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Consensus in Directed Networks of Agents With Nonlinear Dynamics

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***Abstract*—This technical note studies the consensus problem for cooperative agents with nonlinear dynamics in a directed network. Both local and global consensus are defined and investigated. Techniques for studying the synchronization in such complex networks are exploited to establish various sufficient conditions for reaching consensus. The local consensus problem is first studied via a combination of the tools of complex analysis, local consensus manifold approach, and Lyapunov methods. A generalized algebraic connectivity is then proposed to study the global consensus problem in strongly connected networks and also in a broad class of networks containing spanning trees, for which ideas from algebraic graph theory, matrix theory, and Lyapunov methods are utilized.**

***Index Terms*—Algebraic graph theory, complex network, consensus, Lyapunov function, synchronization.**

I. Introduction

Cooperative collective behavior in networks of autonomous agents has received considerable attention in recent years due to the growing interest in understanding intriguing animal group behaviors, such as flocking and swarming, and also due to their emerging broad applications in sensor networks [1], unmanned air vehicles (UAV) formations, robotic teams, to name a few. To coordinate with other agents in a network, agents need to share information with their adjacent peers and agree on a certain value of interest. In this context, the consensus problem usually refers to the problem of how to reach an agreement among a group of autonomous agents in a dynamically changing environment [2]. One of the main challenges of solving such a consensus problem is that an agreement has to be reached by all the agents in the whole dynamic network while the information of each agent is shared only locally.

Various models have been used to study the consensus problem and some of the theoretical results obtained recently are closely related to what is presented in this technical note. In [3], Vicsek *et al.* studied a discrete-time system that models a group of autonomous agents moving in the plane with the same speed but different headings, which in essence is a simplified version of the model proposed earlier by Reynolds [4]. Analysis on Vicsek's model or its continuous-time version [5]–[9] shows that the connectivity of the time-varying graph that describes the neighbor relationships within the group is key in reaching consensus. In particular, in [6], Olfati-Saber and Murray established the relationship between the algebraic connectivity [10] (also called the Fiedler eigenvalue) and the speed of convergence when the directed graph is balanced. A broader class of directed graphs that may lead to consensus are those that contain spanning trees [8], [11], which are also called rooted graphs [9], [12]. Second-order and higher-order consensus in linear multi-agent systems was studied in [13], [33].

Vicsek's model is similar to a class of models discussed in studying the synchronization of complex networks [14]–[20], [26], [27]. In 1998, Pecora and Carroll made use of a master stability function to study the synchronization of coupled complex networks [17]. Thereafter, stability and synchronization of small-world and scale-free networks have been investigated extensively using this master stability function method. In [14], [15], local synchronization was studied using the transverse stability to the synchronization manifold, where synchronization was discussed with respect to small-world and scale-free networks. In [18], a distance from the collective states to the synchronization manifold was defined and then utilized to obtain conditions for global synchronization of coupled systems [19], [20]. It is clear that most of the real-world complex networks, e.g., World Wide Web and mobile communication networks, are directed networks. However, many existing tools developed for the study of synchronization in complex networks can only be applied to undirected networks. This is partly due to the fact that algebraic graph theory especially the algebraic connectivity has not been well developed for directed graphs. For example, there are no standard definitions for the algebraic connectivity and consensus convergence rate for directed graphs, while its counterparts for undirected graphs have been extensively used to study the synchronization problem.

Very recently, the consensus problem in directed networks with nonlinear dynamics has been discussed [21]–[25]. In [22], a class of feedback rules was used and a passivity-based design framework was developed to reach the velocity consensus among agents. Under the assumption that the vector fields satisfy a subtangentiality condition, it was proved in [23] that agents can reach consensus if and only if the network is connected sufficiently frequently over time. In [24], distributed algorithms for reaching network consensus was proposed based on nonsmooth analysis and many results assumed that the network is weight-balanced. In contrast, in this technical note, we consider the multi-agent system in which the dynamics of each agent consist of two terms:

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one is determined by an intrinsic nonlinear function which governs the asymptotical state and the other is a simple linear communication protocol relying only on information about its neighbors. A new framework based on matrix theory was proposed to design cooperative controls for a group of autonomous agents in an intermittent, dynamically changing, and local environment in [25]. Different from previous works [22]–[25], a simple linear protocol is designed in this technical note to generalize the tools developed for undirected complex networks in order to make them applicable to directed networks.

In this technical note, local stability properties of the consensus states in a directed network of agents with nonlinear dynamics are investigated via complex analysis, local synchronization manifold approach, and Lyapunov method. It is found that the real part of the second smallest eigenvalue of the Laplacian matrix plays a key role in deriving the consensus conditions. By assuming that the nonlinear intrinsic function is Lipschitz and introducing a generalized algebraic connectivity in the directed network, some sufficient conditions for reaching network consensus are established. It is found that the general algebraic connectivity is very critical in reaching network consensus and can be used to describe the consensus ability in a directed network which is similar to the role of the algebraic connectivity (Fiedler number [10]) for undirected networks.

II. Preliminaries

Let $G = (\mathcal{V}, \mathcal{E}, A)$ be a weighted directed graph of order N, with the set of nodes $\mathcal{V} = \{v_1, v_2, \cdots, v_N\}$, the set of directed edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, and a weighted adjacent matrix $A = (a_{ij})_{N\times N}$. A directed edge in graph G is denoted by $e_{ij} = (v_i, v_j)$. If there is an edge from node v_j to node v_i, then it is said that node v_j can reach node v_i and $a_{ij} > 0$ is the weight associated with the edge e_{ij}; otherwise, $a_{ij} = 0$. As usual, we assume there is no self-loop in G. The Laplacian matrix $L = (L_{ij})_{N\times N}$ of graph G is defined by $L_{ij} = -a_{ij}$ for $i \neq j$, $i, j \in \{1, \ldots, N\}$ and $L_{ii} = k_i^{in}$ for $i \in \{1, \ldots, N\}$, where $k_i^{in} = \sum_{j=1, j\neq i}^{N} G_{ij}$ is the sum of the weights of the edges ending at node v_i. It is easy to check that $\sum_{j=1}^{N} L_{ij} = 0$ for all $i = 1, 2, \ldots, N$.

