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# Small Gain Theorem and Optimal Robust Stabilization in a Behavioral Framework

H.L. Trentelman\*, Shaik Fiaz\*, K. Takaba\*\*

**Abstract**—Given a nominal plant, together with a fixed neighborhood of this plant, the problem of robust stabilization is to find a controller that stabilizes all plants in that neighborhood (in an appropriate sense). If a controller achieves this design objective, we say that it robustly stabilizes the nominal plant. In this paper we formulate the robust stabilization problem in a behavioral framework, with control as interconnection. We use both rational as well as polynomial representations for the behaviors under consideration. We obtain a behavioral version of the ‘small gain theorem’ and then obtain necessary and sufficient conditions for the existence of robustly stabilizing controllers using the theory of dissipative systems. We will also find the smallest upper bound on the radii of the neighborhoods for which there exists a robustly stabilizing controller. This smallest upper bound is expressed in terms of certain storage functions associated with the nominal control system.

## I. INTRODUCTION

This paper deals with control in a behavioral context. We consider the problem of finding, for a given nominal plant behavior, a controller such that the interconnection of the controller with *any* plant in a given neighborhood of the nominal plant is stable. In other words, we consider the problem of *robust stabilization* in a behavioral framework.

In [1], the problem of robust stabilization was formulated in an input output framework (using normalized coprime factor plant descriptions), and controllers considered for robust stabilization were of feedback form. In contrast to [1], we work in the generality, where we view systems in a behavioral sense, that is, as families of trajectories, and control is viewed as restricting the plant behavior by intersecting it with a controller behavior.

In this paper we formulate the robust stabilization problem using rational representations of behaviors recently introduced in [5]. We use the theory of dissipative systems, first to obtain a behavioral version of the ‘small gain theorem’ and then to obtain necessary and sufficient conditions for the existence of a robustly stabilizing controller. We will also find the smallest upper bound on the radii of the neighborhoods for which there exists a robustly stabilizing controller.

We note that due to space limitations, some of the proofs in this paper have been omitted. For these, the reader is referred to a future, full version of the paper.

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A few words about the notation and nomenclature used. We use standard symbols for the sets  $\mathbb{R}$  and  $\mathbb{C}$ .  $\bar{\mathbb{C}}_+ := \{\lambda \in \mathbb{C} \mid \text{Re}(\lambda) \geq 0\}$  will denote the closed right half complex plane.  $\mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^q)$  denotes the set of infinitely often differentiable functions from  $\mathbb{R}$  to  $\mathbb{R}^q$ , and its subspace consisting of the compact support elements is denoted by  $\mathcal{D}(\mathbb{R}, \mathbb{R}^q)$ , or sometimes simply by  $\mathcal{D}$ .

$\mathbb{R}[\xi]$  denotes the set of polynomials with real coefficients in the indeterminate  $\xi$  and  $\mathbb{R}(\xi)$  denotes the set of real rational functions in the indeterminate  $\xi$ . A square nonsingular real polynomial matrix  $R$  is called Hurwitz if all roots of  $\det(R)$  lie in the open left half complex plane  $\mathbb{C}^-$ . It is called anti-Hurwitz if all roots of  $\det(R)$  lie in the open right half complex plane  $\mathbb{C}^+$ . The rational function  $f \in \mathbb{R}(\xi)$ ,  $f = n/d$ , with  $n, d \in \mathbb{R}[\xi]$  coprime, is said to be stable if  $d$  is Hurwitz, and minimum phase if  $n$  and  $d$  are both Hurwitz. A real rational matrix  $G$  is called stable if all its poles are in  $\mathbb{C}^-$ . A proper, stable real rational matrix  $G$  is called *left prime* if there exists a proper, stable rational matrix  $G^\dagger$  such that  $GG^\dagger = I$ . A proper, stable real rational matrix  $G$  is called *co-inner* if  $G(\xi)G^\top(-\xi) = I$ .

It is called co-inner if  $G$  is stable and  $G(\xi)G^\top(-\xi) = I$ . The  $\mathcal{H}_\infty$ -norm of a proper stable real rational matrix  $G$  is denoted by  $\|G\|_\infty$ . We use the notation  $\text{col}(w_1, w_2)$  to represent a column vector formed by  $w_1$  over  $w_2$ . We call  $B^+$  the Moore-Penrose inverse of a given matrix  $B$  if it satisfies the following properties:  $BB^+B = B$ ,  $B^+BB^+ = B^+$ ,  $(BB^+)^T = BB^+$ , and  $(B^+B)^T = B^+B$ .

## II. LINEAR DIFFERENTIAL SYSTEMS AND RATIONAL REPRESENTATIONS

In the behavioral approach to linear systems, a dynamical system is given by a triple  $\Sigma = (\mathbb{R}, \mathbb{R}^q, \mathfrak{B})$ , where  $\mathbb{R}$  is the time axis,  $\mathbb{R}^q$  is the signal space, and the behavior  $\mathfrak{B}$  is a linear subspace of  $\mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^q)$ , consisting of all solutions of a set of higher order, linear, constant coefficient differential equations. Such a triple is called a linear differential system. The set of all linear differential systems with  $q$  variables is denoted by  $\mathcal{L}^q$ .

For any linear differential system, there exists a polynomial matrix  $R$  such that  $\mathfrak{B}$  is the solution set of

$$R\left(\frac{d}{dt}\right)w = 0. \quad (1)$$

If a behavior  $\mathfrak{B}$  is represented by  $R\left(\frac{d}{dt}\right)w = 0$  (or:  $\mathfrak{B} = \ker\left(R\left(\frac{d}{dt}\right)\right)$ ), with  $R(\xi)$  a real polynomial matrix, then we call this a polynomial kernel representation of  $\mathfrak{B}$ . Suppose  $R$  has  $p$  rows. Then the kernel representation is said to be minimal if every other kernel representation of  $\mathfrak{B}$  has at

least  $p$  rows. A given kernel representation  $\mathfrak{B} = \ker(R)$  is minimal if and only if the polynomial matrix  $R$  has full row rank (see [8], Theorem 3.6.4). The number of rows in any minimal kernel representation of  $\mathfrak{B}$  is denoted by  $p(\mathfrak{B})$ . This number is called the *output cardinality* of  $\mathfrak{B}$ . It corresponds to the number of outputs in any input/output representation of  $\mathfrak{B}$ .

