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Nusse, H E

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ASYMPTOTICALLY PERIODIC BEHAVIOUR IN THE DYNAMICS OF CHAOTIC MAPPINGS*

H. E. NUSSE†

Abstract. We will prove that, for a chaotic mapping f belonging to a suitable class of $C^{1+\alpha}$ functions, the set $A_\infty(f)$ has Lebesgue measure zero, with $A_\infty(f)$ a nonempty set consisting of points whose orbits do not converge to an asymptotically stable periodic orbit of f or to the absorbing boundary. Moreover almost every point is asymptotically periodic with period p , for some positive integer p .

Further, we will show that the same conclusions hold for maps with nonpositive Schwarzian derivative under some additional assumptions.

Key words. iteration of chaotic mappings, asymptotically stable periodic orbit, absorbing boundary point, (direct) domain of attraction, asymptotically periodic point, critical point, Schwarzian derivative

AMS(MOS) subject classifications. 26A16, 26A18, 28A75, 58F13, 58F15, 58F21, 58F22

1. Introduction and statement of the results. Many established results in the study of iteration of mappings, initiated by Lorenz [11], May [13], [14] and Li and Yorke [10], can be found in e.g. the monographs by Collet and Eckmann [4] and Preston [21] and in the article by Nitecki [19].

There exist a lot of numerical investigations dealing with iterations of mappings, see e.g. Stein and Ulam [24], Metropolis, Stein and Stein [16], Hoppensteadt and Hyman [9], Gumowski and Mira [7] and Coste [5]. The numerical results of density functions suggest, for some examples, that almost every point in the interval approach to an asymptotically stable periodic orbit.

In the paper "Periodic three implies chaos," Li and Yorke [10] mentioned the question, whether (for some nice class of functions) the existence of an asymptotically stable periodic point implies that almost every point is asymptotically periodic.

First, we shall reformulate a result obtained by Guckenheimer [6], Misiurewicz [17] and van Strien [25], see also Collet and Eckmann [4] and Preston [21].

Let f be a chaotic C^3 -mapping from a compact interval $[a, b]$ into itself. Assume that f satisfies the following conditions: (i) f has one critical point c which is nondegenerate, f is increasing on $[a, c]$ and f is decreasing on $[c, b]$, (ii) $f(a) = f(b) = a$, (iii) f has a negative Schwarzian derivative, i.e. $f'''(x)/f'(x) - 3/2[f''(x)/f'(x)]^2 < 0$ for all $x \in [a, b] \setminus \{c\}$, (iv) f has an asymptotically stable periodic orbit, (v) $f'(a) > 1$. Then the set of points, whose orbits do not converge to the asymptotically stable periodic orbit, has Lebesgue measure zero.

Remark. If one omits condition (v), then the conclusion above need not be true, e.g. if $f'(a) < 1$ and $f(f^2(c)) = f^2(c)$, $f'(f^2(c)) > 1$ (cf. Guckenheimer [6]).

Now we will recall a result which is due to Henry [8]: Let $f:]0, 1[\rightarrow \mathbb{R}$ be defined by $f(x) = \lambda x(1-x)$ for some $\lambda > 4$. For almost all (in the sense of Lebesgue measure) $x \in]0, 1[$, some iterate of x under f is not in $]0, 1[$.

This paper deals with iteration of chaotic maps from a nontrivial interval into itself which may have many critical points. It will turn out that the two results mentioned above are special cases of the obtained results in this paper.

Our first result is the following theorem.

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† State University at Groningen, Institute of Econometrics, 9700 AV Groningen, The Netherlands.

THEOREM A. *Let f be a mapping, of class $C^{1+\alpha}$ for some positive real number α , from a nontrivial interval X into itself. Assume that f satisfies the following conditions:*

(i) *The set of asymptotically stable periodic points for f is compact (if this set is empty, then there exists at least one absorbing boundary point of X for f).*

(ii) *The set of points whose orbits do not converge to an asymptotically stable periodic orbit of f or to an (or the) absorbing boundary point(s) of X for f is a nonempty compact set, and f is an expanding map on this set.*

Then we have

(1) *The set of points whose orbits do not converge to an asymptotically stable periodic orbit of f or to an (or the) absorbing boundary point(s) of X for f , has Lebesgue measure zero.*

(2) *There exists a positive integer p such that almost every point in X is asymptotically periodic with (not necessarily primitive) period p , provided that $f(X)$ is bounded.*

Consequently, the set of aperiodic points for f , or equivalently, the set on which the dynamical behaviour of f is chaotic, has Lebesgue measure zero.

Further we will show that the conditions in Theorem A are invariant under the conjugation with a diffeomorphism; hence the conditions seem to be satisfactory for theoretical purposes. On the other hand, we observe that the conditions (i) and the second part of (ii) cannot be checked a priori; hence the conditions seem to be unsatisfactory for practical purposes.

Now we will state our second result.

THEOREM B. *Assume that f is a chaotic C^3 -mapping from a nontrivial interval X into itself satisfying the following conditions:*

(i) *f has a nonpositive Schwarzian derivative, i.e.,*

$$\frac{f'''(x)}{f'(x)} - \frac{3}{2} \left[\frac{f''(x)}{f'(x)} \right]^2 \leq 0 \quad \text{for all } x \in X \quad \text{with } f'(x) \neq 0;$$

(ii) *The set of points, whose orbits do not converge to an (or the) absorbing boundary point(s) of X for f , is a nonempty compact set;*

(iii) *The orbit of each critical point for f converges to an asymptotically stable periodic orbit of f or to an (or the) absorbing boundary point(s) of X for f ;*

(iv) *The fixed points of f^2 are isolated.*

Then we have

(1) *The set of points whose orbits do not converge to an asymptotically stable periodic orbit of f or to an (or the) absorbing boundary point(s) of X for f , has Lebesgue measure zero;*

(2) *There exists a positive integer p such that almost every point in X is asymptotically periodic with (not necessarily primitive) period p , provided that $f(X)$ is bounded.*

COROLLARY. *Assume that $f: \mathbb{R} \rightarrow \mathbb{R}$ is a chaotic polynomial mapping satisfying the following conditions:*

(i) *The orbit of each critical point of f converges to an asymptotically stable periodic orbit of f or to an (or the) absorbing boundary point(s) for f ;*

(ii) *Each critical point of f is real.*

Then we have

f satisfies the assumptions (i)–(iv) of Theorem B.

We note that the Schwarzian derivative can be computed for any C^3 mapping, and that the condition “nonpositive Schwarzian derivative” is not invariant under conjugation with a diffeomorphism.

Finally, we note that, for any fixed mapping from a nontrivial interval into itself with finitely many critical points, it may be difficult to check condition (iii) of Theorem

B by using a calculator, since the periods of the asymptotically stable periodic points may be very large.

2. Simple examples. This section is devoted to some very simple examples; we restrict ourselves to those examples appearing in the literature.

Example 1. $X = [-1, 1]$, $f: X \rightarrow X$ is defined by $f(x) = 3.701x^3 - 2.701x$. It can be verified that f has two asymptotically stable periodic orbits with period three.

Since f has a negative Schwarzian derivative and $f'(1) = f'(-1) > 1$, we have by Theorem B (or by the corollary) that almost every point in X is asymptotically periodic with period three.

Example 2. $X = \mathbb{R}$, $f: X \rightarrow X$ is defined by $f(x) = ax^3 + (1-a)x$ with $a > 4$. Note that f has no asymptotically stable periodic points and note that $]-\infty, -1[\cup]1, \infty[$ is the union of the direct domains of attraction of the two absorbing boundary points of X for f . By the theorem, the set of points whose orbits are bounded has Lebesgue measure zero.

Example 3. Let f be a chaotic map of class C^3 from a compact interval $[a, b]$ into itself with the following properties (cf. Collet and Eckmann [4, p. 119]): (1) f has one critical point c which is nondegenerate, f is strictly increasing on $[a, c]$ and strictly decreasing on $[c, b]$; (2) f has a negative Schwarzian derivative; (3) the orbit of c converges to an asymptotically stable periodic orbit of f with smallest period p , for some positive integer p . Since the existence of an asymptotically stable fixed point in $[a, f^2(c)[$ has not been excluded (see Collet and Eckmann [4, p. 95]) the following cases can occur:

(a) f has an asymptotically stable fixed point in $[a, f^2(c)[$ and f has an asymptotically stable periodic orbit which contains the critical point in its direct domain of attraction.

(b) f has an asymptotically stable fixed point in $[a, f^2(c)[$ and the orbit of the critical point converges to this stable fixed point. Furthermore f has 2^n unstable periodic points with period n for each positive integer n .

(c) f has no asymptotically stable fixed point in $[a, f^2(c)[$; consequently, the critical point is in the direct domain of attraction of the asymptotically stable periodic orbit. (This occurs if e.g. f has at most one fixed point in $[a, c]$.)

The map f satisfies the conditions of Theorem B and we have that almost each point in the interval $[a, b]$ is asymptotically periodic with period p .

3. Definitions and notation. Fix a nontrivial interval $X \subset \mathbb{R}$. Let $f: X \rightarrow X$ be a differentiable (noninvertible) mapping. For any positive integer n , the n th iterate of f , denoted by f^n , is inductively defined by $f^n = f \circ f^{n-1}$, with f^0 as the identity mapping. For any point $x \in X$ the orbit of x under f is the set $\{f^n(x); n \in \mathbb{N} \cup \{0\}\}$.

