})} \\
&\qquad \times \left[\prod_{x\in G_i}(1_{m_x\notin U}+w(m_x)) - \prod_{x\in G_i} 1_{m_x\notin U}\right]. \qquad (2.17)
\end{aligned}
$$

We write $I_G^{\tilde m_{\partial\Lambda},m_{\partial G},\eta_G,\sigma_G}=1$ for $G=\emptyset$. So we have obtained the following representation for the non-normalized Ising-weights:

**Lemma 2.3.** *With the above notations we have*

$$
\begin{aligned}
e^{b|\Lambda|} Z_\Lambda^{\tilde m_{\partial\Lambda},\eta_\Lambda}(\sigma_\Lambda) &= e^{-\inf_{m'_\Lambda} H_\Lambda^{\tilde m_{\partial\Lambda},\eta_\Lambda,\sigma_\Lambda}(m'_\Lambda)} \\
&\quad\times \sum_{G:\emptyset\subset G\subset\Lambda} (2\pi)^{\frac{|\Lambda\setminus\overline{G}|}{2}}(\det(a-q\Delta_{\Lambda\setminus\overline{G}}))^{-\frac{1}{2}} \\
&\quad\times \int dm_{\partial G} e^{-\Delta H_{\partial G,\Lambda}^{\tilde m_{\partial\Lambda},\eta_\Lambda,\sigma_\Lambda}(m_{\partial G})} \prod_{x\in\partial G} 1_{m_x\in U} \\
&\quad\times \prod_{\substack{G_i \\ \text{conn.cp of } G}} I_{G_i}^{\tilde m_{\partial\Lambda},m_{\partial G_i},\eta_{G_i},\sigma_{G_i}}. \qquad (2.18)
\end{aligned}
$$

Let us pause for a minute and comment on what we have obtained. For the purely Gaussian model (i.e. the $w$-terms are identically zero) the contributions for $G\neq\emptyset$ vanish. So the above formula is a good starting point for the derivation of the signed-contour representation whose main contributions are provided by the minimum of the Gaussian Hamiltonians in the first line. The main other non-trivial ingredient are the preliminary anharmonic activities $I_G^{\tilde m_{\partial\Lambda}, m_{\partial G}, \eta_G, \sigma_G}$. First of all, the whole construction only makes sense, if we are able to prove a suitable Peierls estimate for them, to be discussed soon. They factorize over connected components $G_i$ of the set $G$. The conditioning on $m_{\partial G}$ has allowed us to have them *local* in the sense that they depend only on random fields and Ising-spins *inside* $G_i$. Note that such a factorization does not hold for the remaining integral over $\partial G$ (that would mean: over connected components of $\partial G$), as it is clear from (2.8). Indeed, the fields $m_{\partial G}$ fluctuate according to the covariance matrix in the total volume $\Lambda$. So to speak, their (stochastic) dependence is mediated by the Gaussian local specification defined with (2.8). Furthermore, the dependence of their mean-value in this local specification is (weakly) on *all* Ising-spins and random fields in $\Lambda$. Both kinds of dependence will have to be expanded later in Sec. 3 when the integral over $\partial G$ is carried out. This will be done by enlarging the "polymers" $G$ and performing a high-temperature expansion. Finally, the determinants provide only trivial modifications of the weights that we will obtain; they can easily be handled by a random walk expansion.

Let us stress the following nice feature of the above representation: "Low-temperature contours" (see Sec. 4) will be created only by the global energy-minimum in the first line. Consequently there will be no complicated boundary terms for these "low-temperature" terms (that could be easily produced by a careless expansion).

Our further treatment of the expansion will be done under the assumption of the following two properties:

**Positivity of anharmonic weights.**

$$I_G^{\tilde m_{\partial\Lambda}, m_{\partial G}, \eta_G, \sigma_G} \geq 0 \tag{2.19}$$

*for all connected* $G$, *and* $\tilde m_{\partial\Lambda} \in U^{\partial\Lambda}$, $m_{\partial G} \in U^{\partial G}$, $\eta_G \in [-\delta, \delta]^G$, $\sigma_G \in \{-1, 1\}^G$.

**Uniform Peierls Condition for anharmonic weights.**

$$I_G^{\tilde m_{\partial\Lambda}, m_{\partial G}, \eta_G, \sigma_G} \leq \epsilon^{|G|} \tag{2.20}$$

*for all connected* $G$, *and* $\tilde m_{\partial\Lambda} \in U^{\partial\Lambda}$, $m_{\partial G} \in U^{\partial G}$, $\eta_G \in [-\delta, \delta]^G$, $\sigma_G \in \{-1, 1\}^G$ *with* $\epsilon > 0$.

Rather than trying to be exhaustive in the description of potentials that satisfy these conditions we will use the rest of this section to fix some properties that imply them and discuss in detail the explicit example of the $\phi^4$-theory in Lemma 2.6. This should however indicate how the above two conditions can be achieved in concrete cases by suitable choices of the neighborhood $U$ and the constants $a$ and $b$ occuring in the quadratic potential. The expansion will be continued in Sec. 3.

Let us start by fixing the following almost trivial one-site criterion. It makes sense if we are assuming the nearest neighbor coupling $q$ to be small.

**Lemma 2.4.** *Suppose that $w(m_x) \geq 0$ for $m_x \in U$.*

(i) *Assume that we have uniformly for all choices of superindices*

$$\int dm_x e^{-\frac{a+4dq}{2}(m_x - m_x^{\tilde m_{\partial\Lambda}, m_{\partial G}, \eta_G, \sigma_G})^2} w(m_x) 1_{m_x \in U}$$

$$\geq \int dm_x e^{-\frac{a}{2}(m_x - m_x^{\tilde m_{\partial\Lambda}, m_{\partial G}, \eta_G, \sigma_G})^2} 1_{m_x \notin U} \,. \tag{2.21}$$

*Then we have the positivity* (2.19).

(ii) *Assume that*

$$\int dm_x e^{-\frac{a}{2}(m_x - m_x^{\tilde m_{\partial\Lambda}, m_{\partial G}, \eta_G, \sigma_G})^2} (w(m_x) 1_{m_x \in U} + (1 + w(m_x)) 1_{m_x \notin U}) \leq \epsilon \,. \tag{2.22}$$

*Then we have the uniform Peierls estimate* (2.20) *with the same $\epsilon$.*

**Proof.** Since we always have $-1 \leq w(m_x) < \infty$ the assumption $1_{m_x \in U} w(m_x) \geq 0$ implies that

$$\int dm_G e^{-\Delta H_G^{\tilde m_{\partial\Lambda}, m_{\partial G}, \eta_G, \sigma_G}(m_G)} \prod_{x \in G} (1_{m_x \notin U} + w(m_x))$$

$$\geq \int dm_G e^{-\Delta H_G^{\tilde m_{\partial\Lambda}, m_{\partial G}, \eta_G, \sigma_G}(m_G)} \prod_{x \in G} w(m_x) 1_{m_x \in U} \geq 0 \,. \tag{2.23}$$

We reduce the estimation of the integrals to product integration by the pointwise estimate on the quadratic form

$$a\|v_G\|_2^2 \leq \langle v_G, (a - q\Delta_G^D) v_G \rangle_G \leq (a + 4dq)\|v_G\|_2^2 \,. \tag{2.24}$$

This gives

$$\int dm_G e^{-\Delta H_G^{\tilde m_{\partial\Lambda}, m_{\partial G}, \eta_G, \sigma_G}(m_G)} \prod_{x \in G} w(m_x) 1_{m_x \in U}$$

$$\geq \prod_{x \in G} \int dm_x e^{-\frac{a+4dq}{2}(m_x - m_x^{\tilde m_{\partial\Lambda}, m_{\partial G}, \eta_G, \sigma_G})^2} w(m_x) 1_{m_x \in U} \tag{2.25}$$

and, on the other hand,

$$\int dm_G e^{-\Delta H_G^{\tilde m_{\partial\Lambda}, m_{\partial G}, \eta_G, \sigma_G}(m_G)} \prod_{x \in G} 1_{m_x \notin U}$$

$$\leq \prod_{x \in G} \int dm_x e^{-\frac{a}{2}\left(m_x - m_x^{\tilde m_{\partial\Lambda}, m_{\partial G}, \eta_G, \sigma_G}\right)^2} 1_{m_x \notin U} \,. \tag{2.26}$$

This proves (i).

The Peierls estimate (ii) follows from dropping the second product in the definition of $I$ and using (2.24) to write

$$
\begin{aligned}
&I_G^{\tilde m_{\partial\Lambda}, m_{\partial G}, \eta_G, \sigma_G} \\
&\quad \le \int dm_G e^{-\Delta H_G^{\tilde m_{\partial\Lambda}, m_{\partial G}, \eta_G, \sigma_G}(m_G)} \prod_{x\in G}((1+w(m_x))1_{m_x\notin U} + w(m_x)1_{m_x\in U}) \\
&\quad \le \prod_{x\in G}\int dm_x e^{-\frac{a}{2}(m_x - m_x^{\tilde m_{\partial\Lambda}, m_{\partial G}, \eta_G, \sigma_G})^2}((1+w(m_x))1_{m_x\notin U} + w(m_x)1_{m_x\in U})
\end{aligned}
\tag{2.27}
$$

□

Next we compute how big the nearest neighbor coupling $q$ and size of the random fields $\delta$ can be in order that any boundary condition in $U$ yields a minimizer of the Gaussian Hamiltonian on $G$ that is "well inside" $U$. We have:

**Lemma 2.5.** *Let $0 < A_1 \le A_2$ and $U^+ = [m^* - A_2, m^* + A_2]$, $U = U^+ \cup (-U^+)$. Assume that $q \le \frac{a}{2d}\left(\frac{2m^*+A_2}{A_1} - 1\right)^{-1}$ and $\delta \le \frac{aA_1}{2}$. Then we have that*

$$\left|m_x^{\tilde m_{\partial\Lambda}, m_{\partial G}, \eta_G, \sigma_G} - m^*\sigma_x\right| \le A_1 \tag{2.28}$$

*for all $G$, $\tilde m_{\partial\Lambda} \in U^{\partial\Lambda}$, $m_{\partial G} \in U^{\partial G}$, $\eta_G \in [-\delta, \delta]^G$, $\sigma_G \in \{-1, 1\}^G$.*

**Proof.** Note the linear dependence $m_x^{\tilde m_{\partial\Lambda}, m_{\partial G}, \eta_G, \sigma_G} = m_x^{\tilde m_{\partial\Lambda}, m_{\partial G}, \eta_G = 0, \sigma_G} + \left(R_G \frac{\eta_G}{q}\right)_x$. Let us thus choose the condition for $q$ s.t.

$$\left|m_x^{\tilde m_{\partial\Lambda}, m_{\partial G}, \eta_G = 0, \sigma_G} - m^*\sigma_x\right| \le \frac{A_1}{2}\,. \tag{2.29}$$

This condition is in fact achieved for a one-point $G = \{x\}$ and the boundary conditions having the "wrong sign" with modulus $m^* + A_2$ as we will formally see as follows. Let us assume that $\sigma_x = -1$ and write this time for simplicity $\partial G$ for the boundary in $\mathbb{Z}^d$ (including possible sites in the outer boundary of $\Lambda$ in $\mathbb{Z}^d$). Then we have, due to the positivity of the matrix elements of $R_G$ that

$$
\begin{aligned}
&m_x^{\tilde m_{\partial\Lambda}, m_{\partial G}, \eta_G = 0, \sigma_G} \\
&\quad \le -R_{G;x,x}cm^* + \sum_{y\in G\setminus\{x\}} R_{G;x,y}cm^* + \left(R_G \partial_{G,\partial G} 1_{\partial G}(m^* + A_2)\right)_x\,.
\end{aligned}
\tag{2.30}
$$

We employ the equation $R_G(c1_G + \partial_{G,\partial G}1_{\partial G}) = 1_G$ to write the last line of (2.30) as

$$m^* - 2R_{G;x,x}cm^* + A_2 - A_2(R_G c 1_G)_x\,. \tag{2.31}$$

We note that $R_{G;x,x}$ is an increasing function in the sets $G \ni x$ (which can be seen by the random walk representation, see Appendix (A.9)). Further $(R_G 1_G)_x$ is an increasing function in $G$. So the maximum over $G$ of (2.31) is achieved for $G = \{x\}$.

With $R_{\{x\};x,x} = \frac{1}{c+2d}$ the value of (2.31) becomes $-m^* + (2m^* + A_2)\,\frac{2d}{c+2d}$ which gives the upper bound

$$m_x^{\tilde m_{\partial\Lambda}, m_{\partial G}, \eta_G = 0, \sigma_{G\setminus\{x\}}, \sigma_x = -1} + m^* \le (2m^* + A_2)\,\frac{2d}{c+2d}\,. \tag{2.32}$$

In the same way we obtain

$$m_x^{\tilde m_{\partial\Lambda}, m_{\partial G}, \eta_G = 0, \sigma_{G\setminus\{x\}}, \sigma_x = -1} + m^* \ge -A_2\frac{2d}{c+2d}\,. \tag{2.33}$$

Equating the r.h.s. with $A_1/2$ gives the r.h.s. of the condition on $q$ stated in the hypothesis.

For the estimate of the random field term note that $0 \le R_{G;x,y} \le R_{\mathbb{Z}^d;x,y}$ and $\sum_{y\in\mathbb{Z}^d} R_{\mathbb{Z}^d;x,y} = \frac{1}{c}$ which give us

$$\left|\sum_{y\in G} R_{G;x,y}\frac{\eta_y}{q}\right| \le \frac{\delta}{q}\sum_{y\in\mathbb{Z}^d} R_{\mathbb{Z}^d;x,y} = \frac{\delta}{a} \le \frac{A_1}{2}\,. \tag{2.34}$$

$\square$

At this stage the treatment has to be made specific to the concrete potential and we specialize to our example, the $\phi^4$-theory with potentials given by (1.2). The following Lemma summarizes how we can produce positivity and an arbitrarily small anharmonic Peierls constant. More specific information can be found in the proof.

**Lemma 2.6.** *For fixed $\epsilon_0 > 0$ we put*

$$U^+ = [m^* - (\epsilon_0 m^*)^{\frac13}, m^* + (\epsilon_0 m^*)^{\frac13}]\,. \tag{2.35}$$

*Then we have*

(i) *For any value of $\epsilon_0, m^*, q, \delta$ there exists a choice of parameters $a$ and $b$ such that the anharmonic weights obey the positivity* (2.19).

*Furthermore there exist strictly positive constants $a(m^*,\epsilon_0)$, $b(m^*,\epsilon_0)$, $q_0(m^*,\epsilon_0)$, and $\delta_0(m^*,\epsilon_0)$ such that the following is true.*

(ii) *For all $q \le q_0(m^*,\epsilon_0)$ and $\delta \le \delta_0(m^*,\epsilon_0)$ we have the Peierls estimate* (2.20) *with a constant $\epsilon(\epsilon_0, m^*)$ that is independent of $q, \delta$.*

(iii) *If $\epsilon_0$ is small enough this constant obeys the estimate $\epsilon(\epsilon_0, m^*) \le \frac{\epsilon_0}{10}$ whenever $m^* \ge m_0^*(\epsilon_0)$ is large enough.*

*The above constants can be chosen like*

$$a(m^*,\epsilon_0) = \frac{(2 + {\epsilon_0}^{\frac13} {m^*}^{-\frac23})^2}{4} \sim 1$$

$$q(m^*,\epsilon_0) = \frac{a(m^*,\epsilon_0)}{2d}\left(20{\epsilon_0}^{-\frac13}{m^*}^{\frac23} + 9\right)^{-1}\,, \quad \delta_0(m^*,\epsilon_0) = \frac{a(m^*,\epsilon_0)\,(\epsilon_0 m^*)^{\frac13}}{20} \tag{2.36}$$

*and $b(m^*,\epsilon_0) \sim e^{-\mathrm{const}\, {m^*}^{\frac23}}$ with $m^* \uparrow \infty$.*

**Proof.** We will take time to motivate our choices of the parameters that are made to ensure the validity of the assumptions of Lemma 2.4. Let us write the neighborhood $U^+$ in the form $U^+ = [(1-\epsilon_1)m^*, (1+\epsilon_1)m^*]$ and show why the choice of $\epsilon_1$ given in (2.35) comes up. The zeroth requirement on $a$ and $b$ we have to meet is $w(m_x)1_{m_x \in U} \geq 0$. So, let us choose the Gaussian curvature $a > 1$ to be the smallest number s.t. we have, for all $m_x \in U^+$, that the Gaussian centered around $m^*$ is dominated by the true potential i.e.

$$e^{-\frac{a(m_x-m^*)^2}{2}} \leq e^{-V(m_x)} \tag{2.37}$$

with equality for $m_x = (1+\epsilon_1)m^*$. This amounts to $a = \frac{(2+\epsilon_1)^2}{4}$, as in (2.36). Then we have on $U^+$ for the Gaussian centered around $-m^*$

$$e^{-\frac{a(m_x+m^*)^2}{2}} \leq e^{-\frac{(2+\epsilon_1)^2(2-\epsilon_1)^2+1-(1+\epsilon_1)^2}{8}m^{*2}} e^{-V(m_x)} \tag{2.38}$$

which gives us the estimate

$$1 + w(m_x) \geq e^b \left[1 + e^{-\frac{(2+\epsilon_1)^2(2-\epsilon_1)^2+1-(1+\epsilon_1)^2}{8}m^{*2}}\right]^{-1} \tag{2.39}$$

on $U^+$. Any choice of $e^b$ bigger than the denominator thus ensures $w(m_x)$ $1_{m_x \in U} \geq 0$.

