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Realisations of W_3 symmetry

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We perform a systematic investigation of free-scalar realisations of the Zamolodchikov W_3 algebra in which the operator product of two spin-three generators contains a non-zero operator of spin four which has vanishing norm. This generalises earlier work where such an operator was required to be absent. By allowing this spin-four null operator we obtain several realisations of the W_3 algebra both in terms of two scalars as well as in terms of an arbitrary number n of free scalars. Our analysis is complete for the case of two-scalar realisations.

1. Introduction

In recent years, there has been a lot of activity in the study of extended conformal symmetries, better known under the name "W-symmetries". These symmetries constitute extensions of the Virasoro algebra which are generically denoted by "W-algebras". W-symmetries can be used to clarify the structure of conformal field theory. They also occur as a "natural" symmetry in a variety of physical models. Another approach is to use W-symmetries for the construction of higher-spin extensions of two-dimensional gravity ("W-gravity") or new string models ("W-strings").

In view of the above-mentioned applications, it is important to have a good understanding of all possible realisations of W-symmetries. The simplest example of a W-algebra is the W_3-algebra of ref. [1] which contains, in addition to the spin-two Virasoro generator T, a spin-three generator W. Using the language of operator product expansions (OPEs), the algebra is given by

$$T(z)T(w)=\frac{c}{2(z-w)^4}+\frac{2T(w)}{(z-w)^2}+\frac{\partial T(w)}{z-w}+\text{regular part}\,,$$

$$T(z)W(w)=\frac{3W(w)}{(z-w)^2}+\frac{\partial W(w)}{z-w}+\text{regular part}\,,$$

$$W(z)W(w)=\frac{c}{3(z-w)^6}+\frac{2T(w)}{(z-w)^4}+\frac{\partial T(w)}{(z-w)^3}+\frac{3}{10}\frac{\partial^2 T(w)}{10(z-w)^2}+\frac{1}{15}\frac{\partial^3 T(w)}{z-w}$$
$$+\frac{16}{22+5c}\left(\frac{2\Lambda(w)}{(z-w)^2}+\frac{\partial\Lambda(w)}{z-w}\right)+\text{regular part}\,. \qquad (1)$$

with $\Lambda=(TT)-\frac{3}{10}\partial^2 T$. The round brackets in (TT) indicate a natural normal ordering in terms of the Laurent modes of the generators (see, e.g., ref. [2]). The first equation in (1) gives the Virasoro algebra, while the second equation expresses the fact that W is a primary field of spin three. The last equation tells us that the OPE of two spin-three generators gives rise to the conformal family of the unit operator. The particular coefficients arising in this equation can all be fixed by the requirement of conformal invariance. Note that in (1) we have used a particular normalisation of the W-generator, i.e. $\langle WW\rangle=\frac{1}{3}c$, in agreement with the common convention.

In order to construct W-algebras and to obtain realisations of them one can follow different strategies. One

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approach is to develop a specific construction procedure, like e.g. the Miura transformation of ref. [3] or the coset construction of ref. [4]. Another approach is to start from an Ansatz for the OPEs of a given set of abstract generators and to require closure of the algebra (see, e.g., ref. [5]). Alternatively, one could start from an Ansatz for the generators of the W-algebra in terms of scalar fields and then impose closure. This has been the approach of refs. [6–8], where a systematic search for free field realisations of the W_3 algebra was undertaken. In particular, starting from certain Ansätze for the generators [7,8], the following n-scalar realisation was found [8]:

$$\begin{aligned} T&=\tfrac{1}{2}(A_0A_0)+\sqrt{2}\,a_0A_0'+T_\mu\,,\\ W&=-\tfrac{1}{3}(A_0A_0A_0)-\sqrt{2}\,a_0(A_0A_0')-\tfrac{2}{3}a_0^2A_0''+2(A_0T_\mu)+\sqrt{2}\,a_0T'_\mu\,, \end{aligned} \tag{2}$$

