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Braaksma, B L J; Schuitman, A

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SOME CLASSES OF WATSON TRANSFORMS AND RELATED INTEGRAL EQUATIONS FOR GENERALIZED FUNCTIONS*

B. L. J. BRAAKSMA† AND A. SCHUITMAN‡

Abstract. Spaces of testing functions which are isomorphically mapped onto one another by the Mellin and the inverse Mellin transform are used to prove that certain spaces are also mapped isomorphically onto one another by the so-called Watson transform. Then Watson transforms for generalized functions are defined. Applications on Hankel transforms, fractional integrals and integral equations of Love involving hypergeometric functions and of Fox involving H-functions are given. Furthermore, dual integral equations for generalized functions with Hankel transforms and H-functions are treated.

Introduction. In this paper we define Watson transforms and other convolution transforms for generalized functions. To this end we introduce spaces of testing functions which are mapped isomorphically onto each other by means of the Mellin transform (§ 1). Using the connection of Watson transforms and Mellin transforms (cf. Titchmarsh [13]) we show that Watson transforms map these function spaces continuously into function spaces of the same type (§ 2). Then these transforms are generalized to generalized functions in the dual spaces. Also the inverses of these transforms are considered. In § 3 the same analysis is done on certain subspaces of the spaces of testing functions of § 1. Examples including Hankel transforms are given in § 4.

Another type of product convolutions is treated in § 5. In particular, operators of fractional integration are considered including the so-called cut fractional integrals. Using these fractional integrals we extend the definition of the Hankel transform in § 6. Here also the cut Hankel transform appears which is useful for the inversion of Hankel transforms. Furthermore relations between Hankel transforms and fractional integrals of generalized functions are given. In §§ 7 and 8 we give applications to dual integral equations for generalized functions involving Hankel transforms and, more generally, transforms with H-functions of Fox which contain many special integral transforms (cf. Fox [6]). Here we use a method of Erdélyi and Sneddon [5]. We give precise conditions for the existence of solutions of the dual integral equations, which were obtained formally by Fox. In § 9 we consider a special case of product convolutions involving hypergeometric functions and related integral equations, which have been studied among others by Love [11a] and [11b]. Some of the results of Love are also extended for ordinary functions.

Other applications to differential equations may be given analogous to those in Zemanian's study of generalized integral transformations [14]. Our approach to Mellin and Hankel transforms is different from Fung Kang's [7] and from Zemanian's approach. Fractional integrals for distributions have been studied recently by Erdélyi and McBride [4] and Erdélyi [3]. Our treatment is similar to theirs, though we do not assume that the testing functions have compact support. Watson transforms for generalized functions have been considered also by

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† Mathematical Institute, University of Groningen, Groningen, Netherlands.

‡ Department of Mathematics, Technological University Delft, Delft, Netherlands.

Hsing-Yuan Hsu [8], starting from testing function spaces closely related to those of Zemanian.

1. The spaces $T(\lambda, \mu)$ and $S(\lambda, \mu)$. Throughout this paper $\mathbb{R}$ denotes the set of the real numbers and $\mathbb{R}^* = \mathbb{R} \cup \{-\infty, \infty\}$. $\mathbb{C}$ is the set of complex numbers. $\mathbb{N} = \{0, 1, 2, \cdots\}$.

Let $\lambda, \mu \in \mathbb{R}^*$, $\lambda < \mu$. Let $(\lambda_n)_{n=0}^\infty$ and $(\mu_n)_{n=0}^\infty$ be sequences of real numbers with $\lambda_n \downarrow \lambda$, $\mu_n \uparrow \mu$ and $\lambda_n < \mu_n$ for all $n \in \mathbb{N}$. $T(\lambda, \mu)$ is the space of all functions $\phi \in C^\infty(0, \infty)$ with the property

$$\tau_n(\phi) = \sup_{\substack{t>0 \\ p=0,1,\cdots,n \\ \lambda_n \leqq c \leqq \mu_n}} |t^{c+p}\phi^{(p)}(t)| < \infty \quad \text{for all } n \in \mathbb{N}. \tag{1.1}$$

$T(\lambda, \mu)$ is a locally convex vector space with the topology generated by the sequence of norms (τ_n). Related spaces have been considered by Zemanian [14, § 4.2].

Let λ, μ, λ_n and μ_n be as above. $S(\lambda, \mu)$ is the space of all functions Φ, analytic on $\lambda < \operatorname{Re} s < \mu$, with the property

$$\sigma_n(\Phi) = \sup_{\substack{\lambda_n \leqq \operatorname{Re} s \leqq \mu_n \\ p=0,1,\cdots,n}} |s^p\Phi(s)| < \infty \quad \text{for all } n \in \mathbb{N}. \tag{1.2}$$

With the topology generated by the sequence of norms (σ_n), $S(\lambda, \mu)$ is a locally convex vector space.

The topologies of $T(\lambda, \mu)$ and $S(\lambda, \mu)$ are independent of the particular choice of the sequences (λ_n) and (μ_n). Using standard arguments it may be shown that both spaces are Fréchet spaces. In the following, isomorphisms and automorphisms between spaces are interpreted as linear continuous mappings onto with continuous inverses.

If ϕ is some function, we denote its Mellin transform by $\mathscr{M}\phi$,

$$(\mathscr{M}\phi)(s) = \int_0^\infty t^{s-1}\phi(t)\,dt. \tag{1.3}$$

If Φ is some function we denote its inverse Mellin transform by $\mathscr{M}^{-1}\Phi$,

$$(\mathscr{M}^{-1}\Phi)(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Phi(s)t^{-s}\,ds.$$

We prove the following theorem.

THEOREM 1. *The Mellin transform $\mathscr{M}$ defines an isomorphism of $T(\lambda, \mu)$ onto $S(\lambda, \mu)$. The adjoint Mellin transform $\mathscr{M}'$ defines an isomorphism of $S'(\lambda, \mu)$ onto $T'(\lambda, \mu)$.*

Proof. If $\phi \in T(\lambda, \mu)$ and $\lambda < \operatorname{Re} s < \mu$, $p \in \mathbb{N}$, then

$$(\mathscr{M}\phi)(s) = \Phi(s) = \int_0^\infty t^{s-1}\phi(t)\,dt = \frac{(-1)^p}{(s)_p}\int_0^\infty t^{s+p-1}\phi^{(p)}(t)\,dt$$

by virtue of (1.1). (Notation: $(s)_0 = 1$, $(s)_p = (s+p-1)(s)_{p-1}$, $p \geqq 1$). Note that $\int_0^\infty t^{s+p-1}\phi^{(p)}(t)\,dt$ has a zero in $s = -h$, $h \in \mathbb{N}$, if $\lambda < -h < \mu$. It follows that $\Phi \in S(\lambda, \mu)$.

If $\Phi \in S(\lambda, \mu)$, $\lambda < c < \mu$, $t > 0$ and $p \in \mathbb{N}$, then

$$(\mathscr{M}^{-1}\Phi)(t) = \phi(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Phi(s)t^{-s}\,ds,$$

$$\phi^{(p)}(t) = \frac{(-1)^p}{2\pi i}\int_{c-i\infty}^{c+i\infty} t^{-s-p}(s)_p\Phi(s)\,ds,$$

where the integrals are absolutely convergent. It follows that $\phi \in T(\lambda, \mu)$. From the well-known inversion theorem for Mellin transforms it follows that $\mathscr{M} \circ \mathscr{M}^{-1}$ and $\mathscr{M}^{-1} \circ \mathscr{M}$ are the identity maps on $S(\lambda, \mu)$ and $T(\lambda, \mu)$. It remains to prove the continuity.

We may assume that the sequences (λ_n) and (μ_n) are chosen in such a way that $\lambda_n, \mu_n \neq 0, -1, -2, \cdots$. Consider the strip $\lambda_n \leqq \operatorname{Re} s \leqq \mu_n$. For each integer $h \leqq 0$ with $\lambda_n < h < \mu_n$, let D_h be the interior of a disc with center h and which lies entirely in the strip. We omit all the sets D_h from the strip and denote the remaining "reduced" strip by S. Let $\phi \in T(\lambda, \mu)$ and $\Phi = \mathscr{M}\phi$. Then

$$\sigma_n(\Phi) = \sup_{\substack{0 \leqq p \leqq n \\ \lambda_n \leqq \operatorname{Re} s \leqq \mu_n}} |s^p\Phi(s)| = \sup_{\substack{0 \leqq p \leqq n \\ s \in S}} \left|\frac{s^p}{(s)_p}\int_0^\infty t^{s+p-1}\phi^{(p)}(t)\,dt\right|.$$

Now

$$K_0 = \sup_{\substack{0 \leqq p \leqq n \\ s \in S}} |s^p/(s)_p| < \infty$$

and with

$$\varepsilon = \min\{\lambda_n - \lambda_{n+1}, \mu_{n+1} - \mu_n\}, \qquad c = \operatorname{Re} s,$$

we have

$$\sigma_n(\Phi) \leqq K_0 \sup_{\substack{0 \leqq p \leqq n \\ \lambda_n \leqq c \leqq \mu_n}} \left\{\int_0^1 |t^{c+p-\varepsilon}\phi^{(p)}(t)|t^{-1+\varepsilon}\,dt + \int_1^\infty |t^{c+p+\varepsilon}\phi^{(p)}(t)|t^{-1-\varepsilon}\,dt\right\} \leqq \frac{2K_0}{\varepsilon}\tau_{n+1}(\phi).$$

This proves the continuity of $\mathscr{M}$.

Let $\Phi \in S(\lambda, \mu)$ and let $\phi = \mathscr{M}^{-1}\Phi$. Then

$$\begin{aligned}
\tau_n(\phi) &= \sup_{\substack{0 \leqq p \leqq n \\ \lambda_n \leqq c \leqq \mu_n \\ t > 0}} |t^{c+p}\phi^{(p)}(t)| \\
&= \sup_{t,c,p} \left| t^{c+p}\frac{1}{2\pi}\int_{c-i\infty}^{c+i\infty} (s)_p\Phi(s)t^{-s-p}\,ds\right| \\
&\leqq \sup_{t,c,p} \frac{1}{2\pi}\left\{\left(\int_{c-i\infty}^{c-i} + \int_{c+i}^{c+i\infty}\right)|s^{p+2}\Phi(s)|\cdot\left|\frac{(s)_p}{s^{p+2}}\right|\cdot|ds| + \left|\int_{c-i}^{c+i}(s)_p\Phi(s)t^{c-s}\,ds\right|\right\} \\
&\leqq K\sigma_{n+2}(\Phi),
\end{aligned}$$

where K depends only on n. Thus $\mathscr{M}^{-1}$ is continuous. The second assertion of the theorem follows at once.

2. Watson transforms on $T(\lambda, \mu)$. In this section we will consider a Watson transformation between two spaces of type $T(\lambda, \mu)$. Formally such a transformation is described by a pair of reciprocal formulas

$$\psi(x) = \int_0^\infty k(xt)\phi(t)\,dt, \qquad \phi(x) = \int_0^\infty h(xt)\psi(t)\,dt.$$

By applying the Mellin transform to these formulas we may formally show that the Mellin transforms $K(s)$ and $H(s)$ of $k(t)$ and $h(t)$ satisfy $K(s)H(1-s) = 1$, (cf. Titchmarsh [13]). We prove two theorems on these transforms in spaces $T(\lambda, \mu)$.

THEOREM 2. *Let $\lambda, \mu \in \mathbb{R}^*$, $\lambda < \mu$. Let $K(s)$ be an analytic function on $\lambda < \operatorname{Re} s < \mu$ such that $K(c + it) \in L(-\infty, \infty)$ for some c with $\lambda < c < \mu$. Assume moreover that for every pair (a, b) such that $\lambda < a \leqq b < \mu$ there exists a real number γ such that*

$$K(s) = O(s^\gamma) \quad \text{as } s \to \infty, \text{uniformly on } a \leqq \operatorname{Re} s \leqq b. \tag{2.1}$$

Let

$$k(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} K(s)t^{-s}\,ds, \qquad t > 0. \tag{2.2}$$

Then the map $A: T(1-\mu, 1-\lambda) \to T(\lambda, \mu)$, defined by

$$\psi(x) = (A\phi)(x) = \int_0^\infty k(xt)\phi(t)\,dt \tag{2.3}$$

is linear and continuous. The adjoint operator A' is continuous from $T'(\lambda, \mu)$ into $T'(1-\mu, 1-\lambda)$.

Proof. The integral in (2.2) is absolutely convergent, hence $k(t)$ exists for $t > 0$. It follows from the definition that if $\phi \in T(1-\mu, 1-\lambda)$, then $t^{-c}\phi(t) \in L(0, \infty)$. Then the reversion of the order of integration in the following computation is allowed:

$$\begin{aligned} \psi(x) &= (A\phi)(x) = \frac{1}{2\pi i}\int_0^\infty dt\,\phi(t)\int_{c-i\infty}^{c+i\infty} K(s)x^{-s}t^{-s}\,ds, \\ \psi(x) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} K(s)\Phi(1-s)x^{-s}\,ds, \end{aligned} \tag{2.4}$$

where $\Phi = \mathscr{M}\phi$. Since $\Phi \in S(1-\mu, 1-\lambda)$, we have $\Phi(1-s) \in S(\lambda, \mu)$. Moreover, from (2.1) we obtain

$$s^q K(s)\Phi(1-s) = O(s^{q+\gamma-p}) \quad \text{as } s \to \infty, \text{on } a \leqq \operatorname{Re} s \leqq b \text{ if } p, q \in \mathbb{N},$$

and we see that $K(s)\Phi(1-s) \in S(\lambda, \mu)$. Define the map

$$\mathscr{K}: S(1-\mu, 1-\lambda) \to S(\lambda, \mu)$$

by

$$(\mathscr{K}\Phi)(s) = K(s)\Phi(1-s).$$

It is clear that $\mathscr{K}$ is linear and continuous. Now (2.4) reads as

$$(A\phi)(x) = (\mathscr{M}^{-1} \circ \mathscr{K} \circ \mathscr{M}\phi)(x) \tag{2.5}$$

and the desired properties of A follow from the corresponding ones of the factors.

