university of groningen

University Medical Center Groningen

University of Groningen

On the characterization of the Duhem hysteresis operator with clockwise input-output dynamics

Ouyang, Ruiyue; Andrieu, Vincent; Jayawardhana, Bayu

Published in:
Systems & Control Letters

DOI:
10.1016/j.sysconle.2012.11.022

IMPORTANT NOTE: You are advised to consult the publisher's version (publisher's PDF) if you wish to cite from it. Please check the document version below.

Document Version
Publisher's PDF, also known as Version of record

Publication date:
2013

Link to publication in University of Groningen/UMCG research database

Citation for published version (APA):
Ouyang, R., Andrieu, V., & Jayawardhana, B. (2013). On the characterization of the Duhem hysteresis operator with clockwise input-output dynamics. Systems & Control Letters, 62(3), 286-293. DOI: 10.1016/j.sysconle.2012.11.022

Copyright
Other than for strictly personal use, it is not permitted to download or to forward/distribute the text or part of it without the consent of the author(s) and/or copyright holder(s), unless the work is under an open content license (like Creative Commons).

Take-down policy
If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

Downloaded from the University of Groningen/UMCG research database (Pure): http://www.rug.nl/research/portal. For technical reasons the number of authors shown on this cover page is limited to 10 maximum.

Download date: 11-02-2018

Systems & Control Letters 62 (2013) 286–293

Contents lists available at SciVerse ScienceDirect

Systems & Control Letters

journal homepage: www.elsevier.com/locate/sysconle

On the characterization of the Duhem hysteresis operator with clockwise input–output dynamics

Ruiyue Ouyang [a], Vincent Andrieu [b], Bayu Jayawardhana [a,*]

[a] Department Discrete Technology and Production Automation, University of Groningen, Groningen 9747AG, The Netherlands

[b] Université Lyon 1, Villeurbanne; CNRS, UMR 5007, LAGEP. 43 bd. du 11 Novembre, 69100 Villeurbanne, France

ARTICLE INFO

Article history:
Received 6 January 2012
Received in revised form
25 October 2012
Accepted 28 November 2012

Keywords:
Hysteresis
Clockwise I/O dynamics
Dissipative systems

ABSTRACT

In this paper we investigate the dissipativity property of a certain class of Duhem hysteresis operator, which has clockwise (CW) input–output (I/O) behavior. In particular, we provide sufficient conditions on the Duhem operator such that it is CW and propose an explicit construction of the corresponding function satisfying dissipation inequality of CW systems. The result is used to analyze the stability of a second order system with hysteretic friction which is described by a Dahl model.

© 2012 Elsevier B.V. All rights reserved.

1. Introduction

Hysteresis is a common nonlinear phenomenon that is present in diverse physical systems, such as piezo-actuator, ferromagnetic material and mechanical systems. From the perspective of input–output behavior, the hysteretic phenomena can be characterized into counterclockwise (CCW) input–output (I/O) dynamics [1], clockwise (CW) I/O dynamics [2], or even more complex I/O map (such as, butterfly map [3]). For example, backlash operator generates CCW I/O dynamics; elastic–plastic operator generates CW I/O dynamics and Preisach operator can have either CCW or CW I/O dynamics depending on the weight of the hysterons which are used in the Preisach model [4–6]. In the recent work by Angeli [1,7], the counterclockwise (CCW) I/O dynamics of a single-input single-output system is characterized by the following inequality

$$\liminf_{T\to\infty} \int_0^T \dot{y}(t)u(t)\mathrm{d}t > -\infty, \tag{1}$$

where u is the input signal and y is the corresponding output signal. It is assumed that $u \in U$ where U is the set of input signals for which y exists and is well defined for all positive time. Compared with the classical definition of passivity [8,9], it can be interpreted that the system is passive from the input u to the time derivative of the corresponding output y. In particular, (1) holds if there exists a function $H : \mathbb{R}^2 \to \mathbb{R}_+$ such that

$$\frac{\mathrm{d}H(y(t),u(t))}{\mathrm{d}t} \le \dot{y}(t)u(t). \tag{2}$$

Indeed, integrating (2) from 0 to ∞ we obtain (1).

Correspondingly, clockwise (CW) I/O dynamics can be described by the following dissipation inequality

$$\liminf_{T\to\infty} \int_0^T \dot{u}(t)y(t)\mathrm{d}t > -\infty. \tag{3}$$

The notions of counterclockwise (CCW) I/O and clockwise (CW) I/O are also discussed in [10–14].

In our previous results in [15,16], we show that for a certain class of Duhem hysteresis operator $\Phi : u \mapsto \Phi(u, y_0) := y$, we can construct a function $H_{\circlearrowleft} : \mathbb{R}^2 \to \mathbb{R}_+$ which satisfies

$$\frac{\mathrm{d}H_{\circlearrowleft}(y(t),u(t))}{\mathrm{d}t} \le \dot{y}(t)u(t). \tag{4}$$

This inequality immediately implies that such Duhem hysteresis operator is dissipative with respect to the supply rate $\dot{y}(t)u(t)$ and has CCW input–output dynamics. The symbol $\circlearrowleft$ in $H_{\circlearrowleft}$ indicates the counterclockwise behavior of Φ.

In this paper, as a dual extension to [15,16], we focus on the clockwise (CW) hysteresis operator where the supply rate is given by $\dot{u}y$ which is dual to the supply rate $u\dot{y}$ considered in [15,16]. This is motivated by the friction induced hysteresis phenomenon in the mechanical system which has CW I/O behavior from the input relative displacement to the output friction force. One may

* Corresponding author. Tel.: +31 503637156.
E-mail addresses: r.ouyang@rug.nl (R. Ouyang), vincent.andrieu@gmail.com (V. Andrieu), bayujw@ieee.org (B. Jayawardhana).

0167-6911/$ – see front matter © 2012 Elsevier B.V. All rights reserved.
doi:10.1016/j.sysconle.2012.11.022

intuitively consider to reverse the input–output relation of the CW hysteresis operator for getting the CCW I/O behavior in the reverse I/O setting. However, this consideration has two drawbacks: (1) the reverse input–output pair may not be physically realizable (this is related to the causality problem in the port-based modeling, such as, the bond graph modeling framework [17]); (2) the operator itself may not be invertible (for example, if the output of the hysteresis operator can be saturated).

In Theorem 1, we provide sufficient conditions on the underlying functions f_1 and f_2 of the Duhem operator, such that it has CW I/O dynamics. Roughly speaking, the functions f_1 and f_2 (as defined later in Section 2) determine two possible different directions (y, u) depending on whether the input u is increasing or decreasing. By evaluating these two functions on two disjoint domains (which are separated by an anhysteresis curve), we can determine whether it has CW I/O dynamics using Theorem 1. This is shown by constructing a function $H_\circlearrowright : \mathbb{R}^2 \to \mathbb{R}_+$ such that the following inequality

$$\frac{\mathrm{d}H_\circlearrowright(y(t), u(t))}{\mathrm{d}t} \le y(t)\dot{u}(t) \tag{5}$$

holds. The function $H_\circlearrowright$ can also be related to the concept of available storage function from [8,9] where, instead of using the standard supply rate yu, we use the CW supply rate $y\dot{u}$ as shown in Proposition 1 in this paper.