Assume that Y is a nonempty subset of X . Fix a positive integer n . The image of Y under f^n , denoted by $f^n(Y)$, is the set $\{f^n(x); x \in Y\}$, the pre-image of Y under f^n , denoted by $f^{-n}(Y)$ or by $(f^n)^{-1}(Y)$, is the set $\{x \in X; f^n(x) \in Y\}$. Y is called a positively f -invariant set if $f(Y) \subset Y$; Y is called a negatively f -invariant set if $f^{-1}(Y) \subset Y$; and Y is called a (completely) f -invariant set if $f(Y) \subset Y$ and $f^{-1}(Y) \subset Y$. We write $\text{Cl}(Y)$ for the closure of Y , $\text{Bd}(Y)$ for the boundary of Y , and $\text{Int}(Y)$ for the interior of Y .

Let D be any subset of Y . We denote the complement of D in Y by $Y \setminus D$, and we denote the Lebesgue measure of D , when D is a Lebesgue measurable set, by $\mu(D)$. D is a component of Y , if it is a maximally connected subset of Y . Assuming that D is a component of Y , then D is called trivial when $\mu(D) = 0$, i.e., D consists of one point.

A fixed point x of f is called *Lyapunov stable* if for every open (in X) neighbourhood V of x , there exists a neighbourhood U of x such that $f^k(U) \subset V$ for each positive integer k . A fixed point x of f is called *asymptotically stable* if the following conditions hold: (i) x is a Lyapunov stable fixed point of f ; (ii) there exists an open (in X) neighbourhood U of x such that $U \setminus \{x\}$ does not contain fixed points of the mapping f or f^2 . A fixed point x is called *unstable* if it is not asymptotically stable.

Assume that $x_0 \in X$ is an asymptotically stable fixed point of f . The *domain of attraction* of x_0 is the set of points whose orbits converge to x_0 ; this set is open in X . The *direct domain of attraction* of x_0 is the component of the domain of attraction of x_0 containing x_0 . If the closure of the direct domain of attraction is contained in the interior of X , then we have: (i) the direct domain of attraction of x_0 is mapped into itself under the map f ; (ii) the boundary of the direct domain of attraction is mapped into itself under the map f ; moreover, for u in the boundary of the direct domain of attraction of x_0 we have: either u is a fixed point of f , or $f(u)$ is a fixed point of f , or u is a fixed point of f^2 .

A point $x \in X$ is called a *periodic point* for f with *period* p , for some positive integer p , if $f^p(x) = x$ (i.e. if x is a fixed point of the mapping f^p). The period is called a *primitive period*, if it is the smallest one. A point $x \in X$ is called an *asymptotically stable periodic point* for f with period p , if x is an asymptotically stable fixed point of the mapping f^p .

A periodic point x for f with period p in the interior of X is called *one-sided asymptotically stable* if there exists a positive real number ϵ such that either $\lim_{n \rightarrow \infty} f^{np}(y) = x$ for all $y \in [x, x + \epsilon[$ and $|f^p(y) - x| > |y - x|$ for all $y \in]x - \epsilon, x[$, or $\lim_{n \rightarrow \infty} f^{np}(y) = x$ for all $y \in]x - \epsilon, x[$ and $|f^p(y) - x| > |y - x|$ for all $y \in]x, x + \epsilon[$.

A point $x \in X$ is called an *eventually periodic point* for f with (*eventually*) *period* p , for some positive integer p , if there exists a periodic point q for f with period p and a positive integer n such that $f^n(x) = q$.

A point $x \in X$ is an *asymptotically periodic point* for f if $\lim_{n \rightarrow \infty} f^{nm}(x)$ exists for some $m \in \mathbb{N}$; a point x is an *aperiodic point* for f if the following two conditions are satisfied: (1) x is not an asymptotically periodic point and (2) the orbit of x is bounded. The map f is called *chaotic* if there exists at least one aperiodic point for f .

A point $x \in (\mathbb{R} \cup \{+\infty\}) \cup \{-\infty\} \setminus X$ is called an *absorbing boundary point* of X for f with period p , for some $p \in \{1, 2\}$, if there exists an open set $U \subset X$ such that $f^{pk}(y) \rightarrow x$ for $k \rightarrow \infty$ for all $y \in U$. Assume that x_0 is an absorbing boundary point of X for f with primitive period p with $p = 1$ or $p = 2$. The set $\{y \in X; f^{pk}(y) \rightarrow x_0 \text{ for } k \rightarrow \infty\}$ is called the *domain of attraction* of x_0 . The component of the domain of attraction of x_0 which has x_0 as a boundary point is called the *direct domain of attraction* of x_0 .

A point $x \in X$ is called a *critical point* for f if $f'(x) = 0$. For any critical point x for f , the value $f(x)$ is called a *critical value* for f . The set consisting of all critical points for f is called the *critical set* of f .

We write the symbol \sqcup for disjoint union, the symbol \square to indicate the end of a proof, and finally

- Per (f): the set of periodic points for f ,
- Crit (f): the set of critical points for f ,
- $p(q)$: primitive period of q , with $q \in \text{Per}(f)$.

4. Preliminaries.

A. We fix a nontrivial interval $X \subset \mathbb{R}$; let $f: X \rightarrow X$ be a fixed chaotic mapping. In this subsection we assume that f satisfies the following conditions:

(4.1) (i) $f \in C^{1+\alpha}(X, X)$ for some $\alpha > 0$;

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- (ii) the set of asymptotically stable periodic points for f is compact;
- (iii) the boundary of X has a neighbourhood, denoted by $U_{Bd(X)}$, consisting of two components of the union of the domains of attraction of the asymptotically stable periodic points for f and the absorbing boundary points of X for f .

The set of asymptotically stable periodic points for a mapping is not necessarily finite, even if the periods are bounded or the set of asymptotically stable periodic points has a compact closure in X . The set consisting of the asymptotically stable periodic points for f and the possible present absorbing boundary points is finite, because (1) if the set of asymptotically stable periodic points is not empty then it is finite since it is discrete and it is assumed to be compact and (2) there are at most two absorbing boundary points. Consequently, the smallest common multiple of the primitive periods of the asymptotically stable periodic points is well defined, provided that f has at least one asymptotically stable periodic point.

We will study the set of points in X , whose orbits do not converge to an asymptotically stable periodic orbit of f , or to an (or the) absorbing boundary point(s) of X for f . In other words, we will study the complement of the union of the domains of attraction of the asymptotically stable periodic points for f and the domain(s) of attraction of the absorbing boundary point(s) of X for f . We define:

- (4.2) Let D_0 be the union of the direct domains of attraction of all asymptotically stable periodic points for f and the absorbing boundary points of X for f . We write A_0 for X and we write A_1 for the complement of D_0 in X . We define by induction $A_{k+1} = \{x \in A_k; f^k(x) \in A_1\}$ for each positive integer k . For each $k \in \mathbb{N}$ we define $D_k = \{x \in A_k; f^k(x) \in D_0\}$. Finally we set $A_\infty(f) = \bigcap_{k=0}^\infty A_k$.

We note that $A_\infty(f)$ includes the aperiodic points for f , and that the sets $A_\infty(f)$ and $X \setminus A_\infty(f)$ are completely f -invariant sets. If X is compact, and f is a chaotic mapping then $A_\infty(f)$ is a nonempty compact set.

Now we will investigate some properties of the defined sets $A_\infty(f)$ and D_k for $k \in \mathbb{N} \cup \{0\}$.

LEMMA 4.3. For any nonnegative integer k , we have

- (i) $A_k = A_{k+1} \sqcup D_k$,
- (ii) $A_0 = A_{k+1} \sqcup \bigsqcup_{j=0}^k D_j$.

Proof. Let k be any fixed nonnegative integer. Then

$$\begin{aligned} A_{k+1} &= \{x \in A_k; f^k(x) \in A_1\} \\ &= \{x \in A_0; f^j(x) \in A_1 \text{ for all } j, 0 \leq j \leq k\}, \end{aligned}$$

i.e., A_{k+1} is the set of points, which will be mapped into A_1 under the map f^k .

$$\begin{aligned} D_k &= \{x \in A_k; f^k(x) \in D_0\} \\ &= \{x \in A_0; f^j(x) \notin D_0 \text{ for all } j, 0 \leq j \leq k-1, f^k(x) \in D_0\}, \end{aligned}$$

i.e., D_k is the set of points in A_k that will be mapped into D_0 under the map f^k .

- (i) $A_{k+1} \cap D_k = \emptyset$, trivial. Splitting the set A_k in the following way:

$$A_k = \{x \in A_k; f^k(x) \in A_1\} \cup \{x \in A_k; f^k(x) \in D_0\},$$

we get $A_k = A_{k+1} \cup D_k$.