To have property (i) in Lemma 2.4. we have to choose $e^b$ even bigger. Obviously it is implied by

$$\inf_{m_x \in U^+} w(m_x) \geq \frac{\int dm_x e^{-\frac{a}{2}(m_x - m_x^{\tilde{m}_{\partial\Lambda}, m_{\partial G}, \eta_G, \sigma_G})^2} 1_{m_x \notin U}}{\int dm_x e^{-\frac{a+4dq}{2}(m_x - m_x^{\tilde{m}_{\partial\Lambda}, m_{\partial G}, \eta_G, \sigma_G})^2} 1_{m_x \in U}}. \tag{2.40}$$

But note that we always have

$$\left| |m_x^{\tilde{m}_{\partial\Lambda}, m_{\partial G}, \eta_G, \sigma_G}| - m^* \right| \leq \hat{m}^{\max}(m^*, \delta, q, a) \tag{2.41}$$

with a constant $\hat{m}^{\max}(m^*, \delta, q, a)$ that is finite for any fixed $m^*, \delta, q, a$ and that is estimated by Lemma 2.5. So the trivial choice

$$\begin{aligned} e^{b(m^*,\delta,q,a)} :=& \left(1 + e^{-\frac{(2+\epsilon_1)^2(2-\epsilon_1)^2+1-(1+\epsilon_1)^2}{8}m^{*2}}\right) \\ &\times \left(1 + \sup_{\widehat{m}: |\widehat{m}| \leq \widehat{m}^{\max}(m^*,\delta,q,a)} \frac{\int dm_x e^{-\frac{a}{2}(m_x - \widehat{m})^2} 1_{m_x \notin U}}{\int dm_x e^{-\frac{a+4dq}{2}(m_x - \widehat{m})^2} 1_{m_x \in U}}\right) \end{aligned} \tag{2.42}$$

gives some finite number and ensures the positivity of the anharmonic activities. This proves (i).

Let us now turn to quantitative estimates on the Peierls constant. To start with, the above definition of $b$ is of course only useful if $b$ will be small. Now, the r.h.s. of (2.42) is small whenever the centering of the Gaussian integrals is “safe” inside $U$ and the neighborhood $U$ is big enough to carry most of the Gaussian integral. We apply Lemma 2.5 with $A_2 = \epsilon_1 m^*$ and $A_1 = \frac{A_2}{10}$. The hypotheses of the Lemma then give us the conditions $q \le q_0$ and $\delta \le \delta_0$ with

$$q_0 = \frac{a}{2d}\left(\frac{20}{\epsilon_1} + 9\right)^{-1}, \quad \delta_0 = \frac{a\epsilon_1 m^*}{20}. \tag{2.43}$$

Then we have

$$\begin{aligned} &\frac{\int dm_x e^{-\frac{a}{2}(m_x - m_x^{\tilde m_{\partial\Lambda}, m_{\partial G}, \eta_G, \sigma_G})^2} 1_{m_x \notin U}}{\int dm_x e^{-\frac{a+4dq}{2}(m_x - m_x^{\tilde m_{\partial\Lambda}, m_{\partial G}, \eta_G, \sigma_G})^2} 1_{m_x \in U}} \\ &\le \sqrt{\frac{a+4dq}{a}} \frac{\mathbb{P}\left[|G| \ge \sqrt{a}\frac{9\epsilon_1 m^*}{10}\right]}{1 - \mathbb{P}\left[|G| \ge \sqrt{a+2dq}\frac{9\epsilon_1 m^*}{10}\right]}. \end{aligned} \tag{2.44}$$

This shows that $b \sim e^{-\text{const}\cdot(\epsilon_1 m^*)^2}$ tends to zero rapidly if $\epsilon_1 m^*$ is getting large.

Let us now see what Peierls constant we get according to Lemma 2.4 (ii). This will explain why the neighborhood $U^+$ should in fact be of the form (2.35).

Our choice of $U$ and $a$ yields that we have, for all $m_x \in U^+$, that

$$e^{-V(m_x) + \frac{a(m_x - m^*)^2}{2}} \le e^{\epsilon_1 (m_x - m^*)^2}. \tag{2.45}$$

This gives $1 + w(m_x) \le e^{b + \epsilon_1 (m_x - m^*)^2}$. From this we have

$$\begin{aligned} &\int dm_x e^{-\frac{a}{2}\left(m_x - m_x^{\tilde m_{\partial\Lambda}, m_{\partial G}, \eta_G, \sigma_G}\right)^2} (1 + w(m_x)) 1_{m_x \in U^{\sigma_x}} \\ &\le e^b \int dm_x e^{-\frac{a}{2}(m_x - m_x^{\tilde m_{\partial\Lambda}, m_{\partial G}, \eta_G, \sigma_G})^2 + \epsilon_1 (m_x - m^*)^2} \\ &= e^b \sqrt{\frac{2\pi}{a - 2\epsilon_1}} e^{\frac{a\epsilon_1}{a - 2\epsilon_1}(m_x^{\tilde m_{\partial\Lambda}, m_{\partial G}, \eta_G, \sigma_G} - m^*)^2} \\ &\le e^b \sqrt{\frac{2\pi}{a - 2\epsilon_1}} e^{\frac{a\epsilon_1^3 m^{*2}}{100(a - 2\epsilon_1)}} \end{aligned} \tag{2.46}$$

and hence

$$\begin{aligned} &\int dm_x e^{-\frac{a}{2}(m_x - m_x^{\tilde m_{\partial\Lambda}, m_{\partial G}, \eta_G, \sigma_G})^2} (1 + w(m_x)) 1_{m_x \in U} \\ &\le 2e^b \left[\sqrt{\frac{2\pi}{a - 2\epsilon_1}} e^{\frac{a\epsilon_1^3 m^{*2}}{100(a - 2\epsilon_1)}} - \sqrt{\frac{2\pi}{a}} \mathbb{P}\left[|G| \le \sqrt{a}\frac{9\epsilon_1 m^*}{10}\right]\right]. \end{aligned} \tag{2.47}$$

Indeed, the l.h.s. is $\mathcal{O}(\epsilon_1^3 m^{*2}) + \mathcal{O}(\epsilon_1)$ and thus imposes the condition that $\epsilon_1^3 m^{*2}$ be small! This estimate essentially cannot be improved upon. It determines the dependence of the Peierls constant $\epsilon$ on $\epsilon_1$ and $m^*$.

Finally, the integrals over $U^c$ are much smaller: Indeed, for the bounded part of $U^c$ we estimate

$$\begin{aligned}
&\int_0^{(1-\epsilon_1)m^*} dm_x e^{-\frac{a}{2}(m_x - m_x^{\tilde m_{\partial\Lambda}, m_{\partial G}, \eta_G, \sigma_G})^2}(1+w(m_x)) \\
&\quad \le \int_0^{(1-\epsilon_1)m^*} dm_x e^{-\frac{a}{2}(m_x - m_x^{\tilde m_{\partial\Lambda}, m_{\partial G}, \eta_G, \sigma_G})^2} e^{-V(m_x)+\frac{a(m_x-m^*)^2}{2}} \\
&\quad = e^{-\frac{a}{2}(m^* - m_x^{\tilde m_{\partial\Lambda}, m_{\partial G}, \eta_G, \sigma_G})^2} \\
&\qquad \times \int_0^{(1-\epsilon_1)m^*} dm_x e^{-a(m^* - m_x^{\tilde m_{\partial\Lambda}, m_{\partial G}, \eta_G, \sigma_G})(m_x - m^*)} e^{-V(m_x)} . \qquad (2.48)
\end{aligned}$$

We have for the last integral

$$\begin{aligned}
&\int_0^{(1-\epsilon_1)m^*} dm_x e^{-a(m^* - m_x^{\tilde m_{\partial\Lambda}, m_{\partial G}, \eta_G, \sigma_G})(m_x - m^*)} e^{-V(m_x)} \\
&\quad \le \int_0^{(1-\epsilon_1)m^*} dm_x e^{\frac{a\epsilon_1 m^*}{10}(m_x - m^*)} e^{-V(m_x)} \\
&\quad \le \int_0^{(1-\epsilon_1)m^*} dm_x e^{\frac{a\epsilon_1 m^*}{10}(m_x - m^*)} e^{-\frac{(m_x-m^*)^2}{8}} . \qquad (2.49)
\end{aligned}$$

The maximizer of the last exponent is $m_x = m^* + \frac{2a\epsilon_1 m^*}{10}$ which is outside the range of integration (due to our choice of the 10 before (2.43)). Estimating for simplicity the integral by the value of the integrand at $(1-\epsilon_1)m^*$ just gives

$$\int_0^{(1-\epsilon_1)m^*} dm_x e^{-\frac{a}{2}(m_x - m_x^{\tilde m_{\partial\Lambda}, m_{\partial G}, \eta_G, \sigma_G})^2}(1+w(m_x)) \le m^* e^{-(\frac{1}{8}-\frac{a}{10})(\epsilon_1 m^*)^2} . \qquad (2.50)$$

For the unbounded part of $U^c$ where $m \ge m^*(1+\epsilon_1)$ we have with our choice of $a$ that $1 + w(m_x) \le 1$. This gives us

$$\begin{aligned}
&\int_{(1+\epsilon_1)m^*}^{\infty} dm_x e^{-\frac{a}{2}(m_x - m_x^{\tilde m_{\partial\Lambda}, m_{\partial G}, \eta_G, \sigma_G})^2}(1+w(m_x)) \\
&\quad \le \sqrt{\frac{2\pi}{a}} \mathbb{P}\left[G \ge \sqrt{a}\frac{9\epsilon_1 m^*}{10}\right] \le e^{-\text{const}(\epsilon_1 m^*)^2} \qquad (2.51)
\end{aligned}$$

Collecting the terms gives our final estimate on the Peierls constant

$$\begin{aligned}
\epsilon \le 2e^b \Bigg[ &\sqrt{\frac{2\pi}{a-2\epsilon_1}} e^{\frac{a\epsilon_1^3 m^{*2}}{100(a-2\epsilon_1)}} - \sqrt{\frac{2\pi}{a}} \\
&+ m^* e^{-(\frac{1}{8}-\frac{a}{10})(\epsilon_1 m^*)^2} + 3\sqrt{\frac{2\pi}{a}} \mathbb{P}\left[G \ge \sqrt{a}\frac{9\epsilon_1 m^*}{10}\right] \Bigg] . \qquad (2.52)
\end{aligned}$$

From here the lemma follows. □

**Proof of Lemma 2.1.** We expand $\prod_{x\in\Lambda}(1+w_x)=1+\sum_{\Lambda_0:\emptyset\neq\Lambda_0\subset\Lambda}\prod_{x\in\Lambda_0}w_x$. Let $A(\Lambda_0)\subset(\Lambda\backslash\mathcal{U})\backslash\Lambda_0$ denote the maximal set amongst the sets $A\subset(\Lambda\backslash\mathcal{U})\backslash\Lambda_0$ that are connected to $\Lambda_0$. (We say that a set $A$ is connected to a set $\Lambda_0$ iff, for each point $u$ in $A$, there exists a nearest neighbor path inside $A\cup\Lambda_0$ that joins $u$ and some point in $\Lambda_0$.) Equivalently, this $A(\Lambda_0)$ is the unique set $A\subset\Lambda\backslash\Lambda_0$ s.t. $x\notin\mathcal{U}$ for all $x\in A$ and $x\in\mathcal{U}$ for all $x\in\partial(\Lambda_0\cup A)$.

We collect terms according to the sets $G=\Lambda_0\cup A(\Lambda_0)$. Denoting by $G_i$ the connected components of $G$ and by $L_i=\Lambda_0\cap G_i$ we have then

$$\begin{aligned}\prod_{x\in\Lambda}(1+w_x)&=1+\sum_{\Lambda_0:\emptyset\neq\Lambda_0\subset\Lambda}\prod_{x\in A(\Lambda_0)}1_{x\notin\mathcal{U}}\prod_{x\in\partial(\Lambda_0\cup A(\Lambda_0))}1_{x\in\mathcal{U}}\prod_{x\in\Lambda_0}w_x\\&=1+\sum_{G:\emptyset\neq G\subset\Lambda}\prod_{\substack{G_i\\ \text{conn.cp of } G}}\sum_{L_i:\emptyset\neq L_i\subset G_i}\prod_{x\in G_i\backslash L_i}1_{x\notin\mathcal{U}}\prod_{x\in\partial G_i}1_{x\in\mathcal{U}}\prod_{x\in L_i}w_x\,.\end{aligned}\tag{2.53}$$

Adding and subtracting the term for $L_i=\emptyset$ we have

$$\sum_{L_i:\emptyset\neq L_i\subset G_i}\prod_{x\in G_i\backslash L_i}1_{x\notin\mathcal{U}}\prod_{x\in L_i}w_x=\prod_{x\in G_i}(1_{x\notin\mathcal{U}}+w_x)-\prod_{x\in G_i}1_{x\notin\mathcal{U}}\tag{2.54}$$

which proves the lemma. □

## 3. Control of Anharmonicity

We start from the representation of Lemma 2.3 for the non-normalized Ising weights. We assume positivity and Peierls condition for the anharmonic ($I$-) weights as discussed in Sec. 2 and verified for the $\phi^4$-potential. Carrying out the last remaining continuous spin-integral we express the last two lines in (2.18) in terms of activities that are positive, obey a Peierls estimate and depend in a *local way* on the Ising-spin configuration $\sigma_\Lambda$ and the realization of the random fields $\eta_\Lambda$. We stress that all estimates that follow will be uniform in the Ising-spin configuration and the configuration of the random field.

The result of this is:

**Proposition 3.1.** *Assume that the anharmonic $I$-weights* (2.17) *satisfy the Positivity* (2.19) *and the uniform Peierls Condition* (2.20) *with a constant $\epsilon$. Suppose that $\epsilon$ is sufficiently small, $q$ is sufficiently small, $a$ is of the order one, $q(m^*)^2$ sufficiently large. Suppose that $\delta\le\text{Const}\,m^*$ and $|U|\le\text{Const}\,m^*$ with constants of the order unity.*

*Then, for any continuous-spin boundary condition $\tilde m_{\partial\Lambda}\in U^{\partial\Lambda}$ and any realization of the random fields $\eta_\Lambda\in[-\delta,\delta]^\Lambda$, the non-normalized Ising weights* (2.1) *have the representation*

$$\begin{aligned}Z_\Lambda^{\tilde m_{\partial\Lambda},\eta_\Lambda}(\sigma_\Lambda)&=e^{-b|\Lambda|}(2\pi)^{\frac{|\Lambda|}{2}}(\det(a-q\Delta_\Lambda))^{-\frac12}e^{-\inf_{m'_\Lambda}H_\Lambda^{\tilde m_{\partial\Lambda},\eta_\Lambda,\sigma_\Lambda}(m'_\Lambda)}\\&\quad\times\sum_{G:\emptyset\subset G\subset\Lambda}\bar\rho^{\tilde m_{\partial_{\partial\Lambda}G}}(G;\sigma_G,\eta_G)\end{aligned}\tag{3.1}$$

*where the activity $\bar\rho$ appearing under the $G$-sum is non-negative and depends only on the indicated arguments. $\bar\rho$ factorizes over the connected components $G_i$ of its support $G$, i.e.*

$$\bar\rho^{\tilde m_{\partial_{\partial\Lambda}G}}(G;\sigma_G,\eta_G)=\prod_i \bar\rho^{\tilde m_{\partial_{\partial\Lambda}G_i}}(G_i;\sigma_{G_i},\eta_{G_i}) \tag{3.2}$$

*and we have $\bar\rho^{\tilde m_{\partial_{\partial\Lambda}G}}(G=\emptyset;\sigma_G,\eta_G)=1$.*

*$\bar\rho$ has the "infinite volume symmetries" of:*

(a) *Invariance under joint flips of spins and random fields $\bar\rho(G;\sigma_G,\eta_G)=\bar\rho(G;-\sigma_G,-\eta_G)$ if $G$ does not touch the boundary (i.e. $\partial_{\partial\Lambda}G=\emptyset$)*
(b) *Invariance under lattice shifts $\bar\rho(G;\sigma_G,\eta_G)=\bar\rho(G+t;\sigma_{G+t},\eta_{G+t})$ if $G, G+t\subset\Lambda$ do not touch the boundary.*
*We have the uniform Peierls estimate*

$$\bar\rho^{\tilde m_{\partial_{\partial\Lambda}\bar G}}(\bar G;\sigma_{\bar G},\eta_{\bar G})\le e^{-\alpha|\bar G|} \tag{3.3}$$

*with*

$$\alpha=\text{const}\times\min\left\{\log\frac{1}{q},\log\frac{1}{\epsilon}\left(\frac{\log\frac{1}{q}}{\log m^*}\right)^d\right\}.$$

**Remark 1.** Note that the first line of (3.1) gives the value for vanishing anharmonicity (i.e. $w(m_x)\equiv 0$).

**Remark 2.** For any fixed Ising-spin $\sigma_\Lambda$ and realization of random fields $\eta_\Lambda$ the sum in the last line is the partition function of a non-translation invariant polymer model for polymers $G$. Note that there is no suppression of the activities $\bar\rho$ in the above bounds in terms of the Ising-spins. From the point of view of the polymers $G$ the Ising spins and random fields play the similar role of describing an "external disorder".

**Proof of Proposition 3.1.** To yield this representation we must treat the last two lines of (2.18). We cannot carry out the $m_{\partial G}$-integral directly but need some further preparation that allows us to treat the "long range" parts of the exponent by a high-temperature expansion. Depending on the parameters of the model (to be discussed below) we will then have to enlarge and glue together connected components of the support $G$. For any set $G\subset\Lambda$ we write

$$G^r=\{x\in\Lambda; d(x,G)\le r\} \tag{3.4}$$

for the $r$-hull of $G$ in $\Lambda$. Then we have, under the assumptions on the parameters as in Proposition 3.1.