*Definition 2.1:* Let  $\mathfrak{B} \in \mathcal{L}^{q_1+q_2}$  with system variable  $(w_1, w_2)$ . We will call  $w_2$  free in  $\mathfrak{B}$  if for any choice of  $w_2 \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^{q_2})$  there exists  $w_1$  such that  $(w_1, w_2) \in \mathfrak{B}$ .

*Proposition 2.2:* Let  $\mathfrak{B} \in \mathcal{L}^{q_1+q_2}$  with system variable  $(w_1, w_2)$ , have minimal representation  $R_1(\frac{d}{dt})w_1 + R_2(\frac{d}{dt})w_2 = 0$ . Then  $w_2$  is free in  $\mathfrak{B}$  if and only if  $R_1$  has full row rank.

Often we are only interested in the behavior of, say, the variable  $w_1$ . This behavior is denoted by  $\mathfrak{B}_{w_1}$  and is obtained by projecting all  $(w_1, w_2) \in \mathfrak{B}$  onto the first component  $w_1$ , in other words:  $\mathfrak{B}_{w_1} := \{w_1 \mid \exists w_2 \text{ such that } (w_1, w_2) \in \mathfrak{B}\}$ .

For a detailed exposition of polynomial representations of behaviors the reader is referred to [8].

Recently, in [5], representations of linear differential systems using *rational matrices* instead of polynomial matrices were introduced. Thus, for a given rational matrix  $G(\xi)$ , a meaning was given to the equation  $G(\frac{d}{dt})w = 0$ . In order to do this, we need the concept of left coprime factorization.

*Definition 2.3:* Let  $G$  be a real rational matrix. The pair  $(P, Q)$  is said to be a left coprime factorization over  $\mathbb{R}[\xi]$  of  $G$  if 1)  $\det(P) \neq 0$ , 2)  $G = P^{-1}Q$ , 3) the matrix  $(P(\lambda) \ Q(\lambda))$  has full row rank for all  $\lambda \in \mathbb{C}$ .

Let  $G$  be a real rational matrix, and consider the equation

$$G(\frac{d}{dt})w = 0. \quad (2)$$

The question of course is to give a meaning to this expression. In [5] this is done as follows. Let  $(P, Q)$  be a left coprime factorization over  $\mathbb{R}[\xi]$  of  $G$ . Then we define:

$$[w : \mathbb{R} \rightarrow \mathbb{R}^q \text{ is a solution of (2)}] \Leftrightarrow [Q(\frac{d}{dt})w = 0].$$

It can be proven that the set of solutions is independent of the particular left coprime factorization. Hence (2) represents the linear differential system  $\Sigma = (\mathbb{R}, \mathbb{R}^q, \ker(Q(\frac{d}{dt}))) \in \mathcal{L}^q$ .

Since the behavior  $\mathfrak{B}$  of the system  $\Sigma$  is the central item, we will mostly speak about the system  $\mathfrak{B} \in \mathcal{L}^q$  (instead of  $\Sigma \in \mathcal{L}^q$ ). If a behavior  $\mathfrak{B}$  is represented by  $G(\frac{d}{dt})w = 0$  (or:  $\mathfrak{B} = \ker(G(\frac{d}{dt}))$ ), with  $G(\xi)$  a real rational matrix, then we call this a rational kernel representation of  $\mathfrak{B}$ .

The following proposition was obtained in [5].

*Proposition 2.4:* Let  $G$  be a real rational matrix, and let  $G = P^{-1}Q$  be a left coprime factorization over  $\mathbb{R}[\xi]$ . Then for any pair of functions  $w_1, w_2$  we have  $w_2 = G(\frac{d}{dt})w_1$  (in the sense that  $(I - G(\frac{d}{dt}))\text{col}(w_2, w_1) = 0$ ) if and only if  $P(\frac{d}{dt})w_2 = Q(\frac{d}{dt})w_1$ .

Many models obtained from first principles models by interconnection and state models include auxiliary variables in addition to the variables the model aims at. We call the latter manifest variables denoted by  $w$ , and the auxiliary variables latent variables denoted by  $\ell$ . In the context of

rational representations, this leads to representations of the form

$$R(\frac{d}{dt})w = M(\frac{d}{dt})\ell \quad (3)$$

with  $R, M$  real rational matrices. This should of course be interpreted as  $(R(\frac{d}{dt}) - M(\frac{d}{dt}))\text{col}(w, \ell) = 0$ . The set of  $w$  for which there exists a  $\ell$  such that (3) holds is called the manifest behavior. If  $R = I$ , the identity matrix, then (3) is called an *image representation* of the manifest behavior.

*Definition 2.5:* A behavior  $\mathfrak{B} \in \mathcal{L}^q$  is said to be *controllable* if for all  $w_1, w_2 \in \mathfrak{B}$ , there exists  $T \geq 0$  and  $w \in \mathfrak{B}$ , such that  $w(t) = w_1(t)$  for  $t < 0$ , and  $w(t) = w_2(t - T)$  for  $t \geq T$ . It is *stabilizable* if for every  $w \in \mathfrak{B}$ , there exists  $w' \in \mathfrak{B}$  such that  $w'(t) = w(t)$  for  $t \leq 0$ , and  $\lim_{t \rightarrow \infty} w'(t) = 0$ .

It can be shown that for a given controllable behavior  $\mathfrak{B} \in \mathcal{L}^q$ , with out loss of generality we can take  $G(\xi)$  such that (2) holds to be proper, stable, left prime and co-inner.

*Definition 2.6:* Let  $\mathfrak{B} \in \mathcal{L}^q$  has system variable  $w$  partitioned as  $(w_1, w_2)$ . We say  $w_2$  is *observable* from  $w_1$  in  $\mathfrak{B}$  if, whenever  $(w_1, w'_2), (w_1, w''_2) \in \mathfrak{B}$ , then  $w'_2 = w''_2$ , and we say  $w_2$  is *detectable* from  $w_1$  in  $\mathfrak{B}$  if, whenever  $(w_1, w'_2), (w_1, w''_2) \in \mathfrak{B}$ , then  $\lim_{t \rightarrow \infty} (w'_2 - w''_2)(t) = 0$ .

*Definition 2.7:* A behavior  $\mathfrak{B} \in \mathcal{L}^q$  is called *stable* if for all  $w \in \mathfrak{B}$  we have  $\lim_{t \rightarrow \infty} w(t) = 0$ .

