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Duality orbits of non-geometric fluxes

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Compactifications in duality covariant constructions such as generalised geometry and double field theory have proven to be suitable frameworks to reproduce gauged supergravities containing non-geometric fluxes. However, it is a priori unclear whether these approaches only provide a reformulation of old results, or also contain new physics. To address this question, we classify the T- and U-duality orbits of gaugings of (half-)maximal supergravities in dimensions seven and higher. It turns out that all orbits have a geometric supergravity origin in the maximal case, while there are non-geometric orbits in the half-maximal case. We show how the latter are obtained from compactifications of double field theory.

1 Introduction

When compactifying heterotic, type II or eleven-dimensional supergravity on a given background, one obtains lower-dimensional effective theories whose features depend on the fluxes included in the compactification procedure and, in particular, on the amount of supersymmetry preserved by the chosen background. When some supersymmetry is preserved during the compactification, the effective theories under consideration are then gauged supergravities.

In particular, in the context of half-maximal [1] and maximal [2] gauged supergravities, not only does supersymmetry tightly organise the ungauged theory, but also it strictly determines the set of possible deformations (*i.e.* gaugings). The development of the so-called embedding tensor formalism has enabled one to formally describe all the possible deformations in a single universal formulation, which therefore completely restores duality covariance. Unfortunately, not all the deformations have a clear higher-dimensional origin, in the sense that they can be obtained by means of a certain compactification of ten or eleven dimensional supergravity.

One of the most interesting open problems concerning flux compactifications is to reproduce, by means of a suitable flux configuration, a given lower-dimensional gauged supergravity theory. Although this was done in particular cases (see for example [3, 4]), an exhaustive analysis remains to be done. This is due to fact that, on the one hand we lack a classification of the possible gauging configurations allowed in gauged supergravities and, on the other hand, only a limited set of compactification scenarios are known. Typically, to go beyond the simplest setups one appeals to dualities. The paradigmatic example [5] starts by

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applying T-dualities to a simple toroidal background with a non-trivial two-form generating a single H_{abc} flux. By T-dualizing this setup, one can construct a chain of T-dualities leading to new backgrounds (like twisted-tori or T-folds) and generating new (dual) fluxes, like the so-called ${Q_a}^{bc}$ and R^{abc}. It is precisely by following duality covariance arguments in the lower-dimensional effective description that non-geometric fluxes [5] were first introduced in order to explain the mismatch between particular flux compactifications and generic gauged supergravities.

Here we would like to emphasise that all these (a priori) different T-duality connected flux configurations by definition lie in the same orbit of gaugings, and therefore give rise to the same lower-dimensional physics. In order to obtain a different gauged supergavity, one should consider more general configurations of fluxes, involving for example combinations of geometric and non-geometric fluxes, that can never be T-dualised to a frame in which the non-geometric fluxes vanish. For the sake of clarity, we depict this concept in Fig. 1.

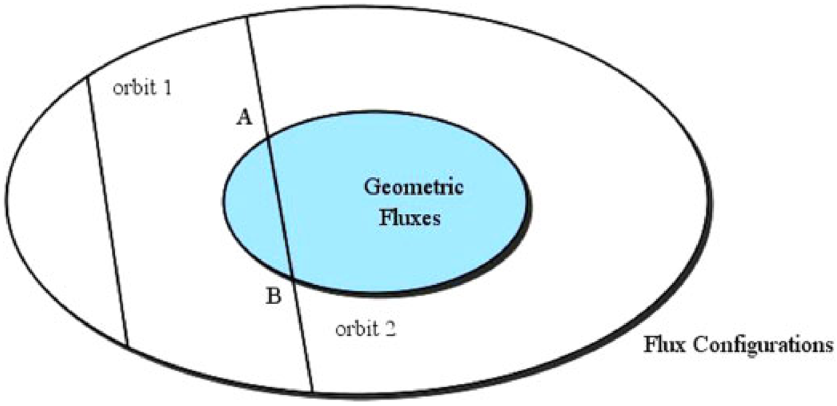

Fig. 1 (online colour at: www.fp-journal.org) The space of flux configurations sliced into duality orbits (vertical lines). Moving along a given orbit corresponds to applying dualities to a certain flux configuration and hence it does not imply any physical changes in the lower-dimensional effective description. Geometric fluxes only constitute a subset of the full configuration space. Given an orbit, the physically relevant question is whether (orbit 2 between A and B) or not (orbit 1) this intersects the geometric subspace. We refer to a given point in an orbit as a *representative*.

Non-geometric fluxes are the inevitable consequence of string dualities, and only a theory which promotes such dualities to symmetries could have a chance to describe them together with geometric fluxes and to understand their origin in a unified way. From the viewpoint of the lower-dimensional effective theory, it turns out that half-maximal and maximal gauged supergravities give descriptions which are explicitly covariant with respect to T- and U-duality respectively. This is schematically depicted in Table 1, even though only restricted to the cases we will address in this work.

Table 1 The various T- and U-duality groups in $D > 6$. These turn out to coincide with the global symmetry groups of half-maximal and maximal supergravities respectively.

D	T-duality	U-duality
9	$\mathrm{O}(1,1)$	$\mathbb{R}^+ \times \mathrm{SL}(2)$
8	$\mathrm{O}(2,2) = \mathrm{SL}(2) \times \mathrm{SL}(2)$	$\mathrm{SL}(2) \times \mathrm{SL}(3)$
7	$\mathrm{O}(3,3) = \mathrm{SL}(4)$	$\mathrm{SL}(5)$

In recent years, a new proposal aiming to promote T-duality to a fundamental symmetry in field theory has received increasing interest. It is named Double Field Theory (DFT) [6] since T-duality invariance requires a doubling of the spacetime coordinates, by supplementing them with dual coordinates associated to the stringy winding modes, whose dynamics can become important in the compactified theory. Recently it has been pointed out how to obtain gaugings of $\mathcal{N} = D = 4$ supergravity by means of twisted double torus reductions of DFT [7,8], even though at that stage, the so-called *weak* and *strong* constraints imposed for consistency of DFT represented a further restriction that prevented one from describing the most general gaugings that solve the Quadratic Constraints (QC) of gauged supergravity.

Subsequently, an indication has been given that gauge consistency of DFT does not need the weak and strong constraints [9]. Following this direction, we could wonder whether relaxing these constraints can provide a higher-dimensional origin for all gaugings of extended supergravity in DFT. Our aim in the present work is to assess to what extent DFT can improve our description of non-geometric fluxes by giving a higher-dimensional origin to orbits which do not follow from standard supergravity compactifications. We will call such orbits of gaugings *non-geometric* (in Fig. 1 they are represented by orbit 1).

As a starting point for this investigation, we will address the problem in the context of maximal and half-maximal gauged supergravities in seven dimensions and higher, where the global symmetry groups are small enough to allow for a general classification of orbits, without needing to consider truncated sectors. We will show that in the half-maximal supergravities in seven and higher-dimensions, where the classifications of orbits can be done exhaustively, *all* the orbits (including geometric and non-geometric) admit an uplift to DFT, through Scherk-Schwarz (SS) [10] compactifications on appropriate backgrounds. We provide explicit backgrounds for every orbit, and discuss their (un)doubled nature. The result is that truly doubled DFT provides the appropriate framework to deal with orbits that can not be obtained from supergravity. In contrast, in maximal supergravities in eight and higher-dimensions, all orbits are geometric and hence can be obtained without resorting to DFT.

The paper is organised as follows. In Sect. 2 we start with a brief review and motivation of DFT. We will make particular emphasis in discussing aspects of its SS compactifications, and the constraints arising from gauge consistency. We will explicitly show how the gaugings in the effective theory are related to the compactification ansatz, in order to make a link with the results of the following sections. In Sect. 3 we present the classification of consistent gaugings in maximal supergravity in terms of U-duality orbits. In particular, in Sect. 3.1 and 3.2, we work out the $D = 9$ and $D = 8$ orbits. In both cases we are able to show that all the duality orbits have a geometric origin in compactifications of ten dimensional supergravity. In Sect. 4 we classify the consistent gaugings in half-maximal supergravity in terms of T-duality orbits. In particular, in Sect. 4.1 and 4.2, we work out the $D = 8$ and $D = 7$ orbits. Here we encounter the first orbits lacking a geometric higher-dimensional origin. We show that such orbits do follow from dimensional reductions of DFT. Finally, our conclusions are presented in Sect. 5. We defer a number of technical details on gauge algebras and 't Hooft symbols to the appendices.

2 Orbits from double field theory

When two configurations of gaugings are connected by a duality transformation, the physics they give rise to is the same. In this direction, we have defined an *orbit* of gaugings as a set of gauged theories that are related by dualities. One can then state that physically distinct theories are labeled by orbits, rather than by generic solutions to the QC. In this section we will provide the link between orbits in gauged supergravities and SS compactifications of DFT.

DFT is a recent proposal that promotes T-duality to a symmetry in field theory [6, 11], and is currently defined in terms of a background independent action [12]. The theory is defined on a double space [13], and its original version was created to describe the dynamics of closed strings on tori, the dual coordinates being associated to the winding modes of the strings. However, the background independent action allows for more general spaces, and SS compactifications of DFT were shown to formally reproduce the bosonic (electric) sector of half-maximal gauged supergravities [7, 8]. The gauge invariance of DFT and closure of its gauge algebra gives rise to a set of constraints that restrict the coordinate dependence of the fields. A possible set of solutions to such constraints is given by restricting the fields and gauge parameters to satisfy the so-called weak and strong constraints. In such a situation, they can always be T-dualised to a frame in which the dependence on dual coordinates is cancelled. This restriction arises naturally in the context of toroidal compactifications and has a close relation to the level-matching condition in the sigma model. In such case, DFT provides an interesting framework in which ten-dimensional supergravity can be rotated to T-dual frames [14]. Many other interesting works on the subject can be found in refs [15, 16].

While toroidal compactifications of DFT lead to half-maximal ungauged supergravities, SS compactifications on more general double spaces are effectively described by gauged supergravities like the ones we will analyse in the next sections. If the internal space is restricted in such a way that there always exists a frame without dual coordinate dependence, the only orbits allowed in the effective theory are those admitting representatives that can be obtained from compactifications of ten dimensional supergravity. This is not the most general case, and we will show that some orbits require the compact space to be truly doubled, capturing information of both momentum and winding modes.

Recently in [9], a new set of solutions to the constraints for DFT has been found. For these solutions the internal dependence of the fields is not dynamical, but fixed. The constraints of DFT restrict the dynamical external space to be undoubled, but allows for a doubling of the internal coordinates as long as the QC for the gaugings are satisfied. Interestingly, these are exactly the constraints needed for consistency of gauged supergravity, so there is a priori no impediment to uplift any orbit to DFT in this situation. In fact, in the following sections we show that all the orbits in half-maximal $D = 7, 8$ gauged supergravities can be reached from twisted double tori compactifications of DFT.

2.1 DFT and (half-)maximal gauged supergravities

DFT is a field theory with manifest invariance under the $\mathrm{O}(d, d)$ T-duality group, and therefore captures stringy features. The coordinates form fundamental vectors $X^M = (\tilde{x}_i, x^i)$, containing d space-time coordinates x^i and d dual coordinates $\tilde{x}_i$, $i = 1, \ldots, d$. The field content is that of the NS-NS sector, but defined on the double space. The generalised metric is a symmetric element of $\mathrm{O}(d, d)$

$$\mathcal{H}_{MN} = \begin{pmatrix} g^{ij} & -g^{ik}b_{kj} \\ b_{ik}g^{kj} & g_{ij} - b_{ik}g^{kl}b_{lj} \end{pmatrix} \tag{2.1}$$

and includes the d-dimensional metric g_{ij}, and the d-dimensional Kalb-Rammond field b_{ij}. The metric of the global symmetry group

$$\eta_{MN} = \begin{pmatrix} 0^{ij} & \delta^i{}_j \\ \delta_i{}^j & 0_{ij} \end{pmatrix} \tag{2.2}$$

raises and lowers the indices of $\mathcal{H}$, such that $\mathcal{H}_{MP}\mathcal{H}^{PN} = \delta_M^N$. On the other hand, the dilaton ϕ is combined with the determinant of g in a T-invariant way $e^{-2d} = \sqrt{g}e^{-2\phi}$.

