Can strategizing in round-robin subtournaments be avoided?

Pauly, Marc

Published in:
Social Choice and Welfare

DOI:
10.1007/s00355-013-0767-6

Publication date:
2014

Soc Choice Welf (2014) 43:29–46
DOI 10.1007/s00355-013-0767-6

ORIGINAL PAPER

Can strategizing in round-robin subtournaments be avoided?

Marc Pauly

Received: 20 December 2012 / Accepted: 5 September 2013 / Published online: 4 October 2013

Abstract This paper develops a mathematical model of strategic manipulation in complex sports competition formats such as the soccer world cup or the Olympic games. Strategic manipulation refers here to the possibility that a team may lose a match on purpose in order to increase its prospects of winning the competition. In particular, the paper looks at round-robin tournaments where both first- and second-ranked players proceed to the next round. This standard format used in many sports gives rise to the possibility of strategic manipulation, as exhibited recently in the 2012 Olympic games. An impossibility theorem is proved which demonstrates that under a number of reasonable side-constraints, strategy-proofness is impossible to obtain.

1 Introduction

In the 2012 London Olympics, eight female badminton players were expelled from the doubles competition. The twin charge was "not using one's best efforts to win a match" and "conducting oneself in a manner that is clearly abusive or detrimental to the sport" (Dillman 2012). The reason these players wanted to lose their match was that this would have resulted in them being ranked second rather than first in their round-robin competition group. The format of this competition required the second-ranked team to play against the first-ranked team of another group, and the disqualified teams preferred to play against this first-ranked team rather than the second-ranked team

Electronic supplementary material The online version of this article (doi:10.1007/s00355-013-0767-6) contains supplementary material, which is available to authorized users.

M. Pauly (✉)
Faculty of Philosophy, University of Groningen,
Oude Boteringestraat 52, 9712 GL Groningen, The Netherlands
e-mail: m.pauly@rug.nl

which they would have had to play had they won their match. Sports commentators of this "badminton scandal" faulted the expelled players for their lack of sportsmanship, few if any comments considered the competition format to be at fault. In academic circles, these events caused some thinking in particular among economists on questions of mechanism design (Ely 2012; Kleinberg 2012; Hartline and Kleinberg 2012). For the problem illustrated by these events at the 2012 Olympics is a pervasive one: the competition format used there which gave rise to strategic play is also used in many other prominent competitions, perhaps most notably the FIFA soccer world cup. So leaving aside the question of whether the expelled badminton players were really morally at fault for their behavior, this paper takes the view that it would be desirable to design competition formats which do not even create room for this kind of behavior, competition formats where "not using one's best efforts to win a match" will never be to one's advantage. Such competition formats would align what is moral (or at least considered to be moral within a particular sports community) with what is expedient.

How to solve this mechanism design problem of Olympic badminton and other similar competition formats? A solution inspired by Parikh's example of the two horsemen (2002) who want to find out whose horse is the slowest (solution: exchange horses and try to ride as fast as possible): we might consider a rule which awards the losing team with the points usually given to the winner. Better yet, let the winning team decide who gets the points associated with a win. The problem is that this solution is asymmetric, in that it confers an advantage on one round-robin group over the other (where it is already clear who is first-ranked and second-ranked). A second solution is to let matches take place simultaneously, so that no team knows yet who will be ranked first or second. This solution was also adopted in the 2012 European soccer cup competition. A problem with this second solution is that the earlier problem may repeat itself on a smaller scale: in one match, a team may have a big lead so that it is clear that it will end up being first-ranked. The teams in the other match may learn of this situation and change their play accordingly. Other problems with this solution are that viewers are forced to choose which match to watch (they cannot watch both), and that advertising revenues are reduced (probably the most relevant argument in today's sports world). A third solution not using simultaneous matches would be to use a random device such as a coin toss to choose with equal probability among the two first-ranked players of their respective groups which of the two gets to choose the opponent from the other group. So if the coin toss selects the first-ranked player of group A, she gets to decide whether to play the first-ranked or the second-ranked player of group B, and the remaining two players (second-ranked player of group A and either the first-ranked or second-ranked player of group B) are also paired in a match. A problem with this third solution is that it makes use of a random device which complicates the required mathematical model and also introduces a problem of practical manipulation (Is the device really fair?). Furthermore, it introduces an element of choice into competition formats, further complicating the required model, and practically yielding the problem of who (in case of a team) should make that choice and bear the responsibilities associated with that choice. Given these problems, we shall not adopt this solution, or any of the other solutions just mentioned.

The model developed in this paper simply models competitions as functions which take as inputs the players competing and information about who beats who. The function returns the winner of the competition. Requirements like anonymity, independence of irrelevant alternatives, etc. can readily be formulated in this model. Theorem 1 shows, however, that a function satisfying these constraints will not be strategy-proof. This is the main result of this paper. A noteworthy feature of the proof of Theorem 1 is that it involves the use of a computer program (see Appendix as online supplementary material). At a certain point in the proof, the aim is to find which functions satisfy a certain number of properties. Given a particular function, these properties are easy to verify, but given that there are 4^8 functions to consider, a computer program is used to carry out the verification automatically. By now, computer-assisted proofs have become more common in mathematics, and Sect. 2 provides references to results and literature in this area.

This paper starts by relating the model used and the results obtained to previous models of tournaments in Sect. 2. In Sect. 3, the formal model for complex competition formats is presented. The notion of monotonicity will be the formal analogue of our informal notion of non-manipulability. Section 4 will introduce knockout and round-robin competition formats, since these are the basic ingredients which make up more complex competition formats like the ones used in Olympic games or the soccer world cup. Section 5 then presents examples of more complex formats, both examples which allow for strategic manipulation and examples which do not. Section 6, finally, will present the main result of this paper, Theorem 1, which imposes definite limits on any attempt to obtain strategy-proofness while using second-ranked round-robin players.

2 Related work

The work in this paper makes use of tournaments which have been well-studied in social choice theory. Laslier (1997) provides a comprehensive overview of the use of tournaments in social choice theory. One of the earliest papers to consider tournament solutions which select one or more elements from a tournament was by Moulin (1986). His tournament model as well as the tournament model employed in this paper does not allow for ties. Tournament solutions for tournaments which do allow for ties have been considered by Peris and Subiza (1999). Rubinstein (1980) provides an axiomatization of a particular tournament solution, the Copeland solution.

Besides social choice theory, also artificial intelligence and computer science have contributed to our knowledge about tournaments. The computational complexity of computing tournament solutions has been investigated by various authors, Hudry (2009) provides a survey. The computational complexity of manipulating round-robin and knockout tournaments is considered by Russell and Walsh (2009). Closest to the results presented in this paper is the work of Altman et al. (2009). They also consider the issue of strategic manipulation in tournaments, but consider a requirement they call pairwise non-manipulability: a tournament solution is pairwise manipulable if two alternatives can make one of them a winner by reversing the result of their match. When coupled with Condorcet consistency, the requirement of pairwise non-manipulability yields an impossibility result, but when Condorcet consistency is weakened to non-imposition, they prove a possibility result which yields tournament solutions that also satisfy monotonicity, i.e. non-manipulability as considered in this paper. Altman and Klienberg (2010) extend this framework to randomized tournament solutions which produce lotteries over alternatives as outcomes.