*Definition 2.8:* Let  $\mathfrak{B}_f = \{(w, \ell) \mid w, \ell \text{ satisfy (3)}\}$ . Then the latent variable representation (3) is said to be *observable* if  $\ell$  is observable from  $w$  in  $\mathfrak{B}_f$ . It is said to be *detectable* if  $\ell$  is detectable from  $w$  in  $\mathfrak{B}_f$ .

#### A. Stabilizing controller

$\mathcal{P} \in \mathcal{L}^q$ ,  $\mathcal{C} \in \mathcal{L}^q$ , and their interconnection  $\mathcal{P} \cap \mathcal{C}$ , the controlled system, which is defined as follows:

$$\mathcal{P} \cap \mathcal{C} := \{w \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^q) \mid w \in \mathcal{P} \text{ and } w \in \mathcal{C}\}.$$

The interconnection  $\mathcal{P} \cap \mathcal{C}$  is said to be *regular* if  $p(\mathcal{P} \cap \mathcal{C}) = p(\mathcal{P}) + p(\mathcal{C})$ .

*Definition 2.9:* A controller  $\mathcal{C}$  is said to be a *stabilizing controller* for  $\mathcal{P}$  if  $\mathcal{P} \cap \mathcal{C}$  is stable and the interconnection is regular. If  $\mathcal{C}$  is a stabilizing controller then we say that  $\mathcal{C}$  *stabilizes*  $\mathcal{P}$ .

The following Lemma was shown in [10].

*Lemma 2.10:* There exists a stabilizing controller  $\mathcal{C}$  for  $\mathfrak{B}$  if and only if  $\mathfrak{B}$  is stabilizable.

In the next section we will formulate the robust stabilization problems studied in this paper.

### III. ROBUST STABILIZATION BY INTERCONNECTION: PROBLEM FORMULATION

Given a controllable nominal plant  $\mathcal{P} \in \mathcal{L}^q$ , together with a fixed neighborhood of this plant, find a controller  $\mathcal{C} \in \mathcal{L}^q$  that stabilizes all plants in that neighborhood. Of course, the concept of neighbourhood should be made explicit. We do this in the following way. Assume that  $\mathcal{P}$  is represented in rational kernel representation by  $R(\frac{d}{dt})w = 0$ , where  $R$  is proper, stable, left prime and co-inner. As noted in the previous section, for a given  $\mathcal{P}$  such  $R$  always exists. For a

given  $\gamma > 0$  we define the ball  $B(\mathcal{P}, \gamma)$  with radius  $\gamma$  around  $\mathcal{P}$  as follows:

$$B(\mathcal{P}, \gamma) := \{ \mathcal{P}_\Delta \in \mathcal{L}_{\text{cont}}^q \mid \text{there exists a proper, stable, real rational } R_\Delta \text{ of full row rank such that } \mathcal{P}_\Delta = \ker(R_\Delta) \text{ and } \|R - R_\Delta\|_\infty \leq \gamma \}. \quad (4)$$

**Problem 1 :** Find necessary and sufficient conditions for the existence of a controller  $\mathcal{C} \in \mathcal{L}^q$  that regularly stabilizes all plants  $\mathcal{P}_\Delta$  in the ball with radius  $\gamma$  around  $\mathcal{P}$ , i.e. for all  $\mathcal{P}_\Delta \in B(\mathcal{P}, \gamma)$ ,  $\mathcal{P}_\Delta \cap \mathcal{C}$  is stable and  $\mathcal{P}_\Delta \cap \mathcal{C}$  is a regular interconnection.

Of course, for a given nominal plant  $\mathcal{P}$  we would like to know the *smallest upper bound* (if it exists) of those  $\gamma$ 's for which there exists a controller  $\mathcal{C}$  that stabilizes all perturbed plants  $\mathcal{P}_\Delta$  in the ball with radius  $\gamma$  around  $\mathcal{P}$ . This is the problem of *optimal robust stabilization*.

**Problem 2 :** Find

$$\gamma^* := \sup \{ \gamma > 0 \mid \exists \mathcal{C} \in \mathcal{L}^q \text{ that stabilizes all perturbed plants } \mathcal{P}_\Delta \in B(\mathcal{P}, \gamma) \}. \quad (5)$$

In the next subsection tests to verify properties of linear systems from its representations are discussed.

#### A. Tests to verify properties of linear differential systems

There are tests to verify certain properties of linear differential systems in terms of the rational matrices appearing in the kernel representation and their zeros. In order to formulate these, we first recall the notions of poles and zeros of rational matrices.

*Proposition 3.1:* Let  $M = P^{-1}Q$  be a left coprime factorization of the rational matrix  $M$ . Then the zeros of  $P$  (disregarding the multiplicity issue) are called the *poles* of  $M$ , and the zeros of  $Q$  (disregarding the multiplicity issue) the *zeros* of  $M$ .

The following proposition can be found in [5].

*Proposition 3.2:* 1) The representation (2) defines a controllable system if and only if  $G$  has no zeros.

2) The representation (2) defines a stabilizable system if and only if  $G$  has no zeros in  $\bar{\mathbb{C}}_+$ .

3) The latent variable representation (3) is observable if and only if  $M$  has full column rank and has no zeros. It is detectable if and only if  $M$  has full column rank, and has no zeros in  $\bar{\mathbb{C}}_+$ .

4)  $\mathfrak{B} \in \mathcal{L}^q$  is controllable if and only if  $\mathfrak{B}$  admits an observable image representation  $w = H(\frac{d}{dt})l$  with  $H$  a real rational matrix.

5)  $\mathfrak{B} \in \mathcal{L}^q$  is controllable if and only if  $\mathfrak{B}$  admits an observable image representation  $w = H(\frac{d}{dt})l$  with  $H$  a real polynomial matrix.

*Definition 3.3:* A real square rational matrix  $M$  is called *minimum phase* if it has no poles and zeros in  $\bar{\mathbb{C}}_+$ .

The following proposition was shown in [13].

*Proposition 3.4:* Let  $\mathfrak{B} \in \mathcal{L}^q$  be described by  $G(\frac{d}{dt})w = 0$ . Assume  $\mathfrak{B}$  is stabilizable. A  $\mathcal{C} \in \mathcal{L}^q$  is a stabilizing controller for  $\mathfrak{B}$  if and only if the following conditions hold:

- 1)  $\mathcal{C}$  admits a representation  $C(\frac{d}{dt})w = 0$  such that  $C$  has no poles and zeros in  $\bar{\mathbb{C}}_+$ , and
- 2)  $\begin{pmatrix} G \\ C \end{pmatrix}$  is square and minimum phase.

