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E_{10} and gauged maximal supergravity

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Abstract: We compare the dynamics of maximal three-dimensional gauged supergravity in appropriate truncations with the equations of motion that follow from a one-dimensional $E_{10}/K(E_{10})$ coset model at the first few levels. The constant embedding tensor, which describes gauge deformations and also constitutes an M-theoretic degree of freedom beyond eleven-dimensional supergravity, arises naturally as an integration constant of the geodesic model. In a detailed analysis, we find complete agreement at the lowest levels. At higher levels there appear mismatches, as in previous studies. We discuss the origin of these mismatches.

Keywords: Extended Supersymmetry, Gauge Symmetry, Supergravity Models, Global Symmetries.

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Contents

1. Introduction **1**

2. The $E_{10}/K(E_{10})$ coset model **4**
- 2.1 Generalities about E_{10} 4
- 2.2 Decomposition under SL(2, $\mathbb{R}$) $\times$ E_8 5
- 2.3 The non-linear sigma model 9
- 2.4 Equations of motion 13

3. Gauged supergravity in three dimensions **15**
- 3.1 The lagrangian 15
  - 3.1.1 Equations of motion 17
  - 3.1.2 Constraints 18
  - 3.1.3 Reformulation with deformation and top-form potentials 18
- 3.2 Dimensional reduction to $D = 1$ 19
- 3.3 Beyond dimensional reduction 20

4. The supergravity/E_{10} correspondence **21**
- 4.1 Ungauged supergravity in $D = 1$ 21
- 4.2 Level $\ell = 1$ as spatial gradient 22
- 4.3 Level $\ell = 2$ and gauged supergravity 23
- 4.4 Quadratic and gauge constraints 26

5. Discussion and outlook **28**

A. Conventions for E_8 **29**

B. Level decomposition of E_{10} **31**

1. Introduction

It is well-known that the highest space-time dimension that allows a supergravity theory is eleven [1]. Upon a torus reduction to lower dimensions, eleven-dimensional supergravity [2] leads, in each space-time dimension $3 \leq D \leq 10$, to a maximal supergravity theory in which the scalars parametrize a coset manifold $G/K(G)$, where $K(G)$ is the maximal compact subgroup of G [3]. For maximal supergravity in $D = 3$ dimensions, the rigid symmetry group is the non-compact split real form of the largest exceptional Lie group E_8; all physical bosonic degrees of freedom reside in the coset space, with no propagating gravitational degrees of freedom left. This theory was already constructed long ago [4, 5]; however, its

JHEP01(2009)020

gauged versions, whose relation with the infinite-dimensional $E_{10}/K(E_{10})$ coset model will be the focus of the present paper, were obtained only much more recently [6, 7].

The different duality groups G characterizing the coset manifolds are described by Dynkin diagrams that are related to each other by deleting nodes (going up in dimension) or adding nodes (going down in dimension). The three-dimensional case corresponds to the group $G = E_8$ which has a Dynkin diagram with 8 nodes. It has been suggested that by reducing to even lower dimensions, $0 \leq D \leq 2$, larger symmetry algebras may emerge that correspond to Dynkin diagrams which are obtained by adding nodes to the E_8 diagram [8]. Such diagrams do not correspond to a finite number of symmetries, as in the case of ordinary Lie groups, but instead lead to an infinite number of symmetries corresponding to the infinite-dimensional groups E_9 ($D = 2$), E_{10} ($D = 1$) and E_{11} ($D = 0$), respectively.

It has been conjectured that maximal supergravity in any dimension $D \leq 11$, independent of any torus reduction, can be described in terms of E_{11} [9 – 11]. While this conjecture works well (at low levels) as far as the kinematics is concerned, yielding the correct bosonic multiplets of various maximal supergravities upon decomposition of E_{11} under its finite-dimensional subalgebras, the underlying dynamics is much less understood. In this paper, we will therefore follow a different route, based on a conjecture proposed and elaborated in [12, 13], according to which the dynamics of any maximal supergravity theory (or some M-theoretic extension thereof) is described by the equations of motion of a one-dimensional sigma model over the coset space $E_{10}/K(E_{10})$. If these equations are supplemented by coset constraints [14], one can establish a correspondence between truncated versions of the coset equations on the one hand, and of the supergravity equations on the other. This correspondence can also be extended to the fermionic sector such that the fermionic field equations can be reformulated to be covariant under the coset model 'R symmetry' $K(E_{10})$ [15 – 17].[1]

For carrying out the comparison one has to formulate both sides of the correspondence appropriately. On the one hand one has to truncate the supergravity fields and break space-time covariance by choosing an ADM gauge, in order to be amenable to a one-dimensional language. On the E_{10} side, on the other hand, one has to perform a so-called level decomposition with respect to the subgroup $GL(D-1) \times G_D$, where G_D denotes the duality group in D dimensions. At low levels, the equations of motion of the E_{10} model precisely match the equations of motion of (pure) supergravity truncated to only a time-dependent, that is, one-dimensional system. This matching is in accord with the (duality) symmetries expected to appear in lower dimensions. However, the main challenge is to go beyond these low levels and to find an interpretation for the infinite tower of representations appearing in the level decomposition of E_{10} and E_{11} (see e.g. [20, 11]) also on the supergravity side.

As one attractive scenario it has been suggested [12, 20, 13] that the higher levels encode the spatial gradients of the supergravity fields, and so by including all of these states one should finally recover the full unrestricted supergravity in D dimensions or

[1] An approach combining ideas of the E_{10} and E_{11} approaches has been explored in [18, 19].

JHEP01(2009)020

an M-theoretic extension thereof.[2] While some intriguing confirmation has been found, certain mismatches remain, such that a conclusive picture of how to identify the spatial dependence within E_{10} and how to understand the emergence of a space-time field theory from the one-dimensional sigma model is still lacking.

Another interpretation for part of the higher levels concerns certain mass deformations of pure maximal supergravity. In [22] it has been shown that the massive Romans supergravity in ten dimensions [23], which deforms type IIA supergravity by a mass parameter m, is contained in the E_{10} model, upon taking a certain 9-form representation into account (see also [24]). For the realization of massive type IIA supergravity within the E_{11} approach see [25].

Apart from switching on spatial gradients and/or mass parameters, another direction will be explored in this paper, namely that of turning on gauge couplings. This possibility relies on the recent realization that E_{11} and E_{10} contain information about gauged supergravity via D- and $(D-1)$-form representations [26 – 29].[3] We will focus on gauged supergravity in three dimensions, but our conclusions are expected to be of general validity. The advantage of this case is that E_8 is the largest finite-dimensional duality group. As a consequence, the E_{10} equations of motion truncated to level $\ell = 0$ already match ungauged supergravity reduced to a one-dimensional system. Thus, this model allows a clear distinction between the 'manifest' aspects of the E_{10} conjecture at level $\ell = 0$ and the more speculative features related to higher levels, as spatial gradients or gauge couplings. We will find surprising correpondences between both sides, but also mismatches, which remain to be investigated further.

Let us emphasize the main features of our results, also reflecting the differences with the E_{11} approach [9, 26, 29]. These are:

- There is *no need to deform the E_{10} Lie algebra* or the E_{10} Cartan form (e.g. by modifying the derivative) in order to obtain agreement (as far as it goes) between the equations of gauged $D = 3$ supergravity and the $E_{10}/K(E_{10})$ coset model. Rather, the gauging appears exclusively as a consequence of 'switching on' certain higher level degrees of freedom in the level expansion of the Cartan form and the coset equations of motion. The relevant components of the embedding tensor are in part beyond level $\ell = 3$ in the SL(10) decomposition, hence cannot be understood via Kaluza-Klein-type compactification from $D = 11$ supergravity (as also emphasized in [26]).

- The absence of any deformation in the original coset model, in turn, is a direct consequence of the fact that the correspondence works only if we adopt the *temporal gauge* for all gauge fields, and in particular for the Chern-Simons gauge potential $\mathtt{A}_\mu{}^{\mathcal{M}}$ (generalizing the pseudo-Gaussian gauge, i.e. vanishing shift, for the gravitational degrees of freedom).

[2] In the E_{11} approach some of the higher level states can be interpreted as dual representations of lower level states [21].

[3] The D-form representations only occur in the E_{11} approach.

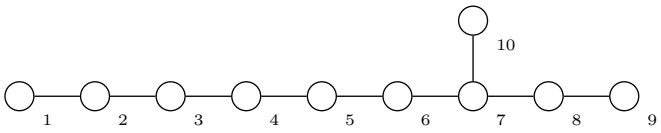

Figure 1: The Dynkin diagram of $E_{10} = {E_8}^{++}$

- We are here working in a Hamiltonian framework. This means that in addition to the coset equations of motion (which are related to the evolution equations involving time derivatives on the supergravity side) we need to impose certain canonical constraints on the coset dynamics (corresponding to constraints on the initial data on the supergravity side). The structure of these constraints was studied in [14], and we here likewise find that the constraints can be written in a Sugawara-like form in terms of the coset variables. One can also show that under (part of) E_{10} the constraints transform into one another, such that duality relates for instance the diffeomorphism constraint and the quadratic constraint of gauged supergravity. This feature is somewhat reminiscent of the $L(\Lambda_1)$ representation found in [29], but the precise relation (if any) is not clear (e.g. in [14] the constraints were found *not* to transform as a highest or lowest weight representation of the whole E_{10}).

The paper is organized as follows. In section 2 we first summarize the $E_{10}/K(E_{10})$ coset model. In particular, we derive the equations of motion at the lowest levels. In section 3 we consider maximal gauged supergravity in three dimensions and its torus reduction to one (time) dimension. Next, in section 4 we discuss the supergravity/E_{10} correspondence: its matches and mismatches. Finally, in section 5 we give our outlook on the status of the E_{10} conjecture. We include two appendices summarizing some basic properties of E_8 and the details about the level decomposition of E_{10}.

2. The $E_{10}/K(E_{10})$ coset model

In this section we introduce the $E_{10}/K(E_{10})$ coset model. In order to make contact with three-dimensional gauged supergravity it proves convenient to write the generators of E_{10} in a $\mathrm{SL}(2,\mathbb{R}) \times E_{8(8)}$ covariant form. We then analyze the one-dimensional coset model in this language and derive the associated geodesic equations.

By $\mathfrak{e}_8$ and $\mathfrak{e}_{10}$ we always mean the split real forms (also denoted $\mathfrak{e}_{8(8)}$ and $\mathfrak{e}_{10(10)}$) of the corresponding complex Lie algebras. The Lie groups obtained by exponentiation of the algebra elements are denoted E_8 and E_{10}. Sometimes the notation ${\mathfrak{e}_8}^{++}$ and ${E_8}^{++}$ is used, indicating that $\mathfrak{e}_{10}$ is the 'over-extension' of $\mathfrak{e}_8$ — the Dynkin diagram of $\mathfrak{e}_{10}$ is obtained by adding two extra nodes to that of $\mathfrak{e}_8$, as can be seen from figure 1.

2.1 Generalities about E_{10}

We first briefly summarize some basic facts about E_{10}. Its Lie algebra is characterized by the Dynkin diagram given in figure 1.

More precisely, the Lie algebra $\mathfrak{e}_{10}$ of E_{10} is defined in terms of a 10×10 Cartan matrix A_{ij} ($i, j = 1, \ldots, 10$), which can be read off from the Dynkin diagram as

$$A_{ij} = \begin{cases} 2 & \text{if } i = j, \\ -1 & \text{if there is a line between nodes } i \text{ and } j, \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

The Lie algebra is then generated by multiple commutators of the ten basic triples of generators $\{h_i, e_i, f_i\}$. The h_i are elements of the abelian Cartan subalgebra. The e_i and f_i are the positive and negative step operators. Their commutation relations (the *Chevalley relations*) read

$$[h_i, e_j] = A_{ij} e_j \ , \quad [h_i, f_j] = -A_{ij} f_j \ , \quad [e_i, f_j] = \delta_{ij} h_i \tag{2.2}$$

(no summation). The multiple commutators are constrained by the *Serre relations*

$$(\mathrm{ad}_{e_i})^{1-A_{ij}} e_j = 0 \ , \qquad (\mathrm{ad}_{f_i})^{1-A_{ij}} f_j = 0 \ . \tag{2.3}$$

Each Kac-Moody algebra admits an invariant Cartan-Killing form, which in the basis introduced above reads

$$\langle e_i | f_j \rangle \ = \ \delta_{ij} \ , \qquad \langle h_i | h_j \rangle \ = \ A_{ij} \ . \tag{2.4}$$

We note that the Cartan matrix A_{ij}, and thereby the Cartan-Killing form on the Cartan subalgebra, is of Lorentzian signature. This will later be used to define a *null-geodesic* motion on the coset space $E_{10}/K(E_{10})$. We also need the Chevalley involution ω in order to define the maximal compact subgroup $K(E_{10})$ and its Lie algebra $\mathfrak{k}(\mathfrak{e}_{10})$. The Chevalley involution is defined by

$$\omega(e_i) = -f_i \ , \quad \omega(f_i) = -e_i \ , \quad \omega(h_i) = -h_i \ . \tag{2.5}$$

One then defines the (generalized) transpose of an $\mathfrak{e}_{10}$ element x as $x^T = -\omega(x)$. The maximal compact subalgebra $\mathfrak{k}(\mathfrak{e}_{10})$ is defined as the subalgebra of $\mathfrak{e}_{10}$ that is pointwise fixed by the Chevalley involution. Thus it consists of all elements $x - x^T$. Similarly, we define the coset $\mathfrak{e}_{10} \ominus \mathfrak{k}(\mathfrak{e}_{10})$ to be the subspace consisting of all elements $x + x^T$. With respect to the Cartan-Killing form, the maximal compact subalgebra $\mathfrak{k}(\mathfrak{e}_{10})$ is negative-definite, the coset $\mathfrak{e}_{10} \ominus \mathfrak{k}(\mathfrak{e}_{10})$ is almost positive-definite (there is one negative eigenvalue of the Cartan-Killing metric in the Cartan subalgebra), and these two subspaces of $\mathfrak{e}_{10}$ are orthogonal complements to each other.