**Lemma 3.2.** *There is a choice of $r\sim \mathrm{Const}\frac{\log m^*}{\log(\frac{1}{q})}$ such that the following is true. For each fixed subset $G\subset\Lambda$, continuous-spin boundary condition $\tilde m_{\partial\Lambda}\in U^{\partial\Lambda}$,* *fixed Ising-configuration* $\sigma_\Lambda \in \{-1,1\}^\Lambda$ *and random fields* $\eta_\Lambda \in [-\delta,\delta]^\Lambda$ *we can write*

$$\int dm_{\partial G} e^{-\Delta H^{\tilde m_{\partial\Lambda},\eta_\Lambda,\sigma_\Lambda}_{\partial G,\Lambda}(m_{\partial G})}\, 1_{m_{\partial G}\in U^{\partial G}}\, I_G^{\tilde m_{\partial\Lambda},m_{\partial G},\eta_G,\sigma_G}$$

$$= (2\pi)^{\frac{|\partial G|}{2}}\sqrt{\det(\Pi_{\partial G}(a-q\Delta_{G^r})^{-1}\Pi_{\partial G})}\sum_{\substack{\tilde G:\tilde G\subset\Lambda\\ G^r\subset\tilde G}}\rho^{\tilde m_{\partial_{\partial\Lambda}\tilde G}}(G,\tilde G;\sigma_{\tilde G},\eta_{\tilde G}), \quad (3.5)$$

*where the activity appearing under the* $\tilde G$*-sum depends only on the indicated arguments and obeys the uniform bounds*

$$0 \le \rho^{\tilde m_{\partial_{\partial\Lambda}\tilde G}}(G,\tilde G;\sigma_{\tilde G},\eta_{\tilde G}) \le e^{-\bar\alpha|\tilde G|} \quad (3.6)$$

*with*

$$\bar\alpha = \text{Const}\times\min\left\{\log\frac{1}{q},\log\frac{1}{\epsilon}\left(\frac{\log\frac{1}{q}}{\log m^*}\right)^d\right\}.$$

*It factorizes over the connected components* $\tilde G_i$ *of the set* $\tilde G$*, i.e.*

$$\rho^{\tilde m_{\partial_{\partial\Lambda}\tilde G}}(G,\tilde G;\sigma_{\tilde G},\eta_{\tilde G}) = \prod_i \rho^{\tilde m_{\partial_{\partial\Lambda}\tilde G_i}}(G\cap\tilde G_i,\tilde G_i;\sigma_{\tilde G_i},\eta_{\tilde G_i}). \quad (3.7)$$

*For* $\tilde G$ *not touching the boundary (i.e.* $\partial_{\partial\Lambda}\tilde G=\emptyset$*)* $\rho$ *is invariant under joint flips of spins and random fields and lattice shifts.*

**Remark.** Later it will be convenient to have the determinant appearing on the r.h.s.; in fact it could also be absorbed in the activities under the $\tilde G$-sum.

**Proof of Lemma 3.2.** Let us recall definition (2.8) of the "fluctuation-Hamiltonian" (involving the global minimizer (2.10)) which gives the Hamiltonian of the projection onto $\partial G$ of an Ising-spin and random-field dependent Gaussian field in $\Lambda$. Our first step is to decompose this projection from $\Lambda$ onto $\partial G$ into a "low temperature-part" and a "high temperature-part". For fixed $G$ we will consider definition (2.8) where $\Lambda$ will be replaced by $G^r$; for $r$ large enough the resulting term "low-temperature"- term is close enough to the full expression, so that the rest can be treated by a high-temperature expansion.

We write $\partial_B A := \{x\in B; d(x,A)=1\}$ for the outer boundary in a set $B\subset\mathbb{Z}^d$. Recall that, with this notation $\partial A = \partial_\Lambda A$, so that $\partial_{\mathbb{Z}^d}(G^r) = \partial_{\partial\Lambda}(G^r)\cup\partial(G^r)$.

Then the precise form of the decomposition we will use reads:

**Lemma 3.3.** *With a suitable choice of* $r\sim\text{Const}\frac{\log m^*}{\log(\frac{1}{q})}$ *we have*

$$\Delta H^{\tilde m_{\partial\Lambda},\eta_\Lambda,\sigma_\Lambda}_{\partial G,\Lambda}(m_{\partial G}) = \Delta H^{(\tilde m_{\partial_{\partial\Lambda}G^r},0_{\partial_\Lambda G^r}),\eta_{G^r},\sigma_{G^r}}_{\partial G,G^r}(m_{\partial G})$$

$$+\sum_{\substack{C\subset\Lambda\\ C\cap\partial G\neq\emptyset;C\cap(G^r)^c\neq\emptyset}}\bar H^{\mathrm{HT}}_{\partial G,G^r}(m_{\partial G},\sigma_{G^r},\eta_{G^r};C,\sigma_C,\eta_C), \quad (3.8)$$

*where the functions appearing under the $C$-sum depend only on the indicated arguments and obey the uniform bound*

$$\left|\bar{H}^{\mathrm{HT}}_{\partial G, G^r}(m_{\partial G}, \sigma_{G^r}, \eta_{G^r}; C, \sigma_C, \eta_C)\right| \leq e^{-\tilde{\alpha}|C|} \tag{3.9}$$

*uniformly in $m_{\partial G} \in U^{\partial G}$ and all other quantities for the $C$'s occuring in the sum in* (3.8). *Here* $\tilde{\alpha} = \mathrm{const} \log \frac{1}{q}$.

**Remark.** Note that the first part ("low temperature-part") decomposes of course over the connected components $(G^r)_i$ of $G^r$, i.e.

$$\begin{aligned}
&\Delta H^{(\tilde{m}_{\partial_{\partial\Lambda} G^r}, 0_{\partial_\Lambda G^r}), \eta_{G^r}, \sigma_{G^r}}_{\partial G, G^r}(m_{\partial G}) \\
&\quad = \sum_i \Delta H^{(\tilde{m}_{\partial_{\partial\Lambda}(G^r)_i}, 0_{\partial_\Lambda (G^r)_i}), \eta_{(G^r)_i}, \sigma_{(G^r)_i}}_{\partial G \cap (G^r)_i, (G^r)_i}(m_{\partial G \cap (G^r)_i}) .
\end{aligned} \tag{3.10}$$

**Proof of Lemma 3.3.** The l.h.s. and the first term on the r.h.s. of (3.8) differ in two places: The matrix and the centerings. We expand both differences using the random walk representation.

The decomposition of the matrix into the matrix where $\Lambda$ is replaced by $G^r$ and a remainder term can be written as

$$\begin{aligned}
(\Pi_{\partial G} R_\Lambda \Pi_{\partial G})^{-1} &= (\Pi_{\partial G} R_{G^r} \Pi_{\partial G})^{-1} \\
&\quad - \sum_{\substack{C \subset \Lambda \setminus \partial G \\ C \cap (G^r)^c \neq \emptyset, C \cap G^2 \neq \emptyset}} \partial_{\partial G, \Lambda \setminus \partial G} \mathcal{R}(\cdot \to \cdot\,; C) \partial_{\Lambda \setminus \partial G, \partial G} ,
\end{aligned} \tag{3.11}$$

where the $\Lambda \times \Lambda$-matrix $\mathcal{R}(\cdot \to \cdot\,; C)$ has non-zero entries only for $x, y \in C$ that are given by

$$\mathcal{R}(x \to y\,; C) = \sum_{\substack{\text{paths } \gamma \text{ from } x \text{ to } y \\ \mathrm{Range}(\gamma) = C}} \left(\frac{1}{c + 2d}\right)^{|\gamma|+1} . \tag{3.12}$$

For the proof of this formula see the Appendix (A.8) ff. and (A.13). where more details about the random walk expansion can be found.

Simply from the decomposition of the resolvent $R_\Lambda = R_{G^r} + (R_\Lambda - R_{G^r})$ and the random walk representation for the second term follows the formula for the centerings

$$m_\Lambda^{\tilde{m}_{\partial\Lambda}, \eta_\Lambda, \sigma_\Lambda} = m_{G^r}^{\tilde{m}_{\partial\Lambda}, \eta_{G^r}, \sigma_{G^r}} + \sum_{\substack{C \subset \Lambda \\ C \cap (G^r)^c \neq \emptyset}} \bar{m}(C; \sigma_C, \eta_C) \tag{3.13}$$

with

$$m_{G^r}^{\tilde{m}_{\partial\Lambda}, \eta_{G^r}, \sigma_{G^r}} := R_{G^r}\left(c m^* \sigma_{G^r} + \frac{\eta_{G^r}}{q} + \partial_{G^r, \partial\Lambda} \tilde{m}_{\partial\Lambda}\right) \tag{3.14}$$

and "high-temperature" terms given by the matrix product

$$\bar{m}(C; \sigma_C, \eta_C) = \mathcal{R}(\cdot \to \cdot\,; C)\left(c m^* \sigma_\Lambda + \frac{\eta_\Lambda}{q} + \partial_{C, \partial\Lambda} \tilde{m}_{\partial\Lambda}\right) . \tag{3.15}$$

From the bound on the resolvent (A.12) we have uniformly

$$|\bar{m}_x(C;\sigma_C,\eta_C)| \leq \mathrm{Const}(m^*+\delta)\left(1+\frac{a}{2dq}\right)^{-|C|}. \tag{3.16}$$

This quantity is in turn bounded by, say, $(1+\frac{a}{2dq})^{-|C|/2}$ if we have that $|C| \geq r$ with $r := \mathrm{Const}\frac{\log m^*}{\log(1+\frac{a}{2dq})}$. So we have $r \sim \mathrm{Const}\frac{\log m^*}{\log(\frac{1}{q})}$ for small $q$.

To write both type of summations over connected sets $C$ in the same form we note that

$$\begin{aligned}
&\sum_{\substack{C_1\subset\Lambda\setminus\partial G\\ C_1\cap(G^r)^c\neq\emptyset, C_1\cap G^2\neq\emptyset}} \partial_{\partial G,\Lambda\setminus\partial G}\mathcal{R}(\cdot\to\cdot;C_1)\partial_{\Lambda\setminus\partial G,\partial G}\\
&\quad=\sum_{\substack{C_2\subset\Lambda\\ C_2\cap\partial G\neq\emptyset; C_2\cap(G^r)^c\neq\emptyset}} \partial_{\partial G,\Lambda\setminus\partial G}\mathcal{R}(\cdot\to\cdot;C_2\setminus\partial G)\partial_{\Lambda\setminus\partial G,\partial G}1_{C_2\setminus\partial G\,\mathrm{conn.}}
\end{aligned}\tag{3.17}$$

which gives us the same range of summation for both sort of terms. The expansion then produces triple sums over connected sets $C$. Collecting terms according to the union of the occuring $C$'s we obtain the desired decomposition with

$$\begin{aligned}
&\bar{H}^{\mathrm{HT}}_{\partial G,G^r}(m_{\partial G},\sigma_{G^r},\eta_{G^r};C,\sigma_C,\eta_C)\\
&\quad=-\frac{q}{2}\langle(m_{\partial G}-\bar{m}^{\sigma_{G^r}}_{\partial G}),\partial_{\partial G,\Lambda\setminus\partial G}\mathcal{R}(\cdot\to\cdot;C\setminus\partial G)\partial_{\Lambda\setminus\partial G,\partial G}1_{C\setminus\partial G\,\mathrm{conn.}}\\
&\qquad\times(m_{\partial G}-\bar{m}^{\sigma_{G^r}}_{\partial G})\rangle+q\langle(m_{\partial G}-\bar{m}^{\sigma_{G^r}}_{\partial G}),(\Pi_{\partial G}R_{G^r}\Pi_{\partial G})^{-1}\bar{m}(C;\sigma_C,\eta_C)\rangle\\
&\qquad-q\sum_{\substack{C_1,C_2\subset\Lambda;C_1\cup C_2=C\\ C_i\cap\partial G\neq\emptyset;C_i\cap(G^r)^c\neq\emptyset}}\langle(m_{\partial G}-\bar{m}^{\sigma_{G^r}}_{\partial G}),\partial_{\partial G,\Lambda\setminus\partial G}\mathcal{R}(\cdot\to\cdot;C_1\setminus\partial G)\\
&\qquad\times\partial_{\Lambda\setminus\partial G,\partial G}1_{C_1\setminus\partial G\,\mathrm{conn.}}\bar{m}(C_2;\sigma_{C_2},\eta_{C_2})\rangle+\frac{q}{2}\sum_{\substack{C_2,C_3\subset\Lambda;C_2\cup C_3=C\\ C_i\cap\partial G\neq\emptyset;C_i\cap(G^r)^c\neq\emptyset}}\\
&\qquad\times\langle\bar{m}(C_2;\sigma_{C_2},\eta_{C_2}),(\Pi_{\partial G}R_{G^r}\Pi_{\partial G})^{-1}\bar{m}(C_3;\sigma_{C_3},\eta_{C_3})\rangle\\
&\qquad-\frac{q}{2}\sum_{\substack{C_1,C_2,C_3\subset\Lambda;C_1\cup C_2\cup C_3=C\\ C_i\cap\partial G\neq\emptyset;C_i\cap(G^r)^c\neq\emptyset}}\langle\bar{m}(C_2;\sigma_{C_2},\eta_{C_2}),\partial_{\partial G,\Lambda\setminus\partial G}\\
&\qquad\times\mathcal{R}(\cdot\to\cdot;C_1\setminus\partial G)\partial_{\Lambda\setminus\partial G,\partial G}1_{C_1\setminus\partial G\,\mathrm{conn.}}\bar{m}(C_3;\sigma_{C_3},\eta_{C_3})\rangle
\end{aligned}\tag{3.18}$$

with the short notation $\bar{m}^{\sigma_{G^r}}_{\partial G}=m^{\tilde{m}_{\partial\Lambda},\eta_{G^r},\sigma_{G^r}}_{G^r}\Big|_{\partial G}$. The bounds are clear now from the bounds on the resolvent, the choice of $r$ and the (trivial) control of the $C_i$-sums, i.e. provided by

$$\sum_{\substack{\text{all subsets } S_1,S_2,S_3\subset C\\ \cup_i S_i=C}} e^{-\alpha(|S_1|+|S_2|+|S_3|)}=(3e^{-\alpha}+3e^{-2\alpha}+e^{-3\alpha})^{|C|}\leq e^{-\mathrm{const}\,\alpha|C|}. \tag{3.19}$$

□

To proceed with the proof of Proposition 3.1 and high temperature-expand the $\bar{H}^{\mathrm{HT}}$-terms we use the subtraction of bounds-trick to ensure the positivity of the resulting activities. We thus write for fixed $G$

$$\begin{aligned}
&e^{-\sum_{\substack{C\subset\Lambda\\ C\cap\partial G\neq\emptyset;C\cap(G^r)^c\neq\emptyset}}\bar{H}^{\mathrm{HT}}_{\partial G,G^r}(m_{\partial G},\sigma_{G^r},\eta_{G^r};C,\sigma_C,\eta_C)}\\
&\quad=\prod_{(G^r)_i\text{conn. cp. of }G^r} e^{-\sum_{\substack{C\subset\Lambda;C\text{ conn. to }(G^r)_i\\ C\cap\partial G\neq\emptyset;C\cap(G^r)^c\neq\emptyset}}e^{-\tilde{\alpha}|C|}}\\
&\qquad\times e^{\sum_{\substack{C\subset\Lambda\\ C\cap\partial G\neq\emptyset;C\cap(G^r)^c\neq\emptyset}}\left(n(G^r,C)e^{-\tilde{\alpha}|C|}-\bar{H}^{\mathrm{HT}}_{\partial G,G^r}(m_{\partial G},\sigma_{G^r},\eta_{G^r};C,\sigma_C,\eta_C)\right)},
\end{aligned}\tag{3.20}$$

where $n(G^r,C)$ is the number of connected components of $G^r$ that are connected to $C$ (i.e. have $(G^r)_i\cap C\neq\emptyset$). The exponential in the last line can then be cluster-expanded and gives

$$\begin{aligned}
&e^{\sum_{\substack{C\subset\Lambda\\ C\cap\partial G\neq\emptyset;C\cap(G^r)^c\neq\emptyset}}\left(n(G^r,C)e^{-\tilde{\alpha}|C|}-\bar{H}^{\mathrm{HT}}_{\partial G,G^r}(m_{\partial G},\sigma_{G^r},\eta_{G^r};C,\sigma_C,\eta_C)\right)}\\
&\quad=\sum_{\substack{K\subset\Lambda;K=\emptyset\text{ or}\\ K\cap\partial G\neq\emptyset,K\cap(G^r)^c\neq\emptyset}}\rho^{\mathrm{HT}}_{\partial G,G^r}(m_{\partial G},\sigma_{G^r},\eta_{G^r};K,\sigma_K,\eta_K)
\end{aligned}\tag{3.21}$$

with $0\leq\rho^{\mathrm{HT}}_{\partial G,G^r}(m_{\partial G},\sigma_{G^r},\eta_{G^r};K,\sigma_K,\eta_K)\leq e^{-\tilde{\alpha}|K|}$. Here we use the convention that $\rho^{\mathrm{HT}}_{\partial G,G^r}(m_{\partial G},\sigma_{G^r},\eta_{G^r};K=\emptyset,\sigma_K,\eta_K)=1$.