The consensus protocol in a multi-agents system considered in [6] is as follows:

$$\dot{x}_i(t) = \sum_{j\neq i} a_{ij}\left(x_j(t) - x_i(t)\right) = -\sum_{j=1}^{N} L_{ij} x_j(t). \tag{1}$$

If the network topology changes with time [5], [7]–[9], then in (1) $L = L(t)$ is a time-varying matrix. Let $\mathcal{S} = \{(x_1, x_2, \cdots, x_N) : x_1 = x_2 = \cdots = x_N\}$ be the *consensus manifold*. It is clear that since $\sum_{j=1}^{N} L_{ij} = 0$ for all $i = 1, \ldots, N$, x_i must be time-invariant on the consensus manifold in (1). In other words, the values of x_i will not change with time once the consensus $x_1(t) = x_2(t) = \ldots = x_N(t)$ is achieved. However, as has been repeatedly demonstrated in physical complex networks, the state of each agent is, generally speaking, not a constant after getting synchronized, but a dynamical variable because of the intrinsic nonlinear dynamics of each agent and the possible complicated ways in which the network is evolving. To study the synchronization of complex networks for more general cases particularly when the synchronized state is a time-varying function rather than a constant equilibrium, we consider the following general consensus protocol:

$$\dot{x}_i(t) = f\left(x_i(t)\right) - c\sum_{j=1}^{N} L_{ij}\Gamma x_j(t) \tag{2}$$

where $x_i \in R^n$ is the state of agent i, $f(x_i) = (f_1(x_i), f_2(x_i), \ldots, f_n(x_i))^T$ is a nonlinear function, c is the coupling strength, and $\Gamma = \mathrm{diag}(\gamma_1, \ldots, \gamma_n) \in R^{n\times n}$ is a semi-positive definite diagonal matrix where $\gamma_j > 0$ means that the agents can communicate through their jth state [14], [15]. Here, the state of each agent is an n-dimensional vector as compared to 1-dimensional variables considered in previous works, e.g. [5], [7], [8], [28]. Note also that it is straightforward to construct the discrete-time counterpart of system (2), but only the continuous-time case is investigated in this technical note.

Clearly, since $\sum_{j=1}^{N} L_{ij} = 0$, if a consensus can be achieved, the solution $s(t)$ of system (2) is expected to be a possible trajectory of an isolated node satisfying

$$\dot{s}(t) = f\left(s(t)\right). \tag{3}$$

Here, $s(t)$ may be an isolated equilibrium point [5], [7], [8], [28], a periodic orbit, or even a chaotic orbit [19].

Let $A \otimes B$ denote the Kroneker product [29] of matrices A and B, $x(t) = (x_1^T(t), x_2^T(t), \ldots, x_N^T(t))^T$, $f(x(t)) = (f^T(x_1(t)), f^T(x_2(t)), \ldots, f^T(x_N(t)))^T$, I_n be the n-dimensional identity matrix, $\mathbf{1}_N$ be the N-dimensional column vector with all entries being 1, and $A^s = (1/2)(A + A^T)$. Then, system (2) can be written as

$$\dot{x}(t) = f\left(x(t)\right) - c(L\otimes\Gamma)x(t). \tag{4}$$

A few definitions and some results are given here, which will be useful in the development of the next few sections.

Definition 1: The consensus in system (4) is said to be local if, for any $\varepsilon > 0$, there exist a $\delta(\varepsilon)$ and a $T > 0$, such that $\|x_i(0) - x_j(0)\| \leq \delta(\varepsilon)$ implies $\|x_i(t) - x_j(t)\| \leq \varepsilon$ for all $t > T$ and $i, j = 1, 2, \ldots, N$.

Definition 2: The consensus in system (4) is said to be global if, for any $\varepsilon > 0$, there exists a $T > 0$ such that $\|x_i(t) - x_j(t)\| \leq \varepsilon$ for any initial conditions and all $t > T$, $i, j = 1, 2, \ldots, N$.

Definition 3: The graph G is said to have a spanning tree if there is a node that can reach all the other nodes following the edge directions in graph G.

Lemma 1: [8] Assume that there is a spanning tree in graph G. Then the Laplacian matrix L of G has eigenvalue 0 with algebraic multiplicity one, and the real parts of all the other eigenvalues are positive, i.e., the eigenvalues of L satisfy $0 = \lambda_1(G) < \mathcal{R}(\lambda_2(G)) \leq \ldots \leq \mathcal{R}(\lambda_N(G))$. In addition, $\mathrm{rank}(L) = N - 1$.

Lemma 2: [29] The Kronecker product has the following properties: For matrices A, B, C and D of appropriate dimensions,

1) $(A + B) \otimes C = A \otimes C + B \otimes C$;
2) $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$.