If we impose further conditions on $K(s)$ in Theorem 2, then the map A is even an isomorphism. From Fig. 1 it is seen that we have to choose $K(s)$ in such a way that $\mathscr{K}$ is an isomorphism. The following theorem gives the precise conditions.

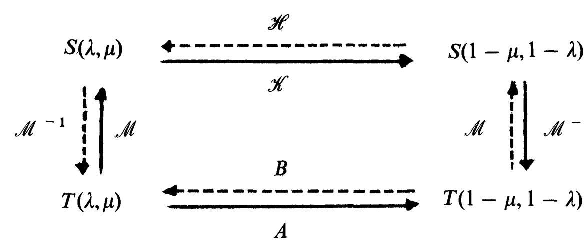

FIG. 1.

THEOREM 3. *Let λ, μ and $K(s)$ be as in Theorem 2 and let $K(s)$ have no zeros in $\lambda < \operatorname{Re} s < \mu$. Define $H(s) = K^{-1}(1-s)$, $1-\mu < \operatorname{Re} s < 1-\lambda$. Assume $H(c_1 + it) \in L(-\infty, \infty)$ for some c_1 with $1-\mu < c_1 < 1-\lambda$. Moreover, assume that to every pair (a_1, b_1), $1-\mu < a_1 \leqq b_1 < 1-\lambda$, there exists a constant γ_1 such that*

$$H(s) = O(s^{\gamma_1}) \quad \text{as } s \to \infty, \text{uniformly on } a_1 \leqq \operatorname{Re} s \leqq b_1. \tag{2.6}$$

Then the map A in Theorem 2 is an isomorphism on $T(1-\mu, 1-\lambda)$ onto $T(\lambda, \mu)$ and the inverse B of A is given by

$$\phi(x) = (B\psi)(x) = \int_0^\infty h(xt)\psi(t)\,dt, \qquad \psi \in T(\lambda, \mu), \tag{2.7}$$

where

$$h(t) = \frac{1}{2\pi i}\int_{c_1 - i\infty}^{c_1 + i\infty} H(s)t^{-s}\,ds. \tag{2.8}$$

The adjoint operator A' is an isomorphism from $T'(\lambda, \mu)$ onto $T'(1-\mu, 1-\lambda)$ with $(A')^{-1} = B'$.

Proof. Define the map $\mathscr{H} : S(\lambda, \mu) \to S(1-\mu, 1-\lambda)$ by

$$(\mathscr{H}\Psi)(s) = H(s)\Psi(1-s), \qquad \Psi \in S(\lambda, \mu).$$

It is easy to see that $\mathscr{H}$ is the continuous inverse of $\mathscr{K}$. If B is defined by (2.7), then

$$B = \mathscr{M}^{-1} \circ \mathscr{H} \circ \mathscr{M}. \tag{2.9}$$

This may be proved in the same way as (2.5). Combining (2.5) and (2.9) we see that A and B are inverses of one another.

Remark 1. The conditions $K(c + it) \in L(-\infty, \infty)$ and $H(c_1 + it) \in L(-\infty, \infty)$ in Theorems 2 and 3 may be omitted provided (2.2), (2.3), (2.7) and (2.8) are modified as follows.

From the assumptions on $K(s)$ we deduce that there are numbers $d \in \mathbb{R}$, $\varepsilon > 0$ and a positive integer n such that

$$K(s) = O(s^{n-1-\varepsilon}) \quad \text{as } s \to \infty \text{ on } \operatorname{Re} s = d, \quad \lambda < d < \mu, \quad d \neq 1, 2, \cdots, n.$$

Then define

$$k_n(t) = \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} \frac{K(s)}{(1-s)_n} t^{n-s}\, ds$$

and

$$(2.10) \qquad (A\phi)(x) = \frac{d^n}{dx^n} \int_0^\infty k_n(xt)\phi(t)t^{-n}\, dt, \quad \text{if } \phi \in T(1-\mu, 1-\lambda).$$

Now

$$\begin{aligned}(A\phi)(x) &= \frac{d^n}{dx^n} \int_0^\infty dt\, \phi(t)t^{-n} \cdot \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} \frac{K(s)}{(1-s)_n} (xt)^{n-s}\, ds \\ &= \frac{1}{2\pi i} \frac{d^n}{dx^n} \int_{d-i\infty}^{d+i\infty} ds \frac{K(s)}{(1-s)_n} x^{n-s} \int_0^\infty \phi(t)t^{-s}\, dt \\ &= \frac{1}{2\pi i} \frac{d^n}{dx^n} \int_{d-i\infty}^{d+i\infty} \frac{K(s)}{(1-s)_n} (\mathcal{M}\phi)(1-s) x^{n-s}\, ds \\ &= \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} K(s)(\mathcal{M}\phi)(1-s) x^{-s}\, ds.\end{aligned}$$

Similarly, $h_m(t)$ and B are defined. Fig. 1 remains valid.

3. Watson transforms on the subspaces T_m and S_m. In this section we shall take $\lambda \in \mathbb{C}$ and $\mu \in \mathbb{R}^*$, $\operatorname{Re} \lambda < \mu$. We want to define subspaces of $T(\operatorname{Re} \lambda, \mu)$ and $S(\operatorname{Re} \lambda, \mu)$ which are mapped onto one another by the maps A and B of § 2. The motivation will become clear in the next section.

Let m be a positive number and $\operatorname{Re} \lambda < \mu$. Then $T_m(\lambda, \mu)$ is the linear space of functions $\phi \in T(\operatorname{Re} \lambda, \mu)$ such that

$$\phi(t) = t^{-\lambda}\tilde{\phi}(t^m), \qquad t > 0, \quad \tilde{\phi} \in C^\infty[0, \infty).$$

We choose a topology on $T_m(\lambda, \mu)$ which is finer than the induced topology of $T(\operatorname{Re} \lambda, \mu)$. If μ_n tends monotonically to μ from below we define

$$\tilde{\tau}_n(\phi) = \sup_{\substack{t \geq 0 \\ p = 0, 1, \cdots, n}} (1 + t^{((\mu_n - \operatorname{Re} \lambda)/m) + p})|\tilde{\phi}^{(p)}(t)|,$$

and we take the topology generated by the norms $\tilde{\tau}_n$, $n \in \mathbb{N}$ on $T_m(\lambda, \mu)$. Then $T_m(\lambda, \mu)$ is a Fréchet space.

Furthermore $S_m(\lambda, \mu)$ is the linear space of elements $\Phi \in S(\operatorname{Re}\lambda, \mu)$ such that

(i) $\Phi(s)$ is analytic if $\operatorname{Re} s < \mu$ except for at most simple poles in the points $s = \lambda - mj, j \in \mathbb{N}$.

(ii) $\Phi(s) = O(s^{-p})$ as $s \to \infty$ uniformly on any strip $a \leqq \operatorname{Re} s \leqq b < \mu$ for any $p \in \mathbb{N}$.

We choose on $S_m(\lambda, \mu)$ the topology generated by the norms $\tilde{\sigma}_n$, $n \in \mathbb{N}$, where

$$\tilde{\sigma}_n(\Phi) = \sup_{s \in G_n} |\Phi(s)| \prod_{j=0}^{n} |s - \lambda + mj|,$$

$$G_n = \{s \in \mathbb{C} : \operatorname{Re}\lambda - mn + m \leqq \operatorname{Re} s \leqq \mu_n\}.$$

It is very easy to prove that $S_m(\lambda, \mu)$ is a Fréchet space.

THEOREM 4. *The Mellin transform* $\mathscr{M}$ *is an isomorphism from* $T_m(\lambda, \mu)$ *onto* $S_m(\lambda, \mu)$. *Its adjoint* $\mathscr{M}'$ *is an isomorphism from* $S'_m(\lambda, \mu)$ *onto* $T'_m(\lambda, \mu)$.

Proof. If $\phi \in T_m(\lambda, \mu)$, $\operatorname{Re}\lambda < \operatorname{Re} s < \mu$ and $p \in \mathbb{N}$, then

$$\mathscr{M}\phi(s) = \frac{1}{m}\int_0^\infty \tau^{((s-\lambda)/m)-1}\tilde{\phi}(\tau)\,d\tau$$

$$= \frac{(-1)^p}{m\left(\dfrac{s-\lambda}{m}\right)_p}\int_0^\infty \tau^{((s-\lambda)/m)+p-1}\tilde{\phi}^{(p)}(\tau)\,d\tau.$$

The last integral is analytic in s if $\operatorname{Re}\lambda - mp < \operatorname{Re} s < \mu$. Hence $\mathscr{M}\phi \in S_m(\lambda, \mu)$ and it easily follows that $\mathscr{M}$ is continuous.

If $\Phi \in S_m(\lambda, \mu)$, $\phi = \mathscr{M}^{-1}\Phi$, $\operatorname{Re}\lambda < c < \mu$, then

$$\tilde{\phi}(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Phi(s) t^{(\lambda - s)/m}\,ds, \qquad t > 0.$$

Consequently,

$$\tilde{\phi}^{(p)}(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Phi(s)\left(\frac{\lambda - s}{m}\right)\cdots\left(\frac{\lambda - s}{m} - p + 1\right)t^{((\lambda-s)/m)-p}\,ds$$

$$= p!\operatorname*{Res}_{s=\lambda-mp}\Phi(s) + \frac{1}{2\pi i}\int_{c_1-i\infty}^{c_1+i\infty}\Phi(s)\left(\frac{\lambda-s}{m}\right)$$

$$\cdots\left(\frac{\lambda-s}{m} - p + 1\right)t^{((\lambda-s)/m)-p}\,ds,$$

where $\operatorname{Res}_{s=\lambda-mp}$ denotes "residue at $s = \lambda - mp$ of"; if $\operatorname{Re}\lambda - m(p+1) < c_1 < \operatorname{Re}\lambda - mp$, $t > 0$. Therefore $\tilde{\phi} \in C^p\,[0, \infty)$. Further it is easily seen that $\phi \in T_m(\lambda, \mu)$ and that $\mathscr{M}^{-1}$ is continuous.

We now follow the method of § 2 to derive some further theorems.

THEOREM 5. *Let* $\lambda \in \mathbb{C}$, $\mu \in \mathbb{R}^*$, $\operatorname{Re}\lambda < \mu$ *and* m *be a positive number. Assume that* $K(s)$ *is analytic for* $\operatorname{Re} s < \mu$ *except for simple poles at* $s = \lambda - jm$, $j \in \mathbb{N}$. *Assume moreover that for each pair* (a, b), $a \leqq b < \mu$, *there exists a constant* γ *such that* (2.1) *holds. Let* $K(c + it) \in L(-\infty, \infty)$ *for some* c *with* $\operatorname{Re}\lambda < c < \mu$. *Then the* *map A of Theorem 2 maps $T(1-\mu,\infty)$ linearly and continuously into $T_m(\lambda,\mu)$. The adjoint map A' is a continuous operator on $T'_m(\lambda,\mu)$ into $T'(1-\mu,\infty)$.*

Proof. The map $\mathscr{K}$ used in the proof of Theorem 2 is a continuous map on $S(1-\mu,\infty)$ into $S_m(\lambda,\mu)$.

In the same way we have the following.

THEOREM 6. *Let λ, μ and m be as in Theorem 5. Assume $H(s)$ is analytic for $\operatorname{Re} s > 1-\mu$ and $H(s)=0$ if $s = 1-\lambda+jm$, $j=0,1,2,\cdots$. Assume that for each pair (a_1,b_1) such that $1-\mu < a_1 \leqq b_1$ there exists a constant γ_1 such that* (2.6) *holds. Moreover let $H(c_1+it)\in L(-\infty,\infty)$ for some c_1 with $1-\mu<c_1<1-\operatorname{Re}\lambda$. Then the map B defined by* (2.7) *maps $T_m(\lambda,\mu)$ linearly and continuously into $T(1-\mu,\infty)$ and B' is a continuous operator on $T'(1-\mu,\infty)$ into $T'_m(\lambda,\mu)$.*

If $H(s)K(1-s)=1$ and $\operatorname{Re} s>1-\mu$, then A is an isomorphism of $T(1-\mu,\infty)$ onto $T_m(\lambda,\mu)$ with inverse B.

Remark 2. Here also we may omit the conditions $K(c+it)\in L(-\infty,\infty)$ and $H(c+it)\in L(-\infty,\infty)$ as in Remark 1 of § 2.

4. Examples.

Example 1. Let m be a positive number, $\lambda\in\mathbb{C}$, λ_0, $\mu\in\mathbb{R}$, $\operatorname{Re}\lambda\leqq\lambda_0<\mu\leqq 1-\operatorname{Re}\lambda$ and let $K_1(s)$ be analytic on $\operatorname{Re}s<\mu$ and on $\operatorname{Re}s>1-\mu$, whereas $K_1(s)=K_1^{-1}(1-s)$. Assume that (2.1) holds for $K_1(s)$ on any set $\lambda'\leqq\operatorname{Re}s\leqq\mu'<\mu$ and any set $1-\mu<1-\mu'\leqq\operatorname{Re}s\leqq 1-\lambda'$. Assume

$$K_1(c+it)=O(t^{-((2c-1)/m)-1-\varepsilon})\quad\text{as } t\to\infty \tag{4.1}$$

for some c with $\lambda_0<c<\mu$ and for some c with $1-\mu<c<1-\lambda_0$, and some $\varepsilon>0$. Define

$$K(s)=\frac{\Gamma((s-\lambda)/m)}{\Gamma((1-\lambda-s)/m)}K_1(s).$$

Then $K(s)=K^{-1}(1-s)$ and Theorems 2 and 6 imply that A is a homeomorphism from $T(1-\mu,1-\lambda_0)$ onto $T(\lambda_0,\mu)$ and from $T(1-\mu,\infty)$ onto $T_m(\lambda,\mu)$, whereas $A=A^{-1}$. Condition (4.1) may be omitted if A is interpreted as in Remark 1.