Detailed reviews of DFT can be found in refs [17]. Here we will only provide a discussion of its constraints and some aspects of its SS reductions, just the minimal ingredients with the corresponding references to make contact with the results of the following sections.

In the SS procedure, the coordinates X^M are split into external directions $\mathbb{X}$ and compact internal $\mathbb{Y}$ coordinates. The former set contains pairs of $\mathrm{O}(D, D)$ dual coordinates, while the latter one contains pairs of $\mathrm{O}(n, n)$ dual coordinates, with $d = D + n$. This means that if a given coordinate is external (internal), its dual must also be external (internal), so the effective theory is formally a (gauged) DFT. The SS procedure is then defined in terms of a reduction ansatz, that specifies the dependence of the fields in $(\mathbb{X}, \mathbb{Y})$

$$\mathcal{H}_{MN}(\mathbb{X}, \mathbb{Y}) = U(\mathbb{Y})^A{}_M \, \widehat{\mathcal{H}}(\mathbb{X})_{AB} \, U(\mathbb{Y})^B{}_N \, , \quad d(\mathbb{X}, \mathbb{Y}) = \widehat{d}(\mathbb{X}) + \lambda(\mathbb{Y}) \, . \tag{2.3}$$

Here the hatted fields $\widehat{\mathcal{H}}$ and $\widehat{d}$ are the dynamical fields in the effective theory, parameterizing perturbations around the background, which is defined by $U(\mathbb{Y})$ and $\lambda(\mathbb{Y})$. The matrix U is referred to as the *twist matrix*, and must be an element of $\mathrm{O}(n,n)$. It contains a DFT T-duality index M, and another index A corresponding to the T-duality group of the effective theory. When DFT is evaluated on the reduction ansatz, the twists generate the gaugings of the effective theory

$$f_{ABC} = 3\eta_{D[A}\,(U^{-1})^M{}_B(U^{-1})^N{}_{C]}\partial_M U^D{}_M\ , \tag{2.4}$$

$$\xi_A = \partial_M (U^{-1})^M{}_A - 2(U^{-1})^M{}_A\partial_M\lambda\ , \tag{2.5}$$

where f_{ABC} and ξ_A build the generalised structure constants of the gauge group in the lower-dimensional theory.

Although U and λ are $\mathbb{Y}$ dependent quantities, the gaugings are forced to be constants in order to eliminate the $\mathbb{Y}$ dependence from the lower dimensional theory. When the external-internal splitting is performed, namely $d = D + n$, the dynamical fields are written in terms of their components which are a D-dimensional metric, a D-dimensional 2-form, $2n$ D-dimensional vectors and n^2 scalars. These are the degrees of freedom of half-maximal supergravities. Since these fields are contracted with the gaugings, one must make sure that after the splitting the gaugings have vanishing Lorentzian indices, and this is achieved by stating that the twist matrix is only non-trivial in the internal directions. Therefore, although formally everything is covariantly written in terms of $\mathrm{O}(d,d)$ indices $A, B, C, \ldots$, the global symmetry group is actually broken to $\mathrm{O}(n,n)$. We will not explicitly show how this splitting takes place, and refer to [7] for more details. In this work, for the sake of simplicity, we will restrict to the case $\xi_A = 0$, which should be viewed as a constraint for λ. Also we will restrict to $O(n,n)$ global symmetry groups, without additional vector fields.

There are two possible known ways to restrict the fields and gauge parameters in DFT, such that the action is gauge invariant and the gauge algebra closes. On the one hand, the so-called weak and strong constraints can be imposed

$$\partial_M\partial^M A = 0\ , \quad \partial_M A\,\partial^M B = 0\ , \tag{2.6}$$

where A and B generically denote products of (derivatives of) fields and gauge parameters. When this is the case, one can argue [12] that there is always a frame in which the fields do not depend on the dual coordinates. On the other hand, in the SS compactification scenario, it is enough to impose the weak and strong constraints only on the external space (*i.e.*, on hatted quantities)

$$\partial_M\partial^M \widehat{A} = 0\ , \quad \partial_M \widehat{A}\,\partial^M \widehat{B} = 0\ , \tag{2.7}$$

and impose QC for the gaugings

$$f_{E[AB}f^E{}_{C]D} = 0\ . \tag{2.8}$$

This second option is more natural for our purposes, since these constraints exactly coincide with those of half-maximal gauged supergravities[1] (which are undoubled theories in the external space, and contain gaugings satisfying the QC).

Notice that if a given U produces a solution to the QC, any T-dual U will also. Therefore, it is natural to define the notion of *twist orbits* as the sets of twist matrices connected through T-duality transformations. If a representative of a twist orbit generates a representative of an orbit of gaugings, one can claim that the twist orbit will generate the entire orbit of gaugings. Also, notice that if a twist matrix satisfies the weak and strong constraints, any representative of its orbit will, so one can define the notions of undoubled and truly doubled twist orbits.

[1] We are working under the assumption that the structure constants not only specify the gauging, but all couplings of the theory. Reproducing the correct structure constants therefore implies reproducing the full theory correctly, as has been proven in $D = 4$ and $D = 10$ [7, 8, 15].

Non-geometry VS weak and strong constraint violation

Any half-maximal supergravity can be uplifted to the maximal theory whenever the following constraint holds[2]

$$f_{ABC}\, f^{ABC} \,=\, 0\,. \tag{2.9}$$

This constraint plays the role of an orthogonality condition between geometric and non-geometric fluxes. Interestingly, the constraint (2.9) evaluated in terms of the twist matrix U and λ can be rewritten as follows (by taking relations (2.4) and (2.5) into account)

$$f_{ABC}\, f^{ABC} \,=\, -3\,\partial_D U^A{}_P\, \partial^D \big(U^{-1}\big)^P{}_A - 24\,\partial_D\lambda\,\partial^D\lambda \,+\, 24\,\partial_D\partial^D\lambda\,. \tag{2.10}$$

The RHS of this equation is zero whenever the background defined by U and λ satisfies the weak and strong constraints. This immediately implies that any background satisfying weak and strong constraints defines a gauging which is upliftable to the maximal theory. Conversely, if an orbit of gaugings in half-maximal supergravity does not satisfy the extra constraint (2.9), the RHS of this equation must be non-vanishing, and then the strong and weak constraint must be relaxed. In conclusion, the orbits of half-maximal supergravity that do not obey the QC of the maximal theory require truly doubled twist orbits, and are therefore genuinely non-geometric. This point provides a concrete criterion to label these orbits as non-geometric. Also, notice that these orbits will never be captured by non-geometric flux configurations obtained by T-dualizing a geometric background[3].

For the sake of clarity, let us briefly review the definitions that we use. A twist orbit is non-geometric if it doesn't satisfy the weak/strong constraint, and geometric if it does. Therefore, the notion of geometry that we consider is local, and we will not worry about global issues (given that the twist matrix is taken to be an element of the global symmetry group, the transition functions between coordinate patches are automatically elements of $O(n,n)$). On the other hand an orbit of gaugings is geometric if it contains a representative that can be obtained from 10 dimensional supergravity (or equivalently from a geometric twist orbit), and it is non-geometric it does not satisfy the constraints of maximal supergravity.

We have now described all the necessary ingredients to formally relate dimensional reductions of DFT and the orbits of half-maximal gauged supergravities. In particular, in what follows we will:

1. Provide a classification of all the orbits of gaugings in maximal and half-maximal supergravities in $D \geq 7$.
2. Explore mechanisms to generate orbits of gaugings from twists, satisfying
   - $U(\mathbb{Y}) \in \mathrm{O}(n,n)$
   - Constant f_{ABC}
   - $f_{E[AB}f^E{}_{C]D} = 0$
3. Show that in the half-maximal theories all the orbits of gaugings can be obtained from twist orbits in DFT.
4. Show that in the half-maximal theories the orbits that satisfy the QC of maximal supergravity admit a representative with a higher-dimensional supergravity origin. For these we provide concrete realisations in terms of unboubled backgrounds in DFT. Instead, the orbits that fail to satisfy (2.9) require, as we argued, truly doubled twist orbits for which we also provide concrete examples.

[2] $D = 4$ half-maximal supergravity is slightly different because its global symmetry group features an extra SL(2) factor; for full details, see [18, 19].

[3] However, we would like to stress that, in general, it is not true that an orbit satisfying the QC constraints of maximal supergravity (2.9) is necessarily generated by an undoubled twist orbit. An example can be found at the end of Sect. 4.

5. Show that there is a degeneracy in the space of twist orbits giving rise to the same orbit of gaugings. Interestingly, in some cases a given orbit can be obtained either from undoubled or truly doubled twist orbits.

In the next sections we will classify all the orbits in (half-)maximal $D \geq 7$ supergravities, and provide the half-maximal ones with concrete uplifts to DFT, explicitly proving the above points.

2.2 Parametrisations of the duality twists

Here we would like to introduce some notation that will turn out to be useful in the uplift of orbits to DFT. We start by noting the double internal coordinates as $\mathbb{Y}^A = (\tilde{y}_a, y^a)$ with $a = 1, ..., n$. As we saw, the SS compactification of DFT is defined by the twists $U(\mathbb{Y})$ and $\lambda(\mathbb{Y})$. The duality twist $U(\mathbb{Y})$ is not generic, but forced to be an element of O(n, n), so we should provide suitable parameterisations. One option is the *light-cone* parameterisation, where the metric of the (internal) global symmetry group is taken to be of the form (2.2)

$$\eta_{AB} = \begin{pmatrix} 0 & \mathbb{1}_n \\ \mathbb{1}_n & 0 \end{pmatrix} . \tag{2.11}$$

The most general form of the twist matrix is then given by

$$U(\mathbb{Y}) = \begin{pmatrix} e & 0 \\ 0 & e^{-T} \end{pmatrix} \begin{pmatrix} \mathbb{1}_n & 0 \\ -B & \mathbb{1}_n \end{pmatrix} \begin{pmatrix} \mathbb{1}_n & \beta \\ 0 & \mathbb{1}_n \end{pmatrix} , \tag{2.12}$$

with $e \in$ GL(n) and B and β are generic $n \times n$ antisymmetric matrices. When $\beta = 0$, $e = e(y^a)$ and $B = B(y^a)$, the matrix e can be interpreted as a n-dimensional internal vielbein and B as a background 2-form for the n-dimensional internal Kalb-Ramond field b. Whenever the background is of this form, we will refer to it as geometric (notice that this still does not determine completely the background, which receives deformations from scalar fluctuations). In this case the gaugings take the simple form

$$\begin{aligned} f_{abc} &= 3(e^{-1})^{\alpha}{}_{[a}(e^{-1})^{\beta}{}_{b}(e^{-1})^{\gamma}{}_{c]}\partial_{[\alpha}B_{\beta\gamma]} , \\ f^{a}{}_{bc} &= 2(e^{-1})^{\beta}{}_{[b}(e^{-1})^{\gamma}{}_{c]}\partial_{\beta}e^{a}{}_{\gamma} , \\ f^{ab}{}_{c} &= f^{abc} = 0 . \end{aligned} \tag{2.13}$$

If we also turn on a $\beta(y^a)$, the relation of e, B and β with the internal g and b is less trivial, and typically the background will be globally well defined up to O(n, n) transformations mixing the metric and the two-form (this is typically called a T-fold). In this case, we refer to the background as locally geometric but globally non-geometric, and this situation formally allows for non-vanishing $f^{ab}{}_{c}$ and f^{abc}. Finally, if the twist matrix is a function of $\tilde{y}_a$, we refer to the background as locally non-geometric. Notice however, that if it satisfies the weak and strong constraints, one would always be able to rotate it to a frame in which it is locally geometric, and would therefore belong to an undoubled orbit.