III. Local Consensus of Multi-Agent Systems

In this section, local consensus of multi-agent systems is investigated. Subtracting (3) from (2) yields the following error dynamical system:

$$\dot{y}_i(t) = f\left(x_i(t)\right) - f\left(s(t)\right) - c\sum_{j=1}^{N} L_{ij}\Gamma y_j(t) \tag{5}$$

where $y_i = x_i - s$, $i = 1, 2, \ldots, N$. Linearizing (5) around $s(t)$ leads to

$$\dot{y}(t) = \left(I_N \otimes Df\left(s(t)\right)\right)y(t) - c(L\otimes\Gamma)y(t) \tag{6}$$

where $y(t) = (y_1^T(t), y_2^T(t), \ldots, y_N^T(t))$ and $Df(s) \in R^{n\times n}$ is the Jacobian matrix of f at $s(t)$. Let P be the Jordan form associated with the Laplacian matrix L, i.e., $L = PJP^{-1}$ where J is the Jordan form of L. Then one has

$$\dot{z}(t) = \left(I_N \otimes Df\left(s(t)\right)\right)z(t) - c(J\otimes\Gamma)z(t) \tag{7}$$

where $z(t) = (P^{-1} \otimes I_n)y(t)$. If L is symmetric, i.e., graph G is undirected, then J is a diagonal matrix with real eigenvalues. However, when G is directed, some eigenvalues of L may be complex, and $J = \mathrm{diag}(J_1, J_2, \ldots, J_r)$, where

$$J_l = \begin{pmatrix} \lambda_l & 0 & 0 & 0 \\ 1 & \ddots & 0 & 0 \\ 0 & \ddots & \ddots & 0 \\ 0 & 0 & 1 & \lambda_l \end{pmatrix}_{N_l \times N_l}. \tag{8}$$

Here, it is assumed that the Laplacian matrix L has eigenvalues λ_l with multiplicity $N_l, l = 1, 2, \ldots, r, N_1 + N_2 + \ldots + N_r = N$. Let $\mathcal{R}(u)$ and $\mathcal{I}(u)$ be the real and imaginary parts of a complex number u, and $\mathcal{R}(A)$ and $\mathcal{I}(A)$ be the real and imaginary parts of matrix $A = (A_{ij})$, where $\mathcal{R}(A)_{ij} = \mathcal{R}(A_{ij})$ and $\mathcal{I}(A)_{ij} = \mathcal{I}(A_{ij})$, respectively. Let $\widetilde{N}_i = N_1 + \ldots + N_i, i = 1, 2, \ldots, r$. Then, separating the real and imaginary parts of (7), one obtains

$$\begin{aligned} \mathcal{R}\left(\dot{\widetilde{z}}_i(t)\right) &= (I_{N_i} \otimes Df(s(t)))\mathcal{R}(\widetilde{z}_i(t)) \\ &\quad - c(\mathcal{R}(J_i) \otimes \Gamma)\mathcal{R}(\widetilde{z}_i(t)) \\ &\quad + c(\mathcal{I}(J_i) \otimes \Gamma)\mathcal{I}(\widetilde{z}_i(t)), \\ \mathcal{I}\left(\dot{\widetilde{z}}_i(t)\right) &= (I_{N_i} \otimes Df(s(t)))\mathcal{I}(\widetilde{z}_i(t)) \\ &\quad - c(\mathcal{R}(J_i) \otimes \Gamma)\mathcal{I}(\widetilde{z}_i(t)) \\ &\quad - c(\mathcal{I}(J_i) \otimes \Gamma)\mathcal{R}(\widetilde{z}_i(t)) \end{aligned} \tag{9}$$

where $\widetilde{z}_i \in R^{N_i}, i = 1, \ldots, r$.

Lemma 3: Suppose that graph G has a spanning tree. If system (9) is asymptotically stable for $i = 2, \ldots, r$, then local consensus can be reached in system (4).

Proof: According to Lemma 1, zero is a simple eigenvalue of the Laplacian matrix L. From (7), one has $y(t) = (P \otimes I_n)z(t)$, where $LP = PJ$. Let $P = (p_1, p_2, \ldots, p_N)$. Then p_1 is the right eigenvector of L associated with eigenvalue 0, i.e., $Lp_1 = 0$. Since $\sum_{j=1}^N L_{ij} = 0$ and $\mathrm{rank}(L) = N - 1$, one has $p_1 = \eta(1, 1, \ldots, 1)^T$, where η is a constant. If system (9) is asymptotically stable for $p = 2, \ldots, N_i$ and $i = 2, \ldots, r$, then $\|\mathcal{R}(z_i(t))\| \to 0$ and $\|\mathcal{I}(z_i(t))\| \to 0$ as $t \to \infty$, $i = 2, \ldots, N$. Therefore

$$y(t) \to \eta\,(z_1(t), z_1(t), \ldots, z_1(t))^T$$

where $\dot{z}_1(t) = Df(s(t))z_1$. The proof is thus completed.

Lemma 4: [21] Let

$$\widetilde{L}^* = \begin{pmatrix} \widetilde{L}^*_{11} & O & \cdots & O \\ \widetilde{L}^*_{21} & \widetilde{L}^*_{22} & \cdots & O \\ \vdots & \vdots & \ddots & O \\ \widetilde{L}^*_{p1} & \widetilde{L}^*_{p2} & \cdots & \widetilde{L}^*_{pp} \end{pmatrix}. \tag{10}$$

where O is a zero matrix with appropriate dimension, $\widetilde{L}_{kk} \in R^{m_k m_k}$, and m_k are positive integers for all $k = 1, 2, \ldots, p$. If there exist positive definite diagonal matrices $Q^*_k \in R^{m_k m_k}$, such that

$$Q^*_k \widetilde{L}^*_{kk} + \widetilde{L}^{*T}_{kk} Q^*_k < 0 \tag{11}$$

then, there exists a positive definite diagonal matrix $\Delta = \mathrm{diag}(\Delta_1 I_{m_1}, \ldots, \Delta_p I_{m_p})$, such that

$$\Delta \widetilde{Q}^* \widetilde{L}^* + \widetilde{L}^{*T} \widetilde{Q}^* \Delta < 0 \tag{12}$$

where $\widetilde{Q}^* = \mathrm{diag}(\widetilde{Q}^*_1, \ldots, \widetilde{Q}^*_p)$.