The work in this paper differs from these results in a number of ways. First, the tournaments we employ do not allow for ties. Second, this paper uses tournaments to construct more complex competition formats which are built up from various simple tournaments. The aim is to be able to model and analyze real competition formats as used in practical sports competitions such as the Olympic games. Examples will be given in Sect. 5 and will typically consist of a number of different rounds and combinations of knockout and round-robin phases. When formally modeled, such complex competition formats will actually involve not one but multiple tournaments, one per round, which may interact in various ways. This requires a more complex mathematical model, which is what is developed here. Second, this paper focuses on a specific problem of strategic manipulation that is a real problem in actual sports competition, the problem of round-robin competitions where players deliberately lose a game to become the second-ranked player. To the best of my knowledge, this problem has not been addressed so far in the academic literature.

Finally, as mentioned above, the proof of Theorem 1 makes use of a computer program. Probably the most famous proof where a computer was used is the proof of the four-color theorem, proved in 1977 (Appel and Haken 1977; Appel et al. 1977). For the proof of the four-color theorem, a computer was used to check whether each one of 1,936 maps had a particular property. Similarly, in this paper, a computer program is used to check which functions of the 65,536 possible ones satisfy a combination of four properties. For a discussion of the history of mechanized proof and the philosophical issues involved, the reader is referred to MacKenzie (2001). More recently, Gonthier (2008) has provided a proof of the four-color theorem formulated in the Coq formal system.

3 Tournaments and competition formats

In this section we will formally model competition formats by functions which take as inputs the competing players (or teams) on the one hand and a tournament relation expressing who beats who in a direct encounter on the other hand. As output, the function returns again a player, e.g. the player who wins the competition, or the player who wins the silver medal. Formally we work with a finite nonempty set of players or teams X.

The notion of a tournament has been investigated in graph theory and social choice theory (see e.g. Laslier 1997). A tournament is a complete asymmetric binary relation over some set. Formally, a *tournament* over X is any set $T \subseteq X \times X$ such that (1) $\forall x, y \in X$, $(x, y) \in T$ implies $(y, x) \notin T$, and (2) $\forall x, y \in X$, if $x \neq y$ then $(x, y) \in T$ or $(y, x) \in T$. We shall sometimes write xTy for $(x, y) \in T$ and refer to this as "x beats y." Alternatively, we will write $T(x, y)$ for the winner of the match between x and y, i.e., $T(x, y) = x$ if xTy and $T(x, y) = y$ otherwise. Note that this implies that while $(x, x) \notin T$ for any T, we do have $T(x, x) = x$. Let T_X refer to the set of all tournaments over X.

Let $\pi : X \rightarrow X$ be a permutation of X, i.e. a bijection onto X. We say that tournaments T and T' are *isomorphic under* π iff for all $x, y \in X$, $(x, y) \in T$ iff $(\pi(x), \pi(y)) \in T'$. T and T' are *isomorphic* iff they are isomorphic under some permutation.

We now introduce the notions of monotonicity and modulo-identity for tournaments. The notion of monotonicity is adopted from the tournament literature (see Laslier 1997, def. 2.3.1.). Given two tournaments T and T' over X and an element $a \in X$, we say that T' *monotonically improves* T for a, denoted as $T' \geq_a T$, iff $\forall x, y \in X$, (1) if $x \neq a$ and $y \neq a$ then $(x, y) \in T$ iff $(x, y) \in T'$, and (2) $(a, x) \in T$ implies $(a, x) \in T'$. Given two tournaments T and T' over set X and some $Y \subseteq X$, we say that T and T' are *identical for* Y, denoted $T =_Y T'$, iff they agree on everything in Y, i.e. $\forall x, y \in Y$ we have $(x, y) \in T$ iff $(x, y) \in T'$. We shall also write $T =_{-a} T'$ for $T =_{X-\{a\}} T'$ and say in this case that T and T' are *identical modulo* a. Note the following relation between the two notions just introduced: if T' monotonically improves T for a then T and T' are identical modulo a. In short, $T' \geq_a T$ implies $T' =_{-a} T$.

Given a set of players or teams X and tournaments T_X, a *competition format* is a function $F : X^n \times T_X \rightarrow X$. The following properties of these functions will be needed later: we say that F is *independent of irrelevant alternatives* (IIR, for short) iff for all $x_1, \ldots, x_n \in X$ and all tournaments T and T' such that $T =_{\{x_1,\ldots,x_n\}} T'$ we have $F(x_1, \ldots, x_n, T) = F(x_1, \ldots, x_n, T')$. F is *input-selecting* iff for all $x_1, \ldots, x_n \in X$ and $T \in T_X$ we have $F(x_1, \ldots, x_n, T) \in \{x_1, \ldots, x_n\}$. We call F *monotonic* iff for all $a, x_1, \ldots, x_n \in X$ and $T, T' \in T_X$, whenever $F(x_1, \ldots, x_n, T) = a$ and $T' \geq_a T$ then $F(x_1, \ldots, x_n, T') = a$. F is called *anonymous*, if whenever T and T' are isomorphic under π, we have $\pi(F(x_1, \ldots, x_n, T)) = F(\pi(x_1), \ldots, \pi(x_n), T')$. A competition format F of arity $2n$ is *symmetric* iff for all $x_1, \ldots, x_{2n} \in X$ and $T \in T_X$ we have $F(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{2n}, T) = F(x_{n+1}, \ldots, x_{2n}, x_1, \ldots, x_n, T)$. Finally, we call F *non-imposed* iff for all $x_1, \ldots, x_n \in X$ and for all $i \leq n$ there is some $T \in T_X$ such that $F(x_1, \ldots, x_n, T) = x_i$.

On some occasions we will also make use of competition formats $F : X^n \times (T_X)^m \rightarrow X$ that take multiple tournament arguments. The notion of monotonicity is easily extended to this case: F is *monotonic* iff for all $a, x_1, \ldots, x_n \in X$ and $T_1, \ldots, T_m, T'_1, \ldots, T'_m \in T_X$, whenever $F(x_1, \ldots, x_n, T_1, \ldots, T_m) = a$ and for all i, $T'_i \geq_a T_i$, then $F(x_1, \ldots, x_n, T'_1, \ldots, T'_m) = a$.