2.2 Decomposition under $\mathrm{SL}(2, \mathbb{R}) \times E_8$

Any Kac-Moody algebra can be written as a direct sum of subspaces $\mathfrak{g}_\ell$ for all integers ℓ such that

$$[\mathfrak{g}_k, \, \mathfrak{g}_\ell] \subseteq \mathfrak{g}_{k+\ell}. \tag{2.6}$$

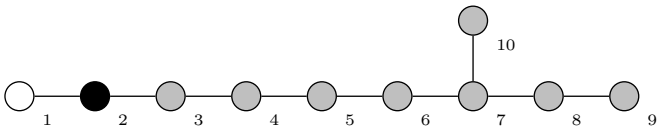

Figure 2: Level decomposition of $E_{10} = {E_8}^{++}$. The grey nodes denote the duality group E_8, the black node is the deleted one and the white node denotes the $\mathrm{SL}(2,\mathbb{R})$ spacetime subgroup.

Level ℓ	$\mathrm{SL}(2,\mathbb{R}) \times E_8$ representation	Generator	Interpretation
0	$(\mathbf{1}\oplus\mathbf{3},\mathbf{1})$	${K^a}_b$	spatial zweibein
	$(\mathbf{1},\mathbf{248})$	$t^{\mathcal{A}}$	scalars
1	$(\mathbf{2},\mathbf{248})$	${E^a}_{\mathcal{A}}$	gauge vectors
2	$(\mathbf{1},\mathbf{1})$	E	θ
	$(\mathbf{1},\mathbf{3875})$	$E_{\mathcal{AB}} = E_{(\mathcal{AB})}$	$\tilde{\Theta}_{\mathcal{MN}}$
	$(\mathbf{3},\mathbf{248})$	${E^{ab}}_{\mathcal{A}} = {E^{(ab)}}_{\mathcal{A}}$	trombone gauging?

Table 1: $\mathrm{SL}(2,\mathbb{R}) \times E_8$ representations within E_{10} up to level 2.

For $k = 0$, this gives a level decomposition of the adjoint representation of $\mathfrak{e}_{10}$ under a subalgebra $\mathfrak{g}_0$, where we call ℓ the *level* of the elements in $\mathfrak{g}_\ell$, and of the corresponding $\mathfrak{g}_0$ representation.

In order to make contact with three-dimensional supergravity we perform a level decomposition of E_{10} with respect to the subgroup of spatial diffeomorphisms and the duality group:

$$E_{10} \supset \mathrm{SL}(2,\mathbb{R}) \times E_8 \ . \tag{2.7}$$

This corresponds to deleting the black node numbered 2 in the Dynkin diagram in figure 2.

Thus we consider the case where $\mathfrak{g}_0 = \mathfrak{gl}(2,\mathbb{R}) \oplus \mathfrak{e}_8$, where the enhancement from $\mathfrak{sl}(2,\mathbb{R})$ to $\mathfrak{gl}(2,\mathbb{R})$ is due to the Cartan generator associated with the deleted node 2. The representations occurring in this level decomposition can be calculated using the computer program SimpLie [30]. Up to level $\ell < 3$ we find the $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{e}_8$ representations in table 1, where we indicated the corresponding generators with their symmetries. We denote by $a,\, b = 1,\, 2$ the fundamental indices of $GL(2,\mathbb{R})$ and by $\mathcal{A},\, \mathcal{B} = 1,\, 2 \ldots, 248$ the adjoint indices of E_8. The fields associated to the $\ell = 0$ generators are the spatial zweibein and the coset scalars. The $\ell = 1$ fields can be interpreted as gauge vectors. The interpretation of the $\ell = 2$ fields will be discussed in section 4.3 (concerning the embedding tensor components θ and $\tilde{\Theta}$), where also some speculations will be made on trombone gaugings. At the negative levels we have the conjugate representations, i.e., the transposed generators of those at the positive levels.

Later we will split the E_8 indices as

$$\mathcal{A} \ \rightarrow \ [IJ],\ A, \tag{2.8}$$

where $I,\, J = 1,\, 2,\, \ldots,\, 16$ and $A = 1,\, 2,\, \ldots,\, 128$ are vector and spinor indices, respectively, of the maximal compact subalgebra $\mathfrak{k}(\mathfrak{e}_8) = \mathfrak{so}(16)$. This is in accordance with the following

decomposition of the adjoint $\mathfrak{e}_8$ representation under the $\mathfrak{so}(16)$ subalgebra

$$\mathbf{248} \to \mathbf{120} + \mathbf{128}\,. \tag{2.9}$$

As indicated in table 1, the generator $E_{\mathcal{AB}}$ is symmetric in the two adjoint E_8 indices. However, it also has to satisfy further conditions in order to belong to the **3875** representation; in particular it must be traceless. The necessary and sufficient condition for this can be expressed as

$$\mathbb{P}_{\mathcal{AB}}{}^{\mathcal{CD}} E_{\mathcal{CD}} = E_{\mathcal{AB}}, \tag{2.10}$$

where the explicit form of the projector $\mathbb{P}_{\mathcal{AB}}{}^{\mathcal{CD}}$ has been determined in [31] and reads

$$\mathbb{P}_{\mathcal{AB}}{}^{\mathcal{CD}} = \frac{1}{7}\delta_{(\mathcal{A}}{}^{\mathcal{C}}\delta_{\mathcal{B})}{}^{\mathcal{D}} - \frac{1}{56}\eta_{\mathcal{AB}}\eta^{\mathcal{CD}} - \frac{1}{14} f^{\mathcal{E}}{}_{\mathcal{A}}{}^{(\mathcal{C}} f_{\mathcal{EB}}{}^{\mathcal{D})}. \tag{2.11}$$

Here f and η denote the E_8 structure constants and the components of the Killing form, respectively. These are given explicitly in appendix A.

At level $\ell = 0$ we find a singlet plus the adjoint of $\mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{e}_8$. The first part, $(\mathbf{1}\oplus\mathbf{3},\mathbf{1})$, can be seen as the adjoint of $\mathfrak{gl}(2,\mathbb{R})$. The $\ell = 0$ subalgebra reads

$$[t^{\mathcal{A}},\, t^{\mathcal{B}}] = f^{\mathcal{AB}}{}_{\mathcal{C}} t^{\mathcal{C}}, \qquad\qquad [K^a{}_b,\, K^c{}_d] = \delta^c{}_b K^a{}_d - \delta^a{}_d K^c{}_b. \tag{2.12}$$

The Lie brackets that do not mix between positive and negative levels are entirely fixed by representation theory and the graded structure (2.6). The commutators involving the $\ell = 0$ generators just give the transformation character of the $|\ell| = 1, 2$ generators under $\mathfrak{gl}(2,\mathbb{R}) \oplus \mathfrak{e}_8$. Since the generators at the negative levels transform in the conjugate representations compared to the positive levels, they have their $\mathfrak{sl}(2,\mathbb{R})$ indices downstairs instead. However, the position of the E_8 indices is arbitrary in the definition of the generators, since they can be raised and lowered by means of the $\mathfrak{e}_8$ Killing form η, which we describe in (A.2). We here define the generators on the negative levels by the following action of the Chevalley involution:

$$\omega(E^a{}_{\mathcal{A}}) = -F_a{}^{\mathcal{A}} \tag{2.13}$$

at level $\ell = -1$ and

$$\omega(E_{\mathcal{AB}}) = -F^{\mathcal{AB}}, \qquad \omega(E) = -F, \qquad \omega(E^{ab}{}_{\mathcal{A}}) = -F_{ab}{}^{\mathcal{A}} \tag{2.14}$$

at level $\ell = -2$. We recall that the transpose then is defined as $x^T = -\omega(x)$.

The commutators involving level zero are now given by

$$\begin{aligned}
[t^{\mathcal{A}},\, E^a{}_{\mathcal{B}}] &= f^{\mathcal{A}}{}_{\mathcal{B}}{}^{\mathcal{C}} E^a{}_{\mathcal{C}}, & [t^{\mathcal{A}},\, F_a{}^{\mathcal{B}}] &= f^{\mathcal{AB}}{}_{\mathcal{C}} F_a{}^{\mathcal{C}},\\
[K^a{}_b,\, E^c{}_{\mathcal{A}}] &= \delta^c{}_b E^a{}_{\mathcal{A}}, & [K^a{}_b,\, F_c{}^{\mathcal{A}}] &= -\delta^a{}_c F_b{}^{\mathcal{A}},\\
[t^{\mathcal{A}},\, E_{\mathcal{BC}}] &= 2 f^{\mathcal{A}}{}_{\mathcal{B}}{}^{\mathcal{D}} E_{\mathcal{CD}}, & [t^{\mathcal{A}},\, F^{\mathcal{BC}}] &= 2 f^{\mathcal{AB}}{}_{\mathcal{D}} F^{\mathcal{CD}},\\
[t^{\mathcal{A}},\, E^{cd}{}_{\mathcal{B}}] &= f^{\mathcal{A}}{}_{\mathcal{B}}{}^{\mathcal{D}} E^{cd}{}_{\mathcal{D}}, & [t^{\mathcal{A}},\, F_{cd}{}^{\mathcal{B}}] &= f^{\mathcal{AB}}{}_{\mathcal{D}} F_{cd}{}^{\mathcal{D}},\\
[K^a{}_b,\, E] &= \delta^a{}_b E, & [K^a{}_b,\, F] &= -\delta^a{}_b F,\\
[K^a{}_b,\, E_{\mathcal{AB}}] &= \delta^a{}_b E_{\mathcal{AB}}, & [K^a{}_b,\, F^{\mathcal{AB}}] &= -\delta^a{}_b F^{\mathcal{AB}},\\
[K^a{}_b,\, E^{cd}{}_{\mathcal{A}}] &= 2\delta^c{}_b E^{ad}{}_{\mathcal{A}}, & [K^a{}_b,\, F_{cd}{}^{\mathcal{A}}] &= -2\delta^a{}_c F_{bd}{}^{\mathcal{A}}.
\end{aligned} \tag{2.15}$$

Here and troughout this paper, we use the convention of implicit (anti-)symmetrization in indices. This means that the right hand side of any equation is always assumed to be (anti-)symmetrized according to the left hand side. In (2.15) this convention concerns the generators $E_{\mathcal{AB}}$ and $E^{ab}{}_{\mathcal{A}}$ at level $\ell = 2$ (and their transposes at level $\ell = -2$), which are symmetric in the E_8 and SL(2, $\mathbb{R}$) indices, respectively (cf. table 1). For example, the last equation in (2.15) should be read as

$$[K^a{}_b,\, F_{cd}{}^{\mathcal{A}}] = -\delta^a{}_c F_{bd}{}^{\mathcal{A}} - \delta^a{}_d F_{bc}{}^{\mathcal{A}}. \tag{2.16}$$

Later, when we split the E_8 indices as in (2.8), this convention will also concern antisymmetric pairs $[IJ]$ of SO(16) vector indices.

We define the generators at level $|\ell| = 2$ by the commutation relations

$$\begin{aligned}
[E^a{}_{\mathcal{A}},\, E^b{}_{\mathcal{B}}] &= \frac{1}{2}\varepsilon^{ab}\eta_{\mathcal{AB}}E + \varepsilon^{ab}E_{\mathcal{AB}} - f_{\mathcal{AB}}{}^{\mathcal{C}}E^{ab}{}_{\mathcal{C}}, \\
[F_a{}^{\mathcal{A}},\, F_b{}^{\mathcal{B}}] &= -\frac{1}{2}\varepsilon_{ab}\eta^{\mathcal{AB}}F - \varepsilon_{ab}F^{\mathcal{AB}} - f^{\mathcal{AB}}{}_{\mathcal{C}}F_{ab}{}^{\mathcal{C}}.
\end{aligned} \tag{2.17}$$

We will see below that this normalization is a convenient choice. Note that both equations have a minus sign on the last term, but otherwise opposite signs on the right hand side. This is necessary if we want $F^{\mathcal{AB}}$ to be the transpose of $E_{\mathcal{AB}}$, that is, if we want to obtain (2.14) from (2.13) using the homomorphism property of ω. The reason is that $f^{\mathcal{AB}}{}_{\mathcal{C}} = -f_{\mathcal{AB}}{}^{\mathcal{C}}$ for the $\mathfrak{e}_8$ structure constants (see appendix A), whereas $\eta^{\mathcal{AB}} = \eta_{\mathcal{AB}}$ and $\delta^{\mathcal{A}}{}_{\mathcal{C}}\delta^{\mathcal{B}}{}_{\mathcal{D}} = \delta_{\mathcal{A}}{}^{\mathcal{C}}\delta_{\mathcal{B}}{}^{\mathcal{D}}$.

As we show in appendix B the Chevalley-Serre relations (2.2) and (2.3) lead to

$$[E^a{}_{\mathcal{A}},\, F_b{}^{\mathcal{B}}] = \delta^a{}_b f_{\mathcal{A}}{}^{\mathcal{B}}{}_{\mathcal{C}}t^{\mathcal{C}} + \delta_{\mathcal{A}}{}^{\mathcal{B}}K^a{}_b - \delta_{\mathcal{A}}{}^{\mathcal{B}}\delta^a{}_b K, \tag{2.18}$$

where we have set

$$K = K^a{}_a = K^1{}_1 + K^2{}_2. \tag{2.19}$$

The remaining non-zero commutation relations up to level $|\ell| = 2$ can be derived from those above by the Jacobi identity. For completeness they are also given in appendix B.