Theorem 1: Suppose that graph G has a spanning tree. Then the local consensus of system (4) can be reached if

$$(Df(s(t)))^s - c\mathcal{R}(\lambda_2)\Gamma < 0,\ \forall\, t > 0. \tag{13}$$

Proof: In view of Lemma 3, one only needs to prove that under condition (13), system (9) is asymptotically stable for $i = 2, \ldots, r$.

Consider the Lyapunov function candidate

$$V(t) = \frac{1}{2}\sum_{i=2}^r \Delta_i \left\{\mathcal{R}^T(\widetilde{z}_i(t))\mathcal{R}(\widetilde{z}_i(t)) + \mathcal{I}^T(\widetilde{z}_i(t))\mathcal{I}(\widetilde{z}_i(t))\right\} \tag{14}$$

where Δ_i is positive for $i = 2, \ldots, r$.

Taking the derivative of $V(t)$ along the trajectories of (9) gives

$$\begin{aligned} \dot{V} = &\sum_{i=2}^r \Delta_i \mathcal{R}^T(\widetilde{z}_i(t))\{(I_{N_i} \otimes Df(s(t))) - c(\mathcal{R}(J_i) \otimes \Gamma)\} \\ &\times \mathcal{R}(\widetilde{z}_i(t)) + \sum_{i=2}^r \Delta_i \mathcal{I}^T(\widetilde{z}_i(t)) \\ &\times \{(I_{N_i} \otimes Df(s(t))) - c(\mathcal{R}(J_i) \otimes \Gamma)\}\mathcal{I}(\widetilde{z}_i(t)). \end{aligned} \tag{15}$$

From Lemma 4, it is easy to see that if $(Df(s(t)))^s - c\mathcal{R}(\lambda_i)\Gamma < 0$, then by choosing appropriate positive constants Δ_i, one can obtain that $[(I_{N_i} \otimes Df(s(t))) - c(\mathcal{R}(J_i) \otimes \Gamma)]^s < 0$. Under condition (13), system (9) is asymptotically stable for $i = 2, \ldots, r$. Therefore, by Lemmas 3 and 4, the local consensus of system (4) can be reached. The proof is completed.

Note that system (4) is linearlized around the state of a single node $s(t)$ to obtain system (6). Thus, only local consensus is ensured. If $s(t)$ does not contain any asymptotical attractor or the state x_i of each agent system is not in the neighborhood of $s(t)$, then local consensus may not be reached. The limitation of the result in Theorem 1 motivates the following study of the global properties of system (4).

IV. Global Consensus of Multi-Agent Systems in General Networks

The following result is widely used to compute the algebraic connectivity of an undirected graph.

Lemma 5: [10], [30] For an undirected graph with Laplacian matrix L, the algebraic connectivity of the network is given by

$$\lambda_2(L) = \min_{x^T \mathbf{1}_N = 0, x \neq 0} \frac{x^T L x}{x^T x}. \tag{16}$$

Let the generalized in-degree and out-degree of a node in a network be the sum of the weights of the edges pointing to or leaving from the node, respectively.

Definition 4: (Balanced Graphs [6]) The node in a directed graph G is said to be balanced if its in-degree is equal to its out-degree. A graph G is called balanced if and only if all its nodes are balanced, i.e., $\sum_{j=1}^N L_{ji} = \sum_{j=1}^N L_{ij} = 0, i = 1, 2, \ldots, N$.

In [6], the consensus problem of strongly connected balanced graphs was investigated. Let $\widehat{L} = (L + L^T)/2$. Then $\widehat{L}$ is symmetric and the sums of the entries in each row and each column are 0. Thus, $\mathbf{1}_N$ is the eigenvector associated with the simple eigenvalue 0. From Lemma 4, one has $x^T L x = x^T \widehat{L} x \geq \lambda_2(\widehat{L}) x^T x$, where $x^T \mathbf{1}_N = 0$. Now the notion of algebraic connectivity is generalized to directed graphs.

Definition 5: For a strongly connected network G with Laplacian matrix L, the general algebraic connectivity is defined to be the real number

$$a_\xi(L) = \min_{x^T \boldsymbol{\xi} = 0, x \neq 0} \frac{x^T \widehat{L} x}{x^T \Xi x} \tag{17}$$

where $\widehat{L} = (\Xi L + L^T \Xi)/2$, $\Xi = \mathrm{diag}(\xi_1, \ldots, \xi_N)$, $\xi = (\xi_1, \xi_2, \ldots, \xi_N)^T$ with $\xi_i > 0$ for $i = 1, 2, \ldots, N$ and $\sum_{i=1}^N \xi_i = 1$.

Note that if $\Xi = \eta I_N$, then $a_\xi(L) = \lambda_2(\widehat{L})$.

Definition 6: [18], [19] Let $T(\epsilon)$ be the set of matrices with real entries such that the sum of the entries in each row is equal to the real number ϵ. The set $M \in M^N(1)$ if and only if $M = I_N - \mathbf{1}_N \xi^T$ and $M \in T(0)$. The set $M^N(n) = \{\mathrm{M} = M \otimes I_n : M \in M^N(1), I_n \text{ is the } n-\text{dimensional identity matrix}\}$.

Lemma 6: [19], [20] Let $x = (x_1, x_2, \cdots, x_N)^T$, where $x_i \in R^n$, $i = 1, 2, \cdots, N$. Then the global consensus in system (4) can be reached if there exists an $\mathbf{M} \in M^N(n)$ satisfying $\|\mathbf{M}x\| \to 0$ as $t \to \infty$.

Now, define a nonnegative distance function by

$$d(x) = \|\mathbf{M}x\|^2 = x^T \mathbf{M}^T \mathbf{M} x, \mathbf{M} \in M^N(n). \tag{18}$$

From the assumptions on $\mathbf{M}$, one has $d(x) \to 0$ if and only if $\|x_i(t) - \bar{x}\| \to 0$ for all $i = 1, 2, \cdots, N$, where $\bar{x} = \sum_{j=1}^N \xi_j x_j(t)$ is the objective consensus state.