where A_0 is the derivative of a free scalar field, i.e. $A_0\equiv\partial\phi_0$. The other $n-1$ scalars are represented by T_μ which commutes with A_0 and satisfies a Virasoro algebra with central charge given by $c_\mu=\frac{1}{4}c+\frac{1}{2}$. The parameter a_0 is the background charge and is related to the central charge parameter c via $c=2(1-16a_0^2)$. The resulting realisation coincides for $n=2$ with the Fateev–Zamolodchikov (FZ) two-scalar realisation [6]. It can be viewed as a natural generalisation of the FZ realisation to an arbitrary number n of scalar fields. Note that in the definition of the nonlinear term (TT) in the W_3 algebra we use a normal ordering in terms of the Laurent modes of the generators. A normal ordering of this term with respect to the modes of the free scalar fields was considered in ref. [9].

The aim of this letter is to generalise the analysis of refs. [6–8] by allowing spin-four null operators in the operator product of two spin-three generators. To be more precise, instead of the third equation in (1) we require that the following OPE holds:

$$W(z)\,W(w)=\text{as in (1)}+\frac{V(w)}{(z-w)^2}+\frac{\frac{1}{2}\partial V(w)}{z-w}\,, \tag{3}$$

where V is a spin-four null operator, i.e. $\langle VV\rangle=0$. Of course, strictly speaking, the algebra corresponding to (3) is not the same as the W_3 algebra given in (1). However, since V is a null operator, it can only generate other null fields in its OPE. The full set of null operators constitutes an ideal of the algebra. It is therefore consistent to set all these null operators equal to zero and one thus obtains a representation of the W_3 algebra.

Realisations of W-symmetries modulo null fields have been considered before in the literature. For instance, they occur in the coset construction of ref. [4] and also, in a supersymmetric context, in ref. [10]. More recently, in refs. [11,12], such realisations were obtained, for specific values of the central charge, from a certain nonlinear W_∞ algebra [12] based upon the coset SL(2, ℝ)/U(1). This algebra is related to the parafermion current algebra of ref. [13]. From a somewhat different point of view, extensions of the W_3 algebra with null generators have occurred recently in a study of certain singular contractions of W-algebras [14].

In our analysis of the W_3 algebra, we have restricted ourselves in the following two ways. First of all, we only consider spin-four null operators. In principle, one could also allow for spin-two null operators in the OPE of two spin-three generators. However, since in most formulations of W-algebras every spin occurs only once, it is less natural to allow for spin-two operators in addition to the Virasoro generator. Secondly, we only consider free field realisations. We will not consider the inclusion of vertex operators in the Ansatz as was done in ref. [15].

2. Ansätze

Our starting point is the following free field Ansatz for the spin-two and spin-three generators of the W_3-algebra [7,8]:

$$T=\tfrac{1}{2}g_{ij}(A^iA^j)+\sqrt{2}\,a_iA^{i\prime}\,, \tag{4}$$

$$W=\tfrac{1}{3}d_{ijk}(A^iA^jA^k)+2\sqrt{2}\,e_{ij}(A^iA^{j\prime})+2f_iA^{i\prime\prime}\,, \tag{5}$$

where $A^i\equiv\partial\phi^i$ and the ϕ^i ($i=0, ..., n-1$) represent a set of n free scalar fields and g_{ij}, a_i, d_{ijk}, e_{ij} and f_i are yet undetermined coefficients. The A_i satisfy the OPE

$$A^i(z)A^j(w)=\frac{g^{ij}}{(z-w)^2}+\text{regular part}\,, \tag{6}$$

where g^{ij} is the inverse of g_{ij}. Our conventions are slightly different from those of ref. [8]. Note that with the above Ansatz the spin-two generator $T(z)$ already satisfies the Virasoro algebra with central $c=n-24a_ia^i$.