We must define the Cartan-Killing form for the generators at level $|\ell| \leq 2$ in a way such that (2.4) is satisfied after identifying the generators in the Chevalley basis (see appendix B). This is achieved by the following normalization at level zero:

$$\langle K^a{}_b | K^c{}_d\rangle = \delta^a{}_d\delta^c{}_b - \delta^a{}_b\delta^c{}_d, \qquad \langle t^{\mathcal{A}} | t^{\mathcal{B}}\rangle = \eta^{\mathcal{AB}}, \qquad \langle K^a{}_b | t^{\mathcal{A}}\rangle = 0, \tag{2.20}$$

which gives back the Cartan-Killing form for $\mathfrak{e}_8$. For the levels $|\ell| = 1, 2$ we now get

$$\begin{aligned}
\langle E^a{}_{\mathcal{A}} | F_b{}^{\mathcal{B}}\rangle &= \delta^a{}_b\delta_{\mathcal{A}}{}^{\mathcal{B}}, & \langle E_{\mathcal{AB}} | F^{\mathcal{CD}}\rangle &= 14\,\mathbb{P}_{\mathcal{AB}}{}^{\mathcal{CD}}, \\
\langle E | F\rangle &= 1, & \langle E^{ab}{}_{\mathcal{A}} | F_{cd}{}^{\mathcal{B}}\rangle &= \delta^a{}_c\delta^b{}_d\delta_{\mathcal{A}}{}^{\mathcal{B}},
\end{aligned} \tag{2.21}$$

and zero elsewhere, using the invariance of the bilinear form.

Taking x to be a basis element of $\mathfrak{e}_{10}$ in the expressions $x - x^T$ and $x + x^T$, we obtain bases of $\mathfrak{k}(\mathfrak{e}_{10})$ and the coset $\mathfrak{e}_{10} \ominus \mathfrak{k}(\mathfrak{e}_{10})$, respectively. On the $\mathfrak{e}_8$ subalgebra the Chevalley involution acts as $\omega(t^{\mathcal{A}}) = -t_{\mathcal{A}} = -\eta_{\mathcal{AB}}t^{\mathcal{B}}$. The transpose is then given by

$$(t_{\mathcal{A}})^T = t^{\mathcal{A}}. \tag{2.22}$$

On the $\mathfrak{sl}(2,\mathbb{R})$ subalgebra, the transpose is just the ordinary transpose,

$$(K^a{}_b)^T = K^b{}_a. \tag{2.23}$$

Thus at level zero we define

$$\mathcal{J}_{IJ} = t_{IJ} - t^{IJ} = -2t^{IJ}, \qquad \mathcal{J}^{ab} = K^a{}_b - K^b{}_a \tag{2.24}$$

as basis elements of $\mathfrak{k}(\mathfrak{e}_{10}) = \mathfrak{so}(16)$ and $\mathfrak{k}(\mathfrak{sl}(2,\mathbb{R})) = \mathfrak{so}(2)$, respectively, which are the level zero subalgebras of $\mathfrak{k}(\mathfrak{e}_{10})$. Likewise, we define

$$\mathcal{S}_A = t_A + t^A = 2t^A, \qquad \mathcal{S}^{ab} = K^a{}_b + K^b{}_a \tag{2.25}$$

as basis elements of the coset $\mathfrak{e}_{10} \ominus \mathfrak{k}(\mathfrak{e}_{10})$ at level zero. Note that there is no $\mathcal{J}_A$ or $\mathcal{S}_{IJ}$; *the indices on $\mathcal{J}_{IJ}$ and $\mathcal{S}_A$ should not be considered as split E_8 indices, but as pure* SO(16) *indices.* This means that we raise the vector indices $I, J, \ldots$ with the invariant SO(16) metric δ^{IJ}, so that $\mathcal{J}_{IJ} = \mathcal{J}^{IJ}$. On the other hand, $t_{IJ} = -t^{IJ}$, since we consider t^{IJ} as an $\mathfrak{e}_8$ element. (For the spinor indices A, B, ..., upstairs and downstairs does not matter.)

Leaving level zero, the basis elements of $\mathfrak{k}(\mathfrak{e}_{10})$ and the coset will mix between positive and negative levels so the graded structure (2.6) will not be preserved,

$$\begin{aligned}
\mathcal{S}^a{}_{\mathcal{A}} &= E^a{}_{\mathcal{A}} + F_a{}^{\mathcal{A}}, & \mathcal{S} &= E + F, \\
\mathcal{S}^{ab}{}_{\mathcal{A}} &= E^{ab}{}_{\mathcal{A}} + F_{ab}{}^{\mathcal{A}} & \mathcal{S}_{\mathcal{AB}} &= E_{\mathcal{AB}} + F^{\mathcal{AB}}. \\
\mathcal{J}^a{}_{\mathcal{A}} &= E^a{}_{\mathcal{A}} - F_a{}^{\mathcal{A}}, & \mathcal{J} &= E - F, \\
\mathcal{J}^{ab}{}_{\mathcal{A}} &= E^{ab}{}_{\mathcal{A}} - F_{ab}{}^{\mathcal{A}} & \mathcal{S}_{\mathcal{AB}} &= E_{\mathcal{AB}} - F^{\mathcal{AB}}.
\end{aligned} \tag{2.26}$$

Computing the Cartan-Killing norm for these basis elements,

$$\begin{aligned}
\langle \mathcal{S}_A | \mathcal{S}_B \rangle &= 4\delta_{AB}, & \langle \mathcal{J}_{IJ} | \mathcal{J}_{KL} \rangle &= -8\delta_{IK}\delta_{JL}, \\
\langle \mathcal{S}^{ab} | \mathcal{S}^{cd} \rangle &= 4(\delta^{ac}\delta^{bd} - \delta^{ab}\delta^{cd}), & \langle \mathcal{J}^{ab} | \mathcal{J}^{cd} \rangle &= -4\delta^{ac}\delta^{bd}, \\
\langle \mathcal{S}^a{}_{\mathcal{A}} | \mathcal{S}^b{}_{\mathcal{B}} \rangle &= -\langle \mathcal{J}^a{}_{\mathcal{A}} | \mathcal{J}^b{}_{\mathcal{B}} \rangle = 2\delta^{ab}\delta_{\mathcal{A}}{}^{\mathcal{B}}, & \langle \mathcal{S}_{\mathcal{AB}} | \mathcal{S}_{\mathcal{CD}} \rangle &= -\langle \mathcal{J}_{\mathcal{AB}} | \mathcal{J}_{\mathcal{CD}} \rangle = 28\mathbb{P}_{\mathcal{AB}}{}^{\mathcal{CD}}, \\
\langle \mathcal{S}^{ab}{}_{\mathcal{A}} | \mathcal{S}^{cd}{}_{\mathcal{B}} \rangle &= -\langle \mathcal{J}^{ab}{}_{\mathcal{A}} | \mathcal{J}^{cd}{}_{\mathcal{B}} \rangle = 2\delta^{ac}\delta^{bd}\delta_{\mathcal{A}}{}^{\mathcal{B}}, & \langle \mathcal{S} | \mathcal{S} \rangle &= -\langle \mathcal{J} | \mathcal{J} \rangle = 2,
\end{aligned} \tag{2.27}$$

we see that the subspace $\mathfrak{k}(\mathfrak{e}_{10})$ is negative-definite and that $\mathfrak{e}_{10} \ominus \mathfrak{k}(\mathfrak{e}_{10})$ is positive-definite away from level zero. Although some of the equations above are written in E_8 indices, for convenience, the position of the indices shows that they are in fact not E_8 covariant. The E_8 indices must be split into SO(16) indices in order to give covariant equations.

2.3 The non-linear sigma model

Following [12, 13] we now introduce a one-dimensional non-linear sigma-model based on the coset $E_{10}/K(E_{10})$. The fields are represented by an E_{10} valued group element $\mathcal{V}(t)$, depending on a parameter t. This group element is subject to global E_{10} transformations from the left and to the local subgroup $K(E_{10})$ from the right:

$$\mathcal{V} \longrightarrow g\,\mathcal{V}\,h(t)\,, \qquad g \in E_{10}\,, \quad h(t) \in K(E_{10})\,. \tag{2.28}$$

Consequently, the E_{10} invariant Maurer-Cartan forms are given by $\mathcal{V}^{-1}\partial_t\mathcal{V}$. These can be decomposed into compact and non-compact parts,

$$\mathcal{V}^{-1}\partial_t\mathcal{V} \;=\; \mathcal{P}(t) + \mathcal{Q}(t)\;, \qquad \mathcal{P}\in\mathfrak{e}_{10}\ominus\mathfrak{k}(\mathfrak{e}_{10})\;, \qquad \mathcal{Q}\in\mathfrak{k}(\mathfrak{e}_{10})\;. \tag{2.29}$$

While $\mathcal{P}$ and $\mathcal{Q}$ are E_{10} invariant, they transform under an infinitesimal local transformation $\delta\mathcal{V} = \mathcal{V}\hat{h}$, where $\hat{h}\in\mathfrak{k}(\mathfrak{e}_{10})$, as

$$\delta\mathcal{Q} \;=\; \partial_t\hat{h} + [\mathcal{Q},\hat{h}]\;, \qquad \delta\mathcal{P} \;=\; [\mathcal{P},\hat{h}]\;, \tag{2.30}$$

i.e. $\mathcal{Q}$ is a (composite) gauge connection, while $\mathcal{P}$ transforms covariantly. The invariant action is then given by

$$S \;=\; \frac{1}{4}\int dt\, n(t)^{-1}\langle\mathcal{P}(t)|\mathcal{P}(t)\rangle\;, \tag{2.31}$$

where $\langle\ |\ \rangle$ denotes the Cartan-Killing form on $\mathfrak{e}_{10}$. Here, $n(t)$ is the lapse function establishing invariance under the one-dimensional diffeomorphisms

$$\delta_\xi n \;=\; \xi\partial_t n + (\partial_t\xi)n\;, \qquad \delta_\xi\mathcal{P} \;=\; \xi\partial_t\mathcal{P} + (\partial_t\xi)\mathcal{P}\;. \tag{2.32}$$

The equations of motion obtained from (2.31) are

$$n\partial_t(n^{-1}\mathcal{P}(t)) + [\mathcal{Q}(t),\mathcal{P}(t)] \;=\; 0\;, \tag{2.33}$$

and the Hamiltonian constraint

$$\langle\mathcal{P}(t)|\mathcal{P}(t)\rangle \;=\; 0\;, \tag{2.34}$$

which imply together that the motion follows a null geodesic.

So far our discussion was rather general. We are now going to evaluate (2.31) for the case we are interested in, namely maximal supergravity in $D = 3$. For this we use the level decomposition of $\mathfrak{e}_{10}$ with respect to $\mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{e}_8$ that we described in the preceding section.

The local $K(E_{10})$ invariance allows us to choose a suitable gauge for the E_{10}-valued group element $\mathcal{V}$. In the *Borel gauge*, we can write $\mathcal{V}$ as a product

$$\mathcal{V} = \mathcal{V}_\ell\mathcal{V}_0 = e^X(e^h e^{\mathcal{H}}), \tag{2.35}$$

where $\mathcal{V}_\ell$ and $\mathcal{V}_0$ are group elements corresponding to $\ell > 0$ and $\ell = 0$, respectively. Thus we can expand the corresponding algebra elements in the basis of $\mathfrak{e}_{10}$ as

$$X = A_m{}^{\mathcal{M}}E^m{}_{\mathcal{M}} + B_{mn}{}^{\mathcal{M}}E^{mn}{}_{\mathcal{M}} + BE + B^{\mathcal{M}\mathcal{N}}E_{\mathcal{M}\mathcal{N}} + \cdots\;, \tag{2.36}$$

$$h = h_a{}^b K^a{}_b\;,$$

$$\mathcal{H} = \mathcal{H}_{\mathcal{A}}t^{\mathcal{A}}\;. \tag{2.37}$$

Here and in the following, $m,n,\ldots = 1,2$ and $\mathcal{M},\mathcal{N}\ldots = 1,2,\ldots,248$ denote curved $GL(2)$ and E_8 indices, respectively. This means that they are 'world' indices indicating rigid transformations from the left, while $\mathcal{A}$ and a are flat indices.