Alternatively, one could also define the *cartesian* parametrisation of the twist matrix, by taking the metric of the (internal) global symmetry group to be of the form

$$\eta_{AB} = \begin{pmatrix} \mathbb{1}_n & 0 \\ 0 & -\mathbb{1}_n \end{pmatrix} . \tag{2.14}$$

This formulation is related to the light-cone parametrisation through a SO$(2n)$ transformation, that must also rotate the coordinates. In this case the relation between the components of the twist matrix and the internal g and b is non-trivial. We will consider the O(n, n) twist matrix to contain a smaller O$(n-1, n-1)$

matrix in the directions $(y^2, ..., y^n, \tilde{y}_2, ..., \tilde{y}_n)$ fibred over the flat directions $(y^1, \tilde{y}_1)$. We have seen that this typically leads to constant gaugings.

Of course these are not the most general parameterisations and ansatz, but they will serve our purposes of uplifting all the orbits of half-maximal supergravity to DFT. Interesting works on how to generate gaugings from twists are [20].

3 U-duality orbits of maximal supergravities

Following the previous discussion of DFT and its relevance for generating duality orbits, we turn to the actual classification of these. In particular, we start with orbits under U-duality of gaugings of maximal supergravity. Moreover, we will demonstrate that all such orbits do have a higher-dimensional supergravity origin.

Starting with the highest dimension for maximal supergravity, $D = 11$, no known deformation is possible here. Moreover, in $D = 10$ maximal supergravities, the only possible deformation occurs in what is known as massive IIA supergravity[4] [25]. It consists of a Stückelberg-like way of giving a mass to the 2-form B_2. Therefore, such a deformation cannot be interpreted as a gauging. The string theory origin of this so-called Romans' mass parameter is nowadays well understood as arising from D8-branes [26]. Furthermore, its DFT uplift has been constructed in [27]. Naturally, the structure of possible orbits becomes richer when going to lower dimensions. In what follows we will perform the explicit classification in dimensions nine and eight.

3.1 Orbits and origin of the $D = 9$ maximal case

Maximal $D = 9$ gauged supergravity

The maximal (ungauged) supergravity in $D = 9$ [28] can be obtained by reducing either massless type IIA or type IIB supergravity in ten dimensions on a circle. The global symmetry group of this theory is

$$G_0 = \mathbb{R}^+ \times \mathrm{SL}(2) \, .$$

Note that G_0 is the global symmetry of the action and hence it is realised off-shell, whereas the on-shell symmetry has an extra $\mathbb{R}^+$ with respect to which the Lagrangian has a non-trivial scaling weight. This is normally referred to as the *trombone symmetry*. As a consequence, the on-shell symmetry contains three independent rescalings [22, 29], which we summarise in Table 2. The full field content consists of the following objects which arrange themselves into irrep's of $\mathbb{R}^+ \times \mathrm{SL}(2)$:

$$\text{9D}: \qquad \underbrace{e_\mu{}^a \, , \, A_\mu \, , \, A_\mu{}^i \, , \, B_{\mu\nu}{}^i \, , \, C_{\mu\nu\rho} \, , \, \varphi \, , \, \tau = \chi + i\, e^{-\phi}}_{\text{bosonic dof's}} \; ; \; \underbrace{\psi_\mu \, , \, \lambda \, , \, \tilde{\lambda}}_{\text{fermionic dof's}} \, , \tag{3.1}$$

where $\mu, \nu, \cdots$ denote nine-dimensional curved spacetime, $a, b, \cdots$ nine-dimensional flat spacetime and $i, j, \cdots$ fundamental SL(2) indices respectively.

The general deformations of this theory have been studied in detail in [30], where both embedding tensor deformations and gaugings of the trombone symmetry have been considered. For the present scope we shall restrict ourselves to the first ones. The latter ones would correspond to the additional mass parameters m_{IIB} and $(m_{11}, m_{\mathrm{IIA}})$ in refs [22, 30], which give rise to theories without an action principle.

The vectors of the theory $\{A_\mu \, , \, A_\mu{}^i\}$ transform in the $V' = \mathbf{1}_{(+4)} \oplus \mathbf{2}_{(-3)}$ of $\mathbb{R}^+ \times \mathrm{SL}(2)$, where the $\mathbb{R}^+$ scaling weights are included as well[5]. The resulting embedding tensor deformations live in the

[4] Throughout this paper we will not consider the trombone gaugings giving rise to theories without an action principle, as discussed in *e.g.* [21–24].

[5] The $\mathbb{R}^+$ factor in the global symmetry is precisely the combination $\left(\frac{4}{3}\,\alpha - \frac{3}{2}\,\delta\right)$ of the different rescalings introduced in [22].

Table 2 The scaling weights of the nine-dimensional fields. As already anticipated, only three rescalings are independent since they are subject to the following constraint: $8\alpha - 48\beta - 18\gamma - 9\delta = 0$. As the scaling weight of the Lagrangian $\mathcal{L}$ shows, β and γ belong to the off-shell symmetries, whereas α and δ can be combined into a trombone symmetry and an off-shell symmetry.

ID	$e_\mu{}^a$	A_μ	$A_\mu{}^1$	$A_\mu{}^2$	$B_{\mu\nu}{}^1$	$B_{\mu\nu}{}^2$	$C_{\mu\nu\rho}$	e^φ	χ	e^ϕ	ψ_μ	$\lambda\,,\,\tilde{\lambda}$	$\mathcal{L}$
α	$\frac{9}{7}$	3	0	0	3	3	3	$\frac{6}{\sqrt{7}}$	0	0	$\frac{9}{14}$	$-\frac{9}{14}$	9
β	0	$\frac{1}{2}$	$-\frac{3}{4}$	0	$-\frac{1}{4}$	$\frac{1}{2}$	$-\frac{1}{4}$	$\frac{\sqrt{7}}{4}$	$-\frac{3}{4}$	$\frac{3}{4}$	0	0	0
γ	0	0	1	-1	1	-1	0	0	2	-2	0	0	0
δ	$\frac{8}{7}$	0	2	2	2	2	4	$-\frac{4}{\sqrt{7}}$	0	0	$\frac{4}{7}$	$-\frac{4}{7}$	8

following tensor product

$$\mathfrak{g}_0 \otimes V = \mathbf{1}_{(-4)} \oplus 2 \cdot \mathbf{2}_{(+3)} \oplus \mathbf{3}_{(-4)} \oplus \mathbf{4}_{(+3)}\ . \tag{3.2}$$

The Linear Constraint (LC) projects out the $\mathbf{4}_{(+3)}$, the $\mathbf{1}_{(-4)}$ and one copy of the $\mathbf{2}_{(+3)}$ since they would give rise to inconsistent deformations. As a consequence, the consistent gaugings are parameterised by embedding tensor components in the $\mathbf{2}_{(+3)} \oplus \mathbf{3}_{(-4)}$. We will denote these allowed deformations by θ^i and $\kappa^{(ij)}$.

The closure of the gauge algebra and the antisymmetry of the brackets impose the following Quadratic Constraints (QC)

$$\epsilon_{ij}\,\theta^i\,\kappa^{jk} = 0\,, \qquad \mathbf{2}_{(-1)} \tag{3.3}$$

$$\theta^{(i}\,\kappa^{jk)} = 0\,. \qquad \mathbf{4}_{(-1)} \tag{3.4}$$

The $\mathbb{R}^+ \times \mathrm{SL}(2)$ orbits of solutions to the QC

The QC (3.3) and (3.4) turns out to be very simple to solve; after finding all the solutions, we studied the duality orbits, *i.e.* classes of those solutions which are connected via a duality transformation. The resulting orbits of consistent gaugings in this case are presented in Table 3.

Table 3 All the U-duality orbits of consistent gaugings in maximal supergravity in $D = 9$. For each of them, the simplest representative is given. The subscripts β and γ refer to the rescalings summarised in Table 2.

ID	θ^i	κ^{ij}	gauging
1	$(0,0)$	$\mathrm{diag}(1,1)$	SO(2)
2		$\mathrm{diag}(1,-1)$	$\mathrm{SO}(1,1)$
3		$\mathrm{diag}(1,0)$	$\mathbb{R}^+_\gamma$
4	$(1,0)$	$\mathrm{diag}(0,0)$	$\mathbb{R}^+_\beta$

Higher-dimensional geometric origin

The four different orbits of maximal $D = 9$ theory have the following higher-dimensional origin in terms of geometric compactifications [31]:

- **Orbits 1 – 3:** These come from reductions of type IIB supergravity on a circle with an SL(2) twist.
- **Orbit 4:** This can be obtained from a reduction of type IIA supergravity on a circle with the inclusion of an $\mathbb{R}^+_\beta$ twist.

3.2 Orbits and origin of the $D = 8$ maximal case

Maximal $D = 8$ gauged supergravity

The maximal (ungauged) supergravity in $D = 8$ [32] can be obtained by reducing eleven-dimensional supergravity on a T^3. The global symmetry group of this theory is

$$G_0 \;=\; \mathrm{SL}(2)\;\times\;\mathrm{SL}(3)\;.$$

The full field content consists of the following objects which arrange themselves into irrep's of SL(2) $\times$ SL(3):

$$\text{8D}: \qquad \underbrace{{e_\mu}^a\,,\,{A_\mu}^{\alpha m}\,,\,B_{\mu\nu m}\,,\,C_{\mu\nu\rho}\,,\,{L_m}^I\,,\,\phi\,,\,\chi}_{\text{bosonic dof's}}\;;\;\underbrace{\psi_\mu\,,\,\chi_I}_{\text{fermionic dof's}}\;, \tag{3.5}$$

where $\mu,\nu,\cdots$ denote eight-dimensional curved spacetime, $a,b,\cdots$ eight-dimensional flat spacetime, $m,n,\cdots$ fundamental SL(3), $I,J,\cdots$ fundamental SO(3) and $\alpha,\beta,\cdots$ fundamental SL(2) indices respectively. The six vector fields ${A_\mu}^{\alpha m}$ in (3.5) transform in the $V' = \left(\mathbf{2},\mathbf{3}'\right)$. There are eleven group generators, which can be expressed in the adjoint representation $\mathfrak{g}_0$.

The embedding tensor Θ then lives in the representation $\mathfrak{g}_0 \otimes V$, which can be decomposed into irreducible representations as

$$\mathfrak{g}_0 \otimes V = 2\,\cdot(\mathbf{2},\mathbf{3}) \oplus \left(\mathbf{2},\mathbf{6}'\right) \oplus (\mathbf{2},\mathbf{15}) \oplus (\mathbf{4},\mathbf{3})\;. \tag{3.6}$$

The LC restricts the embedding tensor to the $(\mathbf{2},\mathbf{3}) \oplus \left(\mathbf{2},\mathbf{6}'\right)$ [33]. It is worth noticing that there are two copies of the $(\mathbf{2},\mathbf{3})$ irrep in the above composition; the LC imposes a relation between them [34]. This shows that, for consistency, gauging some SL(2) generators implies the necessity of gauging some SL(3) generators as well. Let us denote the allowed embedding tensor irrep's by $\xi_{\alpha m}$ and ${f_\alpha}^{(mn)}$ respectively.