Following ref. [8], we split the n-component index i into "0" and an ($n-1$)-component index μ and take the coefficients d_{ijk} to be

$$d_{000}=s,\quad d_{0\mu\nu}=-sg_{\mu\nu}\,, \tag{7}$$

where the parameter s is fixed by the choice of normalisation of the W generator. The expression for the d-coefficients is a solution to

$$d_{(ij}{}^{m}d_{kl)m}=s^2g_{(ij}g_{kl)}\,. \tag{8}$$

The latter equation guarantees the closure of the classical version w_3 of the W_3 algebra [7]. In the analysis of ref. [8], equations for the unknown coefficients in (4) and (5) were found by demanding that the generators satisfy the W_3 algebra given in (1), i.e. without spin-four null operators. It was subsequently shown that these equations were solved by the n-scalar realisation given in (2).

We now consider the same Ansatz (4,5), but instead require that the generators satisfy the W_3 algebra modulo a spin-four null operator as indicated in (3). This allows us to take the following less restrictive Ansatz for the coefficients d_{ijk}:

$$d_{000}=s,\quad d_{0\mu\nu}=tg_{\mu\nu}\,, \tag{9}$$

with s and t free parameters (although one of them may be fixed by choosing a normalisation for W).

The following three equations have to be satisfied in order that W is primary with respect to T:

$$d^j{}_{ji}-24e_{ij}a^j+12f_i=0\,, \tag{10}$$

$$2e_{(ij)}-d_{ijk}a^k=0\,, \tag{11}$$

$$3f_i-2a^je_{ji}=0\,. \tag{12}$$

For more details, see ref. [8]. On the fourth order pole of the OPE of W with itself a primary spin-two operator shows up besides a multiple of the energy–momentum tensor. We require that this operator vanishes because we want T to be only spin-two operator in the algebra. This leads to the following equation:

$$d_i{}^{kl}d_{jkl}+12d_{ijk}f^k-24e_i{}^ke_{jk}=\frac{3}{2c}N_3g_{ij}\,. \tag{13}$$

In (13) N_3 is the norm of the operator W, which we prefer not to fix for the moment:

$$N_3\equiv\langle WW\rangle=\tfrac{2}{3}(d_{ijk}d^{ijk}-72e_{ij}e^{ji}+720f_if^i)\,. \tag{14}$$

In ref. [8] two more equations were used, which guaranteed the vanishing of a primary spin-four operator V in the OPE of $W(z)W(w)$. Instead, we will allow such a spin-four operator, but only if it is null. This requirement leads to one more, rather complicated, equation which we gave given in appendix A. We will refer to this equation as the spin-four equation.

We conclude that the full set of equations that has to be satisfied by the Ansatz (4), (5) and (9) is given by eqs. (10)–(13) and the spin-four equation which can be found in appendix A. The general analysis of these equations is rather complicated. Among the solutions one should of course find, as a special case, those of ref. [8] which are characterized by taking $s=-t$ in (9) and $V\equiv 0$, i.e. no spin-four null operator. We will now discuss the new solutions we obtained.

3. Solutions

Our strategy is to first solve eqs. (10)–(13) and afterwards impose the the spin-four equation. It is convenient to distinguish between the two cases corresponding to $a_0\neq 0$ and $a_0=0$. From now on we will take $t=1$ as a choice of normalisation. Note that in general this differs from the standard normalisation $\langle WW\rangle=\frac{1}{3}c$. For $a_0\neq 0$ (case I) we find

$$\text{Case I:}\quad e_{00}=\tfrac{1}{2}sa_0,\quad e_{\mu 0}=0,\quad e_{0\mu}=a_\mu,\quad e_{(\mu\nu)}=\tfrac{1}{2}a_0 g_{\mu\nu},\quad e_{[\mu\nu]}=0,\quad f_0=\tfrac{1}{3}sa_0^2,\quad f_\mu=a_0a_\mu\,. \tag{15}$$