Note that the resulting activities factorize over connected components of $K\cup G^r$; this is due to the (trivial) fact that the number $n(G^r,C)$ that enters the definition of the contour activities depends only on those components of $G^r$ that $C$ is connected to. We put

$$\rho^{\mathrm{geo}}(\partial G,G^r):=\prod_{(G^r)_i\ \text{conn. cp. of }G^r} e^{-\sum_{\substack{C\subset\Lambda;C\text{ conn. to }(G^r)_i\\ C\cap\partial G\neq\emptyset;C\cap(G^r)^c\neq\emptyset}}e^{-\tilde{\alpha}|C|}}\tag{3.22}$$

and note that

$$1\geq\rho^{\mathrm{geo}}(\partial G,G^r)\geq e^{-|G^r|e^{-\mathrm{const}\,\tilde{\alpha}}}.\tag{3.23}$$

We can finally carry out the integral on $\partial G$ to get the form as promised in the proposition. In doing so it is convenient to pull out a normalization constant and introduce the normalized Gaussian measures on $\partial G$ corresponding to the Hamiltonian on the r.h.s. of (3.8), given by

$$\begin{aligned}
&\int\mu^{(\tilde{m}_{\partial_{\partial\Lambda}G^r},0_{\partial_\Lambda G^r}),\eta_{G^r},\sigma_{G^r}}_{\partial G,G^r}(dm_{\partial G})f(m_{\partial G})\\
&\quad:=\frac{\int dm_{\partial G}e^{-\Delta H^{(\tilde{m}_{\partial_{\partial\Lambda}G^r},0_{\partial_\Lambda G^r}),\eta_{G^r},\sigma_{G^r}}_{\partial G,G^r}(m_{\partial G})}f(m_{\partial G})}{\int dm'_{\partial G}e^{-\Delta H^{(\tilde{m}_{\partial_{\partial\Lambda}G^r},0_{\partial_\Lambda G^r}),\eta_{G^r},\sigma_{G^r}}_{\partial G,G^r}(m'_{\partial G})}}.
\end{aligned}\tag{3.24}$$

So we can write

$$\int dm_{\partial G} e^{-\Delta H^{\tilde m_{\partial\Lambda},\eta_\Lambda,\sigma_\Lambda}_{\partial G,\Lambda}(m_{\partial G})} 1_{m_{\partial G}\in U^{\partial G}} I_G^{\tilde m_{\partial\Lambda},m_{\partial G},\eta_G,\sigma_G}$$

$$= (2\pi)^{\frac{|\partial G|}{2}} \sqrt{\det(\Pi_{\partial G}(a-q\Delta_{G^r})^{-1}\Pi_{\partial G})} \sum_{\substack{K\subset\Lambda; K=\emptyset \text{ or} \\ K\cap\partial G\neq\emptyset, K\cap(G^r)^c\neq\emptyset}} \rho^{\text{geo}}(\partial G, G^r)$$

$$\times \int \mu^{(\tilde m_{\partial_{\partial\Lambda}G^r}, 0_{\partial_\Lambda G^r}),\eta_{G^r},\sigma_{G^r}}_{\partial G, G^r}(dm_{\partial G})\, 1_{m_{\partial G}\in U^{\partial G}}$$

$$\times \rho^{\text{HT}}_{\partial G, G^r}(m_{\partial G},\sigma_{G^r},\eta_{G^r};K,\sigma_K,\eta_K) I_G^{\tilde m_{\partial\Lambda},m_{\partial G},\eta_G,\sigma_G}. \tag{3.25}$$

This has in fact the desired form (3.5) with the obvious definition

$$\rho^{\tilde m_{\partial_{\partial\Lambda}\tilde G}}(G,\tilde G;\sigma_{\tilde G},\eta_{\tilde G}) := \rho^{\text{geo}}(\partial G, G^r)$$
$$\times \int \mu^{(\tilde m_{\partial_{\partial\Lambda}G^r}, 0_{\partial_\Lambda G^r}),\eta_{G^r},\sigma_{G^r}}_{\partial G, G^r}(dm_{\partial G})\, 1_{m_{\partial G}\in U^{\partial G}}$$
$$\times \rho^{\text{HT}}_{\partial G, G^r}(m_{\partial G},\sigma_{G^r},\eta_{G^r};K,\sigma_K,\eta_K) I_G^{\tilde m_{\partial\Lambda},m_{\partial G},\eta_G,\sigma_G} \tag{3.26}$$

with $K = \tilde G\backslash G^r$ on the r.h.s. Note that these activities factorize over connected components of $\tilde G$.

In view of the trivial bound (3.23) on the geometric activity (3.22) and the normalization of the measure, the bounds follows from the HT-bounds and the bounds on the anharmonic activities $I$. The value of the "Peierls constant" $\bar\alpha$ is now clear from $\bar\alpha = \text{Const}\min\{(2r+1)^{-d}\log\frac{1}{\epsilon},\tilde\alpha\}$, assuming that both terms in the minimum are sufficiently large. □

To finish with the proof of Proposition 3.1 is now an easy matter. Using the formula for the determinant from Appendix (A.3) we can write

$$\frac{1}{\det(a-q\Delta_{\Lambda\backslash\overline{G}})}\det\left(\Pi_{\partial G}(a-q\Delta_{G^r})^{-1}\Pi_{\partial G}\right)$$
$$= \frac{1}{\det(a-q\Delta_\Lambda)} \times \frac{\det(\Pi_{\partial G}(a-q\Delta_{G^r})^{-1}\Pi_{\partial G})}{\det(\Pi_{\partial G}(a-q\Delta_\Lambda)^{-1}\Pi_{\partial G})} \times \det(a-q\Delta_G). \tag{3.27}$$

Remember that the correction given by the middle term on the r.h.s. stems from the lack of terms with range longer than $r$ in the quadratic form of (3.24) that we had cut off. The random walk representation then gives the following expansion whose proof is given in the Appendix.

**Lemma 3.4.**

$$\frac{\det(\Pi_{\partial G}(a-q\Delta_{G^r})^{-1}\Pi_{\partial G})}{\det(\Pi_{\partial G}(a-q\Delta_\Lambda)^{-1}\Pi_{\partial G})} = e^{-2\sum_{\substack{C\subset\Lambda \\ C\cap\partial G\neq\emptyset; C\cap(G^r)^c\neq\emptyset}} \epsilon^{\det}(C)}, \tag{3.28}$$

*where* $0\le\epsilon^{\det}(C)\le e^{-\alpha|C|}$ *with* $\alpha\sim\text{const}\log\frac{1}{q}$.

Next we use subtraction of bounds as in (3.20) to write

$$e^{-\sum_{\substack{C\subset\Lambda\\ C\cap\partial G\neq\emptyset; C\cap(G^r)^c\neq\emptyset}}\epsilon^{\det}(C)}$$
$$= \rho^{\mathrm{geo,det}}(\partial G, G^r) \sum_{\substack{K\subset\Lambda; K=\emptyset \text{ or}\\ K\cap\partial G\neq\emptyset, K\cap(G^r)^c\neq\emptyset}} \rho^{\det}_{\partial G, G^r}(K)\,, \tag{3.29}$$

where $1 \geq \rho^{\mathrm{geo,det}} \geq e^{-|G^r|e^{-\mathrm{const}\,\tilde{\alpha}}}$ and $0 \leq \rho^{\det}_{\partial G,G^r}(K) \leq e^{-\mathrm{const}\,\tilde{\alpha}|K|}$. So we get

$$\begin{aligned} e^{b|\Lambda|} Z^{\tilde{m}_{\partial\Lambda}}_{\Lambda}(\sigma_\Lambda) &= (2\pi)^{\frac{|\Lambda|}{2}}(\det(a-q\Delta_\Lambda))^{-\frac{1}{2}} e^{-\inf_{m'_\Lambda} H^{\tilde{m}_{\partial\Lambda},\eta_\Lambda,\sigma_\Lambda}_{\Lambda}(m'_\Lambda)} \\ &\quad\times \sum_{G:\emptyset\subset G\subset\Lambda} (2\pi)^{-\frac{|G|}{2}} \sqrt{\det(a-q\Delta_G)} \sum_{\substack{\tilde{G}:\tilde{G}\subset\Lambda\\ G^r\subset\tilde{G}}} \rho^{\tilde{m}_{\partial_{\partial\Lambda}\tilde{G}}}(G,\tilde{G};\sigma_{\tilde{G}},\eta_{\tilde{G}}) \\ &\quad\times \rho^{\mathrm{geo,det}}(\partial G, G^r) \sum_{\substack{K\subset\Lambda; K=\emptyset \text{ or}\\ K\cap\partial G\neq\emptyset, K\cap(G^r)^c\neq\emptyset}} \rho^{\det}_{\partial G,G^r}(K)\,. \end{aligned} \tag{3.30}$$

This can be summed over $G, \tilde{G}, K$ (collecting terms that give the same $\tilde{G}\cup K$) to yield the claims of Proposition 3.1. □

## 4. The Effective Contour Model: Gaussian Case

It is instructive to make explicit the result of our transformation to an effective Ising-contour model at first without the presence of anharmonic potentials where the proof is easy. In fact, as we will explain in Sec. 5, the work done in Secs. 2 and 3 will then imply that a weak anharmonicity can be absorbed in essentially the same type of contour activities we encountered in the purely Gaussian model.

We remind the reader that in the purely Gaussian case the Ising-weights $(T(\mu^{\tilde{m}_{\partial\Lambda},\eta_\Lambda}_{\Lambda}))(\sigma_\Lambda)$ are obtained by normalizing $\exp(-\inf_{m_\Lambda\in\mathbb{R}^\Lambda} H^{\tilde{m}_{\partial\Lambda},\eta_\Lambda,\sigma_\Lambda}_{\Lambda}(m_\Lambda))$ by its $\sigma_\Lambda$-sum. For simplicity we restrict now to the boundary condition $\tilde{m}_x = m^*$ for all $x$ (that is everywhere in the minimum of the positive wells).

We will now express the latter exponential as a sum over contour-weights. To do so we use the following (by now standard) definition of a signed contour model, including +-boundary conditions.

**Definition.** A *contour* in $\Lambda$ is a pair $\Gamma = (\underline{\Gamma}, \sigma_\Lambda)$ where $\underline{\Gamma}\subset\Lambda$ (the support of $\Gamma$) and the spin-configuration $\sigma_\Lambda \in \{-1,1\}^\Lambda$ are such that the extended configuration $(\sigma_\Lambda, +1_{\mathbb{Z}^d\setminus\Lambda})$ is constant on connected components of $\mathbb{Z}^d\setminus\underline{\Gamma}$.

The *connected components* of a contour $\Gamma$ are the contours $\Gamma_i$ whose supports are the connected components $\underline{\Gamma}_i$ of $\underline{\Gamma}$ and whose sign is determined by the requirement that it be the same as that of $\Gamma$ on $\overline{\underline{\Gamma}_i}$.

A *contour model representation* for a probability measure $\nu$ on the space $\{-1,1\}^\Lambda$ of Ising-spins in $\Lambda$ is a probability measure $N$ on the space of contours in $\Lambda$ s.t. the marginal on the spin reproduces $\nu$, i.e. we have

$$\nu(\{\sigma_\Lambda\}) = \sum_{\substack{\Gamma\\ \sigma_\Lambda(\Gamma)=\sigma_\Lambda}} N(\{\Gamma\})\,. \tag{4.1}$$

Recall that, in the simplest low-temperature contour model, arising from the standard nearest neighbor ferromagnetic Ising model, $N(\{\Gamma\}) = \text{Const} \times \rho(\Gamma)$ is proportional to a (non-negative) activity $\rho(\Gamma)$ that factorizes over connected components of the contour and obeys a Peierls estimate of the form $\rho(\Gamma) \leq e^{-\tau|\underline{\Gamma}|}$. There is a satisfying theory for the treatment of deterministic models with additional volume terms for activities that are not necessarily symmetric under spin-flip, known as Pirogov–Sinai theory. For random models then, while the activities will be random, there also have to be additional random volume-contributions to $N(\{\Gamma\})$, even when the distribution of the disorder is symmetric, caused by local fluctuations in the free energies of the different states. The fluctuations of these volume terms are responsible for the fact that, even in situations where the disorder is "irrelevant", not all contours carry exponentially small mass but the formation of some contours (depending on the specific realization) is favorable. It is the control of this phenomenon that poses the difficulties in the analysis of the stability of disordered contour models and necessitates RG (or possibly some related multiscale method).

To write down the Peierls-type estimates to come for the present model we introduce the "naive contour-energy" (i.e. the $d-1$-dimensional volume of the plaquettes separating plus- and minus-regions in $\mathbb{Z}^d$) putting

$$E_s(\Gamma) = \sum_{\{x,y\}\subset\underline{\Gamma}, d(x,y)=1} 1_{\sigma_x \neq \sigma_y} + \sum_{\substack{x\in\underline{\Gamma}, y\in\partial\Lambda \\ d(x,y)=1}} 1_{\sigma_x=-1} \tag{4.2}$$

again taking into account the interaction with the positive boundary condition.

Then the result of the transformation of the purely Gaussian continuous spin model to an effective Ising-contour model is given by the following:

**Proposition 4.1.** *Suppose that $q$ is sufficiently small, $q(m^*)^2$ sufficiently large, $a$ is of the order 1 and $\delta \leq \text{Const}\, m^*$ with a constant of the order 1.*

*Then there is a $\sigma_\Lambda$-independent constant $K_\Lambda(\eta_\Lambda)$ s.t. we have the representation*

$$\begin{aligned} e^{-\inf_{m_\Lambda\in\mathbb{R}^\Lambda} H_\Lambda^{+m^*1_{\partial\Lambda},\eta_\Lambda,\sigma_\Lambda}(m_\Lambda)} &= K_\Lambda(\eta_\Lambda) e^{\sum_{C\subset V^+(\sigma_\Lambda)} S_C^{\text{Gauß}}(\eta_C) - \sum_{C\subset V^-(\sigma_\Lambda)} S_C^{\text{Gauß}}(\eta_C)} \\ &\quad \times \sum_{\substack{\Gamma \\ \sigma_\Lambda(\Gamma)=\sigma_\Lambda}} \rho_0(\Gamma;\eta_{\underline{\Gamma}}) \end{aligned} \tag{4.3}$$

*for any $\sigma_\Lambda$, with $V^\pm(\sigma_\Lambda) = \{x\in\Lambda; \sigma_x = \pm 1\}$. Here*

(i) *$\eta_C \mapsto S_C^{\text{Gauß}}(\eta_C)$ are functions of the random fields indexed by the connected sets $C\subset\Lambda$. They are symmetric, i.e. $S_C^{\text{Gauß}}(-\eta_C) = -S_C^{\text{Gauß}}(\eta_C)$ and invariant under lattice-shifts. For $C = \{x\}$ we have in particular $S_x^{\text{Gauß}}(\eta_x) = \frac{am^*}{a+2dq}\eta_x$.*

(ii) *The activity $\rho_0(\Gamma;\eta_{\underline{\Gamma}})$ is non-negative. It factorizes over the connected components of $\Gamma$, i.e.*

$$\rho_0(\Gamma;\eta_{\underline{\Gamma}}) = \prod_{\Gamma_i \,\text{conn cp. of}\, \Gamma} \rho_0(\Gamma_i;\eta_{\Gamma_i}). \tag{4.4}$$

*For $\underline{\Gamma}$ not touching the boundary (i.e. $\partial_{\partial\Lambda}\underline{\Gamma} = \emptyset$) the value of $\rho_0(\Gamma; \eta_{\underline{\Gamma}})$ is independent of $\Lambda$. We then have the "infinite volume properties" of*

(a) *Spin-flip symmetry, i.e.* $\rho_0((\underline{\Gamma}, \sigma_\Lambda); \eta_{\underline{\Gamma}}) = \rho_0((\underline{\Gamma}, -\sigma_\Lambda); -\eta_{\underline{\Gamma}})$.
(b) *Invariance under joint lattice shifts of spins and random fields.*

**Peierls-type bounds.** *There exist positive constants $\tilde{\beta}_{\mathrm{Gauß}}, \beta$ s.t. we have the bounds*

$$0 \le \rho_0(\Gamma; \eta_{\underline{\Gamma}}) \le e^{-\beta E^s(\Gamma) - \tilde{\beta}_{\mathrm{Gauß}}|\underline{\Gamma}|} \tag{4.5}$$

*uniformly in $\eta_{\underline{\Gamma}} \in [-\delta, \delta]^{\underline{\Gamma}}$ where the "Peierls-constants" can be chosen like*

$$\begin{aligned} \beta &= \frac{q(m^*)^2}{2} \frac{a^2}{(a+2dq)^2 - q^2}\,, \\ \tilde{\beta}_{\mathrm{Gauß}} &= \mathrm{Const} \times \min\left\{ \log\frac{1}{q}, qm^{*2} \left( \frac{\log\frac{1}{q}}{\log m^*} \right)^d \right\} - m^*\delta\,. \end{aligned} \tag{4.6}$$

*The non-local random fields obey the estimate*

$$|S_C^{\mathrm{Gauß}}(\eta_C)| \le \delta m^* e^{-\tilde{\beta}_0 |C|} \tag{4.7}$$

*for all $|C| \ge 1$ with $\tilde{\beta}_0 = \mathrm{Const} \times \log\frac{1}{q}$.*

**Remark 1.** This structure will be familiar to the reader familiar with [5] or [8] (see p. 457). Indeed, the above model falls in the class of contour models given in (5.1) of [5] (as written therein for the partition function). This form was then shown to be of sufficient generality to describe the contour models arising from the random field Ising model under any iteration of the contour-RG that was constructed in [5]. (The additional non-local interaction $W(\Gamma)$ encountered in [5] is not necessary and can be expanded by subtraction-of-bounds as in (3.20), giving rise to enlarged supports $\Gamma$, as it was done in [8].)

**Remark 2.** There is some freedom in the precise formulation of contours and contour activities, resp. the question of keeping information additional to the support and the spins on the contours. [5] speak of inner and outer supports, while in [8] it was preferred to define contours with activities containing interactions. The latter is motivated by the limit of the temperature going to zero (making the interactions vanish). Since we do not perform such a limit here, we present the simplest possible choice and do not make such distinctions here, simply collecting all interactions from different sources into "the support".

**Remark 3.** The magnitude of $\beta \sim \mathrm{Const}\ qm^{*2}$ is easily understood since it gives the true order of magnitude of the minimal energetic contribution to the original Hamiltonian of a nearest neighbor pair of continuous spins sitting in potential wells with opposite signs. This term appears again in the estimate on $\tilde{\beta}_{\mathrm{Gauß}}$ (up to logarithmic corrections) together with a contribution of the same form as $\tilde{\beta}_0$. The latter comes from a straight-forward expansion of long-range contributions. The last term in (4.6), $m^*\delta$, is a trivial control on the worst realization of the random fields; it could easily be avoided by the introduction of so-called "bad regions". These are regions of space where the realizations of the random fields are exceptionally (and dangerously) large in some sense and, while comparing with [5] or [8], the reader might have already missed them. Indeed, a renormalization of the present model will immediately produce such bad regions in the next steps. Of course, we could have started, here and also in the presence of anharmonicity, with an unbounded distribution of the $\eta_x$. In the latter case we would have to single out regions of space where the behavior of our transformation to the Ising-model gets exceptional (i.e. because we lose Lemma 2.5.) We chose however not to treat this case here in order to keep the technicalities down.