The dissipativity property (5) can be further used in the stability analysis of the systems with CW hysteresis, such as, a second-order mechanical system with hysteretic friction [18] as discussed in Section 4.2. As an illustrative example on the application of (5), let us consider a mechanical system described by

$$m\ddot{x} = F - F_{\text{friction}},$$
$$F_{\text{friction}} = \Phi(x, y_0),$$

with the hysteresis operator Φ satisfying the Dahl model as follows

$$\dot{F}_{\text{friction}} = \rho\left(1 - \frac{F_{\text{friction}}}{F_C}\right)\max\{0, \dot{x}\} + \rho\left(1 + \frac{F_{\text{friction}}}{F_C}\right)\min\{0, \dot{x}\},$$

where m refers to the mass, x refers to the displacement, F is the applied force (control input), $\rho > 0$ describes the stiffness constant, $F_C > 0$ represents the Coulomb friction constant and y_0 is the initial condition of the Dahl model (see, for example, [10]). By taking $x_1 = x$, $x_2 = \dot{x}$ and $x_3 = F_{\text{friction}}$ as the state variables, we can rewrite this hysteretic system into state–space form as follows

$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = \frac{F}{m} - \frac{x_3}{m},$$
$$\dot{x}_3 = \rho\left(1 - \frac{x_3}{F_C}\right)\max\{0, x_2\} + \rho\left(1 + \frac{x_3}{F_C}\right)\min\{0, x_2\}.$$

In Section 4.1, we obtain the function $H_\circlearrowright$ satisfying (5) explicitly and it is parameterized by ρ and F_C. Using $V(x_1, x_2, x_3) = \frac{1}{2}mx_2^2 + H_\circlearrowright(x_3, x_1)$ as a Lyapunov function we have

$$\begin{aligned}\dot{V} &= m\dot{x}_2x_2 + \frac{\mathrm{d}H_\circlearrowright(x_3, x_1)}{\mathrm{d}t}\\ &= -x_3x_2 + Fx_2 + \frac{\mathrm{d}H_\circlearrowright(x_3, x_1)}{\mathrm{d}t}\\ &\le Fx_2.\end{aligned}$$

This inequality establishes that the closed-loop system is passive from the applied force F to the velocity x_2. Thus a simple proportional feedback $F = -dx_2$, where $d > 0$, can guarantee the asymptotic convergence of the velocity x_2 to zero without having to know precisely the parameters ρ and F_C.

2. Duhem operator and clockwise hysteresis operators

Denote $C^1(\mathbb{R}_+)$ the space of continuously differentiable functions $f : \mathbb{R}_+ \to \mathbb{R}$ and $AC(\mathbb{R}_+)$ the space of absolutely continuous functions $f : \mathbb{R}_+ \to \mathbb{R}$. Define $\frac{\mathrm{d}z(t)}{\mathrm{d}t} := \lim_{h\searrow 0^+} \frac{z(t+h)-z(t)}{h}$.

The Duhem operator $\Phi : AC(\mathbb{R}_+) \times \mathbb{R} \to AC(\mathbb{R}_+)$, $(u, y_0) \mapsto \Phi(u, y_0) =: y$ is described by [5,10,19]

$$\dot{y}(t) = f_1(y(t), u(t))\dot{u}_+(t) + f_2(y(t), u(t))\dot{u}_-(t), \quad y(0) = y_0, \tag{6}$$

where $\dot{u}_+(t) := \max\{0, \dot{u}(t)\}$, $\dot{u}_-(t) := \min\{0, \dot{u}(t)\}$. The functions f_1 and f_2 are assumed to be C^1.

The existence of solutions to (6) has been reviewed in [5]. In particular, if for every $\xi \in \mathbb{R}$, f_1 and f_2 satisfy

$$\begin{aligned}&(\sigma_1 - \sigma_2)[f_1(\sigma_1, \xi) - f_1(\sigma_2, \xi)] \le \lambda_1(\xi)(\sigma_1 - \sigma_2)^2,\\ &(\sigma_1 - \sigma_2)[f_2(\sigma_1, \xi) - f_2(\sigma_2, \xi)] \ge -\lambda_2(\xi)(\sigma_1 - \sigma_2)^2,\end{aligned} \tag{7}$$

for all $\sigma_1, \sigma_2 \in \mathbb{R}$, where λ_1 and λ_2 are nonnegative, then the solution to (6) exist and Φ maps $AC(\mathbb{R}_+) \times \mathbb{R} \to AC(\mathbb{R}_+)$. We will assume throughout the paper that the solution to (6) exists for all $u \in AC(\mathbb{R}_+)$ and $y_0 \in \mathbb{R}$.

As a dual definition to counterclockwise (CCW) I/O behavior [1], we define the clockwise (CW) I/O dynamics as follows

Definition 1. An operator Q is *clockwise* (CW) if for every $u \in U$ with the corresponding output map $y := Qu$, where U is the space of input signals such that y is well-defined for all positive time, the following inequality holds

$$\liminf_{T\to\infty} \int_0^T y(t)\dot{u}(t)\mathrm{d}t > -\infty. \tag{8}$$

For the Duhem operator Φ, inequality (8) holds if there exists a function $H_\circlearrowright : \mathbb{R}^2 \to \mathbb{R}_+$ such that for every $u \in AC(\mathbb{R}_+)$ and $y_0 \in \mathbb{R}$, the inequality

$$\frac{\mathrm{d}H_\circlearrowright(y(t), u(t))}{\mathrm{d}t} \le y(t)\dot{u}(t), \tag{9}$$

holds for all t where $y := \Phi(u, y_0)$.

In the following subsections, we describe several well-known hysteresis operators which generate clockwise I/O dynamics and we recast these operators into the Duhem operator as in (6).

2.1. Dahl model

The Dahl model [20,21] is commonly used in mechanical systems, which represents the friction force with respect to the relative displacement between two surfaces in contact. The general representation of the Dahl model is given by

$$\dot{y}(t) = \rho\left|1 - \frac{y(t)}{F_c}\mathrm{sgn}(\dot{u}(t))\right|^r \mathrm{sgn}\left(1 - \frac{y(t)}{F_c}\mathrm{sgn}(\dot{u}(t))\right)\dot{u}(t), \tag{10}$$

where y denotes the friction force, u denotes the relative displacement, $F_c > 0$ denotes the Coulomb friction force, $\rho > 0$ denotes the rest stiffness and $r \ge 1$ is a parameter that determines the shape of the hysteresis loops.

The Dahl model can be described by the Duhem hysteresis operator (6) with

$$f_1(\sigma, \xi) = \rho\left|1 - \frac{\sigma}{F_c}\right|^r \mathrm{sgn}\left(1 - \frac{\sigma}{F_c}\right), \tag{11}$$

$$f_2(\sigma, \xi) = \rho\left|1 + \frac{\sigma}{F_c}\right|^r \mathrm{sgn}\left(1 + \frac{\sigma}{F_c}\right). \tag{12}$$

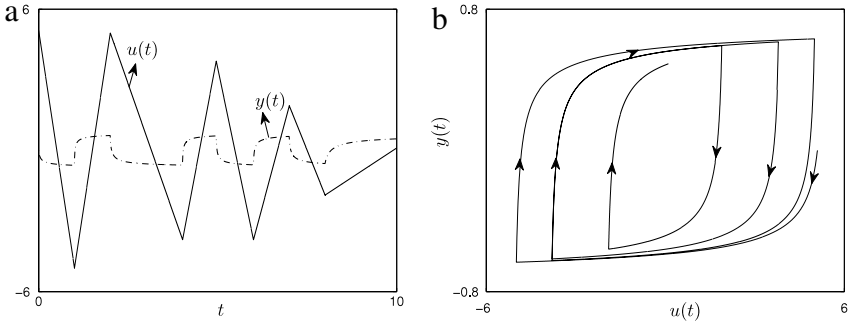

Fig. 1. The input–output dynamics of the Dahl model with $F_c = 0.75$, $\rho = 1.5$ and $r = 3$.

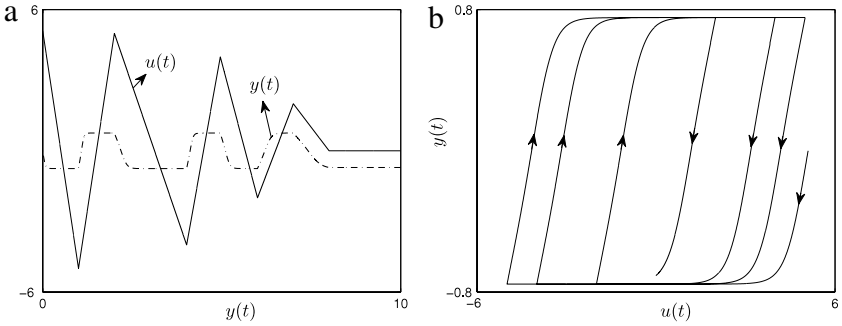

Fig. 2. The input–output dynamics of the Bouc–Wen model with $\alpha = 1$, $\beta = 1$, $\zeta = 1$ and $n = 3$.

