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Published in:
Journal of Mathematical Physics

DOI:
10.1063/1.524807

Publication date:
1981

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Citation: Journal of Mathematical Physics **22**, 2482 (1981); doi: 10.1063/1.524807
View online: https://doi.org/10.1063/1.524807
View Table of Contents: http://aip.scitation.org/toc/jmp/22/11
Published by the American Institute of Physics

The Coulomb unitarity relation and some series of products of three Legendre functions

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(Received 29 April 1981; accepted for publication 12 June 1981)

We obtain from the off-shell Coulomb unitarity relation a closed expression for $\Sigma_{l=0}^{\infty}(2l+1)P_l(x) \times Q_l^{i\gamma}(y)\, Q_l^{-i\gamma}(z)$, and we consider some related series of products of Legendre functions.

PACS numbers: 02.30.Lt, 02.30.Gp, 03.65.Nk

In this paper we shall consider the Coulomb unitarity relation[1–4] and derive from this relation a closed expression for an infinite series of products of three Legendre functions, P_l, $Q_l^{i\gamma}$, and $Q_l^{-i\gamma}$ [see Eq. (12)]. By taking the limit $\gamma \to 0$ we obtain agreement with an expression[5] for the corresponding series, which exists in the literature. However, our expression has a much simpler form, which means that we have obtained a substantial reduction of the expression given in.[5] After the derivation of our main result, Eq. (12), we shall briefly consider some related series of products of Legendre functions [see Eqs. (14)–(25)].

The unitarity relation, or generalized optical theorem, or Low equation, in quantum-mechanical scattering theory establishes a simple relation between the imaginary part of the off-shell T matrix and its half-off-shell elements.[1,2] Suppressing the energy, $E = k^2 + i\eta$, $\eta \downarrow 0$, we have

$$\langle \mathbf{p} | T - T^\dagger | \mathbf{p}' \rangle = -i\pi k \int \langle \mathbf{p} | T | \mathbf{k} \rangle \, \langle \mathbf{k} | T^\dagger | \mathbf{p}' \rangle \, d\hat{k}, \quad (1)$$

where the integration is over the unit sphere. Equation (1) is valid when the potential associated with T has a short range. However, for the Coulomb potential V_c Eq. (1) has to be modified because the half-shell limit of the off-shell Coulomb T matrix T_c does not exist. Instead we have[4]

$$\begin{aligned}\langle \mathbf{p} | T_c - T_c^\dagger | \mathbf{p}' \rangle &= -i\pi k \int \langle \mathbf{p} | T_c | \mathbf{k}\infty \rangle \, \langle \mathbf{k}\infty | T_c^\dagger | \mathbf{p}' \rangle \, d\hat{k} \\ &= -i\pi k \int \langle \mathbf{p} | V_c | \mathbf{k}+ \rangle_c \; {}_c\langle \mathbf{k}+ | V_c | \mathbf{p}' \rangle \, d\hat{k}, \end{aligned} \quad (2)$$

where $|\mathbf{k}\infty\rangle$ is the so-called Coulombian asymptotic state and $|\mathbf{k}+\rangle_c$ is the Coulomb scattering state with energy $(k+i\epsilon)^2$, $\epsilon \downarrow 0$. The left-hand side of Eq. (2) is known in closed form (Ref. 4). We rewrite the right-hand side by inserting

$$\langle \mathbf{p} | V_c | \mathbf{k}+ \rangle_c = \sum_{l=0}^{\infty} (4\pi)^{-1}(2l+1) P_l(\hat{p}\cdot\hat{k}) \langle p | V_{cl} | kl+ \rangle_c, \quad (3)$$

and using the orthogonality relation

$$\int P_l(\hat{p}\cdot\hat{k}) P_{l'}(\hat{p}'\cdot\hat{k}) \, d\hat{k} = 4\pi(2l+1)^{-1} P_l(\hat{p}\cdot\hat{p}') \delta_{ll'}. \quad (4)$$

In Eq. (3), $|kl+\rangle_c$ is the partial-wave Coulomb scattering state. Denoting $(p^2+k^2)/(2pk)$ by y and assuming $p > k$, we have[4]

$$\langle p | V_{cl} | kl+ \rangle_c = 2\gamma(\pi p)^{-1} e^{(1/2)\pi\gamma} Q_l^{i\gamma}(y), \quad (5)$$

where γ is Sommerfeld's parameter, which is real ($k > 0$). It is important to note that $Q_l^{i\gamma}(y)$ is not real-analytic: For the complex conjugate of both members of Eq. (5) we obtain

$$\langle p | V_c | kl+ \rangle_c^* = 2\gamma(\pi p)^{-1} e^{-(3/2)\pi\gamma} Q_l^{-i\gamma}(y). \quad (6)$$