Besides the Romans solution, corresponding to $s-1$, these equations have the following other solutions a well:

$$\text{Case I:}\quad a_0^2=\frac{s-2}{2(s-3)},\quad a_\mu a^\mu=\frac{-3s^2+4s+3+n(s-3)}{24(s-3)}, \tag{16}$$

$$c=3s-7\,, \tag{17}$$

where the parameter s is still undetermined. We now substitute these solutions into the spin-four equation. It turns out that this equation is satisfied only for the values $s=\frac{7}{3}$, $\frac{5}{3}$, -1, $\frac{5}{2}$ and $\frac{13}{5}$. For $s=\frac{7}{3}$ and $s=\frac{5}{2}$, corresponding to $c=0$ and $c=\frac{1}{2}$, respectively, W turns out to be a null field as well, and we will not consider these cases further. For $s=-1$ we get the Romans solution for $c=-10$. The two new solutions we find are given by $s=\frac{5}{3}$ $(c=-2)$ and $s=\frac{13}{5}$ $(c=\frac{4}{5})$. In appendix B the explicit form of these realisations is given for $n=2$.

We note that the $c=\frac{4}{5}$ realisation has an imaginary background charge a_0. In order to obtain real coefficients in the realisation it is necessary to perform the redefinitions $A_0\to \mathrm{i}A_0$ and $W\to \mathrm{i}W$. The result is a "noncompact" realisation where the quadratic A_0 part in T has a minus sign.

A general feature of the case I solutions is that the W_3 generators take on the form

$$T=\tfrac{1}{2}(A_0A_0)+\sqrt{2}\,a_0A_0'+T_\mu\,, \tag{18}$$

$$W=\tfrac{1}{3}s(A_0A_0A_0)+\sqrt{2}\,sa_0(A_0A_0')+\tfrac{2}{3}sa_0^2A_0''+2(A_0T_\mu)+\sqrt{2}\,a_0T'_\mu\,, \tag{19}$$

where T_μ is the energy–momentum tensor corresponding to the $n-1$ fields A_μ with central charge

$$c_\mu=-s(1-8a_0^2)\,. \tag{20}$$

The total central charge is

$$c=c_0+c_\mu=1-24a_0^2-s(1-8a_0^2)\,. \tag{21}$$

So there is one scalar that appears explicitly in (18), (19), and the rest enters only via their energy-momentum tensor T_μ. This situation also occurs in the Romans solution (2). We note that for both the Romans solution (2) at $c=-2$ as well as the case I $c=-2$ solution given in (18), (19), T_μ is null and the A_0 part becomes the one scalar realisation of W_3 [10].

We next consider solutions of eqs. (10)–(13) for $a_0=0$ (case II). From eqs. (10)–(13) we deduce that

$$\text{Case II:}\quad e_{00}=0,\quad e_{0\mu}+e_{\mu 0}=a_\mu,\quad e_{0\mu}a^\mu=\tfrac{1}{4}a_\mu a^\mu+\tfrac{1}{32}(s+n-1),\quad e_{(\mu\nu)}=0,\quad e_{[\mu\nu]}a^\nu=0,\quad e_{[\mu\nu]}e_0^\nu=0\,,$$
$$f_0=\tfrac{1}{2}a_\mu a^\mu-\tfrac{1}{48}(s+n-1),\quad f_\mu=0\,. \tag{22}$$

Furthermore, the background charges and the central charge are given by

$$\text{Case II:}\quad a_\mu a^\mu=\tfrac{1}{24}(n-3s+7),\quad c=3s-7\,. \tag{23}$$

We also obtain expressions for the contractions $e_{0\mu}e^{0\mu}$ and $e_{[\mu\nu]}e^{[\mu\nu]}$. Since they are rather involved we will not give them here. We still have to impose the spin-four equation. We were able to simplify this equation only for $n=2$ and have not analysed it for general values of n. For $n=2$ the spin-four equation becomes, rewritten in terms of c using (23)

$$\langle VV\rangle=\frac{16c(2+c)(7+c)(10+c)^2(-\frac{1}{2}+c)(-4+5c)}{27(-2+c)^2(22+5c)}=0\,. \tag{24}$$