Assumption 1: There exist constants θ and $\varepsilon > 0$ such that

$$(x-y)^T (f(x) - f(y)) - \theta (x-y)^T \Gamma (x-y) \leq -\varepsilon (x-y)^T (x-y), \ \forall\, x, y \in R^n. \tag{19}$$

Note that the condition (19) is very mild: If $\partial f_j / \partial x_{ij}$, $i = 1, 2, \ldots, N$, $j = 1, 2, \ldots, n$, are bounded, then this condition is automatically satisfied. So systems satisfying (19) include many well-known systems, such as the Lorenz system, Chen system, Lü system, various neural networks, Chua's circuit, to name just a few.

Theorem 2: Under Assumption 1, the global consensus of system (4) can be reached if

$$\theta - c a_\xi(L) < 0. \tag{20}$$

Proof: Consider the following Lyapunov function candidate defined by the distance function in Lemma 6:

$$V(t) = \frac{1}{2} x^T \mathbf{M}^T \Xi \mathbf{M} x$$

where $\mathbf{M} = (I_N - \mathbf{1}_N \xi^T) \otimes I_n$, $\Xi = \mathrm{diag}(\xi_1, \ldots, \xi_N)$, $\xi_i > 0$, $i = 1, 2, \ldots, N$, with $\sum_{i=1}^N \xi_i = 1$, and $\mathbf{\Xi} = \Xi \otimes I_n$. Let $\bar{x} = \sum_{j=1}^N \xi_j x_j(t)$ and $\mathbf{M}x = x - \mathbf{1}_N \otimes \bar{x}$.

Taking the derivative of $V(t)$ along the trajectories of (4) gives

$$\begin{aligned}
\dot{V} &= x^T \mathbf{M}^T \mathbf{\Xi} \mathbf{M} \left[f(x(t)) - c(L \otimes \Gamma) x(t) \right] \\
&= x^T \mathbf{M}^T \mathbf{\Xi} \left[f(x(t)) - \left((\mathbf{1}_N \xi^T) \otimes I_n \right) f(x(t)) \right] \\
&\quad - c x^T \mathbf{M}^T \mathbf{\Xi} \left[(I_N - \mathbf{1}_N \xi^T) \otimes I_n \right] (L \otimes \Gamma) x(t) \\
&= x^T \mathbf{M}^T \mathbf{\Xi} \left[f(x(t)) - \mathbf{1}_N \otimes f(\bar{x}) \right] - c x^T \mathbf{M}^T \mathbf{\Xi} (L \otimes \Gamma) \\
&\quad \times \left[(I_N - \mathbf{1}_N \xi^T) \otimes I_n \right] x(t) + x^T \mathbf{M}^T \mathbf{\Xi} \\
&\quad \times \left[\mathbf{1}_N \otimes f(\bar{x}) - \left((\mathbf{1}_N \xi^T) \otimes I_n \right) f(x(t)) \right] \\
&\quad + c x^T \mathbf{M}^T \mathbf{\Xi} \left[(\mathbf{1}_N \xi^T) \otimes I_n \right] (L \otimes \Gamma) x(t).
\end{aligned} \tag{21}$$

The third equality is satisfied due to the fact that $(L \otimes \Gamma)[(\mathbf{1}_N \xi^T) \otimes I_n] = 0$, since $\mathbf{1}_N$ is the right eigenvector of L associated with eigenvalue 0. From $\xi^T \mathbf{1}_N = 1$, one has

$$\begin{aligned}
&x^T \mathbf{M}^T \mathbf{\Xi} \left[\mathbf{1}_N \otimes f(\bar{x}) \right] \\
&= \left[\mathbf{1}_N^T \otimes f^T(\bar{x}) \right] (\Xi \otimes I_n) \left[(I_N - \mathbf{1}_N \xi^T) \otimes I_n \right] x(t) \\
&= \left\{ \left[\xi^T (I_N - \mathbf{1}_N \xi^T) \right] \otimes f^T(\bar{x}) \right\} x(t) = 0
\end{aligned} \tag{22}$$

$$\begin{aligned}
&x^T \mathbf{M}^T \mathbf{\Xi} \left((\mathbf{1}_N \xi^T) \otimes I_n \right) f(x(t)) \\
&= \left\{ \left[\xi \xi^T (I_N - \mathbf{1}_N \xi^T) \right] \otimes f^T(\bar{x}) \right\} x(t) = 0
\end{aligned} \tag{23}$$

and

$$x^T \mathbf{M}^T \mathbf{\Xi} \left((\mathbf{1}_N \xi^T) \otimes I_n \right) (L \otimes \Gamma) x(t) = 0. \tag{24}$$

Combining (21)–(24), one obtains

$$\begin{aligned}
\dot{V} &= x^T \mathbf{M}^T \mathbf{\Xi} \left[f(x(t)) - \mathbf{1}_N \otimes f(\bar{x}) \right] \\
&\quad - c x^T \mathbf{M}^T \mathbf{\Xi} (L \otimes \Gamma) \mathbf{M} x(t) \\
&\leq -\varepsilon x^T \mathbf{M}^T \mathbf{\Xi} \mathbf{M} x(t) + \theta x^T \mathbf{M}^T \mathbf{\Xi} \mathbf{\Gamma} \mathbf{M} x(t) \\
&\quad - c x^T \mathbf{M}^T \mathbf{\Xi} (L \otimes \Gamma) \mathbf{M} x(t) \\
&= -\varepsilon x^T \mathbf{M}^T \mathbf{\Xi} \mathbf{M} x(t) + x^T \mathbf{M}^T \left[(\theta \Xi - c \Xi L) \otimes \Gamma \right] \mathbf{M} x(t) \\
&\leq -\varepsilon x^T \mathbf{M}^T \mathbf{\Xi} \mathbf{M} x(t) + (\theta - c a_\xi(L)) \\
&\quad \times x^T \mathbf{M}^T \mathbf{\Xi} \mathbf{\Gamma} \mathbf{M} x(t).
\end{aligned} \tag{25}$$

Under condition (20), global consensus of system (4) is reached. This completes the proof.