The quadratic constraints (QC) then read [35, 36]

$$\epsilon^{\alpha\beta}\,\xi_{\alpha p}\xi_{\beta q} = 0\,, \qquad \left(\mathbf{1},\mathbf{3}'\right) \tag{3.7}$$

$$f_{(\alpha}{}^{np}\xi_{\beta)p} = 0\,, \qquad \left(\mathbf{3},\mathbf{3}'\right) \tag{3.8}$$

$$\epsilon^{\alpha\beta}\left(\epsilon_{mqr}{f_\alpha}^{qn}{f_\beta}^{rp} + {f_\alpha}^{np}\xi_{\beta m}\right) = 0\,. \qquad \left(\mathbf{1},\mathbf{3}'\right)\oplus(\mathbf{1},\mathbf{15}) \tag{3.9}$$

Any solution to the QC (3.7), (3.8) and (3.9) specifies a consistent gauging of a subgroup of SL(2) $\times$ SL(3) where the corresponding generators are given by

$$(X_{\alpha m})_\beta{}^\gamma = \delta^\gamma_\alpha\,\xi_{\beta m} \,-\, \frac{1}{2}\,\delta^\gamma_\beta\,\xi_{\alpha m}\;, \tag{3.10}$$

$$(X_{\alpha m})_n{}^p = \epsilon_{mnq}\,{f_\alpha}^{qp} \,-\, \frac{3}{4}\left(\delta^p_m\,\xi_{\alpha n} \,-\, \frac{1}{3}\,\delta^p_n\,\xi_{\alpha m}\right)\;. \tag{3.11}$$

The SL(2) $\times$ SL(3) orbits of solutions to the QC

We exploited an algebraic geometry tool called the Gianni-Trager-Zacharias (GTZ) algorithm [37]. This algorithm has been computationally implemented by the SINGULAR project [38] and it consists in the primary decomposition of ideals of polynomials. After finding all the solutions to the QC by means of the algorithm mentioned above, one has to group together all the solutions which are connected through a duality transformation, thus obtaining a classification of such solutions in terms of duality orbits. The resulting orbits of consistent gaugings[6] in this case are presented in Table 4.

Table 4 All the U-duality orbits of consistent gaugings in maximal supergravity in $D = 8$. For each of them, the simplest representative is given. We denote by $\mathrm{Solv}_2 \subset \mathrm{SL}(2)$ and $\mathrm{Solv}_3 \subset \mathrm{SL}(3)$ a solvable algebra of dimension 2 and 3 respectively. To be more precise, Solv_2 identifies the Borel subgroup of SL(2) consisting of 2×2 upper-triangular matrices. Solv_3, instead, is a Bianchi type V algebra.

ID	${f_+}^{mn}$	${f_-}^{mn}$	ξ_{+m}	ξ_{-m}	gauging
1	$\mathrm{diag}(1,1,1)$	$\mathrm{diag}(0,0,0)$	$(0,0,0)$	$(0,0,0)$	SO(3)
2	$\mathrm{diag}(1,1,-1)$				$\mathrm{SO}(2,1)$
3	$\mathrm{diag}(1,1,0)$				ISO(2)
4	$\mathrm{diag}(1,-1,0)$				$\mathrm{ISO}(1,1)$
5	$\mathrm{diag}(1,0,0)$				$\mathrm{CSO}(1,0,2)$
6	$\mathrm{diag}(0,0,0)$	$\mathrm{diag}(0,0,0)$	$(1,0,0)$	$(0,0,0)$	$\mathrm{Solv}_2 \times \mathrm{Solv}_3$
7	$\mathrm{diag}(1,1,0)$	$\mathrm{diag}(0,0,0)$	$(0,0,1)$	$(0,0,0)$	$\mathrm{Solv}_2 \times \mathrm{Solv}_3$
8	$\mathrm{diag}(1,-1,0)$				
9	$\mathrm{diag}(1,0,0)$				
10	$\mathrm{diag}(1,-1,0)$	$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\frac{2}{9}(0,0,1)$	$(0,0,0)$	$\mathrm{Solv}_2 \times \mathrm{SO}(2) \ltimes \mathrm{Nil}_3(2)$

Higher-dimensional geometric origin

- **Orbits 1 – 5:** These stem from reductions of eleven-dimensional supergravity on a three-dimensional group manifold of type A in the Bianchi classification [39]. The special case in orbit 1 corresponds to a reduction over an SO(3) group manifold and it was already studied in [32].
- **Orbit 6:** This can be obtained from a reduction of maximal nine-dimensional supergravity on a circle with the inclusion of an $\mathbb{R}^+$ twist inside the global symmetry group.
- **Orbits 7 – 9:** These can come from the same reduction from $D = 9$ but upon inclusion of a more general $\mathbb{R}^+ \times \mathrm{SL}(2)$ twist.
- **Orbit 10:** This orbit seems at first sight more complicated to be obtained from a dimensional reduction owing to its non-trivial SL(2) angles. Nevertheless, it turns out that one can land on this orbit by compactifying type IIB supergravity on a circle with an SL(2) twist and then further reducing on another circle with $\mathbb{R}^+ \times \mathrm{SL}(2)$ twist given by the residual little group leaving invariant the intermediate nine-dimensional deformation.

Remarks on the $D = 7$ maximal case

The general deformations of the maximal theory in $D = 7$ are constructed and presented in full detail in [40]. For the present aim we only summarise here a few relevant facts.

[6] Recently, also the possible vacua of the different theories have been analysed [36]. It was found that only **orbit 3** has maximally symmetric vacua.

The global symmetry group of the theory is SL(5). The vector fields ${A_\mu}^{MN} = {A_\mu}^{[MN]}$ transform in the $\mathbf{10}'$ of SL(5), where we denote by M a fundamental SL(5) index. The embedding tensor Θ takes values in the following irreducible components

$$\mathbf{10} \otimes \mathbf{24} \,=\, \mathbf{10} \oplus \mathbf{15} \oplus \mathbf{40}' \oplus \mathbf{175}\,. \tag{3.12}$$

The LC restricts the embedding tensor to the $\mathbf{15} \oplus \mathbf{40}'$, which can be parameterised by the following objects

$$Y_{(MN)}\,, \qquad \text{and} \qquad Z^{[MN],P} \quad \text{with} \quad Z^{[MN,P]} = 0\,. \tag{3.13}$$

The generators of the gauge algebra can be written as follows

$$(X_{MN})_P{}^Q \,=\, \delta^Q_{[M}\, Y_{N]P} \,-\, 2\,\epsilon_{MNPRS}\, Z^{RS,Q}\,, \tag{3.14}$$

or, identically, if one wants to express them in the $\mathbf{10}$,

$$(X_{MN})_{PQ}{}^{RS} \,=\, 2\,(X_{MN})_{[P}{}^{[R}\, \delta^{S]}_{Q]}\,. \tag{3.15}$$

The closure of the gauge algebra and the antisymmetry of the brackets imply the following QC

$$Y_{MQ}\, Z^{QN,P} \,+\, 2\,\epsilon_{MRSTU}\, Z^{RS,N}\, Z^{TU,P} \,=\, 0\,, \tag{3.16}$$

which have different irreducible pieces in the $\mathbf{5}' \oplus \mathbf{45}' \oplus \mathbf{70}'$. Unfortunately, in this case, both the embedding tensor deformations and the quadratic constraints reach a level of complexity that makes an exhaustive and general analysis difficult. Such analysis lies beyond the scope of our work.

4 T-duality orbits of half-maximal supergravities

After the previous section on maximal supergravities, we turn our attention to theories with half-maximal supersymmetry. In particular, in this section we will classify the orbits under T-duality of all gaugings of half-maximal supergravity. We will only consider the theories with duality groups $\mathbb{R}^+ \times \mathrm{SO}(d,d)$ in $D = 10 - d$, which places a restriction on the number of vector multiplets. For these theories we will classify all duality orbits, and find a number of non-geometric orbits. Furthermore, we demonstrate that double field theory does yield a higher-dimensional origin for all of them.

Starting from $D = 10$ half-maximal supergravity without vector multiplets, it can be seen that there is no freedom to deform this theory, rendering this case trivial. In $D = 9$, instead, we have the possibility of performing an Abelian gauging inside $\mathbb{R}^+ \times \mathrm{SO}(1,1)$, which will depend on one deformation parameter. However, this is precisely the parameter that one expects to generate by means of a twisted reduction from $D = 10$. This immediately tells us that non-geometric fluxes do not yet appear in this theory. In order to find the first non-trivial case, we will have to consider the $D = 8$ case.

4.1 Orbits and origin of the $D = 8$ half-maximal case

Half-maximal $D = 8$ gauged supergravity

Half-maximal supergravity in $D = 8$ is related to the maximal theory analysed in the previous section by means of a $\mathbb{Z}_2$ truncation. The action of such a $\mathbb{Z}_2$ breaks $\mathrm{SL}(2) \times \mathrm{SL}(3)$ into $\mathbb{R}^+ \times \mathrm{SL}(2) \times \mathrm{SL}(2)$, where $\mathrm{SL}(2) \times \mathrm{SL}(2) = \mathrm{O}(2,2)$ can be interpreted as the T-duality group in $D = 8$ as shown in Table 1. The embedding of $\mathbb{R}^+ \times \mathrm{SL}(2)$ inside SL(3) is unique and it determines the following branching of the fundamental representation

$$\begin{aligned} \mathbf{3} \;&\longrightarrow\; \mathbf{1}_{(+2)} \,\oplus\, \mathbf{2}_{(-1)}\,, \\ m \;&\longrightarrow\; (\bullet\,,\, i)\,, \end{aligned}$$

where the $\mathbb{R}^+$ direction labeled by $\bullet$ is parity even, whereas i is parity odd, such as the other SL(2) index α. In the following we will omit all the $\mathbb{R}^+$ weights since they do not play any role in the truncation.

The embedding tensor of the maximal theory splits in the following way

$$
\begin{aligned}
(\mathbf{2},\mathbf{3}) &\longrightarrow \cancel{(\mathbf{2},\mathbf{1})} \oplus (\mathbf{2},\mathbf{2}) \,, \\
(\mathbf{2},\mathbf{6}') &\longrightarrow \cancel{(\mathbf{2},\mathbf{1})} \oplus (\mathbf{2},\mathbf{2}) \oplus \cancel{(\mathbf{2},\mathbf{3})} \,,
\end{aligned}
$$

where all the crossed irrep's are projected out because of $\mathbb{Z}_2$ parity. This implies that the consistent embedding tensor deformations of the half-maximal theory can be described by two objects which are doublets with respect to both SL(2)'s. Let us denote them by $a_{\alpha i}$ and $b_{\alpha i}$. This statement is in perfect agreement with the Kac-Moody analysis performed in [41]. The explicit way of embedding $a_{\alpha i}$ and $b_{\alpha i}$ inside $\xi_{\alpha m}$ and $f_\alpha{}^{mn}$ is given by

$$
f_\alpha{}^{i\bullet} = f_\alpha{}^{\bullet i} = \epsilon^{ij}\, a_{\alpha j} \,, \tag{4.1}
$$

$$
\xi_{\alpha i} = 4\, b_{\alpha i} \,. \tag{4.2}
$$

The QC given in (3.7), (3.8) and (3.9) are decomposed according to the following branching

$$
\begin{aligned}
(\mathbf{1},\mathbf{3}') &\longrightarrow (\mathbf{1},\mathbf{1}) \oplus \cancel{(\mathbf{1},\mathbf{2})} \,, \\
(\mathbf{3},\mathbf{3}') &\longrightarrow (\mathbf{3},\mathbf{1}) \oplus \cancel{(\mathbf{3},\mathbf{2})} \,, \\
(\mathbf{1},\mathbf{15}) &\longrightarrow (\mathbf{1},\mathbf{1}) \oplus \cancel{2\cdot(\mathbf{1},\mathbf{2})} \oplus 2\cdot(\mathbf{1},\mathbf{3}) \oplus \cancel{(\mathbf{1},\mathbf{4})} \,.
\end{aligned}
$$

As a consequence, one expects the set of $\mathbb{Z}_2$ even QC to consist of 3 singlets, a $(\mathbf{3},\mathbf{1})$ and 2 copies of the $(\mathbf{1},\mathbf{3})$. By plugging (4.1) and (4.2) into (3.7), (3.8) and (3.9), one finds

$$
\epsilon^{\alpha\beta}\,\epsilon^{ij}\, b_{\alpha i}\, b_{\beta j} = 0 \,, \qquad (\mathbf{1},\mathbf{1}) \tag{4.3}
$$

$$
\epsilon^{\alpha\beta}\,\epsilon^{ij}\, a_{\alpha i}\, b_{\beta j} = 0 \,, \qquad (\mathbf{1},\mathbf{1}) \tag{4.4}
$$

$$
\epsilon^{\alpha\beta}\,\epsilon^{ij}\, a_{\alpha i}\, a_{\beta j} = 0 \,, \qquad (\mathbf{1},\mathbf{1}) \tag{4.5}
$$

$$
\epsilon^{ij}\, a_{(\alpha i}\, b_{\beta) j} = 0 \,, \qquad (\mathbf{3},\mathbf{1}) \tag{4.6}
$$

$$
\epsilon^{\alpha\beta}\, a_{\alpha (i}\, b_{\beta j)} = 0 \,. \qquad (\mathbf{1},\mathbf{3}) \tag{4.7}
$$

With respect to what we expected from group theory, we seem to be finding a $(\mathbf{1},\mathbf{3})$ less amongst the even QC. This could be due to the fact that $\mathbb{Z}_2$ even QC can be sourced by quadratic expressions in the odd embedding tensor components that we truncated away. After the procedure of turning off all of them, the two $(\mathbf{1},\mathbf{3})$'s probably collapse to the same constraint or one of them vanishes directly.