(ii) $A_{k+1} \cap (\bigcup_{j=0}^k D_j) = \emptyset$, trivial. Splitting the set A_0 as follows:

$$A_0 = \{x \in A_0; f^k(x) \in A_1\} \cup \{x \in A_0; f^k(x) \in A_0 \setminus A_1\},$$

we obtain

$$\begin{aligned} A_0 &= A_{k+1} \cup D_0 \cup \bigcup_{i=1}^k \{x \in A_0; f^i(x) \in A_1 \text{ for all } j, 0 \leq j \leq i-1, f^i(x) \in D_0\} \\ &= A_{k+1} \cup D_0 \cup \bigcup_{i=1}^k \{x \in A_i; f^i(x) \in D_0\} = A_{k+1} \cup \bigcup_{i=0}^k D_i. \end{aligned} \quad \square$$

LEMMA 4.4. For any positive integer k and any nonnegative integer n , we have:

(i) $f^k(A_{n+k}) = A_n \cap f(X)$,

(ii) $f^k(D_{n+k}) = D_n \cap f(X)$.

Proof. Fix any n, k as in the lemma.

(i) By definition we have $A_{n+k} = \{x \in A_{n+k-1}; f^{n+k-1}(x) \in A_1\}$. It follows that $A_{n+k} = \{x \in A_n; f^k(x) \in A_n\}$. Since $f^k(A_{n+k}) \subset f(X)$ we obtain $f^k(A_{n+k}) = A_n \cap f(X)$.

(ii) By definition we have $D_{n+k} = \{x \in A_{n+k}; f^{n+k}(x) \in D_0\}$. It follows that $D_{n+k} = \{x \in A_n; f^k(x) \in A_n \text{ and } f^n(f^k(x)) \in D_0\}$. Since $f^k(D_{n+k}) \subset f(X)$, we get $f^k(D_{n+k}) = D_n \cap f(X)$. \square

Note that, for each positive integer k , the set A_{k+1} is the pre-image of A_1 under f^k , but the set D_k is a subset of the pre-image of D_0 under f^k .

LEMMA 4.5. There exists a positive integer R such that A_R is compact.

Proof. Evidently, there exist $m \in \mathbb{N}$ such that $U_{\text{Bd}(X)} \subset \bigcup_{k=0}^m D_k$. Then the set A_{m+1} is bounded. By definition, the set A_{m+1} is closed. Hence, A_R is compact with $R = m+1$. \square

LEMMA 4.6. Let n be any fixed positive integer. Let S be a component of A_n . We assume that the interior of S is nonempty. For any $k \in \mathbb{N}$ we have

$$\text{If } S \cap \bigcup_{j=0}^{k-1} D_{n+j} = \emptyset, \text{ then } S \text{ is a component of } A_{n+k}.$$

Proof. Fix any $n \in \mathbb{N}$. Assume that S is a nontrivial component of A_n . Recall from Lemma 4.3 the following properties: $A_0 = A_{m+1} \sqcup \bigsqcup_{j=0}^m D_j$ and $A_m = A_{m+1} \sqcup D_m$ for each $m \in \mathbb{N} \cup \{0\}$. Obviously, we have $A_n = A_{n+k} \sqcup \bigsqcup_{j=0}^{k-1} D_{n+j}$ for every $k \in \mathbb{N}$. Assume that $k \in \mathbb{N}$ for which hold that $S \cap \bigsqcup_{j=0}^{k-1} D_{n+j} = \emptyset$. Then S is a component of A_{n+k} , since it is a nontrivial component of A_n . \square

LEMMA 4.7. Let S be a nontrivial component of A_n , for some fixed $n \in \mathbb{N}$, such that $S \cap U_{\text{Bd}(X)} = \emptyset$. Assume that there exists a nonnegative integer k such that $S \cap D_{n+k} \neq \emptyset$ and $S \cap \bigsqcup_{j=0}^k D_{n+j-1} = \emptyset$. Then S is a component of A_{n+k} and there exists a component D of D_{n+k} such that $D \subset \text{Int}(S)$.

Proof. Fix any $n \in \mathbb{N}$. Let S be a nontrivial component of A_n , such that $S \cap U_{\text{Bd}(X)} = \emptyset$. Then the boundary points of S are pre-images of unstable periodic points for f . In other words, the boundary points of S are eventually periodic points for f . Hence $\text{Bd}(S) \cap X \setminus A_\infty(f) = \emptyset$.

Let $k \in \mathbb{N} \cup \{0\}$ be given such that $S \cap \bigsqcup_{j=0}^k D_{n+j-1} = \emptyset$ and $S \cap D_{n+k} \neq \emptyset$. By Lemma 4.6 we have that S is a component of A_{n+k} . Since $A_{n+k} = A_{n+k+1} \sqcup D_{n+k}$, we have that S is not a component of A_{n+k+1} .

We conclude that S is a component of A_{n+k} and that $D \subset \text{Int}(S)$ for some component D of D_{n+k} . \square

LEMMA 4.8. For any positive integer k we have:

(i) $\text{Bd}(D_k) = \{x \in A_k; f^k(x) \in \text{Bd}(D_0)\}$,

- (ii) $\text{Bd}(A_{k+1}) = \text{Bd}(A_k) \cup \text{Bd}(D_k)$,
- (iii) $\text{Bd}(A_{k+1}) = \bigcup_{j=0}^k \text{Bd}(D_j)$,
- (iv) $\text{Bd}(A_k)$ is positively f -invariant.

Proof. Apply Lemmas 4.3, 4.4, 4.6 and 4.7. \square

Remark 4.9. (i) Assume that f has at least one asymptotically stable periodic point. Let the integer p be the smallest common multiple of the periods of the asymptotically stable periodic points for f . For any component S of A_n for $n \geq R$ with R as in Lemma 4.5 we have that each boundary point of S is an eventually periodic point for f with (eventual) period $2p$.

(ii) If f has no asymptotically stable periodic points, then each boundary point of such a set S as in (i) is eventually periodic with period two.

In order to be able to determine the Lebesgue measure of the complement of the union of the domains of attraction of the asymptotically stable periodic points for f and the absorbing boundary points of X for f , we give the following definition.

DEFINITION 4.10. The mapping f is called *eventually expanding*, if we can find positive integers N and M such that $|(f^N)'(x)| > 1$ for all $x \in A_M$.

We first formulate a lemma which says that the property “eventually expanding” is invariant under conjugation with a diffeomorphism.

LEMMA 4.11. Let $h : X \rightarrow \mathbb{R}$ be a diffeomorphism. If f is eventually expanding, then the mapping $g : h(X) \rightarrow \mathbb{R}$ defined by $g(x) = h \circ f \circ h^{-1}(x)$ is eventually expanding.

Proof. Assume that $h : X \rightarrow \mathbb{R}$ is any fixed diffeomorphism, and that f is eventually expanding. Let the positive integers N and M be as in Definition 4.10. By Lemma 4.5 let k be the smallest nonnegative integer such that A_{M+k} is compact. We put $K = \min \{|f^N)'(x)|; x \in A_{M+k}\}$, $c_1 = \min \{|h'(x)|; x \in A_{M+k}\}$ and $c_2 = \min \{|h^{-1}'(x)|; x \in h(A_{M+k})\}$. Select the smallest $s \in \mathbb{N}$ such that $c_1 K^s c_2 > 1$. For each $x \in h(A_{M+k})$ we now get, using the chain rule:

$$\begin{aligned} |(g^{sN})'(x)| &= |((h \circ f \circ h^{-1})^{sN})'(x)| = |(h \circ f^{sN} \circ h^{-1})'(x)| \\ &= |h'(f^{sN} \circ h^{-1}(x))| \cdot \prod_{j=0}^{s-1} |(f^N)'(f^{jN} \circ h^{-1}(x))| \cdot |(h^{-1})'(x)| \\ &\geq c_1 K^s c_2 > 1. \end{aligned}$$

Hence g is eventually expanding. \square

We will give a second definition of an expanding mapping on suitable subsets of X . This definition is similar to one commonly used for expanding diffeomorphisms in the theory of dynamical systems. Recall that a subset Λ of X is called positively f -invariant if $f(\Lambda) \subset \Lambda$.

DEFINITION 4.12. Assume that $\Lambda \subset X$ is a closed positively f -invariant set. f is called an *expanding map* on Λ , if we can find a constant $C > 0$ and a constant $K > 1$ such that $|(f^n)'(x)| \geq C \cdot K^n$ for every positive integer n and each point $x \in \Lambda$.

In spite of the fact that the set A_M in Definition 4.10 is not a positively f -invariant set, we will show that Definitions 4.10 and 4.12 have something to do with each other by taking $\Lambda = A_\infty(f)$ in Definition 4.12. Before we can do this, we have to prove that $A_\infty(f)$ is a compact positively f -invariant set.

LEMMA 4.13. $A_\infty(f)$ is a compact positively f -invariant set.

Proof. The positive invariance follows from the definition (4.2) of $A_\infty(f)$; the compactness follows from Lemma 4.5. \square

Note that $A_\infty(f)$ is a compact (completely) f -invariant set.

LEMMA 4.14. The following conditions are equivalent:

- (i) f is an expanding map on $A_\infty(f)$.

(ii) f is eventually expanding.