JHEP01(2009)020

In (2.35), the ordering of the exponentials is fixed by the requirement that the fields $A_m{}^{\mathcal{M}}$, etc. transform under the SL(2, $\mathbb{R}$) according to their world indices m, n. In fact, under (2.28) we have

$$\mathcal{V}_0 \to g\mathcal{V}_0 h(t)\ , \qquad \mathcal{V}_\ell \to g\mathcal{V}_\ell g^{-1}\ . \tag{2.38}$$

Therefore, parameterizing $g = \exp(R_m{}^n K^m{}_n)$ and using $gA^m g^{-1} = A^n R_n{}^m$, one finds

$$A_m{}' \ = \ R_m{}^n A_n\ , \quad \text{etc.}\ , \tag{2.39}$$

as required (where we have omitted the E_8 indices). In the Borel gauge, $\mathcal{P}$ and $\mathcal{Q}$ have the same components in the bases of $\mathfrak{k}(\mathfrak{e}_{10})$ and the coset, except at level zero,

$$\begin{aligned}\mathcal{P} &= P^A\mathcal{S}_A + \frac{1}{2}P_{ab}\mathcal{S}^{ab} + P_a{}^{\mathcal{A}}\mathcal{S}^a{}_{\mathcal{A}} + P_{ab}{}^{\mathcal{A}}\mathcal{S}^{ab}{}_{\mathcal{A}} + P\mathcal{S} + P^{\mathcal{AB}}\mathcal{S}_{\mathcal{AB}},\\ \mathcal{Q} &= \frac{1}{2}Q^{IJ}\mathcal{J}_{IJ} + \frac{1}{2}Q_{ab}\mathcal{J}^{ab} + P_a{}^{\mathcal{A}}\mathcal{J}^a{}_{\mathcal{A}} + P_{ab}{}^{\mathcal{A}}\mathcal{J}^{ab}{}_{\mathcal{A}} + P\mathcal{J} + P^{\mathcal{AB}}\mathcal{J}_{\mathcal{AB}}.\end{aligned} \tag{2.40}$$

We write $\mathcal{V}_0$ as a product of two 'vielbeine' $\exp h$ and $\exp \mathcal{H}$, which are group elements of $\mathfrak{gl}(2, \mathbb{R})$ and $\mathfrak{e}_8$, respectively. We denote the components of these group elements by $e_m{}^a$ and $\mathcal{E}^{\mathcal{M}}{}_{\mathcal{A}}$. Occasionally, we will denote the components of the inverses by $e_a{}^m$ and $\mathcal{E}^{\mathcal{A}}{}_{\mathcal{M}}$. (The position of flat and curved indices thus keeps this notation unambiguous.) Now we can write the components of $\mathcal{P}$ and $\mathcal{Q}$, defined by (2.40), at level zero as

$$\begin{aligned} P_{ab} &= \frac{1}{2}(e_a{}^m\partial_t e_m{}^b + e_b{}^m\partial_t e_m{}^a), & P^A &= \frac{1}{2}(\mathcal{E}^{-1}\partial_t\mathcal{E})^A,\\ Q_{ab} &= \frac{1}{2}(e_a{}^m\partial_t e_m{}^b - e_b{}^m\partial_t e_m{}^a), & Q^{IJ} &= \frac{1}{4}(\mathcal{E}^{-1}\partial_t\mathcal{E})^{IJ}, \end{aligned} \tag{2.41}$$

and we obtain the level zero part of the Lagrangian,

$$\mathcal{L}_0 = n^{-1}P^A P^A + \frac{1}{4}n^{-1}(P_{ab}P_{ab} - P_{aa}P_{bb}). \tag{2.42}$$

As we will see below, this precisely coincides with the truncation of ungauged supergravity to a one-dimensional time-like system.

We now turn to the computation of the full Maurer-Cartan form, including also the $\ell > 0$ part. We then have

$$\mathcal{V}^{-1}\partial_t\mathcal{V} = \mathcal{V}_0{}^{-1}\partial_t\mathcal{V}_0 + \mathcal{V}_0{}^{-1}(\mathcal{V}_\ell{}^{-1}\partial_t\mathcal{V}_\ell)\mathcal{V}_0. \tag{2.43}$$

The first term is the $\ell = 0$ contribution which we used above. To evaluate the second term we make use of the Baker-Campbell-Hausdorff formulas

$$\begin{aligned} e^{-A}de^A \ &= \ dA + \frac{1}{2!}[dA, A] + \frac{1}{3!}[[dA, A], A]] + \cdots\ ,\\ e^{-A}Be^A \ &= \ B + [B, A] + \frac{1}{2!}[[B, A], A] + \cdots\ , \end{aligned} \tag{2.44}$$

and find

$$\begin{aligned}\mathcal{V}_0{}^{-1}(\mathcal{V}_\ell{}^{-1}\partial_t\mathcal{V}_\ell)\mathcal{V}_0 \ =\ & e_a{}^m\mathcal{E}^{\mathcal{A}}{}_{\mathcal{M}}D_t A_m{}^{\mathcal{M}}E^a{}_{\mathcal{A}} + e_a{}^m e_b{}^n\mathcal{E}^{\mathcal{A}}{}_{\mathcal{M}}D_t B_{mn}{}^{\mathcal{M}}E^{ab}{}_{\mathcal{C}}\\ & +(\det e)^{-1}(D_t BE + 14\mathcal{E}^{\mathcal{A}}{}_{\mathcal{M}}\mathcal{E}^{\mathcal{B}}{}_{\mathcal{N}}D_t B^{\mathcal{MN}}E^{\mathcal{AB}}).\end{aligned} \tag{2.45}$$

JHEP01(2009)020

The determinant of the vielbein $e_m{}^a$ appears since the level two fields B and $B^{\mathcal{MN}}$ transform with a nonzero weight under $\mathfrak{gl}(2,\mathbb{R})$.[4] In (2.45) we have introduced the 'covariant derivatives'

$$\begin{aligned} D_t A_m{}^{\mathcal{M}} &= \partial_t A_m{}^{\mathcal{M}}, \\ D_t B_{mn}{}^{\mathcal{P}} &= \partial_t B_{mn}{}^{\mathcal{P}} + \frac{1}{2} f_{\mathcal{MN}}{}^{\mathcal{P}} A_{(m}{}^{\mathcal{M}} \partial_t A_{n)}{}^{\mathcal{N}}, \\ D_t B &= \partial_t B - \frac{1}{4} \varepsilon^{ab} \eta_{\mathcal{MN}} A_m{}^{\mathcal{M}} \partial_t A_n{}^{\mathcal{N}}, \\ D_t B^{\mathcal{MN}} &= \partial_t B^{\mathcal{MN}} - \frac{1}{2} \varepsilon^{mn} \mathbb{P}_{\mathcal{PQ}}{}^{\mathcal{MN}} A_m{}^{\mathcal{P}} \partial_t A_n{}^{\mathcal{Q}}. \end{aligned} \tag{2.46}$$

Note that the $\mathfrak{e}_{10}$ algebra leads to non-trivial Chern-Simons like terms inside the covariant derivatives. For instance, acting with the group element

$$g = \exp\left(\Lambda_m{}^{\mathcal{M}} E^m{}_{\mathcal{M}} + \Lambda^{\mathcal{MN}} E_{\mathcal{MN}} + \cdots\right) \tag{2.47}$$

on the coset representative (2.35) yields the following global symmetry transformation on the fields

$$\delta_\Lambda A_m{}^{\mathcal{M}} = \Lambda_m{}^{\mathcal{M}}\,, \qquad \delta_\Lambda B^{\mathcal{MN}} = \Lambda^{\mathcal{MN}} + \frac{1}{2}\varepsilon^{mn} \Lambda_m{}^{\mathcal{P}} A_n{}^{\mathcal{Q}} \mathbb{P}_{\mathcal{PQ}}{}^{\mathcal{MN}}\,, \tag{2.48}$$

which leaves (2.46) invariant.

In order to project onto the non-compact part $\mathcal{P}(t)$, we have to replace x by $\frac{1}{2}(x+x^T)$. Then using (2.21) and inserting into (2.31) yields the sigma model Lagrangian

$$\begin{aligned} \mathcal{L} = \mathcal{L}_0 &+ \frac{1}{8} n^{-1} (g^{mn} \mathcal{G}_{\mathcal{MN}} D_t A_m{}^{\mathcal{M}} D_t A_n{}^{\mathcal{N}} + g^{mp} g^{nq} \mathcal{G}_{\mathcal{MN}} D_t B_{mn}{}^{\mathcal{M}} D_t B_{pq}{}^{\mathcal{N}}) \\ &+ \frac{1}{8} n^{-1} (\det g)^{-1} (D_t B D_t B + 14 \mathcal{G}_{\mathcal{MP}} \mathcal{G}_{\mathcal{NQ}} D_t B^{\mathcal{MN}} D_t B^{\mathcal{PQ}}), \end{aligned} \tag{2.49}$$

where $\mathcal{L}_0$ now can be written as

$$\mathcal{L}_0 = \frac{1}{960} n^{-1} \partial_t \mathcal{G}^{\mathcal{MN}} \partial_t \mathcal{G}^{\mathcal{PQ}} \mathcal{G}_{\mathcal{MP}} \mathcal{G}_{\mathcal{NQ}} + \frac{1}{16} n^{-1} \partial_t g_{mn} \partial_t g_{pq} (g^{mp} g^{nq} - g^{mn} g^{pq}) \tag{2.50}$$

and we introduced the (inverse) 'metrics'

$$g^{mn} = e_a{}^m e_a{}^n, \qquad \mathcal{G}_{\mathcal{MN}} = \mathcal{E}^{\mathcal{A}}{}_{\mathcal{M}} \mathcal{E}^{\mathcal{A}}{}_{\mathcal{N}}\,. \tag{2.51}$$

We stress that for the 'E_8 metric', the contraction is not performed by means of the E_8 invariant Cartan-Killing form, but instead with the ordinary delta symbol. Specifically, in

[4] More explicitly, the expansion gives

$$\begin{aligned} \mathcal{V}_0^{-1}(D_t B E)\mathcal{V}_0 &= D_t B \left(E - h^a{}_b [K^b{}_a, E] + \frac{1}{2} h^a{}_b h^c{}_d [K^b{}_a, [K^d{}_c, E]] + \ldots\right) \\ &= D_t B E \left(1 - h^a{}_a + \frac{1}{2}(h^a{}_a)^2 + \ldots\right) = (\det e)^{-1} D_t B E\,. \end{aligned}$$

JHEP01(2009)020

the SO(16) decomposition, this 'metric' (2.51) and its inverse read

$$\begin{aligned} \mathcal{G}_{\mathcal{M}\mathcal{N}} &= \frac{1}{2}\mathcal{E}^{IJ}{}_{\mathcal{M}}\mathcal{E}^{IJ}{}_{\mathcal{N}} + \mathcal{E}^{A}{}_{\mathcal{M}}\mathcal{E}^{A}{}_{\mathcal{N}}\,, \\ \mathcal{G}^{\mathcal{M}\mathcal{N}} &= \frac{1}{2}\mathcal{E}^{\mathcal{M}}{}_{IJ}\mathcal{E}^{\mathcal{N}}{}_{IJ} + \mathcal{E}^{\mathcal{M}}{}_{A}\mathcal{E}^{\mathcal{N}}{}_{A}\,, \end{aligned} \tag{2.52}$$

whereas the contraction with the (indefinite) Cartan-Killing metric (A.2) would give rise to a relative minus sign between the two terms on the r.h.s., and simply reproduce the Cartan-Killing metric: $\mathcal{E}^{\mathcal{M}}{}_{\mathcal{A}}\mathcal{E}^{\mathcal{N}}{}_{\mathcal{B}}\,\eta^{\mathcal{A}\mathcal{B}} = \eta^{\mathcal{M}\mathcal{N}}$. The equation (2.52) is consistent with the local SO(16) symmetry, in accordance with the contraction over *flat* indices. Likewise, the first equation in (2.51) is consistent with the local SO(2) symmetry.

We compare (2.49) with the expression for the Lagrangian that we get directly from (2.31) and (2.40),

$$\mathcal{L} = \frac{1}{4}n^{-1}\langle\mathcal{P}|\mathcal{P}\rangle = \mathcal{L}_0 + \frac{1}{2}n^{-1}(P_a{}^{\mathcal{A}}P_a{}^{\mathcal{A}} + P_{ab}{}^{\mathcal{A}}P_{ab}{}^{\mathcal{A}} + PP + 14P^{\mathcal{A}\mathcal{B}}P^{\mathcal{A}\mathcal{B}}). \tag{2.53}$$

Here the contraction of E_8 indices is again made with the delta symbol, as in (2.52). Comparing the expressions (2.53) and (2.49), we see that the components of $\mathcal{P}$ are the 'covariant derivatives' in (2.46) converted to flat indices,

$$\begin{aligned} P_a{}^{\mathcal{A}} &= \frac{1}{2}e_a{}^m\mathcal{E}^{\mathcal{A}}{}_{\mathcal{M}}D_t A_m{}^{\mathcal{M}}, & P_{ab}{}^{\mathcal{A}} &= \frac{1}{2}e_a{}^m e_b{}^n\mathcal{E}^{\mathcal{A}}{}_{\mathcal{M}}D_t B_{mn}{}^{\mathcal{M}}, \\ P &= \frac{1}{2}(\det e)^{-1}D_t B, & P^{\mathcal{A}\mathcal{B}} &= \frac{1}{2}(\det e)^{-1}\mathcal{E}^{\mathcal{A}}{}_{\mathcal{M}}\mathcal{E}^{\mathcal{B}}{}_{\mathcal{N}}D_t B^{\mathcal{M}\mathcal{N}}. \end{aligned} \tag{2.54}$$