Finally, we review the notion of orthogonal complement of a behavior. Let  $\mathfrak{B} \in \mathcal{L}^q$  be a controllable behavior. Then we define its orthogonal complement  $\mathfrak{B}^\perp$  by

$$\mathfrak{B}^\perp := \{ w \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^q) \mid \int_{-\infty}^{\infty} w^\top w' dt = 0 \text{ for all } w' \in \mathfrak{B} \cap \mathcal{D} \}.$$

$\mathfrak{B}^\perp$  is again controllable. If  $R(\frac{d}{dt})w = 0$  is a rational (polynomial) kernel representation of  $\mathfrak{B}$ , then  $\tilde{w} = R^\top(-\frac{d}{dt})\ell$  is a rational (polynomial) image representation of  $\mathfrak{B}^\perp$ .

## IV. TWO-VARIABLE POLYNOMIAL MATRICES AND QUADRATIC DIFFERENTIAL FORMS

In this paper, an important role will be played by two-variable polynomial matrices and *quadratic differential forms* (QDFs). A QDF  $Q_\Phi(w)$  is defined as a quadratic form of  $w : \mathbb{R} \rightarrow \mathbb{R}^q$  and its derivatives. Namely  $Q_\Phi(w) = \sum_{h,k=0}^n \left( \frac{d^h w}{dt^h} \right)^\top \Phi_{h,k} \frac{d^k w}{dt^k}$  where  $\Phi_{h,k} \in \mathbb{R}^{q \times q}$  and  $\Phi_{h,k} = \Phi_{k,h}^\top$ . We can associate  $Q_\Phi$  with a symmetric two-variable polynomial matrix  $\Phi(\zeta, \eta) = \sum_{h,k=0}^n \Phi_{h,k} \zeta^h \eta^k$ . Note that the indeterminates  $\zeta$  and  $\eta$  correspond to the differentiations on  $w^\top$  and  $w$  respectively.

The properties of the two-variable polynomial matrix  $\Phi(\zeta, \eta)$  are completely determined by the real constant  $(n+1)q \times (n+1)q$  matrix  $\tilde{\Phi}$  whose  $(h, k)$ th block is equal to  $\Phi_{h,k}$ . This matrix will be called the *coefficient matrix* associated with  $\Phi(\zeta, \eta)$ . For a detailed exposition of QDF's the reader is referred to [7].

## V. DISSIPATIVE SYSTEMS AND STORAGE FUNCTIONS

In this paper, our aim is to establish conditions on the plant  $\mathcal{P}$  for the existence of a robustly stabilizing controller. An important role in our development will be played by the notions of dissipativeness, strict dissipativeness, and storage functions. These notions have been studied before in [11]. In the present section we review these notions in the framework of linear differential systems.

Consider, in general, a controllable differential system  $\mathfrak{B}$  given by the observable image representation

$$w = W\left(\frac{d}{dt}\right)\ell \quad (6)$$

with  $W \in \mathbb{R}^{q \times 1}[\xi]$ . In addition, let  $Q_\Phi : \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^q) \rightarrow \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$ ;  $w \mapsto Q_\Phi(w)$ , be the QDF associated with a given symmetric two-variable polynomial matrix  $\Phi \in \mathbb{R}^{q \times q}[\zeta, \eta]$ .  $Q_\Phi$  will be called the *supply rate*. The system  $\mathfrak{B}$  will be called *dissipative* with respect to the supply rate  $Q_\Phi$  if for all  $w \in \mathfrak{B} \cap \mathcal{D}(\mathbb{R}, \mathbb{R}^q)$  there holds

$$\int_{-\infty}^{\infty} Q_\Phi(w) dt \geq 0. \quad (7)$$

$\mathfrak{B}$  is called *strictly dissipative* with respect to the supply rate  $Q_\Phi$  if there exists  $\epsilon > 0$  such that for all  $w \in \mathfrak{B} \cap \mathcal{D}(\mathbb{R}, \mathbb{R}^q)$

$$\int_{-\infty}^{\infty} Q_\Phi(w) dt \geq \epsilon^2 \int_{-\infty}^{\infty} \|w(t)\|^2 dt. \quad (8)$$

In this paper the supply rate will often be given by a constant real symmetric matrix, say  $\Sigma$ . In that case we have  $Q_\Sigma(w) = w^\top \Sigma w$ . We say that the system  $\mathfrak{B}$  is (strictly)  $\Sigma$ -dissipative if it is (strictly) dissipative with respect to the supply rate  $Q_\Sigma(w)$ .

Given the image representation (6) and the polynomial matrix  $\Phi(\zeta, \eta)$ , define  $\Phi' \in \mathbb{R}^{1 \times 1}[\zeta, \eta]$  by  $\Phi'(\zeta, \eta) := W^\top(\zeta) \Phi(\zeta, \eta) W(\eta)$ . It is easily verified that if  $w$  and  $\ell$  are related by (6), then  $Q_\Phi(w) = Q_{\Phi'}(\ell)$ . Therefore, the system is dissipative iff for all  $\ell \in \mathcal{D}(\mathbb{R}, \mathbb{R}^1)$  we have  $\int_{-\infty}^{\infty} Q_{\Phi'}(\ell) dt \geq 0$ , and strictly dissipative iff there exists  $\epsilon > 0$  such that for all  $\ell \in \mathcal{D}(\mathbb{R}, \mathbb{R}^1)$  we have

$$\int_{-\infty}^{\infty} Q_{\Phi'}(\ell) dt \geq \epsilon^2 \int_{-\infty}^{\infty} \|W(\frac{d}{dt})\ell\|^2 dt.$$

