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A New Perspective on Nonrelativistic Gravity[1]

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Abstract—The geometric reformulation of Newton's gravity is known as Newton–Cartan theory. We compare the traditional derivation of this theory with a new, algebraic derivation, based on the gauging of a centrally extended Galilean symmetry algebra. In this comparison the role of the central charge gauge field will be explained. In particular, we show that the scalar potential following from this procedure coincides with the one given by the theory of Cartan. Our procedure can be generalized to describe other nonrelativistic limits of gravity involving gravitating strings.

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1. THE TRADITIONAL DERIVATION

Newton–Cartan gravity is a geometric reformulation of Newtonian gravity in which the curved trajectories of particles in flat space $\mathbb{R}^d$ are replaced by geodesics in curved Newtonian spacetime. This spacetime has a singular metric structure [2, 3] due to the absolute time t. The spatial metric $h^{\mu\nu}$ and temporal metric $\tau_{\mu\nu}$, with $h^{\mu\nu}\tau_{\nu\rho} = 0$, are covariantly constant: $\nabla_\rho h^{\mu\nu} = \nabla_\rho \tau_{\nu\nu} = 0$. This defines a class of connections Γ. These can be solved for by introducing $h_{\mu\nu}$ and $\tau^{\mu\nu}$ with $h_{\mu\rho}\tau^{\rho\nu} = 0$, and the projective relation $h^{\mu\rho}h_{\mu\nu} + \tau^{\mu\rho}\tau_{\mu\nu} = \delta^\rho_\nu$. Unlike in general relativity, Γ is not uniquely determined from metric compatibility; there is an ambiguity which is parametrized by an arbitrary two-form $K_{\mu\nu}$.

Writing $\tau_{\mu\nu} = \tau_\mu\tau_\nu$, the metric compatibility of $\tau_{\mu\nu}$ implies in adapted coordinates $x^0 = t$ that $\tau_\mu = \partial_\mu t = \delta^0_\mu$ and $h^{\mu 0} = 0$. This leaves us with nonzero connection components $\{\Gamma^i_{jk},\ \Gamma^i_{0j},\ \Gamma^i_{00}\}$ with $i = 1, \dots d$. The form of the connection suggests the definitions $\Phi_i = \partial_0 h_{i0} - 1/2\partial_i h_{00} + K_{i0}$, and $\Omega_{ij} = \partial[_i h_j]_0 - 1/2K_{ij}$. To identify the Newtonian potential, a number of constraints needs to be imposed on the Riemann tensor. These are summarized by what are known as the Ehlers conditions [4, 5]

$$h^{\sigma[\lambda}R^{\mu]}{}_{\nu\rho\sigma}(\Gamma) = 0. \tag{1}$$

In adapted coordinates, these conditions imply $R^m{}_{ijk}(\Gamma^i_{jk}) = 0$. This means that space is flat, so that we can choose spatial coordinates in which $h_{ij} = \delta_{ij}$, and thus obtain $\Gamma^i_{jk} = 0$ and $\Gamma^i_{0j} \propto \Omega_{ij}$. In this coordinate system we are still allowed to perform time dependent spatial rotations, which preserve $h_{ij} = \delta_{ij}$. The symmetric part $(\nu\rho)$ of the Ehlers conditions (1) implies $\partial[_\rho K_{\mu\nu}] = 0$. The full expression leads to the restrictions

$$\partial_0\Omega_{ij} - \partial_{[i}\Phi_{j]} = 0, \quad \partial_k\Omega_{ij = 0.} \tag{2}$$

The second restriction shows that Ω_{ij} only depends on time, so that it can be set to zero via a time-dependent rotation. This means that $\Gamma^i_{j0} = 0$, which in terms of K is equivalent to $K_{ij} = 2\partial_{[i}h_{j]0}$. Since the curl of K vanishes, we can introduce a vector M_μ such that

$$K_{\mu\nu} = 2\partial_{[\mu}M_{\nu]}, \tag{3}$$

which means that $h_{j0} = M_j + \partial_j f$, where f is an arbitary function. This enables us to write the only remaining non-zero connection coefficients Γ^i_{00} as

$$\Gamma^i_{00} = \partial_i\left(M_0 - \frac{1}{2}h_{00} + \partial_0 f\right) \equiv \partial_i\phi, \tag{4}$$

which defines the Newtonian potential ϕ. This identification is motivated by the form of the geodesic equation of a particle in our singular metric and by considering the Einstein equations, in which only $R_{00} = \partial^i\partial_i\phi$ is nonzero.