Note that the functions f_1, f_2 for the Dahl model given in (11) and (12) depend only on σ. This model can be considered as a special case of the Duhem operator as given in [21].

In Fig. 1, we illustrate the behavior of the Dahl model where $F_c = 0.75$, $\rho = 1.5$ and $r = 3$.

2.2. Bouc–Wen model

The Bouc–Wen model [22,23] is commonly used to model the elastic stress–strain relationships in structures. Moreover, it is also used to represent the magnetorheological behavior in the MR damper [24]. The general representation of the Bouc–Wen model is given by

$$\dot{y}(t) = \alpha\dot{u}(t) - \beta\dot{u}(t)|y(t)|^n - \gamma|\dot{u}(t)|y(t)|y(t)|^{n-1},$$

where u denotes the displacement, y denotes the elastic strain, $n \geq 1$ and β, ζ are the parameters determine the shape of the hysteresis curve.

The Bouc–Wen model can be described by the Duhem hysteresis operator (6) with

$$f_1(\sigma, \xi) = \alpha - \beta|\sigma|^n - \zeta\sigma|\sigma|^{n-1}, \tag{13}$$

$$f_2(\sigma, \xi) = \alpha - \beta|\sigma|^n + \zeta\sigma|\sigma|^{n-1}. \tag{14}$$

Similar to the Dahl model, the Bouc–Wen model is also a special case of the Duhem operator, since the functions f_1, f_2 depend only on σ. In Fig. 2, we illustrate the behavior of the Bouc–Wen model where $\alpha = 1$, $\beta = 1$, $\zeta = 1$ and $n = 3$.

3. Main result

Before stating our main contribution, we need to introduce three functions in the following subsections: an anhysteresis function f_{an}, a traversing function ω_Φ and an intersecting function Λ; these functions will play an important role in the characterization of dissipativity and in the construction of the storage function. These three functions are defined based on the knowledge of f_1 and f_2. Generally speaking, the anhysteresis function f_{an} defines the curve where $f_1 = f_2$, the function ω_Φ describes the trajectory of Φ when a monotone increasing u or a monotone decreasing u is applied from a given point in the hysteresis phase plot, and the intersecting function Λ defines the intersection of the anhysteresis function f_{an} and function ω_Φ from a given point. The anhysteresis function f_{an} and the traversing function ω_Φ have the same definitions as given in our previous results in [25].

3.1. Anhysteresis function

In order to define the anhysteresis function, we rewrite f_1 and f_2 as follows

$$\left.\begin{aligned} f_1(y(t), u(t)) &= F(y(t), u(t)) + G(y(t), u(t)), \\ f_2(y(t), u(t)) &= -F(y(t), u(t)) + G(y(t), u(t)), \end{aligned}\right\} \tag{15}$$

where $F, G : \mathbb{R}^2 \to \mathbb{R}$. We assume that the implicit function $F(\sigma, \xi) = 0$ can be represented by an explicit function $\sigma = f_{an}(\xi)$ or $\xi = g_{an}(\sigma)$. Such a function f_{an} (or g_{an}) is called an *anhysteresis function* and the corresponding graph $\{(\xi, f_{an}(\xi))|\xi \in \mathbb{R}\}$ is called an *anhysteresis curve*. Using f_{an}, it can be checked that $f_1(f_{an}(\xi), \xi) = f_2(f_{an}(\xi), \xi)$ holds. Note also that the functions F and G in (15) are defined by

$$F = \frac{f_1 - f_2}{2} \qquad G = \frac{f_1 + f_2}{2}.$$

3.2. Traversing function ω_Φ

For every given point $(\sigma, \xi) \in \mathbb{R}^2$ in the hysteresis phase plot, let $\omega_{\Phi,1}(\cdot, \sigma, \xi) : [\xi, \infty) \to \mathbb{R}$ be the solution x of

$$x(\tau) - x(\xi) = \int_\xi^\tau f_1(x(\lambda), \lambda)\mathrm{d}\lambda \quad x(\xi) = \sigma \ \forall \tau \in [\xi, \infty),$$

and let $\omega_{\Phi,2}(\cdot,\sigma,\xi):(-\infty,\xi]\to\mathbb{R}$ be the solution x of

$$x(\tau)-x(\xi)=\int_{\xi}^{\tau} f_2(x(\lambda),\lambda)\,\mathrm{d}\lambda \quad x(\xi)=\sigma\ \forall\tau\in(-\infty,\xi].$$

Using the above definitions, for every point $(\sigma,\xi)\in\mathbb{R}^2$ in the hysteresis phase plot, the *traversing function* $\omega_\Phi(\cdot,\sigma,\xi):\mathbb{R}\to\mathbb{R}$ is defined by the concatenation of $\omega_{\Phi,2}(\cdot,\sigma,\xi)$ and $\omega_{\Phi,1}(\cdot,\sigma,\xi)$:

$$\omega_\Phi(\tau,\sigma,\xi)=\begin{cases}\omega_{\Phi,2}(\tau,\sigma,\xi) & \forall\tau\in(-\infty,\xi),\\ \omega_{\Phi,1}(\tau,\sigma,\xi) & \forall\tau\in[\xi,\infty).\end{cases}\tag{16}$$

We remark that the function $\omega_\Phi(\cdot,\sigma,\xi)$ defines the (unique) hysteresis curve where the curve $\{(\tau,\omega_\Phi(\tau,\sigma,\xi))\mid\tau\in(-\infty,\xi]\}$ is obtained by applying a monotone decreasing u to $\Phi(\cdot,\sigma)$ with $u(0)=\xi$, $\lim_{t\to\infty}u(t)=-\infty$ and, similarly, the curve $\{(\tau,\omega_\Phi(\tau,\sigma,\xi))\mid\tau\in[\xi,\infty)\}$ is obtained by introducing a monotone increasing u to $\Phi(\cdot,\sigma)$ with $u(0)=\xi$ and $\lim_{t\to\infty}u(t)=\infty$.

3.3. Intersecting function Λ

The intersecting function Λ describes the intersection between the anhysteresis curve f_{an} and the curve ω_Φ. The function $\Lambda:\mathbb{R}^2\to\mathbb{R}$ is an *intersecting function* (corresponding to ω_Φ and f_{an}) if: (i) $\omega_\Phi(\Lambda(\sigma,\xi),\sigma,\xi)=f_{an}(\Lambda(\sigma,\xi))$ for all $(\sigma,\xi)\in\mathbb{R}^2$ and; (ii) $\Lambda(\sigma,\xi)\le\xi$ whenever $\sigma\ge f_{an}(\xi)$ and $\Lambda(\sigma,\xi)>\xi$ otherwise. This implies that the two functions $\omega_\Phi(\cdot,\sigma,\xi)$ and $f_{an}(\cdot)$ intersect at a unique point larger or smaller than ξ depending on the sign of $\sigma-f_{an}(\xi)$. In our main result, we also need that $\frac{\mathrm{d}\Lambda(y(t),u(t))}{\mathrm{d}t}$ exists for every solutions (y,u) of (6).

In the following lemma we give sufficient conditions for the existence of such an intersecting function Λ.

Lemma 1. *Assume that f_1 and f_2 in (15) be such that f_1,f_2 are C^1. Moreover, assume that f_{an} is strictly increasing and there exists a positive real constant $\epsilon>0$ such that for all $(\sigma,\xi)\in\mathbb{R}^2$ the following inequality holds*

$$f_1(\sigma,\xi)>\frac{\mathrm{d}f_{an}(\xi)}{\mathrm{d}\xi}+\epsilon \quad \text{whenever } \sigma>f_{an}(\xi),\tag{17}$$

$$f_2(\sigma,\xi)>\frac{\mathrm{d}f_{an}(\xi)}{\mathrm{d}\xi}+\epsilon \quad \text{whenever } \sigma<f_{an}(\xi).\tag{18}$$

Then there exists an intersecting function $\Lambda\in C^1(\mathbb{R}^2,\mathbb{R})$ such that

(1) *$\Lambda(\sigma,\xi)\le\xi$ whenever $\sigma\ge f_{an}(\xi)$ and $\Lambda(\sigma,\xi)>\xi$ otherwise.*
(2)

$$\omega_\Phi(\Lambda(\sigma,\xi),\sigma,\xi)=f_{an}(\Lambda(\sigma,\xi)).\tag{19}$$

(3) *Moreover, for all $u\in C^1, y:=\Phi(u,y_0)$, we have that $\frac{\mathrm{d}}{\mathrm{d}t}\Lambda(y(t),u(t))$ exists.*

The proof of Lemma 1 is given in the Appendix A.