Lemma 1 *If* $|X| \geq n+2$ *then any n-ary competition format* (i.e., *taking n players and a tournament as inputs*) *which satisfies IIR and anonymity is input-selecting.*

Proof Suppose F takes n players as arguments and satisfies IIR and anonymity and suppose that $|X| \geq n+2$. Suppose by reductio that F is not input-selecting, so we have some tournament T and $x_1, \ldots, x_n, y \in X$ for which $F(x_1, \ldots, x_n, T) = y$ and $y \neq x_i$ for all i. Now given the size of X we know there is a $y' \in X$ different from $x_1, \ldots, x_n, y$. Define function $\pi : X \rightarrow X$ such that $\pi(y) = y'$, $\pi(y') = y$ and for all other $x \in X$ we let $\pi(x) = x$. Let T' be the tournament isomorphic to T under π. By anonymity, we must have $F(x_1, \ldots, x_n, T') = F(\pi(x_1), \ldots, \pi(x_n), T') = \pi(F(x_1, \ldots, x_n, T)) = y'$. But by IIR, we must also have $F(x_1, \ldots, x_n, T') = F(x_1, \ldots, x_n, T) = y$, a contradiction. □

Note that the lower bound on the set of players is a true lower bound: for $|X| = n+1$, there are n-ary competition formats satisfying IIR and anonymity without being input-selecting. To see this, consider the binary function G over $X = \{a, b, c\}$ defined as follows: $G(x_1, x_2, T) = x_1$ if $x_1 = x_2$, and otherwise we define $G(x_1, x_2, T)$ to be the element of X not among the inputs. G is not input-selecting while satisfying IIR and anonymity.

4 Basic competition formats

Arguably, the two most basic competition formats used in sporting events are round-robin (all-play-all) and knockout (single-elimination). Let the function $R_n^1 : X^n \times T_X \to X$ return the winner of a round-robin competition among the arguments according to the given tournament. Formally, for $Y \subseteq X$, $a \in X$ and tournament $T \in T_X$, the (*Copeland*) *score* of a in T restricted to Y is defined as $score(a, Y, T) = |\{b \in Y | aTb\}|$. Then we define the *round-robin competition format* by $R_n^1(x_1, \ldots, x_n, T) = x_i$ iff i is the smallest index for which $score(x_i, \{x_1, \ldots, x_n\}, T) \geq score(x_j, \{x_1, \ldots, x_n\}, T)$ for all $j \leq n$. Note that such a smallest index always exists, so R_n^1 is well-defined. The need to refer to the smallest index arises because of the possibility of ties, i.e. there may be multiple players beating the same number of opponents. It is easily verified that R_n^1 is monotonic, independent of irrelevant alternatives and input-selecting. On the other hand R_n^1 is not symmetric due to the tie-breaking mechanism. Note also that $R_2^1(x_1, x_2, T) = T(x_1, x_2)$. When the arity of the function is clear from the context or arbitrary, we usually omit it from the function, writing R^1 instead of R_n^1.

We will later also use the second-ranked player of a round-robin competition. For this purpose, let $R_n^2(x_1, \ldots, x_n, T) = x_i$ iff there is some k such that $R_n^1(x_1, \ldots, x_n, T) = x_k$ with $k \neq i$ and i is the smallest index for which $score(x_i, \{x_1, \ldots, x_n\}, T) \geq score(x_j, \{x_1, \ldots, x_n\}, T)$ for all $j \neq k$. Note that for $n \geq 2$, such an i always exists, e.g., $R_2^2(a, a, T) = R_2^1(a, a, T) = a$. Note also that R^2 fails to be symmetric and also fails to be monotonic: let $T = \{(a, b)\}$, then $R_2^2(a, b, T) = b$ but $R_2^2(a, b, T') = a$ if $T' = \{(b, a)\}$. Intuitively, by winning more matches, a silver medal winner may lose her silver medal (and instead win a gold medal).

While round-robin competitions can involve an arbitrary number of players, knockout competitions are restricted to $n = 2^k$ players for $k \geq 1$. Formally, we define the *knockout competition format* $K_n^1 : X^n \times T_X \to X$ recursively by repeated application of round-robins as follows:

$$\begin{aligned}
K_2^1(x_1, x_2, T) &= R_2^1(x_1, x_2, T) = T(x_1, x_2),\\
K_{2^{j+1}}^1\left(x_1, \ldots, x_{2^{j+1}}, T\right) &= R_2^1\left(K_{2^j}^1\left(x_1, \ldots, x_{2^j}, T\right), K_{2^j}^1\left(x_{2^j+1}, \ldots, x_{2^{j+1}}, T\right), T\right),\\
&= T\left(K_{2^j}^1\left(x_1, \ldots, x_{2^j}, T\right), K_{2^j}^1\left(x_{2^j+1}, \ldots, x_{2^{j+1}}, T\right)\right),
\end{aligned}$$

for $j \geq 1$. Thus, in the case of K_8^1, the arguments $x_1, \ldots, x_8$ are distributed over the tree underlying the knockout competition as depicted in Fig. 1.

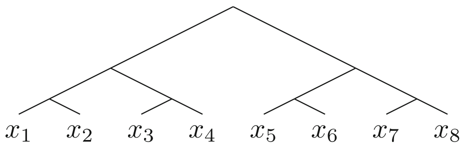


Fig. 1 The knockout competition format corresponding to K_8^1

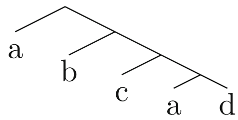


Fig. 2 A knockout competition K_{16}^1 in abbreviated form

Note that K_n^1 is independent of irrelevant alternatives and input-selecting for all n. However, knockout competitions are not necessarily monotonic. Consider $X = \{a, b, c, d\}$, tournament $T = \{(a, b), (b, d), (d, c), (d, a), (c, a), (c, b)\}$ and the knockout competition $K_{16}^1(a, a, a, a, a, a, a, a, b, b, b, b, c, c, a, d)$ depicted in an abbreviated form in Fig. 2.

According to T, a will be the winner of the competition and will be returned by the function. Now consider T' which is just like T, except that (d, a) is replaced by (a, d). Thus, $T' \geq_a T$. But in spite of a winning more matches in T', it now fails to win the competition as a whole, since the function now returns c. The reason is that in T, a avoids meeting its nemesis c, but in T', due to winning against d, a faces c and loses not only the match against c but also the competition as a whole.

The properties of round-robin and knockout competitions are summarized in Lemma 2.

5 Strategizing in complex competition formats

This section will illustrate the problem of strategic manipulation in some complex competition formats. As a first example, consider the competition format

$$F(x_1, \ldots, x_8, T_1, T_2) = K^1\Big(R^1(x_1, x_2, x_3, x_4, T_1), R^2(x_5, x_6, x_7, x_8, T_1), R^1(x_5, x_6, x_7, x_8, T_1), R^2(x_1, x_2, x_3, x_4, T_1), T_2\Big),$$

under tournaments

$$\begin{aligned} T_1 &\supseteq \{(x_1, x_3), (x_1, x_4), (x_2, x_1), (x_2, x_3), (x_3, x_4), (x_4, x_2)\} \\ &\quad \cup \{(x_5, x_6), (x_5, x_7), (x_5, x_8), (x_6, x_7), (x_6, x_8), (x_7, x_8)\}, \\ T_2 &\supseteq \{(x_1, x_6), (x_2, x_5), (x_2, x_1), (x_6, x_2)\}. \end{aligned}$$

Given these tournaments, the two semi-finals pair x_1 against x_6 and x_5 against x_2, resulting in $F(x_1, \ldots, x_8, T_1, T_2) = x_2$. But modifying T_1 by simply reversing the relationship between x_2 and x_4 yielding T_1', we have $T_1' \geq_{x_2} T_1$ and new semifinals

x_2 against x_6 and x_5 against x_1, yielding $F(x_1, \ldots, x_8, T_1', T_2) \neq x_2$. Hence, this example demonstrates that monotonicity can fail once second-ranked players of round-robin tournaments are allowed to continue in a competition. The source of the problem illustrated by the example is that by moving from second rank to first rank, x_2 will face in the semifinal a different opponent from the other group to which it may lose in the knockout competition (even though that other opponent was ranked lower in the other group). It was precisely the fear of this scenario which caused strategic behavior in the badminton matches of the 2012 Olympics.