Example 2. A special case of Example 2 is the following. Let $K_1(s)=2^{s-(1/2)}$ and

$$K(s)=\frac{\Gamma((\nu+\frac12+s)/2)}{\Gamma((\nu+\frac32-s)/2)}2^{s-(1/2)}.$$

Now $K(s)=K^{-1}(1-s)$, $K_1(s)=O(1)$, $K(s)=O(s^{\operatorname{Re}s-(1/2)})$ as $s\to\infty$ on any strip $a\leqq\operatorname{Re}s\leqq b$, $m=2$, $c<\frac12$ and

$$k(t)=t^{1/2}J_\nu(t)$$

(cf. [13, p. 214]). Suppose $\operatorname{Re}\nu>-1$ and choose λ and μ such that $-\operatorname{Re}\nu-\frac12\leqq\lambda<\mu\leqq\operatorname{Re}\nu+\frac32$. If $\operatorname{Re}\nu>0$, $\lambda<-\frac12$, $\frac32<\mu$ we may choose c and c_1 such that $-\operatorname{Re}\nu-\frac12<c<-\frac12$, $c<\mu$, $1-\mu<c_1<-\frac12$, $c_1<1-\lambda$. Then $K(c+it)$, $K(c_1+it)\in L(-\infty,\infty)$. Hence, if $\operatorname{Re}\nu>0$, $-\operatorname{Re}\nu-\frac12\leqq\lambda<-\frac12$, $\frac32<\mu\leqq\operatorname{Re}\nu+\frac32$, the Hankel transform H_ν defined by

$$(H_\nu\phi)(x)=\int_0^\infty (xt)^{1/2}J_\nu(xt)\phi(t)\,dt \tag{4.2}$$

is a homeomorphism of $T(1-\mu, 1-\lambda)$ onto $T(\lambda, \mu)$ and of $T(1-\mu, \infty)$ onto $T_2(-\nu-\frac{1}{2}, \mu)$. Furthermore $H_\nu = H_\nu^{-1}$.

We may weaken these conditions by extending the definition of the Hankel transform as in Remark 1 (see for a modification of this extension § 6). Then we see that the extended Hankel transform is a continuous operator from $T(1-\mu, 1-\lambda)$ into $T(\lambda, \mu)$, if $-\operatorname{Re}\nu - \frac{1}{2} \leqq \lambda < \mu$. However, if

$$-\operatorname{Re}\nu - \tfrac{1}{2} \leqq \lambda < 1, \qquad \lambda < \mu, \tag{4.3}$$

then the extended transform and the transform given by (4.2) coincide, since the differentiations in (2.10) may be performed under the integral sign. This follows from the asymptotic behavior of the Bessel function near the origin and ∞. It is now easy to prove the following result for $T(1-\mu, 1-\lambda)$ and some of its subspaces.

THEOREM 7. *The Hankel transform H_ν defined by* (4.2) *is a continuous operator of*

(i) $T(1-\mu, 1-\lambda)$ *into* $T(\lambda, \mu)$ *if* (4.3) *holds*;

(ii) $T(1-\mu, \infty)$ *into* $T_2(-\nu-\frac{1}{2}, \mu)$ *if* $-\operatorname{Re}\nu - \frac{1}{2} < \mu$;

(iii) $T_2(-\nu-\frac{1}{2}-2h, \mu)$ *into* $T(1-\mu, \infty)$ *if* $-\operatorname{Re}\nu - \frac{1}{2} - 2h < \mu \leqq \operatorname{Re}\nu + \frac{3}{2}$, $h \in \mathbb{N}$;

(iv) $T_2(-\nu-\frac{1}{2}-2h, \infty)$ *into itself if* $\operatorname{Re}\nu > -h-1$, $h \in \mathbb{N}$.

In the cases (iii) *and* (iv) *with $h = 0$ it is an involutory isomorphism. It is also an involutory isomorphism of $T(1-\mu, 1-\lambda)$ if*

$$-\operatorname{Re}\nu - \tfrac{1}{2} \leqq \lambda < \mu \leqq \operatorname{Re}\nu + \tfrac{3}{2}, \qquad \lambda < 1, \quad \mu > 0.$$

In all these cases,

$$\mathscr{M}H_\nu\phi(s) = \frac{\Gamma(\frac{1}{2}\nu + \frac{1}{4} + \frac{1}{2}s)}{\Gamma(\frac{1}{2}\nu + \frac{3}{4} - \frac{1}{2}s)} 2^{s-(1/2)}(\mathscr{M}\phi)(1-s). \tag{4.4}$$

Remark 3. Let $K(s)$ be as in Example 1 with $K_1(s)$ an entire function, $K_1(s) = K_1^{-1}(1-s)$, $K_1(s) = O(s^\gamma)$, $s \to \infty$ on any set $a \leqq \operatorname{Re}s \leqq b$, where γ depends on a and b. Assume (4.1) holds for some c with $\operatorname{Re}\lambda < c$ and for some c with $c < 1 - \operatorname{Re}\lambda$, ($\operatorname{Re}\lambda < \frac{1}{2}$). Then $A = A^{-1}$ is an automorphism on $T_m(\lambda, \infty)$.

Example 3. Let $m, n, p, q \in \mathbb{N}$, $n \leqq p$, $m \leqq q$. Let $\mathbf{a}, \boldsymbol{\alpha} \in \mathbb{C}^p$, $\mathbf{b}, \boldsymbol{\beta} \in \mathbb{C}^q$; $a_j > 0$, $j = 1, \cdots, p$; $b_j > 0$, $j = 1, \cdots, q$. Suppose

$$\frac{\operatorname{Re}\alpha_j - 1}{a_j} < c < \frac{\operatorname{Re}\beta_h}{b_h}, \qquad j = 1, \cdots, n; \quad h = 1, \cdots, m. \tag{4.5}$$

Suppose

$$\sum_{j=1}^{n} a_j - \sum_{j=m+1}^{q} b_j > \sum_{j=n+1}^{p} a_j - \sum_{j=1}^{m} b_j \tag{4.6}$$

or the following two conditions are satisfied:

$$\sum_{j=1}^{n} a_j - \sum_{j=m+1}^{q} b_j = \sum_{j=n+1}^{p} a_j - \sum_{j=1}^{m} b_j \tag{4.7}$$

and

$$c\left(\sum_{j=1}^{p} a_j - \sum_{j=1}^{q} b_j\right) < -1 + \frac{q-p}{2} + \sum_{1}^{p} \operatorname{Re}\alpha_j - \sum_{1}^{q} \operatorname{Re}\beta_j. \tag{4.8}$$

Then we define according to Fox [6]:

$$H_{p,q}^{m,n}\left(x\,\middle|\,\begin{matrix}\mathbf{a},\boldsymbol{\alpha}\\ \mathbf{b},\boldsymbol{\beta}\end{matrix}\right) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\prod_1^n \Gamma(1-\alpha_j+a_js)\prod_1^m \Gamma(\beta_j - b_js)x^{-s}}{\prod_{m+1}^q \Gamma(1-\beta_j+b_js)\prod_{n+1}^p \Gamma(\alpha_j - a_js)}\,ds \tag{4.9}$$

if $x > 0$. This integral is easily seen to be absolutely convergent.

Suppose

$$\frac{\operatorname{Re}\alpha_j - 1}{a_j} \leqq \lambda < \mu \leqq \frac{\operatorname{Re}\beta_h}{b_h}, \qquad j = 1, \cdots, n; \quad h = 1, \cdots, m. \tag{4.10}$$

Then the map A defined by

$$(A\phi)(x) = \int_0^\infty H_{p,q}^{m,n}\left(xt\,\middle|\,\begin{matrix}\mathbf{a},\boldsymbol{\alpha}\\ \mathbf{b},\boldsymbol{\beta}\end{matrix}\right)\phi(t)\,dt \tag{4.11}$$

is a continuous linear map of $T(1-\mu, 1-\lambda)$ into $T(\lambda, \mu)$.

A is an isomorphism of $T(1-\mu, 1-\lambda)$ into $T(\lambda, \mu)$ with

$$(A^{-1}\psi)(x) = \int_0^\infty H_{p,q}^{q-m,p-n}\left(xt\,\middle|\,\begin{matrix}\tilde{\mathbf{a}},\tilde{\boldsymbol{\alpha}}\\ \tilde{\mathbf{b}},\tilde{\boldsymbol{\beta}}\end{matrix}\right)\psi(t)\,dt, \tag{4.12}$$

where

$$\begin{aligned}\tilde{\mathbf{a}} &= (a_{n+1}, \cdots, a_p, a_1, \cdots, a_n),\\ \tilde{\mathbf{b}} &= (b_{m+1}, \cdots, b_q, b_1, \cdots, b_m),\end{aligned} \tag{4.13}$$

$$\begin{aligned}\tilde{\boldsymbol{\alpha}} &= (1 + a_{n+1} - \alpha_{n+1}, \cdots, 1 + a_p - \alpha_p, 1 + a_1 - \alpha_1, \cdots, 1 + a_n - \alpha_n),\\ \tilde{\boldsymbol{\beta}} &= (1 + b_{m+1} - \beta_{m+1}, \cdots, 1 + b_q - \beta_q, 1 + b_1 - \beta_1, \cdots, 1 + b_m - \beta_m)\end{aligned} \tag{4.14}$$

if the following conditions are satisfied:

(i) (4.7), (4.8) and (4.10);

$$\text{(ii)}\quad \begin{cases}\mu \leqq \operatorname{Re}(\alpha_j/a_j), & j = n+1, \cdots, p,\\ \operatorname{Re}((\beta_j - 1)/b_j) \leqq \lambda, & j = m+1, \cdots, q;\end{cases} \tag{4.15}$$

(iii) there exists a real number c_1 such that $1 - \mu < c_1 < 1 - \lambda$ and

$$(c_1 - 1)\left(\sum_1^p a_j - \sum_1^q b_j\right) < -1 + \frac{p-q}{2} + \sum_1^q \operatorname{Re}\beta_j - \sum_1^p \operatorname{Re}\alpha_j. \tag{4.16}$$

Proceeding as in Remark 1, § 2, we may extend the definition of A and A^{-1} in cases where (4.8) and (4.16) are not satisfied (cf. also § 8).

Since the G-function and many other special functions are special cases of the H-function, many integral transforms are contained in this example. Especially the Hankel transform of Example 2 may be considered as a special case of Example 3.

5. Other product convolutions; fractional integrals. The Watson transforms of §§ 2 and 3 have the so-called "product-kernel" $k(xt)$. Another integral transform arises if we replace $k(xt)$ by $k(x/t)$ and $\phi(t)$ by $(1/t)\phi(1/t)$. Both integral transforms are called product convolutions. Since

$$\int_0^\infty k\left(\frac{x}{t}\right)\phi(t)\frac{dt}{t} = \int_0^\infty k(xt)\phi\left(\frac{1}{t}\right)\frac{dt}{t},$$

the new integral transform is a Watson transform applied to $(1/t)\phi(1/t)$. If $\phi \in T(\lambda, \mu)$, then $(1/t)\phi(1/t) \in T(1-\mu, 1-\lambda)$ and conversely. Hence we have the following.

THEOREM 8. *If $k(x)$ satisfies the assumptions of Theorem 2, then the map A_1 of $T(\lambda, \mu)$ defined by*

$$A_1\phi(x) = \int_0^\infty k\left(\frac{x}{t}\right)\phi(t)\frac{dt}{t}, \qquad \phi \in T(\lambda, \mu), \tag{5.1}$$

is linear and continuous into $T(\lambda, \mu)$.

Moreover, if $K(s)$ does not have zeros in $\lambda < \operatorname{Re} s < \mu$ and $H(s) = K^{-1}(s)$ satisfies (2.6) *uniformly on any strip $\lambda < a_1 \leqq \operatorname{Re} s \leqq b_1 < \mu$ with some constant γ_1 depending on a_1 and b_1 and if $H(c_1 + it) \in L(-\infty, \infty)$ for some c_1 with $\lambda < c_1 < \mu$, then A_1 is an isomorphism of $T(\lambda, \mu)$ onto $T(\lambda, \mu)$ and*

$$(A_1^{-1}\phi)(x) = \int_0^\infty h\left(\frac{x}{t}\right)\phi(t)\frac{dt}{t}, \tag{5.2}$$

where h is defined by (2.8).

Remark 4. The maps A_1 and A_1^{-1} are given in Fig. 2;

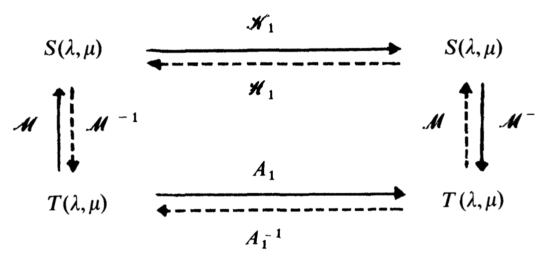

FIG. 2.

where $(\mathscr{K}_1\Phi)(s) = K(s)\Phi(s)$, $(\mathscr{H}_1\Phi)(s) = H(s)\Phi(s)$. The conditions on $K(s)$ and $H(s)$ may be weakened as in Remark 1. If we define k_n and h_m as in Remark 1, then

$$(A_1\phi)(x) = \frac{d^n}{dx^n}\int_0^\infty k_n\left(\frac{x}{t}\right)\phi(t)t^{n-1}\,dt,$$

$$(A_1^{-1}\phi)(x) = \frac{d^m}{dx^m}\int_0^\infty h_m\left(\frac{x}{t}\right)\phi(t)t^{m-1}\,dt,$$

where $\phi \in T(\lambda, \mu)$. It is easy to formulate and to prove the analogues of Theorems 5 and 6 for the transform A_1.

As an application of this type of product convolution we consider the operators of fractional integration, studied among others by Kober [9] and Erdélyi [3].