The above set of QC characterises the consistent gaugings of the half-maximal theory which are liftable to the maximal theory, and hence they are more restrictive than the pure consistency requirements of the half-maximal theory. In order to single out only these we need to write down the expression of the gauge generators and impose the closure of the algebra. The gauge generators in the $(\mathbf{2},\mathbf{2})$ read

$$
(X_{\alpha i})_{\beta j}{}^{\gamma k} = \frac{1}{2}\,\delta^\gamma_\beta\,\epsilon_{ij}\,\epsilon^{kl}\,a_{\alpha l} + \delta^\gamma_\alpha\,\delta^k_j\,b_{\beta i} - \frac{3}{2}\,\delta^\gamma_\beta\,\delta^k_i\,b_{\alpha j} + \frac{1}{2}\,\delta^\gamma_\beta\,\delta^k_j\,b_{\alpha i} + \epsilon_{\alpha\beta}\,\epsilon^{\gamma\delta}\,\delta^k_j\,b_{\delta i} \,. \tag{4.8}
$$

The closure of the algebra generated by (4.8) implies the following QC

$$
\epsilon^{\alpha\beta}\,\epsilon^{ij}\,\left(a_{\alpha i}\,a_{\beta j} - b_{\alpha i}\,b_{\beta j}\right) = 0 \,, \qquad (\mathbf{1},\mathbf{1}) \tag{4.9}
$$

$$
\epsilon^{\alpha\beta}\,\epsilon^{ij}\,\left(a_{\alpha i}\,b_{\beta j} + b_{\alpha i}\,b_{\beta j}\right) = 0 \,, \qquad (\mathbf{1},\mathbf{1}) \tag{4.10}
$$

$$
\epsilon^{ij}\, a_{(\alpha i}\, b_{\beta) j} = 0 \,, \qquad (\mathbf{3},\mathbf{1}) \tag{4.11}
$$

$$
\epsilon^{\alpha\beta}\, a_{\alpha (i}\, b_{\beta j)} = 0 \,. \qquad (\mathbf{1},\mathbf{3}) \tag{4.12}
$$

To facilitate the mapping of gaugings $a_{\alpha i}$ and $b_{\alpha i}$ with the more familiar f_{ABC} and ξ_A in the DFT language, we have written a special section in the Appendix B. The mapping is explicitly given in (B.6).

The O$(2,2)$ orbits of solutions to the QC

After solving the QC given in (4.9), (4.10), (4.11) and (4.12) again with the aid of SINGULAR, we find a 1-parameter family of T-duality orbits plus two discrete ones. The results are all collected in Table 5.

Table 5 All the T-duality orbits of consistent gaugings in half-maximal supergravity in $D = 8$. For each of them, the simplest representative is given. Solv$_2$ refers again to the solvable subgroup of SL(2) as already explained in the caption of Table 4.

ID	$a_{\alpha i}$	$b_{\alpha i}$	gauging
1	diag$(\cos\alpha, 0)$	diag$(\sin\alpha, 0)$	Solv$_2 \times$ SO$(1,1)$
2	diag$(1,1)$	diag$(-1,-1)$	SL$(2) \times$ SO$(1,1)$
3	diag$(1,-1)$	diag$(-1,1)$	

Higher-dimensional geometric origin

The possible higher-dimensional origin of the three different orbits is as follows:

- **Orbit 1:** This orbit can be obtained by performing a two-step reduction of type I supergravity. In the first step, by reducing a circle, we can generate an $\mathbb{R}^+ \times$ SO$(1,1)$ gauging of half-maximal $D = 9$ supergravity. Subsequently, we reduce such a theory again on a circle with the inclusion of a new twist commuting with the previous deformation. Also, these orbits include a non-trivial ξ_A gauging, so we will not address it from a DFT perspective.
- **Orbits 2 – 3:** These do not seem to have any obvious geometric higher-dimensional origin in supergravity. In fact, they do not satisfy the extra constraints (2.9), so one can only hope to reproduce them from truly doubled twist orbits in DFT.

Therefore we find that, while the half-maximal orbits in $D = 9$ all have a known geometric higher-dimensional origin, this is not the case for the latter two orbits in $D = 8$. We have finally detected the first signals of non-geometric orbits.

Higher-dimensional DFT origin

As mentioned, the **orbits 2** and **3** lack of a clear higher-dimensional origin. Here we would like to provide a particular twist matrix giving rise to these gaugings. We chose to start in the cartesian framework, and propose the following form for the SO$(2,2)$ twist matrix

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cosh(m\, y^1 + n\, \tilde{y}_1) & 0 & \sinh(m\, y^1 + n\, \tilde{y}_1) \\ 0 & 0 & 1 & 0 \\ 0 & \sinh(m\, y^1 + n\, \tilde{y}_1) & 0 & \cosh(m\, y^1 + n\, \tilde{y}_1) \end{pmatrix} . \tag{4.13}$$

This is in fact an element of SO$(1,1)$ lying in the directions $(\tilde{y}_2, y^2)$, fibred over the double torus $(\tilde{y}_1, y^1)$. Here, the coordinates are written in the cartesian formulation, so we must rotate this in order to make contact with the light-cone case.

For this twist matrix, the weak and strong constraints in the light-cone formulation read $(m+n)(m-n)=0$, while the QC are always satisfied. The gaugings are constant, and when written in terms of $a_{\alpha i}$ and $b_{\alpha i}$ we find

$$a_{\alpha i} = -b_{\alpha i} = \mathrm{diag}\left(-\frac{m+n}{2\sqrt{2}},\ \frac{m-n}{2\sqrt{2}}\right)\,, \tag{4.14}$$

so **orbit 2** is obtained by choosing $m=0$, $n=-2\sqrt{2}$, and **orbit 3** by choosing $m=-2\sqrt{2}$, $n=0$. Notice that in both cases the twist orbit is truly doubled, so we find the first example of an orbit of gaugings without a clear supergravity origin, that finds an uplift to DFT in a truly doubled background.

4.2 Orbits and origin of the $D=7$ half-maximal case

Half-maximal $D=7$ gauged supergravity

A subset of half-maximal gauged supergravities is obtained from the maximal theory introduced in Sect. 3.2 by means of a $\mathbb{Z}_2$ truncation. Thus, we will in this section perform this truncation and carry out the orbit analysis in the half-maximal theory. As we already argued before, this case is not only simpler, but also much more insightful from the point of view of understanding T-duality in gauged supergravities and its relation to DFT.

The action of our $\mathbb{Z}_2$ breaks[7] SL(5) into $\mathbb{R}^+\times$ SL(4). Its embedding inside SL(5) is unique and it is such that the fundamental representation splits as follows

$$\mathbf{5}\ \longrightarrow\ \mathbf{1}_{(+4)}\ \oplus\ \mathbf{4}_{(-1)}\,. \tag{4.15}$$

After introducing the following notation for the indices in the $\mathbb{R}^+$ and in the SL(4) directions

$$M\ \longrightarrow\ (\diamond\,,\,m)\,, \tag{4.16}$$

we assign an even parity to the $\diamond$ direction and odd parity to m directions.

The embedding tensor of the maximal theory splits according to

$$\mathbf{15}\longrightarrow \mathbf{1}\oplus \cancel{\mathbf{4}}\oplus \mathbf{10}\,, \tag{4.17}$$

$$\mathbf{40'}\longrightarrow \cancel{\mathbf{4}}\oplus \mathbf{6}\oplus \mathbf{10'}\oplus \cancel{\mathbf{20}}\,, \tag{4.18}$$

where again, as in Sect. 4.1, all the crossed irrep's are projected out because of $\mathbb{Z}_2$ parity. This implies that the embedding tensor of the half-maximal theory lives in the $\mathbf{1}\oplus\mathbf{6}\oplus\mathbf{10}\oplus\mathbf{10'}$ and hence it is described by the following objects

$$\theta\ ,\ \xi_{[mn]}\ ,\ M_{(mn)}\ ,\ \tilde{M}^{(mn)}\,. \tag{4.19}$$

[7] The $\mathbb{Z}_2$ element with respect to which we are truncating is the following USp(4) $=$ SO(5) element

$$\alpha = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}$$

projecting out half of the supercharges.

This set of deformations agrees with the decomposition $\mathrm{D}_8^{+++} \rightarrow \mathrm{A}_3 \times \mathrm{A}_6$ given in [41]. The objects in (4.19) are embedded in Y and Z in the following way

$$Y_{\diamond\diamond} = \theta\,, \tag{4.20}$$

$$Y_{mn} = \frac{1}{2}\,M_{mn}\,, \tag{4.21}$$

$$Z^{mn,\diamond} = \frac{1}{8}\,\xi^{mn}\,, \tag{4.22}$$

$$Z^{m\diamond,n} = -Z^{\diamond m,n} = \frac{1}{16}\,\tilde{M}^{mn} + \frac{1}{16}\,\xi^{mn}\,, \tag{4.23}$$

where for convenience we defined $\xi^{mn} = \frac{1}{2}\,\epsilon^{mnpq}\,\xi_{pq}$.

Now we will obtain the expression of the gauge generators of the half-maximal theory by plugging the expressions (4.20) – (4.23) into (3.14). We find

$$(X_{mn})_p{}^q = \frac{1}{2}\,\delta^q_{[m}\,M_{n]p} - \frac{1}{4}\,\epsilon_{mnpr}\,\left(\tilde{M} + \xi\right)^{rq}\,, \tag{4.24}$$

which extends the expression given in [42] by adding an antisymmetric part to $\tilde{M}$ proportional to ξ. Note that the ξ term is also the only one responsible for the trace of the gauge generators which has to be non-vanishing in order to account for $\mathbb{R}^+$ gaugings.

The presence of such a term in the expression (4.24) has another consequence: the associated structure constants that one writes by expressing the generators in the **6** $(X_{mn})_{pq}{}^{rs}$ will not be automatically antisymmetric in the exchange between mn and pq. This implies the necessity of imposing the antisymmetry by means of some extra QC[8].