From the series of roots of (24) the values $c=0$, -7, $\frac{1}{2}$ make W a null field as well, and for $c=-10$ ($s=-1$) we get a FZ realisation. The new solutions occur again for $c=-2$ and $c=\frac{4}{5}$. The case II $c=-2$ and $c=\frac{4}{5}$ realisations also appear in ref. [12] as specific truncations of a non-linear W_∞ algebra. They can also be derived from the second realisation mentioned in a footnote of the paper by Fateev and Zamolodchikov [6]. The explicit form of the solutions can be found in appendix B.

Unlike the case I realisations the case II realisations are not of the form (18), (19), i.e. there exists no SO(2) redefinition of the fields such that (18), (19) is obtained. It is therefore not clear whether these solutions can be generalised to $n\geqslant 2$ scalars.

4. Generalisations

We now discuss generalisations of the case I and case II realisations. First, consider the $c=-2$ one-scalar realisation of ref. [10]:

$$T_0=\tfrac{1}{2}(A_0A_0)+\tfrac{1}{2}A_0'\,, \tag{25}$$

$$W_0=-\tfrac{2}{3}(A_0T_0)-\tfrac{1}{6}T_0'\,. \tag{26}$$

We now add an extra energy–momentum tensor, denoted by $\tilde{T}$, to the above system that commutes with A_0 and which is null. We then make the following Ansatz for W:

$$T=T_0+\tilde{T}\,, \tag{27}$$

$$W=W_0+d_1(A_0\tilde{T})+d_2\tilde{T}'\,. \tag{28}$$

Since $\tilde{T}$ commutes with T_0 the total central charge is given by $c=-2$. The requirement that W is primary with respect to T can be shown to imply $d_1=4d_2$. Next, in order to get rid of a primary spin-two field in the OPE $W(z)W(w)$, which occurs in addition to T, the following quadratic equation has to be satisfied:

$$20d_2^2-4d_2-3=0\,, \tag{29}$$

with roots $\frac{1}{2}$ and $-\frac{3}{10}$. If we represent $\tilde{T}$ in terms of $n-1$ scalar fields ($n\geqslant 2$) then we obtain for $d_2=\frac{1}{2}$ the Romans realisation at $c=-2$ and for $d_2=-\frac{3}{10}$ the case I $c=-2$ realisation (cf. (19)). Note that if, in the above example, we do not modify W_0 (i.e. $d_1=d_2=0$ in (28)), the algebra also closes modulo null operators. However, in this case also a spin-two null operator is present in $W(z)W(w)$.

We now perform the same procedure starting from the Romans realisation (2) for arbitrary c. Again we add a null energy–momentum tensor $\tilde{T}$ to the generators in such a way that they remain primary. We thus obtain

$$T=T_0+T_\mu+\tilde{T}\,, \tag{30}$$

$$W=W_0+2(a_0T_\mu)+\sqrt{2}\,a_0T_\mu'+d[(A_0\tilde{T})+\tfrac{1}{2}\sqrt{2}\,a_0\tilde{T}']\,, \tag{31}$$

$$T_0=\tfrac{1}{2}(A_0A_0)+\sqrt{2}\,a_0A_0'\,, \tag{32}$$

$$W_0=-\tfrac{1}{3}[2(A_0T_0)+\sqrt{2}\,a_0T_0']\,. \tag{33}$$

The central charges are given by (takes $s=-1$ in (20) and (21))

$$c_0 = 1-24a_0^2 = \tfrac{3}{4}c - \tfrac{1}{2}\,, \tag{34}$$

$$c_\mu = 1-8a_0^2 = \tfrac{1}{4}c + \tfrac{1}{2}\,. \tag{35}$$

The requirement that the additional spin-two primary field that occurs in WW vanishes, now leads to the equation