In the above indicated way we obtain from Eqs. (2)–(6),

$$\begin{aligned}&\sum_{l=0}^{\infty} (2l+1) P_l(x) Q_l^{i\gamma}(y) Q_l^{-i\gamma}(z) \\ &\quad - [\Gamma(1+i\gamma)\Gamma(1-i\gamma)/2i\gamma(\alpha_+\alpha_-)^{1/2}] \\ &\quad \times (Y^{i\gamma} - Y^{-i\gamma}) \\ &\quad = -\pi \sin(\gamma \ln Y)/(\alpha_+\alpha_-)^{1/2} \sinh \pi\gamma. \end{aligned} \quad (7)$$

Here $x = \hat{p}\cdot\hat{p}'$, $z = (p'^2 + k^2)/(2p'k)$, $p' > k$,

$$\alpha_\pm = yz - x \pm (y^2-1)^{1/2}(z^2-1)^{1/2}, \quad (8)$$

$$Y = (\alpha_+^{1/2} - \alpha_-^{1/2})/(\alpha_+^{1/2} + \alpha_-^{1/2}). \quad (9)$$

For convenience we introduce the quantity W,

$$W = W(x,y,z) = x^2 + y^2 + z^2 - 2xyz - 1. \quad (10)$$

Then we have $\alpha_+\alpha_- = W \geqslant 0$,

$$Y^2 = (yz - x - W^{1/2})/(yz - x + W^{1/2}), \quad (11)$$

so that Eq. (7) can be rewritten as

$$\begin{aligned}&\sum_{l=0}^{\infty} (2l+1) P_l(x) Q_l^{i\gamma}(y) Q_l^{-i\gamma}(z) \\ &\quad = -\pi \sin(\tfrac{1}{2}\gamma \ln Y^2)/W^{1/2} \sinh \pi\gamma. \end{aligned} \quad (12)$$

By analytic continuation it follows that Eq. (12) is valid for complex x, y, z, and γ. The series in Eq. (12) is convergent if $\mathrm{Re}\,x > 0$, $\mathrm{Re}\,y > 0$, $\mathrm{Re}\,z > 0$, and

$$|x + (x^2-1)^{1/2}| < |y + (y^2-1)^{1/2}| \cdot |z + (z^2-1)^{1/2}|. \quad (13)$$

When $\mathrm{Re}\,x < 0$, one should replace x by $-x$ in Eq. (13), and similarly for y and z. It may be noted that

$$P_l(-y) = (-1)^l P_l(y),$$

$$Q_l^{i\gamma}(-z) = (-1)^{l+1} Q_l^{i\gamma}(z).$$

Now we are going to consider the more general expression

$$\begin{aligned}&F_{mn}(x_1,\ldots,x_m; z_1,\ldots,z_n) \\ &\quad = \sum_{l=0}^{\infty} (2l+1) P_l(x_1)\cdots P_l(x_m) Q_l(z_1)\cdots Q_l(z_n) \end{aligned} \quad (14)$$

(cf. Ref. 5) for $n, m = 0,1,2,3$, $x_i \in \mathbb{C}$, $y_i \in \mathbb{C}\setminus[-1,1]$. When $\operatorname{Re} x_i > 0$, $\operatorname{Re} z_j > 0$, this series is convergent if

$$\prod_{i=1}^{m} |x_i + (x_i^2 - 1)^{1/2}| < \prod_{j=1}^{n} |z_j + (z_j^2 - 1)^{1/2}|. \tag{15}$$

Let us first consider F_{12}. By taking the limit for $\gamma \to 0$ in Eq. (12) we obtain

$$F_{12}(x;y,z) = \sum_{l=0}^{\infty} (2l+1) P_l(x) Q_l(y) Q_l(z) = \tfrac{1}{2} W^{-1/2} \ln \frac{yz - x + W^{1/2}}{yz - x - W^{1/2}}. \tag{16}$$

It is interesting to note that

$$F_{12}(x;y,z) = \frac{1}{4\pi} \int (y - \hat{p}\cdot\hat{q})^{-1} (z - \hat{p}'\cdot\hat{q})^{-1}\, d\hat{q},$$

where $\hat{p}\cdot\hat{p}' = x$ and the integration is over the unit sphere. In Ref. 5 an expression has been given for $F_{12}(x;z,z)$. Our result given by Eq. (16) means a considerable reduction of that expression. Indeed, in the notation of Ref. 5 we have $d = z^2 - 1 + t^2$, $2t^2 = 1 - y$, and Eq. (16) gives

$$F_{12}(y;z,z) = \frac{1}{4td^{1/2}} \ln\left(\frac{t + d^{1/2}}{t - d^{1/2}}\right)^2, \tag{17}$$

whereas in Ref. 5 the following result is given,

$$F_{12}(y;z,z) = \frac{1}{4td^{1/2}} \times \ln\left[\frac{1 + t\,[z + 2 + 2t^2/(z-1)]d^{-1/2} + 2t^2/(z-1)}{1 + t\,[z - 2 + 2t^2/(z+1)]d^{-1/2} - 2t^2/(z+1)}\right].$$