On first sight, it may seem that this problem of strategic manipulation can be solved by replacing the second-stage knockout competition by a round-robin competition. This, however, is not so. Consider function

$$F'(x_1, \ldots, x_8, T_1, T_2) = R^1\Big(R^1(x_1, x_2, x_3, x_4, T_1), R^2(x_5, x_6, x_7, x_8, T_1),$$
$$R^1(x_5, x_6, x_7, x_8, T_1), R^2(x_1, x_2, x_3, x_4, T_1), T_2\Big),$$

under tournaments

$$\begin{aligned} T_1 \supseteq\ & \{(x_1, x_3), (x_2, x_1), (x_3, x_2), (x_3, x_4), (x_4, x_1), (x_4, x_2)\} \\ & \cup \{(x_5, x_6), (x_5, x_7), (x_5, x_8), (x_6, x_7), (x_6, x_8), (x_7, x_8)\}, \\ T_2 \supseteq\ & \{(x_1, x_2), (x_1, x_3), (x_1, x_4), (x_1, x_5), (x_1, x_6)\} \\ & \cup \{(x_4, x_2), (x_4, x_3), (x_4, x_5), (x_4, x_6)\}. \end{aligned}$$

Given these tournaments, we have $F'(x_1, \ldots, x_8, T_1, T_2) = x_4$. But modifying T_1 by reversing the relationship between x_3 and x_4 yielding T_1', we have $T_1' \geq_{x_4} T_1$ but also $F'(x_1, \ldots, x_8, T_1', T_2) = x_1$, so monotonicity fails. With second-round round-robin, the source of the manipulability is different from what we encountered with second-round knockout. Now, the problem arises because when x_4 moves from second rank to first rank, the previously first-ranked alternative x_3 moves out of the first two ranks and is hence not present anymore for the second round-robin round. Instead, a new alternative x_1 moves into second place which was not present before in the second round-robin and which defeats x_4 to become the winner.

Both of the examples just presented pertain to situations where a second-ranked player may move into first rank when winning honestly but chooses to remain in second place. There is, however, also a possibility of manipulation for a first-ranked player who by losing remains in first place but can change who becomes the second-ranked player. Consider again the complex knockout function F considered earlier in this section, and the following situation:

$$\begin{aligned} T_1 \supseteq\ & \{(x_2, x_1), (x_1, x_3), (x_1, x_4), (x_2, x_3), (x_4, x_2), (x_4, x_3)\} \\ & \cup \{(x_5, x_6), (x_5, x_7), (x_5, x_8), (x_6, x_7), (x_6, x_8), (x_7, x_8)\}, \\ T_2 \supseteq\ & \{(x_1, x_6), (x_1, x_5), (x_5, x_2), (x_4, x_5), (x_4, x_1)\}. \end{aligned}$$

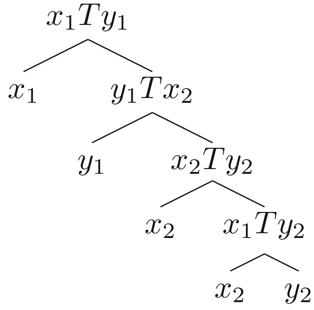
Given these tournaments, we have $F(x_1, \ldots, x_8,\ T_1,\ T_2) = x_1$. But modifying T_1 by reversing the relationship between x_1 and x_2 yielding T_1', we have $T_1' \geq_{x_1} T_1$ but also $F'(x_1, \ldots, x_8,\ T_1',\ T_2) = x_4$, so monotonicity fails. So in this case, if x_1 strategically loses against x_2, this will keep x_1 first-ranked in the round-robin group, but it changes who continues on to the next round as a second-ranked player. By losing, x_1 can make sure that an easier opponent continues on to the next round, x_2 rather than x_4.

Since the commonly used complex competition formats are thus vulnerable to different kinds of strategic manipulation, the question arises which complex competition formats can avoid manipulation and preserve monotonicity. Consider the new *mix* function $M :\ X^4 \times T_X \rightarrow X$ defined in Fig. 3. Intuitively, this function can be visualized by the decision tree depicted in Fig. 4. There are various note-worthy features of this function: the function is asymmetric, input-selecting, monotonic and satisfies IIR.

x_1Ty_1	x_1Ty_2	x_2Ty_1	x_2Ty_2	$M(x_1, x_2, y_1, y_2, T)$
1	1	1	1	x_1
1	1	1	0	x_1
1	1	0	1	x_1
1	1	0	0	x_1
1	0	1	1	x_1
1	0	1	0	x_1
1	0	0	1	x_1
1	0	0	0	x_1
0	1	1	1	x_2
0	1	1	0	x_2
0	1	0	1	y_1
0	1	0	0	y_1
0	0	1	1	x_2
0	0	1	0	y_2
0	0	0	1	y_1
0	0	0	0	y_1

Fig. 3 The mix function *M* (where 1 indicates that the given tournament relation holds, 0 that it does not hold)

Fig. 4 The mix function *M* visualized as a decision tree



The mix function can now be used to obtain complex competition formats that make use of second-ranked players of round-robin tournaments. We define the *cross-play* function C^i_j as follows:

$$C^i_j\left(x_1,\ldots,x_i,\, x_{i+1},\ldots,x_{i+j},\, T\right) = M\Big(R^1\left(x_1,\ldots,x_i,\, T\right),\, R^2\left(x_1,\ldots,x_i,\, T\right),\, R^1\left(x_{i+1},\ldots,x_{i+j},\, T\right),\, R^2\left(x_{i+1},\ldots,x_{i+j},\, T\right),\, T\Big),$$

where $i,\ j \geq 2$. Note that this function can do with just one tournament argument, since in the second round of the mix function, players will not play anyone they played in the first round-robin round. As will be shown in Lemma 2, this function does satisfy monotonicity, so it is an example of a competition format that uses second-ranked round-robin players while preserving strategy-proofness. There are, however, two problems with this function: first, as the decision tree in Fig. 4 shows, if matches take place sequentially and if it comes to a game between x_1 and y_2, x_1 has no incentive to win the game, since its own success in no way depends on it. This does not contradict strategy-proofness, as x_1 has no incentive to lose the game, either, but it is certainly not very desirable for a competition format. Second, and more importantly, the competition format is asymmetric. This means that players in the first round-robin group have a big advantage over players in the second group. x_1 wins in eight cases, y_1 only in four, and x_2 wins in three cases, whereas y_2 only wins in one case. This raises the question whether we cannot find a strategy-proof use of second-ranked round-robin players which is symmetric. As Theorem 1 will show, the answer turns out to be negative. The following lemma summarizes the properties of not just the cross-play function, put also of the other functions we have encountered so far.