$$-2+\tfrac{1}{2}d^2 + a_0^2(-\tfrac{3}{2}d^2 - 2d + 10) = 0\,, \tag{36}$$

with roots

$$d=2, \quad d=-2\frac{1-5a_0^2}{1-3a_0^2}\,. \tag{37}$$

For the $d=2$ solution, $\tilde{T}$ can be absorbed into T_μ and we obtain the Romans realisation. The second solution for d does not fit within the Ansatz (9), which is why we did not find this solution before. For $c=-2$, $a_0^2=\frac{1}{8}$, T_μ is null and can be consistently put to zero, and we find that for this value of c the second solution reduces to the case I $c=-2$ realisation.

In principle, one could generalise the case II realisation from $n=2$ to arbitrary n by the same procedure. One adds a null field $\tilde{T}$ to T_0 and adds $d_1(A_0\tilde{T}) + d_2(A_1\tilde{T}) + d_3\tilde{T}'$ to W_0. Making W primary fixes one parameter, and the spin-two absence implies a quadratic equation in the two remaining parameters. We have not attempted to investigate systematically the solutions to this equation.

5. Comments

We have performed a systematic investigation of free-field realisations of the W_3 algebra where we allow in the OPE of two spin-three generators a spin-four null field. Our starting point was a free field Ansatz for the generators. Closure of the algebra then led to a set of equations for the coefficients occurring in the Ansatz. We analyzed these equations and gave several solutions to them. Besides the Romans solution (see (2)), we found further two-scalar solutions (case II) as well as n-scalar solutions (case I and the second solution of eqs. (30)–(33).

Since we used a specific Ansatz, our analysis is not exhaustive. Only in the case of two-scalar realisations were we able to verify that our analysis is complete. Besides the FZ realisation we found four more realisations whose explicit form can be found in appendix B. Two of these solutions also occur in the work of refs. [6,11,12]. It would be interesting to see whether the other two solutions could be understood from other construction procedures as well.

Finally, one could consider the classical limit of our results. In the case of the Romans realisation one obtains in this limit a realisation of a classical version w_3 of the W_3 algebra. This is consistent with the fact that the Ansatz of refs. [7,8] satisfies the identity (8) which guarantees the closure of the classical w_3 algebra [7]. Our Ansatz does not satisfy (8) and therefore, to obtain closure in the classical limit, one should include the whole ideal of null operators generated by the spin-four operator V. For the case II solutions, this leads to the classical limit of the nonlinear W_∞ algebra of ref. [12]. It would be interesting to see which classical algebras the case I realisations lead to.

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Appendix A. The Spin-four equation

To determine the spin-four equation mentioned in section 2, we must first calculate the expression for the spin-four operator V. This expression can be found from eq. (3.12) in ref. [8] by subtracting the descendents of the energy–momentum tensor. Next, it is a straightforward exercise to calculate the norm of V and require it to be zero. We thus find the following spin-four equation:

$$\begin{aligned}\langle VV\rangle =& 24S^{ijkl}S_{ijkl}+30S_i{}^{ikl}S_j{}^j{}_{kl}-280S_i{}^{ikl}S_j{}^j{}_k{}^m a_l a_m-60\sqrt{2}\,S_i{}^{ikl}T_{klm}a^m+24\sqrt{2}\,S_i{}^{ikl}T_{mkl}a^m\\ &+\tfrac{560}{3}\sqrt{2}\,S_i{}^{ikl}(T_{mnl}+2T_{lmn})a_k a^m a^n-12T^{ijk}T_{ijk}-16T^{ijk}T_{ikj}+60T_{ijk}T^{ij}{}_l a^k a^l-48T_{ijk}T_l{}^{ij}a^k a^l\\ &+\tfrac{328}{3}T_{kij}T_l{}^{ij}a^k a^l+\tfrac{104}{3}T_{kij}T_l{}^{ji}a^k a^l-\tfrac{560}{9}(T_{ijm}T_{kl}{}^m+4T_{imj}T_{kl}{}^m+4T_{imj}T_k{}^m{}_l)a^i a^j a^k a^l\\ =&0\,,\end{aligned}$$