These conditions are equivalent to

$$\Phi'(-i\omega, i\omega) \geq 0 \quad \text{for all } \omega \in \mathbb{R}, \quad (9)$$

$$\Phi'(-i\omega, i\omega) \geq \epsilon^2 W^\top(-i\omega) W(i\omega) \quad \text{for all } \omega \in \mathbb{R} \quad (10)$$

respectively (see [7]). It is well known that if (9) holds then we can factorize  $\partial\Phi'(\xi) := \Phi'(-\xi, \xi) = F^\top(-\xi)F(\xi)$ , with  $F \in \mathbb{R}^{1 \times 1}[\xi]$ . If (10) holds, then  $F$  can be chosen Hurwitz, and also anti-Hurwitz. Introduce now the two-variable polynomial  $\Delta$  defined by  $\Delta(\zeta, \eta) := \Phi'(\zeta, \eta) - F^\top(\zeta)F(\eta)$ . Since  $\partial\Delta = 0$ , the two-variable polynomial  $\Delta$  must contain a factor  $\zeta + \eta$  (see [7], theorem 3.1), and therefore we can define the new two-variable polynomial  $\Psi$  by

$$\Psi(\zeta, \eta) := (\zeta + \eta)^{-1} \Delta(\zeta, \eta). \quad (11)$$

Consider now the QDF's  $Q_\Psi$  and  $Q_\Delta$  associated with  $\Psi$  and  $\Delta$ , respectively. We have  $Q_\Delta(\ell) = Q_{\Phi'}(\ell) - \|F(\frac{d}{dt})\ell\|^2$ . Furthermore, (11) is equivalent to:  $\frac{dQ_\Psi(\ell)}{dt} = Q_\Delta(\ell)$  for all  $\ell \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^1)$ . Thus we obtain

$$\frac{dQ_\Psi(\ell)}{dt}(t) \leq Q_{\Phi'}(\ell)(t), \quad (12)$$

for all  $\ell \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^1)$ , for all  $t \in \mathbb{R}$ . If we interpret  $Q_\Psi(\ell)(t)$  as the amount of supply (e.g., energy) stored inside the system at time  $t$ , then (12) expresses the fact that the rate at which the internal storage increases does not exceed the rate at which supply flows into the system. The inequality (12) is called the *dissipation inequality*. Any quadratic differential form  $Q_\Psi(\ell)$  that satisfies this inequality is called a *storage function* for  $\mathfrak{B}$ . It can be shown that  $\mathfrak{B}$  is dissipative if and only if there exists a symmetric two-variable polynomial matrix  $\Psi(\zeta, \eta)$  such that the corresponding QDF  $Q_\Psi$  satisfies (12). In general, storage functions are not unique. In fact, we quote [[7], Theorem 5.7].

*Proposition 5.1:* Let  $\mathfrak{B}$  be represented by the observable image representation (6). Assume  $\mathfrak{B}$  is dissipative with respect to  $Q_\Phi$ . Then there exist storage functions  $Q_{\Psi_-}$

and  $Q_{\Psi_+}$  such that any other storage function  $Q_\Psi$  satisfies  $Q_{\Psi_-} \leq Q_\Psi \leq Q_{\Psi_+}$ . If  $\mathfrak{B}$  is strictly dissipative then  $\Psi_-$  and  $\Psi_+$  may be constructed as follows. Let  $H$  and  $A$  be respectively Hurwitz and anti-Hurwitz factorizations of  $\partial\Phi'$ . Then  $\Psi_+(\zeta, \eta) = \frac{\Phi'(\zeta, \eta) - A^\top(\zeta)A(\eta)}{\zeta + \eta}$  and  $\Psi_-(\zeta, \eta) = \frac{\Phi'(\zeta, \eta) - H^\top(\zeta)H(\eta)}{\zeta + \eta}$ .

Finally, we spend some words on state maps, and on positive and negative definiteness of storage functions. Again, let  $\mathfrak{B}$  be given in image representation  $w = W(\frac{d}{dt})\ell$ . An  $n \times 1$  polynomial matrix  $X$  is said to define a *state map* for  $\mathfrak{B}$  if  $x := X(\frac{d}{dt})\ell$  is a state variable for  $\mathfrak{B}$  (see [12]). The dimension of the state space of a state-minimal representation of  $\mathfrak{B} \in \mathcal{L}^q$  is called the *McMillan degree* of  $\mathfrak{B}$  and is denoted by  $n(\mathfrak{B})$ . Often,  $n(\mathfrak{B})$  is denoted by  $n$ . A state map  $X$  for  $\mathfrak{B}$  is called a *minimal state map* if its number of rows is equal to  $n(\mathfrak{B})$ . If the polynomial matrix  $X$  yields a minimal state map, then its coefficient matrix  $\tilde{X}$  is a full row rank real matrix. It can be shown that we can always choose  $X$  which defines a minimal state map for  $\mathfrak{B}$  such that  $\tilde{X}\tilde{X}^\top = I$ . The following proposition obtained in [7] (also see [14]) will play an important role in the sequel.

*Proposition 5.2:* Let  $\mathfrak{B}$  be represented by the observable image representation (6). Assume  $\mathfrak{B}$  is dissipative with respect to  $Q_\Sigma(w) = w^\top \Sigma w$ , where  $\Sigma \in \mathbb{R}^{q \times q}$ , and let  $Q_\Psi(\ell)$  be a storage function. Let  $X \in \mathbb{R}^{n \times 1}[\xi]$  defines a minimal state map of  $\mathfrak{B}$ . Then there exists a real symmetric matrix  $K \in \mathbb{R}^{n \times n}$  such that  $\Psi(\zeta, \eta) = X^\top(\zeta)KX(\eta)$ , equivalently,  $Q_\Psi(\ell) = (X(\frac{d}{dt})\ell)^\top KX(\frac{d}{dt})\ell$  for all  $\ell \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^1)$ .

*Definition 5.3:* A storage function  $Q_\Psi$  for  $\mathfrak{B}$  is called *positive (negative) definite* if there exists a state map  $X$  for  $\mathfrak{B}$  and a matrix  $K > 0$  ( $K < 0$ ) such that  $Q_\Psi(\ell) = (X(\frac{d}{dt})\ell)^\top KX(\frac{d}{dt})\ell$  for all  $\ell \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^1)$ .