**Proof.** An elementary computation yields the important fact that the minimum of the quadratic Hamiltonian (2.6) with any boundary condition $\tilde{m}$ is given by

$$\begin{aligned} -\inf_{m_\Lambda\in\mathbb{R}^\Lambda} H_\Lambda^{\tilde{m}_{\partial\Lambda},\eta_\Lambda,\sigma_\Lambda}(m_\Lambda) = & -\frac{a^2(m^*)^2}{2q}\langle\sigma_\Lambda, R_\Lambda\sigma_\Lambda\rangle_\Lambda + \frac{a(m^*)^2}{2}|\Lambda| \\ & -\frac{am^*}{q}\langle\eta+\tilde{\eta}_{\partial(\Lambda^c)}(q\tilde{m}), R_\Lambda\sigma_\Lambda\rangle_\Lambda \\ & -\frac{1}{2q}\langle\eta+\tilde{\eta}_{\partial(\Lambda^c)}(q\tilde{m}), R_\Lambda(\eta+\tilde{\eta}_{\partial(\Lambda^c)}(q\tilde{m}))\rangle_\Lambda \\ & +\frac{q}{2}\sum_{\substack{x\in\Lambda;y\in\partial\Lambda\\ d(x,y)=1}}\tilde{m}_y^2 \end{aligned} \tag{4.8}$$

with $\tilde{\eta}_{\partial(\Lambda^c)}(\tilde{m}) := \partial_{\Lambda,\partial\Lambda}\tilde{m}_{\partial\Lambda}$ denoting the field created by the boundary condition. We subtract a term that is constant for $\sigma_\Lambda$ (and thus of no interest) and write

$$\begin{aligned} &\left(\inf_{m_\Lambda} H_\Lambda^{\tilde{m}_{\partial\Lambda},\eta_\Lambda,\sigma_\Lambda}(m_\Lambda) - \inf_{m_\Lambda} H^{\tilde{m}_{\partial\Lambda},\eta_\Lambda,1_\Lambda}(m_\Lambda)\right) - \frac{am^*}{q}\langle\eta_\Lambda, R_\Lambda 1_\Lambda\rangle_\Lambda \\ &\quad = -\frac{a^2(m^*)^2}{2q}(\langle\sigma_\Lambda, R_\Lambda\sigma_\Lambda\rangle_\Lambda - \langle 1_\Lambda, R_\Lambda 1_\Lambda\rangle_\Lambda) - \frac{am^*}{q}\langle\eta_\Lambda, R_\Lambda\sigma_\Lambda\rangle_\Lambda \\ &\qquad - am^*\langle\tilde{\eta}_{\partial(\Lambda^c)}(\tilde{m}), R_\Lambda(\sigma_\Lambda - 1_\Lambda)\rangle_\Lambda\,. \end{aligned} \tag{4.9}$$

The first term on the r.h.s. gives rise to the low-temp. Peierls constant; the next term is a weakly nonlocal random field term (suppressed by the decay of the resolvent) and the last term the symmetry-breaking coupling to the boundary. As in Sec. 3 we use the random walk representation $R_\Lambda = \sum_{C\subset\Lambda}\mathcal{R}(\cdot\ \to\ \cdot\,;C)$ (see Appendix (A.11)) and decompose according to the size of $C$'s.

As the first step for the contour representation we associate to any spin-configuration $\sigma_\Lambda \in \{-1,1\}^\Lambda$ a preliminary (or "inner") support in the following way. Choose some finite integer $r\geq 1$, to be determined below, and put

$$\begin{aligned} \underline{\Gamma}_\Lambda^+(\sigma_\Lambda) := & \{x\in\Lambda; \exists y\in\Lambda \text{ s.t. } d(x,y)\leq r \text{ where } \sigma_x\neq\sigma_y\} \\ & \cup\{x\in\Lambda; d(x,\partial\Lambda)\leq r+1 \text{ where } \sigma_x = -1\}\,. \end{aligned} \tag{4.10}$$

The second term makes this definition $\Lambda$-dependent by taking into account the interaction with the boundary leading to the (desired) symmetry breaking for contours touching the boundary. For given $\sigma_\Lambda$ the activities $\rho_0(\Gamma;\eta_{\underline{\Gamma}})$ to be defined will be non-zero only for supports $\underline{\Gamma} \supset \underline{\Gamma}^+(\sigma_\Lambda)$. The range $r$ will be chosen below in such a way that the terms corresponding to interactions with range larger than $r$ have decayed sufficiently so that they can be high-temperature expanded in a straightforward way. This choice then also determines the value of the Peierls-constant for the low-temperature contributions.

Keeping the small $C$'s of diameter up to $r$ define the (preliminary) "low-temperature activities"

$$\begin{aligned}&\rho^{\mathrm{LT},\tilde m_{\partial\Lambda}}(\sigma_\Lambda)\\ &:= e^{\sum_{C\subset\Lambda;\mathrm{diam}(C)\le r}\left[\frac{a^2(m^*)^2}{2q}\left(\langle\sigma_C,\mathcal{R}(\cdot\to\cdot;C)\sigma_C\rangle-\langle 1_C,\mathcal{R}(\cdot\to\cdot;C)1_C\rangle\right)+am^*\langle\tilde\eta_{\partial(\Lambda^c)}(\tilde m),\mathcal{R}(\cdot\to\cdot;C)(\sigma_C-1_C)\rangle\right]}\,.\end{aligned} \tag{4.11}$$

Note that the "inner support" (4.10) can be trivially rewritten as

$$\underline{\Gamma}^+(\sigma_\Lambda) = \bigcup_{\substack{C\subset\Lambda;\mathrm{diam}(C)\le r\\ \sigma_C\neq 1_C \text{ and } \sigma_C\neq -1_C}} C \cup \bigcup_{\substack{C \text{ conn. to } \partial\Lambda\\ \mathrm{diam}(C)\le r;\sigma_C\neq 1_C}} C \tag{4.12}$$

which shows that it is just the union of all connected $C$'s with diameter less or equal $r$ that give any contribution to the sum occuring in the exponent of (4.11). So we can rewrite

$$\begin{aligned}&e^{-\inf_{m_\Lambda} H_\Lambda^{\tilde m_{\partial\Lambda},\eta_\Lambda,\sigma_\Lambda}(m_\Lambda)+\inf_{m_\Lambda} H_\Lambda^{\tilde m_{\partial\Lambda},\eta_\Lambda,1_\Lambda}(m_\Lambda)+\frac{am^*}{q}\langle\eta_\Lambda,R_\Lambda 1_\Lambda\rangle_\Lambda}\\ &\quad=\rho^{\mathrm{LT},\tilde m_{\partial\Lambda}}(\sigma_\Lambda)e^{\frac{am^*}{q}\sum_{\substack{C\subset\Lambda;\mathrm{diam}(C)\le r\\ \sigma_C\neq\mathrm{const}}}\langle\eta_C,\mathcal{R}(\cdot\to\cdot;C)\sigma_C\rangle}\\ &\qquad e^{\frac{am^*}{q}\sum_{C\subset V^+(\Gamma)}\langle\eta_C,\mathcal{R}(\cdot\to\cdot;C)1_C\rangle-\frac{am^*}{q}\sum_{C\subset V^-(\Gamma)}\langle\eta_C,\mathcal{R}(\cdot\to\cdot;C)1_C\rangle}\\ &\qquad e^{\sum_{\substack{C\subset\Lambda;\mathrm{diam}(C)>r\\ \sigma_C\neq\mathrm{const}}}\left[\frac{a^2(m^*)^2}{2q}\left(\langle\sigma_C,\mathcal{R}(\cdot\to\cdot;C)\sigma_C\rangle-\langle 1_C,\mathcal{R}(\cdot\to\cdot;C)1_C\rangle\right)+\frac{am^*}{q}\langle\eta_C,\mathcal{R}(\cdot\to\cdot;C)\sigma_C\rangle\right]}\\ &\qquad e^{\sum_{\substack{C\subset\Lambda;\mathrm{diam}(C)>r\\ C\cap\partial(\Lambda)^c\neq\emptyset}} am^*\langle\tilde\eta_{\partial(\Lambda^c)}(\tilde m),\mathcal{R}(\cdot\to\cdot;C)(\sigma_C-1_C)\rangle}\,.\end{aligned} \tag{4.13}$$

The terms in the first line depend only on quantities on $\underline{\Gamma}^+(\sigma_\Lambda)$ and factorize over its connected components. They will give contributions to the activities $\rho_0$. The terms in the second line are the small-field contributions to the vacua given by

$$S_C^{\mathrm{Gauß}}(\eta_C) := \frac{am^*}{q}\langle\eta_C,\mathcal{R}(\cdot\to\cdot;C)1_C\rangle\,. \tag{4.14}$$

The terms in the last two lines are small (since only $C$'s with sufficiently large diameter contribute) and only non-zero for $C$'s intersecting with $\underline{\Gamma}^+(\sigma_\Lambda)$ or touching the boundary. They can be expanded.

Let us see now what explicit bounds we get on the low-temperature activity (4.11). Keeping only $C$'s made of two nearest neighbors $x, y = x + e$ we have the upper bound

$$\sum_{C\subset\Lambda;\mathrm{diam}(C)\le r}\frac{a^2(m^*)^2}{2q}(\langle\sigma_C,\mathcal{R}(\cdot\,\to\cdot\,;C)\sigma_C\rangle-\langle 1_C,\mathcal{R}(\cdot\,\to\cdot\,;C)1_C\rangle)$$
$$\le -\frac{a^2(m^*)^2}{q}\sum_{\{x,y\}\subset\underline{\Gamma}^+(\sigma_\Lambda),d(x,y)=1}\mathcal{R}(x\,\to\,y;C=\{x,y\})1_{\sigma_x\ne\sigma_y}\,. \quad (4.15)$$

Computing

$$\mathcal{R}(x\,\to\,x+e;C=\{x,x+e\})=\frac{1}{c+2d}\sum_{k=1,3,5,\dots}^{\infty}\left(\frac{1}{c+2d}\right)^k=\frac{1}{(c+2d)^2-1} \quad (4.16)$$

with $c=a/q$ we get an upper bound on the l.h.s. of (4.15) of

$$-2\beta\sum_{\{x,y\}\subset\underline{\Gamma}^+(\sigma_\Lambda),d(x,y)=1}1_{\sigma_x\ne\sigma_y}$$

where $\beta$ is given by (4.6). Applying a similar reasoning on the boundary term, thereby using that $\mathcal{R}(x\,\to;C=\{x\})=\frac{1}{c+2d}$, gives the bound

$$\sum_{C\subset\Lambda;\mathrm{diam}(C)\le r}am^*\langle\tilde\eta_{\partial(\Lambda^c)}(\tilde m),\mathcal{R}(\cdot\,\to\cdot\,;C)(\sigma_C-1_C)\rangle$$
$$\le -q(m^*)^2\frac{2a}{a+2dq}\sum_{\substack{x\in\underline{\Gamma}^+(\sigma_\Lambda),y\in\partial\Lambda\\ d(x,y)=1}}1_{\sigma_x=-1}\,. \quad (4.17)$$

Since the modulus of the prefactor in the last line is larger than $2\beta$ we get an energetic suppression of

$$\rho^{\mathrm{LT},\tilde m_{\partial\Lambda}}(\sigma_\Lambda)\le e^{-2\beta E_s(\Gamma^+_\Lambda(\sigma_\Lambda),\sigma_\Lambda)}\le e^{-\beta E_s(\Gamma^+_\Lambda(\sigma_\Lambda),\sigma_\Lambda)-\beta(2r+1)^{-d}|\Gamma^+_\Lambda(\sigma_\Lambda)|}\,. \quad (4.18)$$

Using $\sum_y R_{\Lambda;x,y}\le 1/c$ for the next term in (4.13) we have immediately

$$\frac{am^*}{q}\sum_{\substack{C\subset\Lambda;\mathrm{diam}(C)\le r\\ \sigma_C\ne\mathrm{const}}}\langle\eta_C,\mathcal{R}(\cdot\,\to\cdot\,;C)\sigma_C\rangle\le m^*\delta|\Gamma^+_\Lambda(\sigma_\Lambda)|\,. \quad (4.19)$$

This finishes the Peierls estimate for the low-temperature contributions.

Let us come to the treatment of the "high-temperature parts" in (4.13) now, proceeding algebraically at first. Using subtraction-of-bounds as in Sec. 3 (3.20) we get the high-temperature expansion

$$e^{\sum_{\substack{C\subset\Lambda;\mathrm{diam}(C)>r\\ \sigma_C\ne\mathrm{const}}}\left[\frac{a^2(m^*)^2}{2q}(\langle\sigma_C,\mathcal{R}(\cdot\,\to\cdot\,;C)\sigma_C\rangle-\langle 1_C,\mathcal{R}(\cdot\,\to\cdot\,;C)1_C\rangle)+\frac{am^*}{q}\langle\eta_C,\mathcal{R}(\cdot\,\to\cdot\,;C)\sigma_C\rangle\right]}$$
$$=\tilde\rho^{\mathrm{geo}}(\underline{\Gamma}^+(\sigma_\Lambda))\sum_{\substack{K\subset\Lambda;\mathrm{diam}(K)>r\text{ or }K=\emptyset\\ \sigma_K\ne\mathrm{const}}}\rho^{\mathrm{HT1}}(K,\sigma_K,\eta_K) \quad (4.20)$$

if the terms in the exponential on the l.h.s. are sufficiently small. To control them we just use the bound (A.12)

$$\sum_{y\in\mathbb{Z}^d}\mathcal{R}(x\ \to y\,;C)\le\frac{1}{c}\left(\frac{2d}{c+2d}\right)^{|C|-1}\,. \tag{4.21}$$

This gives the deterministic bound upper bound on the first two terms in (4.13) of

$$\frac{a^2(m^*)^2}{2q}|\langle\sigma_C,\mathcal{R}(\cdot\ \to\cdot\,;C)\sigma_C\rangle-\langle 1_C,\mathcal{R}(\cdot\ \to\cdot\,;C)1_C\rangle|\le e^{-\alpha|C|} \tag{4.22}$$

if we have

$$\alpha\le\log\left(1+\frac{a}{2dq}\right)\left[1-\frac{\log(am^{*2})}{(|C|-1)\log\left(1+\frac{a}{2dq}\right)}\right]-\frac{1}{e} \tag{4.23}$$

which is in turn bounded by $\alpha_0:=\frac{1}{2}\log(1+\frac{a}{2dq})-\frac{1}{e}$ for the $C$'s in the above sum if we put

$$r=\left[2\frac{\log(am^{*2})}{\log\left(1+\frac{a}{2dq}\right)}\right]+1\sim 4\frac{\log m^*}{\log\frac{1}{q}}\,. \tag{4.24}$$

Remember here that we are interested in the regime of $\frac{1}{q}$ small and $m^{*2}$ even larger.

Assuming (4.24), with $|\eta_x|\le\delta$ the random field contribution is estimated by

$$\frac{am^*}{q}|\langle\eta_C,\mathcal{R}(\cdot\ \to\cdot\,;C)\sigma_C\rangle|\le\frac{\delta}{am^*}e^{-\alpha_0|C|}\,, \tag{4.25}$$

where can use that $\frac{\delta}{am^*}\le$ Const. The estimates on $S_C^{\mathrm{Gauß}}(\eta_C)$ are obtained in the very same way.