Remark 1: Note that the condition (20) in Theorem 2 can also be used to check consensus in networks with time-varying topologies. In this case, condition (20) can be modified as $\theta - c a_\xi(L(t)) < 0, \forall t > 0$. Taking control input $u_i = -c \sum_{j=1}^N L_{ij} \Gamma x_j(t)$ and from Assumption 1 in (19), system (2) can reach consensus by controlling some states with positive $\gamma_j > 0, j = 1, \ldots, N$ and choosing a control gain θ. The derived condition (20) in Theorem 2 reveals how the network topology can resist the nonlinear dynamics and thus affects the group collective behavior in networks.

It is still not straightforward to verify whether the condition in (20) is satisfied by a properly chosen positive vector ξ. If $\theta < 0$, then it is possible that $a_\xi(L) = 0$; if $\theta = 0$, the condition depends only on $a_\xi(L)$; and if $\theta > 0$, then $a_\xi(L) > 0$ must be satisfied. From condition (20), global consensus can be reached even if the network is disconnected with $\theta < 0$. For periodic and chaotic nodes with $\theta > 0$, one may be interested in the condition under which $a_\xi(L) > 0$. However, when is it possible to have $a_\xi(L) > 0$? In what follows, an answer is given to this question.

Lemma 7: (Theorem 8.4.4 in [31]) Suppose that A is irreducible and nonnegative. Then there is a positive vector x such that $Ax = \rho(A)x$, where $\rho(A)$ is the spectral radius of matrix A.

Lemma 8: Suppose that the Laplacian matrix L is irreducible. Then there is a positive vector x such that $L^T x = 0$.

Proof: Choose a positive integer l such that $l - \lambda_N(L) > 0$ and $l - L_{ii} > 0$ for all $i = 1, 2, \ldots, N$. Then matrix $lI_N - L$ is positive definite and, from Lemma 1, $\rho(lI_N - L) = l$. The matrix $(lI_N - L)^T = lI_N - L^T$ is also positive definite and $\rho(lI_N - L^T) = l$. By Lemma 7, there is a positive vector x such that $(lI_N - L^T)x = lx$, and one obtains $L^T x = 0$. The proof is thus completed.

Lemma 9: Suppose that the Laplacian matrix L is irreducible. Then there exists a positive-definite diagonal matrix $\Xi = \mathrm{diag}(\xi_1, \xi_2, \ldots, \xi_N)$, such that $\widehat{L} = (1/2)(\Xi L + L^T \Xi)$ is symmetric and $\sum_{j=1}^N \widehat{L}_{ij} = 0, i = 1, 2, \ldots, N$.

Proof: By Lemma 8, the proof can be completed by similar analysis in [21].

Lemma 10: [21] Suppose that the matrix $\widehat{L}$ is symmetric and irreducible, and satisfies $\sum_{j=1}^N \widehat{L}_{ij} = 0$ with $\widehat{L}_{ij} \leq 0, i \neq j, i, j = 1, 2, \ldots, N$. Let

$$\widehat{a}_\xi(\widehat{L}) = \min_{x^T \boldsymbol{\xi} = 0, x \neq 0} \frac{x^T \widehat{L} x}{x^T x}. \tag{26}$$

Then $\lambda_2(\widehat{L}) \geq \widehat{a}_\xi(\widehat{L}) \geq 0$. In addition, $\widehat{a}_\xi(\widehat{L}) = 0$ if and only if ξ is orthogonal to the left eigenvector of $\widehat{L}$ associated with eigenvalue 0; $\widehat{a}_\xi(\widehat{L}) = \lambda_2(\widehat{L})$ if ξ is the left eigenvector of $\widehat{L}$ associated with eigenvalue 0.

Corollary 2: [21] If the Laplacian matrix L is irreducible, then $a_\xi(L) > 0$, where the chosen positive vector ξ satisfies $\xi^T L = 0$.

Lemma 11: [21] The general algebraic connectivity of a strongly connected network can be computed by the following LMI:

$$\max \delta \qquad \text{Subject to } Q^T(\widehat{L} - \delta \Xi)Q \geq 0 \tag{27}$$

where $Q = \begin{pmatrix} I_{N-1} \\ -(\widehat{\xi}^T/\xi_N) \end{pmatrix} \in R^{N\times(N-1)}$ and $\widehat{\xi} = (\xi_1, \ldots, \xi_{N-1})^T$.

Remark 2: The definition of the algebraic connectivity presented here is motivated by a similar definition in [11], where it is assumed that $\xi^T L = 0$. In this technical note, a more general case is considered, where ξ is a positive vector. Based on this general definition of the algebraic connectivity, the above theoretical analysis for reaching global consensus can be carried out.

Next, analysis on the global consensus is presented assuming that graph G has a spanning tree. Let the Laplacian matrix L of graph G be in its Frobenius normal form [32]

$$L = \begin{pmatrix} \widetilde{L}_{11} & 0 & \ldots & 0 \\ \widetilde{L}_{21} & \widetilde{L}_{22} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \widetilde{L}_{p1} & \widetilde{L}_{p2} & \ldots & \widetilde{L}_{pp} \end{pmatrix} \tag{28}$$

where $\widetilde{L}_{kk} \in R^{m_k m_k}$ is irreducible for all $k = 1, 2, \ldots, p$. Matrix (28) can be interpreted as follows: the nodes and their adjacent edges in $\widetilde{L}_{kk}$ constitute an irreducible subgraph of G, and $\widetilde{L}_{kj} (j < k)$ represents the influence from subgraph $\widetilde{L}_{jj}$ to subgraph $\widetilde{L}_{kk}$.