Proof. (i) \Rightarrow (ii): Assume that f is an expanding map on $A_\infty(f)$. Select constants C and K as in Definition 4.12. Let N be a positive integer satisfying $C \cdot K^N > 1$; assume N is minimal. We write U_N for an open neighbourhood of $A_\infty(f)$ such that $|(f^N)'(x)| > 1$ for all $x \in U_N$. Choose the positive integer M minimally such that $A_M \subset U_N$, which exists because the A_k form, from a certain nonnegative integer, a decreasing sequence of compact sets. We conclude that f is eventually expanding.

(ii) \Rightarrow (i): Assume that f is eventually expanding. Using (4.1) (iii) and Definition 4.10, we find positive integers N and M such that A_M is closed and $|(f^N)'(x)| > 1$ for all $x \in A_M$. We write $c = \min \{|f'(x)|; x \in A_M\}$ and $K_0 = \min \{|(f^N)'(x)|; x \in A_M\}$. Note that $c > 0$ and $K_0 > 1$. If $c > 1$ it is obvious that f is an expanding map on $A_\infty(f)$. So we assume from now on $c \leq 1$.

Let a positive integer n be given. We write $n = sN + t$ for some nonnegative integers s, t with $0 \leq t \leq N - 1$. Recalling that $A_\infty(f)$ is a positively f -invariant set, we have for each $x \in A_\infty(f)$:

$$|(f^n)'(x)| \geq K_0^s c^t \geq [K_0^{1/N}]^{sN} \cdot c^N \geq [K_0^{1/N}]^n \left[\frac{c}{K_0} \right]^N.$$

Choosing $C = [c/K_0]^N$ and $K = K_0^{1/N}$, we get the result that f is an expanding map on $A_\infty(f)$. \square

B. Let $f: X \rightarrow X$ be a chaotic mapping of class C^3 from an interval X into itself. In this subsection we assume that f has a nonpositive Schwarzian derivative Sf , i.e.,

$$Sf(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2} \left[\frac{f''(x)}{f'(x)} \right]^2 \leq 0 \quad \text{for all } x \in X \setminus \text{Crit}(f).$$

From results due to Singer [22] (see also Collet and Eckmann [4]) we have:

- (i) The n th iterate f^n of f has a nonpositive Schwarzian derivative, for each $n \in \mathbb{N}$;
- (ii) If the Schwarzian derivative of f is negative, then $|f'|$ has no positive local minima in the interior of X (this follows from the fact that f' and f''' must have opposite signs at a critical point of f'). This second result also holds for maps with nonpositive Schwarzian derivative.

LEMMA 4.15. Assume that $Y \subset X$ is a nontrivial closed interval such that $\text{Int}(Y) \cap \text{Crit}(f) = \emptyset$. Then $|f'|$ assumes its minimum value at a point of the boundary of Y .

Proof. Let Y be as in the lemma. We have either $f'(x) > 0$ for all $x \in \text{Int}(Y)$ or $f'(x) < 0$ for all $x \in \text{Int}(Y)$.

First we consider the case $f'(x) > 0$ for all $x \in \text{Int}(Y)$. We write $m = \min \{f'(x); x \in Y\}$. Suppose that $f'(z) > m$ for each $z \in \text{Bd}(Y)$. Let $[a, b]$ be a component of the set $\{x \in Y: f'(x) = m\}$.

Let $\varepsilon > 0$ be such that $[a - \varepsilon, b + \varepsilon] \subset \text{Int}(Y)$, and $m < f'(x) < f'(z)$ and $f''(x) \neq 0$ for all $x \in [a - \varepsilon, a] \cup [b, b + \varepsilon]$, $z \in \text{Bd}(Y)$. Such an ε exists, because otherwise $[a, b]$ would not be a component of $\{x \in Y: f'(x) = m\}$.

Let α_1 be the slope of the straight line l_1 through the points $(a - \varepsilon, f(a - \varepsilon))$ and $((a + b)/2, f((a + b)/2))$; let α_2 be the slope of the line l_2 through the points $((a + b)/2, f((a + b)/2))$ and $(b + \varepsilon, f(b + \varepsilon))$. We write $\alpha = \min \{\alpha_1, \alpha_2\}$. Let $(x_1, f(x_1))$ and $(x_4, f(x_4))$ with $a - \varepsilon \leq x_1 < x_4 \leq b + \varepsilon$ be the intersection points of l_α with the graph of f with $l_\alpha = l_1$ if $\alpha = \alpha_1$ and $l_\alpha = l_2$ if $\alpha = \alpha_2$.

Let $\delta > 0$ be such that $m + 2\delta \leq \alpha$. We write l_3 for the line through $((a + b)/2, f((a + b)/2))$ with slope $m + \delta$. Let $(x_2, f(x_2))$ and $(x_3, f(x_3))$ with $x_1 < x_2 < x_3 < x_4$ be the intersection points of l_3 with the graph of f .

From the construction above we have:

$$\frac{f(x_4) - f(x_1)}{x_4 - x_1} = \alpha, \quad \frac{f(x_3) - f(x_2)}{x_3 - x_2} = m + \delta, \quad \frac{f(x_4) - f(x_3)}{x_4 - x_3} > \alpha \quad \text{and} \quad \frac{f(x_2) - f(x_1)}{x_2 - x_1} > \alpha.$$

Since $m + \delta < m + 2\delta \leq \alpha$ we obtain

$$\frac{f(x_4) - f(x_1)}{x_4 - x_1} \cdot \frac{f(x_3) - f(x_2)}{x_3 - x_2} \cdot \frac{f(x_4) - f(x_3)}{x_4 - x_3} \cdot \frac{f(x_2) - f(x_1)}{x_2 - x_1} < \alpha(m + \delta) - \alpha \cdot \alpha < \alpha^2 - \alpha^2 = 0;$$

this implies

$$(*) \quad \frac{(x_4 - x_1)(x_3 - x_2)}{(x_4 - x_3)(x_2 - x_1)} > \frac{f(x_4) - f(x_1)}{f(x_4) - f(x_3)} \cdot \frac{f(x_3) - f(x_2)}{f(x_2) - f(x_1)}.$$

Following Allwright [1] (see also Collet and Eckmann [4] and Preston [21]) we have, since $Sf(x) \leq 0$ for all $x \in \text{Int}(Y)$,

$$(**) \quad \frac{(y_4 - y_1)(y_3 - y_2)}{(y_4 - y_3)(y_2 - y_1)} \leq \frac{f(y_4) - f(y_1)}{f(y_4) - f(y_3)} \cdot \frac{f(y_3) - f(y_2)}{f(y_2) - f(y_1)}$$

for all $y_1, y_2, y_3, y_4 \in \text{Int}(Y)$ with $y_1 < y_2 < y_3 < y_4$. Since (*) contradicts (**), we conclude that f' assumes its minimum value at $\text{Bd}(Y)$.

Now we consider the case $f'(x) < 0$ for all $x \in \text{Int}(Y)$. Because $Sf(x) = Sg(x)$ for all $x \in \text{Int}(Y)$ with $g = -f$, we get that f' assumes its maximum value at $\text{Bd}(Y)$ using the obtained result for $-f'$.

Conclusion: $|f'|$ assumes its minimum value at the boundary of Y . \square

The direct domain of attraction of each (one-sided) asymptotically stable periodic orbit of f with period ≥ 3 contains at least one critical point of the map f . This property follows from the next lemma.

LEMMA 4.16. *For each (one-sided) asymptotically stable periodic point q for f with $\inf\{x: x \in \text{Per}(f)\} < q < \sup\{x: x \in \text{Per}(f)\}$ we have: the direct domain of attraction of the orbit of q contains at least one critical point of f .*

Proof. If f has a negative Schwarzian derivative, then it is the result due to Singer [22]. In the case that $Sf(x) = 0$ for at least one x in $X \setminus \text{Crit}(f)$, combine Singer's proof and Lemma 4.15. \square

COROLLARY 4.17. *Let q be an (one-sided) asymptotically stable periodic point for f with period $p(q)$.*

(i) *If the direct domain of attraction of the orbit of q contains no critical point of f , then $p(q) = 1$ or $p(q) = 2$; moreover q is one of the end-points of the set of periodic points for f .*

(ii) *Fix some positive integer N . If f has N critical points, then f has at most $N + 2$ (one-sided) asymptotically stable periodic orbits.*

Proof. The proof is left to the reader. \square

Remark. If the orbit of each critical point of f converges to an asymptotically stable periodic orbit of f or to the possible present absorbing boundary of X for f , then it follows from Corollary 4.17 that f has at most two one-sided asymptotically stable periodic points.

LEMMA 4.18. *For each periodic point q for f with period $p(q) \geq 3$ and $(f^{2p(q)})'(q) = 1$ we have: either q is one-sided asymptotically stable or q is asymptotically stable.*

Proof. Let q be as in the lemma. Suppose that q is neither one-sided asymptotically stable nor asymptotically stable. Then $(f^{2p(q)})'$ does not assume its minimum value at the boundary of small interval around q . This contradicts Lemma 4.15. \square

5. Proof of Theorem A. Since the proof of the theorem is quite long, we will do it in several steps. Let f be a chaotic map that satisfies conditions (4.1). Assuming that f is an expanding map on $A_\infty(f)$, we now can formulate some lemmas that lead to the result that the Lebesgue measure of the set $A_\infty(f)$ is zero.