2.4 Equations of motion

We now work out the equations of motion that follow from the Lagrangian (2.33). In the truncation to $|\ell| \leq 2$, they read

$$\begin{aligned} n\partial_t(n^{-1}P_{ab}) = &-2P_{ac}Q_{bc} - P_a{}^{IJ}P_b{}^{IJ} - 2P_a{}^A P_b{}^A - 2P_{ac}{}^{IJ}P_{bc}{}^{IJ} - 4P_{ac}{}^A P_{bc}{}^A \\ &+ \delta_{ab}\Big[P_c{}^{IJ}P_c{}^{IJ} + 2P_c{}^A P_c{}^A + 2P_{cd}{}^{IJ}P_{cd}{}^{IJ} + 4P_{cd}{}^A P_{cd}{}^A + 2PP \\ &\quad + 7(P^{IJKL}P^{IJKL} + 4P^{AIJ}P^{AIJ} + 4P^{AB}P^{AB})\Big], \end{aligned} \tag{2.55a}$$

$$\begin{aligned} n\partial_t(n^{-1}P^A) = &\frac{1}{2}\Gamma^{IJ}{}_{AB}(P^B Q^{IJ} + P_a{}^B P_a{}^{IJ} + P_{ab}{}^B P_{ab}{}^{IJ} \\ &\quad + 28P^{BC}P^{IJC} + 14P^{BKL}P^{IJKL}), \end{aligned} \tag{2.55b}$$

$$\begin{aligned} n\partial_t(n^{-1}P_a{}^A) = &(P_{ab} - Q_{ab})P_b{}^A + \frac{1}{2}\Gamma^{IJ}{}_{AB}(Q^{IJ}P_a{}^B + P_a{}^{IJ}P^B) \\ &- \frac{1}{2}\Gamma^{IJ}{}_{AB}(P_{ab}{}^B P_b{}^{IJ} + P_{ab}{}^{IJ}P_b{}^B) \\ &- \varepsilon_{ab}(28P^{AB}P_b{}^B + 14P^{AIJ}P_b{}^{IJ} + PP_b{}^A), \end{aligned} \tag{2.55c}$$

$$\begin{aligned} n\partial_t(n^{-1}P_a{}^{IJ}) = &(P_{ab} - Q_{ab})P_b{}^{IJ} - 4Q^{IK}P_a{}^{JK} + \Gamma^{IJ}{}_{AB}P_a{}^A P^B \\ &- 4P_{ab}{}^{IK}P_b{}^{JK} - \Gamma^{IJ}{}_{AB}P_{ab}{}^A P_b{}^B \\ &- \varepsilon_{ab}(28P^{IJA}P_b{}^A + 14P^{IJKL}P_b{}^{KL} - PP_b{}^{IJ}), \end{aligned} \tag{2.55d}$$

$$n\partial_t(n^{-1}P_{ab}{}^A) = 2(P_{ac} - Q_{ac})P_{cb}{}^A + \frac{1}{2}Q^{IJ}\Gamma^{IJ}{}_{AB}P_{ab}{}^B + \frac{1}{2}P_{ab}{}^{IJ}\Gamma^{IJ}{}_{AB}P^B, \tag{2.55e}$$

$$n\partial_t(n^{-1}P_{ab}{}^{IJ}) = 2(P_{ac} - Q_{ac})P_{cb}{}^{IJ} - 4Q^{IK}P_{ab}{}^{JK} + \Gamma^{IJ}{}_{AB}P_{ab}{}^A P^B, \tag{2.55f}$$

$$n\partial_t(n^{-1}P^{AB}) = P_{aa}P^{AB} + Q^{IJ}\Gamma^{IJ}{}_{AC}P^{BC} + P^{BIJ}\Gamma^{IJ}{}_{AC}P^C, \tag{2.55g}$$

$$\begin{aligned} n\partial_t(n^{-1}P^{AIJ}) = {} & P_{aa}P^{AIJ} + \frac{1}{2}Q^{KL}\Gamma^{KL}{}_{AB}P^{BIJ} - 4Q^{KI}P^{AKJ} \\ & + \frac{1}{2}P^{IJKL}\Gamma^{KL}{}_{AB}P^B + \Gamma^{IJ}{}_{BC}P^C P^{AB}, \end{aligned} \tag{2.55h}$$

$$\begin{aligned} n\partial_t(n^{-1}P^{IJKL}) = {} & P_{aa}P^{IJKL} - 4Q^{MK}P^{MLIJ} - 4Q^{MI}P^{MJKL} \\ & + \Gamma^{IJ}{}_{AB}P^{AKL}P^B + \Gamma^{KL}{}_{AB}P^{AIJ}P^B, \end{aligned} \tag{2.55i}$$

$$n\partial_t(n^{-1}P) = P_{aa}P. \tag{2.55j}$$

In the above equations the irreducibility constraint (2.10) on the level two field $P^{\mathcal{AB}}$ is not spelled out explicitly, but see (3.9) and (3.10) below.

The equations of motion can of course also be computed directly from the Lagrangian (2.49), without using the commutation relations. By varying the level two fields, we get

$$\begin{aligned} 0 &= \partial_t(n^{-1}g^{mp}g^{nq}\mathcal{G}_{\mathcal{MN}}D_t B_{pq}{}^{\mathcal{N}}), \\ 0 &= \partial_t(n^{-1}(\det g)^{-1}D_t B), \\ 0 &= \partial_t(n^{-1}(\det g)^{-1}\mathcal{G}_{\mathcal{MP}}\mathcal{G}_{\mathcal{NQ}}D_t B^{\mathcal{PQ}}), \end{aligned} \tag{2.56}$$

and for the first level,

$$\begin{aligned} 0 = {} & \frac{1}{2}n^{-1}\Big(g^{mp}g^{nq}\mathcal{G}_{\mathcal{PQ}}f_{\mathcal{MN}}{}^{\mathcal{P}}D_t A_n{}^{\mathcal{N}}D_t B_{pq}{}^{\mathcal{Q}} \\ & - \frac{1}{2}\varepsilon^{mn}\eta_{\mathcal{MN}}(\det g)^{-1}D_t A_n{}^{\mathcal{N}}D_t B \\ & - 14\varepsilon^{mn}(\det g)^{-1}\mathcal{G}_{\mathcal{MP}}\mathcal{G}_{\mathcal{NQ}}D_t A_n{}^{\mathcal{N}}D_t B^{\mathcal{PQ}}\Big) \\ & - \frac{1}{2}\partial_t\Big[n^{-1}\Big(2g^{mn}\mathcal{G}_{\mathcal{MN}}D_t A_n{}^{\mathcal{N}} - g^{mp}g^{nq}\mathcal{G}_{\mathcal{PQ}}f_{\mathcal{MN}}{}^{\mathcal{P}}A_n{}^{\mathcal{N}}D_t B_{pq}{}^{\mathcal{Q}} \\ & \qquad + \frac{1}{2}\varepsilon^{mn}\eta_{\mathcal{MN}}(\det g)^{-1}A_n{}^{\mathcal{N}}D_t B \\ & \qquad + 14\varepsilon^{mn}(\det g)^{-1}\mathcal{G}_{\mathcal{MP}}\mathcal{G}_{\mathcal{NQ}}A_n{}^{\mathcal{N}}D_t B^{\mathcal{PQ}}\Big)\Big]. \end{aligned} \tag{2.57}$$

We use the equations (2.56) to rewrite the second half of (2.57),

$$\begin{aligned} n\partial_t(n^{-1}g^{mn}\mathcal{G}_{\mathcal{MN}}D_t A_n{}^{\mathcal{N}}) = {} & g^{mp}g^{nq}\mathcal{G}_{\mathcal{PQ}}f_{\mathcal{MN}}{}^{\mathcal{P}}D_t A_n{}^{\mathcal{N}}D_t B_{pq}{}^{\mathcal{Q}} \\ & - \frac{1}{2}\varepsilon^{mn}\eta_{\mathcal{MN}}(\det g)^{-1}D_t A_n{}^{\mathcal{N}}D_t B \\ & - 14\varepsilon^{mn}(\det g)^{-1}\mathcal{G}_{\mathcal{MP}}\mathcal{G}_{\mathcal{NQ}}D_t A_n{}^{\mathcal{N}}D_t B^{\mathcal{PQ}}. \end{aligned} \tag{2.58}$$

It is then straightforward to show that we get the same equations as above. The equations (2.56) can also be used to rewrite the first half of (2.57), as we will see in section 4.3.

JHEP01(2009)020

3. Gauged supergravity in three dimensions

In this section we review gauged three-dimensional supergravity in a formulation suitable for comparison with the E_{10} analysis of the preceding section. The comparison will be carried out in the next section.

The bosonic sector of ungauged maximal supergravity in three dimensions contains 128 propagating scalars transforming in the coset $E_8/(\mathrm{Spin}(16)/\mathbb{Z}_2)$ and a vielbein $\mathtt{e}_\mu{}^\alpha$ that carries no dynamical degrees of freedom [4, 5]. The scalars can also be described by an (internal) vielbein which we denote by $\mathtt{E}^{\mathcal{M}}{}_{\mathcal{A}}$ (which was denoted $\mathcal{V}^{\mathcal{M}}{}_{\mathcal{A}}$ in [7]).[5] The inverses will be written as $\mathtt{e}_\alpha{}^\mu$ and $\mathtt{E}^{\mathcal{A}}{}_{\mathcal{M}}$. The curved indices are written as Greek indices $\mu, \nu, \ldots = (t, m)$ and the flat indices are $\alpha, \beta, \ldots = 0,\ 1,\ 2$. The E_8 indices follow the same conventions as before. We ignore fermions throughout the paper.

3.1 The lagrangian

The construction of gauged three-dimensional supergravity where a subgroup G_0 of the global symmetry group E_8 has been gauged proceeds via the introduction of gauge fields $\mathtt{A}_\mu{}^{\mathcal{M}}$ in the adjoint of $\mathfrak{e}_8$ such that one has the modified Maurer-Cartan forms [6, 7][6]

$$\mathtt{E}^{-1}\mathcal{D}_\mu\mathtt{E} \ = \mathtt{Q}_\mu + \mathtt{P}_\mu = \frac{1}{2}\mathtt{Q}_\mu^{IJ}\mathcal{J}^{IJ} + \mathtt{P}_\mu^A\mathcal{S}^A \ , \tag{3.1}$$

where the gauge-covariant derivative is given by

$$\mathtt{E}^{-1}\mathcal{D}_\mu\mathtt{E} \ = \ \mathtt{E}^{-1}\partial_\mu\mathtt{E} + g\mathtt{A}_\mu{}^{\mathcal{M}}\Theta_{\mathcal{M}\mathcal{N}}(\mathtt{E}^{-1}t^{\mathcal{N}}\mathtt{E}) \ . \tag{3.2}$$

The quantity $\Theta_{\mathcal{M}\mathcal{N}}$ is the constant embedding tensor describing the generators of the Lie algebra $\mathfrak{g}_0 \subset \mathfrak{e}_8$ in terms of $\mathfrak{e}_8$ generators: $X_{\mathcal{M}} = \Theta_{\mathcal{M}\mathcal{N}}t^{\mathcal{N}}$. There are only $\dim(\mathfrak{g}_0)$ many non-vanishing $X_{\mathcal{M}}$ but it is convenient to maintain an E_8 covariant notation. In such a notation, the embedding tensor is symmetric in its indices and transforms in the $\mathbf{3875} \oplus \mathbf{1}$ representation of E_8. We will sometimes split it into its irreducible parts as

$$\Theta_{\mathcal{M}\mathcal{N}} \ = \ \tilde{\Theta}_{\mathcal{M}\mathcal{N}} + \theta\,\eta_{\mathcal{M}\mathcal{N}} \ , \tag{3.3}$$

where $\tilde{\Theta}$ transform in the $\mathbf{3875}$, and θ is the singlet part.

Under infinitesimal local G_0 transformations with parameter $\Lambda^{\mathcal{M}}X_{\mathcal{M}}$ one has

$$\delta\,\mathtt{A}_\mu{}^{\mathcal{M}} = \mathcal{D}_\mu\Lambda^{\mathcal{M}} \ \equiv \ \partial_\mu\Lambda^{\mathcal{M}} + gf^{\mathcal{M}\mathcal{N}}{}_{\mathcal{K}}\Theta_{\mathcal{N}\mathcal{L}}A_\mu{}^{\mathcal{L}}\Lambda^{\mathcal{K}} \ , \tag{3.4}$$

$$\delta\,\mathtt{E} = g\Lambda^{\mathcal{M}}X_{\mathcal{M}}\,\mathtt{E} \ , \tag{3.5}$$

and the Maurer-Cartan form is invariant.

[5] Generally, we will use the 'typewriter' font for supergravity variables in order to distinguish them from the corresponding E_{10} quantities.

[6] We reiterate that we have changed the normalization of the generators of the coset generators $\mathcal{S}^A = 2Y^A$, $\mathcal{J}^{IJ} = -2X^{IJ}$ compared to the generators used in [6, 7]. Also the space-time signature here is $(-++)$, opposite to that used there. The convention for the Levi-Civita symbol is $\varepsilon^{012} = +1$.