Definition 7: [32] Let $\mathcal{G}$ be a directed network and let $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_p$ be the strongly connected components of $\mathcal{G}$ with connection matrices $\widetilde{L}_{11}, \widetilde{L}_{22}, \ldots, \widetilde{L}_{pp}$. $\mathcal{G}^*$ is a *condensation network* of $\mathcal{G}$ if there is a connection from a vertex in $\mathcal{V}(\mathcal{G}_j)$ to a vertex in $\mathcal{V}(\mathcal{G}_i) (i \neq j)$, then the weight $G^*_{ij} > 0$; otherwise, $G^*_{ij} = 0$ for $i, j = 1, 2, \ldots, p$; $G^*_{ii} = 0$ for $i = 1, 2, \ldots, p$.

Note that the condensation network $\mathcal{G}^*$ of a directed network $\mathcal{G}$ has no closed directed walks [32].

Lemma 12: [21] For every $i = 2, 3, \ldots, p$, there is an integer $j < i$ such that $G^*_{ij} > 0$ if and only if the directed network $\mathcal{G}$ contains a directed spanning tree.

Let $\widetilde{L}_{kk} = \widetilde{A}^k + \widetilde{D}^k$, where $\sum_{j=1}^{m_k} \widetilde{A}^k_{ij} = 0$ and $\widetilde{D}^k$ is a diagonal matrix for all $i = 1, 2, \ldots, m_k$. From Lemma 12, it is easy to see that $\widetilde{D}^k \geq 0$ and $\widetilde{D}^k \neq 0$ for all $k = 2, \ldots, p$.

Lemma 13: [26] If L is irreducible, $L_{ij} = L_{ji} \geq 0$ for $i \neq j$, and $\sum_{j=1}^N L_{ij} = 0$, for all $i = 1, 2, \ldots, N$, then all eigenvalues of the matrix

$$\begin{pmatrix} L_{11} + \varepsilon & L_{12} & \ldots & L_{1N} \\ L_{21} & L_{22} & \ldots & L_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ L_{N1} & L_{N2} & \ldots & L_{NN} \end{pmatrix}$$

are positive for any positive constant ε.

Definition 8: For a network with a directed spanning tree and the Laplacian matrix in the form of (28), the general algebraic connectivity of the ith strongly connected component $(2 \leq i \leq p)$ is defined to be the real number

$$\begin{aligned} b_{\widetilde{\xi}_i}(\widetilde{L}_{ii}) &= \min_{x\neq 0} \frac{x^T \widehat{\widetilde{L}}_{ii} x}{x^T \widetilde{\Xi}_i x} \\ &= \min_{x \neq 0} \frac{(\sqrt{\widetilde{\Xi}_i} x)^T \sqrt{\widetilde{\Xi}_i}^{-1} \widehat{\widetilde{L}}_{ii} \sqrt{\widetilde{\Xi}_i}^{-1} (\sqrt{\widetilde{\Xi}_i} x)}{(\sqrt{\widetilde{\Xi}} x)^T (\sqrt{\widetilde{\Xi}_i} x)} \\ &= \lambda_{\min} \sqrt{\widetilde{\Xi}_i}^{-1} \widehat{\widetilde{L}}_{ii} \sqrt{\widetilde{\Xi}_i}^{-1} \end{aligned} \tag{29}$$

where $\widehat{\widetilde{L}}_{ii} = (\widetilde{\Xi}_i \widetilde{L}_i + \widetilde{L}_i^T \widetilde{\Xi}_i)/2$, $\widetilde{\Xi}_i = \mathrm{diag}(\widetilde{\xi}_{i1}, \ldots, \widetilde{\xi}_{im_i})$, $\sqrt{\widetilde{\Xi}_i} = \mathrm{diag}(\sqrt{\widetilde{\xi}_{i1}}, \ldots, \sqrt{\widetilde{\xi}_{im_i}}) \widetilde{\xi}_i = (\widetilde{\xi}_{i1}, \ldots, \widetilde{\xi}_{im_i})^T > 0$, and $\widetilde{\xi}_i^T \widetilde{A}^i = 0$, $\sum_{j=1}^{m_i} \widetilde{\xi}_{ij} = 1$.

Lemma 14: If the Laplacian matrix L has a directed spanning tree, then $\min_{2\leq j\leq p}\{a_\xi(\widetilde{L}_{11}), b_{\widetilde{\xi}_i}(\widetilde{L}_{ii})\} > 0$, where the chosen positive vector ξ in $a_\xi(\widetilde{L}_{11})$ satisfies $\xi^T \widetilde{L}_{11} = 0$ and the positive vectors $\widetilde{\xi}_i$ in $b_{\widetilde{\xi}_i}(\widetilde{L}_{ii})$ satisfy $\widetilde{\xi}_i^T \widetilde{A}^i = 0$ for $i = 2, \ldots, p$.

Proof: From Corollary 2, one knows that $a_\xi(\widetilde{L}_{11}) > 0$. It suffices to prove that $\min_{2\leq j \leq p} b_{\widetilde{\xi}_j}(\widetilde{L}_{jj}) > 0$. Note that

$$\begin{aligned} b_{\widetilde{\xi}_i}(\widetilde{L}_{ii}) &= \lambda_{\min} \sqrt{\widetilde{\Xi}_i}^{-1} \widehat{\widetilde{L}}_{ii} \sqrt{\widetilde{\Xi}_i}^{-1} \\ &= \lambda_{\min} \sqrt{\widetilde{\Xi}_i}^{-1} \left(\frac{\widetilde{\Xi}_i \widetilde{A}^i + \widetilde{A}^{iT} \widetilde{\Xi}_i}{2} + \widetilde{D}^i \widetilde{\Xi}_i \right) \sqrt{\widetilde{\Xi}_i}^{-1} \end{aligned}$$

where $(1/2)(\widetilde{\Xi}_i \widetilde{A}^i + \widetilde{A}^{iT} \widetilde{\Xi}_i)$ is a zero sums symmetric matrix and $\widetilde{D}^i \geq 0$. By Lemma 12, there is at least one positive diagonal entry in $\widetilde{D}^i$. According to Lemma 13, $b_{\widetilde{\xi}_i}(\widetilde{L}_{ii}) > 0$ for $2 \leq i \leq p$.