where S and T are given by

$$S_{ijkl}=d_{(ij}{}^m d_{kl)m}-\frac{24N_3}{c(22+5c)}g_{(ij}g_{kl)}\,,\quad T_{ijk}=4\sqrt{2}\left(-2d_{ij}{}^l e_{[kl]}+2e_{(i}{}^l d_{j)kl}-\frac{24N_3}{c(22+5c)}g_{ij}a_k\right).$$

Appendix B. Two-scalar realisations

For the case of two scalars ($n=2$) we find all possible realisations of W_3 that close modulo a non-zero spin-four null field. We find four different realisations. Firstly, the case I $c=-2$ realisation is given by

$$T=\tfrac{1}{2}(A_0A_0)+\tfrac{1}{2}(A_1A_1)+\tfrac{1}{2}A_0'+\tfrac{1}{6}\sqrt{3}\,A_1'\,,$$

$$W=\tfrac{5}{9}(A_0A_0A_0)+\tfrac{5}{6}(A_0A_0')+\tfrac{5}{36}A_0''+(A_0A_1A_1)+\tfrac{1}{3}\sqrt{3}\,(A_0A_1')+\tfrac{1}{2}(A_1A_1')+\tfrac{1}{12}\sqrt{3}\,A_1''\,.$$

Secondly, the case I $c=\frac{4}{5}$ realisation in a real basis is given by

$$T=-\tfrac{1}{2}(A_0A_0)+\tfrac{1}{2}(A_1A_1)+\tfrac{1}{2}\sqrt{6}\,A_0'+\tfrac{2}{5}\sqrt{10}\,A_1'\,,$$

$$W=\tfrac{13}{15}(A_0A_0A_0)-\tfrac{13}{10}\sqrt{6}\,(A_0A_0')+\tfrac{13}{10}A_0''-(A_0A_1A_1)-\tfrac{4}{5}\sqrt{10}\,(A_0A_1')+\tfrac{1}{2}\sqrt{6}\,(A_1A_1')+\tfrac{1}{5}\sqrt{60}\,A_1''\,.$$

The above realisations are obtained from (18), (19) by substituting the appropriate values for the parameters and by realising T_μ in terms of A_1, the derivative of the scalar field ϕ_1.

Next, the case II $c=-2$ realisation is given by

$$T=\tfrac{1}{2}(A_0A_0)+\tfrac{1}{2}(A_1A_1)+\tfrac{1}{3}\sqrt{3}\,A_1'\,,$$

$$W=\tfrac{9}{5}\,(A_0A_0A_0)+(A_0A_1A_1)+\tfrac{1}{2}\sqrt{3}\,(A_0A_1')+\tfrac{1}{6}\sqrt{3}\,(A_0'A_1)+\tfrac{1}{18}A_0''\,.$$

Finally, the case II $c=\frac{4}{5}$ realisation is given by

$$T=\tfrac{1}{2}(A_0A_0)+\tfrac{1}{2}(A_1A_1)+\tfrac{1}{10}\sqrt{10}\,A_1'\,,$$

$$W=\tfrac{13}{15}(A_0A_0A_0)+(A_0A_1A_1)+\tfrac{1}{2}\sqrt{10}\,(A_0A_1')-\tfrac{3}{10}\sqrt{10}\,(A_0'A_1)-\tfrac{1}{10}A_0''\,.$$

For the $n=2$, $c=-2$ realisations $\langle WW\rangle=-\frac{25}{9}$. The $c=\frac{4}{5}$ realisations have $\langle WW\rangle=\frac{52}{75}$ (case I) and $\langle WW\rangle=-\frac{52}{75}$ (case II).

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