The bosonic Lagrangian of three-dimensional maximal gauged supergravity is [6, 7]

$$\mathcal{L} = \mathtt{e}\left(\frac{1}{4}R - \mathtt{P}_\mu{}^A\mathtt{P}^{\mu A} - V\right) + \mathcal{L}_{CS}, \tag{3.6}$$

with $\mathtt{e} = \det(\mathtt{e}_\mu{}^\alpha)$ and the Chern-Simons term

$$\mathcal{L}_{CS} = -\frac{1}{4}g\varepsilon^{\mu\nu\rho}\Theta_{\mathcal{M}\mathcal{N}}\mathtt{A}_\mu{}^{\mathcal{M}}\partial_\nu\mathtt{A}_\rho^{\mathcal{N}} - \frac{1}{12}g^2\varepsilon^{\mu\nu\rho}\Theta_{\mathcal{M}\mathcal{N}}\Theta_{\mathcal{P}\mathcal{Q}}f^{\mathcal{M}\mathcal{P}}{}_{\mathcal{R}}\mathtt{A}_\mu{}^{\mathcal{N}}\mathtt{A}_\nu{}^{\mathcal{Q}}\mathtt{A}_\rho{}^{\mathcal{R}}\,. \tag{3.7}$$

Since there is no kinetic term for them, the gauge fields $\mathtt{A}_\mu{}^{\mathcal{M}}$ do not contain propagating degrees of freedom. The gauging also introduces an indefinite scalar potential. In order to write it out, one introduces the so-called T-tensor that transforms in the **3875** of E_8, and is defined by

$$\tilde{T}_{\mathcal{A}\mathcal{B}} = \mathtt{E}^{\mathcal{M}}{}_{\mathcal{A}}\mathtt{E}^{\mathcal{N}}{}_{\mathcal{B}}\tilde{\Theta}_{\mathcal{M}\mathcal{N}}\,. \tag{3.8}$$

The field dependent T-tensor is thus the E_8 rotated version of the (constant) embedding tensor $\tilde{\Theta}_{\mathcal{M}\mathcal{N}}$. Note that here we have defined the T-tensor only with respect to **3875**, in contrast to [7]. The fact that $\tilde{T}$ transforms in the **3875** implies that it has the components

$$\begin{aligned}
A_1{}^{IJ} &= -\delta_{IJ}\theta + \frac{1}{7}\tilde{T}_{IK\,JK}, \\
A_2{}^{I\dot{A}} &= -\frac{1}{7}\Gamma^J{}_{A\dot{A}}\tilde{T}_{IJ\,A}, \\
A_3{}^{\dot{A}\dot{B}} &= 2\delta_{\dot{A}\dot{B}}\theta + \frac{1}{48}\Gamma^{IJKL}{}_{\dot{A}\dot{B}}\tilde{T}_{IJ\,KL},
\end{aligned} \tag{3.9}$$

corresponding to the decomposition

$$\mathbf{3875} \to \mathbf{135} \oplus \mathbf{1820} \oplus \mathbf{1920} \tag{3.10}$$

of this E_8 representation under SO(16) [7]. Here $A_1{}^{IJ}$ is symmetric, $A_1{}^{IJ} = A_1{}^{(IJ)}$ and $A_2{}^{I\dot{A}}$ traceless, that is $\Gamma^I{}_{A\dot{A}}A_2{}^{I\dot{A}} = 0$. The potential then is the sum of two parts [7], one negative-definite and the other positive-definite,

$$V = \frac{1}{8}g^2\Big(- A_1{}^{IJ}A_1{}^{IJ} + \frac{1}{2}A_2{}^{I\dot{A}}A_2{}^{I\dot{A}}\Big). \tag{3.11}$$

Note that there is no contribution involving $A_3^{\dot{A}\dot{B}}$. Alternatively, the potential can be written in the form

$$V = \frac{1}{32}g^2 G^{\mathcal{M}\mathcal{N},\mathcal{K}\mathcal{L}}\Theta_{\mathcal{M}\mathcal{N}}\Theta_{\mathcal{K}\mathcal{L}}\,, \tag{3.12}$$

where [32]

$$G^{\mathcal{M}\mathcal{N},\mathcal{K}\mathcal{L}} \;=\; \frac{1}{14}\mathtt{G}^{\mathcal{M}\mathcal{K}}\mathtt{G}^{\mathcal{N}\mathcal{L}} + \mathtt{G}^{\mathcal{M}\mathcal{K}}\eta^{\mathcal{N}\mathcal{L}} - \frac{3}{14}\eta^{\mathcal{M}\mathcal{K}}\eta^{\mathcal{N}\mathcal{L}} - \frac{4}{6727}\eta^{\mathcal{M}\mathcal{N}}\eta^{\mathcal{K}\mathcal{L}} \tag{3.13}$$

with the metric $\mathtt{G}^{\mathcal{M}\mathcal{N}}$ defined in (2.52), but here with respect to the supergravity E_8 vielbein $\mathtt{E}^{\mathcal{M}}{}_{\mathcal{A}}$. Inserting (3.9) into (3.11), and using the relations (A.7) (which follow from

the fact that $\tilde{T}$ transform in the **3875** representation) we get yet another expression for the potential,

$$V = \frac{1}{112} g^2 (3\tilde{T}_{AB}\tilde{T}_{AB} + \tilde{T}_{A\,IJ}\tilde{T}_{A\,IJ} - \tilde{T}_{IJKL}\tilde{T}_{IJKL}) - 2g^2\theta^2. \tag{3.14}$$

Both (3.12) and (3.14) will be used for the comparison with the E_{10} sigma model. Note, however, that in this form the decomposition (3.10) is only implicit.

3.1.1 Equations of motion

Varying (3.6) with respect to the gauge field one obtains the following non-abelian duality relation

$$\mathtt{e}^{-1}\varepsilon^{\mu\nu\rho}\Theta_{\mathcal{M}\mathcal{N}}\mathtt{F}_{\nu\rho}{}^{\mathcal{N}} \;=\; -4\Theta_{\mathcal{M}\mathcal{N}}\mathtt{E}^{\mathcal{N}}{}_{A}\mathtt{P}^{\mu A}\,, \tag{3.15}$$

in terms of the non-abelian field strength

$$\mathtt{F}_{\mu\nu}{}^{\mathcal{M}} = \partial_\mu \mathtt{A}_\nu{}^{\mathcal{M}} - \partial_\nu \mathtt{A}_\mu{}^{\mathcal{M}} + g\Theta_{\mathcal{P}\mathcal{Q}} f^{\mathcal{M}\mathcal{P}}{}_{\mathcal{R}} \mathtt{A}_\mu{}^{\mathcal{Q}} \mathtt{A}_\nu{}^{\mathcal{R}}\,. \tag{3.16}$$

We stress that the summation in (3.15) is only over the coset indices A and not over the whole E_8. The Einstein equation can be written as

$$R_{\mu\nu} = 4\,\mathtt{P}_\mu{}^{A}\mathtt{P}_\nu{}^{A} + 4\,\mathtt{g}_{\mu\nu}V\,, \tag{3.17}$$

where, again, the summation only is over the SO(16) spinor indices.

For the scalars, we first consider only the positive term in (3.11), and its variation along the coset,

$$\delta(A_2{}^{I\dot{A}}A_2{}^{I\dot{A}}) = \frac{1}{14}\Gamma^{IJ}{}_{AB}(2\tilde{T}_{AC}\tilde{T}_{IJC} + \tilde{T}_{AKL}\tilde{T}_{IJKL})(\mathtt{E}^{-1}\delta\mathtt{E})^B. \tag{3.18}$$

Since we also have

$$\delta(\mathtt{P}_\mu{}^{A}\mathtt{P}^{\mu A}) = \mathtt{P}_\mu{}^{A}\partial^\mu(\mathtt{E}^{-1}\delta\mathtt{E})^A + \frac{1}{2}\mathtt{Q}_\mu{}^{IJ}\Gamma^{IJ}{}_{AB}\mathtt{P}^{\mu B}(\mathtt{E}^{-1}\delta\mathtt{E})^A, \tag{3.19}$$

it follows that the scalar equation of motion, without the contribution from the negative term in the potential, becomes

$$\mathtt{e}^{-1}\partial_\mu(\mathtt{e}\mathtt{P}^{\mu A}) = \frac{1}{2}\Gamma^{IJ}{}_{AB}\left(\mathtt{Q}_\mu{}^{IJ}\mathtt{P}^{\mu B} - \frac{1}{56}g^2\tilde{T}_{BC}\tilde{T}_{IJC} - \frac{1}{112}g^2\tilde{T}_{BKL}\tilde{T}_{IJKL}\right) + \ldots \tag{3.20}$$

For the negative-definite part in (3.11) we have

$$\delta(A_1{}^{IJ}A_1{}^{IJ}) = \frac{1}{14}\Gamma^{IJ}{}_{AB}\Big(- 3\tilde{T}_{AC}\tilde{T}_{IJC} + 2\tilde{T}_{AKL}\tilde{T}_{IJKL}\Big)(\mathtt{E}^{-1}\delta\mathtt{E})^B. \tag{3.21}$$

Thus the full equation of motion for the scalars reads

$$\mathtt{e}^{-1}\partial_\mu(\mathtt{e}\mathtt{P}^{\mu A}) = \frac{1}{2}\Gamma^{IJ}{}_{AB}\left(\mathtt{Q}_\mu{}^{IJ}\mathtt{P}^{\mu B} + \frac{1}{14}g^2\tilde{T}_{BC}\tilde{T}_{IJC} - \frac{3}{112}g^2\tilde{T}_{BKL}\tilde{T}_{IJKL}\right). \tag{3.22}$$

This rewritten form of the equations of motion of [6, 7] is convenient for the comparison with the E_{10} sigma model.

JHEP01(2009)020

3.1.2 Constraints

From the form of the Maurer-Cartan form (3.1) one deduces the following integrability relations

$$g\mathtt{F}_{\mu\nu}{}^{\mathcal{M}}\Theta_{\mathcal{M}\mathcal{N}}\mathbb{E}^{\mathcal{N}}{}_{\mathcal{A}}t^{\mathcal{A}} = 2\partial_{[\mu}\mathtt{P}_{\nu]} + 2\partial_{[\mu}\mathtt{Q}_{\nu]} + [\mathtt{Q}_\mu + \mathtt{P}_\mu, \mathtt{Q}_\nu + \mathtt{P}_\nu] \,. \tag{3.23}$$

Using the duality relation (3.15) this can be rewritten as a relation expressed solely in terms of $\mathtt{P}$, $\mathtt{Q}$ and the embedding tensor as

$$\begin{aligned} 2\partial_{[\mu}\mathtt{P}_{\nu]} + 2\partial_{[\mu}\mathtt{Q}_{\nu]} = &- [\mathtt{Q}_\mu + \mathtt{P}_\mu, \mathtt{Q}_\nu + \mathtt{P}_\nu] \\ &+ \mathtt{e}g\varepsilon^{\rho}{}_{\mu\nu}\tilde{T}_{A\,IJ}\mathtt{P}_\rho{}^{A}t^{IJ} + 2\mathtt{e}g\varepsilon^{\rho}{}_{\mu\nu}\tilde{T}_{AB}\mathtt{P}_\rho{}^{A}t^{B} + 2\mathtt{e}g\varepsilon^{\rho}{}_{\mu\nu}\theta\mathtt{P}_\rho{}^{A}t^{A} \,. \end{aligned} \tag{3.24}$$

The equation (3.24) is the deformation of the usual integrability constraint of non-linear sigma models in the presence of gauging. In addition there are three-dimensional Bianchi constraints, *viz.*

$$\mathcal{D}_{[\mu}\mathtt{F}_{\nu\rho]}{}^{\mathcal{M}}\Theta_{\mathcal{M}\mathcal{N}} = 0 \tag{3.25}$$

for the gauge field and for the gravity sector

$$R_{[\mu\nu\,\rho]\sigma} = 0 \,. \tag{3.26}$$

Finally, the embedding tensor is subject to linear and quadratic constraints [6, 7]. The linear constraint arises from supersymmetry and implies that it transforms in the $\mathbf{1} \oplus \mathbf{3875}$ part of the symmetric tensor product of two $\mathbf{248}$ representations, so that the $\mathbf{27000}$ is absent. This constraint leads to the relations (A.7) that we already used in (3.14) and (3.22) to simplify expressions involving the T tensor. The quadratic constraint reads

$$\mathcal{Q}_{\mathcal{M}\mathcal{N},\mathcal{P}} \;\equiv\; \Theta_{\mathcal{K}\mathcal{P}}\Theta_{\mathcal{L}(\mathcal{M}}f^{\mathcal{K}\mathcal{L}}{}_{\mathcal{N})} \;=\; 0 \,. \tag{3.27}$$

As we will see in section 3.2, further constraints on the fields arise when some of the gauge freedom has been fixed.

3.1.3 Reformulation with deformation and top-form potentials

Here we briefly introduce a reformulation of gauged supergravity with so-called deformation and top-form potentials [33, 28], which will be useful for the interpretation of the E_{10} equations below. These potentials are part of a tensor hierarchy introduced in [34] and can be viewed as Lagrange multipliers enforcing the constancy of the embedding tensor and the quadratic constraint. Denoting the deformation two-form by $\mathtt{B}_{\mu\nu}{}^{\mathcal{M}\mathcal{N}}$ and the top-form by $\mathtt{C}_{\mu\nu\rho}{}^{\mathcal{M}\mathcal{N},\mathcal{P}}$, which respectively transform in the $\mathbf{1} \oplus \mathbf{3875}$ and $\mathbf{3875} \oplus \mathbf{147250}$ representations of E_8 [33, 28], one has

$$\mathcal{L}_{\rm tot} \;=\; \mathcal{L}_g + \frac{1}{4}g\varepsilon^{\mu\nu\rho}\mathcal{D}_\mu\Theta_{\mathcal{M}\mathcal{N}}\mathtt{B}_{\nu\rho}{}^{\mathcal{M}\mathcal{N}} - \frac{1}{6}g^2\Theta_{\mathcal{K}\mathcal{P}}\Theta_{\mathcal{L}(\mathcal{M}}f^{\mathcal{K}\mathcal{L}}{}_{\mathcal{N})}\varepsilon^{\mu\nu\rho}\mathtt{C}_{\mu\nu\rho}{}^{\mathcal{M}\mathcal{N},\mathcal{P}} \,, \tag{3.28}$$

where the embedding tensor now satisfies only the linear constraint. Here we have written a covariant derivative on $\Theta_{\mathcal{M}\mathcal{N}}$,

$$\mathcal{D}_\mu\Theta_{\mathcal{M}\mathcal{N}} \;=\; \partial_\mu\Theta_{\mathcal{M}\mathcal{N}} + 2g\mathtt{A}_\mu{}^{\mathcal{P}}\Theta_{\mathcal{K}\mathcal{P}}\Theta_{\mathcal{L}(\mathcal{M}}f^{\mathcal{K}\mathcal{L}}{}_{\mathcal{N})} \,. \tag{3.29}$$

The second term vanishes identically upon use of the quadratic constraint, whence the equations of motion imply that Θ is constant (and not just covariantly constant). Since the space-time dependent embedding tensor is now a dynamical field, it possesses its own equations of motions, which can be viewed as duality relations between the 2-form potential and the embedding tensor [33, 32]. Below we will see that an analogous relation follows naturally from the sigma model equations of motion, with the E_{10} field $B^{\mathcal{MN}}$ interpreted as (the Hodge dual of) the spatial part of the deformation potential. By contrast, in the E_{11} approach of [29] both $B_{\mu\nu}{}^{\mathcal{MN}}$ and $C_{\mu\nu\rho}{}^{\mathcal{MN},\mathcal{P}}$ appear in the decomposition of E_{11}, whereas the embedding tensor must be introduced as an 'extraneous' object to parametrize the deformation of the derivative in the Cartan form.