First, we will show that for each nonnegative integer n , we may write D_n as a disjoint union of finitely many intervals, that are open in X , and A_n as a disjoint union of finitely many (trivial or nontrivial) intervals that are closed in X .

LEMMA 5.1. *Assume that f is eventually expanding. For any nonnegative integer n , there exist nonnegative integers $N(D_n)$, $N(A_n)$ and $T(A_n)$ such that the set D_n consists of $N(D_n)$ components and the set A_n consists of the union of $N(A_n)$ nontrivial components and $T(A_n)$ trivial components (points).*

Proof. We assume that f is eventually expanding. Since f is chaotic, we have that $\text{Int}(A_n) \neq \emptyset$ for each $n \in \mathbb{N}$ and $\text{Crit}(f) \neq \emptyset$.

We write $U_{\text{Bd}(X)}$ for the open neighbourhood of the boundary of X as in (4.1) (iii). Assume that m is a nonnegative integer such that $U_{\text{Bd}(X)} \subset \cup_{i=0}^m D_i$. We fix positive integers N and M as in Definition 4.10. It is no restriction to assume that $M \geq m + 1$.

Write $U = \{D; D \text{ component of } \sqcup_{i=0}^{M-1} D_i, D \cap \text{Crit}(f) \neq \emptyset\}$. The set U consists of finitely many components of the set $\sqcup_{i=0}^{M-1} D_i$, since $\text{Crit}(f) \cap X \setminus U_{\text{Bd}(X)}$ is compact and $\text{Crit}(f) \neq \emptyset$. Let $N(U)$ denote the number of elements of U .

Let n be any nonnegative integer. Let $N(D_0)$ denote the number of components of the set D_0 . Recall that D_0 has finitely many components, and that f is a monotone mapping on the components of $X \setminus U$. Using that $f(D_{n+1}) = D_n \cap f(X)$ (see Lemma 4.4), one obtains that $(N(U) + 1)^n \cdot N(D_n)$ an upper bound for the number of components of the set D_n . Consequently, the set A_n has finitely many components. \square

According to Lemma 5.1, we write for each nonnegative integer n :

$$(5.2) \quad D_n = \bigsqcup_{i=1}^{N(D_n)} D_{n;i}, \quad A_n = \bigsqcup_{i=1}^{N(A_n)} A_{n;i} \sqcup W(A_n) \quad \text{with } W(A_n)$$

is the set consisting of the $T(A_n)$ trivial components of A_n .

Further, we write

(5.3) The number p for the smallest common multiple of the periods of the asymptotically stable periodic points for f multiplied by two, provided that f has at least one asymptotically stable periodic point, and $p = 2$ if f has no asymptotically stable periodic points.

In the next lemma we prove that there exists a positive integer P such that, for each positive integer n that is sufficiently large, we have the property: “if A is a nontrivial component of A_n , then A is not a component of A_{n+P} ”; or putting this in another way: “if A is a nontrivial component of A_n , then $A \cap \cup_{i=0}^{n+P} D_i \neq \emptyset$.”

LEMMA 5.4. *Let f be eventually expanding, and let M be as in Definition 4.10 and p as in (5.3). Then for each integer $n \geq M$, and for each nontrivial component A of A_n , there exists an integer j , $0 \leq j \leq p$ and a component D of D_{n+j} such that $D \subset A$.*

Proof. Let the mapping f and the integers M and p be as in the lemma. Let the integer N be as in Definition 4.10. Fix any integer $n \geq M$. Pick some nontrivial component S of A_n . It is no restriction to assume that $S \cap U_{\text{Bd}(X)} = \emptyset$, where $U_{\text{Bd}(X)}$ denotes the open neighbourhood of the boundary of X as in (4.1)(iii).

There exists an integer i , $0 \leq i \leq n$, and a point $x_i \in \text{Bd}(D_0)$ such that $x_i \in \text{Bd}(f^i(S))$. Choose i minimal with this property. Assume that $u \in \text{Bd}(S)$ satisfies $f^i(u) = x_i$. Let $v \in \text{Bd}(S)$ be such that $v \neq u$.

We write $f^i(S) = [\tilde{\delta}, x_i]$ where $\delta = f^i(v)$. Recall that f^p is a monotone mapping on a neighbourhood of x_i and from Remark 4.9 that $f^p(f^p(x_i)) = f^p(x_i)$. For each positive integer m , there exists a one-sided neighbourhood of x_i , say $U_m(x_i)$, such that $f^r(x) \in A_1$ for all $x \in U_m(x_i)$, for all $0 \leq r \leq m$; consequently $f^i(v) \neq x_i$.

By a similar argument as above there exists an integer k , $i \leq k \leq n$ (because i was chosen minimally), and a point $y_k \in \text{Bd}(D_0)$ such that $f^k(v) = y_k$. Note that $x_i \neq y_k$.

For real numbers α, β we write $[\alpha, \beta] = [\alpha, \beta]$ if $\alpha \leq \beta$ and $[\alpha, \beta] = [\beta, \alpha]$ if $\alpha \geq \beta$.

Suppose that $f^j([\tilde{y}_k, f^{k-i}(x_i)]) \cap D_0 = \emptyset$ for all integers j , $0 \leq j \leq p$ then $f^j([\tilde{y}_k, f^{k-i}(x_i)]) \cap \bigsqcup_{i=0}^M D_i = \emptyset$ for all nonnegative integers j . In particular we now have

- (i) $f^p(f^p(y_k)) = f^p(y_k)$;
- (ii) $f^p(f^{p+k-i}(x_i)) = f^{p+k-i}(x_i)$;
- (iii) f^p restricted to $[\tilde{y}_k, f^{p+k-i}(x_i)]$ is strictly monotone.

This implies $f^{2p+k}(S) \cap D_0 \neq \emptyset$; and this contradicts the above result that $f^{j+k}(S) \cap D_0 = \emptyset$ for all nonnegative integers j . We conclude that there exists an integer j , $0 \leq j \leq p$ such that $f^j([\tilde{y}_k, f^{k-i}(x_i)]) \cap D_0 \neq \emptyset$.

From the facts $y_k \in \text{Bd}(D_0)$, $y_k \notin D_0$, $f^{k-i}(x_i) \in \text{Bd}(D_0)$, and $f^{k-i}(x_i) \notin D_0$ we conclude that there is a component of D_0 , say D , such that $D \subset f^j([\tilde{y}_k, f^{k-i}(x_i)])$ for some integer j , $0 \leq j \leq p$. \square

Assume that D and A are as in Lemma 5.4. If we could show that $\mu(D)/\mu(A) \geq \varepsilon$ for some fixed $\varepsilon > 0$, then from the properties (i) $\mu(A_n) \leq \varepsilon^{-1} \cdot \sum_{m=n}^{n+p} \mu(D_m)$ and (ii) $\lim_{n \rightarrow \infty} \sum_{m=n}^{n+p} \mu(D_m) = 0$ (see proof of Lemma 5.6) the desired result follows. In the lemma that is stated below, we show that there exists such a uniformly defined lower bound.

LEMMA 5.5. *If f is eventually expanding, then we can find a number $\varepsilon > 0$ such that for each nonnegative integer n , for each nontrivial component A of A_n , and any component D of D_n satisfying $D \subset A$, we have $\mu(D)/\mu(A) \geq \varepsilon$.*

Proof. Assume that f is eventually expanding. Let the integers N and M be as in Definition 4.10; using (4.1)(iii) without any restriction, we may assume that the set A_M is compact. We set $K = \min \{|(f^N)'(x)|; x \in A_M\}$. Note that $K > 1$.

Let the integer $n > M$ be fixed. By Lemmas 4.6 and 5.4 we can fix an integer $l(n)$, $1 \leq l(n) \leq N(D_n)$, and an integer $k(n)$, $1 \leq k(n) \leq N(A_n)$, such that $D_{n;l(n)} \subset A_{n;k(n)}$, where $N(D_n)$ and $N(A_n)$ are as in (5.2).