The QC of the maximal theory are branched into

$$\mathbf{5'} \longrightarrow \mathbf{1} \oplus \cancel{\mathbf{4'}}\,, \tag{4.25}$$

$$\mathbf{45'} \longrightarrow \cancel{\mathbf{4}} \oplus \mathbf{6} \oplus \mathbf{15} \oplus \cancel{\mathbf{20'}}\,, \tag{4.26}$$

$$\mathbf{70'} \longrightarrow \mathbf{1} \oplus \cancel{\mathbf{4}} \oplus \cancel{\mathbf{4'}} \oplus \mathbf{10'} \oplus \mathbf{15} \oplus \cancel{\mathbf{36'}}\,. \tag{4.27}$$

By substituting the expressions (4.20) – (4.23) into the QC (3.16), one finds

$$\theta\,\xi_{mn} = 0\,, \qquad (\mathbf{6}) \tag{4.28}$$

$$\left(\tilde{M}^{mp} + \xi^{mp}\right)\,M_{pq} = 0\,, \qquad (\mathbf{1} \oplus \mathbf{15}) \tag{4.29}$$

$$M_{mp}\,\xi^{pn} - \xi_{mp}\,\left(\tilde{M}^{pn} + \xi^{pn}\right) = 0\,, \qquad (\mathbf{1} \oplus \mathbf{15}) \tag{4.30}$$

$$\theta\,\tilde{M}^{mn} = 0\,. \qquad (\mathbf{10'}) \tag{4.31}$$

Based on the Kac-Moody analysis performed in [41], the QC constraints of the half-maximal theory should only impose conditions living in the $\mathbf{1} \oplus \mathbf{6} \oplus \mathbf{15} \oplus \mathbf{15}$. The problem is then determining which constraint in the **1** is already required by the half-maximal theory and which is not.

[8] The QC which ensure the antisymmetry of the gauge brackets are given by $(X_{mn})_{pq}{}^{rs}\,X_{rs} + (mn \leftrightarrow pq) = 0$, where X is given in an arbitrary representation.

By looking more carefully at the constraints (4.28) – (4.31), we realise that the traceless part of (4.29) exactly corresponds to the Jacobi identities that one gets from the closure of the algebra spanned by the generators (4.24), whereas the full (4.30) has to be imposed to ensure antisymmetry of the gauge brackets. Since there is only one constraint in the **6**, we do not have ambiguities there[9].

We are now able to write down the set of QC of the half-maximal theory:

$$\theta\,\xi_{mn} = 0\,, \qquad (\mathbf{6}) \tag{4.32}$$

$$\left(\tilde{M}^{mp} + \xi^{mp}\right) M_{pq} - \frac{1}{4}\left(\tilde{M}^{np} M_{np}\right)\delta^m_q = 0\,, \qquad (\mathbf{15}) \tag{4.33}$$

$$M_{mp}\,\xi^{pn} + \xi_{mp}\,\tilde{M}^{pn} = 0\,, \qquad (\mathbf{15}) \tag{4.34}$$

$$\epsilon^{mnpq}\,\xi_{mn}\,\xi_{pq} = 0\,. \qquad (\mathbf{1}) \tag{4.35}$$

We are not really able to confirm whether (4.32) is part of the QC of the half-maximal theory, in the sense that there appears a top-form in the **6** from the D_8^{+++} decomposition but it could either be a tadpole or a QC. This will however not affect our further discussion, in that we only consider orbits of gaugings in which $\theta = 0$. The extra QC required in order for the gauging to admit an uplift to maximal supergravity are

$$\tilde{M}^{mn}\,M_{mn} = 0\,, \qquad (\mathbf{1}) \tag{4.36}$$

$$\theta\,\tilde{M}^{mn} = 0\,. \qquad (\mathbf{10'}) \tag{4.37}$$

The O$(3,3)$ orbits of solutions to the QC in the $\mathbf{10} \oplus \mathbf{10'}$

The aim of this section is to solve the constraints summarised in (4.32), (4.33), (4.34) and (4.35). We will start by considering the case of gaugings only involving the $\mathbf{10} \oplus \mathbf{10'}$. This restriction is motivated by flux compactification, as we will try to argue later on.

The only non-trivial QC are the following

$$\tilde{M}^{mp}\,M_{pn} - \frac{1}{4}\left(\tilde{M}^{pq}\,M_{pq}\right)\delta^m_n = 0\,, \tag{4.38}$$

which basically implies that the matrix product between M and $\tilde{M}$, which in principle lives in the $\mathbf{1} \oplus \mathbf{15}$, has to be pure trace. We made use of a GL(4) transformation in order to reduce M to pure signature; as a consequence, the QC (4.38) imply that $\tilde{M}$ is diagonal as well [43]. This results in a set of eleven 1-parameter orbits[10] of solutions to the QC which are given in Table 6.

As we will see later, some of these consistent gaugings in general include non-zero non-geometric fluxes, but at least in some of these cases one will be able to dualise the given configuration to a perfectly geometric background.

Higher-dimensional geometric origin

Ten-dimensional heterotic string theory compactified on a T^3 gives rise to a half-maximal supergravity in $D = 7$ where the SL$(4) = $ SO$(3,3)$ factor in the global symmetry of this theory can be interpreted as the

[9] We would like to stress that the parameter θ within the half-maximal theory is a consistent deformation, but it does not correspond to any gauging and hence QC involving it cannot be derived as Jacobi identities or other consistency constraints coming from the gauge algebra.

[10] We would like to point out that the extra discrete generator η of O$(3,3)$ makes sure that, given a certain gauging with M and $\tilde{M}$, it lies in the same orbit as its partner with the role of M and $-\tilde{M}$ interchanged.

Table 6 All the T-duality orbits of consistent gaugings in half-maximal supergravity in $D = 7$. Any value of α parameterises inequivalent orbits. More details about the non-semisimple gauge algebras $\mathfrak{f}_1$, $\mathfrak{f}_2$, $\mathfrak{h}_1$, $\mathfrak{h}_2$, $\mathfrak{g}_0$ and $\mathfrak{l}$ are given in Appendix A.

ID	$M_{mn}/\cos\alpha$	$\tilde{M}^{mn}/\sin\alpha$	range of α	gauging
1	$\text{diag}(1,1,1,1)$	$\text{diag}(1,1,1,1)$	$-\frac{\pi}{4} < \alpha \leq \frac{\pi}{4}$	$\begin{cases} \text{SO}(4)\,, & \alpha \neq \frac{\pi}{4}\,, \\ \text{SO}(3)\,, & \alpha = \frac{\pi}{4}\,. \end{cases}$
2	$\text{diag}(1,1,1,-1)$	$\text{diag}(1,1,1,-1)$	$-\frac{\pi}{4} < \alpha \leq \frac{\pi}{4}$	$\text{SO}(3,1)$
3	$\text{diag}(1,1,-1,-1)$	$\text{diag}(1,1,-1,-1)$	$-\frac{\pi}{4} < \alpha \leq \frac{\pi}{4}$	$\begin{cases} \text{SO(2,2)}\,, & \alpha \neq \frac{\pi}{4}\,, \\ \text{SO}(2,1)\,, & \alpha = \frac{\pi}{4}\,. \end{cases}$
4	$\text{diag}(1,1,1,0)$	$\text{diag}(0,0,0,1)$	$-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$	$\text{ISO}(3)$
5	$\text{diag}(1,1,-1,0)$	$\text{diag}(0,0,0,1)$	$-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$	$\text{ISO}(2,1)$
6	$\text{diag}(1,1,0,0)$	$\text{diag}(0,0,1,1)$	$-\frac{\pi}{4} < \alpha \leq \frac{\pi}{4}$	$\begin{cases} \text{CSO}(2,0,2)\,, & \alpha \neq \frac{\pi}{4}\,, \\ \mathfrak{f}_1 \quad (\text{Solv}_6)\,, & \alpha = \frac{\pi}{4}\,. \end{cases}$
7	$\text{diag}(1,1,0,0)$	$\text{diag}(0,0,1,-1)$	$-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$	$\begin{cases} \text{CSO}(2,0,2)\,, & \lvert\alpha\rvert < \frac{\pi}{4}\,, \\ \text{CSO}(1,1,2)\,, & \lvert\alpha\rvert > \frac{\pi}{4}\,, \\ \mathfrak{g}_0 \quad (\text{Solv}_6)\,, & \lvert\alpha\rvert = \frac{\pi}{4}\,. \end{cases}$
8	$\text{diag}(1,1,0,0)$	$\text{diag}(0,0,0,1)$	$-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$	$\mathfrak{h}_1 \quad (\text{Solv}_6)$
9	$\text{diag}(1,-1,0,0)$	$\text{diag}(0,0,1,-1)$	$-\frac{\pi}{4} < \alpha \leq \frac{\pi}{4}$	$\begin{cases} \text{CSO}(1,1,2)\,, & \alpha \neq \frac{\pi}{4}\,, \\ \mathfrak{f}_2 \quad (\text{Solv}_6)\,, & \alpha = \frac{\pi}{4}\,. \end{cases}$
10	$\text{diag}(1,-1,0,0)$	$\text{diag}(0,0,0,1)$	$-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$	$\mathfrak{h}_2 \quad (\text{Solv}_6)$
11	$\text{diag}(1,0,0,0)$	$\text{diag}(0,0,0,1)$	$-\frac{\pi}{4} < \alpha \leq \frac{\pi}{4}$	$\begin{cases} \mathfrak{l} \quad (\text{Nil}_6(3))\,, & \alpha \neq 0\,, \\ \text{CSO}(1,0,3)\,, & \alpha = 0\,. \end{cases}$

T-duality group. The set of generalised fluxes which can be turned on here is given by

$$\left\{ f_{abc},\, f_{ab}{}^{c},\, f_{a}{}^{bc},\, f^{abc} \right\} \equiv \left\{ H_{abc},\, \omega_{ab}{}^{c},\, Q_{a}{}^{bc},\, R^{abc} \right\} \,, \tag{4.39}$$

where $a, b, c = 1, 2, 3$.

These are exactly the objects that one obtains by decomposing a three-form of $\text{SO}(3,3)$ with respect to its $\text{GL}(3)$ subgroup. The number of independent components of the above fluxes (including traces of ω and Q) amounts to $1 + 9 + 9 + 1 = 20$, which is the number of independent components of a three-form of $\text{SO}(3,3)$. Nevertheless, the three-form representation is not irreducible since the Hodge duality operator in 3+3 dimensions squares to 1. This implies that one can always decompose it in a self-dual (SD) and anti-self-dual (ASD) part

$$\mathbf{10} \oplus \mathbf{10}' \quad \text{of } \text{SL}(4) \quad \longleftrightarrow \quad \mathbf{10}_{\text{SD}} \oplus \mathbf{10}_{\text{ASD}} \quad \text{of } \text{SO}(3,3)\,, \tag{4.40}$$

such that the matching between the embedding tensor deformations $(M_{mn},\, \tilde{M}^{mn})$ and the generalised fluxes given in (4.39) now perfectly works. The explicit mapping between vectors of SO$(3,3)$ expressed in light-cone coordinates and two-forms of SL(4) can be worked out by means of the SO$(3,3)$ 't Hooft symbols $(G_A)^{mn}$ (see Appendix B). This gives rise to the following dictionary between the M and $\tilde{M}$-components and the fluxes given in (4.39)

$$M \,=\, \mathrm{diag}\,\left(H_{123},\, Q_1{}^{23},\, Q_2{}^{31},\, Q_3{}^{12}\right)\,, \quad \tilde{M} \,=\, \mathrm{diag}\,\left(R^{123},\, \omega_{23}{}^{1},\, \omega_{31}{}^{2},\, \omega_{12}{}^{3}\right)\,. \tag{4.41}$$

The QC given in equations (4.32)-(4.35) enjoy a symmetry in the exchange

$$(M,\, \xi) \overset{\eta}{\leftrightarrow} (-\tilde{M},\, -\xi)\,. \tag{4.42}$$

The discrete $\mathbb{Z}_2$ transformation η corresponds to the following O$(3,3)$ element with determinant -1

$$\eta = \begin{pmatrix} 0 & \mathbb{1}_3 \\ \mathbb{1}_3 & 0 \end{pmatrix}\,, \tag{4.43}$$

which can be interpreted as a triple T-duality exchanging the three compact coordinates y^a with the corresponding winding coordinates $\tilde{y}_a$ in the language of DFT.