To demonstrate the equivalence of this result and that in Eq. (17) is *not* completely trivial. It can be done by dividing out the common factor $(1 + tzd^{-1/2})(z^2 - 1)^{-1}$ from the numerator and the denominator of the fraction which forms the argument of the logarithm. By this procedure Eq. (17) is retrieved.

We shall briefly consider some other interesting particular cases of the general function F_{mn}. By taking $x = 1$ in Eq. (16) we obtain the well-known result

$$F_{02}(y,z) = \tfrac{1}{2}(y - z)^{-1} \ln\left(\frac{y-1}{y+1}\,\frac{z+1}{z-1}\right). \tag{18}$$

Other well-known formulas are[6]

$$F_{21}(x,y;z) = W^{-1/2}, \tag{19}$$

$$F_{11}(y;z) = (z - y)^{-1}, \tag{20}$$

$$F_{01}(z) = (z - 1)^{-1}. \tag{21}$$

Eq. (20) is called Heine's formula.

When $n = 0$ we shall restrict x, y, and z in F_{m0} to the interval $[-1,1]$. According to Ref. 6, p. 307 we have

$$F_{30}(x,y,z) = \begin{Bmatrix} 0 & \text{if } W > 0 \\ 2\pi^{-1}(-W)^{-1/2} & \text{if } W < 0 \end{Bmatrix}. \tag{22}$$

Furthermore, we have [cf. Eq. (4)]

$$F_{20}(x,y) = 2\delta(y - x), \tag{23}$$

$$F_{10}(x) = 2\delta(1 - x), \tag{24}$$

where δ is Dirac's delta distribution.

Finally we shall briefly consider F_{03}. In virtue of Eq. (16) we have

$$F_{03}(p,y,z) = \sum_{l=0}^{\infty} (2l+1) Q_l(p) Q_l(y) Q_l(z) = \frac{1}{2}\int_{-1}^{1} \frac{dx}{p - x} \sum_{l=0}^{\infty} (2l+1) P_l(p) Q_l(y) Q_l(z) = \frac{1}{4}\int_{-1}^{1} \frac{W^{-1/2}}{p - x} \ln \frac{yz - x + W^{1/2}}{yz - x - W^{1/2}}\, dx.$$

Putting $a = (y^2 - 1)^{1/2}(z^2 - 1)^{1/2}$, $v = \operatorname{arcosh}((yz - x)/a)$, $v_\pm = \operatorname{arcosh}((yz \pm 1)/a)$ we get $W^{1/2} = a \sinh v$ and

$$F_{03}(p,y,z) = (1/2)a \int_{v_-}^{v_+} \frac{v\, dv}{\cosh v - (yz - p)/a}. \tag{25}$$

According to formula 2.478.7 of Ref. 7 we have

$$\int \frac{x\, dx}{\cosh 2x - \cos 2t} = \frac{1}{2 \sin 2t} [L(u + t) - L(u - t) - 2L(t)], \tag{26}$$

where $u = \arctan(\tanh x \cot t)$ and L is Lobachevski's function, defined by

$$L(x) = -\int_0^x \ln(\cos t)\, dt. \tag{27}$$

This implies that F_{03} cannot be expressed in terms of elementary functions.

By using the series representation

$$L(x) = -x \ln 2 + (1/2) \sum_{n=1}^{\infty} (-1)^n n^{-2} \sin 2nx, \tag{28}$$

the right member of Eq. (26) can be rewritten as

$$\frac{1}{4 \sin 2t} \sum_{n=0}^{\infty} (-1)^n n^{-2} \sin 2nt \cos^2 nu. \tag{29}$$

We point out that on p. 377 of Ref. 6, Eq. (56.8.1), a closed formula is given for the series

$$\sum_{l=0}^{\infty} (2l+1) P_l(x) P_l^{\,m}(y) P_l^{\,-m}(z), \tag{30}$$

where $m \in \mathbb{N}$ and $x,y,z \in [-1,1]$.

ACKNOWLEDGMENT

This work is part of the project "The Coulomb Potential in Quantum Mechanics and Related Topics."

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