Lemma 2 *The following table expresses the properties of various competition formats, where* 1 *represents true and* 0 *false.*

	R^1_n	R^2_n	K^1_n	M	C^i_j
IIR	1	1	1	1	1
Input-selecting	1	1	1	1	1
Anonymous	1	1	1	1	1
Symmetric	0	0	1	0	0
Non-imposed	1	1	1	1	1
Monotonic	1	0	0	1	1

Proof Most of the properties are easily verified, and some have been proved already earlier in this paper. A few comments on less obvious aspects of these proofs:

R^1_n is not symmetric: consider $R^1_4(x_1,\, x_2,\, x_3,\, x_4,\, T)$ where x_4 is beaten by all other players and $x_1Tx_2,\ x_2Tx_3$ and x_3Tx_1. Then $R^1_4(x_1,\, x_2,\, x_3,\, x_4,\, T) = x_1$ whereas $R^1_4(x_3,\, x_4,\, x_1,\, x_2,\, T) = x_3$.

R_n^2 is not symmetric: consider $R_4^2(x_1, x_2, x_3, x_4, T)$ where x_1 beats all other players and $x_2 T x_3$, $x_3 T x_4$ and $x_4 T x_2$. Then we have $R_4^2(x_1, x_2, x_3, x_4, T) = x_2$ whereas $R_4^2(x_3, x_4, x_1, x_2, T) = x_3$.

C_j^i is monotonic: let $a_1 = R^1(x_1, \ldots, x_i, T)$, $a_2 = R^2(x_1, \ldots, x_i, T)$, $b_1 = R^1(x_{i+1}, \ldots, x_{i+j}, T)$ and $b_2 = R^2(x_{i+1}, \ldots, x_{i+j}, T)$. Suppose further that $C_j^i(x_1, \ldots, x_{i+j}, T) = c$ and that $T' \geq_c T$. There are four cases to consider:

(i) $c = a_1$: making a_1 win more matches in the round-robin round will keep him first-ranked, and M will also still return a_1 if it wins more matches.

(ii) $c = a_2$: this means that a_2 wins against b_1 in the second round. Now there are two possibilities. Either a_2 still remains second-ranked in the round-robin phase under T', or it becomes first-ranked. If it becomes first-ranked, then simply by beating b_1, it will remain the winner also under T'. If it remains second-ranked, then the first-ranked player of the round-robin round must also remain unchanged, including its tournament results. The only possible difference between T and T' is then that a_2 may now beat b_2 as well (whereas it did not do so in T), but this will keep a_2 as the winner of the competition.

(iii) $c = b_1$: this means that b_1 in T wins against both a_1 and a_2, and making b_1 win more matches (in the round-robin phase) will not change this fact.

(iv) $c = b_2$: this can only occur in one situation, where b_2 beats both a_1 and a_2. If b_2 remains second-ranked in T', also the round-robin winner will remain unchanged and the relevant tournament relations in T' will be as in T. If b_2 becomes first-ranked in T', then since it wins both its matches against the other group it will also be the winner of the competition under T'. □

6 An impossibility result

The problem with the cross-play function is that since it is not symmetric it treats the two round-robin groups differently. The following impossibility result shows, however, that we cannot hope to do much better than that.

Theorem 1 *If* $|X| \geq 6$, *there is no function* $G : X^4 \times T_X \to X$ *which is symmetric, non-imposed, anonymous and IIR and for which the function*

$$\begin{aligned} &H_j^i\left(x_1, \ldots, x_i, x_{i+1}, \ldots, x_{i+j}, T_A, T_B, T\right) \\ &= G\Big(R^1\left(x_1, \ldots, x_i, T_A\right), R^2\left(x_1, \ldots, x_i, T_A\right), \\ &\quad R^1\left(x_{i+1}, \ldots, x_{i+j}, T_B\right), R^2\left(x_{i+1}, \ldots, x_{i+j}, T_B\right), T\Big), \end{aligned}$$

is monotonic for all $i, j \geq 2$.

Proof For a proof by contradiction, suppose there is such a function G. Note that by Lemma 1, we know that G is input-selecting. The following four claims hold for all pairwise distinct $x_1, x_2, y_1, y_2, z \in X$ and $T \in T_X$:

Claim 1a: if $G(x_1, x_2, y_1, y_2, T) = x_1$ then $G(x_1, z, y_1, y_2, T) = x_1$.
Claim 1b: if $G(x_1, x_2, y_1, y_2, T) = y_1$ then $G(x_1, x_2, y_1, z, T) = y_1$.
Claim 2a: if $G(x_1, x_2, y_1, y_2, T) = x_2$ then $G(x_2, z, y_1, y_2, T) = x_2$.
Claim 2b: if $G(x_1, x_2, y_1, y_2, T) = y_2$ then $G(x_1, x_2, y_2, z, T) = y_2$.

We shall only prove claims 1a and 2a, the other two claims are proved analogously. As for claim 1a, suppose $G(x_1, x_2, y_1, y_2, T) = x_1$ and consider tournaments $T_B = \{(y_1, y_2)\}$ and $T_A = \{(x_1, z), (z, x_2), (x_2, x_1)\}$. Then we have

$$G\Big(R^1(x_1, x_2, z, T_A), R^2(x_1, x_2, z, T_A), R^1(y_1, y_2, T_B), R^2(y_1, y_2, T_B), T\Big),$$

equal to $G(x_1, x_2, y_1, y_2, T) = x_1$. Now consider $T'_A = \{(x_1, z), (z, x_2), (x_1, x_2)\}$. Note that $T'_A \geq_{x_1} T_A$. Hence, since H is monotonic, we must have

$$G\Big(R^1\big(x_1, x_2, z, T'_A\big), R^2\big(x_1, x_2, z, T'_A\big), R^1(y_1, y_2, T_B), R^2(y_1, y_2, T_B), T\Big),$$

equal to $G(x_1, z, y_1, y_2, T) = x_1$ which proves claim 1a. As for claim 2a, suppose $G(x_1, x_2, y_1, y_2, T) = x_2$, and consider the tournaments $T_B = \{(y_1, y_2)\}$ and $T_A = \{(x_1, x_2), (x_2, z), (z, x_1)\}$. Then we have

$$G\Big(R^1(x_1, x_2, z, T_A), R^2(x_1, x_2, z, T_A), R^1(y_1, y_2, T_B), R^2(y_1, y_2, T_B), T\Big),$$

equal to $G(x_1, x_2, y_1, y_2, T) = x_2$. Now consider $T'_A = \{(x_2, x_1), (x_2, z), (z, x_1)\}$. Note that $T'_A \geq_{x_2} T_A$. Hence, since H is monotonic, we must have

$$G\Big(R^1\big(x_1, x_2, z, T'_A\big), R^2\big(x_1, x_2, z, T'_A\big), R^1(y_1, y_2, T_B), R^2(y_1, y_2, T_B), T\Big),$$

equal to $G(x_2, z, y_1, y_2, T) = x_2$ which proves claim 2a. In what follows, we will use the following variants of these claims, together with a variant of the symmetry condition: for all pairwise distinct $x_1, x_2, y_1, y_2 \in X$ and $T \in T_X$:

1a: $\forall T' =_{\{x_1,y_1,y_2\}} T$: if $G(x_1, x_2, y_1, y_2, T) = x_1$ then $G(x_1, x_2, y_1, y_2, T') = x_1$.
1b: $\forall T' =_{\{x_1,x_2,y_1\}} T$: if $G(x_1, x_2, y_1, y_2, T) = y_1$ then $G(x_1, x_2, y_1, y_2, T') = y_1$.
2a: $\forall T' =_{\{y_1,y_2\}} T$ such that for all i, $x_1 T' y_i$ iff $x_2 T y_i$: if $G(x_1, x_2, y_1, y_2, T) = x_2$ then $G(x_1, x_2, y_1, y_2, T') = x_1$.
2b: $\forall T' =_{\{x_1,x_2\}} T$ such that for all i, $y_1 T' x_i$ iff $y_2 T x_i$: if $G(x_1, x_2, y_1, y_2, T) = y_2$ then $G(x_1, x_2, y_1, y_2, T') = y_1$.
Sym: $G(x_1, x_2, y_1, y_2, T') = \pi(G(x_1, x_2, y_1, y_2, T))$, where $\pi(x_1) = y_1$, $\pi(x_2) = y_2$, $\pi(y_1) = x_1$, $\pi(y_2) = x_2$ and aTb iff $\pi(a)T'\pi(b)$.

To verify this last claim, since T and T' are isomorphic under π, note that by anonymity we have $\pi(G(x_1, x_2, y_1, y_2, T)) = G(y_1, y_2, x_1, x_2, T')$ which by symmetry equals $G(x_1, x_2, y_1, y_2, T')$.

As for the other four claims, we will prove only the variants of claims 1a and 2a. As for variant 1a, suppose $T' =_{\{x_1,y_1,y_2\}} T$ and $G(x_1, x_2, y_1, y_2, T) = x_1$. Now take any $x_3 \notin \{x_1, x_2, y_1, y_2\}$ and define tournament T'' such that $T'' =_{\{x_1,x_2,y_1,y_2\}} T$ and for all $z \in \{x_1, y_1, y_2\}$, $x_3 T'' z$ iff $x_2 T' z$. By IIR, $G(x_1, x_2, y_1, y_2, T'') = x_1$, and by claim 1a, $G(x_1, x_3, y_1, y_2, T'') = x_1$. But G is anonymous and satisfies IIR, so since T'' and T' are isomorphic wrt the relevant arguments, we have $G(x_1, x_2, y_1, y_2, T') = x_1$.

As for variant 2a, suppose $T' =_{\{y_1,y_2\}} T$ and $x_1 T' y_i$ iff $x_2 T y_i$ for all i. Let $G(x_1, x_2, y_1, y_2, T) = x_2$. Now take any $x_3 \notin \{x_1, x_2, y_1, y_2\}$ and define tournament T'' such that for all z, $x_3 T'' z$ iff $x_2 T' z$, and for all $z, z' \in \{x_1, x_2, y_1, y_2\}$, $z T'' z'$ iff $\pi(z) T \pi(z')$ where $\pi(y_i) = y_i$ and $\pi(x_1) = x_2$ and $\pi(x_2) = x_1$. Now by anonymity and IIR, $G(x_2, x_1, y_1, y_2, T'') = x_1$. Applying claim 2a, we have $G(x_1, x_3, y_1, y_2, T'') = x_1$ and by definition of T'', anonymity and IIR, we have $G(x_1, x_3, y_1, y_2, T'') = G(x_1, x_2, y_1, y_2, T')$.

Now consider any $x_1, x_2, y_1, y_2 \in X$ which are pairwise distinct. Since G satisfies IIR, the value of $G(x_1, x_2, y_1, y_2, T)$ will depend only on the six tournament pairs concerning x_1, x_2, y_1, y_2. Hence, since G is input-selecting, there is a function $g : \{0, 1\}^6 \to \{x_1, x_2, y_1, y_2\}$ such that for all T we have

$$g(b_1, b_2, b_3, b_4, b_5, b_6) = G(x_1, x_2, y_1, y_2, T),$$

where

$$\begin{aligned} b_1 &= 1 \text{ iff } x_1 T x_2, \\ b_2 &= 1 \text{ iff } y_1 T y_2, \\ b_3 &= 1 \text{ iff } x_1 T y_1, \\ b_4 &= 1 \text{ iff } x_1 T y_2, \\ b_5 &= 1 \text{ iff } x_2 T y_1, \\ b_6 &= 1 \text{ iff } x_2 T y_2. \end{aligned}$$

Now we can translate the five earlier conditions into this new format, using the g function, obtaining the following: for all $b_1, \ldots, b_6, c, c', c'' \in \{0, 1\}$:

Condition 1a: if $g(b_1, \ldots, b_6) = x_1$ then $g(c, b_2, b_3, b_4, c', c'') = x_1$.
Condition 1b: if $g(b_1, \ldots, b_6) = y_1$ then $g(b_1, c, b_3, c', b_5, c'') = y_1$.
Condition 2a: if $g(b_1, \ldots, b_6) = x_2$ then $g(c, b_2, b_5, b_6, c', c'') = x_1$.
Condition 2b: if $g(b_1, \ldots, b_6) = y_2$ then $g(b_1, c, b_4, c', b_6, c'') = y_1$.
Symmetry: $\pi(g(b_1, \ldots, b_6)) = g(b_2, b_1, 1-b_3, 1-b_5, 1-b_4, 1-b_6)$.