Now we have all the elements to analyse the higher dimensional origin of the orbits classified in Table 6.

- **Orbits 1 – 3:** These gaugings are non-geometric for every $\alpha \neq 0$; for $\alpha = 0$, they correspond to coset reductions of heterotic string theory. See *e.g.* the S^3 compactification in [44] giving rise to the SO(4) gauging. This theory was previously obtained in [45] as $\mathcal{N} = 2$ truncation of a maximal supergravity in $D = 7$.
- **Orbits 4 – 5:** For any value of α we can always dualise these representatives to the one obtained by means of a twisted T^3 reduction with H and ω fluxes.
- **Orbits 6 – 7:** For any $\alpha \neq 0$ these orbits could be obtained from supergravity compactifications on locally-geometric T-folds, whereas for $\alpha = 0$ it falls again in a special case of the reductions described for orbits 4 and 5.
- **Orbits 8 – 11:** For any value of α, these orbits always contain a geometric representative involving less general H and ω fluxes.

To summarise, in the half-maximal $D = 7$ case, we encounter a number of orbits which do not have an obvious higher-dimensional origin. To be more precise, these are orbits 1, 2 and 3 for $\alpha \neq 0$. The challenge in the next subsection will be to establish what DFT can do for us in order to give these orbits a higher-dimensional origin. Again, before reading the following subsections we refer to the Sect. 2.2 for a discussion of what we mean by light-cone and cartesian formulations.

Higher-dimensional DFT origin

First of all we would like to show here how to capture the gaugings that only involve (up to duality rotations) fluxes H_{abc} and $\omega_{ab}{}^{c}$. For this, we start from the light-cone formulation, and propose the following Ansatz for a *globally geometric twist* (involving e and B and physical coordinates y)

$$e = \begin{pmatrix} 1 & 0 & \frac{\omega_1}{\omega_3}\sin(\omega_1\,\omega_3\,y^2) \\ 0 & \cos(\omega_2\,\omega_3\,y^1) & -\frac{\omega_2}{\omega_3}\cos(\omega_1\,\omega_3\,y^2)\sin(\omega_2\,\omega_3\,y^1) \\ 0 & \frac{\omega_3}{\omega_2}\sin(\omega_2\,\omega_3\,y^1) & \cos(\omega_1\,\omega_3\,y^2)\cos(\omega_2\,\omega_3\,y^1) \end{pmatrix} , \tag{4.44}$$

$$B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & H\,y^1\,\cos(\omega_1\,\omega_3\,y^2) \\ 0 & -H\,y^1\,\cos(\omega_1\,\omega_3\,y^2) & 0 \end{pmatrix} , \tag{4.45}$$

$$\lambda = -\frac{1}{2}\log(\cos(\omega_1\omega_3 y^2))\ . \tag{4.46}$$

This is far from being the most general ansatz, but it serves our purposes of reaching a large family of geometric orbits. The parameters ω_i can be real, vanishing or imaginary, since U is real and well-behaved in these cases. The QC, weak and strong constraints are all automatically satisfied, and the gaugings read

$$M = \mathrm{diag}(H\ ,\ 0\ ,\ 0\ ,\ 0)\ ,\quad \tilde{M} = \mathrm{diag}(0\ ,\ \omega_1^2\ ,\ \omega_2^2\ ,\ \omega_3^2)\ . \tag{4.47}$$

From here, by choosing appropriate values of the parameters the **orbits 4, 5, 8, 10** and **11** can be obtained. Indeed these are geometric as they only involve gauge and (geo)metric fluxes.

Secondly, in order to address the remaining orbits, we consider an SO$(2,2)$ twist U_4 embedded in O$(3,3)$ in the following way

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & A & 0 & B \\ 0 & 0 & 1 & 0 \\ 0 & C & 0 & D \end{pmatrix} ,\quad U_4 = \begin{pmatrix} A & B \\ C & D \end{pmatrix} ,\quad \lambda = 0\ . \tag{4.48}$$

This situation is analog to the SO$(1,1)$ twist considered in the $D = 8$ case, but with a more general twist. Working in the cartesian formulation, one can define the generators and elements of SO$(2,2)$ as

$$[t_{IJ}]_K{}^L = \delta^L_{[I}\eta_{J]K}\ ,\quad U_4 = \exp\left(t_{IJ}\phi^{IJ}\right)\ , \tag{4.49}$$

where the rotations are generated by t_{12} and t_{34}, and the boosts by the other generators. Also, we take $\phi^{IJ} = \alpha^{IJ}y^1 + \beta^{IJ}\tilde{y}_1$ to be linear.

From the above SO$(2,2)$ duality element one can reproduce the following orbits employing a *locally geometric twist* (including e, B and β but only depending on y, usually referred to as a T-fold):

- **Orbit 6** can be obtained by taking

  $$(\mathbf{6})\quad \alpha^{12} = -\beta^{12} = -\frac{1}{\sqrt{2}}\left(\cos\alpha + \sin\alpha\right)\ ,\quad \alpha^{34} = -\beta^{34} = -\frac{1}{\sqrt{2}}\left(\cos\alpha + \sin\alpha\right)\ .$$

  and all other vanishing.

- **Orbits 7** and **9** can be obtained by the following particular identifications

  $$\phi^{14} = \phi^{23}\quad ,\quad \phi^{12} = \phi^{34}\quad \text{and}\quad \phi^{13} = \phi^{24}\quad .$$

  $$(\mathbf{7})\quad \alpha^{14} = -\beta^{14} = -\frac{1}{\sqrt{2}}\sin\alpha\ ,\quad \alpha^{12} = -\beta^{12} = -\frac{1}{\sqrt{2}}\cos\alpha\ ,\quad \alpha^{13} = \beta^{13} = 0\ ,$$

  $$(\mathbf{9})\quad \alpha^{14} = -\beta^{14} = -\frac{1}{\sqrt{2}}\sin\alpha\ ,\quad \alpha^{12} = \beta^{12} = 0\ ,\quad \alpha^{13} = \beta^{13} = -\frac{1}{\sqrt{2}}\cos\alpha\ .$$

All these backgrounds satisfy both the weak and the strong constraints and hence they admit a locally geometric description. This is in agreement with the fact that the simplest representative of **orbits 6, 7** and **9** given in Table 6 contains H, ω and Q fluxes but no R flux.

Finally, one can employ the same SO(2, 2) duality elements with different identifications to generate the remaining orbits with a *non-geometric twist* (involving both y and $\tilde{y}$ coordinates):

- **Orbits 1, 3** can be again obtained by considering an SO(2) $\times$ SO(2) twist with arbitrary ϕ^{12} and ϕ^{34}:

$$(\mathbf{1})\quad \alpha^{12} = -2\sqrt{2}\,(\cos\alpha + \sin\alpha)\;,\quad \beta^{34} = 2\sqrt{2}\,(\cos\alpha - \sin\alpha)\;,\quad \alpha^{34} = \beta^{12} = 0\;,$$

$$(\mathbf{3})\quad \alpha^{34} = -2\sqrt{2}\,(\cos\alpha + \sin\alpha)\;,\quad \beta^{12} = 2\sqrt{2}\,(\cos\alpha - \sin\alpha)\;,\quad \alpha^{12} = \beta^{34} = 0\;.$$

- **Orbit 2** can be obtained by means of a different SO(2, 2) twist built out of the two rotations and two boosts subject to the following identification

$$\phi^{14} = \phi^{23}\;,\quad \phi^{12} = \phi^{34}\;. \tag{4.50}$$

$$(\mathbf{2})\quad \alpha^{14} = \beta^{12} = \frac{1}{\sqrt{2}}\,(\cos\alpha - \sin\alpha)\;,\quad \alpha^{12} = -\beta^{14} = -\frac{1}{\sqrt{2}}\,(\cos\alpha + \sin\alpha)\;.$$

These backgrounds violate both the weak and the strong constraints for $\alpha \neq 0$. This implies that these backgrounds are truly doubled and they do not even admit a locally geometric description.

Finally, let us also give an example of degeneracy in twist orbits-space reproducing the same orbit of gaugings. The following twist

$$\phi^{12} = \phi^{13}\;,\quad \phi^{34} = \phi^{24}\;,\quad \phi^{23} = \phi^{14} = 0 \tag{4.51}$$

$$(\mathbf{6})\quad \alpha^{13} = -\frac{1}{\sqrt{2}}(\cos\alpha + \sin\alpha)\;,\quad \beta^{24} = \frac{1}{\sqrt{2}}(\cos\alpha - \sin\alpha)\;,\quad \alpha^{24} = \beta^{13} = 0\;,$$

also reproduces the **orbit 6**, but in this case through a non-geometric twist. What happens in this case is that although the twist matrix does not satisfy the weak/strong constraints, the contractions in (2.10) cancel.

5 Conclusions

In this paper we have provided a litmus test to the notion of non-geometry, by classifying the explicit orbits of consistent gaugings of different supergravity theories, and considering the possible higher-dimensional origins of these. The results turn out to be fundamentally different for the cases of U-duality orbits of maximal supergravities, and T-duality orbits of half-maximal theories.

In the former case we have managed to explicitly classify all U-duality orbits in dimensions $8 \leq D \leq 11$. This led to zero, one, four and ten discrete orbits in dimensions $D = 11, 10, 9$ and 8, respectively, with different associated gauge groups. Remarkably, we have found that all of these orbits have a higher-dimensional origin via some geometric compactification, be it twisted reductions or compactifications on group manifolds or coset spaces. In our parlance, we have therefore found that all U-duality orbits are geometric. The structure of U-duality orbits is therefore dramatically different from the sketch of Fig. 1 in the introduction. Although a full classification of all orbits in lower-dimensional cases becomes increasingly cumbersome, we are not aware of any examples that are known to be non-geometric. It could therefore hold in full generality that all U-duality orbits are necessarily geometric.

This is certainly not the case for T-duality orbits of gaugings of half-maximal supergravities. In this case, we have provided the explicit classification in dimensions $7 \leq D \leq 10$ (where in $D = 7$ we have only included three-form fluxes). The numbers of distinct families of orbits in this case are zero, one, three and eleven in dimensions $D = 10, 9, 8$ and 7, respectively, which includes both discrete and one-parameter orbits. A number of these orbits do not have a higher-dimensional origin in terms of a geometric compactification. Such cases are **orbits 2** and **3** in $D = 8$ and **orbits 1, 2** and **3** in $D = 7$ for $\alpha \neq 0$. Indeed, these are exactly the orbits that do not admit an uplift to the maximal theory. As proven in Sect. 2.1, all such orbits necessarily violate the weak and/or strong constraints, and therefore need truly doubled backgrounds. Thus, the structure of T-duality orbits is very reminiscent of Fig. 1 in the introduction. Given the complications that already arise in these simpler higher-dimensional variants, one can anticipate that the situation will be similar in four-dimensional half-maximal supergravity.

Fortunately, the formalism of double field theory seems tailor-made to generate additional T-duality orbits of half-maximal supergravity. Building on the recent generalisation of the definition of double field theory [9], we have demonstrated that all T-duality orbits, including the non-geometric ones in $D = 7, 8$, can be generated by a twisted reduction of double field theory. We have explicitly provided duality twists for all orbits. For locally-geometric orbits the twists only depend on the physical coordinates y, while for the non-geometric orbits these necessarily also include $\tilde{y}$. Again, based on our exhaustive analysis in higher-dimensions, one could conjecture that also in lower-dimensional theories, all T-duality orbits follow from this generalised notion of double field theory.

At this point we would like to stress once more that a given orbit of gaugings can be generated from different twist orbits. Therefore, there is a degeneracy in the space of twist orbits giving rise to a particular orbit of gaugings. Interestingly, as it is the case of **orbit 6** in $D = 7$ for instance, one might find two different twist orbits reproducing the same orbit of gaugings, one violating weak and strong constraints, the other one satisfying both. Our notion of a locally geometric orbit of gaugings is related to the existence of at least one undoubled background giving rise to it. However, this ambiguity seems to be peculiar of gaugings containing Q flux. These can, in principle, be independently obtained by either adding a β but no $\tilde{y}$ dependence (locally geometric choice, usually called T-fold), or by including non-trivial $\tilde{y}$ dependence but no β (non-geometric choice) [7].