In passing we verify that all activities constructed so far are invariant under joint flips of spins and random fields (inside $\Lambda$). The boundary terms can be expanded similarly giving

$$\begin{aligned}
&e^{\sum_{\substack{C\subset\Lambda;\mathrm{diam}(C)>r\\ C\cap\partial(\Lambda)^c\neq\emptyset}}am^*\langle\tilde\eta_{\partial(\Lambda^c)}(\tilde m),\mathcal{R}(\cdot\to\cdot\,;C)(\sigma_C-1_C)\rangle}\\
&\quad=e^{-2\sum_{\substack{C\subset\Lambda;\mathrm{diam}(C)>r\\ C\cap\partial(\Lambda)^c\neq\emptyset}}am^*\langle\tilde\eta_{\partial(\Lambda^c)}(\tilde m),\mathcal{R}(\cdot\to\cdot\,;C)1_C\rangle}\\
&\quad\quad\times e^{\sum_{\substack{C\subset\Lambda;\mathrm{diam}(C)>r\\ C\cap\partial(\Lambda)^c\neq\emptyset}}am^*\langle\tilde\eta_{\partial(\Lambda^c)}(\tilde m),\mathcal{R}(\cdot\to\cdot\,;C)(\sigma_C+1_C)\rangle}\\
&\quad=e^{-2\sum_{\substack{C\subset\Lambda;\mathrm{diam}(C)>r\\ C\cap\partial(\Lambda)^c\neq\emptyset}}am^*\langle\tilde\eta_{\partial(\Lambda^c)}(\tilde m),\mathcal{R}(\cdot\to\cdot\,;C)1_C\rangle}\\
&\quad\quad\times\sum_{\substack{K\subset\Lambda;\mathrm{diam}(K)>r\text{ or }K=\emptyset\\ K\cap\partial(\Lambda)^c\neq\emptyset}}\rho^{\mathrm{HT2}}(K,\sigma_K,\eta_K)
\end{aligned} \tag{4.26}$$

This gives

$$
\begin{aligned}
&e^{-\inf_{m_\Lambda} H_\Lambda^{\tilde m_{\partial\Lambda},\eta_\Lambda,\sigma_\Lambda}(m_\Lambda)+\inf_{m_\Lambda} H_\Lambda^{\tilde m_{\partial\Lambda},\eta_\Lambda,1_\Lambda}(m_\Lambda)+\frac{am^*}{q}\langle\eta_\Lambda,R_\Lambda 1_\Lambda\rangle_\Lambda} \\
&\quad\times e^{+2\sum_{\substack{C\subset\Lambda;\mathrm{diam}(C)>r\\ C\cap\partial(\Lambda^c)\neq\emptyset}} am^*\langle\tilde\eta_{\partial(\Lambda^c)}(\tilde m),\mathcal{R}(\cdot\to\cdot;C)1_C\rangle} \\
&\qquad = \rho^{\mathrm{LT},\tilde m_{\partial\Lambda}}(\sigma_\Lambda)e^{\frac{am^*}{q}\sum_{\substack{C\subset\Lambda;\mathrm{diam}(C)\le r\\ \sigma_C\neq\mathrm{const}}}\langle\eta_C,\mathcal{R}(\cdot\to\cdot;C)\sigma_C\rangle} \\
&\qquad\quad\times e^{\frac{am^*}{q}\sum_{C\subset V^+(\Gamma)}\langle\eta_C,\mathcal{R}(\cdot\to\cdot;C)1_C\rangle-\frac{am^*}{q}\sum_{C\subset V^-(\Gamma)}\langle\eta_C,\mathcal{R}(\cdot\to\cdot;C)1_C\rangle} \\
&\qquad\quad\times\tilde\rho^{\mathrm{geo}}(\underline{\Gamma}^+(\sigma_\Lambda))\sum_{\substack{K\subset\Lambda;\mathrm{diam}(K)>r\text{ or }K=\emptyset\\ \sigma_K\neq\mathrm{const}}}\rho^{\mathrm{HT1}}(K,\sigma_K,\eta_K) \\
&\qquad\quad\times\sum_{\substack{K_1\subset\Lambda;\mathrm{diam}(K_1)>r\text{ or }K_1=\emptyset\\ K_1\cap\partial(\Lambda)^c\neq\emptyset}}\rho^{\mathrm{HT2}}(K_1,\sigma_{K_1},\eta_{K_1})
\end{aligned}\tag{4.27}
$$

which proves the desired representation (4.3) with the obvious definition

$$
\begin{aligned}
\rho_0(\Gamma;\eta_{\underline{\Gamma}}) :=\ & \rho^{\mathrm{LT},\tilde m_{\partial\Lambda}}(\sigma_\Lambda)e^{\frac{am^*}{q}\sum_{\substack{C\subset\Lambda;\mathrm{diam}(C)\le r\\ \sigma_C\neq\mathrm{const}}}\langle\eta_C,\mathcal{R}(\cdot\to\cdot;C)\sigma_C\rangle}\tilde\rho^{\mathrm{geo}}(\underline{\Gamma}^+(\sigma_\Lambda)) \\
&\times\sum_{\substack{K_0,K_1\subset\Lambda;K_0\cup K_1\cup\underline{\Gamma}^+(\sigma_\Lambda)=\underline{\Gamma};\mathrm{diam}(K_i)>r\text{or }K_i=\emptyset\\ \sigma_{K_0}\neq\mathrm{const},\ K_1\cap\partial(\Lambda)^c\neq\emptyset}} \\
&\times\rho^{\mathrm{HT1}}(K_0,\sigma_{K_0},\eta_{K_0})\rho^{\mathrm{HT2}}(K_1,\sigma_{K_1},\eta_{K_1})\,.
\end{aligned}\tag{4.28}
$$

The form (4.6) of the Peierls constant $\tilde\beta_{\mathrm{Gauß}}$ is now clear from $\tilde\beta_{\mathrm{Gauß}} = \mathrm{Const}\ \min\{\beta(2r+1)^{-d},\alpha_0\} - m^*\delta$, assuming that both terms in the minimum are sufficiently large to control the entropy in (4.28) and the slight modification in the exponential bounds on $\rho^{\mathrm{HT1}}$ arising from the subtraction of bounds. □

## 5. The Final Contour Model — Proof of Phase Transition

We put together the results of Secs. 3 and 4 to obtain the contour representation of the full model. It is of the same form as the Gaussian model of Sec. 4, while a modifaction of the Peierls constant $\tilde\beta$ accounts for the anharmonic contributions. More precisely we have:

**Proposition 5.1.** *Assume that the anharmonic $I$-weights* (2.17) *satisfy the Positivity* (2.19) *and the uniform Peierls Condition* (2.20) *with a constant $\epsilon$. Suppose that $\epsilon$ is sufficiently small, $q$ is sufficiently small, $a$ is of the order one, $q(m^*)^2$ sufficiently large. Suppose that $\delta \le$ Const $m^*$ and $|U| \le$ Const $m^*$ with constants that are sufficiently small.*

*Then the measures $T(\mu_\Lambda^{+m^*1_{\partial\Lambda},\eta_\Lambda})$ on $\{-1,1\}^\Lambda$ have the contour representation*

$$
\begin{aligned}
&T(\mu_\Lambda^{+m^*1_{\partial\Lambda},\eta_\Lambda})(\sigma_\Lambda) \\
&\quad=\frac{1}{Z^{+,\eta_\Lambda}_{\mathrm{contour},\Lambda}}e^{\sum_{C\subset V^+(\sigma_\Lambda)}S_C(\eta_C)-\sum_{C\subset V^-(\sigma_\Lambda)}S_C(\eta_C)}\sum_{\substack{\Gamma\\ \sigma_\Lambda(\Gamma)=\sigma_\Lambda}}\rho(\Gamma;\eta_{\underline{\Gamma}})
\end{aligned}\tag{5.1}
$$

*with the contour-model partition function*

$$Z^{+,\eta_\Lambda}_{\text{contour},\Lambda} = \sum_{\Gamma} e^{\sum_{C\subset V^+(\Gamma_\Lambda)} S_C(\eta_C) - \sum_{C\subset V^-(\Gamma_\Lambda)} S_C(\eta_C)} \rho(\Gamma;\eta_{\underline{\Gamma}})\,. \tag{5.2}$$

*For the partition function* (1.4) *we have* $Z^{+m^* 1_{\partial\Lambda},\eta_\Lambda}_{\Lambda} = C^{+,\eta_\Lambda}_{\Lambda} Z^{+,\eta_\Lambda}_{\text{contour},\Lambda}$ *with a trivial constant containing the contributions of Gaussian fluctuations that satisfies, a.s.*

$$\lim_{\Lambda\uparrow Z^d} \frac{1}{|\Lambda|}\log C^{+,\eta_\Lambda}_{\Lambda} = -\frac{\mathbb{E}\eta_0^2}{2}[(a-q\Delta_{\mathbb{Z}^d})^{-1}]_{0,0} - \frac{1}{2}[\log(a-q\Delta_{\mathbb{Z}^d})]_{0,0} - b + \frac{1}{2}\log(2\pi)\,. \tag{5.3}$$

*The quantities appearing in* (5.1) *are as follows.*

(i) $\eta_C \mapsto S_C(\eta_C)$ *are functions of the random fields indexed by the connected sets* $C\subset\Lambda$ *that are symmetric, i.e.* $S_C(-\eta_C) = -S_C(\eta_C)$. *In particular we have* $S_x(\eta_x) = \frac{am^*}{a+2dq}\eta_x$. *They obey the uniform bound*

$$|S_C(\eta_C)| \le m^*\delta e^{-\alpha_{\text{final}}|C|} \tag{5.4}$$

*for all* $C$ *with* $\alpha_{\text{final}} = \text{const}\min\{\log\frac{1}{q}, \log\frac{1}{\epsilon}(\frac{\log\frac{1}{q}}{\log m^*})^d\}$.

(ii) *The activity* $\rho_{\text{Ising}}(\Gamma;\eta_{\underline{\Gamma}})$ *is non-negative and depends only on the indicated arguments. It factorizes over the connected components* (*as in* (4.4)). *For* $\underline{\Gamma}$ *not touching the boundary it does not depend on* $\Lambda$ *and has the infinite volume symmetries of* (a) *invariance under joint flips of spins and random fields and* (b) *invariance under lattice shifts.*

*There exist* (*large*) *positive constants* $\tilde\beta, \beta$ *s.t. we have the Peierls-type bounds*:

$$\rho_{\text{Ising}}(\Gamma;\eta_{\underline{\Gamma}}) \le e^{-\beta E^s(\Gamma) - \tilde\beta|\underline{\Gamma}|} \tag{5.5}$$

*uniformly in* $\eta_G$. *Here* $\beta = \frac{q(m^*)^2}{2}\frac{a^2}{(a+2dq)^2-q^2}$ *is the same as in* (4.8) *and*

$$\tilde\beta = \text{Const}\times\min\left\{\log\frac{1}{q}, qm^{*2}\left(\frac{\log\frac{1}{q}}{\log m^*}\right)^d, \log\frac{1}{\epsilon}\left(\frac{\log\frac{1}{q}}{\log m^*}\right)^d\right\} - m^*\delta\,. \tag{5.6}$$

**Proof.** Assuming the control of the anharmonicity, summarized in Proposition 3.1, the proof is easy. For any fixed $\sigma_\Lambda$ we can cluster-expand the last sum in (3.1). Dropping now the dependence on the boundary condition $\tilde m_{\partial\Lambda} = +m^* 1_{\partial\Lambda}$ in the notation we have

$$\begin{aligned}\log\sum_{G:\emptyset\subset G\subset\Lambda}\bar\rho(G;\sigma_G,\eta_G) &= \sum_{C:\emptyset\subset C\subset\Lambda}\bar\epsilon\,(C;\sigma_C,\eta_C)\\ &= \sum_{C\subset V^+(\sigma_\Lambda)}\bar\epsilon(C;1_C,\eta_C) + \sum_{C\subset V^-(\sigma_\Lambda)}\bar\epsilon(C;-1_C,\eta_C)\\ &\quad + \sum_{C\subset\Lambda;\sigma_C\neq\text{const}}\bar\epsilon(C;\sigma_C,\eta_C)\,,\end{aligned} \tag{5.7}$$

where the sum is over connected sets $C$ and we have the bounds $|\epsilon(C;\sigma_C,\eta_C)| \leq e^{-\mathrm{const}\,\alpha|C|}$ with $\alpha$ given in Proposition 3.1. Together with the representation (4.3) for the purely Gaussian model this gives

$$\begin{aligned}
&e^{-\inf_{m_\Lambda\in\mathbb{R}^\Lambda} H_\Lambda^{\tilde m_{\partial\Lambda},\eta_\Lambda,\sigma_\Lambda}(m_\Lambda)} \sum_{G:\emptyset\subset G\subset\Lambda} \bar\rho(G;\sigma_G,\eta_G)\\
&\quad = K_\Lambda(\eta_\Lambda) e^{\sum_{C\subset V^+(\sigma_\Lambda)}(S_C^{\text{Gauß}}(\eta_C)+\bar\epsilon(C;1_C,\eta_C))-\sum_{C\subset V^-(\sigma_\Lambda)}(S_C^{\text{Gauß}}(\eta_C)+\bar\epsilon(C;1_C,\eta_C))}\\
&\qquad e^{\sum_{C\subset\Lambda;\sigma_C\neq\mathrm{const}}\bar\epsilon(C;\sigma_C,\eta_C)} \sum_{\substack{\Gamma\\ \sigma_\Lambda(\Gamma)=\sigma_\Lambda}} \rho_0(\Gamma;\eta_{\underline{\Gamma}})\,. \qquad (5.8)
\end{aligned}$$

Note that the $C$'s in the exponential in the last line are in particular connected to $\Gamma$. Using subtraction-of-bounds as before we can expand those terms and, as we did before in Secs. 3 and 4, rewrite the last line in terms of a new (and final) contour summation as

$$e^{\sum_{C\subset\Lambda;\sigma_C\neq\mathrm{const}}\bar\epsilon(C;\sigma_C,\eta_C)} \sum_{\substack{\Gamma\\ \sigma_\Lambda(\Gamma)=\sigma_\Lambda}} \rho_0(\Gamma;\eta_{\underline{\Gamma}}) = \sum_{\substack{\Gamma\\ \sigma_\Lambda(\Gamma)=\sigma_\Lambda}} \rho(\Gamma;\eta_{\underline{\Gamma}})\,. \qquad (5.9)$$

The values of the Peierls constants for the final activities on the r.h.s. follow from the statements of the Propositions 3.1 and 4.1 with a slight loss due to the control of entropy.

Finally, to see the statement for the free energy, we start from (3.1) and recall the construction of the activities in the purely Gaussian case, using the explicit expression (4.11) for the energy minimum in the Gaussian model in terms of the resolvent we obtain, with some trivial control on boundary terms, using the SLLN applied on the random fields the desired formula

$$\begin{aligned}
\lim_{\Lambda\uparrow Z^d}\frac{1}{|\Lambda|}\log C_\Lambda^{+,\eta_\Lambda} &= -\lim_{\Lambda\uparrow Z^d}\frac{1}{|\Lambda|}\mathbb{E}\inf_{m_\Lambda\in\mathbb{R}^\Lambda} H_\Lambda^{\tilde m_{\partial\Lambda},\eta_\Lambda,1_\Lambda}(m_\Lambda)\\
&\quad -\lim_{\Lambda\uparrow Z^d}\frac{1}{2|\Lambda|}\log\det(a-q\Delta_\Lambda) - b + \frac12\log(2\pi)\\
&= -\frac{\mathbb{E}\eta_0^2}{2}\left[(a-q\Delta_{\mathbb{Z}^d})^{-1}\right]_{0,0} - \frac12\left[\log(a-q\Delta_{\mathbb{Z}^d})\right]_{0,0}\\
&\quad - b + \frac12\log(2\pi)\,. \qquad (5.10)
\end{aligned}$$

□

The following result provides control of the original measure in terms of the coarse-grained one up to two corrections:

**Proposition 5.2.** *Assume the conditions of Proposition* 5.1 *and suppose that* $\tilde m_{\partial\Lambda}\in(U^+)^{\partial\Lambda}$. *Then we have*

$$\mu_\Lambda^{\tilde m_{\partial\Lambda},\eta_\Lambda}\left[m_{x_0}\leq\frac{m^*}{2}\right] \leq (T(\mu_\Lambda^{\tilde m_{\partial\Lambda},\eta_\Lambda}))[\sigma_{x_0}=-1] + e^{-\mathrm{const}\,\alpha} + e^{-\mathrm{const}(m^*)^2}\,, \qquad (5.11)$$

*where*

$$\alpha = \text{const} \times \min \left\{ \log \frac{1}{q}, \log \frac{1}{\epsilon} \left( \frac{\log \frac{1}{q}}{\log m^*} \right)^d \right\}$$

*is given in Proposition* 3.1.

**Remark.** The first error term on the r.h.s. accounts for the anharmonicity, the next one for the Gaussian fluctuations.

**Proof.** We carry out the transformation that led to Lemma 2.3 while carrying through the indicator function $1_{m_{x_0} \le \frac{m^*}{2}}$ to get

$$\begin{aligned}
&\int_{\mathbb{R}^\Lambda} dm_\Lambda 1_{m_{x_0} \le \frac{m^*}{2}} e^{-E_\Lambda^{\tilde m_{\partial\Lambda}, \eta_\Lambda}(m_\Lambda)} \\
&\quad = e^{-b|\Lambda|} \sum_{\sigma_\Lambda} e^{-\inf_{m'_\Lambda} H_\Lambda^{\tilde m_{\partial\Lambda}, \eta_\Lambda, \sigma_\Lambda}(m'_\Lambda)} \\
&\quad\quad \times \Bigg[ \sum_{\substack{G : G \subset \Lambda \\ G \ni x_0}} (2\pi)^{\frac{|\Lambda \setminus \overline{G}|}{2}} (\det(a - q\Delta_{\Lambda\setminus\overline{G}}))^{-\frac{1}{2}} \\
&\quad\quad \times \int dm_{\partial G} e^{-\Delta H_{\partial G, \Lambda}^{\tilde m_{\partial\Lambda}, \eta_\Lambda, \sigma_\Lambda}(m_{\partial G})} 1_{m_{\partial G} \in U^{\partial G}} I_{G;x_0}^{\tilde m_{\partial\Lambda}, m_{\partial G}, \eta_G, \sigma_G} \\
&\quad\quad + \sum_{\substack{G : G \subset \Lambda \\ \partial G \ni x_0}} (2\pi)^{\frac{|\Lambda \setminus \overline{G}|}{2}} (\det(a - q\Delta_{\Lambda\setminus\overline{G}}))^{-\frac{1}{2}} \\
&\quad\quad \times \int dm_{\partial G} e^{-\Delta H_{\partial G, \Lambda}^{\tilde m_{\partial\Lambda}, \eta_\Lambda, \sigma_\Lambda}(m_{\partial G})} 1_{m_{\partial G} \in U^{\partial G}} 1_{m_{x_0} \le \frac{m^*}{2}} I_G^{\tilde m_{\partial\Lambda}, m_{\partial G}, \eta_G, \sigma_G} \\
&\quad\quad + \sum_{\substack{G : G \subset \Lambda \\ \overline{G} \not\ni x_0}} \int dm_{\partial G} e^{-\Delta H_{\partial G, \Lambda}^{\tilde m_{\partial\Lambda}, \eta_\Lambda, \sigma_\Lambda}(m_{\partial G})} 1_{m_{\partial G} \in U^{\partial G}} \\
&\quad\quad \times \int dm_{\Lambda\setminus\overline{G}} e^{-\Delta H_{\Lambda\setminus\overline{G}}^{\tilde m_{\partial\Lambda}, m_{\partial G}, \eta_{\Lambda\setminus\overline{G}}, \sigma_{\Lambda\setminus\overline{G}}}(m_{\Lambda\setminus\overline{G}})} 1_{m_{x_0} \le \frac{m^*}{2}} \\
&\quad\quad \times I_G^{\tilde m_{\partial\Lambda}, m_{\partial G}, \eta_G, \sigma_G} \Bigg]
\end{aligned} \tag{5.12}$$

with $I_{G;x_0} = I_{G;x_0}^{(1)} - I_{G;x_0}^{(2)}$ (superscripts are dropped now) where we have defined

$$I_{G;x_0}^{(1)} := \int dm_G e^{-\Delta H_G^{\tilde m_{\partial\Lambda}, m_{\partial G}, \eta_G, \sigma_G}(m_G)} \prod_{x \in G} (1_{m_x \notin U} + w(m_x)) 1_{m_{x_0} \le \frac{m^*}{2}}$$

$$I_{G;x_0}^{(2)} := \int dm_G e^{-\Delta H_G^{\tilde m_{\partial\Lambda}, m_{\partial G}, \eta_G, \sigma_G}(m_G)} \prod_{x \in G} 1_{m_x \notin U} \; 1_{m_{x_0} \le \frac{m^*}{2}} \, . \tag{5.13}$$