Set for each integer j , $M + 1 \leq j \leq n$:

$$A_{j-1;k(j-1)} = f(A_{j;k(j)}) \quad \text{and} \quad D_{j-1;l(j-1)} = f(D_{j;l(j)}).$$

Applying the mean value theorem, we can find for each integer j , $M + 1 \leq j \leq n$, real numbers $a_j \in A_{j;k(j)}$ and $d_j \in D_{j;l(j)}$, such that

$$\begin{aligned} |f'(a_j)| \cdot \mu(A_{j;k(j)}) &= \mu(A_{j-1;k(j-1)}), \\ |f'(d_j)| \cdot \mu(D_{j;l(j)}) &= \mu(D_{j-1;l(j-1)}). \end{aligned}$$

This leads to

$$(5.5.1) \quad \frac{\mu(D_{n;l(n)})}{\mu(A_{n;k(n)})} = \prod_{j=M+1}^n \left| \frac{f'(a_j)}{f'(d_j)} \right| \cdot \frac{\mu(D_{M;l(M)})}{\mu(A_{M;k(M)})}$$

or

$$(5.5.1') \quad \frac{\mu(D_{n;l(n)})}{\mu(A_{n;k(n)})} = \prod_{j=M+1}^n \left| 1 - \left\{ 1 - \frac{|f'(a_j)|}{|f'(d_j)|} \right\} \right| \cdot \frac{\mu(D_{M;l(M)})}{\mu(A_{M;k(M)})}.$$

Let $\alpha > 0$ be fixed. Assume that the mapping f' is Hölder continuous with positive constant H and positive exponent α . We get for $M + 1 \leq j \leq n$

$$(5.5.2) \quad \left| 1 - \frac{f'(a_j)}{f'(d_j)} \right| = \left| \frac{f'(d_j) - f'(a_j)}{f'(d_j)} \right| \leq \frac{H|d_j - a_j|^\alpha}{\min \{|f'(x)|; x \in A_j\}} \leq \frac{H \cdot [\mu(A_{j;k(j)})]^\alpha}{\min \{|f'(x)|; x \in A_M\}}$$

From the property $|(f^N)'(x)| \geq K$ for all $x \in A_M$ it follows that

$$(5.5.3) \quad \mu(A_{n;k(n)}) \leq K^{1-(n-M)/N} \cdot \mu(A_M).$$

Using (5.5.2) and (5.5.3), we obtain

$$(5.5.4) \quad \begin{aligned} \left| 1 - \frac{f'(a_n)}{f'(d_n)} \right| &\leq \frac{H}{\min \{|f'(x)|; x \in A_M\}} \cdot [\mu(A_{n;k(n)})]^\alpha \\ &\leq \frac{H}{\min \{|f'(x)|; x \in A_M\}} [\mu(A_M)]^\alpha \cdot K^{\alpha(1-(n-M)/N)}. \end{aligned}$$

We write

$$K^\alpha = \tilde{K}, \frac{H}{\min \{|f'(x)|; x \in A_M\}} \cdot [\mu(A_M)]^\alpha = \tilde{C},$$

and $M = sN + t$ for some nonnegative integers $s, t, 0 \leq t \leq N - 1$. We assume that $\beta \geq s + 2$; then using (5.5.4), we get

$$\begin{aligned} \sum_{j=(s+1)N+1}^{\beta N} \left| 1 - \frac{f'(a_j)}{f'(d_j)} \right| &= \sum_{j=s+1}^{\beta-1} \sum_{i=1}^N \left| 1 - \frac{f'(a_{jN+i})}{f'(d_{jN+i})} \right| \leq \sum_{j=s+1}^{\beta-1} \sum_{i=1}^N \tilde{C} \tilde{K}^{1-(jN+i-M)/N} \\ &\leq N \tilde{C} \sum_{j=s+1}^{\beta-1} \tilde{K}^{2+s-j} < N \tilde{C} \tilde{K} \sum_{j=s+1}^{\infty} \tilde{K}^{s+1-j} = \frac{N \tilde{C} \tilde{K}^2}{\tilde{K} - 1}. \end{aligned}$$

Conclusion:

$$(5.5.5) \quad \text{The series } \sum_{j=M+1}^{\infty} \left| 1 - \frac{f'(a_j)}{f'(d_j)} \right| \text{ is bounded.}$$

Choose an integer $L > M$ such that $\tilde{C} \tilde{K}^{2-(L-M)/N} \leq 1$. From (5.5.4) we conclude:

$$(5.5.6) \quad \left| 1 - \frac{f'(a_m)}{f'(d_m)} \right| \leq \tilde{K}^{-1} \quad \text{for each integer } m \geq L.$$

Put $p_m = |1 - f'(a_m)/f'(d_m)|$ for each integer $m \geq L$. Note that $0 \leq p_m < 1$ for $m \geq L$. For each integer $m \geq L$ we have:

$$(5.5.7) \quad 0 \leq |p_m - \log(1 - p_m)| = \sum_{k=2}^{\infty} \frac{(p_m)^k}{k} < p_m \sum_{k=1}^{\infty} (p_m)^k \leq p_m \sum_{k=1}^{\infty} \tilde{K}^{-k} = \frac{p_m}{\tilde{K} - 1}.$$

Using (5.5.5) and (5.5.7), we have $\sum_{m=L}^{\infty} |p_m - \log(1 - p_m)|$ is a convergent series. Now we apply (5.5.5) again and we get that the series $\sum_{m=L}^{\infty} \log(1 - p_m)$ is convergent. This implies $\log \prod_{m=L}^{\infty} (1 - p_m)$ is convergent; consequently $\prod_{m=L}^{\infty} (1 - p_m) > 0$. We conclude:

$$(5.5.8) \quad \lim_{m \rightarrow \infty} \prod_{j=M+1}^m \left| \frac{f'(a_j)}{f'(d_j)} \right| > 0.$$

Thus from (5.5.1) and (5.5.8) we have the following:

$$(5.5.9) \quad \text{There exists a real number } \varepsilon > 0 \text{ such that for all integers } m \geq M + 1, \text{ we have } \mu(D)/\mu(A) \geq \varepsilon \text{ for all components } A \subset A_m, D \subset D_m \text{ with } D \subset A. \quad \square$$

LEMMA 5.6. *If f is eventually expanding, then $\lim_{n \rightarrow \infty} \mu(A_n) = 0$.*

Proof. We assume that f is eventually expanding. Let the integers N and M be as in Definition 4.10 such that A_M is compact. We put $\varepsilon_0 > 0$ as in (5.5.9).

Using Lemma 5.4 and (5.5.9), we get

$$(5.6.1) \quad \mu(A_n) \leq \varepsilon_0^{-1} \sum_{m=n}^{n+p} \mu(D_m) \quad \text{for each integer } n \geq M.$$

By (4.3)(ii) we obtain for each positive integer n :

$$A_0 = A_n \sqcup \bigsqcup_{i=0}^{n-1} D_i \quad \text{and} \quad A_0 = A_{n+p+1} \sqcup \bigsqcup_{i=0}^{n+p} D_i.$$

Consequently

$$A_n \sqcup \bigsqcup_{i=0}^{n-1} D_i = A_{n+p+1} \sqcup \bigsqcup_{i=0}^{n+p} D_i \quad \text{for each } n \in \mathbb{N}.$$

We conclude that

$$(5.6.2) \quad A_n = A_{n+p+1} \sqcup \bigsqcup_{i=n}^{n+p} D_i, \quad \text{for each positive integer } n.$$

The sequence $\{\mu(A_n)\}_{n=1}^\infty$ is monotone decreasing and bounded. So $\lim_{n \rightarrow \infty} \mu(A_n)$ exists; call it γ . Then, for each fixed integer k , we have: $\lim_{n \rightarrow \infty} \mu(A_n) = \gamma = \lim_{n \rightarrow \infty} \mu(A_{n+k})$. From (5.6.2) we get $\mu(A_n) = \mu(A_{n+p+1}) + \sum_{i=n}^{n+p} \mu(D_i)$ for each $n \in \mathbb{N}$. Hence

$$(5.6.3) \quad \lim_{n \rightarrow \infty} \sum_{i=n}^{n+p} \mu(D_i) = 0.$$

Now we use (5.6.1) and (5.6.3); this gives

$$0 \leq \gamma = \lim_{n \rightarrow \infty} \mu(A_n) \leq \lim_{n \rightarrow \infty} \left\{ \varepsilon_0^{-1} \sum_{i=n}^{n+p} \mu(D_i) \right\} = 0.$$

Conclusion: $\lim_{n \rightarrow \infty} \mu(A_n) = 0$. \square

Proof of Theorem A: (1) We assume that $X \subset \mathbb{R}$ is a nontrivial interval. Fix some positive real number α . Let $f \in C^{1+\alpha}(X, X)$ satisfy the properties (i) and (ii) of the theorem.

For each $n \in \mathbb{N}$ we have that the compact set $A_\infty(f)$ is a subset of A_n , with $A_\infty(f)$ and A_n as in (4.2).

We write $x_0 = \min \{x \in X; x \in A_\infty(f)\}$, $y_0 = \max \{x \in X; x \in A_\infty(f)\}$. If $\text{Crit}(f) \cap [x_0, y_0] = \emptyset$, then it is clear that $\mu(A_\infty(f)) = 0$. From now on we assume that $\text{Crit}(f) \cap [x_0, y_0] \neq \emptyset$, and that $x_0 \neq y_0$.

We consider the following cases:

- (a) $\text{Bd}(X) \cap A_\infty(f) = \emptyset$;
- (b) $\text{Bd}(X) \cap A_\infty(f) = \{x_0\}$;
- (c) $\text{Bd}(X) \cap A_\infty(f) = \{y_0\}$;
- (d) $\text{Bd}(X) \cap A_\infty(f) = \{x_0, y_0\}$.

Case (a): Assume that $\text{Bd}(X) \cap A_\infty(f) = \emptyset$; then we have $A_\infty(f) \subset \text{Int}(X)$. Applying (4.1), Lemma 4.14 and Lemma 5.6, we obtain the result.