2. THE ALGEBRAIC DERIVATION

Our starting point in the algebraic approach [1] is the Lagrangian of a non-relativistic free particle with is invariant under the centrally extended Galilei algebra known as the Bargmann algebra. A gauging procedure is applied to this Bargmann algebra giving rise to a gauge field for every generator with corresponding curvatures that satisfy Bianchi identities.

In a next step we impose a set of constraints on the curvatures such that the gauge fields $e_\mu{}^i$ and τ_μ corresponding to the spatial and temporal translations, respectively, transform as vielbeins. These constraints have the additional effect that the spin connections $\omega_\mu{}^{ij}$ and $\omega_\mu{}^{i0}$ corresponding to the rotations and to

[1] The article is published in the original.

the Galilean boosts can be expressed in terms of the independent fields e_μ^i and τ_μ together with the gauge field m_μ corresponding to the central charge transformations. Without the central charge, the spin connections could not have been solved for.

To identify the Newtonian potential we need to impose a further curvature constraint. To motivate this constraint we first construct a connection by the requirement that $\nabla_\rho e_\mu{}^i = \nabla_\rho \tau_\mu = 0$. These vielbein postulates imply that Γ can be expressed *uniquely* in terms of the independent fields, in contrast to the traditional approach. Then in adapted coordinates one has again that $\Gamma^0_{\mu\nu} = 0$, leaving us with nonzero connection components $\{\Gamma^i_{jk},\ \Gamma^i_{j0},\ \Gamma^i_{00}\}$.

We now impose that the curvature of the rotational symmetries vanishes, $R^{ij}_{\mu\nu}(J) = 0$. By expressing the Riemann tensor in terms of the curvatures of the Bargmann algebra, this constraint implies that space is flat, $R^m{}_{ijk}(\Gamma^i_{jk}) = 0$. We can therefore again choose spatial coordinates such that $h_{ij} = \delta_{ij}$ and hence $\Gamma^i_{jk} = 0$. The fact that $R_{\mu\nu}{}^{ij}(J) = 0$ also implies that $\omega_\mu{}^{ij}$ is a pure gauge, and can be set to zero by a local rotation. Physically, these coordinates describe a non-rotating observer. Since is a pure gauge, and can be set to zero by a local rotij is a dependent field, this implies the following relation between the central charge gauge field m_μ and the time-component of the vielbein $e_\mu{}^i$:

$$\omega_0{}^{ij} \propto \Gamma^i_{0j} \propto \partial_{[i}(m_{j]} - e_{0j}] = 0. \tag{5}$$

A further analysis shows that the only remaining connection coefficients Γ^i_{00} are given by

$$\Gamma^i_{00} = \partial_i\left(m_0 - \frac{1}{2}h_{00} + \partial_0 f\right) \equiv \partial_i\phi, \tag{6}$$

where $h_{00} = e_0{}^k e_{0k}$ and ϕ is the Newtonian potential. Here f arises from (5), which implies $e_{0k} - m_k = \partial_k f$. Just as in the previous analysis we find that $\Gamma^i_{00} = \partial_i\phi$, and that R_{00} is therefore the only nonzero component of the Ricci tensor.

3. CONCLUSION AND DISCUSSION

We conclude that the expression (6) for the Newtonian potential derived in the algebraic analysis coincides with the expression (4) derived in the traditional analysis. Therefore, the two procedures lead to the same final result. The central charge gauge field m_μ in the gauge algebra analysis corresponds in the traditional derivation to the field M_μ in (3).

The advantage of the algebraic procedure is that it can be applied to define other nonrelativistic limits of general relativity, e.g. by switching to nonrelativistic strings [9]. There, the starting point is the symmetry group of the nonrelativistic string action which is a deformation of the stringy Galilei group. Gauging this algebra and imposing a set of curvature constraints similarly to the point particle leads to a new nonrelativistic gravity theory. The foliation space then becomes a two-dimensional Minkowski foliation, $\tau_{ab} = \eta_{ab}$, $(a = 0, 1)$. The extra foliation direction corresponds to the longitudinal direction along the string. We thus end up with a generalization of the Poisson equation and the geodesic equation, involving a gravitational tensor potential ϕ_{ab}. By changing the curvature constraint for the foliation space this nonrelativistic string can be placed in a spacetime exhibiting stringy Newton—Hooke symmetry [6], where the foliation space is AdS_2 and the transverse space is flat. These new nonrelativistic limits of general relativity, in the case of strings, could have applications in the context of nonrelativistic versions of AdS/CFT [7]. Although most of the literature on this topic concerns background solutions of general relativity with nonrelativistic isometries, such as the Schrodinger and Lifshitz symmetries, there are a number of intriguing field theories with Galilei symmetries as well. It would be interesting to see whether the gravitational description of this class of field theories can be understood by using the stringy nonrelativistic limit of general relativity that follows from our procedure.

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