## VI. A RELEVANT DISSIPATIVITY SYNTHESIS PROBLEM

Let  $\mathcal{P}_{\text{full}} \in \mathcal{L}^{q+v+c}$  be a controllable system with system variable  $(w, v, c)$ .  $v$  is assumed to be free in  $\mathcal{P}_{\text{full}}$ . The variable  $c$  has the interpretation of interconnection variable, through which we are allowed to interconnect  $\mathcal{P}_{\text{full}}$  with controller  $\mathcal{C} \in \mathcal{L}^c$ , and is assumed to be free in  $\mathcal{P}_{\text{full}}$ . Interconnection leads to the full controlled behavior  $\mathcal{P}_{\text{full}} \wedge_c \mathcal{C} := \{(w, v, c) | (w, v, c) \in \mathcal{P}_{\text{full}} \text{ and } c \in \mathcal{C}\}$ . In addition, the *manifest controlled behavior* is defined as  $(\mathcal{P}_{\text{full}} \wedge_c \mathcal{C})_{(w, v)}$ , the full controlled behavior projected on the variable  $(w, v)$ . Let  $\mathcal{N}_{(w, c)}(\mathcal{P}_{\text{full}} \wedge_c \mathcal{C}) := \{(w, c) | (w, 0, c) \in \mathcal{P}_{\text{full}} \wedge_c \mathcal{C}\}$ .

In the context of  $\mathcal{H}_\infty$  control problem, we have the following definition of stabilizing controller.

*Definition 6.1:* For  $\mathcal{P}_{\text{full}} \in \mathcal{L}^{q+v+c}$ , a controller  $\mathcal{C} \in \mathcal{L}^c$  is called *disturbance-free stabilizing* if  $v$  is free in  $\mathcal{P}_{\text{full}} \wedge_c \mathcal{C}$ , and  $[(w, 0, c) \in \mathcal{P}_{\text{full}} \wedge_c \mathcal{C}] \Rightarrow [\lim_{t \rightarrow \infty} (w(t), c(t)) = 0]$  i.e.,  $\mathcal{N}_{(w, c)}(\mathcal{P}_{\text{full}} \wedge_c \mathcal{C})$  is stable.

It can be easily verified that the interconnection of a disturbance-free stabilizing controller with  $\mathcal{P}_{\text{full}}$  will be regular.

Define

$$\Sigma_\gamma := \begin{pmatrix} -I & 0 \\ 0 & \frac{1}{\gamma^2} I \end{pmatrix}, \quad \text{so} \quad -\Sigma_\gamma^{-1} = \begin{pmatrix} I & 0 \\ 0 & -\gamma^2 I \end{pmatrix}. \quad (13)$$

For  $\gamma > 0$ , a controller  $\mathcal{C} \in \mathcal{L}^c$  is called *strictly  $\gamma$ -contracting* if the manifest controlled behavior  $(\mathcal{P}_{\text{full}} \wedge_c \mathcal{C})_{(w,v)}$  is strictly dissipative with respect to the supply rate  $-|w|^2 + \frac{1}{\gamma^2}|v|^2$ , i.e., strictly  $\Sigma_\gamma$ -dissipative.

The following Theorem provides necessary and sufficient conditions for the existence of a disturbance-free stabilizing, strictly  $\gamma$ -contracting controller for  $\mathcal{P}_{\text{full}}$ .

**Theorem 6.2:** Let  $\mathcal{P}_{\text{full}} \in \mathcal{L}^{q+v+c}$  be controllable. Assume  $v$  is free, and  $(w, v)$  is detectable from  $c$  in  $\mathcal{P}_{\text{full}}$ . Let  $\gamma > 0$ . Then there exists a disturbance-free stabilizing, strictly  $\gamma$ -contracting controller for  $\mathcal{P}_{\text{full}}$  if and only if  $(\mathcal{P}_{\text{full}})_{(w,v)}^\perp$  is  $-\Sigma_\gamma^{-1}$ -dissipative and has a negative definite storage function.

*Proof:* The proof is omitted.  $\square$

## VII. ROBUST STABILIZATION

In this section we study first Problem 1 and then Problem 2 introduced in section III.

### A. Solution to Problem 1

Let  $\mathcal{P} \in \mathcal{L}^q$  be controllable, and let it be represented in rational kernel representation by  $R(\frac{d}{dt})w = 0$ , where  $R$  is proper, stable, real rational, left prime and co-inner. Define the auxiliary system  $\mathcal{P}_{\text{aux}} \in \mathcal{L}^{q+v+q}$  as  $\mathcal{P}_{\text{aux}} = \{(w, v, c) \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^{q+v+q}) \mid R(\frac{d}{dt})w + v = 0, c = w\}$ . Let  $R(\xi) = P^{-1}(\xi)Q(\xi)$  be a left coprime factorization over  $\mathbb{R}[\xi]$  where  $P$  is Hurwitz. Then by definition  $\mathcal{P}_{\text{aux}} = \{(w, v, c) \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^{q+v+q}) \mid Q(\frac{d}{dt})w + P(\frac{d}{dt})v = 0, c = w\}$ .

Let  $\mathcal{C} \in \mathcal{L}^q$  be given by  $C'(\frac{d}{dt})w = 0$ , where  $C'$  is rational, proper and has no poles and zeros in  $\bar{\mathbb{C}}^+$ . Let  $C'(\xi) = S^{-1}(\xi)C(\xi)$  be a left coprime factorization over  $\mathbb{R}[\xi]$  where  $S$  is Hurwitz. Then by definition  $\mathcal{C} := \{w \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^q) \mid C(\frac{d}{dt})w = 0\}$ . We have the following Theorem:

**Theorem 7.1:** Let  $\gamma > 0$ . Then the following statements are equivalent:

1)  $\mathcal{C}$  stabilizes  $\mathcal{P}_\Delta$  for all  $\mathcal{P}_\Delta \in B(\mathcal{P}, \gamma)$ , i.e.,  $\mathcal{P}_\Delta \cap \mathcal{C}$  is stable, and  $\mathcal{P}_\Delta \cap \mathcal{C}$  is a regular interconnection for all  $\mathcal{P}_\Delta \in B(\mathcal{P}, \gamma)$ .

2)  $\begin{pmatrix} Q \\ C \end{pmatrix}$  is Hurwitz and  $\left\| \begin{pmatrix} Q \\ C \end{pmatrix}^{-1} \begin{pmatrix} P \\ 0 \end{pmatrix} \right\|_\infty < \frac{1}{\gamma}$ .

*Proof:* The proof is omitted.  $\square$

The following Lemma characterizes a disturbance-free stabilizing and strictly  $\gamma$ -contracting controller  $\mathcal{C}$  for  $\mathcal{P}_{\text{aux}}$  in terms of their representations.