Case (b): Assume that $\text{Bd}(X) \cap [x_0, y_0] = \{x_0\}$. We define $u_1 = \min \{x \in [x_0, y_0]; x \in \text{Crit}(f)\}$, $u_2 = \min \{f(x); u_1 \leq x \leq y_0\}$, and $u_3 = \min \{x \in [x_0, y_0]; x \text{ is asymptotically stable periodic point of } f\}$ provided that f has at least one asymptotically stable periodic point; further we write $u_m = \min \{u_1, u_2, u_3\}$. Note that $u_m > x_0$ and that $f(u_m) \geq u_m$. The set Y_1 , defined $Y_1 = \{x \in X; x \geq u_m\}$ is a positively f -invariant set. Moreover, we have that the set $A_\infty(f) \cap Y_1$ is a subset of $\text{Int}(Y_1)$. As in case (a) we obtain that $\mu(A_\infty(f) \cap Y_1) = 0$. Since $A_\infty(f) = \bigcup_{n=0}^\infty f^{-n}(A_\infty(f) \cap Y_1) \cup \{x_0\}$ and the fact that the restriction of f to the set $X \setminus Y_1$ is a homeomorphism, we get $\mu(A_\infty(f)) = 0$.

Case (c): Assume that $\text{Bd}(X) \cap [x_0, y_0] = \{y_0\}$. We define $v_1 = \max \{x \in [x_0, y_0]; x \in \text{Crit}(f)\}$, $v_2 = \max \{f(x); x_0 \leq x \leq v_1\}$ and $v_3 = \max \{x \in [x_0, y_0]; x \text{ is asymptotically stable periodic point of } f\}$ provided that f has at least one asymptotically stable periodic point; further we write $v_M = \max \{v_1, v_2, v_3\}$. Note that $v_M < y_0$ and that $f(v_M) \leq v_M$. The set Y_2 , defined by $Y_2 = \{x \in X; x \leq v_M\}$ is a positively f -invariant set. As in case (b) we obtain $\mu(A_\infty(f)) = 0$.

Case (d): Assume that $\text{Bd}(X) \cap A_\infty(f) = \{x_0, y_0\}$. We define u_m as in case (b), and we define v_M as in case (c). We write $Y = [u_m, v_M]$. The set Y is a positively f -invariant set, and we obtain $\mu(A_\infty(f) \cap Y) = 0$. Observe that the restriction of f to the set $X \setminus Y$ is locally a homeomorphism. Since $A_\infty(f) = \bigcup_{n=0}^\infty f^{-n}(A_\infty(f) \cap Y) \cup \{x_0, y_0\}$ we conclude that $\mu(A_\infty(f)) = 0$.

(2) Apply (1) of Theorem A and (5.3). \square

6. Proof of Theorem B. In this section we fix a nontrivial interval $X \subset \mathbb{R}$. First, we assume that $f: X \rightarrow X$ is a chaotic mapping, for which the following conditions hold:

- (6.1) (i) $f \in C^3(X, X)$ and $Sf(x) \leq 0$ for all $x \in X \setminus \text{Crit}(f)$;
- (ii) $\text{Crit}(f) \subset \bigcup_{k=0}^\infty D_k$;
- (iii) $U_{\text{Bd}(X)}$ is an open neighbourhood of the boundary of X which consists of two components in the union of the domains of attraction of the asymptotically stable periodic points for f and the absorbing boundary points of X for f .

Remark. From the assumptions (6.1) and Corollary 4.17 it follows that f has no one-sided asymptotically stable periodic points.

The first lemma gives the result that the set consisting of the asymptotically stable periodic points for f and the absorbing boundary points of X for f is a finite set.

LEMMA 6.2. *The set consisting of the union of the asymptotically stable periodic points for f and the absorbing boundary points of X for f is finite.*

Proof. The proof is left to the reader. \square

COROLLARY 6.3. *The number p , the smallest common multiple of the periods of the asymptotically stable periodic points for f multiplied by two, is well defined.*

Remark. (1) The number of critical points of f in Lemma 6.2 need not be finite. Consequently, the result is a generalization of maps having finitely many critical points with negative Schwarzian derivative (see Singer [22] or Collet and Eckmann [4]).

(2) Conditions (6.1) (i) and (ii) are not sufficient to imply that the number of asymptotically stable periodic points is finite.

LEMMA 6.4. *There exists a nonnegative integer $N(f)$ such that, for each $k \in \mathbb{N}$, we have*

- (i) $A_{k+N(f)}$ is compact;
- (ii) $\text{Crit}(f^k) \cap A_{k+N(f)} = \emptyset$.

Proof. By assumption (6.1) we can find a nonnegative integer $N(f)$ such that $\text{Crit}(f) \cup U_{\text{Bd}(X)} \subset \bigcup_{k=0}^{N(f)} D_k$. Assume that $N(f)$ is minimal. Then $A_{1+N(f)}$ is compact and since $A_{1+N(f)}$ and $\bigcup_{k=0}^{N(f)} D_k$ are complementary by Lemma 4.3, it follows that $\text{Crit}(f) \cap A_{1+N(f)} = \emptyset$. By definition we have for each $k \in \mathbb{N}$ that the set $A_{k+N(f)}$ is compact. By induction one obtains $\text{Crit}(f^k) \cap A_{k+N(f)} = \emptyset$ for each $k \in \mathbb{N}$. \square

LEMMA 6.5. *For each $k \in \mathbb{N}$ there exists $N(k) \in \mathbb{N}$ such that $|(f^{N(k)})'(y)| > 1$ for all $y \in \text{Bd}(A_{k+N(f)})$, with $N(f)$ as in Lemma 6.4.*

Proof. Fix the integer $N(f)$ as in Lemma 6.4 and the integer p as in Corollary 6.3. Fix any positive integer k . We write $K = \min \{|(f^p)'(x)|; x \in \text{Bd}(D_0) \cap \text{Per}(f)\}$. From (6.1) and Lemma 4.18 it follows that $K > 1$.

For each point $y \in \text{Bd}(A_{k+N(f)})$ we have:

- (i) $f^{k+N(f)+p}(y)$ is a periodic point for f with period p ;
- (ii) $|(f^p)'(f^{k+N(f)+p}(y))| \geq K > 1$.

Set $s_k = \min \{|(f^{k+N(f)+p})'(y)|; y \in \text{Bd}(A_{k+N(f)})\}$. Choose a positive integer β satisfying $K^\beta \cdot s_k > 1$. Then for each $y \in \text{Bd}(A_{k+N(f)})$ we have $f^{k+N(f)+p}(y) \in \text{Bd}(A_{k+N(f)})$, since $\text{Bd}(A_j) \subset \text{Bd}(A_{j+1})$ and $\text{Bd}(A_j)$ is positively f -invariant for each positive integer j (see Lemma 4.8). We obtain

$$|(f^{\beta p+k+N(f)+p})'(y)| = |(f^{\beta p})'(f^{k+N(f)+p}(y))| \cdot |(f^{k+N(f)+p})'(y)| \geq K^\beta \cdot s_k > 1.$$

We have, by setting $N(k) = (\beta + 1)p + k + N(f)$, $|(f^{N(k)})'(y)| > 1$ for each $y \in \text{Bd}(A_{k+N(f)})$. \square

LEMMA 6.6. *There exists a number $\varepsilon > 0$ such that for each $k \in \mathbb{N}$, and for all $y_k \in \text{Bd}(A_{k+N(f)})$ with the property*

$$|(f^k)'(y_k)| = \min \{|(f^k)'(y)|; y \in \text{Bd}(A_{k+N(f)})\}$$

we have: $\mu(D) > \varepsilon / |(f^k)'(y_k)|$, with D is a component of $\sqcup \{D_i; 0 \leq i \leq k + N(f) - 1\}$ satisfying $y_k \in \text{Bd}(D)$, and $N(f)$ as in Lemma 6.4.

Proof. Select the integer $N(f)$ as in Lemma 6.4, and the integer p as in Corollary 6.3. We write V for the set which consists of the union of the critical set of f together with all its forward iterates and the set of asymptotically stable periodic points for f . Define $V_0 = V \cap [x_0, y_0]$, with $x_0 = \min \{x \in X; x \in A_{1+N(f)}\}$, $y_0 = \max \{x \in X; x \in A_{1+N(f)}\}$.

We set $\delta = \frac{1}{2} \inf \{|x - y|; x \in V_0, y \in \text{Bd}(A_{1+N(f)})\}$. Note that $\delta > 0$ since V_0 and $\text{Bd}(A_{1+N(f)})$ are disjoint nonempty compact sets.

Fix any $k \in \mathbb{N}$. Assume that a point $y_k \in \text{Bd}(A_{k+N(f)})$ is chosen such that $|(f^k)'(y_k)| = \min \{|(f^k)'(y)|; y \in \text{Bd}(A_{k+N(f)})\}$. We set D for the component of the set $\cup \{D_j; 0 \leq j \leq k + N(f) - 1\}$ satisfying $y_k \in \text{Bd}(D)$.

Using (6.1) (iii), and Lemma 4.16 we get that the mapping $f^{k+N(f)+p}$ restricted to D has a critical point.

We write $\varepsilon = \delta / \max \{|(f^{N(f)+p})'(x)|; x \in \text{Bd}(A_{N(f)}) \cup \cup_{k=0}^{N(f)-1} D_k\}$. Let $U \subset D$ be an open interval with the following properties:

- (1) $y_k \in \text{Bd}(U)$,
- (2) $U \cap \text{Crit}(f^{k+N(f)+p}) = \emptyset$,
- (3) $\text{Bd}(U) \cap \text{Crit}(f^{k+N(f)+p}) \neq \emptyset$.