Let $\alpha, \eta \in \mathbb{C}$, $\operatorname{Re}\alpha > 0$, $\lambda, \mu, m \in \mathbb{R}$, $m > 0$, $\lambda < \mu$ and $m(\operatorname{Re}\eta + 1) > \lambda$. Then

$$(5.3)\qquad \begin{aligned} I_m^{\eta,\alpha}\phi(x) &= \frac{m}{\Gamma(\alpha)} x^{-m(\alpha+\eta)} \int_0^x (x^m - t^m)^{\alpha-1} t^{m\eta+m-1}\phi(t)\,dt \\ &= \frac{m}{\Gamma(\alpha)} \int_0^x \left\{\left(\frac{x}{t}\right)^m - 1\right\}^{\alpha-1} \left(\frac{x}{t}\right)^{-m(\alpha+\eta)} \phi(t)\frac{dt}{t}, \end{aligned}$$

if we choose $\phi \in T(\lambda, \mu)$, $x > 0$. So we have the special case of Theorem 8 with

$$k(t) = \frac{m}{\Gamma(\alpha)}(t^m - 1)^{\alpha-1} t^{-m(\alpha+\eta)} \quad \text{if } t > 1, \quad k(t) = 0 \quad \text{if } 0 < t < 1,$$

and

$$(5.4)\qquad K(s) = \Gamma\left(1 + \eta - \frac{s}{m}\right)\left\{\Gamma\left(1 + \eta + \alpha - \frac{s}{m}\right)\right\}^{-1}.$$

Here $K(s) = O(s^{-\alpha})$ as $s \to \infty$ uniformly on any strip $a \leqq \operatorname{Re} s \leqq b$. Hence, $I_m^{\eta,\alpha}$ is an automorphism of $T(\lambda, \mu)$ if $\operatorname{Re}\alpha > 1$ and

$$(5.5)\qquad \lambda < \mu \leqq m(1 + \operatorname{Re}\eta).$$

In order to relax the conditions on η we use the extension of fractional integrals considered by Erdélyi [1]. If $\operatorname{Re}\alpha > 1$, $\phi \in T(\lambda, \mu)$, $h \in \mathbb{N}$ and

$$(5.5)^h\qquad m(\operatorname{Re}\eta + h) \leqq \lambda < \mu \leqq m(1 + \operatorname{Re}\eta + h), \qquad h \neq 0,$$

we define

$$(5.3)^h\qquad \begin{aligned} I_{m,h}^{\eta,\alpha}\phi(x) = \frac{m}{\Gamma(\alpha)} x^{-m(\alpha+\eta)} \Bigg[\int_0^x \left\{ (x^m - t^m)^{\alpha-1} - \sum_{j=0}^{h-1} \binom{\alpha-1}{j} (-x^{-m}t^m)^j x^{m(\alpha-1)} \right\} \\ \cdot\, t^{m(1+\eta)-1}\phi(t)\,dt - \int_x^\infty \sum_{j=0}^{h-1} \binom{\alpha-1}{j} (-x^{-m}t^m)^j x^{m(\alpha-1)} t^{m(1+\eta)-1}\phi(t)\,dt \Bigg]. \end{aligned}$$

It is easy to show that this so-called cut fractional integral operator is a product convolution and that the Mellin transform of the kernel is given by (5.4). Moreover, the operator $I_{m,h}^{\eta,\alpha}$ is continuous on $T(\lambda, \mu)$, $h \in \mathbb{N}$, $h \neq 0$. For convenience we shall use the notation $I_{m,0}^{\eta,\alpha}$ for $I_m^{\eta,\alpha}$ and $(5.5)^0$, $(5.3)^0$ for (5.5), (5.3).

In order to avoid the condition on α we may use Remark 4. However, an adaption of the method in that remark is more useful. The starting point for this extension is the relation

$$(5.6)\qquad x^{-m(\alpha+\eta)}\left(\frac{d}{dx^m}\right)^n x^{m(\alpha+\eta+n)} I_{m,h}^{\eta,\alpha+n} = I_{m,h}^{\eta,\alpha},$$

which is valid on $T(\lambda, \mu)$ if $\operatorname{Re}\alpha > 1$, $n \in \mathbb{N}$ and $(5.5)^h$ is satisfied. For, if we apply the left-hand side of (5.6) to $\phi \in T(\lambda, \mu)$, then we obtain

$$\frac{1}{2\pi i} x^{-m(\alpha+\eta)} \left(\frac{d}{dx^m}\right)^n \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(1 + \eta - (s/m))}{\Gamma(1 + \alpha + \eta + n - (s/m))} \Phi(s)(x^m)^{\alpha+\eta+n-(s/m)}\,ds,$$

where $\lambda < c < \mu$, $\Phi = \mathcal{M}\phi$, and this expression is easily seen to be equal to $I_{m,h}^{\eta,\alpha}\phi$. However, the left-hand side of (5.6) defines a continuous operator on $T(\lambda, \mu)$ if $\operatorname{Re}(\alpha + n) > 1$ and $(5.5)^h$ holds. Therefore we use (5.6) as the definition of $I_{m,h}^{\eta,\alpha}$ on $T(\lambda, \mu)$ if $\operatorname{Re}(\alpha + n) > 1$ and $(5.5)^h$ holds. If $\operatorname{Re}\alpha > 0$ and $(5.5)^h$ holds, then $(5.3)^h$ remains valid.

The operator $I_{m,h}^{\eta,\alpha}$ is continuous on $T(\lambda, \mu)$ and

$$(5.7)\qquad (\mathcal{M}I_{m,h}^{\eta,\alpha}\phi)(s) = \Gamma\left(1 + \eta - \frac{s}{m}\right)\left\{\Gamma\left(1 + \alpha + \eta - \frac{s}{m}\right)\right\}^{-1}(\mathcal{M}\phi)(s),$$

if $(5.5)^h$ is satisfied and $\phi \in T(\lambda, \mu)$. From this it easily follows that

$$(5.8)\qquad I_{m,h_1}^{\eta+\alpha,\beta}I_{m,h}^{\eta,\alpha} = I_{m,h}^{\eta,\alpha}I_{m,h_1}^{\eta+\alpha,\beta} = I_{m,h}^{\eta,\alpha+\beta}$$

on $T(\lambda, \mu)$ if $(5.5)^h$ holds and

$$(5.9)\qquad \begin{aligned} &\lambda < \mu \leqq m(1 + \operatorname{Re}\eta + \operatorname{Re}\alpha) \quad \text{if } h_1 = 0, \\ &m(\operatorname{Re}\eta + \operatorname{Re}\alpha + h_1) \leqq \lambda < \mu \leqq m(1 + \operatorname{Re}\eta + \operatorname{Re}\alpha + h_1) \quad \text{if } h_1 \neq 0. \end{aligned}$$

Then in particular,

$$(5.10)\qquad (I_{m,h}^{\eta,\alpha})^{-1} = I_{m,h_1}^{\eta+\alpha,-\alpha}, \qquad I_{m,h}^{\eta,0} = \text{identity operator},$$

and $I_{m,h}^{\eta,\alpha}$ is a topological automorphism of $T(\lambda, \mu)$.

According to (5.6),

$$(5.11)\qquad I_{m,h}^{\eta,-n} = x^{-m(\eta-n)}\left(\frac{d}{dx^m}\right)^n x^{m\eta}, \qquad n \in \mathbb{N}.$$

We may use this last relation as the definition for arbitrary values of η. Indeed, it is easily verified that the right-hand side of (5.11) represents a continuous operator of $T(\lambda, \mu)$ into itself even if $(5.5)^h$ is not satisfied.

Combining (5.8) and (5.11) we obtain an analogue of (5.6),

$$(5.6)'\qquad I_{m,h}^{\eta,\alpha} = I_{m,h}^{\eta,\alpha+n}x^{-m(\alpha+\eta)}\left(\frac{d}{dx^m}\right)^n x^{m(\alpha+\eta+n)}.$$

From (5.7) we readily deduce that if $\alpha = n \in \mathbb{N}$, then

$$(5.12)\qquad (\mathcal{M}I_{m,h}^{\eta,n}\phi)(s) = \left\{\left(1 + \eta - \frac{s}{m}\right)_n\right\}^{-1}(\mathcal{M}\phi)(s),$$

and consequently, $I_{m,h}^{\eta,n}$ is a continuous operator independent of h on $T(\lambda, \mu)$ if $h \geqq n$,

$$(5.12)'\qquad m(n + \operatorname{Re}\eta) \leqq \lambda < \mu.$$

Finally (5.7) implies

$$(5.13)\qquad I_{m,h}^{\eta,\alpha}x^\beta = x^\beta I_{m,h}^{\eta+(\beta/m),\alpha}$$

on $T(\lambda, \mu)$ if the operators I exist. The above results are collected in the following theorem.

THEOREM 9. *Let $h, n \in \mathbb{N}$, $m > 0$, $\eta, \alpha \in \mathbb{C}$, $\lambda < \mu$. Let the operator $I_{m,h}^{\eta,-n}$ be defined by* (5.11) *on $T(\lambda, \mu)$. It is independent of h. Let the operator $I_{m,h}^{\eta,\alpha}$ be defined by*

$(5.3)^h$ if $\operatorname{Re}\alpha > 0$ *and* $(5.5)^h$ *holds. Here* $I_{m,0}^{\eta,\alpha} = I_m^{\eta,\alpha}$ *and* $(5.3)^\circ$, $(5.5)^\circ$ *denote* (5.3), (5.5). *If* $\alpha = n$, $h \geqq n$, *then condition* $(5.5)^h$ *may be replaced by* $(5.12)'$ *and then* $I_{m,h}^{\eta,n}$ *is independent of h. If* $-n < \operatorname{Re}\alpha \leqq 0$, $-\alpha \notin \mathbb{N}$ *and* $(5.5)^h$ *holds, then* $I_{m,h}^{\eta,\alpha}$ *is defined by* (5.6) *on* $T(\lambda, \mu)$. *This definition does not depend on the choice of n.*

In all these cases the operator $I_{m,h}^{\eta,\alpha}$ *is a continuous operator from* $T(\lambda, \mu)$ *into itself. It satisfies* (5.6), $(5.6)'$, (5.7), (5.8) *and* (5.13) *on* $T(\lambda, \mu)$ *provided the operators I involved exist. In particular,* $I_{m,h}^{\eta,\alpha}$ *is an automorphism on* $T(\lambda, \mu)$ *satisfying* (5.10) *if* $(5.5)^h$ *and* (5.9) *hold.*

A second operator of fractional integration studied a.o. by Kober [9] and Erdélyi is given by

$$K_m^{\eta,\alpha}\phi(x) = \frac{m}{\Gamma(\alpha)} x^{m\eta} \int_x^\infty (t^m - x^m)^{\alpha-1} t^{m(1-\alpha-\eta)-1}\phi(t)\,dt$$

$$(5.14)\qquad = \frac{m}{\Gamma(\alpha)} \int_x^\infty \left\{1 - \left(\frac{x}{t}\right)^m\right\}^{\alpha-1} \left(\frac{x}{t}\right)^{m\eta} \phi(t)\frac{dt}{t}.$$

Here we choose $\phi \in T(\lambda, \mu)$, $\operatorname{Re}\alpha > 0$, $m \operatorname{Re}\eta + \mu > 0$. This is the special case of Theorem 8 with

$$k(t) = \frac{m}{\Gamma(\alpha)}(1 - t^m)^{\alpha-1} t^{m\eta} \quad \text{if } 0 < t < 1, \quad k(t) = 0 \quad \text{if } t > 1,$$

and

$$K(s) = \Gamma\left(\eta + \frac{s}{m}\right)\left\{\Gamma\left(\alpha + \eta + \frac{s}{m}\right)\right\}^{-1}.$$

Now $K(s) = O(s^{-\alpha})$ as $s \to \infty$ uniformly on any strip $a \leqq \operatorname{Re} s \leqq b$. Hence, $K_m^{\eta,\alpha}$ is an automorphism of $T(\lambda, \mu)$ if $\operatorname{Re}\alpha > 1$ and

$$(5.15)\qquad -m \operatorname{Re}\eta \leqq \lambda < \mu.$$

The extension to other values of η is given by

$$(5.14)^h\qquad K_{m,h}^{\eta,\alpha}\phi(x) = \frac{m}{\Gamma(\alpha)} x^{m\eta}\left[\int_x^\infty \left\{(t^m - x^m)^{\alpha-1} - \sum_{j=0}^{h-1}\binom{\alpha-1}{j}(-t^{-m}x^m)^j t^{m(\alpha-1)}\right\} \cdot t^{m(1-\alpha-\eta)-1}\phi(t)\,dt - \int_0^x \sum_{j=0}^{h-1}\binom{\alpha-1}{j}(-t^{-m}x^m)^j t^{-m\eta-1}\phi(t)\,dt\right],$$

where $\phi \in T(\lambda, \mu)$, $\operatorname{Re}\alpha > 0$, $h \in \mathbb{N}$ and

$$(5.15)^h\qquad -m(\operatorname{Re}\eta + h) \leqq \lambda < \mu \leqq -m(\operatorname{Re}\eta + h - 1), \qquad h \neq 0.$$

Then $K_{m,h}^{\eta,\alpha}$ is a continuous operator on $T(\lambda, \mu)$. We use for convenience the notation $K_{m,0}^{\eta,\alpha}$ for $K_m^{\eta,\alpha}$ and $(5.14)^\circ$, $(5.15)^\circ$ for (5.14), (5.15). In all cases we have

$$(5.16)\qquad (\mathscr{M}K_{m,h}^{\eta,\alpha}\phi)(s) = \frac{\Gamma(\eta + (s/m))}{\Gamma(\alpha + \eta + (s/m))}(\mathscr{M}\phi)(s).$$

The analogue of (5.6) is

$$(5.17)\qquad x^{m(\alpha+\eta+n)}\left(-\frac{d}{dx^m}\right)^n x^{-m(\alpha+\eta)} K_{m,h}^{\eta,\alpha+n} = K_{m,h}^{\eta,\alpha}, \qquad n \in \mathbb{N}.$$

This relation will be used as the definition of $K_{m,h}^{\eta,\alpha}$ if $-n < \operatorname{Re}\alpha \leqq 0$. Then this operator does not depend on n, it is continuous on $T(\lambda, \mu)$ and (5.16) remains true if $(5.15)^h$ holds.