We use the same notations *without* the subscript $x_0$ on the l.h.s. to denote the integrals without the $1_{m_{x_0}\leq\frac{m^*}{2}}$ on the r.h.s. so that we have $I_G = I_G^1 - I_G^2$. Note that it is not clear anymore that $I_{G;x_0}$ is positive for any sign $\sigma_{x_0}$ and dominated by $I_G$. To bypass this little inconvenience we argue as follows. Let us slightly enlarge $b$ in Sec. 2 by putting a factor 2 in front of the fraction of integrals in Definition 2.42. This leaves $b$ very small and all subsequent arguments based on a fixed choice of $b$ remain valid. Going back through Lemma 2.4, we see that this definition implies that even $I_G^{(2)} \leq 2^{-|G|} I_G^{(1)}$ (which can be seen as a strengthening of the positivity of $I_G$). But from this we have in particular that

$$I_{G;x_0} = I^{(1)}_{G;x_0} - I^{(2)}_{G;x_0} \leq I^{(1)}_{G;x_0} \leq I^{(1)}_G \leq 2 I_G \,. \tag{5.14}$$

We use this estimate on the last $G$-sum in (5.12) and bound the second $G$-sum in (5.12) by the corresponding expression without the indicator. Carrying out the $m_{\partial G}$-integral as described in Sec. 3 we get from this the bound

$$\begin{aligned}
&\sum_{\substack{G:G\subset\Lambda\\ G\ni x_0}} (2\pi)^{\frac{|\Lambda\setminus\overline{G}|}{2}} (\det(a - q\Delta_{\Lambda\setminus\overline{G}}))^{-\frac12}\\
&\qquad\times \int dm_{\partial G} e^{-\Delta H^{\tilde m_{\partial\Lambda},\eta_\Lambda,\sigma_\Lambda}_{\partial G,\Lambda}(m_{\partial G})} 1_{m_{\partial G}\in U^{\partial G}} I^{\tilde m_{\partial\Lambda}, m_{\partial G},\eta_G,\sigma_G}_{G;x_0}\\
&\qquad + \sum_{\substack{G:G\subset\Lambda\\ \partial G\ni x_0}} (2\pi)^{\frac{|\Lambda\setminus\overline{G}|}{2}} (\det(a - q\Delta_{\Lambda\setminus\overline{G}}))^{-\frac12}\\
&\qquad\times \int dm_{\partial G} e^{-\Delta H^{\tilde m_{\partial\Lambda},\eta_\Lambda,\sigma_\Lambda}_{\partial G,\Lambda}(m_{\partial G})} 1_{m_{\partial G}\in U^{\partial G}} 1_{m_{x_0}\leq\frac{m^*}{2}} I^{\tilde m_{\partial\Lambda}, m_{\partial G},\eta_G,\sigma_G}_{G}\\
&\leq\, 2\cdot(2\pi)^{\frac{|\Lambda|}{2}} (\det(a - q\Delta_\Lambda))^{-\frac12} \sum_{G:x_0\in\overline{G}\subset\Lambda} \bar\rho^{\tilde m_{\partial_{\partial\Lambda}G}}(G;\sigma_G,\eta_G)\,.
\end{aligned} \tag{5.15}$$

Using the positivity of the activities in the last line we can use the usual Peierls argument on the fixed-$\sigma$ contour model appearing in (3.1) that controls the anharmonicity. So we estimate

$$\begin{aligned}
&\sum_{G:x_0\in\overline{G}\subset\Lambda} \bar\rho^{\tilde m_{\partial_{\partial\Lambda}G}}(G;\sigma_G,\eta_G)\\
&\qquad\leq \sum_{G_0:x_0\in\overline{G_0}\subset\Lambda} \bar\rho^{\tilde m_{\partial_{\partial\Lambda}G}}(G_0;\sigma_{G_0},\eta_{G_0}) \sum_{G:G\subset\Lambda} \bar\rho^{\tilde m_{\partial_{\partial\Lambda}G}}(G;\sigma_G,\eta_G)\\
&\qquad\leq e^{-\operatorname{const}\alpha} \sum_{G:G\subset\Lambda} \bar\rho^{\tilde m_{\partial_{\partial\Lambda}G}}(G;\sigma_G,\eta_G)\,,
\end{aligned} \tag{5.16}$$

where the first sum is over connected sets $G_0$ and we have used Proposition 3.1 for its estimation.

To treat the first $G$-sum in (5.12) we note that the expectation outside the anharmonic contours is given by the one-dimensional Gaussian probability:

$$
\begin{aligned}
&\int dm_{\partial G} e^{-\Delta H^{\tilde m_{\partial\Lambda},\eta_\Lambda,\sigma_\Lambda}_{\partial G,\Lambda}(m_{\partial G})} 1_{m_{x_0}\le \frac{m^*}{2}} \\
&\quad = (2\pi)^{\frac{|\Lambda\setminus\overline{G}|}{2}} (\det(a-q\Delta_{\Lambda\setminus\overline{G}}))^{-\frac12} \\
&\qquad \times \mathcal{N}\left[ m^{\tilde m_{\partial\Lambda},m_{\partial G},\eta_{\Lambda\setminus\overline{G}},\sigma_{\Lambda\setminus\overline{G}}}_{\Lambda\setminus\partial G}; \left(a-q\Delta^D_{\Lambda-\overline{G}}\right)^{-1}_{x_0,x_0}\right] \left(1_{m_{x_0}\le\frac{m^*}{2}}\right)
\end{aligned}
\tag{5.17}
$$

with the notation $\mathcal{N}[a;\sigma^2](\phi) = \int_{-\infty}^{\infty} \frac{e^{-\frac{(x-a)^2}{2\sigma^2}}\phi(x)}{\sqrt{2\pi\sigma^2}}$. We use the uniform control on the expectation value given by Lemma 2.5 and the fact that the variance occuring in (5.17) is of the order one, in any volume. If $\sigma_{x_0} = +1$ we have from this, uniformly in all involved quantities that

$$
\mathcal{N}\left[ m^{\tilde m_{\partial\Lambda},m_{\partial G},\eta_{\Lambda\setminus\overline{G}},\sigma_{\Lambda\setminus\overline{G}}}_{\Lambda\setminus\partial G}; \left(a-q\Delta^D_{\Lambda-\overline{G}}\right)^{-1}_{x_0,x_0}\right] \left(1_{m_{x_0}\le\frac{m^*}{2}}\right) \le e^{-\mathrm{const}(m^*)^2}
\tag{5.18}
$$

so it can be pulled out of the $m_{\partial G}$-integral. For $\sigma_{x_0} = -1$ we use the trivial bound 1 to write

$$
\begin{aligned}
&\sum_{\substack{G:G\subset\Lambda\\ \overline{G}\not\ni x_0}} \int dm_{\partial G} e^{-\Delta H^{\tilde m_{\partial\Lambda},\eta_\Lambda,\sigma_\Lambda}_{\partial G,\Lambda}(m_{\partial G})} 1_{m_{\partial G}\in U^{\partial G}} \\
&\qquad \times \int dm_{\Lambda\setminus\overline{G}} e^{-\Delta H^{\tilde m_{\partial\Lambda},m_{\partial G},\eta_{\Lambda\setminus\overline{G}},\sigma_{\Lambda\setminus\overline{G}}}_{\Lambda\setminus\overline{G}}(m_{\Lambda\setminus\overline{G}})} 1_{m_{x_0}\le\frac{m^*}{2}} \\
&\le \left(e^{-\mathrm{const}(m^*)^2} 1_{\sigma_{x_0}=1} + 1_{\sigma_{x_0}=-1}\right) \sum_{\substack{G:G\subset\Lambda\\ \overline{G}\not\ni x_0}} (2\pi)^{\frac{|\Lambda\setminus\overline{G}|}{2}} (\det(a-q\Delta_{\Lambda\setminus\overline{G}}))^{-\frac12} \\
&\qquad \times \int dm_{\partial G} e^{-\Delta H^{\tilde m_{\partial\Lambda},\eta_\Lambda,\sigma_\Lambda}_{\partial G,\Lambda}(m_{\partial G})} 1_{m_{\partial G}\in U^{\partial G}} I^{\tilde m_{\partial\Lambda},m_{\partial G},\eta_G,\sigma_G}_G \\
&\le \left(e^{-\mathrm{const}(m^*)^2} + 1_{\sigma_{x_0}=-1}\right) \sum_{G:G\subset\Lambda} (2\pi)^{\frac{|\Lambda\setminus\overline{G}|}{2}} (\det(a-q\Delta_{\Lambda\setminus\overline{G}}))^{-\frac12} \\
&\qquad \times \int dm_{\partial G} e^{-\Delta H^{\tilde m_{\partial\Lambda},\eta_\Lambda,\sigma_\Lambda}_{\partial G,\Lambda}(m_{\partial G})} 1_{m_{\partial G}\in U^{\partial G}} I^{\tilde m_{\partial\Lambda},m_{\partial G},\eta_G,\sigma_G}_G .
\end{aligned}
\tag{5.19}
$$

Now it is simple to put together (5.12), (5.16)–(5.19) and rerunning the next steps of the transformation yields the claim. □

Applying the information of [5] we obtain the main result of the paper.

**Proof of Theorem 1.** We apply statement Theorem 2.1 [5] on the measure $T(\mu_\Lambda^{+m^* 1_{\partial\Lambda},\eta_\Lambda})$. Indeed, this is justified from Proposition 5.1 which implies that this measure is contained in the class of contour measures described in [5] Chap. 5 "Flow of the RGT", Paragraph 5.1. We note that of the three constants $\tilde\beta$, $\beta$, $\alpha_{\mathrm{final}}$ (controlling the exponential decay of the activities in terms of the volume resp. in terms of the naive contour energy, and the decay of the non-local fields) the constant $\tilde\beta$ is the smallest.

So Statement 2.3 from [5] gives in our case that for $d \geq 3$, $\tilde{\beta}$ large enough and $\sigma^2$ small enough we have that

$$\mathbb{P}[T(\mu_\Lambda^{+m^* 1_{\partial\Lambda},\eta_\Lambda})[\sigma_{x_0} = -1] \geq e^{-\text{const}\,\tilde{\beta}}] \leq e^{-\frac{\text{const}}{\sigma^2}} . \tag{5.20}$$

We apply Proposition 5.2 and note that the two correction terms given therein are also controlled by $e^{-\text{const}\,\tilde{\beta}}$ (with possible modification of const.) From this in particular the estimates of Theorem 1 follow. □

**Remark.** We have not given an estimate on the value of $\gamma$ as a function of $q$ and $m^*$. This would of course follow from a more careful estimate of the best value of the "anharmonicity-constant" $\epsilon$ (which is entering $\tilde{\beta}$) as a function of $q$ and $m^*$ (see Sec. 2) and is left to the reader.

Finally, Theorem 2 for the $\phi^4$-theory follows immediately from:

**Proposition 5.3.** *Assume that the anharmonic I-weights* (2.17) *satisfy the Positivity* (2.19) *and the uniform Peierls Condition* (2.20) *with a constant* $\epsilon$ (*that is sufficiently small*). *Let* $\mu_\infty^\eta$ *be any continuous spin Gibbs-measure obtained as a weak limit of* $\mu_\Lambda^{\tilde{m}_{\partial\Lambda},\eta_\Lambda}$ *along a sequence of cubes* $\Lambda$ *for some* (*not necessarily positive*) *continuous-spin boundary condition* $\tilde{m} \in U^{\mathbb{Z}^d}$.

*Then the measure* $T(\mu_\infty^\eta)$ *on* $\{-1,1\}^{\mathbb{Z}^d}$ *is a Gibbs measure for the absolutely summable Ising–Hamiltonian*

$$\begin{aligned} H_{\text{Ising}}^\eta(\sigma) = &-\frac{a^2(m^*)^2}{2}\sum_{x,y}(a - q\Delta_{\mathbb{Z}^d})^{-1}_{x,y}\sigma_x\sigma_y \\ &- am^*\sum_{x,y}(a - q\Delta_{\mathbb{Z}^d})^{-1}_{x,y}\eta_x\sigma_y - \sum_{C:|C|\geq 2}\Phi_C(\sigma_C;\eta_C)\,, \end{aligned} \tag{5.21}$$

*where the interaction potentials* $\Phi_C(\sigma_C,\eta_C) = \Phi_C(-\sigma_C,-\eta_C)$ *obey the uniform bound*

$$|\Phi_C(\sigma_C,\eta_C)| \leq e^{-\text{const}\,\alpha|C|}$$

*for all* $C$ *with*

$$\alpha = \text{const} \times \min\left\{\log\frac{1}{q}, \log\frac{1}{\epsilon}\left(\frac{\log\frac{1}{q}}{\log m^*}\right)^d\right\}$$

*as in* (3.3).

**Remark.** Note that it follows in particular that the interaction will be the same e.g. also in continuous spin Dobrushin-states [11] (that are believed to exist) one could construct using the boundary condition $+m^*$ in the upper half-space and $-m^*$ in the lower half-space.

**Proof.** Denote by $H_{\text{Ising},V}^{\bar{\sigma}_{\mathbb{Z}^d}}(\sigma_V)$ the usual restriction of (5.21) to the finite volume $V$, obtained by keeping the sums over sets $\{x,y\}$ and $C$ that intersect $V$

and putting the spin equal to $\bar{\sigma}_{\mathbb{Z}^d}$ for $x \notin V$. Following [7] it suffices to show that, for each $\bar{\sigma}_{\mathbb{Z}^d}$ we have that

$$\lim_{\Lambda_2 \uparrow \mathbb{Z}^d} \lim_{\Lambda_1 \uparrow \mathbb{Z}^d} \frac{Z_{\Lambda_1}^{\tilde{m}_{\partial\Lambda_1}, \eta_{\Lambda_1}}(\sigma_V, \bar{\sigma}_{\Lambda_2 \setminus V})}{\sum_{\tilde{\sigma}_V} Z_{\Lambda_1}^{\tilde{m}_{\partial\Lambda_1}, \eta_{\Lambda_1}}(\tilde{\sigma}_V, \bar{\sigma}_{\Lambda_2 \setminus V})} = \frac{e^{-H_{\mathrm{Ising},V}^{\bar{\sigma}_{\mathbb{Z}^d}, \eta}(\sigma_V)}}{\sum_{\tilde{\sigma}_V} e^{-H_{\mathrm{Ising},V}^{\bar{\sigma}_{\mathbb{Z}^d}, \eta}(\tilde{\sigma}_V)}}, \tag{5.22}$$

along (say) sequences of cubes where

$$Z_{\Lambda_1}^{\tilde{m}_{\partial\Lambda_1}, \eta_{\Lambda_1}}(\sigma_{\Lambda_2}) := \int_{\mathbb{R}^{\Lambda_1}} e^{-E_{\Lambda_1}^{\tilde{m}_{\partial\Lambda_1}, \eta_{\Lambda_1}}(m_{\Lambda_1})} \prod_{x \in \Lambda_2} T_x(\sigma_x \big| m_x). \tag{5.23}$$

This is clear, since (according to our assumption of weak convergence) there is a subsequence of cubes $\Lambda_1$ s.t. the inner limit exists and equals $(T(\mu_\infty^\eta))|(\sigma_V|\bar{\sigma}_{\Lambda_2 \setminus V})$. Summing Proposition 3.1 over the spins in $\Lambda_1 \setminus \Lambda_2$ we have then

$$\begin{aligned} Z_{\Lambda_1}^{\tilde{m}_{\partial\Lambda_1}, \eta_{\Lambda_1}}(\sigma_{\Lambda_2}) &= e^{-b|\Lambda_1|}(2\pi)^{\frac{|\Lambda_1|}{2}}(\det(a - q\Delta_{\Lambda_1}))^{-\frac{1}{2}} \\ &\quad \times \sum_{\hat{\sigma}_{\Lambda_1 \setminus \Lambda_2}} e^{-\inf_{m'_{\Lambda_1}} H_{\Lambda_1}^{\tilde{m}_{\partial\Lambda_1}, \eta_{\Lambda_1}, \hat{\sigma}_{\Lambda_1 \setminus \Lambda_2}; \sigma_{\Lambda_2}}(m'_{\Lambda_1})} \\ &\quad \times \sum_{G: \emptyset \subset G \subset \Lambda_1} \bar{\rho}^{\tilde{m}_{\partial_{\partial\Lambda_1} G}}(G; \sigma_{G \cap \Lambda_2}, \hat{\sigma}_{G \cap \Lambda_1 \setminus \Lambda_2}, \eta_G). \end{aligned} \tag{5.24}$$

From here the proof is easy, given the explicit formula (4.8) for the minimum and the absolute summability of the polymer weights, uniformly in the spins and random fields.