Note that there are $4^{(2^6)} = 4^{64} > 10^{37}$ possible g functions to consider. Since this is computationally infeasible, we will analyze the function space in stages. Consider first the following functions $g_1, g_2 \colon \{0, 1\}^4 \to \{x_1, x_2, y_1, y_2\}$:

$$\begin{aligned} g_1(b_3, b_4, b_5, b_6) &= g(1, 1, b_3, b_4, b_5, b_6), \\ g_2(b_3, b_4, b_5, b_6) &= g(0, 0, b_3, b_4, b_5, b_6). \end{aligned}$$

The five conditions for g now yield the following conditions for the these two new functions as consequences (for both $i = 1$ and $i = 2$): for all $b_3, \ldots, b_6, c, c' \in \{0, 1\}$:

Condition 1a: if $g_i(b_3, b_4, b_5, b_6) = x_1$ then $g_i(b_3, b_4, c, c') = x_1$.
Condition 1b: if $g_i(b_3, b_4, b_5, b_6) = y_1$ then $g_i(b_3, c, b_5, c') = y_1$.
Condition 2a: if $g_i(b_3, b_4, b_5, b_6) = x_2$ then $g_i(b_5, b_6, c, c') = x_1$.
Condition 2b: if $g_i(b_3, b_4, b_5, b_6) = y_2$ then $g_i(b_4, c, b_6, c') = y_1$.
Symmetry: $\pi(g_i(b_3, b_4, b_5, b_6)) = g_i(1-b_3, 1-b_5, 1-b_4, 1-b_6)$.

Note that for both g_1 and g_2, we have $4^{(2^4)} = 4^{16}$ possible functions to consider, but if we build symmetry into the functions from the start, we only have $4^{(2^3)} = 4^8 = 65,536$ functions to consider which is easily manageable by computers. A C-program checking which of the 65,536 possible symmetric functions satisfy the four conditions is provided in the Appendix (see online supplementary material). It turns out that of the 65,536 symmetric functions, 322 functions satisfy condition 1a and b and 348 functions satisfy condition 2a and b, but there are only two symmetric functions g_α and g_β satisfying both conditions:

$$g_\alpha(b_3, b_4, b_5, b_6) = x_1 \quad \text{if} \quad b_3 = 0 \quad \text{and} \quad y_1 \text{ otherwise,}$$
$$g_\beta(b_3, b_4, b_5, b_6) = y_1 \quad \text{if} \quad b_3 = 0 \quad \text{and} \quad x_1 \text{ otherwise.}$$

This means that for all $b_3, \ldots, b_6, c \in \{0, 1\}$ we have $g(c, c, b_3, b_4, b_5, b_6) = x_1$ or $g(c, c, b_3, b_4, b_5, b_6) = y_1$. The last step of our proof establishes that for any $b_3, \ldots, b_6, c \in \{0, 1\}$ we have either $g(c, 1-c, b_3, b_4, b_5, b_6) = x_1$ or $g(c, 1-c, b_3, b_4, b_5, b_6) = y_1$. So consider any $b_3, \ldots, b_6, c \in \{0, 1\}$. It suffices to consider the following two cases:

(i) $g(c, c, b_3, b_4, b_5, b_6) = y_1$: then by condition 1b, $g(c, 1-c, b_3, b_4, b_5, b_6) = y_1$ and we are done.
(ii) $g(c, c, b_3, b_4, b_5, b_6) = x_1$: then by condition 1a, $g(1-c, c, b_3, b_4, b_5, b_6) = x_1$. Furthermore, we must also have $g(1-c, 1-c, b_3, b_4, b_5, b_6) = x_1$ (for otherwise we would have $g(1-c, 1-c, b_3, b_4, b_5, b_6) = y_1$, and by condition 1b, $g(1-c, c, b_3, b_4, b_5, b_6) = y_1$, a contradiction). Again applying condition 1a we obtain $g(c, 1-c, b_3, b_4, b_5, b_6) = x_1$.

Hence, we have shown that for all $b_1, \ldots, b_6 \in \{0, 1\}$ we have $g(b_1, \ldots, b_6) = x_1$ or $g(b_1, \ldots, b_6) = y_1$. But this means that for any tournament T, we have $G(x_1, x_2, y_1, y_2, T) = x_1$ or $G(x_1, x_2, y_1, y_2, T) = y_1$ which violates our assumption of non-imposition, hence we have obtained a contradiction. □

In the remainder of this section, I shall comment on the necessity of the various conditions involved in the result obtained. I shall start by considering the various competition format conditions and afterwards I will comment on the size of the player set X.

First, note that symmetry is not implied by the remaining conditions: the mix function M satisfies all conditions except symmetry (see Lemma 2). Second, monotonicity is not implied by the remaining conditions: the standard knockout competition format $G(x_1, x_2, y_1, y_2, T) = K^1(x_1, y_2, y_1, x_2, T)$ satisfies all conditions except monotonicity (recall the discussion in Sect. 5). Third, non-imposition is not implied by the remaining conditions: consider function G defined as $G(x_1, x_2, y_1, y_2, T) = x_1$ if $x_1 T y_1$ and y_1 otherwise. This function satisfies all conditions except non-imposition. Fourth, IIR is not implied by the other conditions: consider the function G defined as follows:

$$G\,(x_1, x_2, y_1, y_2, T) = \begin{cases} x_2 & \text{if for all } z \in X-\{x_2\} \text{ we have } x_2 T z \text{ (1)}, \\ y_2 & \text{if for all } z \in X-\{y_2\} \text{ we have } y_2 T z \text{ (2)}, \\ x_1 & \text{if neither (1) nor (2) holds and } x_1 T y_1, \\ y_1 & \text{if neither (1) nor (2) holds and } y_1 T x_1. \end{cases}$$

This function satisfies all conditions except independence of irrelevant alternatives. Fifth, regarding anonymity, unfortunately the question whether there are any functions satisfying all the conditions except anonymity remains open.

Finally a few remarks concerning the number of players needed for these results, the size of X. The impossibility theorem is formulated for situations where $|X| \geq 6$. In terms of applicability, this lower bound is good enough since in practice, a complex competition format using two or more round-robin groups will only be used with at least six players, at least three players per group. Formally, Theorem 1 makes use of Lemma 1 which induces this lower bound. If we add input-selection as an extra condition to Theorem 1, we can reduce the lower bound to five players, for the proof of the theorem only makes use of the monotonicity of H_2^3 and H_3^2 (the latter for conditions 1b and 2b) to obtain a contradiction.

Also, we actually could have chosen a stronger but less elegant formulation of the theorem referring explicitly to H_2^3 and H_3^2 rather than to all H_j^i. However, there is nothing special about this pair: larger round-robin groups could have been used in the proof, where additional arguments of the round-robin functions could have been filled by dummy players. The pair H_2^3 and H_3^2 does, however, form the lower limit where monotonicity becomes impossible: there are G functions satisfying all the conditions of the theorem for which monotonicity of H_2^2 is achievable. Consider, for instance, the following function G^*:

$$G^*\,(x_1, x_2, y_1, y_2, T) = \begin{cases} x_1 & \text{if } x_1 T y_1 \text{ and } (x_1 T y_2 \text{ or } x_2 T y_1 \text{ or } x_2 T y_2), \\ y_1 & \text{if } y_1 T x_1 \text{ and } (y_1 T x_2 \text{ or } y_2 T x_1 \text{ or } y_2 T x_2), \\ x_2 & \text{if } y_1 T x_1 \text{ and } x_2 T y_1 \text{ and } x_1 T y_2 \text{ and } x_2 T y_2, \\ y_2 & \text{if } x_1 T y_1 \text{ and } y_2 T x_1 \text{ and } y_1 T x_2 \text{ and } y_2 T x_2. \end{cases}$$