Example 1. In order to illustrate these functions, let us consider the Duhem operator Φ with $f_1(\sigma,\xi)=e^{0.5(-1.2\sigma+\xi)}+0.83$ and $f_2(\sigma,\xi)=e^{0.5(1.2\sigma-\xi)}+0.83$ as shown in Fig. 3. It can be checked that the anhysteresis function of the operator is $f_{an}(\xi)=0.83\xi$ and the functions f_1 and f_2 satisfy the hypotheses in Lemma 1. With a reference to Fig. 3, let the current state of Φ be given by $(y(t),u(t))$. In this figure, the traversing function $\omega_\phi(\cdot,y(t),u(t))$ is depicted by the dashed-line and the anhysteresis function f_{an} is shown by the thick solid-line. The point $(y(t),u(t))$ is located above the anhysteresis curve, i.e., $y(t)>f_{an}(u(t))$. It can be seen from the figure that the intersecting point $\Lambda(y(t),u(t))$ (which is shown by the solid circle) is less than $u(t)$, i.e., $\Lambda(y(t),u(t))\le u(t)$. This shows that the property (1) in Lemma 1 holds.

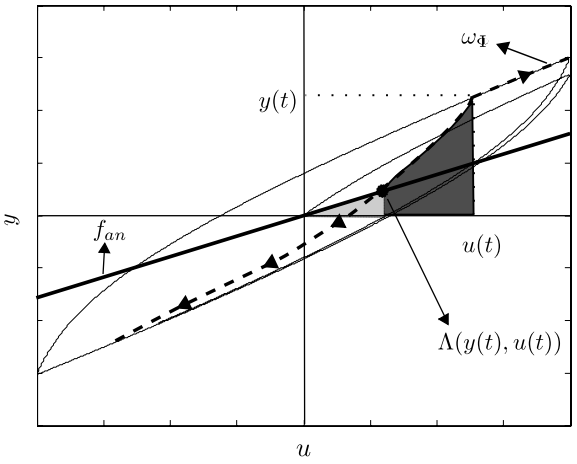

Fig. 3. Illustration of a Duhem operator with $f_1(\sigma,\xi)=e^{0.5(-1.2\sigma+\xi)}+0.83$ and $f_2(\sigma,\xi)=e^{0.5(1.2\sigma-\xi)}+0.83$ for all $(\sigma,\xi)\in\mathbb{R}^2$. The anhysteresis curve $f_{an}(\xi)=0.83\xi$ is shown by the thick solid-line. If the current state be given by $(y(t),u(t))$, the traversing function $\omega_\phi(\cdot,y(t),u(t))$ is depicted by the dashed-line and the intersecting point $\Lambda(y(t),u(t))$ is shown by the solid circle.

3.4. Duhem operator with clockwise hysteresis

Based on the three functions ω_Φ, f_{an} and Λ, we define $H_\circlearrowleft:\mathbb{R}^2\to\mathbb{R}_+$ as follows

$$H_\circlearrowleft(\sigma,\xi)=\int_0^{\Lambda(\sigma,\xi)} f_{an}(\tau)\mathrm{d}\tau-\int_\xi^{\Lambda(\sigma,\xi)}\omega_\Phi(\tau,\sigma,\xi)\mathrm{d}\tau.\tag{20}$$

Theorem 1. *Consider the Duhem hysteresis operator Φ defined in (6) and (15) with C^1 functions $F,G:\mathbb{R}^2\to\mathbb{R}$ and with the traversing function ω_Φ and the anhysteresis function f_{an}. Suppose that there exists an intersecting function Λ (e.g. the hypotheses in Lemma 1 hold). Let the following condition hold for all (σ,ξ) in $\mathbb{R}^2$*

(A) *$F(\sigma,\xi)\ge0$ whenever $\sigma\le f_{an}(\xi)$, and $F(\sigma,\xi)<0$ otherwise.*

Then for every $u\in AC(\mathbb{R}_+)$ and for every $y_0\in\mathbb{R}$, the function $t\to H_\circlearrowleft(y(t),u(t))$ with $H_\circlearrowleft$ as in (20) and $y:=\Phi(u,y_0)$, is right differentiable and satisfies (5). Moreover, if the anhysteresis function f_{an} satisfies $f_{an}(0)=0$, then $H_\circlearrowleft\ge0$ and the Duhem operator is clockwise (CW).

The proof of Theorem 1 is given in the Appendix B.

Remark 1. In addition to the result in Theorem 1, if f_1 and f_2 satisfy the hypotheses given in Theorem 1, then for every $u\in AC(\mathbb{R}_+)$ and $y_0\in\mathbb{R}$, the function $t\to H_\circlearrowleft(y(t),u(t))$ with $H_\circlearrowleft$ as in (20) is left-differentiable and satisfies

$$\lim_{h\nearrow0^-}\frac{H_\circlearrowleft(y(t+h),u(t+h))-H_\circlearrowleft(y(t),u(t))}{h}\le y(t)\dot{u}(t).$$

The proof of this claim follows a similar line as that of Theorem 1.

In order to depict the storage function $H_\circlearrowleft$ that is constructed in Theorem 1, we recall again the example of the Duhem operator Φ in Example 1 where $f_1=e^{0.5(-1.2y+u)}+0.83$ and $f_2=e^{0.5(1.2y-u)}+0.83$, and it is shown in Fig. 3. Based on the functions f_{an} and $\omega_\Phi(\cdot,y(t),u(t))$ as shown in Fig. 3 and following the construction of the storage function $H_\circlearrowleft$ as in (20), the first component on the RHS of (20) corresponds to the light gray area in Fig. 3. Correspondingly, the second component on the RHS of (20) refers to the dark gray area in Fig. 3. The summation of these two areas gives the storage function $H_\circlearrowleft$ for a given state $(y(t),u(t))$ satisfying (5) according to Theorem 1.

The principle of the construction of $H_\circlearrowleft$ in (20) can be described in words as follows. From a given state $(y(0), u(0))$, let us define the trajectory that crosses the anhysteresis curve at a given time T by applying either a monotonically increasing input signal $u(t) = u(0) + t$ or a monotonically decreasing input signal $u(t) = u(0) - t$. Denote this trajectory by y and the intersecting point by $(y(T), u(T))$. Then the storage function $H_\circlearrowleft$ is given by the integral of the anhysteresis function from 0 to $u(T)$ minus the integral of y from 0 to T.

Proposition 1. *Consider the Duhem operator Φ satisfying the hypotheses in Theorem 1. Moreover, we assume that the anhysteresis function f_{an} satisfies $f_{an}(\xi) = 0$ for all $\xi \in \mathbb{R}$. Then for every $y_0, u_0 \in \mathbb{R}$, the function $H_\circlearrowleft$ as in (20) satisfies*

$$H_\circlearrowleft(y_0, u_0) = \sup_{\substack{u\in AC(\mathbb{R}_+)\\ u(0)=u_0}} -\int_0^T y(\tau)\dot{u}(\tau)\mathrm{d}\tau,$$

where $y := \Phi(u, y_0)$. In other words, $H_\circlearrowleft$ defines the available storage function (as discussed in [8,9]) where the supply rate is given by $y\dot{u}$ (instead of yu as in [8,9]).