Using $(5.14)^h$, (5.16) and (5.17) it is easily seen that $K_{m,h}^{\eta,\alpha}$ also defines a continuous operator from $T_m(-m(\alpha + \eta + h_1), \mu)$ into $T_m(-m(\eta + h), \mu)$ if $h_1, h \in \mathbb{N}$ and

$$(5.18)\qquad \begin{aligned} &-m\operatorname{Re}\eta < \mu \quad \text{in case } h = 0,\\ &-m(\operatorname{Re}\eta + h) < \mu \leqq -m(\operatorname{Re}\eta + h - 1) \quad \text{in case } h > 0,\\ &-m\operatorname{Re}(\alpha + \eta + h_1) < \mu. \end{aligned}$$

Analogous to (5.8) we have

$$(5.19)\qquad K_{m,h}^{\eta,\alpha}K_{m,h_1}^{\eta+\alpha,\beta} = K_{m,h}^{\eta,\alpha+\beta},\ (K_{m,h}^{\eta,\alpha})^{-1} = K_{m,h_1}^{\eta+\alpha,-\alpha},$$

$$(5.19)'\qquad K_{m,h_1}^{\eta+\alpha,\beta}K_{m,h}^{\eta,\alpha} = K_{m,h}^{\eta,\alpha+\beta},$$

on $T(\lambda, \mu)$ if $(5.15)^h$ holds and

$$(5.20)\qquad \begin{aligned} &-m\operatorname{Re}(\alpha + \eta) \leqq \lambda < \mu \quad \text{in case } h_1 = 0,\\ &-m\operatorname{Re}(\alpha + \eta + h_1) \leqq \lambda < \mu \leqq -m\operatorname{Re}(\alpha + \eta + h_1 - 1) \quad \text{in case } h_1 > 0, \end{aligned}$$

whereas (5.19) holds on $T_m(-m(\alpha + \beta + \eta + h_2), \mu)$ if (5.18) holds and

$$\mu \leqq -m\operatorname{Re}(\alpha + \eta + h_1 - 1) \quad \text{if } h_1 > 0 \quad \text{and} \quad -m\operatorname{Re}(\alpha + \beta + \eta + h_2) < \mu. \tag{5.21}$$

The operator $K_{m,h}^{\eta,\alpha}$ is an automorphism on $T(\lambda, \mu)$ in the first case and it is an isomorphism between $T_m(-m(\alpha + \eta + h_1), \mu)$ and $T_m(-m(\eta + h), \mu)$ in the second case (with $h_2 = h$).

The analogue of (5.11) is

$$(5.22)\qquad K_{m,h}^{\eta,-n} = x^{m\eta}\left(-\frac{d}{dx^m}\right)^n x^{m(n-\eta)}, \qquad n, h \in \mathbb{N}.$$

This relation may be used as the definition of K on $T(\lambda, \mu)$ if $(5.15)^h$ is not satisfied. The analogue of (5.12) shows that $K_{m,h}^{\eta,n}$ is a continuous operator on $T(\lambda, \mu)$ independent of h if $h \geqq n$ and

$$(5.23)\qquad \lambda < \mu \leqq -m(\operatorname{Re}\eta + n - 1).$$

Then

$$(5.24)\qquad K_{m,h}^{\eta,n} = (-1)^n I_m^{-\eta-n,n}.$$

In the same way we obtain

$$(5.25)\qquad I_{m,h}^{\eta,n} = (-1)^n K_m^{-\eta-n,n}$$

on $T(\lambda, \mu)$ if $(5.12)'$ is satisfied and $h \geqq n$.

We deduce from (5.16),

$$(5.26)\qquad K_{m,h}^{\eta,\alpha}x^{\beta} = x^{\beta}K_{m,h}^{\eta-(\beta/m),\alpha},$$

if the operators K exist.

Combining (5.19) and (5.22) we get the analogue of (5.17):

$$K_{m,h}^{\eta,\alpha} = K_{m,h}^{\eta,\alpha+n} x^{m(\alpha+\eta+n)}\left(-\frac{d}{dx^m}\right)^n x^{-m(\alpha+\eta)}. \tag{5.17$'$}$$

Combining the results above we obtain the following theorem.

THEOREM 10. *Let* $n, h \in \mathbb{N}$, $m > 0$, $\alpha, \eta \in \mathbb{C}$, $\lambda < \mu$. *Let the operator* $K_{m,h}^{\eta,-n}$ *be defined independently of h by* (5.22) *on* $T(\lambda, \mu)$. *Let* $K_{m,h}^{\eta,\alpha}$ *be defined on* $T(\lambda, \mu)$ *by* $(5.14)^h$ *and* (5.17) *respectively if* $(5.15)^h$ *holds and moreover* $\operatorname{Re}\alpha > 0$ *and* $-n < \operatorname{Re}\alpha \leqq 0$ *respectively. Here* $K_{m,0}^{\eta,\alpha} = K_m^{\eta,\alpha}$ *and* $(5.15)^0$ *denotes* (5.15). *If* $\alpha = n$, $h \geqq n$ *the condition* $(5.15)^h$ *may be replaced by* (5.23).

In these cases $K_{m,h}^{\eta,\alpha}$ *is a continuous operator on* $T(\lambda, \mu)$. *It is also a continuous operator from* $T_m(-m(\alpha+\eta+h_1), \mu)$ *into* $T_m(-m(\eta+h), \mu)$ *defined by* $(5.14)^h$ *and* (5.17), *if* $h_1 \in \mathbb{N}$ *and* (5.18) *holds.*

This operator satisfies (5.16), (5.17), (5.17)$'$, (5.19) *and* (5.26) *in all cases where the expressions involved make sense according to the definitions above. In particular,* (5.19) *holds on* $T(\lambda,\mu)$ *if* $(5.15)^h$ *and* (5.20) *are satisfied, and on* $T_m(-m(\alpha+\beta+\eta+h_2),\mu)$ *if* (5.18) *and* (5.21) *are satisfied. In the first case* $K_{m,h}^{\eta,\alpha}$ *is an automorphism on* $T(\lambda, \mu)$, *in the second case* (*with* $h_2 = h$) *it is an isomorphism between* $T_m(-m(\alpha+\eta+h_1), \mu)$ *and* $T_m(-m(\eta+h), \mu)$.

We now define subspaces of $T(\lambda, \mu)$ which have useful properties for operators of fractional integration.

DEFINITION. Let a be a positive number. Then $T([0, a], \lambda)$ is the subspace of $T(\lambda, \infty)$ of functions with support contained in $[0, a]$. In the same way $T_m([0, a], \lambda)$ is the subspace of $T_m(\lambda, \infty)$ consisting of functions with support contained in $[0, a]$. Finally, $T([a, \infty), \mu)$ is the subspace of $T(-\infty, \mu)$ consisting of functions with support contained in $[a, \infty)$. It is clear that in this way really closed subspaces are defined.

From the definitions of I and K it follows that

(i) $I_m^{\eta,\alpha}$ is a continuous operator from $T([a, \infty), \mu)$ into itself if

$$\mu \leqq m(1 + \operatorname{Re}\eta) \tag{5.27}$$

and it is an automorphism if moreover

$$\mu \leqq m(1 + \operatorname{Re}\alpha + \operatorname{Re}\eta); \tag{5.28}$$

(ii) $K_m^{\eta,\alpha}$ is a continuous operator from $T([0, a], \lambda)$ into itself if

$$\lambda \geqq -m \operatorname{Re}\eta \tag{5.29}$$

and it is an automorphism if moreover

$$\lambda \geqq -m \operatorname{Re}(\alpha + \eta); \tag{5.30}$$

(iii) $K_m^{\eta,\alpha}$ is an isomorphism from $T_m([0, a], -m\alpha - m\eta)$ onto $T_m([0, a], -m\eta)$.

The translation of the results above to the dual operators is easy. A simplification of the notation may be obtained as follows. Suppose $T(\lambda_0, \mu_0) \subset T'(1-\mu, 1-\lambda)$. This is the case iff $\lambda_0 < \mu$, $\lambda < \mu_0$. Suppose $f \in T(\lambda_0, \mu_0)$, $\phi \in T(1-\mu, 1-\lambda)$ and $(5.5)^h$ is satisfied, and $(5.5)^h$ also holds with λ and μ replaced by λ_0 and μ_0. Then

$$\int_0^\infty \phi(x) I_{m,h}^{\eta,\alpha} f(x)\,dx = \int_0^\infty f(x) K_{m,h}^{\eta_0,\alpha} \phi(x)\,dx, \qquad \eta_0 = \eta + 1 - \frac{1}{m}. \tag{5.31}$$

Hence$(K_{m,h}^{\eta_0;\alpha})' = I_{m,h}^{\eta;\alpha}$ on any space $T(\lambda_0, \mu_0) \subset T'(1-\mu, 1-\lambda)$ and therefore we use this relation as a notation on $T'(1-\mu, 1-\lambda)$ if $(5.5)^h$ is satisfied. In the same way (5.31) motivates the notation $(I_{m,h}^{\eta_1;\alpha})' = K_{m,h}^{\eta;\alpha}$ on $T'(1-\mu, 1-\lambda)$ if $\eta_1 = \eta - 1 + (1/m)$ and $(5.15)^h$ is satisfied.

THEOREM 11. *Let $n, h, h_1 \in \mathbb{N}$, $m, a \in \mathbb{R}_+$, $\lambda < \mu$, $\alpha, \eta \in \mathbb{C}$, $\eta_1 = \eta - 1 + (1/m)$. Then the adjoint operator of $I_{m,h}^{\eta_1;\alpha}$, to be denoted by $K_{m,h}^{\eta;\alpha}$, is a continuous operator on $T'(1-\mu, 1-\lambda)$ in the following cases:*
(i) $\alpha = -n$; (ii) $\alpha = n$, *$h \geqq n$ and* (5.23) *holds*; (iii) $(5.15)^h$ *holds. The operator $K_m^{\eta;\alpha}$ is a continuous operator on $T'([a, \infty), 1-\lambda)$ if* (5.29) *holds.*

Furthermore, the relations (5.17), (5.17)′, (5.19), (5.19)′, (5.22), (5.24)–(5.26) *hold in all cases where the operators involved make sense according to the definitions above. In particular,* (5.19) *and* (5.19)′ *hold on $T'(1-\mu, 1-\lambda)$ if $(5.15)^h$ and* (5.20) *are satisfied. In this case $K_{m,h}^{\eta;\alpha}$ is an automorphism. Finally, $K_m^{\eta;\alpha}$ is an automorphism on $T'([a, \infty), 1-\lambda)$ if* (5.29) *and* (5.30) *are satisfied.*

THEOREM 12. *Let $n, h, h_1 \in \mathbb{N}$, $m, a \in \mathbb{R}_+$, $\lambda < \mu$, $\alpha, \eta \in \mathbb{C}$, $\eta_1 = \eta + 1 - (1/m)$. Then the adjoint operator of $K_{m,h}^{\eta_1;\alpha}$, to be denoted by $I_{m,h}^{\eta;\alpha}$, is continuous on $T'(1-\mu, 1-\lambda)$ in the following cases:*
(i) $\alpha = -n$; (ii) $\alpha = n$, *$h \geqq n$ and* (5.13) *holds*; (iii) $(5.5)^h$ *holds. It is a continuous operator from $T'_m(1 - m(\eta + h + 1), \mu)$ into $T'_m(1 - m(\alpha + \eta + h_1 + 1), \mu)$ if* (5.18) *with η replaced by η_1 holds.*

Furthermore, $I_m^{\eta;\alpha}$ is continuous from $T'([0, a], 1-\mu)$ into itself if (5.27) *holds and an automorphism if moreover* (5.28) *holds. It is an isomorphism from $T'_m([0, a], 1 - m\eta - m)$ onto $T'_m([0, a], 1 - m\alpha - m\eta - m)$.*

The operator $I_{m,h}^{\eta;\alpha}$ satisfies (5.6), (5.6)′, (5.8), (5.10), (5.11) *and* (5.13) *in all cases where the operators involved exist according to the definitions above. In particular,* (5.8) *holds on $T'(1-\mu, 1-\lambda)$ if $(5.5)^h$ and* (5.9) *are satisfied. In this case, $I_{m,h}^{\eta;\alpha}$ is an automorphism on $T'(1-\mu, 1-\lambda)$, whereas it is an isomorphism from $T'_m(1 - m(\eta + h + 1), \mu)$ into $T'_m(1 - m(\alpha + \eta + h_1 + 1), \mu)$ if* (5.18) *and* (5.21) *with η replaced by η_1 are satisfied.*

6. Extension of the Hankel transform. The extension of the Hankel transform H_ν to arbitrary values of ν has been treated in [10] and [14] by means of auxiliary operators N_ν and M_ν. (For the definitions cf. [14, pp. 135 and 163]). Our approach includes these methods as is easily seen from the behavior of the differential operators N_ν and M_ν with respect to the Mellin transform.

For the extension of the definition of the Hankel transform we use the relation

$$H_\nu = 2^\alpha x^{-\alpha} H_{\nu+\alpha} K_{2,h}^{(1/2)\nu + (1/4) + (1/2)\alpha, -\alpha} x^{-\alpha}. \tag{6.1}$$

This formula is valid on $T(1-\mu, 1-\lambda)$ if the following conditions are satisfied: (4.3),

$$-\operatorname{Re}\nu - \tfrac{1}{2} \leqq \lambda < 1 + \operatorname{Re}\alpha, \tag{6.2}$$

$$\begin{aligned} &\lambda < \mu \leqq \tfrac{3}{2} + \operatorname{Re}(\nu + 2\alpha) \quad \text{in case } h = 0,\\ &\operatorname{Re}(\nu + 2\alpha) + 2h - \tfrac{1}{2} \leqq \lambda < \mu \leqq \operatorname{Re}(\nu + 2\alpha) + 2h + \tfrac{3}{2} \quad \text{in case } h \in \mathbb{N}, h \neq 0. \end{aligned} \tag{6.3}$$

The proof is straightforward using Mellin transforms and Theorems 7 and 10.

In particular, if $n \in \mathbb{N}$ we obtain with (5.22),

$$H_\nu = (-2)^n x^{-n} H_{\nu+n} x^{\nu+n+(1/2)} \left(\frac{d}{dx^2}\right)^n x^{-\nu-(1/2)}. \tag{6.4}$$

The right-hand side exists and is a continuous operator on

(i) $T(1-\mu, 1-\lambda)$ if $-\mathrm{Re}\,\nu - \frac{1}{2} \leqq \lambda < n+1$;
(ii) $T_2(-\nu-\frac{1}{2}, \mu)$ if $-\mathrm{Re}\,\nu - 2n - \frac{1}{2} < \mu \leqq \mathrm{Re}\,\nu + \frac{3}{2}$;
(iii) $T_2(-\nu-\frac{1}{2}, \infty)$ if $\mathrm{Re}\,\nu > -n-1$.