Intuitively, a second-ranked player wins the competition only if it wins both its matches against the other group and each first-ranked player wins exactly one match. If none of the second-ranked players wins, then the winning first-ranked player is the one who wins the match against the other. Note first that G^* is non-imposed, input-selecting, anonymous, IIR and symmetric. Furthermore the function

$$\begin{aligned} &H_2^2\,(x_1,\ x_2,\ y_1,\ y_2,\ T_A,\ T_B,\ T) \\ &\quad = G^*\left(R^1\,(x_1,\ x_2,\ T_A),\ R^2(x_1,\ x_2,\ T_A),\ R^1\,(y_1,\ y_2,\ T_B),\ R^2(y_1,\ y_2,\ T_B),\ T\right), \end{aligned}$$

is monotonic. The reason monotonicity is possible in this case is that the kind of manipulation illustrated in the third (and second) example of Sect. 5 is impossible if the round-robin group has only two players: if a player is first-ranked and wins more matches, the second-ranked player can never be replaced by a third player, since there are only two players in the group. This means that monotonicity is less demanding in trivial round-robin groups.

7 Conclusions

To summarize, this paper has analyzed strategic manipulation in complex competition formats that are used in sports events like the Olympics or the soccer world cup. In the beginning of this paper, the necessary mathematical notions were developed to analyze examples of such competitions, with monotonicity capturing the notion of strategy-proofness. We looked in detail at the competition format that produces a failure of monotonicity, a format that uses second-ranked players of round-robin competitions. For this format, an alternative cross-play function was proposed that does satisfy monotonicity. The disadvantage of this function was, however, that it does not treat the two round-robin groups equally. As the impossibility theorem showed, monotonicity and symmetry are in conflict with each other.

How reasonable are the conditions the impossibility result imposes? The remarks following Theorem 1 can help us to answer this question, since they discuss a number of competition formats which almost meet all the conditions. None of these candidates seems desirable, however. A further option might be to allow for partial functions, competition formats which do not always yield a winner. It can be shown that if one is willing to accept that competitions fail to yield a winner in a small number of cases, the other conditions of Theorem 1 can be met. Again, it seems questionable, however, whether such a competition format is desirable in practice.

This paper raises many further research questions concerning competition formats. On the theoretical level, the open question remaining is the status of anonymity in the impossibility theorem: is anonymity already implied by the other conditions, or are there examples of non-anonymous competition formats which meet all the other requirements? Going beyond the formal model of this paper, we might want to allow for ties (as these are possible in many real-life competitions), and we might generalize competition formats to produce not players but lotteries over players. Also, there are competition formats and competition properties we have not looked at in this paper. An example of such a competition format is the so-called repechage or double elimination contest where players who lose are moved into a repechage bracket and still maintain a small possibility to win (or at least obtain a bronze medal). Such a competition format was used, e.g., for the Olympic Judo competitions in 2012. Furthermore, while this paper has focused on strategy-proofness, there are other important properties of competition formats. Neutrality is an example of such a property: it expresses that a competition format provides equal entries into the competition, in the sense that it does not matter for a player what starting place in the competition the player is assigned to. In this sense, round-robin is more equal than knockout, since starting places in round-robin groups only determine the order in which opponents are played but not the opponents themselves, in contrast to knockout competitions. In general, this line of research is closely related to the characterization of tournament solutions in social choice theory, but looking at more complex solutions (in our terminology: competition formats) and possibly also somewhat different (combinations of) properties. For instance, we might want a competition format to be forgiving (losing one match does not eliminate the player from the competition), or that it does not pair players against each other more than once.

Finally, and maybe most interestingly, we might also decide to weaken our notion of strategy-proofness. As was shown in Sect. 6, there are competition formats that are symmetric and allow for a limited form of strategy-proofness, where all we care about is that the two top-ranked players do not exchange places strategically. In fact, the function G^* introduced in that section would have avoided this kind of manipulation as well as the Badminton scandal of the 2012 Olympics. However, as this paper has demonstrated in Sect. 5, competition formats also allow for more subtle forms of strategic manipulation, and these are not avoided by any reasonable competition format. This is the content of the impossibility result obtained. Hence, the impossibility theorem can be seen as a a theoretical explanation for why frequently used competition formats like the soccer world cup format which use second-ranked players of round-robin tournaments are strategically manipulable.

Acknowledgments I would like to thank two anonymous referees as well as Jesse Alama for their helpful comments regarding an earlier version of this paper.

References

Altman A, Klienberg R (2010) Nonmanipulable randomized tournament selections. In: Proceedings of AAAI-10. AAAI Press, Menlo Park, pp 686–690

Altman A, Procaccia A, Tennenholtz M (2009) Nonmanipulable selections from a tournament. In: Proceedings of IJCAI'09. Morgan Kaufmann Publishers, San Francisco, pp 27–32

Appel K, Haken W (1977) Every planar map is four colorable. Part I. Discharging. Ill J Math 21:429–490

Appel K, Haken W, Koch J (1977) Every planar map is four colorable. Part II. Reducibility. Ill J Math 21:491–567

Dillman L (2012) London olympics: badminton scandal rocks sport; 8 players expelled. Los Angel Times (Internet Ed) 1 Aug 2012 (consulted on 12 Dec 2012)

Ely J (2012) Cheap talk: a blog about economics, politics and the random interests of forty-something professors. http://cheaptalk.org/2012/08/01/lets-write-this-paper/. Accessed 16 Aug 2012

Gonthier G (2008) Formal proof—the four-color theorem. Notices Am Math Soc 55(11):1382–1393

Hartline J, Kleinberg R (2012) Badminton and the science of rule making. Huffington Post 13 Aug 2012. http://www.huffingtonpost.com/jason-hartline/badminton-and-the-science-of-ru. Accessed 10 April 2013

Hudry O (2009) A survey on the complexity of tournament solutions. Math Soc Sci 57:292–303

Kleinberg R (2012) Olympic badminton is not incentive compatible. http://agtb.wordpress.com/2012/08/01/olympic-badminton-is-not-incentive-comp. Accessed 16 Aug 2012

Laslier JF (1997) Tournament solutions and majority voting. Springer, Heidelberg

MacKenzie D (2001) Mechanizing proof: computing, risk and trust. MIT Press, Cambridge

Moulin H (1986) Choosing from a tournament. Soc Choice Welf 3:271–291

Parikh R (2002) Social software. Synthese 132:187–211

Peris JE, Subiza B (1999) Condorcet choice correspondences for weak tournaments. Soc Choice Welf 16:217–231

Rubinstein A (1980) Ranking the participants in a tournament. SIAM J Appl Math 38(1):108–111

Russell T, Walsh T (2009) Manipulating tournaments in cup and round robin competitions. In: Rossi F, Tsoukias A (eds) Algorithmic decision theory: first international conference, ADT 2009. Springer, Berlin, pp 26–37