Another remarkable degeneracy occurs for the case of semi-simple gaugings, corresponding to **orbits 1 – 3** in $D = 7$. For the special case of $\alpha = 0$, we have two possible ways of generating such orbits from higher-dimensions: either a coset reduction over a sphere or analytic continuations thereof, or a duality twist involving non-geometric coordinate dependence. Therefore d-dimensional coset reductions seem to be equivalent to $2d$-dimensional twisted torus reductions (with the latter in fact being more general, as it leads to all values of α). Considering the complications that generally arise in proving the consistency of coset reductions, this is a remarkable reformulation that would be interesting to understand in more detail. Furthermore, when extending the notion of double field theory to type II and M-theory, this relation could also shed new light on the consistency of the notoriously difficult four-, five- and seven-sphere reductions of these theories.

Our results mainly focus on Scherk-Scharz compactifications leading to gauged supergravities with vanishing ξ_M fluxes. In addition, we have restricted to the NSNS sector and ignored α'-effects. Also, we stress once again that relaxing the strong and weak constraints is crucial in part of our analysis. If we kept the weak constraint, typically the Jacobi identities would lead to backgrounds satisfying also the strong constraint[11] [9]. However, from a purely (double) field theoretical analysis the weak constraint is not necessary. A sigma model analysis beyond tori would help us to clarify the relation between DFT without the weak and strong constraints and string field theory on more general backgrounds. We hope to come back to this point in the future.

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[11] We thank O. Hohm for enlightening correspondence on this point.

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A Different solvable and nilpotent gaugings

In Sect. 4.2 we have studied the T-duality orbits of gaugings in half-maximal $D = 7$ supergravity and for each of them, we identified the gauge algebra and presented the results in Table 6. Since there is no exhaustive classification of non-semisimple algebras of dimension 6, we would like to explicitly give the form of the algebras appearing in Table 6.

Solvable algebras

The CSO$(2, 0, 2)$ and CSO$(1, 1, 2)$ algebras

The details about these algebras can be found in [46]; we summarise here some relevant facts.

The six generators are labelled as $\{t_0,\, t_i,\, s_i,\, z\}_{i=1,2}$, where t_0 generates SO(2) (SO$(1,1)$), under which $\{t_i\}$ and $\{s_i\}$ transform as doublets

$$[t_0,\, t_i] \,=\, \epsilon_i{}^j\, t_j \quad , \quad [t_0,\, s_i] \,=\, \epsilon_i{}^j\, s_j \quad , \tag{A.1}$$

where the Levi-Civita symbol $\epsilon_i{}^j$ has one index lowered with the metric η_{ij} $=$diag$(\pm 1, 1)$ depending on the two different signatures. z is a central charge appearing in the following commutators

$$\left[t_i,\, s_j\right] \,=\, \delta_{ij}\, z\ . \tag{A.2}$$

The Cartan-Killing metric is diag$(\mp 1, \underbrace{0, \cdots, 0}_{6\text{ times}})$, where the $\mp$ is again related to the two different signatures.

The $\mathfrak{f}_1$ and $\mathfrak{f}_2$ algebras

These are of the form Solv$_4 \times$ U$(1)^2$. The 4 generators of Solv$_4$ are labeled by $\{t_0,\, t_i,\, z\}_{i=1,2}$, where t_0 generates SO(2) (SO$(1,1)$), under which $\{t_i\}$ transform as a doublet

$$[t_0,\, t_i] \,=\, \epsilon_i{}^j\, t_j\ , \tag{A.3}$$

$$\left[t_i,\, t_j\right] \,=\, \epsilon_{ij}\, z\ . \tag{A.4}$$

The Cartan-Killing metric is diag$(\mp 1, \underbrace{0, \cdots, 0}_{6\text{ times}})$.

The $\mathfrak{h}_1$ and $\mathfrak{h}_2$ algebras

The 6 generators are $\{t_0,\, t_i,\, s_i,\, z\}_{i=1,2}$ and they satisfy the following commutation relations

$$\begin{aligned} &[t_0,\, t_i] \,=\, \epsilon_i{}^j\, t_j \quad , \quad [t_0,\, s_i] \,=\, \epsilon_i{}^j\, s_j \,+\, t_i \quad , \\ &\left[t_i,\, s_j\right] \,=\, \delta_{ij}\, z \quad , \quad \left[s_i,\, s_j\right] \,=\, \epsilon_{ij}\, z \quad\quad . \end{aligned} \tag{A.5}$$

The Cartan-Killing metric is diag$(\mp 1, \underbrace{0, \cdots, 0}_{6\text{ times}})$.

The $\mathfrak{g}_0$ algebra

The 6 generators are $\{t_0,\, t_I,\, z\}_{I=1,\cdots,4}$, where t_0 transforms cyclically the $\{t_I\}$ amongst themselves such that

$$[[[[t_I,\, t_0]\,,\, t_0]\,,\, t_0]\,,\, t_0] \,=\, t_I \;, \tag{A.6}$$

and

$$[t_1,\, t_3] \,=\, [t_2,\, t_4] \,=\, z \;. \tag{A.7}$$

Note that this algebra is solvable and not nilpotent even though its Cartan-Killing metric is *completely zero*.

Nilpotent algebras

The CSO$(1, 0, 3)$ algebra

The details about this algebra can be again found in [46]; briefly summarizing, the 6 generators are given by $\{t_m,\, z^m\}_{m=1,2,3}$ and they satisfy the following commutation relations

$$[t_m,\, t_n] \,=\, \epsilon_{mnp}\, z^p \;, \tag{A.8}$$

with all the other brackets being vanishing. The order of nilpotency of this algebra is 2.

The $\mathfrak{l}$ algebra

The 6 generators $\{t_1, \cdots, t_6\}$ satisfy the following commutation relations

$$[t_1,\, t_2] \,=\, t_4 \quad , \quad [t_1,\, t_4] \,=\, t_5 \quad , \quad [t_2,\, t_4] \,=\, t_6 \quad . \tag{A.9}$$

The corresponding central series reads

$$\{t_1,\, t_2,\, t_3,\, t_4,\, t_5,\, t_6\} \;\supset\; \{t_4,\, t_5,\, t_6\} \;\supset\; \{t_5,\, t_6\} \;\supset\; \{0\} \;, \tag{A.10}$$

from which we can immediately conclude that its nilpotency order is 3.

B SO(2, 2) and SO(3, 3) ’t Hooft symbols

In Sect. 2 we discuss the origin of a given flux configuration from DFT backgrounds specified by twist matrices U. The deformations of half-maximal supergravity in $D = 10 - d$ which can be interpreted as the gauging of a subgroup of the T-duality group O(d, d) can be described by a 3-form of O(d, d) f_{ABC} which represents a certain (non-)geometric flux configuration.

In $D = 8$ and $D = 7$, the T-duality group happens to be isomorphic to SL$(2) \times$ SL(2) and SL(4) respectively. As a consequence, in order to explicitly relate flux configurations and embedding tensor orbits, we need to construct the mapping between T-duality irrep’s and irrep’s of SL$(2) \times$ SL(2) and SL(4) respectively.

From the $(\mathbf{2}, \mathbf{2})$ of SL$(2) \times$ SL(2) to the $\mathbf{4}$ of SO$(2, 2)$

The ’t Hooft symbols $(G_A)^{\alpha i}$ are invariant tensors which map the fundamental representation of SO$(2, 2)$ (here denoted by A), into the $(\mathbf{2}, \mathbf{2})$ of SL$(2) \times$ SL(2)

$$v^{\alpha i} \,=\, (G_A)^{\alpha i}\; v^A \;, \tag{B.1}$$

where v^A denotes a vector of SO$(2,2)$ and the indices α and i are raised and lowered by means of $\epsilon_{\alpha\beta}$ and ϵ_{ij} respectively. $(G_A)^{\alpha i}$ and $(G_A)_{\alpha i}$ satisfy the following identities

$$(G_A)_{\alpha i}\,(G_B)^{\alpha i} = \eta_{AB}\,, \tag{B.2}$$

$$(G_A)^{\alpha i}\,\left(G^A\right)^{\beta j} = \epsilon^{\alpha\beta}\,\epsilon^{ij}\,, \tag{B.3}$$

where η_{AB} is the SO$(2,2)$ metric.

After choosing light-cone coordinates for SO$(2,2)$, our choice for the tensors $(G_A)^{\alpha i}$ is the following

$$(G_1)^{\alpha i} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\,, \qquad (G_2)^{\alpha i} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\,, \tag{B.4}$$

$$(G_{\bar{1}})^{\alpha i} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\,, \qquad (G_{\bar{2}})^{\alpha i} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}\,. \tag{B.5}$$

By making use of the mapping (B.1), we can rewrite the structure constants $(X_{\alpha i})_{\beta j}{}^{\gamma k}$ as a 3-form of SO$(2,2)$ as follows:

$$f_{ABC} = (X_{\alpha i})_{\beta j}{}^{\gamma k}(G_A)^{\alpha i}(G_B)^{\beta j}(G_C)_{\gamma k}\,. \tag{B.6}$$

From the **6** of SL(4) to the **6** of SO$(3,3)$

The 't Hooft symbols $(G_A)^{mn}$ are invariant tensors which map the fundamental representation of SO$(3,3)$, *i.e.* the **6** into the anti-symmetric two-form of SL(4)

$$v^{mn} = (G_A)^{mn}\,v^A\,, \tag{B.7}$$

where v^A denotes a vector of SO$(3,3)$. The two-form irrep of SL(4) is real due to the role of the Levi-Civita tensor relating v^{mn} to v_{mn}

$$v_{mn} = \frac{1}{2}\,\epsilon_{mnpq}\,v^{pq}\,. \tag{B.8}$$

The 't Hooft symbols with lower SL(4) indices $(G_A)_{mn}$ carry out the inverse mapping of the one given in (B.7). The tensors $(G_A)^{mn}$ and $(G_A)_{mn} = \frac{1}{2}\,\epsilon_{mnpq}\,(G_A)^{pq}$ satisfy the following identities

$$(G_A)_{mn}\,(G_B)^{mn} = 2\,\eta_{AB}\,, \tag{B.9}$$

$$(G_A)_{mp}\,(G_B)^{pn} + (G_B)_{mp}\,(G_A)^{pn} = -\delta^n_m\,\eta_{AB}\,, \tag{B.10}$$

$$(G_A)_{mp}\,(G_B)^{pq}\,(G_C)_{qr}\,(G_D)^{rs}\,(G_E)_{st}\,(G_F)^{tn} = \delta^n_m\,\epsilon_{ABCDEF}\,, \tag{B.11}$$

where η_{AB} and ϵ_{ABCDEF} are the SO$(3,3)$ metric and Levi-Civita tensor respectively.

After choosing light-cone coordinates for SO(3, 3) vectors, our choice of the 't Hooft symbols is

$$(G_1)^{mn} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} , \qquad (G_2)^{mn} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} , \tag{B.12}$$

$$(G_3)^{mn} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} , \qquad (G_{\bar{1}})^{mn} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} , \tag{B.13}$$

$$(G_{\bar{2}})^{mn} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} , \qquad (G_{\bar{3}})^{mn} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} . \tag{B.14}$$

Thus, we can rewrite the structure constants in the **6**, $(X_{mn})_{pq}{}^{rs}$, arising from (4.24) as a 3-form of SO(3, 3) as follows:

$$f_{ABC} = (X_{mn})_{pq}{}^{rs}(G_A)^{mn}(G_B)^{pq}(G_C)_{rs} \ . \tag{B.15}$$

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