Therefore we define in these cases H_ν by (4.2) and (6.4). By choosing n suitably we thus obtain a continuous operator H_ν:

(i) from $T(1-\mu, 1-\lambda)$ to $T(\lambda, \mu)$ if $\lambda \geqq -\mathrm{Re}\,\nu - \frac{1}{2}$;
(ii) from $T_2(-\nu-\frac{1}{2}, \mu)$ to $T(1-\mu, \infty)$ if $\mu \leqq \mathrm{Re}\,\nu + \frac{3}{2}$;
(iii) from $T_2(-\nu-\frac{1}{2}, \infty)$ into itself for arbitrary values of ν.

Then (6.1) holds:

(I) on $T(1-\mu, 1-\lambda)$ if $\lambda \geqq -\mathrm{Re}\,\nu - \frac{1}{2}$ and (6.3) is satisfied; if $\alpha \in \mathbb{N}$, we may omit (6.3) and then (6.1) reduces to (6.4) with $\alpha = n$; if $-\alpha \in \mathbb{N}$, $0 < -\alpha \leqq h$, we may replace (6.3) by $\lambda \geqq \mathrm{Re}\,\nu - \frac{1}{2}$ and use (5.24);
(II) on $T_2(-\nu - \frac{1}{2} - 2g, \mu)$ if $g \in \mathbb{N}$,

$$-\mathrm{Re}\,\nu - 2g - \tfrac{1}{2} < \mu \leqq \mathrm{Re}\,\nu + \tfrac{3}{2}, \tag{6.5}$$

$$(6.6)^h \quad \begin{array}{ll} -\mathrm{Re}(\nu+2\alpha) - \frac{1}{2} < \mu & \text{if } h = 0, \\ -\mathrm{Re}(\nu+2\alpha) - 2h - \frac{1}{2} < \mu \leqq -\mathrm{Re}(\nu+2\alpha) - 2h + \frac{3}{2} & \text{if } h \in \mathbb{N}, h \neq 0; \end{array}$$

if $\alpha \in \mathbb{N}$ we may omit (6.6), and now (6.1) reduces to (6.4) with $\alpha = n$; if $-\alpha \in \mathbb{N}$, $h \geqq -\alpha > 0$, then we may replace (6.6) by $\mu \leqq \frac{3}{2} - \mathrm{Re}\,\nu$, and use (5.24).

(III) on $T_2(-\nu-\frac{1}{2}, \infty)$ for arbitrary ν and $h = 0$.

Next we consider the cut Hankel transform (cf. [1]). Suppose $p \in \mathbb{N}$, $p \neq 0$, $\lambda < -\frac{1}{2}$,

$$(6.7)^p \quad -\mathrm{Re}\,\nu - \tfrac{1}{2} - 2p \leqq \lambda < \mu \leqq -\mathrm{Re}\,\nu + \tfrac{3}{2} - 2p.$$

If $\phi \in T(1-\mu, 1-\lambda)$, we define

$$H_{\nu,p}\phi(x) = \int_0^\infty (xt)^{1/2} \left\{ J_\nu(xt) - \sum_{j=0}^{p-1} \frac{(-1)^j (\frac{1}{2}xt)^{\nu+2j}}{j!\,\Gamma(\nu+j+1)} \right\} \phi(t)\,dt. \tag{6.8}$$

Now

$$H_{\nu,p}\phi(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(\frac{1}{2}\nu + \frac{1}{4} + \frac{1}{2}s)}{\Gamma(\frac{1}{2}\nu + \frac{3}{4} - \frac{1}{2}s)} 2^{s-(1/2)} \Phi(1-s) x^{-s}\,ds,$$

if $\Phi = \mathscr{M}\phi$, $x > 0$, $\lambda < c < -\frac{1}{2}$, $c < \mu$. So again,

$$(\mathscr{M}H_{\nu,p}\phi)(s) = \frac{\Gamma(\frac{1}{2}\nu + \frac{1}{4} + \frac{1}{2}s)}{\Gamma(\frac{1}{2}\nu + \frac{3}{4} - \frac{1}{2}s)} 2^{s-(1/2)} \Phi(1-s). \tag{6.9}$$

Analogous to (6.1) we have

$$H_{\nu,p} = 2^\alpha x^{-\alpha} H_{\nu+\alpha,p} K_{2,h}^{(1/2)\nu+(1/4)+(1/2)\alpha,-\alpha} x^{-\alpha}, \tag{6.10}$$

if $\lambda < \operatorname{Re}\alpha - \frac{1}{2}$ and $(6.7)^p$ and (6.3) are satisfied. In particular,

$$H_{\nu,p} = (-2)^n x^{-n} H_{\nu+n,p} x^{\nu+n+(1/2)} \left(\frac{d}{dx^2}\right)^n x^{-\nu-(1/2)}, \tag{6.11}$$

if $\lambda < n - \frac{1}{2}$ and $(6.7)^p$ holds. By means of (6.11) with a suitable value of $n \in \mathbb{N}$ we may extend the definition of $H_{\nu,p}$ on $T(1-\mu, 1-\lambda)$ if $(6.7)^p$ holds. Then (6.10) is valid if (6.3) and $(6.7)^p$ are satisfied, and also if $-\alpha \in \mathbb{N}$, $h \geqq -\alpha$, $\lambda \geqq \operatorname{Re}\nu - \frac{1}{2}$.

Since $H_{\nu,0} = H_\nu$ we conclude that $H_{\nu,p}$ is a continuous operator from $T(1-\mu, 1-\lambda)$ into $T(\lambda, \mu)$ if $p \in \mathbb{N}$ and $(6.7)^p$ holds where $(6.7)^\circ$ is given by

$$-\operatorname{Re}\nu - \tfrac{1}{2} \leqq \lambda < \mu. \tag{6.7°}$$

It follows that $H_{\nu,q}$, $q \in \mathbb{N}$, is a continuous operator from $T(\lambda, \mu)$ into $T(1-\mu, 1-\lambda)$ if

$$(6.12)^q \qquad \begin{aligned} &\lambda < \mu \leqq \operatorname{Re}\nu + \tfrac{3}{2} \quad \text{in case } q = 0,\\ &\operatorname{Re}\nu + 2q - \tfrac{1}{2} \leqq \lambda < \mu \leqq \operatorname{Re}\nu + 2q + \tfrac{3}{2} \quad \text{in case } q > 0. \end{aligned}$$

Using Mellin transforms, (6.9) and the Theorems 9 and 10 we may prove an extension of Theorem 7.

THEOREM 7^a. *The Hankel transform $H_{\nu,p}$ defined by* (6.8) *and* (6.11) *is a continuous operator from $T(1-\mu, 1-\lambda)$ into $T(\lambda, \mu)$ if $p \in \mathbb{N}$ and $(6.7)^p$ holds. It is an isomorphism between these spaces if moreover $(6.12)^q$ is satisfied for some $q \in \mathbb{N}$. Then*

$$(H_{\nu,p})^{-1} = H_{\nu,q}. \tag{6.13}$$

Furthermore, H_ν is an involutory automorphism on $T_2(-\nu - \frac{1}{2}, \infty)$ for arbitrary ν.

The following relations hold whenever the operators involved make sense: (6.1), (6.10), (6.4), (6.11),

$$H_{\nu,p} = 2^\alpha x^{-\alpha} I_{2,h}^{(1/2)\nu-(1/4)+(1/2)\alpha,-\alpha} H_{\nu+\alpha,p} x^{-\alpha}, \tag{6.14}$$

$$H_{\nu,p} = 2^{-\alpha} x^\alpha K_{2,p}^{(1/2)\nu+(1/4)-(1/2)\alpha,\alpha} H_{\nu+\alpha,h} x^\alpha, \tag{6.15}$$

$$H_{\nu,p} = 2^{-\alpha} x^\alpha H_{\nu+\alpha,h} I_{2,p}^{(1/2)\nu-(1/4)-(1/2)\alpha,\alpha} x^\alpha. \tag{6.16}$$

In particular, (6.1) *holds in the cases* (I), (II), (III) *mentioned above. Moreover,* (6.14) *with $p = 0$ holds in case* (I). *Formulas* (6.10) *and* (6.14) *hold on $T(1-\mu, 1-\lambda)$ if* (6.3) *and $(6.7)^p$ are satisfied; if $\alpha \in \mathbb{N}$, we may omit* (6.3) (*then we may use* (5.22) *and* (5.11)), *and if $-\alpha \in \mathbb{N}$, $0 < -\alpha \leqq h$ we may replace* (6.3) *by $\lambda \geqq \operatorname{Re}\nu - \frac{1}{2}$. In the last case we may transform* (6.10) *and* (6.14) *by means of* (5.24) *and* (5.25).

The relations (6.15) *and* (6.16) *hold on $T(1-\mu, 1-\lambda)$ if $(6.7)^p$ and*

$$(6.17)^h \qquad \begin{aligned} &-\operatorname{Re}(\nu + 2\alpha) - \tfrac{1}{2} \leqq \lambda < \mu \quad \textit{in case } h = 0,\\ &-\operatorname{Re}(\nu + 2\alpha) - 2h - \tfrac{1}{2} \leqq \lambda < \mu \leqq -\operatorname{Re}(\nu - 2\alpha) - 2h + \tfrac{3}{2} \\ &\qquad\qquad \textit{in case } h \in \mathbb{N},\ h \neq 0. \end{aligned}$$

Furthermore (6.15) *with $p = h = 0$ is also valid on $T_2(-\nu - \frac{1}{2}, \mu)$ if*

$$-\operatorname{Re}\nu - \tfrac{1}{2} < \mu \leqq \operatorname{Re}\nu + \tfrac{3}{2} + \min(0, 2\operatorname{Re}\alpha), \tag{6.18}$$

whereas it holds on $T_2(-\nu - \frac{1}{2}, \infty)$ for arbitrary ν.

Now we consider the adjoint operator $(H_{\nu,p})'$. We may simplify the notation in view of the Parseval relation

$$\int_0^\infty (H_{\nu,p}\phi)(x)\psi(x)\,dx = \int_0^\infty \phi(x)(H_{\nu,p}\psi)(x)\,dx, \tag{6.19}$$

which holds for example if $\phi, \psi \in T(1-\mu, 1-\lambda)$ and $(6.7)^p$ is satisfied (this may be proved using Mellin transforms). Therefore we may denote $(H_{\nu,p})'$ on $T'(\lambda, \mu)$ by $H_{\nu,p}$.

From Theorems 7 and 7^a we now deduce the following.

THEOREM 13. *The Hankel transform $H_{\nu,p}$ is a continuous operator from $T'(\lambda, \mu)$ into $T'(1-\mu, 1-\lambda)$ if $(6.7)^p$ holds. The operator H_ν is continuous from $T'(1-\mu, \infty)$ into $T'_2(-\nu-\frac{1}{2}-2h, \mu)$ if*

$$-\mathrm{Re}\,\nu - \tfrac{1}{2} - 2h < \mu \leqq \mathrm{Re}\,\nu + \tfrac{3}{2}, \qquad h \in \mathbb{N}. \tag{6.20}$$

This operator is an involutory isomorphism from $T'(1-\mu, \infty)$ onto $T'_2(-\nu-\frac{1}{2}, \mu)$ if (6.18) holds, and an involutory automorphism on $T'_2(-\nu-\frac{1}{2}, \infty)$ for arbitrary ν. The operator $H_{\nu,p}$ is an isomorphism from $T'(\lambda, \mu)$ onto $T'(1-\mu, 1-\lambda)$ satisfying (6.13) if $(6.7)^p$ and $(6.12)^q$ are satisfied.

The relation (6.14) with $p = 0$ holds in the following cases:

(i) *on $T'(\lambda, \mu)$ if $(6.7)^0$ and (6.3) are satisfied. If $\alpha \in \mathbb{N}$ we may omit (6.3) and use (5.11). If $-\alpha \in \mathbb{N}$, $0 < -\alpha \leqq h$, we may replace (6.3) by $\lambda \geqq \mathrm{Re}\,\nu - \frac{1}{2}$ and use (5.25).*

(ii) *on $T'(1-\mu, \infty)$ if (6.5) with some $g \in \mathbb{N}$ and $(6.6)^h$ are satisfied. If $\alpha \in \mathbb{N}$ we may omit (6.6) and use (5.11). If $-\alpha \in \mathbb{N}$, $0 < -\alpha \leqq h$, we may replace (6.6) by $\mu \leqq \frac{3}{2} - \mathrm{Re}\,\nu$ and use (5.25).*

(iii) *on $T'_2(-\nu-\frac{1}{2}, \infty)$ for arbitrary ν and $h = 0$.*

Furthermore, (6.1) is valid on $T'(\lambda, \mu)$ if $\lambda \geqq -\mathrm{Re}\,\nu - \frac{1}{2}$ and (6.3) holds. If $\alpha \in \mathbb{N}$ we may omit (6.3) and then (6.1) reduces to (6.4). If $-\alpha \in \mathbb{N}$, $0 < -\alpha \leqq h$, we may replace (6.3) by $\lambda \geqq \mathrm{Re}\,\nu - \frac{1}{2}$ and use (5.24). The relations (6.10) and (6.14) hold on $T'(\lambda, \mu)$, if (6.3) and $(6.7)^p$ are satisfied; if $\alpha \in \mathbb{N}$, we may omit (6.3) (then we may use (5.22) and (5.11)); if $-\alpha \in \mathbb{N}$, $0 < -\alpha \leqq h$, we may replace (6.3) by $\lambda \geqq \mathrm{Re}\,\nu - \frac{1}{2}$. In the last case we may use (5.24) and (5.25). The relations (6.15) and (6.16) hold on $T'(\lambda, \mu)$ if $(6.7)^p$ and $(6.17)^h$ are satisfied. Finally, (6.16) with $p = h = 0$ is valid on $T'(1-\mu, \infty)$ if (6.20) holds, whereas it holds on $T'_2(-\nu-\frac{1}{2}, \infty)$ for arbitrary ν.