Proof. As given in the first part of the proof of Theorem 1, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}H_\circlearrowleft(y(t), u(t)) = \dot{u}(t)y(t) - \int_{u(t)}^{u^*(t)} \frac{\mathrm{d}}{\mathrm{d}t}\omega_\Phi(\tau, y(t), u(t))\mathrm{d}\tau. \quad (21)$$

Integrating (21) from $t = 0$ to T, we obtain

$$\begin{aligned} &H_\circlearrowleft(T) - H_\circlearrowleft(0)\\ &= \int_0^T y(\tau)\dot{u}(\tau)\mathrm{d}\tau - \int_0^T \int_{u(t)}^{u^*} \frac{\mathrm{d}}{\mathrm{d}t}\omega_\Phi(\tau, y(t), u(t))\mathrm{d}\tau\mathrm{d}t, \end{aligned}$$

where $u^* = \Lambda(y(t), u(t))$ and we have used the shorthand notation of $H_\circlearrowleft(t) := H_\circlearrowleft(y(t), u(t))$.

By rearranging the terms in this equation, we arrive at

$$\begin{aligned} &-\int_0^T y(\tau)\dot{u}(\tau)\mathrm{d}\tau\\ &= H_\circlearrowleft(0) - H_\circlearrowleft(T) - \int_0^T \int_{u(t)}^{u^*} \frac{\mathrm{d}}{\mathrm{d}t}\omega_\Phi(\tau, y(t), u(t))\mathrm{d}\tau\mathrm{d}t. \end{aligned} \quad (22)$$

The supremum of the LHS of (22) over all possible u and T defines the available storage function where the supply rate is $y\dot{u}$. Note that this supply rate is a particular class of the general supply rate as studied in [26,8]. Since the last two terms on the RHS of (22) are non-positive, we will show that we can define u and T such that these two terms are equal to zero, and thus the supremum of the LHS of (22) is equal to $H_\circlearrowleft(y(0), u(0))$, which is equivalent to $H_\circlearrowleft(y_0, u_0)$, i.e., $H_\circlearrowleft$ is the available storage function.

From a given initial condition (y_0, u_0), let us introduce an input signal $u(t) = u_0(T - t) + t\Lambda(y_0, u_0)$ for all $t \in [0, T]$ and $u(t) = \Lambda(y_0, u_0)$ otherwise. This means that we have an input signal u which starts from u_0, ends at $\Lambda(y_0, u_0)$ at $t = T$ and remains there for all $t > T$. Together with the corresponding signal $y = \Phi(u, y_0)$, we have $\Lambda(y(t), u(t)) = \Lambda(y_0, u_0)$ for all t, i.e. the intersecting point is always the same. Indeed, this follows from the fact that $\Lambda(y(t), u(t))$ remains the same along the trajectories that converge to the intersection point $(\omega_\Phi(u^*, y_0, u_0), u^*)$ where $u^* = \Lambda(y_0, u_0)$.

Following the same arguments as in the proof of Theorem 1 (c.f., the arguments that lead to Eq. (B.9)), this input signal ensures that the last term on the RHS of (22) is equal to zero. Since $u(T) = \Lambda(y_0, u_0)$ for all $t > T$, we also have that $H_\circlearrowleft(y(t), u(t)) = 0$ for all $t > T$, i.e. the second term on the RHS of (22) is zero using such an input signal. Hence $H_\circlearrowleft$ as in (20) is an available storage function. □

The results given in Theorem 1 can be slightly generalized in order to incorporate the case when the Duhem hysteresis operator Φ has saturated output. Consider the set $D \subset \mathbb{R}^2$ which contains all relations of Φ, i.e., $\mathcal{R}_{y_0,u} := \{(y(t), u(t)) \in \mathbb{R}^2 | y = \Phi(u, y_0), t \in \mathbb{R}_+\} \subset D$ holds for all $u \in AC(\mathbb{R}_+)$ and $(y_0, u(0)) \in D$. For example, the set D for the Dahl model in Section 2 is given by $D = (-F_c, F_c) \times \mathbb{R}$. Using D, we can generalize Theorem 1 as follows.

Proposition 2. *Consider the Duhem hysteresis operator Φ defined in (6) and (15) with C^1 functions $F, G : D \to \mathbb{R}$ and with the traversing function ω_Φ and the anhysteresis function f_{an}. Assume that the anhysteresis curve is in D and there exists an intersecting function Λ (e.g., the hypotheses in Lemma 1 hold). Assume further that the Assumption **(A)** holds for all (σ, ξ) in D. Then for every $u \in AC(\mathbb{R}_+)$ and $(y_0, u(0)) \in D$, the function $t \to H_\circlearrowleft(y(t), u(t))$ with $H_\circlearrowleft$ as in (20) and $y := \Phi(u, y_0)$ is right differentiable and satisfies (5). Moreover, if the anhysteresis function f_{an} satisfies $f_{an}(0) = 0$, then $H_\circlearrowleft \geq 0$ and the Duhem operator is clockwise (CW).*

The proof follows the same arguments as that of Theorem 1.

4. Examples

4.1. The function $H_\circlearrowleft$ for the Dahl model

Recall the Dahl model as defined in Section 2.1 and consider the case when $r = 1$. In this case, the Dahl model can be reformulated into the Duhem operator as in (6) with

$$f_1(\sigma, \xi) = \rho\left(1 - \frac{\sigma}{F_c}\right), \qquad f_2(\sigma, \xi) = \rho\left(1 + \frac{\sigma}{F_c}\right), \quad (23)$$

where $\rho > 0$ and $F_c > 0$. It is immediate to check that the conditions as given in (7) are satisfied, which means there exists solution for this Duhem operator for all positive time. The anhysteresis function of the Dahl model is $f_{an}(\xi) = 0$.

Calculating the curve ω_Φ, we have

$$\begin{aligned} &\omega_\Phi(\tau, y(t), u(t))\\ &= \begin{cases} F_c + (y(t) - F_c)e^{\frac{\rho}{F_c}(u(t)-\tau)} & \tau \in [u(t), \infty),\\ -F_c + (y(t) + F_c)e^{\frac{\rho}{F_c}(\tau-u(t))} & \tau \in (-\infty, u(t)]. \end{cases} \end{aligned} \quad (24)$$

From (23) and (24), it is immediate to see that the pair (y, u) is well-defined in $D = (-F_c, F_c) \times \mathbb{R}$. The intersecting function $\Lambda(y(t), u(t))$ is given as follows

$$\Lambda(y(t), u(t)) = \begin{cases} u(t) + \dfrac{F_c}{\rho}\ln\dfrac{F_c}{y(t) + F_c} & y(t) \geq 0,\\ u(t) - \dfrac{F_c}{\rho}\ln\dfrac{-F_c}{y(t) - F_c} & y(t) < 0. \end{cases} \quad (25)$$

Since f_1 and f_2 in (23) satisfy the Assumption **(A)** for all $(\sigma, \xi) \in D$, the result in Proposition 2 holds.

By denoting $u^*(t) = \Lambda(y(t), u(t))$, we can compute explicitly the function $H_\circlearrowleft$ as follows

$$\begin{aligned} &H_\circlearrowleft(y(t), u(t))\\ &= \begin{cases} -F_c(u(t) - u^*(t)) + \dfrac{F_c}{\rho}(y(t) + F_c)(1 - e^{\frac{\rho}{F_c}(u^*(t)-u(t))})\\ \qquad y(t) \geq 0\\ F_c(u(t) - u^*(t)) + \dfrac{F_c}{\rho}(y(t) - F_c)(e^{\frac{\rho}{F_c}(u(t)-u^*(t))} - 1)\\ \qquad y(t) < 0 \end{cases}\\ &= \begin{cases} \dfrac{F_c^2}{\rho}\ln\dfrac{F_c}{y(t) + F_c} + \dfrac{F_C}{\rho}y(t) & y(t) \geq 0\\ \dfrac{F_C^2}{\rho}\ln\dfrac{-F_c}{y(t) - F_c} - \dfrac{F_C}{\rho}y(t) & y(t) < 0. \end{cases} \end{aligned} \quad (26)$$

Indeed, it can be checked that $\dot{H}_\circlearrowleft \leq \dot{u}(t)y(t)$.