3.2 Dimensional reduction to $D = 1$

We now effectively reduce the three-dimensional gauged supergravity theory to a one-dimensional time-like system. For this we perform the ADM-like split of the vielbein

$$\mathtt{e}_\mu{}^\alpha \;=\; \begin{pmatrix} N & 0 \\ 0 & \mathtt{e}_m{}^a \end{pmatrix} , \tag{3.30}$$

in which everything depends only on one coordinate $x^0 = t$ and we have split curved indices as $\mu = (t, m)$ and flat ones as $\alpha = (0, a)$ (with signature $(-++)$). Here we have chosen a gauge with vanishing shift N^m, which turns out to be necessary in order to match the E_{10} coset. As stressed before, gauge fixing is crucial for comparing the E_{10} sigma model to supergravity. The field $\mathtt{e}_m{}^a$ denotes the internal 'spatial' vielbein, i.e. an element of $GL(2,\mathbb{R})/\mathrm{SO}(2)$. The three-dimensional Einstein-Hilbert Lagrangian in (3.6) can be rewritten up to a total derivative as

$$\tfrac{1}{4}\mathtt{e}R \;=\; -\tfrac{1}{16}\mathtt{e}\Omega^{\alpha\beta\,\gamma}\Omega_{\alpha\beta\,\gamma} + \tfrac{1}{8}\mathtt{e}\Omega^{\alpha\beta\,\gamma}\Omega_{\beta\gamma\,\alpha} + \tfrac{1}{4}\mathtt{e}\Omega_{\alpha\beta}{}^{\beta}\Omega^{\alpha\gamma}{}_{\gamma} , \tag{3.31}$$

where $\Omega_{\alpha\beta\,\gamma}$ are the coefficients of anholonomy:

$$\Omega_{\alpha\beta\,\gamma} = \mathtt{e}_\alpha{}^\mu \mathtt{e}_\beta{}^\nu (\partial_\mu \mathtt{e}_{\nu\gamma} - \partial_\nu \mathtt{e}_{\mu\gamma}) . \tag{3.32}$$

The only non-vanishing components in the strict reduction to $D = 1$ are

$$\Omega_{a0\,b} = -\Omega_{0a\,b} = -N^{-1}\mathtt{e}_a{}^m \partial_t \mathtt{e}_{mb} =: -N^{-1}\mathtt{h}_{ab} , \tag{3.33}$$

where we have introduced the $\mathfrak{gl}(2,\mathbb{R})$-valued current $\mathtt{h}_{ab}$ converted into flat indices. The current has both a symmetric and an antisymmetric part, $\mathtt{h}_{ab} = \mathtt{P}_{ab} + \mathtt{Q}_{ab}$. Inserting into the Einstein-Hilbert action, one finds that the antisymmetric part cancels and the resulting expression is

$$\mathtt{e}^{-1}\mathcal{L}_{\mathrm{EH}} \;=\; \frac{1}{4}N^{-2}\left(\mathtt{P}_{ab}\mathtt{P}_{ab} - \mathtt{P}_{aa}\mathtt{P}_{bb}\right) . \tag{3.34}$$

On the other hand, the E_8 valued fields are all scalars and trivially reduce according to $\mathtt{E}(x) \to \mathtt{E}(t)$. Using $\mathtt{e} = \det(\mathtt{e}_\mu{}^\alpha) = N\det(\mathtt{e}_m{}^a)$, one finds in total for the case of ungauged supergravity

$$\mathcal{L}_{g=0}^{D=1} \;=\; \mathtt{n}^{-1}\mathtt{P}_t{}^A\mathtt{P}_t{}^A + \frac{1}{4}\mathtt{n}^{-1}\left(\mathtt{P}_{ab}\mathtt{P}_{ab} - \mathtt{P}_{aa}\mathtt{P}_{bb}\right) , \tag{3.35}$$

where we have defined the quantity

$$\mathtt{n} = N(\det(\mathtt{e}_m{}^a))^{-1}\,. \tag{3.36}$$

Evidently, (3.35) has exactly the same form as the level zero Lagrangian (2.42).

We turn now to gauged supergravity. For the reduction of the tensor fields we choose a temporal gauge

$$\mathtt{A}_t{}^{\mathcal{M}} = 0\,, \quad \mathtt{B}_{tm}{}^{\mathcal{MN}} = 0\,, \quad \mathtt{C}_{tmn}{}^{\mathcal{MN},\mathcal{P}} = 0\,. \tag{3.37}$$

Reducing the action (3.28) of gauged supergravity to $D = 1$, we then find

$$\begin{aligned}\mathcal{L}_g^{D=1} = \mathcal{L}_{g=0}^{D=1} - \mathtt{n}^{-1}N^2\mathtt{g}^{mn}[\mathtt{E}^{-1}\mathcal{D}_m\mathtt{E}]^A[\mathtt{E}^{-1}\mathcal{D}_n\mathtt{E}]^A - \mathtt{n}^{-1}N^2V \\ +\frac{1}{4}g\varepsilon^{mn}\mathtt{A}_m{}^{\mathcal{M}}\Theta_{\mathcal{MN}}\partial_t\mathtt{A}_n{}^{\mathcal{N}} + \frac{1}{4}g\varepsilon^{mn}\mathcal{D}_t\Theta_{\mathcal{MN}}\mathtt{B}_{mn}{}^{\mathcal{MN}}\,.\end{aligned} \tag{3.38}$$

Here, $\mathcal{D}_m\mathtt{E}$ denotes the spatial part of the gauge-covariant derivative, which in the case of pure time dependence reads

$$\mathtt{E}^{-1}\mathcal{D}_m\mathtt{E} \;=\; g\mathtt{A}_m{}^{\mathcal{M}}\Theta_{\mathcal{MN}}\mathtt{E}^{-1}t^{\mathcal{N}}\mathtt{E}\,. \tag{3.39}$$

The appearance of the gauge vector here is the only remnant of the gauging in the scalar kinetic terms. In fact, the gauge choices (3.37) have the advantage that the time component of the gauge covariant derivatives in $D = 1$ collapses, e.g.

$$\mathtt{E}^{-1}\mathcal{D}_t\mathtt{E} \;\equiv\; \mathtt{E}^{-1}\partial_t\mathtt{E}\,. \tag{3.40}$$

Similarly, the cubic term in the reduction of the Chern-Simons term disappears as well as the top-form potential term enforcing the quadratic constraint. That the Maurer-Cartan forms are unchanged is essential for the comparison with the E_{10} model in its original form.

When fixing gauges one should not forget the equations of motion (constraints) resulting from varying with respect to the temporal components of the gauge fields in (3.37). They read from (3.15) and (3.28)

$$C_{\mathcal{M}} := \mathtt{n}^{-1}\varepsilon^{mn}\Theta_{\mathcal{MN}}\mathtt{F}_{mn}{}^{\mathcal{N}} + 4\Theta_{\mathcal{MN}}\mathtt{E}^{\mathcal{N}}{}_A\mathtt{P}_t{}^A \;=\; 0\,, \tag{3.41}$$

$$C^m_{\mathcal{MN}} := \mathtt{n}^{-1}g\varepsilon^{mn}D_n\Theta_{\mathcal{MN}} \;=\; 0\,, \tag{3.42}$$

$$C_{\mathcal{MN},\mathcal{P}} := g^2\Theta_{\mathcal{KP}}\Theta_{\mathcal{L}(\mathcal{M}}f^{\mathcal{KL}}{}_{\mathcal{N})} \;=\; 0\,. \tag{3.43}$$

As constructed, the constraints for $\mathtt{B}_{tm}{}^{\mathcal{MN}}$ and $\mathtt{C}_{tmn}{}^{\mathcal{MN},\mathcal{P}}$ correspond to the (spatial) constancy of the embedding tensor and the quadratic constraint. Below we will interpret the temporal constancy of $\Theta_{\mathcal{MN}}$ as an equation of motion rather than as a constraint.

3.3 Beyond dimensional reduction

The E_{10} model also takes into account terms that are beyond dimensional reduction to $D = 1$ [12, 13]. Therefore we also need to keep track of terms that arise from spatial gradients and contribute to the equations of motion. Instead of writing out all the resulting

equations we illustrate the procedure in the example of equation (3.24). Considering the equation in flat spatial indices and split into $\mathfrak{so}(16)$ and coset components we find for the $(\alpha, \beta) = (0, a)$ component

$$\begin{aligned}\partial_0 \mathtt{Q}_a{}^{IJ} - \partial_a \mathtt{Q}_0{}^{IJ} &= -4\mathtt{Q}_0{}^{[I|K}\mathtt{Q}_a^{J]K} - \Gamma^{IJ}{}_{AB}\mathtt{P}_a{}^A\mathtt{P}_0{}^B \\ &\quad - N^{-1}(\mathtt{Q}_{ab} + \mathtt{P}_{ab})\mathtt{Q}_b{}^{IJ} - \mathtt{e}g\varepsilon_{ab}\tilde{T}^{AIJ}\mathtt{P}_b{}^A, \qquad (3.44)\\ \partial_0 \mathtt{P}_a{}^A - \partial_a \mathtt{P}_0{}^A &= \frac{1}{2}\mathtt{Q}_0{}^{IJ}\Gamma^{IJ}{}_{AB}\mathtt{P}_a{}^B - \frac{1}{2}\mathtt{Q}_a{}^{IJ}\Gamma^{IJ}{}_{AB}\mathtt{P}_0{}^B \\ &\quad - N^{-1}(\mathtt{Q}_{ab} + \mathtt{P}_{ab})\mathtt{P}_b{}^A + \mathtt{e}g\varepsilon_{ab}(\tilde{T}^{AB} + \delta^{AB}\theta)\mathtt{P}_b{}^B\,. \qquad (3.45)\end{aligned}$$

In analogy with these equations spatial dependence can be retained systematically in all equations.

4. The supergravity/E_{10} correspondence

In this section we compare (a certain truncation) of supergravity to the E_{10} coset model. First, as a consistency check, we compare the dynamics of ungauged supergravity with only time dependence to the $\ell = 0$ truncation of the E_{10} equations of motion. Then, in section 4.2, we discuss ungauged supergravity with the inclusion of certain spatial gradients, that should be related to the $\ell = 1$ truncation of the E_{10} theory. An alternative interpretation of the $\ell = 1$ state is as a gauge vector and so we discuss a possible relation between gauged supergravity and E_{10} in section 4.3. Finally, we analyze the possible E_{10} interpretation of the gauge constraints and quadratic constraints on the supergravity side in section 4.4.

4.1 Ungauged supergravity in $D = 1$

The equations of motion of ungauged supergravity reduced to only time dependence follow from the Lagrangian displayed in (3.35). As this Lagrangian is identical to the $\ell = 0$ part of the Lagrangian of the E_{10} sigma model derived in (2.42) and depends on the same fields, the associated dynamics agrees trivially. The 'dictionary' which achieves this correspondence at level $\ell = 0$ reads

$$\begin{aligned} n(t) \equiv \mathtt{n}(t)\,, \qquad & P_{ab}(t) \equiv \mathtt{P}_{ab}(t)\,, \quad Q_{ab}(t) \equiv \mathtt{Q}_{ab}(t)\,, \\ & P^A(t) \equiv \mathtt{P}_t{}^A(t)\,, \quad Q^{IJ}(t) \equiv \mathtt{Q}_t{}^{IJ}(t)\,, \end{aligned} \qquad (4.1)$$

where $\mathtt{n}(t)$ is defined in (3.36). Here, we have displayed the coset quantities on the left hand side and the supergravity variables on the right hand side – one can also write the correspondence in terms of the coset elements as

$$e_m{}^a(t) \equiv \mathtt{e}_m{}^a(t)\,, \quad \mathcal{E}^{\mathcal{M}}{}_{\mathcal{A}}(t) \equiv \mathtt{E}^{\mathcal{M}}{}_{\mathcal{A}}(t)\,. \qquad (4.2)$$

The only equation besides the equations of motion here is the Hamiltonian constraint and it is mapped to the null condition of the geodesic.