For the convenience of the reader we give a complete proof for the simplest case of vanishing anharmonicity $w(m_x) \equiv 0$, and vanishing magnetic fields $\eta_x = 0$; it illustrates the way boundary terms are entering. Using (4.8) we have indeed

$$\begin{aligned} &Z_{\Lambda_1}^{\tilde{m}_{\partial\Lambda_1}, \eta_{\Lambda_1}}(\sigma_{\Lambda_2}) \\ &\quad := \mathrm{Const} \sum_{\sigma_{\Lambda_1 \setminus \Lambda_2}} e^{\frac{a^2(m^*)^2}{2q}\langle(\sigma_{\Lambda_2}, \sigma_{\Lambda_1 \setminus \Lambda_2}), R_{\Lambda_1}(\sigma_{\Lambda_2}, \sigma_{\Lambda_1 \setminus \Lambda_2})\rangle_{\Lambda_1}} \\ &\qquad\qquad \times e^{e + am^* \langle \tilde{\eta}_{\partial(\Lambda_1{}^c)}(\tilde{m}), R_{\Lambda_1}(\sigma_{\Lambda_2}, \sigma_{\Lambda_1 \setminus \Lambda_2})\rangle_{\Lambda_1}}. \end{aligned} \tag{5.25}$$

Now, using the exponential decay of the resolvent,

$$\begin{aligned} &Z_{\Lambda_1}^{\tilde{m}_{\partial\Lambda_1}, \eta_{\Lambda_1}}(\sigma_V, \bar{\sigma}_{\Lambda_2 \setminus V}) \\ &\quad = \mathrm{Const} \times e^{\frac{a^2(m^*)^2}{2q}\langle(\sigma_V, \bar{\sigma}_{\Lambda_2 \setminus V}), R_{\Lambda_1}(\sigma_V, \bar{\sigma}_{\Lambda_2 \setminus V})\rangle_{\Lambda_1}} \\ &\qquad \times e^{+am^* \langle \tilde{\eta}_{\partial(\Lambda_1{}^c)}(\tilde{m}), R_{\Lambda_1}(\sigma_V, \bar{\sigma}_{\Lambda_2 \setminus V})\rangle_{\Lambda_1}} \\ &\quad \times \sum_{\sigma_{\Lambda_1 \setminus \Lambda_2}} e^{\frac{a^2(m^*)^2}{q}\langle(\sigma_V, \bar{\sigma}_{\Lambda_2 \setminus V}), R_{\Lambda_1}\sigma_{\Lambda_1 \setminus \Lambda_2}\rangle_{\Lambda_1} + \frac{a^2(m^*)^2}{2q}\langle \sigma_{\Lambda_1 \setminus \Lambda_2}, R_{\Lambda_1}\sigma_{\Lambda_1 \setminus \Lambda_2}\rangle_{\Lambda_1}} \\ &\qquad \times e^{+am^* \langle \tilde{\eta}_{\partial(\Lambda_1{}^c)}(\tilde{m}), R_{\Lambda_1}\sigma_{\Lambda_1 \setminus \Lambda_2}\rangle_{\Lambda_1}} \end{aligned}$$

$$
\begin{aligned}
&= \text{Const} \times e^{\frac{a^2(m^*)^2}{2q}\langle(\sigma_V,\bar\sigma_{\Lambda_2\setminus V}),R_{\Lambda_1}(\sigma_V,\bar\sigma_{\Lambda_2\setminus V})\rangle_{\Lambda_1}\pm\text{Const}|\Lambda_2|e^{-\alpha'\text{dist}(\Lambda_2,\Lambda_1^c)}} \\
&\qquad\times e^{\pm\text{Const}|V|e^{-\alpha'\text{dist}(V,\Lambda_2^c)}} \\
&\quad\times \sum_{\sigma_{\Lambda_1\setminus\Lambda_2}} e^{\frac{a^2(m^*)^2}{q}\langle\bar\sigma_{\Lambda_2\setminus V},R_{\Lambda_1}\sigma_{\Lambda_1\setminus\Lambda_2}\rangle_{\Lambda_1}} \\
&\qquad\times e^{+\frac{a^2(m^*)^2}{2q}\langle\sigma_{\Lambda_1\setminus\Lambda_2},R_{\Lambda_1}\sigma_{\Lambda_1\setminus\Lambda_2}\rangle_{\Lambda_1}} \\
&\qquad\times e^{+am^*\langle\tilde\eta_{\partial(\Lambda_1{}^c)}(\tilde m),R_{\Lambda_1}\sigma_{\Lambda_1\setminus\Lambda_2}\rangle_{\Lambda_1}}\,.
\end{aligned}
\tag{5.26}
$$

The terms in the last sum do not depend on $\sigma_V$ so that we get

$$
\begin{aligned}
&\frac{Z_{\Lambda_1}^{\tilde m_{\partial\Lambda_1},\eta_{\Lambda_1}}(\sigma_V,\bar\sigma_{\Lambda_2\setminus V})}{\sum_{\tilde\sigma_V} Z_{\Lambda_1}^{\tilde m_{\partial\Lambda_1},\eta_{\Lambda_1}}(\tilde\sigma_V,\bar\sigma_{\Lambda_2\setminus V})} \\
&\qquad = e^{\pm\text{Const}|\Lambda_2|e^{-\alpha'\text{dist}(\Lambda_2,\Lambda_1^c)}\pm\text{Const}|V|e^{-\alpha'\text{dist}(V,\Lambda_2^c)}} \\
&\qquad\quad\times\frac{e^{\frac{a^2(m^*)^2}{2q}\langle(\sigma_V,\bar\sigma_{\Lambda_2\setminus V}),R_{\Lambda_1}(\sigma_V,\bar\sigma_{\Lambda_2\setminus V})\rangle_{\Lambda_1}}}{\sum_{\tilde\sigma_V} e^{\frac{a^2(m^*)^2}{2q}\langle(\tilde\sigma_V,\bar\sigma_{\Lambda_2\setminus V}),R_{\Lambda_1}(\tilde\sigma_V,\bar\sigma_{\Lambda_2\setminus V})\rangle_{\Lambda_1}}}
\end{aligned}
\tag{5.27}
$$

with uniform constants. Taking first $\Lambda_1\uparrow\mathbb{Z}^d$ (using that $R_{\Lambda_1}|_{\Lambda_2}\to R_{\mathbb{Z}^d}|_{\Lambda_2}$) and then $\Lambda_2\uparrow\mathbb{Z}^d$ we get in fact the desired result in our special case.

The (random) non-Gaussian case follows easily from the cluster expansion of the $G$-sum in (3.1). Indeed, since we have uniform exponential decay of the activities $\bar\rho^{\tilde m_{\partial_{\partial\Lambda}G}}$, the cluster expansion gives us quantities $\Phi_C^{\tilde m_{\partial_{\partial\Lambda}C}}(\sigma_C;\eta_C)$ that obey a uniform bound of the form as desired s.t. we have

$$
\sum_{G:\emptyset\subset G\subset\Lambda}\bar\rho^{\tilde m_{\partial_{\partial\Lambda}G}}(G;\sigma_G,\eta_G)=e^{\sum_{C\,\text{conn.}:\emptyset\subset C\subset\Lambda}\Phi_C^{\tilde m_{\partial_{\partial\Lambda}C}}(\sigma_C;\eta_C)}\,.
\tag{5.28}
$$

The Gibbs potential in (5.21) is then given by the value of $\Phi_C$ for polymers $C$ that are not touching the boundary. With estimates on boundary terms as in (5.26) the claim (5.22) follows. □

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## Appendix

For easy reference we collect some formulae about quadratic forms and the random walk expansion of determinants and correlation functions we use. We start with:

**Lemma A.1.** *Let $Q_\Lambda$ be symmetric and positive definite. Let $V\subset\Lambda$ and write, with obvious notations,*

$$
Q_\Lambda=\begin{pmatrix} Q_V & Q_{V,\Lambda\setminus V} \\ Q_{\Lambda\setminus V,V} & Q_{\Lambda\setminus V,\Lambda\setminus V}\end{pmatrix}.
\tag{A.1}
$$

*Then we have the following formulae*:

(i) $Q_\Lambda^{-1}$

$$
= \begin{pmatrix} (Q_V - Q_{V,\Lambda\setminus V} Q_{\Lambda\setminus V,\Lambda\setminus V}^{-1} Q_{\Lambda\setminus V,V})^{-1} & -Q_V^{-1}(Q_{\Lambda\setminus V,\Lambda\setminus V} - Q_{\Lambda\setminus V,V} Q_V^{-1} Q_{V,\Lambda\setminus V})^{-1} \\ -Q_{\Lambda\setminus V,\Lambda\setminus V}^{-1}(Q_V - Q_{V,\Lambda\setminus V} Q_{\Lambda\setminus V,\Lambda\setminus V}^{-1} Q_{\Lambda\setminus V,V})^{-1} & (Q_{\Lambda\setminus V,\Lambda\setminus V} - Q_{\Lambda\setminus V,V} Q_V^{-1} Q_{V,\Lambda\setminus V})^{-1} \end{pmatrix} \tag{A.2}
$$

(ii) $\det Q_\Lambda = \det\left(\Pi_V Q_\Lambda^{-1} \Pi_V\right)^{-1} \times \det Q_{\Lambda\setminus V}$ (A.3)

(iii) *For any* $z_\Lambda$ *we can write*

$$
\begin{aligned}
&\frac{1}{2}\langle m_\Lambda, Q_\Lambda m_\Lambda\rangle_\Lambda - \langle m_\Lambda, z_\Lambda\rangle_\Lambda \\
&\quad = \frac{1}{2}\left\langle \left(m_V - m_\Lambda^{z_\Lambda}\Big|_V\right), \left(\Pi_V Q_\Lambda^{-1}\Pi_V\right)^{-1}\left(m_V - m_\Lambda^{z_\Lambda}\Big|_V\right)\right\rangle_\Lambda \\
&\qquad + \frac{1}{2}\left\langle \left(m_{\Lambda-V} - m_{\Lambda\setminus V}^{z_{\Lambda-V},m_V}\right), Q_{\Lambda\setminus V}\left(m_V - m_{\Lambda\setminus V}^{z_{\Lambda\setminus V},m_V}\right)\right\rangle_\Lambda \\
&\qquad - \frac{1}{2}\left\langle z_\Lambda, Q_\Lambda^{-1} z_\Lambda\right\rangle_\Lambda ,
\end{aligned} \tag{A.4}
$$

*where the "global minimizer"* $m_\Lambda^{z_\Lambda} = Q_\Lambda^{-1} z_\Lambda$ *is the minimizer of the total energy, i.e.*

$$
m_\Lambda \mapsto \frac{1}{2}\langle m_\Lambda, Q_\Lambda m_\Lambda\rangle_\Lambda - \frac{1}{2}\langle m_\Lambda, z_\Lambda\rangle_\Lambda\,. \tag{A.5}
$$

*We write* $Q_V = \Pi_V Q_\Lambda \Pi_V$, $Q_{\Lambda\setminus V,V} = \Pi_{\Lambda\setminus V} Q_\Lambda \Pi_V$. *The "conditional minimizer"*

$$
m_{\Lambda\setminus V}^{z_{\Lambda\setminus V},m_V} = Q_{\Lambda\setminus V}^{-1}(z_{\Lambda\setminus V} + Q_{\Lambda\setminus V,V} m_V) \tag{A.6}
$$

*is the minimizer of the function*

$$
m_{\Lambda\setminus V} \mapsto \frac{1}{2}\langle (m_{\Lambda\setminus V}, m_V), Q_\Lambda(m_{\Lambda\setminus V}, m_V)\rangle_\Lambda - \frac{1}{2}\langle (m_{\Lambda\setminus V}, m_V), z_\Lambda\rangle_\Lambda \tag{A.7}
$$

*for fixed* $m_V$.

**Remark.** The quadratic forms on the diagonal of the r.h.s. of (A.2) are automatically positive definite.

The proofs are easy and well-known computations and will not be given here. Next we collect some formulae and introduce notation concerning the random walk representation.

**Lemma A.2.** *Denote by* $\mathcal{R}$ *the* (*non-normalized*) *measure on the set of all finite paths on* $\mathbb{Z}^d$ (*with all possible lengths*), *defined by*

$$
\mathcal{R}(\{\gamma\}) = \left(\frac{1}{c+2d}\right)^{|\gamma|+1} \tag{A.8}
$$

*for a nearest neighbor path* $\gamma$ *of finite length* $|\gamma|$. *Then we have*

$$
R_{\Lambda;x,y} = \mathcal{R}(\gamma \textit{ from } x \textit{ to } y;\ \mathrm{Range}(\gamma)\subset\Lambda) \tag{A.9}
$$

*where* $\mathrm{Range}(\gamma) = \{\gamma_t; t = 0, \dots, k\}$ *is set of sites visited by a path* $\gamma = (\gamma_t)_{t=0,\dots,k}$ *of length* $|\gamma| = k$.

**Proof.** Write $\Delta_\Lambda^D = 2d - T_\Lambda$ where $T_{\Lambda;x,y} = 1$ iff $x, y \in \Lambda$ are nearest neighbors and $T_{\Lambda;x,y} = 0$ otherwise. Then

$$R_{\Lambda;x,y} = (c + 2d - T_\Lambda)^{-1}_{x,y} = \sum_{t=0}^{\infty} \left(\frac{1}{c+2d}\right)^{t+1} (T_\Lambda^t)_{x,y} \tag{A.10}$$

which proves (A.10). □

We will also use the obvious matrix notation

$$(\mathcal{R}(\cdot \to \cdot\,; C))_{x,y} = \mathcal{R}(\gamma \text{ from } x \text{ to } y;\, \mathrm{Range}(\gamma) = C) \tag{A.11}$$

so that one has the matrix equality $R_V = \sum_{C \subset V} \mathcal{R}(\cdot \to \cdot\,; C)$ for any volume $V$. We need to use a bound on its matrix elements at several places. Let us note the simple estimate

$$\sum_{y \in \mathbb{Z}^d} \mathcal{R}(x \to y\,; C) \le \mathcal{R}(\gamma \text{ starting at } x,\, \mathrm{length}(\gamma) \ge |C| - 1) = \frac{1}{c}\left(\frac{2d}{c+2d}\right)^{|C|-1}. \tag{A.12}$$

We will use these notations at many different places. As an example, let us prove formula (3.11). Indeed, we have

$$\begin{aligned}
(\Pi_{\partial G}(c - \Delta_\Lambda)^{-1}\Pi_{\partial G})^{-1} &= c - (\Delta_{\partial G} + \partial_{\partial G, \Lambda\setminus\partial G} R_{\Lambda\setminus\partial G} \partial_{\Lambda\setminus\partial G, \partial G}) \\
&= c - (\Delta_{\partial G} + \partial_{\partial G, \Lambda\setminus\partial G} R_{G^r\setminus\partial G} \partial_{\Lambda\setminus\partial G, \partial G}) \\
&\quad - \partial_{\partial G, \Lambda\setminus\partial G}(R_{\Lambda\setminus\partial G} - R_{G^r\setminus\partial G})\partial_{\Lambda\setminus\partial G, \partial G} \\
&= (\Pi_{\partial G}(c - \Delta_{G^r})^{-1}\Pi_{\partial G})^{-1} \\
&\quad - \partial_{\partial G, \Lambda\setminus\partial G}(R_{\Lambda\setminus\partial G} - R_{G^r\setminus\partial G})\partial_{\Lambda\setminus\partial G, \partial G} \\
&= (\Pi_{\partial G}(c - \Delta_{G^r})^{-1}\Pi_{\partial G})^{-1} \\
&\quad - \sum_{\substack{C \subset \Lambda\setminus\partial G \\ C \cap (G^r)^c \neq \emptyset,\, C \cap G^2 \neq \emptyset}} \partial_{\partial G, \Lambda\setminus\partial G} \mathcal{R}(\cdot \to \cdot\,; C) \partial_{\Lambda\setminus\partial G, \partial G}\,.
\end{aligned} \tag{A.13}$$

Here we have used Lemma A.1(i) in the first and third equality and Lemma A.2 in the last one. Finally we give the:

**Proof of Lemma 3.4.** The random walk representation of the determinant is obtained writing

$$\begin{aligned}
\log\det(c - \Delta_V) &= \log\det\left[(c + 2d)\left(1 + \frac{1}{c+2d} T_V\right)\right] \\
&= |V|\log(c + 2d) + \mathrm{Tr}\log\left(1 + \frac{1}{c+2d} T_V\right)
\end{aligned} \tag{A.14}$$

and expanding the logarithm. Using (A.3) we can then write

$$\begin{aligned}
\log \frac{\det(\Pi_{\partial G}(a-q\Delta_{G^r})^{-1}\Pi_{\partial G})}{\det(\Pi_{\partial G}(a-q\Delta_{\Lambda})^{-1}\Pi_{\partial G})}
&= \log \frac{\det_{G^r-\partial G}(c-\Delta_{G^r-\partial G})}{\det_{G^r}(c-\Delta_{G^r})}\frac{\det_{\Lambda}(c-\Delta_{\Lambda})}{\det_{\Lambda-\partial G}(c-\Delta_{\Lambda-\partial G})} \\
&= \sum_{t=1}^{\infty}\frac{1}{t}\frac{1}{(c+2d)^t}(\mathrm{Tr}_{G^r}(T_{G^r})^t - \mathrm{Tr}_{G^r-\partial G}(T_{G^r-\partial G})^t \\
&\quad - \mathrm{Tr}_{\Lambda}(T_{\Lambda})^t + \mathrm{Tr}_{\Lambda-\partial G}(T_{\Lambda-\partial G})^t)\,. \qquad \text{(A.15)}
\end{aligned}$$

It is not difficult to convince oneself that we have that

$$\begin{aligned}
&\mathrm{Tr}_{G^r}(T_{G^r})^t - \mathrm{Tr}_{G^r-\partial G}(T_{G^r-\partial G})^t - \mathrm{Tr}_{\Lambda}(T_{\Lambda})^t + \mathrm{Tr}_{\Lambda-\partial G}(T_{\Lambda-\partial G})^t \\
&\quad = -\sum_{x\in\Lambda}\#\{\gamma : x\mapsto x;\ \mathrm{Range}(\gamma)\subset\Lambda;\ \mathrm{Range}(\gamma)\cap\partial G\neq\emptyset; \\
&\qquad \mathrm{Range}(\gamma)\cap\Lambda\backslash G^r\neq\emptyset;\ |\gamma|=t\}\,. \qquad \text{(A.16)}
\end{aligned}$$

So we get the form (3.28) putting

$$2\epsilon^{\det}(C) := \sum_{t=2}^{\infty}\frac{1}{t}\frac{1}{(c+2d)^t}\sum_{x\in C}\#\{\gamma : x\mapsto x;\ \mathrm{Range}(\gamma)=C;\ |\gamma|=t\}\,. \qquad \text{(A.17)}$$

From this the bounds of the form $\epsilon^{\det}(C)\leq e^{-\mathrm{const}(\log c)|C|}$ are clear, assuming that $c$ is large. □

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