**Lemma 7.2:** Let  $\mathcal{C} = \{c \mid C(\frac{d}{dt})c = 0\}$ . Then  $\mathcal{C}$  is a disturbance-free stabilizing and strictly  $\gamma$ -contracting controller for  $\mathcal{P}_{\text{aux}}$  if and only if  $\begin{pmatrix} Q \\ C \end{pmatrix}$  is Hurwitz and

$$\left\| \begin{pmatrix} Q \\ C \end{pmatrix}^{-1} \begin{pmatrix} P \\ 0 \end{pmatrix} \right\|_\infty < \frac{1}{\gamma}.$$

*Proof:* The proof is omitted.  $\square$

The following lemma will formulate a behavioral version of the ‘small gain theorem’.

**Lemma 7.3:** Let  $\mathcal{C} \in \mathcal{L}^q$ . Then  $\mathcal{C}$  stabilizes  $\mathcal{P}_\Delta$  for all  $\mathcal{P}_\Delta \in B(\mathcal{P}, \gamma)$  if and only if  $\mathcal{C}$  is a disturbance-free stabilizing and strictly  $\gamma$ -contracting controller for  $\mathcal{P}_{\text{aux}}$ .

*Proof:* The proof is evident from Theorem 7.1 and Lemma 7.2.  $\square$

A solution to Problem 1 is given by the following Theorem.

**Theorem 7.4:** Let  $\gamma > 0$ . There exists a controller  $\mathcal{C} \in \mathcal{L}^q$  such that  $\mathcal{P}_\Delta \cap \mathcal{C}$  is stable, and  $\mathcal{P}_\Delta \cap \mathcal{C}$  is a regular interconnection for all  $\mathcal{P}_\Delta \in B(\mathcal{P}, \gamma)$  if and only if  $(\mathcal{P}_{\text{aux}})_{(w,v)}^\perp$  is strictly  $-\Sigma_\gamma^{-1}$  dissipative and has a negative definite storage function.

*Proof:* Clearly, since  $Q$  has full row rank,  $v$  is free in  $\mathcal{P}_{\text{aux}}$ . Also, from  $c = 0$  it follows that  $w = 0$  and  $\lim_{t \rightarrow \infty} v(t) = 0$  (use that  $P$  is Hurwitz), so  $(w, v)$  is detectable from  $c$  in  $\mathcal{P}_{\text{aux}}$ . Hence we can apply Theorem 6.2 to  $\mathcal{P}_{\text{aux}}$ . The remaining proof is straightforward from Lemma 7.3 and Theorem 6.2.  $\square$

### B. Solution to Problem 2

Consider the system  $\mathcal{P}^\perp$ , which has a rational image representation  $\tilde{w} = R^\top(-\frac{d}{dt})\ell$  and a polynomial image representation  $\tilde{w} = Q^\top(-\frac{d}{dt})\ell$ . Together with  $\mathcal{P}^\perp$  we consider the supply rate  $\|\tilde{w}\|^2$ . Clearly, by the form of this supply rate,  $\mathcal{P}^\perp$  is strictly dissipative. We denote by  $Q_{\Psi_-}(\ell)$  and  $Q_{\Psi_+}(\ell)$  its smallest and largest storage function, respectively. Clearly,  $Q_{\Psi_-} \leq 0$  and  $Q_{\Psi_+} \geq 0$ . We compute the underlying two-variable polynomials  $\Psi_-$  and  $\Psi_+$  as follows. Since the rational matrix  $R$  is co-inner,  $R(\xi)R^\top(-\xi) = I$ , we have

$$Q(\xi)Q^\top(-\xi) = P(\xi)P^\top(-\xi). \quad (14)$$

Note that  $P^\top(-\xi)$  is anti-Hurwitz. Thus (14) displays an anti-Hurwitz polynomial spectral factorization of  $Q(\xi)Q^\top(-\xi)$ . Consequently, by Proposition 5.1,

$$\Psi_+(\zeta, \eta) = \frac{Q(-\zeta)Q^\top(-\eta) - P(-\zeta)P^\top(-\eta)}{\zeta + \eta} \quad (15)$$

yields the largest storage function of  $\mathcal{P}^\perp$  with respect to the supply rate  $\|\tilde{w}\|^2$ . Next, we compute  $\Psi_-(\zeta, \eta)$ . Let

$$Q(\xi)Q^\top(-\xi) = H^\top(-\xi)H(\xi). \quad (16)$$

be a Hurwitz polynomial spectral factorization. Then we have

$$\Psi_-(\zeta, \eta) = \frac{Q(-\zeta)Q^\top(-\eta) - H(\zeta)H^\top(\eta)}{\zeta + \eta} \quad (17)$$

for the smallest storage function of  $\mathcal{P}^\perp$  with respect to the supply rate  $\|\tilde{w}\|^2$ .

Now consider  $(\mathcal{P}_{\text{aux}})_{(w,v)}^\perp$ , which has a polynomial image representation

$$\begin{pmatrix} \tilde{w} \\ \tilde{v} \end{pmatrix} = \begin{pmatrix} Q^\top(-\frac{d}{dt}) \\ P^\top(-\frac{d}{dt}) \end{pmatrix} \ell,$$

together with the supply rate  $\|\tilde{w}\|^2 - \gamma^2\|\tilde{v}\|^2$ , where  $\gamma > 0$ . Recall that this supply rate is associated with the matrix  $\Sigma_\gamma^{-1}$  given by (13). We now investigate strict  $-\Sigma_\gamma^{-1}$ -dissipativity of  $(\mathcal{P}_{\text{aux}})_{(w,v)}^\perp$ . It turns out that the smallest storage function

of  $(\mathcal{P}_{\text{aux}})_{(w,v)}^\perp$  as a  $-\Sigma_\gamma^{-1}$ -dissipative system can be expressed in terms of the smallest and largest storage function of  $\mathcal{P}^\perp$  with respect to the supply rate  $\|\tilde{w}\|^2$ .

**Theorem 7.5:** Let  $\gamma > 0$ . Then  $(\mathcal{P}_{\text{aux}})_{(w,v)}^\perp$  is strictly  $-\Sigma_\gamma^{-1}$ -dissipative if and only if  $0 < \gamma < 1$ . The smallest storage function of  $(\mathcal{P}_{\text{aux}})_{(w,v)}^\perp$  as a  $-\Sigma_\gamma^{-1}$ -dissipative system is induced by the two-variable polynomial matrix

$$\Psi_-^\gamma = (1 - \gamma^2)\Psi_- + \gamma^2\Psi_+, \quad (18)$$

where  $\Psi_-$  and  $\Psi_+$  are given by (17) and (15), respectively.