4.2. Stability analysis of a second-order mechanical system with hysteretic friction

Now, let us consider an example of a mechanical system with the Dahl friction model given by $m\ddot{x} + d\dot{x} + kx + \Phi(x) = 0$, where $m > 0, d > 0, k > 0$, the hysteresis operator Φ is given as in (23) with $\rho > 0$ and $F_c > 0$. As discussed before, the functions f_1 and f_2 satisfy the hypotheses of Proposition 2.

The state–space representation of the system is given as follows

$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = -\frac{k}{m}x_1 - \frac{d}{m}x_2 - \frac{x_3}{m},$$
$$\dot{x}_3 = \rho\left(1 - \frac{x_3}{F_C}\right)x_{2+} + \rho\left(1 + \frac{x_3}{F_C}\right)x_{2-},$$

where $x_{2+} = \max\{0, x_2\}$ and $x_{2-} = \min\{0, x_2\}$. Using $V(x_1, x_2, x_3) = \frac{1}{2}kx_1^2 + \frac{1}{2}mx_2^2 + H_\circlearrowleft(x_3, x_1)$, where $H_\circlearrowleft$ is as in (26) and satisfies (5), a routine calculation shows that

$$\dot{V} \le -x_2x_3 - dx_2^2 + x_3x_2 = -dx_2^2.$$

Since the relations of the corresponding Dahl operator (i.e. the set $\mathcal{R}_{y_0,u} := \{(y(t), u(t)) | y = \Phi(u, y_0)\}$) is contained in $(-F_C, F_C) \times \mathbb{R}$ for all $y_0 \in (-F_C, F_C)$ and $u \in AC(\mathbb{R}_+)$, then it implies that x_3 (which is the output of the Dahl operator) is bounded and lies in the interval $(-F_C, F_C)$. Additionally, we have V which is lower bounded and radially unbounded in the first and second arguments, i.e. $V(x_1, x_2, x_3) \to \infty$ as $\|\begin{smallmatrix}x_1\\x_2\end{smallmatrix}\| \to \infty$. Thus $\dot{V} \le -dx_2^2$ implies that the state trajectory (x_1, x_2) is bounded. Moreover, using the boundedness of (x_1, x_2) and the boundedness of x_3, an application of the Lasalle's invariance principle [27,28] shows that (x_1, x_2, x_3) converges to the largest invariant set where $x_2 = 0$. By analyzing the corresponding state equations, this invariant set is given by $\{(x_1, x_2, x_3) | kx_1 = -x_3,\ x_2 = 0\}$.

5. Conclusion

In this paper, we have investigated the clockwise I/O dynamics of a class of Duhem hysteresis operator by obtaining sufficient conditions for the Duhem operators to be CW. The CW property is obtained via the construction of a suitable function satisfying the CW dissipation inequality which can be useful for studying stability of systems having CW hysteretic element, such as, mechanical systems with hysteretic friction. The sufficient conditions for CW I/O dynamics incorporates also the knowledge of anhysteresis function which is commonly neglected in the literature of hysteretic systems. For systems identification of hysteresis systems, the results provide additional characterization of the Duhem operators that can be used to restrict the class of the Duhem operators which will be fitted with the measurement data.

Appendix A. Proof of Lemma 1

Proof. The proof is similar to the proof of [25, Lemma 3.1]. Consider the continuous function $\varphi : \mathbb{R}^3 \to \mathbb{R}$ defined as $\varphi(\xi, y_0, u_0) = \omega_\Phi(\xi, y_0, u_0) - f_{an}(\xi)$. Consider also A_0 and A_1 the two subsets of $\mathbb{R}^3$ defined as,

$$A_0 = \{(\xi, y_0, u_0) \in \mathbb{R}^3, y_0 > f_{an}(u_0), \xi < u_0\},$$
$$A_1 = \{(\xi, y_0, u_0) \in \mathbb{R}^3, y_0 < f_{an}(u_0), \xi > u_0\}.$$

Note that the function f_{an} being strictly increasing by assumption, implies that these sets are open sets. Also, the function ω_Φ satisfies

$$\frac{\partial \omega_\Phi}{\partial \xi}(\xi, y_0, u_0) = f_2(\omega_\Phi(\xi, y_0, u_0), \xi) \quad \forall (\xi, y_0, u_0) \in A_0,$$
$$\frac{\partial \omega_\Phi}{\partial \xi}(\xi, y_0, u_0) = f_1(\omega_\Phi(\xi, y_0, u_0), \xi) \quad \forall (\xi, y_0, u_0) \in A_1.$$

Consequently, $\omega_\Phi(\xi, y_0, u_0)$ is the solution of ordinary differential equations computed from C^1 vector field. With [29, Theorem V.3.1], it implies that ω_Φ is a C^1 function in $A_0 \cup A_1$. Moreover, the function f_{an} being C^1 implies that the function φ is C^1 in $A_0 \cup A_1$. With (17) and (18), the function φ satisfies,

$$\frac{\partial \varphi}{\partial \xi}(\xi, y_0, u_0) > \epsilon \neq 0, \quad \forall (\xi, y_0, u_0) \in A_0 \cup A_1.$$

Consequently, φ is a strictly increasing function in its first argument in the set $A_0 \cup A_1$. This also implies that,

$$\varphi(\xi, y_0, u_0) < \varphi(u_0, y_0, u_0) + \epsilon(\xi - u_0) \quad \forall (\xi, y_0, u_0) \in A_0,$$
$$\varphi(\xi, y_0, u_0) > \varphi(u_0, y_0, u_0) + \epsilon(\xi - u_0) \quad \forall (\xi, y_0, u_0) \in A_1.$$

Note that if $y_0 > f_{an}(u_0)$, then $\varphi(u_0, y_0, u_0) > 0$ and consequently there exists a unique real number u^* such that $\varphi(u^*, y_0, u_0) = 0$ and $(u^*, y_0, u_0) \in A_0$. On the other hand, if $y_0 < f_{an}(u_0)$, then $\varphi(u_0, y_0, u_0) < 0$ and consequently there exists a unique real number u^* such that $\varphi(u^*, y_0, u_0) = 0$ and $(u^*, y_0, u_0) \in A_1$. Therefore, by denoting $\Lambda(y_0, u_0) = u^*$, by employing the implicit function theorem and using the fact that φ is C^1, it can be shown that Λ is C^1. □

Appendix B. Proof of Theorem 1

Proof. The proof of Theorem 1 follows the same line as in our previous work [25]. In the first part of the proof we will prove that for all $t \in \mathbb{R}_+$, $\dot{H}_\circlearrowleft(y(t), u(t))$ exists and satisfies (5). In the second part we show the non-negativeness of $H_\circlearrowleft(y(t), u(t))$.

To show that $H_\circlearrowleft$ exists, let us denote $u^* := \Lambda(y, u)$. Using the Leibniz derivative rule, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}H_\circlearrowleft(y(t), u(t)) &= \frac{\mathrm{d}}{\mathrm{d}t}\left[\int_0^{\Lambda(y(t),u(t))} f_{an}(\tau)\mathrm{d}\tau - \int_{u(t)}^{\Lambda(y(t),u(t))} \omega_\Phi(\tau, y(t), u(t))\mathrm{d}\tau\right]\\ &= \dot{\Lambda}(y(t), u(t))f_{an}(\Lambda(y(t), u(t)))\\ &\quad - \dot{\Lambda}(y(t), u(t))\omega_\Phi(\Lambda(y(t), u(t)), y(t), u(t))\\ &\quad + \dot{u}(t)\omega_\Phi(u(t), y(t), u(t))\\ &\quad - \int_{u(t)}^{\Lambda(y(t),u(t))} \frac{\mathrm{d}\omega_\Phi(\tau, y(t), u(t))}{\mathrm{d}t}\mathrm{d}\tau = \dot{u}(t)y(t)\\ &\quad - \int_{u(t)}^{u^*(t)} \frac{\mathrm{d}}{\mathrm{d}t}\omega_\Phi(\tau, y(t), u(t))\mathrm{d}\tau, \end{aligned} \tag{B.1}$$

where $u^*(t) = \Lambda(y(t), u(t))$ and the last equation is due to $\omega_\Phi(u(t), y(t), u(t)) = y(t)$ and by the hypothesis given in Lemma 1. The first term on the RHS of (B.1) exists for all $t \ge 0$ since $u(t)$ satisfies (6). In order to get (5), it remains to check whether the last term of (B.1) exists, is finite and satisfies

$$\int_{u(t)}^{u^*(t)} \frac{\mathrm{d}}{\mathrm{d}t}\omega_\Phi(\tau, y(t), u(t))\mathrm{d}\tau \ge 0. \tag{B.2}$$

It suffices to show that, for every $\tau \in [u(t), u^*(t)]$, the following limit

$$\lim_{\epsilon \searrow 0^+} \frac{1}{\epsilon}[\omega_\Phi(\tau, y(t+\epsilon), u(t+\epsilon)) - \omega_\Phi(\tau, y(t), u(t))] \tag{B.3}$$

exist and the limit of (B.3) is greater or equal to zero when $u^*(t) > u(t)$ and the limit is less or equal to zero elsewhere.