7. A dual integral equation involving Hankel functions. Let $c_1, c_2, \nu \in \mathbb{C}$, $a > 0$, $\lambda < \mu$, $p \in \mathbb{N}$ and

$$g_1 \in T'([0, a], 1-\mu-\mathrm{Re}\,c_1), \; g_2 \in T'([a, \infty), 1-\lambda-\mathrm{Re}\,c_2). \tag{7.1}$$

Consider the following dual integral equation:

$$H_\nu x^{c_1} f = g_1, \quad H_{\nu,p} x^{c_2} f = g_2, \tag{7.2}$$

where the left-hand sides have to be interpreted as elements of $T'([0, a], 1-\mu-\mathrm{Re}\,c_1)$ and $T'([a, \infty), 1-\lambda-\mathrm{Re}\,c_2)$ respectively. This is a distributional analogue of a dual integral equation considered by Titchmarsh [13], Erdélyi and Sneddon [5] and others. Erdélyi and Sneddon use fractional integrals in the solution of their equation. We extend their method to the solution of (7.2). Thus we obtain all solutions $f \in T'(\lambda, \mu)$ of (7.2) if the following conditions are satisfied: $h \in \mathbb{N}$,

$$-\mathrm{Re}(\nu + c_1) - \tfrac{1}{2} \leqq \lambda < \mu \leqq \mathrm{Re}(\nu - c_1) + \tfrac{3}{2}, \tag{7.3}$$

$$\begin{aligned} &\mu \leqq \mathrm{Re}(\nu - c_2) + \tfrac{3}{2}, \quad \text{if } h = 0, \\ &\mathrm{Re}(\nu - c_2) + 2h - \tfrac{1}{2} \leqq \lambda < \mu \leqq \mathrm{Re}(\nu - c_2) + 2h + \tfrac{3}{2} \quad \text{if } h > 0, \end{aligned} \tag{7.4^h}$$

$$\begin{aligned} &-\mathrm{Re}(\nu + c_2) - \tfrac{1}{2} \leqq \lambda \quad \text{if } p = 0, \\ &-\mathrm{Re}(\nu + c_2) - 2p - \tfrac{1}{2} \leqq \lambda < \mu \leqq -\mathrm{Re}(\nu + c_2) - 2p + \tfrac{3}{2} \quad \text{if } p > 0. \end{aligned} \tag{7.5^p}$$

First we assume that a solution f of (7.2) exists. Let $c = \frac{1}{2}(c_1 - c_2)$. We apply Theorem 13, formula (6.14) with ν and α replaced by $\nu + c$ and $-c$, $h = p = 0$. Then we get

$$2^{-c} x^c I_2^{(1/2)\nu - (1/4), c} H_\nu x^{c_1} f = H_{\nu + c} x^{(1/2)(c_1 + c_2)} f. \tag{7.6}$$

The conditions $(6.7)^0$ and (6.3) for formula (6.14) are satisfied because of (7.3).

Next we apply Theorem 13, formula (6.15) with ν, α, h and p replaced by $\nu + c$, $-c$, p and 0. Then we obtain

$$2^c x^{-c} K_2^{(1/2)\nu + (1/4) + c, -c} H_{\nu, p} x^{c_2} f = H_{\nu + c} x^{(1/2)(c_1 + c_2)} f. \tag{7.7}$$

The conditions $(6.7)^0$ and $(6.17)^p$ for formula (6.15) are satisfied because of (7.3) and $(7.5)^p$.

Now let

$$F = H_{\nu + c} x^{(1/2)(c_1 + c_2)} f. \tag{7.8}$$

Then (7.2), (7.6) and (7.7) imply

$$\begin{aligned} F &= 2^{-c} x^c I_2^{(1/2)\nu - (1/4), c} g_1 \quad \text{in } T'([0, a], 1 - \mu - \tfrac{1}{2}\mathrm{Re}(c_1 + c_2)), \\ F &= 2^c x^{-c} K_2^{(1/2)\nu + (1/4) + c, -c} g_2 \quad \text{in } T'([a, \infty), 1 - \lambda - \tfrac{1}{2}\mathrm{Re}(c_1 + c_2)), \end{aligned} \tag{7.9}$$

where the right-hand sides exist as elements of these spaces because of Theorems 11 and 12. Hence we know F completely if we can determine F on $\mathscr{D}(\frac{1}{2}a, \frac{3}{2}a)$. However, by (7.9) we know the restriction of F on $\mathscr{D}(\frac{1}{2}a, a)$ and on $\mathscr{D}(a, \frac{3}{2}a)$. Therefore we may write F as the generalized derivative of some order q of regular distributions on these spaces. Consequently F may be extended to a continuous linear functional F_0 on the completions C_1 of $\mathscr{D}(\frac{1}{2}a, a)$ and C_2 of $\mathscr{D}(a, \frac{3}{2}a)$ in $C^q[\frac{1}{2}a, \frac{3}{2}a]$.

Let $\phi \in \mathscr{D}(\frac{1}{2}a, \frac{3}{2}a)$ and $\chi \in \mathscr{D}(\frac{1}{4}a, 2a)$, $\chi(x) = 1$ if $\frac{1}{2}a \leqq x \leqq \frac{3}{2}a$. Then we may write

$$\phi(x) = \sum_{j=0}^{q} \frac{1}{j!} \phi^{(j)}(a)(x - a)^j \chi(x) + \phi_1(x) + \phi_2(x), \tag{7.10}$$

where $\phi_1 \in C_1$, $\phi_2 \in C_2$. Now (F_0, ϕ_1) and (F_0, ϕ_2) may be calculated using (7.9). If $\phi \in \mathscr{D}(\frac{1}{2}a, \frac{3}{2}a)$, then we define

$$(F_0, \phi) = (F_0, \phi_1) + (F_0, \phi_2). \tag{7.11}$$

Now $(F, \phi) = (F_0, \phi)$ if $\phi \in \mathscr{D}(\frac{1}{2}a, \frac{3}{2}a)$ and ϕ vanishes in a neighborhood of a. So $F - F_0 \in \mathscr{D}'(\frac{1}{2}a, \frac{3}{2}a)$ is concentrated in a. Therefore $F - F_0$ is a linear combination of the delta-functional and a finite number of its derivatives concentrated in a. Apart from these terms now F is uniquely determined on $\mathscr{D}(\frac{1}{2}a, \frac{3}{2}a)$, and consequently as an element of $T'(1 - \mu - \frac{1}{2}\operatorname{Re}(c_1 + c_2), 1 - \lambda - \frac{1}{2}\operatorname{Re}(c_1 + c_2))$ by means of g_1 and g_2.

From Theorem 13 and (7.8) we now deduce

$$f = x^{-(1/2)(c_1+c_2)}H_{\nu+c,h}F \in T'(\lambda, \mu). \tag{7.12}$$

So if a solution of (7.2) exists in $T'(\lambda, \mu)$ it is given by (7.12). Conversely, it is easy to check that the distributions f constructed above from g_1 and g_2 by means of (7.9) and (7.12) are solutions of (7.2). Extensions to other dual integral equations as in [2] may be given in an analogous way.

8. Dual integral equations involving *H*-functions. Before considering such integral equations we first extend the definition of the operator A of § 4, Example 3. In what follows we use the notation of that example and

$$I(\eta, \alpha, m) = I_m^{\eta,\alpha}, \quad K(\eta, \alpha, m) = K_m^{\eta,\alpha}. \tag{8.1}$$

Suppose (4.7) and (4.10) are satisfied. If $n < j \leqq p$,

$$\mu \leqq \operatorname{Re} \tilde{\alpha}_j / a_j \tag{8.2}$$

and (4.8), and (4.8) with α_j replaced by $\tilde{\alpha}_j$ are satisfied, then

$$A = \tilde{A}K(\tilde{\alpha}_j - a_j, \alpha_j - \tilde{\alpha}_j, a_j^{-1}) \quad \text{on } T(1 - \mu, 1 - \lambda), \tag{8.3}$$

where $\tilde{A}$ is defined by (4.11) with α_j replaced by $\tilde{\alpha}_j$. This may be shown using Mellin transforms, (5.16) and (4.9).

If $\tilde{\alpha}_j - \alpha_j \in \mathbb{N}$, we may omit (8.2) and use (5.22). Choosing $\tilde{\alpha}_j$ sufficiently large, the right-hand side of (8.3) exists on $T(1 - \mu, 1 - \lambda)$ even if (4.8) does not hold. Hence we may use (8.3) to define A in case only (4.7) and (4.10) are satisfied. It is a continuous operator of $T(1 - \mu, 1 - \lambda)$ into $T(\lambda, \mu)$ satisfying (8.3) if (4.7) and (4.10) are fulfilled.

In the same way we have

$$A = A^*I(b_h - \beta_h^*, \beta_h^* - \beta_h, b_h^{-1}) \quad \text{on } T(1 - \mu, 1 - \lambda), \tag{8.4}$$

if $m < h \leqq q$, A^* denotes the operator A with β_h replaced by β_h^*, (4.7), (4.10), (4.8) and (4.8) with β_h replaced by β_h^* are satisfied and

$$(\operatorname{Re} \beta_h^* - 1)/b_h \leqq \lambda. \tag{8.5}$$

If $\beta_h - \beta_h^* \in \mathbb{N}$ we may omit (8.5) and use (5.11). If $n = p$ and (4.7) holds, then $m < q$. If in this case (4.10) is fulfilled but (4.8) is not satisfied, we may use (8.4) with a suitably chosen β_h^* as definition of A. Hence A is defined as a continuous operator of $T(1 - \mu, 1 - \lambda)$ into $T(\lambda, \mu)$ if (4.10) and either (4.6) or (4.7) hold. The relations (8.3) and (8.4) are valid on $T(1 - \mu, 1 - \lambda)$ if (4.10) and either (4.6) or (4.7) are satisfied and in case of (8.3) also (8.2), in case of (8.4) also (8.5).

In case (4.7), (4.10) and (4.15) are fulfilled, the inverse of A exists as a continuous operator from $T(\lambda, \mu)$ into $T(1 - \mu, 1 - \lambda)$ and it is given by (4.12) with (4.13) and (4.14).

Now we consider the adjoint A' of A. It is a continuous operator from $T'(\lambda, \mu)$ into $T'(1-\mu, 1-\lambda)$ if (4.10) and either (4.6) or (4.7) hold. Using Parseval's formula we may show that

$$(8.6)\qquad (A\phi, \psi) = (\phi, A\psi) \quad \text{if } \psi \in T(1-\mu, 1-\lambda), \quad \phi \in T(\lambda_1, \mu_1),$$
$$\lambda + \lambda_1 < 1 < \mu + \mu_1 .$$

Therefore we denote A' by A on $T'(\lambda, \mu)$. The dual relations of (8.3) and (8.4) are

$$(8.7)\qquad A = I(\tilde{\alpha}_j - 1, \alpha_j - \tilde{\alpha}_j, a_j^{-1})\tilde{A}$$

and

$$(8.8)\qquad A = K(1 - \beta_h^*, \beta_h^* - \beta_h, b_h^{-1})A^*$$

which hold on $T'(\lambda, \mu)$ and on $T(1-\mu, 1-\lambda)$ if either (4.6) or (4.7) holds, (4.10) is satisfied, whereas in case of (8.7) we assume $n < j \leqq p$ and (8.2) and in case of (8.8) we assume $m < h \leqq q$ and (8.5). Also (8.3) and (8.4) are valid on $T'(\lambda, \mu)$ with corresponding conditions.

Let B be the operator which arises from A by replacing α_j and β_h by γ_j and δ_h for $j = 1, \cdots, p$ and $h = 1, \cdots, q$, where

$$(8.9)\qquad \operatorname{Re}\frac{\gamma_j - 1}{a_j} \leqq \lambda < \mu \leqq \operatorname{Re}\frac{\delta_h}{b_h}, \qquad j = 1, \cdots, n; \quad h = 1, \cdots, m.$$

Now we consider the dual integral equation,

$$(8.10)\qquad Af = g_1 \quad \text{in } T'([0, a], 1-\mu), \qquad Bf = g_2 \quad \text{in } T'([a, \infty), 1-\lambda),$$

where $a > 0$ and g_1 and g_2 are given elements in these spaces and (4.7) holds. Integral equations of this type for ordinary functions have been treated by Fox [6] and Saxena [12]. We use here a construction of solutions which is analogous to their formal solution.