*Proof:* The proof is omitted.  $\square$

Now we obtain an explicit formula for the smallest upper bound  $\gamma^*$  of the radii  $\gamma > 0$  of the ball  $B(\mathcal{P}, \gamma)$  that can be stabilized using a single controller  $\mathcal{C} \in \mathcal{L}^q$ .

In the following, let  $\Psi_-(\zeta, \eta)$  and  $\Psi_+(\zeta, \eta)$  be given by (17) and (15), respectively.

According to Lemma 7.3, there exists  $\mathcal{C} \in \mathcal{L}^q$  such that  $\mathcal{P}_\Delta \cap \mathcal{C}$  is stable for all  $P_\Delta \in B(\mathcal{P}, \gamma)$  if and only if  $(\mathcal{P}_{\text{aux}})_{(w,v)}^\perp$  is strictly  $-\Sigma_\gamma^{-1}$ -dissipative and its smallest storage function  $\Psi_-^\gamma$  is negative definite. Let  $X(\xi)$  be a polynomial matrix such that  $X(\frac{d}{dt})$  is a minimal state map for  $\mathcal{P}^\perp$  and  $\tilde{X}\tilde{X}^\top = I$ . It is easily seen that  $X(\frac{d}{dt})$  is also a minimal state map for  $(\mathcal{P}_{\text{aux}})_{(w,v)}^\perp$ . Let  $n$  be the number of rows of  $X$ . Now, there exist real symmetric  $n \times n$  matrices  $K_-$  and  $K_+$  such that  $\Psi_-(\zeta, \eta) = X^\top(\zeta)K_-X(\eta)$  and  $\Psi_+(\zeta, \eta) = X^\top(\zeta)K_+X(\eta)$ . Since, by inspection,  $\mathcal{P}^\perp$  is strictly dissipative both on  $\mathbb{R}_-$  and on  $\mathbb{R}_+$  with respect to the supply rate  $\|\tilde{w}\|^2$ , it follows from ([4], Lemma 6) that  $K_- < 0$  and  $K_+ > 0$ . As a consequence, the smallest storage function  $\Psi_-^\gamma(\zeta, \eta)$  of  $(\mathcal{P}_{\text{aux}})_{(w,v)}^\perp$  is equal to

$$\Psi_-^\gamma(\zeta, \eta) = X^\top(\zeta) \left( (1 - \gamma^2)K_- + \gamma^2K_+ \right) X(\eta).$$

Thus,  $\Psi_-^\gamma$  yields a negative definite storage function for  $(\mathcal{P}_{\text{aux}})_{(w,v)}^\perp$  if and only if  $(1 - \gamma^2)K_- + \gamma^2K_+ < 0$ .

Choose  $\tilde{Y}$  such that  $\begin{pmatrix} \tilde{X} \\ \tilde{Y} \end{pmatrix}$  is orthogonal (such a  $\tilde{Y}$  exists since  $\tilde{X}\tilde{X}^\top = I$ ). Let  $\tilde{\Psi}_-$  and  $\tilde{\Psi}_+$  represent coefficient matrices of  $\Psi_-$  and  $\Psi_+$  respectively. Let  $\tilde{\Psi}_+^\perp$  be the Moore-Penrose inverse of a given matrix  $\tilde{\Psi}_+$ . Then it can be shown that  $\tilde{\Psi}_-\tilde{\Psi}_+^\perp = \begin{pmatrix} \tilde{X} \\ \tilde{Y} \end{pmatrix}^\top \begin{pmatrix} K_-K_+^{-1} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \tilde{X} \\ \tilde{Y} \end{pmatrix}$ . Then we have

$$\lambda_{\max}(\tilde{\Psi}_-\tilde{\Psi}_+^\perp) = \lambda_{\max}(K_-K_+^{-1}). \quad (19)$$

The following theorem gives the optimum  $\gamma^*$  in terms of  $\tilde{\Psi}_-$  and  $\tilde{\Psi}_+$ .

**Theorem 7.6:** Let  $\gamma^*$  be given by (5). Then

$$\gamma^* = \sqrt{\frac{\lambda_{\max}(\tilde{\Psi}_-\tilde{\Psi}_+^\perp)}{\lambda_{\max}(\tilde{\Psi}_-\tilde{\Psi}_+^\perp) - 1}}. \quad (20)$$

In particular, for  $\gamma > 0$  the following holds: there exists  $\mathcal{C} \in \mathcal{L}^q$  such that  $\mathcal{P}_\Delta \cap \mathcal{C}$  is stable, and  $\mathcal{P}_\Delta \cap \mathcal{C}$  is a regular interconnection for all  $P_\Delta \in B(\mathcal{P}, \gamma)$  if and only if  $\gamma < \gamma^*$ .

*Proof:* The following equivalences hold:  $(1 - \gamma^2)K_- + \gamma^2K_+ < 0 \iff (1 - \gamma^2)\lambda_{\max}(K_-K_+^{-1}) < -\gamma^2 \iff$

$\lambda_{\max}(K_-K_+^{-1}) < -[1 - \lambda_{\max}(K_-K_+^{-1})]\gamma^2 \iff \gamma^2 < \frac{\lambda_{\max}(K_-K_+^{-1})}{\lambda_{\max}(K_-K_+^{-1}) - 1}$ . In the above, note that since  $K_- < K_+$ , we have  $\lambda_{\max}(K_-K_+^{-1}) = \lambda_{\max}(\tilde{\Psi}_-\tilde{\Psi}_+^\perp) < 1$ . Thus,  $\Psi_-^\gamma$  yields a negative definite storage function for  $(\mathcal{P}_{\text{aux}})_{(w,v)}^\perp$  if and only if  $\gamma < \sqrt{\frac{\lambda_{\max}(\tilde{\Psi}_-\tilde{\Psi}_+^\perp)}{\lambda_{\max}(\tilde{\Psi}_-\tilde{\Psi}_+^\perp) - 1}}$ .  $\square$

## VIII. CONCLUSION

A formulation of the robust stabilization problem has been presented in the behavioral framework. Necessary and sufficient conditions for the existence of robustly stabilizing controllers have been found in terms of the dual of the plant behavior and its associated smallest storage function. The smallest upper bound for the uncertainty radii for which there still exists a robustly stabilizing controller has been found, and has been expressed in terms of the two-variable polynomial matrices obtained after polynomial spectral factorizations.

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