For any $\epsilon \geq 0$, let us introduce the continuous function $\omega_\epsilon : \mathbb{R} \to \mathbb{R}$ by

$$\omega_\epsilon(\tau) = \omega_\Phi(\tau, y(t+\epsilon), u(t+\epsilon)). \tag{B.4}$$

More precisely, using (16), ω_ϵ is the unique solution of

$$\omega_\epsilon(\tau) = \begin{cases} y(t+\epsilon) + \displaystyle\int_{u(t+\epsilon)}^{\tau} f_1(\omega_\epsilon(s), s)\mathrm{d}s & \forall \tau \geq u(t+\epsilon) \\ y(t+\epsilon) + \displaystyle\int_{u(t+\epsilon)}^{\tau} f_2(\omega_\epsilon(s), s)\mathrm{d}s & \forall \tau \leq u(t+\epsilon). \end{cases} \tag{B.5}$$

Note that $\omega_0(\tau) = \omega_\Phi(\tau, y(t), u(t))$ as in (16) for all $\tau \in \mathbb{R}$ and

$$\omega_\epsilon(u(t+\epsilon)) = y(t+\epsilon) \quad \forall \epsilon \in \mathbb{R}_+ . \tag{B.6}$$

In order to show the existence of (B.3) and the validity of (B.2), we consider several cases depending on the sign of $\dot{u}(t)$ and $F(y(t), u(t))$. It can be checked that the hypothesis **(A)** on F implies that $f_1(y(t), u(t)) \geq f_2(y(t), u(t))$ whenever $y(t) \leq f_{an}(u)$, and $f_1(y(t), u(t)) < f_2(y(t), u(t))$ otherwise.

First, we assume that $\dot{u}(t) > 0$ and $y(t) \geq f_{an}(u(t))$. In this case, according to Lemma 1, we have $u^*(t) < u(t)$. Since $\dot{u}(t) > 0$, there exists $\gamma > 0$ such that $\tau \leq u(t) < u(s)$ for all s in $(t, t+\gamma)$. It follows from (B.5) and assumption **(A)** that for every $\epsilon \in (0, \gamma)$:

$$\begin{aligned} \frac{\mathrm{d}\omega_\epsilon(u(s))}{\mathrm{d}s} &= f_2(\omega_\epsilon(u(s)), u(s))\dot{u}(s) \\ &\geq f_1(\omega_\epsilon(u(s)), u(s))\dot{u}(s) \quad \forall s \in [t, t+\epsilon], \end{aligned}$$

and the function ω_0 satisfies

$$\frac{\mathrm{d}\omega_0(u(s))}{\mathrm{d}s} = f_1(y(s), u(s))\dot{u}(s) \quad \forall s \in [t, t+\epsilon].$$

Since the functions $\epsilon \mapsto w_0(u(t+\epsilon))$ and $\epsilon \mapsto y(t+\epsilon)$ with $\epsilon \in (0, \gamma]$ are two C^1 functions which are solutions of the same locally Lipschitz ODE and with the same initial value. By uniqueness of solution, we get $\omega_0(u(t+\epsilon)) = y(t+\epsilon)$.

This together with the fact that $\omega_\epsilon(u(t+\epsilon)) = y(t+\epsilon)$ and using the comparison principle (in reverse direction), we get that for every $\epsilon \in [0, \gamma)$:

$$\omega_\epsilon(u(s)) \leq \omega_0(u(s)) \quad \forall s \in [t, t+\epsilon].$$

Since the two functions $\omega_\epsilon(\tau)$ and $\omega_0(\tau)$ for $\tau \in [u^*(t), u(t)]$ are two solutions of the same ODE, it follows that[1] $\omega_\epsilon(\tau) \geq \omega_0(\tau)$ and we get that if it exists:

$$\lim_{\epsilon \searrow 0^+} \frac{1}{\epsilon}[\omega_\epsilon(\tau) - \omega_0(\tau)] \leq 0 \quad \forall \tau \in [u^*(t), u(t)]. \tag{B.7}$$

Then it is clear that

$$\lim_{\epsilon \searrow 0^+} \frac{1}{\epsilon}[\omega_\epsilon(\tau) - \omega_0(\tau)] \geq 0 \quad \forall \tau \in [u(t), u^*(t)]. \tag{B.8}$$

In the following, we show the existence of the limit given in (B.7) by computing a bound on the function $\epsilon \mapsto \frac{1}{\epsilon}[\omega_\epsilon(\tau) - \omega_0(\tau)]$.

Note that for every $\epsilon \in [0, \gamma]$,

$$\begin{aligned} |\omega_\epsilon(\tau) - \omega_0(\tau)| &\leq |y(t+\epsilon) - y(t)| + \left| \int_{u(t+\epsilon)}^{u(t)} f_2(\omega_\epsilon(s), s)\mathrm{d}s \right| \\ &\quad + \left| \int_{u(t)}^{\tau} f_2(\omega_\epsilon(s), s) - f_2(\omega_0(s), s)\mathrm{d}s \right| \\ &\leq |y(t+\epsilon) - y(t)| + \int_{u(t)}^{u(t+\epsilon)} |f_2(\omega_\epsilon(s), s)|\mathrm{d}s + \int_{\tau}^{u(t)} |f_2(\omega_\epsilon(s), s) \\ &\quad - f_2(\omega_0(s), s)|\mathrm{d}s, \end{aligned}$$

for all $\tau \in [u^*(t), u(t)]$. By the locally Lipschitz property of f_2 and by the boundedness of ω_ϵ on $[\tau, u(t)]$ for all $\epsilon \in [0, \gamma]$, it can be shown that there exists α, such that α is a bound of f_2 on a compact set. Then

$$\begin{aligned} |\omega_\epsilon(\tau) - \omega_0(\tau)| &\leq |y(t+\epsilon) - y(t)| + \int_{\tau}^{u(t)} L|\omega_\epsilon(s) - \omega_0(s)|\mathrm{d}s \\ &\quad + \alpha|u(t+\epsilon) - u(t)|, \end{aligned}$$

where L is the Lipschitz constant of f_2 on $[\omega_{\min}, \omega_{\max}] \times [\tau, u(t)]$ with

$$\begin{aligned} \omega_{\min} &= \min_{(c,s) \in [0,\gamma] \times [\tau, u(t)]} \omega_c(s), \\ \omega_{\max} &= \max_{(c,s) \in [0,\gamma] \times [\tau, u(t)]} \omega_c(s). \end{aligned}$$

With Gronwall's lemma, this implies that for every $\epsilon \in [0, \gamma]$

$$\begin{aligned} &|\omega_\epsilon(\tau) - \omega_0(\tau)| \\ &\quad \leq \exp((u(t) - \tau)L)[|y(t+\epsilon) - y(t)| + \alpha|u(t+\epsilon) - u(t)|], \end{aligned}$$

for all $\tau \in [u^*(t), u(t)]$. Hence

$$\begin{aligned} &\lim_{\epsilon \searrow 0^+} \frac{1}{\epsilon}|\omega_\epsilon(\tau) - \omega_0(\tau)| \\ &\quad \leq \exp((u(t) - \tau)L)\,[|f_1(y(t), u(t))| + \alpha]\,\dot{u}(t), \end{aligned}$$

for all $\tau \in [u^*(t), u(t)]$. Consequently the limit given in (B.7) exists. It implies that the inequality (B.2) holds when $\dot{u}(t) > 0$ and $y(t) \geq f_{an}(u(t))$.