Let C be the operator A with α_j replaced by $\gamma_j (j = n+1, \cdots, p)$ and β_h replaced by δ_h $(h = 1, \cdots, m)$. Define

$$(8.11)\qquad \begin{aligned} P_1 &= \left\{ \prod_{j=n+1}^{p} I(\alpha_j - 1, \gamma_j - \alpha_j, a_j^{-1}) \right\} \left\{ \prod_{h=1}^{m} I(\delta_h - 1, \beta_h - \delta_h, b_h^{-1}) \right\}, \\ P_2 &= \left\{ \prod_{j=1}^{n} K(1 - \alpha_j, \alpha_j - \gamma_j, a_j^{-1}) \right\} \left\{ \prod_{h=m+1}^{q} K(1 - \delta_h, \delta_h - \beta_h, b_h^{-1}) \right\}. \end{aligned}$$

For the existence of these operators on $T'(1-\mu, 1-\lambda)$ we assume (cf. Theorems 11 and 12) besides (4.10) and (8.9) also

$$(8.12)\qquad \operatorname{Re}\frac{\delta_h - 1}{b_h} \leqq \lambda < \mu \leqq \operatorname{Re}\frac{\alpha_j}{a_j}, \qquad h = m+1, \cdots, q; \quad j = n+1, \cdots, p.$$

Then

$$(8.13)\qquad P_1 A f = Cf = P_2 B f.$$

From this, (8.10) and Theorems 11 and 12 it follows that

$$(8.14)\qquad Cf = P_1 g_1 \quad \text{in } T'([0, a], 1-\mu), \qquad Cf = P_2 g_2 \quad \text{in } T'([a, \infty), 1-\lambda).$$

As in § 7 we may now determine $F = Cf$ in $T'(1 - \mu, 1 - \lambda)$ from (8.14) apart from a linear combination of the delta-functional with center a and a finite number of its derivatives. From F we now obtain the solution f of (8.10) by means of

$$f = C_0 F,$$

where C_0 is the adjoint of the operator defined by

$$\psi \to \int_0^\infty H_{p,q}^{q-m,p-n}\left(xt\middle|\begin{matrix}\tilde{\mathbf{a}}, \boldsymbol{\alpha}^*\\ \tilde{\mathbf{b}}, \boldsymbol{\beta}^*\end{matrix}\right)\psi(t)\,dt$$

with $\tilde{\mathbf{a}}$ and $\tilde{\mathbf{b}}$ given by (4.13) and

$$\boldsymbol{\alpha}^* = (1 + a_{n+1} - \gamma_{n+1}, \cdots, 1 + a_p - \gamma_p, 1 + a_1 - \alpha_1, \cdots, 1 + a_n - \alpha_n),$$

$$\boldsymbol{\beta}^* = (1 + b_{m+1} - \beta_{m+1}, \cdots, 1 + b_q - \beta_q, 1 + b_1 - \delta_1, \cdots, 1 + b_m - \delta_m).$$

This solution exists if (4.7), (4.10), (8.9), (8.12) and

$$\operatorname{Re}\frac{\beta_h - 1}{b_h} \leqq \lambda < \mu \leqq \operatorname{Re}\frac{\gamma_j}{a_j}, \qquad h = m + 1, \cdots, q; j = n + 1, \cdots, p$$

are satisfied.

9. A convolution map involving a hypergeometric function. Finally we consider another special case of the product convolution (5.1), viz. a hypergeometric integral transform considered among others by Love [11a] and [11b]. Let $\operatorname{Re} c > 1$,

$$-\operatorname{Re} a \leqq \lambda, \quad -\operatorname{Re} b \leqq \lambda < \mu. \tag{9.1}$$

Then if $\phi \in T(\lambda, \mu)$, we define

$$(A\phi)(x) = \frac{1}{\Gamma(c)}\int_x^\infty \left(1 - \frac{x}{t}\right)^{c-1} F\left(a, b; c; 1 - \frac{t}{x}\right)\phi(t)\frac{dt}{t}. \tag{9.2}$$

Now we have the special case of Theorem 8 where

$$\begin{aligned} k(x) &= \frac{1}{\Gamma(c)}(1 - x)^{c-1} F\left(a, b; c; 1 - \frac{1}{x}\right) \quad \text{if } 0 < x \leqq 1,\\ k(x) &= 0 \quad \text{if } x > 1. \end{aligned} \tag{9.3}$$

The Mellin transform $K(s)$ of $k(x)$ is given by

$$K(s) = \frac{\Gamma(a + s)\Gamma(b + s)}{\Gamma(c + s)\Gamma(a + b + s)}. \tag{9.4}$$

This may be shown using Euler's integral for the hypergeometric function or Barnes' integral representation for this function and Barnes' lemma.

The condition $\operatorname{Re} c > 1$ may be removed as in Remark 4. However, we may also use a modification of the method in Remark 4. If $\operatorname{Re} c + n > 0$, we define

$$(A\phi)(x) = \frac{(-1)^n x^{c+n}}{\Gamma(c + n)}\frac{d^n}{dx^n}x^{-c}\int_x^\infty \left(1 - \frac{x}{t}\right)^{c+n-1} F\left(a, b; c + n; 1 - \frac{t}{x}\right)\phi(t)\frac{dt}{t}. \tag{9.5}$$

This is consistent with the first definition in (9.2) since (9.5) implies $\mathscr{M}(A\phi)(s) = K(s)(\mathscr{M}\phi)(s)$. Hence A defines a continuous mapping of $T(\lambda, \mu)$ into itself if (9.1) holds.

From Theorem 10, formula (5.16), and (9.4) we deduce

$$A = K_1^{a,c-a} K_1^{b,a} \quad \text{on } T(\lambda, \mu). \tag{9.6}$$

This relation may also be proved directly using the definition (9.2) and Euler's integral for the hypergeometric function.

Now we consider the inverse of A, if it exists. First we assume

$$-\operatorname{Re} c \leqq \lambda, \quad -\operatorname{Re}(a+b) \leqq \lambda. \tag{9.7}$$

Then it follows from (9.6) and Theorem 10 that A is an automorphism on $T(\lambda, \mu)$ with

$$A^{-1} = K_1^{a+b,-a} K_1^{c,a-c}. \tag{9.8}$$

From (9.4) and Theorem 8 we may also deduce that A is an automorphism, and if moreover $\operatorname{Re} c < -1$,

$$A^{-1}\phi(x) = \int_0^\infty h\left(\frac{x}{t}\right)\phi(t)\frac{dt}{t}, \tag{9.9}$$

where

$$h(x) = \left\{ \mathcal{M}^{-1} \frac{\Gamma(c+s)\Gamma(a+b+s)}{\Gamma(a+s)\Gamma(b+s)} \right\}(x). \tag{9.10}$$

Using residue calculus we obtain

$$\begin{aligned} h(x) &= 0 \quad \text{if } x > 1, \\ h(x) &= \frac{1}{\Gamma(-c)} x^c (1-x)^{-c-1} F(-a, -b; -c; 1-x) \quad \text{if } 0 < x < 1. \end{aligned} \tag{9.11}$$

Hence if (9.7) holds and $\operatorname{Re} c < 0$, then the inverse B of A on $T(\lambda, \mu)$ is given by

$$B\phi(x) = \frac{1}{\Gamma(-c)} x^c \int_x^\infty (t-x)^{-c-1} F\left(-a, -b; -c; 1 - \frac{x}{t}\right)\phi(t)\, dt. \tag{9.12}$$

If $\operatorname{Re} c < m$, $m \in \mathbb{N}$, we easily see using (9.10) that

$$A^{-1} = (-1)^m x^c \frac{d^m}{dx^m} x^{m-c} B_m \quad \text{on } T(\lambda, \mu), \tag{9.13}$$

where B_m is defined by (9.12) with c replaced by $c - m$ and B by B_m.

Now we consider cases where (9.7) need not be fulfilled. Then we suppose that λ and μ satisfy the following condition with p and $q \in \mathbb{N}$:

$$\begin{aligned} -p - \operatorname{Re} c &\leqq \lambda < \mu \leqq 1 - p - \operatorname{Re} c, \\ -q - \operatorname{Re}(a+b) &\leqq \lambda < \mu \leqq 1 - q - \operatorname{Re}(a+b). \end{aligned} \tag{9.14}$$

If $p = 0$ or $q = 0$ we may omit the expression "$\leqq 1 - p - \operatorname{Re} c$" or "$\leqq 1 - q - \operatorname{Re}(a+b)$" respectively in this condition. Now (9.6) and Theorem 10 imply that A is an automorphism on $T(\lambda, \mu)$ with

$$A^{-1} = K_{1,q}^{a+b,-a} K_{1,p}^{c,a-c}. \tag{9.15}$$

If $\operatorname{Re} c < -1$ and (9.14) holds, we deduce (9.9) with (9.10) from Theorem 8 and (9.4). Using residue calculus we get

$$h(x) = -P(x) \quad \text{if } x > 1, \tag{9.16}$$
$$h(x) = \frac{1}{\Gamma(-c)} x^c (1-x)^{-c-1} F(-a, -b; -c; 1-x) - P(x)$$

if $0 < x < 1$, where

$$P(x) = \frac{\Gamma(a+b-c)}{\Gamma(a-c)\Gamma(b-c)} \sum_{j=0}^{p-1} \frac{(1+c-a)_j(1+c-b)_j}{j!(1+c-a-b)_j} x^{c+j} \tag{9.17}$$
$$+ \frac{\Gamma(c-a-b)}{\Gamma(-b)\Gamma(-a)} \sum_{j=0}^{q-1} \frac{(1+a)_j(1+b)_j}{j!(1+a+b-c)_j} x^{a+b+j}.$$

Hence if (9.14) is fulfilled and $\operatorname{Re} c < 0$, the inverse B of A on $T(\lambda, \mu)$ is given by

$$B\phi(x) = \int_x^\infty \left\{ \frac{1}{\Gamma(-c)} \frac{t}{x} \left(\frac{t}{x} - 1 \right)^{-c-1} F\left(-a, -b; -c; 1 - \frac{x}{t} \right) - P\left(\frac{x}{t} \right) \right\} \phi(t) \frac{dt}{t} \tag{9.18}$$
$$- \int_0^x P\left(\frac{x}{t} \right) \phi(t) \frac{dt}{t}.$$

If $\operatorname{Re} c < m$, $m \in \mathbb{N}$, we have (9.13) where B_m is defined by (9.18) and (9.17) with B, c and p replaced by B_m, $c - m$ and $p + m$.

Finally, we consider the adjoint A' of A on $T'(\lambda, \mu)$. Assuming (9.1),

$$\phi \in T(\lambda, \mu), \quad f \in T(\lambda', \mu') \subset T'(\lambda, \mu) \quad (\text{hence } \lambda + \lambda' < 1 < \mu + \mu'),$$

we have according to Parseval's formula,

$$\int_0^\infty f(x) A\phi(x)\, dx = \frac{1}{2\pi i} \int_{\nu - i\infty}^{\nu + i\infty} K(s)\Phi(s)F(1-s)\, ds$$
$$= \frac{1}{2\pi i} \int_{1-\nu-i\infty}^{1-\nu+i\infty} K(1-s)F(s)\Phi(1-s)\, ds = \int_0^\infty (\tilde{A}f)(x)\phi(x)\, dx,$$

where $\lambda < \nu < \mu$, $1 - \mu' < \nu < 1 - \lambda'$, $F = \mathcal{M}f$, $\Phi = \mathcal{M}\phi$,

$$(\mathcal{M}\tilde{A}f) = \frac{\Gamma(a+1-s)\Gamma(b+1-s)}{\Gamma(c+1-s)\Gamma(a+b+1-s)} F(s).$$

Hence $A' = \tilde{A}$ on $T(\lambda', \mu')$ where (cf. (9.3) and (9.4))

$$\tilde{A}f(x) = \frac{1}{x\Gamma(c)} \int_0^x \left(1 - \frac{t}{x} \right)^{c-1} F\left(a, b; c; 1 - \frac{x}{t} \right) f(t)\, dt, \tag{9.19}$$

if $f \in T(\lambda', \mu')$, $\lambda' < 1 + \min(\operatorname{Re} a, \operatorname{Re} b)$, $\operatorname{Re} c > 0$,

$$\tilde{A}f(x) = \frac{x^{-c}}{\Gamma(c+n)} \frac{d^n}{dx^n} x^{c+n-1} \int_0^x \left(1 - \frac{t}{x} \right)^{c+n-1} F\left(a, b; c+n; 1 - \frac{x}{t} \right) f(t)\, dt \tag{9.20}$$

if $f \in T(\lambda', \mu')$, $\lambda' < 1 + \min(\operatorname{Re} a, \operatorname{Re} b)$, $\operatorname{Re} c + n > 0$, $n \in \mathbb{N}$. If (9.1) holds, $\tilde{A}$ is a continuous operator of $T(1-\mu, 1-\lambda)$ into itself.

Now A' is a continuous mapping of $T'(\lambda, \mu)$ into itself if (9.1) is satisfied, and according to Theorem 12 and (9.8):

$$A' = I_1^{b,a} I_1^{a,c-a}. \tag{9.21}$$

If moreover (9.14) holds, then A' is an automorphism on $T'(\lambda, \mu)$ with

$$(A')^{-1} = I_{1,p}^{c,a-c} I_{1,q}^{a+b,-a}. \tag{9.22}$$

Analogous to $\tilde{A}$ we define an operator $\tilde{B}$ which plays the same role with respect to $(A')^{-1}$ as $\tilde{A}$ plays with respect to A'.

Suppose $\lambda' < \mu'$, $p, q \in \mathbb{N}$,

$$\lambda' < 1 + \min\{p + \operatorname{Re} c, q + \operatorname{Re}(a + b)\},$$

$$p + \operatorname{Re} c < \mu' \quad \text{if } p \neq 0, \quad q + \operatorname{Re}(a + b) < \mu' \quad \text{if } q \neq 0.$$

Let P be defined by (9.17) and $g \in T(\lambda', \mu')$. If $\operatorname{Re} c < 0$, then

$$\begin{aligned} \tilde{B}g(x) = \int_0^x \left\{ \frac{1}{\Gamma(-c)} \left(\frac{x}{t} - 1 \right)^{-c-1} F\left(-a, -b; -c; 1 - \frac{t}{x} \right) - \frac{t}{x} P\left(\frac{t}{x} \right) \right\} g(t) \frac{dt}{t} \\ - \frac{1}{x} \int_x^\infty P\left(\frac{t}{x} \right) g(t)\, dt. \end{aligned} \tag{9.23}$$

If $\operatorname{Re} c < m$, $m \in \mathbb{N}$, we define

$$\tilde{B} = x^{m-c} \frac{d^m}{dx^m} x^c \tilde{B}_m, \tag{9.24}$$

where $\tilde{B}_m$ is defined by (9.23) and (9.17) with $\tilde{B}$, c and p replaced by $\tilde{B}_m$, $c - m$ and $p + m$. Then the operator $(A')^{-1}$ on $T'(\lambda, \mu)$ coincides with $\tilde{B}$ on $T(\lambda', \mu')$ if $\lambda + \lambda' < 1 < \mu + \mu'$, (9.1) and (9.14) are satisfied. Furthermore, $\tilde{B}$ is a continuous operator on $T(1 - \mu, 1 - \lambda)$ and it is the inverse of $\tilde{A}$ if (9.1) and (9.14) are satisfied.

It is obvious that instead of starting with the transformation A we could also start with B, $\tilde{A}$ or $\tilde{B}$ and apply an analogous reasoning as above. We obtain similar results by extending the definition of A and $\tilde{A}$ in the same way as the definition of B is extended from (9.12) and (9.18).

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