Theorem 3: Suppose that Assumption 1 holds and graph G has a spanning tree. Then global consensus of system (4) can be reached if

$$\theta - c \min_{2\leq j\leq p} \left\{ a_{\widetilde{\xi}}(\widetilde{L}_{11}), b_{\widetilde{\xi}_j}(\widetilde{L}_{jj}) \right\} < 0. \tag{30}$$

Proof: From (28), one knows that $\widetilde{L}_{11}$ is irreducible, and in view of Theorem 2, the consensus of agents in the subgraph $\widetilde{L}_{11}$ can be reached. Suppose that agents $1, \ldots, m_1$ are synchronized to the state of the following system:

$$\dot{s}(t) = f(s(t)) + \mathcal{O}(e^{-\epsilon t}) \tag{31}$$

where ϵ is a positive constant. First, one has

$$\begin{aligned} \dot{x}_i(t) &= f(x_i(t)) - c \sum_{j=1}^N L_{ij} x_j(t) \\ &= f(x_i(t)) - c \sum_{j=1}^{q_{k-1}} L_{ij} \Gamma x_j(t) \\ &\quad - c \sum_{j=q_{k-1}+1}^{q_{k-1}+m_k} L_{ij} \Gamma x_j(t) \end{aligned} \tag{32}$$

where $q_k = m_1 + \ldots + m_k$. Subtracting (31) from (32) yields the following error dynamical system:

$$\dot{e}_i(t) = f(x_i(t)) - f(s(t)) - c \sum_{j=1}^{q_k} L_{ij} \Gamma [x_j(t) - s(t)] + \mathcal{O}(e^{-\epsilon t}) \tag{33}$$

where $i = q_{k-1} + 1, \ldots, q_{k-1} + m_k$ and $e_i(t) = x_i(t) - s(t)$. Choose the following Lyapunov function candidate:

$$V(t) = \frac{1}{2} \sum_{k=2}^p \sum_{i=q_{k-1}+1}^{q_{k-1}+m_k} \Delta_k \widetilde{\xi}_{ki} e_i^T(t) e_i(t) \tag{34}$$

where Δ_k are positive constants to be determined and $\widetilde{\xi}_{ki}$ are defined in Definition 8, $k = 2, \ldots, p$ and $i = q_{k-1} + 1, \ldots, q_{k-1} + m_k$.

The derivative of $V(t)$ along the trajectories (33) gives

$$\begin{aligned}\dot{V}(t) &= \sum_{k=2}^{p}\sum_{i=q_{k-1}+1}^{q_{k-1}+m_k}\Delta_k\widetilde{\xi}_{ki}e_i^T(t)\\ &\quad\times\left[f(x_i(t))-f(s(t))-c\sum_{j=1}^{q_k}L_{ij}\Gamma e_j(t)+\mathcal{O}(e^{-\epsilon t})\right]\\ &\le \sum_{k=2}^{p}\left\{\left(\theta-cb_{\widetilde{\xi}_k}(\widetilde{L}_{kk})\right)\widetilde{e}_k^T(t)(\widetilde{\Xi}_k\otimes\Gamma)\widetilde{e}_k(t)\right.\\ &\qquad\left.-c\Delta_k\sum_{l=2}^{k-1}\widetilde{e}_i^T(t)(\widetilde{\Xi}_k\widetilde{L}_{kl}\otimes\Gamma)\widetilde{e}_j(t)\right\}\\ &\quad+\mathcal{O}(e^{-\epsilon t})-\varepsilon\sum_{k=2}^{p}\widetilde{e}_k^T(t)(\widetilde{\Xi}_k\otimes I_n)\widetilde{e}_k(t)\end{aligned}\tag{35}$$

where $\widetilde{e}_k(t) = (e_{q_{k-1}+1}^T,\ldots,e_{q_{k-1}+m_k}^T)^T$ and $\widetilde{\Xi}_k = \mathrm{diag}(\widetilde{\xi}_{k1},\ldots,\widetilde{\xi}_{km_k})$.

Under condition (30) and by Lemma 4, one has $\dot{V}(t) < 0$. Since graph G has a spanning tree, $\widetilde{L}_{kk} = \widetilde{A}^k + \widetilde{D}^k$, where the sum of the entries in $\widetilde{A}^k$ is zero, and $\widetilde{D}^k \neq 0$. Now, Lemmas 13 and 14 together lead to the conditions given in (30) where $\theta > 0$. The proof is thus completed.

V. Conclusion

In this technical note, both local and global consensus problems for multi-agent systems in directed networks have been investigated. The main contributions of this technical note include the following: i) Local consensus in a directed network of agents with nonlinear dynamics has been studied. It is found that the real part of the second smallest eigenvalue of the Laplacian matrix plays a key role in deriving the consensus conditions. ii) A generalized algebraic connectivity, which can be used to describe the consensus ability of the network, has been proposed to discuss global convergence properties of consensus in strongly connected networks and also in networks containing spanning trees. Some future works are as follows: i) Networks with nonlinear couplings and networks consisting of heterogeneous autonomous agents will be studied. ii) The multi-agent system with switching topology will be considered. iii) Based on the general algebraic connectivity discussed in this technical note, the stable consensus regions will be further investigated in multi-agent systems with nonlinear dynamics and a general inner coupling matrix Γ in the near future.

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