For the next case, we assume that $\dot{u}(t) > 0$ and $y(t) < f_{an}(u(t))$. Again, according to Lemma 1, we have $u^*(t) > u(t)$. Since for every $\epsilon \in (0, \gamma]$ the two functions $\omega_\epsilon(\tau)$ and $\omega_0(\tau)$ satisfy the same ODE for[2] $\tau \in [u(t+\epsilon), u^*(t)]$, we have

$$\omega_\epsilon(\tau) = \omega_0(\tau) \quad \forall \tau \in [u(t+\epsilon), u^*(t)],$$

for all $\epsilon \in [0, \gamma]$. This implies that

$$\lim_{\epsilon \searrow 0^+} \frac{1}{\epsilon}[\omega_\epsilon(\tau) - \omega_0(\tau)] = 0. \tag{B.9}$$

We can use similar arguments to prove that (B.2) is satisfied when $\dot{u}(t) < 0$.

Finally, when $\dot{u}(t) = 0$, we simply get

$$\lim_{\epsilon \searrow 0^+} \frac{1}{\epsilon}|\omega_\epsilon(\tau) - \omega_0(\tau)| = 0,$$

by continuity of the above bound.

For the second step, we need to show that $H_\circlearrowleft$ is non-negative. Consider the case when $y(t) \geq f_{an}(u(t))$, we have $u^*(t) < u(t)$ and

[1] Otherwise there exist $\tau_1 < \tau_2$ such that $\omega_\epsilon(\tau_1) = \omega_0(u(\tau_1))$ and $\omega_\epsilon(\tau_2) > \omega_0(u(\tau_2))$ which contradict the uniqueness of the solution of the locally Lipschitz ODE.

[2] we have for all $\tau \in [u(t+\epsilon), u^*(t)]$:

$$\frac{\mathrm{d}\omega_\epsilon(\tau)}{\mathrm{d}\tau} = f_1(\omega_\epsilon(\tau), \tau), \qquad \frac{\mathrm{d}\omega_0(\tau)}{\mathrm{d}\tau} = f_1(\omega_0(\tau), \tau).$$

$\omega_\Phi(\tau) \geq f_{an}(\tau)$ for all $\tau \in [u^*(t), u(t)]$ by Lemma 1. Since $f_{an}(\tau)$ belongs to the sector $[0, \infty)$ for all $\tau \in \mathbb{R}$, we have

$$H_{\circlearrowleft}(y(t), u(t)) = \int_0^{u(t)} f_{an}(\tau)\mathrm{d}\tau + \int_{u(t)}^{u^*(t)} f_{an}(\tau) - \omega_\Phi(\tau, y(t), u(t))\mathrm{d}\tau \geq 0.$$

In case $y(t) < f_{an}(u(t))$, we can show the non-negativeness of H by using similar arguments. □

References

[1] D. Angeli, Systems with counterclockwise input–output dynamics, IEEE Trans. Automat. Control 51 (7) (2006) 1130–1143.
[2] A.K. Padthe, J. Oh, D.S. Bernstein, Counterclockwise dynamics of a rate-independent semilinear Duhem model, Proc. IEEE Conf. Dec. Contr., Seville, 2005.
[3] G. Bertotti, I.D. Mayergoyz, The Science of Hysteresis: Mathematical Modeling and Applications, Academic press, San Diego, 2006.
[4] M. Brokate, J. Sprekels, Hysteresis and Phase Transitions, Springer-Verlag, New York, 1996.
[5] J.W. Macki, P. Nistri, P. Zecca, Mathematical models for hysteresis, SIAM Rev. 35 (1) (1993) 94–123.
[6] H. Logemann, E.P. Ryan, Systems with hysteresis in the feedback loop: existence, regularity and asymptotic behaviour of solutions, ESAIM Control, Optimi. Calc. Var. 9 (2003) 169–196.
[7] D. Angeli, Multistability in systems with counter-clockwise input–ouput dynamics, IEEE Trans. Automat. Control 52 (4) (2007) 596–609.
[8] J.C. Willems, Dissipative dynamical systems. Part I: general theory. Part II: linear systems with quadratic supply rates, Arch. Ration. Mech. Anal. 45 (5) (1972) 321–393.
[9] A.J. Van der Schaft, L_2-Gain and Passivity Techniques in Nonlinear Control, Springer-Verlag, London, 2000.
[10] J. Oh, D.S. Bernstein, Semilinear Duhem model for rate-independent and rate-dependent hysteresis, IEEE Trans. Automat. Control 50 (5) (2005) 631–645.
[11] R.B. Gorbet, K.A. Morris, Generalized dissipation in hysteretic systems, Proc. IEEE Conf. Dec. Contr. (1998).
[12] I. Mayergoyz, Mathematical Models of Hysteresis and Their Applications, Academic Press, Boston, 2003.
[13] T. Pare, A. Hassabi, J.J. How, A KYP lemma and invariance principle for systems with multiple hysteresis non-linearities, Internat. J. Control 74 (11) (2001) 1140–1157.
[14] I.R. Petersen, A. Lanzon, Feedback control of negative-imaginary system, IEEE Control Syst. Mag. 30 (5) (2010) 54–72.
[15] B. Jayawardhana, R. Ouyang, V. Andrieu, Stability of systems with the Duhem hysteresis operator: the dissipativity approach, Automatica 48 (10) (2012) 2657–2662. http://dx.doi.org/10.1016/j.automatica.2012.06.069.
[16] B. Jayawardhana, V. Andrieu, Sufficient conditions for dissipativity on Duhem hysteresis model, Proc. IEEE Conf. Dec. Contr., Shanghai, 2009.
[17] P.C. Breedveld, Port-based modeling of mechatronic systems, Math. Comput. Simulation 66 (2–3) (2004) 99–127.
[18] C. Canudas de Wit, H. Olsson, K.J. Åström, P. Lischinsky, A new model for control of system with friction, IEEE Trans. Automat. Control 40 (3) (1995) 419–425.
[19] A. Visintin, Differential Models of Hysteresis, Springer-Verlag, New York, 1994.
[20] P. Dahl, Solid friction damping of mechanical vibrations, AIAA J. 14 (2) (1976) 1675–1682.
[21] A.K. Padthe, B. Drincic, J. Oh, D.D. Rizos, S.D. Fassois, D.S. Bernstein, Duhem modeling of friction-induced hysteresis, IEEE Control Syst. Mag. 28 (5) (2008) 90–107.
[22] P.M. Sain, M.K. Sain, B.F. Spencer, Models for hysteresis and application to structural control, Proc. American Control Conference, Albuquerque, 1997.
[23] Y.K. Wen, Method for random vibration of hysteretic systems, J. Eng. Mech. Division, Proc. ASCE 102 (1976) 249–263.
[24] S.J. Dyke, B.F. Spencer, M.K. Sain, J.D. Carlson, Modeling and control of magnetorheological dampers for seismic response reduction, Smart Material 5 (5) (1996) 565–575.
[25] B. Jayawardhana, R. Ouyang, V. Andrieu, Dissipativity of general Duhem hysteresis models, Proc. IEEE Conf. Dec. Contr., Orlando, 2011.
[26] H.L. Trentelmann, J.C. Willems, Every storage functions is a state function, Systems Control Lett. 32 (1997) 249–259.
[27] H.K. Khalil, Nonlinear Systems, third ed., Prentice-Hall, Upper Saddle River, NJ, 2002.
[28] H. Logemann, E.P. Ryan, Asymptotic behaviour of nonlinear systems, Amer. Math. Monthly 111 (10) (2004) 864–889.
[29] P. Hartman, Ordinary Differential Equations, second ed., Birkhauser, 1982.

Further reading

[1] H. Logemann, A.D. Mawby, Extending hysteresis operators to spaces of piecewise continuous functions, J. Math. Anal. Appl. 282 (2003) 107–127.