We write z_k for a point in $\text{Cl}(U)$ at which the map $|(f^{k+N(f)+p})'|$ assumes its maximum value.

Let $V \subset U$ be an open interval which has either the property (i) $V = U$ if $z_k = y_k$, or (ii)

- (1) $z_k \in \text{Bd}(V)$,
- (2) $V \cap \text{Crit}(f^{k+N(f)+p}) = \emptyset$,
- (3) $\text{Bd}(V) \cap \text{Crit}(f^{k+N(f)+p}) \neq \emptyset$ if $z_k \neq y_k$.

From the choice of z_k we obtain $|(f^{k+N(f)+p})'(x)| \leq |(f^{k+N(f)+p})'(z_k)|$ for all $x \in V$.

Let W be the component of $A_{k+N(f)}$ containing y_k . From Lemma 4.15 and the choice of y_k we have $|(f^k)'(x)| \geq |(f^k)'(y_k)|$ for all $x \in W$. Applying Lemma 4.15 again we have $|(f^k)'(x)| \geq |(f^k)'(z_k)|$ for all x in the interval $W \cup U \setminus V$. In particular, we have $|(f^k)'(y_k)| \geq |(f^k)'(z_k)|$.

We conclude:

$$\begin{aligned} \mu(D) > \mu(U) &\geq \mu(V) \geq \delta / |(f^{k+N(f)+p})'(z_k)| \\ &= \delta / |(f^{N(f)+p})'(f^k(z_k))| \cdot |(f^k)'(z_k)| \\ &\geq \varepsilon / |(f^k)'(z_k)| \\ &\geq \varepsilon / |(f^k)'(y_k)|. \end{aligned}$$

\square

LEMMA 6.7. *There exists a positive integer N such that $|(f^N)'(x)| > 1$ for all $x \in A_{N+N(f)}$, with $N(f)$ as in Lemma 6.4.*

Proof. We assume that the integers $N(f)$, respectively p , are as in Lemma 6.4, respectively, Corollary 6.3. Set $\varepsilon > 0$ as in Lemma 6.6. Choose $L \in \mathbb{N}$ such that $\varepsilon \cdot L > \mu(A_{1+N(f)})$. We write $s_k = \min \{|(f^k)'(x)|; x \in \text{Bd}(A_{k+N(f)})\}$ for each $k \in \mathbb{N}$.

If $s_1 > 1$ then $|f'(x)| > 1$ for all $x \in \text{Bd}(A_{1+N(f)})$. Using Lemma 4.15 we have $|f'(x)| > 1$ for all $x \in \text{Bd}(A_{1+N(f)})$ and we are done. So we assume that $s_1 \leq 1$.

We set $N(1) = 1$. Applying Lemma 6.5, we know there exists a nondecreasing sequence $\{N(i)\}_{i=1}^\infty$ of positive integers, defined as follows by induction: if $s_{N(i)} \leq 1$, let $N(i+1) > N(i)$ be minimal such that $|(f^{N(i+1)})'(x)| > 1$ for all $x \in \text{Bd}(A_{N(i)+N(f)})$, and if $s_{N(i)} > 1$ set $N(i+1) = N(i)$.

Suppose that for some positive integer i , we have $s_{N(i)} \leq 1$. Let k be any integer, $1 \leq k \leq i - 1$. We assume that $y_{N(k)} \in \text{Bd}(A_{N(k)+N(f)})$ satisfies $|(f^{N(k)})'(y_{N(k)})| = s_{N(k)}$.

We set $U_{N(k)}$ for the component of $\cup \{D_j; 0 \leq j \leq N(k) + N(f) - 1\}$ such that $y_{N(k)} \in \text{Bd}(U_{N(k)})$. Let $x_{N(k)}$ be a boundary point of $U_{N(k)}$ and $x_{N(k)} \neq y_{N(k)}$.

Then

- (i) $U_{N(k)} \neq U_{N(k+1)}$ since $|(f^{N(k+1)})'(x_{N(k)})| > 1$,
 $|(f^{N(k+1)})'(y_{N(k)})| > 1$ and $|(f^{N(k+1)})'(y_{N(k+1)})| \leq 1$ by using Lemma 6.5.
- (ii) $\mu(U_{N(k)}) \geq \varepsilon / s_{N(k)} \geq \varepsilon$ by using Lemma 6.6.

Observe that $U_{N(k+1)} \subset \cup \{D_j; N(k) + N(f) \leq j \leq N(k+1) + N(f) - 1\}$; hence $U_{N(k+1)} \subset A_{1+N(f)}$.

We obtain: $s_{N(L+1)} > 1$, since $\varepsilon \cdot L > \mu(A_{1+N(f)})$, and consequently $|(f^{N(L+1)})'(x)| > 1$ for all $x \in A_{N(L+1)+N(f)}$ by applying Lemma 4.15. \square

COROLLARY 6.8. *f is eventually expanding.*

Proof of Theorem B. We assume that $X \subset \mathbb{R}$ is a nontrivial interval. Let $f \in C^3(X, X)$ satisfy the assumptions (i)-(iv) of the theorem.

(1) For each $n \in \mathbb{N}$ we have that the compact set $A_\infty(f)$ in X is a subset of A_n with $A_\infty(f)$ and A_n as in (4.2).

We write $x_0 = \min \{x \in X; x \in A_\infty(f)\}$, $y_0 = \max \{x \in X; x \in A_\infty(f)\}$.

We consider the following cases:

- (a) $\text{Bd}(X) \cap A_\infty(f) = \emptyset$,
- (b) $\text{Bd}(X) \cap A_\infty(f) = \{x_0\}$,
- (c) $\text{Bd}(X) \cap A_\infty(f) = \{y_0\}$,
- (d) $\text{Bd}(X) \cap A_\infty(f) = \{x_0, y_0\}$.

Case (a): Assume that $\text{Bd}(X) \cap A_\infty(f) = \emptyset$, then we have $A_\infty(f) \subset \text{Int}(X)$. Then there exists $n \in \mathbb{N}$ such that $A_n \subset \text{Int}(X)$, consequently f has no one-sided asymptotically stable periodic points. Applying (6.1), Corollary 6.8 and Lemmas 4.3, 4.14 and 6.2, we obtain that f satisfies conditions (i) and (ii) of Theorem A. Consequently, $\mu(A_\infty(f)) = 0$.

Cases (b), (c) and (d): Similar to the proof of cases (b), (c) and (d) of the proof of Theorem A, provided that f has no one-sided asymptotically stable periodic points.

Now assume that f has at least 1 one-sided asymptotically stable periodic point. From (6.1) and Corollary 4.17 it follows that the domain of attraction of such a one-sided asymptotically stable periodic point consists of at most two components. Let Y be the complement of the domain of attraction of the one-sided asymptotically stable periodic points. The map f maps Y into itself and the restriction of f to Y has no one-sided asymptotically stable periodic points. Since the fixed points of f^2 are isolated, we can proceed as before. Note that for each $x \in X \setminus Y$ the point x is asymptotically periodic with period two.

(2) This follows from the above result $\mu(A_\infty(f)) = 0$ and Corollary 6.3. \square

Proof of the corollary. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a polynomial mapping of degree $N + 1$ for some $N \in \mathbb{N}$, such that conditions (i) and (ii) hold.

Then f has a negative Schwarzian derivative (see Collet and Eckmann [4]). We conclude: f satisfies conditions (i)–(iv) of Theorem B. \square

Remark. Assume that f is a mapping as in Theorem B. For each $x \in A_\infty(f)$ we have

- (i) $x \in \text{Cl}(\bigcup_{n=0}^\infty \text{Crit}(f^n))$.
- (ii) For every number $K > 1$ there exists $N \in \mathbb{N}$ such that

$$|(f^N)'(x)| > K.$$

These paradoxical properties cause the subtle proof.

7. Some concluding comments. Let $f \in C^1(X, X)$ and $A_\infty(f)$ is defined as in (4.2).

Remark 7.1. (i) The condition “ f is expanding on $A_\infty(f)$ ” does not imply “if $f^n(x) = x$ then $|(f^n)'(x)| \neq 1$.”

(ii) The condition “if $f^n(x) = x$ then $|(f^n)'(x)| \neq 1$ ” does not imply “ f is expanding on $A_\infty(f)$.”

Question 7.2. Assume that f is expanding on $A_\infty(f)$, and that $A_\infty(f)$ is compact in \mathbb{R} . Is it true that the set of asymptotically stable periodic points is compact? The question can be answered affirmatively if periods of the asymptotically stable periodic points are bounded.

Problem 7.3. Assume that f is a chaotic C^3 -map from a nontrivial interval X into itself with the following properties:

- (i) f has a nonpositive Schwarzian derivative.
- (ii) The set of points whose orbits do not converge to an asymptotically stable periodic orbit, to a one-sided asymptotically stable periodic orbit or to the possible present absorbing boundary of X , is a nonempty compact set.
- (iii) The orbit of each critical point converges to an asymptotically stable periodic orbit of f , to a one-sided asymptotically stable periodic orbit of f or to the possible present absorbing boundary of X .
- (iv) The fixed points of f^2 are isolated.

Is it true that the conclusions of Theorem B hold? Preston [21] mentioned a similar question for maps with one critical point and negative Schwarzian derivative.

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