JHEP01(2009)020

When relaxing the strict dimensional reduction we will retain this dictionary except that we will interpret the supergravity variables to be the values at a fixed spatial point $\mathbf{x}_0$, so that the dictionary modifies to

$$\begin{aligned} n(t) \equiv \mathtt{n}(t,\,\mathbf{x}_0)\,, \quad P_{ab}(t) \equiv \mathtt{P}_{ab}(t,\,\mathbf{x}_0)\,, \qquad Q_{ab}(t) \equiv \mathtt{Q}_{ab}(t,\,\mathbf{x}_0)\,, \\ P^A(t) \equiv \mathtt{P}_t{}^A(t,\,\mathbf{x}_0)\,, \quad Q^{IJ}(t) \equiv \mathtt{Q}_t{}^{IJ}(t,\,\mathbf{x}_0)\,, \end{aligned} \tag{4.3}$$

or, in terms of the coset variables,

$$e_m{}^a(t) \equiv \mathtt{e}_m{}^a(t,\,\mathbf{x}_0)\,, \quad \mathcal{E}^{\mathcal{M}}{}_{\mathcal{A}}(t) \equiv \mathtt{E}^{\mathcal{M}}{}_{\mathcal{A}}(t,\,\mathbf{x}_0)\,. \tag{4.4}$$

4.2 Level $\ell = 1$ as spatial gradient

Let us now turn on the fields at level $\ell = 1$ of the coset model. One possible interpretation here is that this corresponds to a spatial gradient — in contrast to the interpretation as a gauge vector, which we will discuss in the next section. For the investigation of spatial gradients it turns out to be useful to compare both sides of the correspondence not at the level of the elementary fields but instead at the level of the derived object $\mathcal{P}$ that carries flat indices. By studying the Einstein equation (2.55a) and the equations of level $\ell = 1$, (2.55c) and (2.55d), one finds after comparison with (3.17), (3.44) and (3.45) that the dictionary on this level is

$$P_a{}^A(t) \equiv N\varepsilon_{ab}\,\mathtt{P}_b{}^A(t,\mathbf{x}_0)\,, \qquad\qquad P_a{}^{IJ}(t) \equiv -N\varepsilon_{ab}\,\mathtt{Q}_b{}^{IJ}(t,\mathbf{x}_0)\,. \tag{4.5}$$

This choice together with (4.1) makes the sigma model equations match largely with the supergravity equations in the absence of gauging, where now the equations of motion at $\ell = 1$ correspond to the integrability constraints (3.44) and (3.45) of the three-dimensional theory. There are, however, terms that do not quite match. First of all, the equation of motion (2.55a) gets translated into

$$R_{ab} = 2\,\mathtt{P}_a{}^A\mathtt{P}_b{}^A + \,\mathtt{Q}_a{}^{IJ}\mathtt{Q}_b{}^{IJ} \tag{4.6}$$

if spatial gradients of the spin connection are truncated as usual in such correspondences [13]. This is not the correct Einstein equation, see (3.17), in that the coefficient of $\mathtt{P}_a{}^A\mathtt{P}_b{}^A$ is 2 rather than 4 and that there is an extra term proportional to $\mathtt{Q}^2$. The first problem is immediately related to a similar discrepancy in the $D = 11$ interpretation of the E_{10} model [13] where one contribution to the only spatial derivatives in the curvature term in $D = 11$ was missing.[7] After reduction to $D = 3$ this problem gets shifted into the scalar sector which explains why the scalar energy-momentum tensor does not have the right coefficient. The $\mathtt{Q}^2$ term arises in a similar way in the sigma model and has no

[7] More precisely, the spatial Ricci tensor R_{ab} in $D = 11$ has contributions (eq. (4.81) in [13]) of the form

$$\frac{1}{4}\Omega_{cd\,a}\Omega_{cd\,b} - \frac{1}{2}\Omega_{ac\,d}\Omega_{bc\,d} - \frac{1}{2}\Omega_{ac\,d}\Omega_{bd\,c} \tag{4.7}$$

and it is the last term which is not reproduced by the sigma model. But it contributes to the scalar energy-momentum tensor in lower dimensions.

counterpart in supergravity (where it would violate the invariance under local SO(16)). The same term was already noticed in [35].

It is noteworthy that there are no difficulties with the spatial curvature in $D = 3$ since the problematic term vanishes completely due to our gauge choice. Indeed, one has that the full spatial anholonomy is given by

$$\Omega_{ab\,c} = -\varepsilon_{ab}\varepsilon_{cd}\Omega_{de\,e}\,. \tag{4.8}$$

Since we always choose the trace $\Omega_{de\,e}$ to vanish, the full spatial anholonomy vanishes in $D = 3$ and gives no contribution to the Ω^2 terms in R_{ab}. In other words, in this gauge choice there is no dual graviton in agreement with its absence in the table of representations of E_{10} under $\mathrm{SL}(2,\mathbb{R}) \times E_8$ (table 1).[8]

The final equation of motion to be compared is the equation of motion for the scalars, (2.55b) on the E_{10} side and (3.22) on the supergravity side. Here, we find agreement in the absence of gauging.

We would like to comment on the interpretation of the dictionary (4.5). One can introduce dual vector fields to the E_8 coset scalars also in the absence of gauging, similar to the duality relation (3.15). These vector fields are the ones that appear in coset element (2.36) at level $\ell = 1$.

4.3 Level $\ell = 2$ and gauged supergravity

In this section we turn to gauged supergravity. First, we employ the interpretation that the level $\ell = 1$ field is not related to (spatial derivatives of) scalars prior to any gauging, but instead the genuine gauge field to be introduced on top of the scalars. According to this picture we will compare to a purely time-like truncation. As the level $\ell = 2$ fields naturally encode the gauging, they will be used at the same time. In a second step we consider the inclusion of spatial gradients in the presence of gauging. For this we will discuss the extension of the dictionary (4.3) and (4.5) to level $\ell = 2$.

We start from the gauged supergravity action (3.38), reduced to one dimension. Since on the E_{10} side there is no analogue of the zero-component of the gauge field $\mathtt{A}_\mu{}^{\mathcal{M}}$, we use the gauge-fixing condition $\mathtt{A}_t{}^{\mathcal{M}} = 0$. Moreover, it turns out to be convenient to rewrite the action entirely in terms of the E_8 'metric' $\mathtt{G}^{\mathcal{MN}}$. For this we use the identity

$$\mathtt{E}^{\mathcal{M}}{}_{A}\mathtt{E}^{\mathcal{N}A} \;=\; \frac{1}{2}\left(\mathtt{G}^{\mathcal{MN}} + \eta^{\mathcal{MN}}\right)\,, \tag{4.9}$$

which follows from the fact that the Cartan-Killing metric $\eta^{\mathcal{MN}}$ differs from $\mathtt{G}^{\mathcal{MN}}$ by a relative sign in the non-compact part. The Lagrangian (3.38) reads

$$\begin{aligned} L_g^{D=1} \;=\; L_{g=0}^{D=1} - \frac{1}{8}g^2\mathtt{e}\mathtt{g}^{mn}(\mathtt{G}^{\mathcal{MN}} + \eta^{\mathcal{MN}})\Theta_{\mathcal{MK}}\Theta_{\mathcal{NL}}\mathtt{A}_m{}^{\mathcal{K}}\mathtt{A}_n{}^{\mathcal{L}} - \mathtt{e}V \\ +\frac{1}{4}g\Theta_{\mathcal{MN}}\varepsilon^{mn}\mathtt{A}_m{}^{\mathcal{M}}\partial_t\mathtt{A}_n{}^{\mathcal{N}}\,. \end{aligned} \tag{4.10}$$

[8] Since gravity in $D = 3$ is not propagating one would not have expected a dual graviton.

For convenience we have here used the conventional formulation without deformation potential, as the field equations merely relate this potential to the embedding tensor. In contrast, the analogous equations on the E_{10} side introduce the embedding tensor.

The 'Einstein' equations obtained by varying with respect to the spatial $\mathtt{g}^{mn}$ read

$$\begin{aligned}&\frac{\delta L_0}{\delta \mathtt{g}^{mn}} + \frac{1}{2}\mathtt{e}\mathtt{g}_{mn}V \\ &\quad + \frac{1}{16}g^2\mathtt{e}(\mathtt{G}^{\mathcal{MN}} + \eta^{\mathcal{MN}})\Theta_{\mathcal{MK}}\Theta_{\mathcal{NL}}\left(\mathtt{g}_{mn}\mathtt{g}^{kl}\mathtt{A}_k{}^{\mathcal{K}}\mathtt{A}_l{}^{\mathcal{L}} - 2\mathtt{A}_m{}^{\mathcal{K}}\mathtt{A}_n{}^{\mathcal{L}}\right) \;=\; 0\,,\end{aligned} \tag{4.11}$$

while for the scalar equations we find

$$\begin{aligned}&\frac{\delta L_0}{\delta \mathtt{G}^{\mathcal{MN}}} - \frac{1}{8}g^2\mathtt{e}\mathtt{g}^{mn}\Theta_{\mathcal{MK}}\Theta_{\mathcal{NL}}\mathtt{A}_m{}^{\mathcal{K}}\mathtt{A}_n{}^{\mathcal{L}} \\ &\qquad - \frac{1}{7\cdot 32}\mathtt{e}g^2\mathtt{G}^{\mathcal{KL}}\Theta_{\mathcal{MK}}\Theta_{\mathcal{NL}} - \frac{1}{16}\mathtt{e}g^2\eta^{\mathcal{KL}}\Theta_{\mathcal{MK}}\Theta_{\mathcal{NL}} \;=\; 0\,,\end{aligned} \tag{4.12}$$

using the explicit form of the scalar potential in (3.13). Here we do not write out the variation of L_0, since we verified already that this Lagrangian coincides on both sides of the correspondence. Finally, varying with respect to the non-propagating vector fields $\mathtt{A}_m{}^{\mathcal{M}}$ yields the one-dimensional form of the duality relation,

$$g\Theta_{\mathcal{MN}}\varepsilon^{mn}\partial_t\mathtt{A}_n{}^{\mathcal{N}} + \frac{1}{2}g^2\mathtt{e}(\mathtt{G}^{\mathcal{KL}} + \eta^{\mathcal{KL}})\mathtt{g}^{mn}\Theta_{\mathcal{MK}}\Theta_{\mathcal{NL}}\mathtt{A}_n{}^{\mathcal{N}} \;=\; 0\,. \tag{4.13}$$

At first sight these equations are rather different from the sigma model equations, which are given by

$$\begin{aligned}&\frac{\delta L_0}{\delta g^{mn}} + \frac{1}{8}n^{-1}\mathcal{G}_{\mathcal{MN}}\partial_t A_m{}^{\mathcal{M}}\partial_t A_n{}^{\mathcal{N}} \\ &\qquad + \frac{1}{8}n^{-1}(\det g)^{-1}g_{mn}\left(D_tBD_tB + 14\mathcal{G}_{\mathcal{MP}}\mathcal{G}_{\mathcal{NQ}}D_tB^{\mathcal{MN}}D_tB^{\mathcal{PQ}}\right) = \;0\,, \\ &\frac{\delta L_0}{\delta G_{\mathcal{MN}}} + \frac{1}{8}n^{-1}g^{mn}\partial_t A_m{}^{\mathcal{M}}\partial_t A_n{}^{\mathcal{N}} + \frac{14}{4}n^{-1}(\det g)^{-1}\mathcal{G}_{\mathcal{KL}}D_tB^{\mathcal{MK}}D_tB^{\mathcal{NL}} = \;0\end{aligned} \tag{4.14}$$

for the $\ell = 0$ fields, and by (2.56) and (2.58) for the higher-level fields. Consistent with the field equations, we set in the following $D_tB_{mn}{}^{\mathcal{M}} = 0$, since their meaning will be discussed below.

We will see that the equations on both sides are more closely related, if one uses the observation that in $D = 1$ second-order equations can be integrated to first-order equations. For instance, the equation (2.56) gives rise to integration constants which can be identified with the components of the embedding tensor,

$$\begin{aligned}n^{-1}(\det g)^{-1}D_tB \;&=\; c_1 g\theta\,, \\ n^{-1}(\det g)^{-1}\mathcal{G}_{\mathcal{MP}}\mathcal{G}_{\mathcal{NQ}}D_tB^{\mathcal{PQ}} \;&=\; c_2 g\tilde{\Theta}_{\mathcal{MN}}\,,\end{aligned} \tag{4.15}$$

where c_1 and c_2 are two arbitrary constants. This allows to almost recover the duality relation (4.13) from the E_{10} equations of motion (2.58). First, (2.58) may be rewritten as

$$\partial_t\left(n^{-1}g^{mn}\mathcal{G}_{\mathcal{MN}}\partial_t A_n{}^{\mathcal{N}} + \frac{1}{2}c_1 g\varepsilon^{mn}\eta_{\mathcal{MN}}\theta A_n{}^{\mathcal{N}} + 14c_2 g\varepsilon^{mn}\tilde{\Theta}_{\mathcal{MN}}A_n{}^{\mathcal{N}}\right) \;=\; 0\,. \tag{4.16}$$

Therefore, it can be integrated to the first-order equation

$$n^{-1}g^{mn}\mathcal{G}_{\mathcal{MN}}\partial_t A_n{}^{\mathcal{N}} \;=\; g\varepsilon^{mn}\Theta_{\mathcal{MN}}A_n{}^{\mathcal{N}} + \Xi^m{}_{\mathcal{M}}\;. \tag{4.17}$$

Here we have chosen the free constants to be $c_1 = 2$ and $c_2 = 1/14$ in order to conveniently combine the irreducible parts of the embedding tensor into $\Theta_{\mathcal{MN}}$ according to [7]. Moreover, $\Xi^m{}_{\mathcal{M}}$ denotes an integration constant. This integration constant cannot be set to zero without breaking the symmetries. The situation is analogous to the integration leading to the embedding tensor $\Theta_{\mathcal{MN}}$ in (4.15), which generically breaks the global E_8 symmetry once $\Theta_{\mathcal{MN}}$ is constant. Correspondingly, the E_{10} shift symmetry leaves this first-order equation only invariant if the integration constant also transforms as a shift,

$$\delta_\Lambda \Xi^m{}_{\mathcal{M}} \;=\; -g\varepsilon^{mn}\Theta_{\mathcal{MN}}\Lambda_n{}^{\mathcal{N}}\;, \tag{4.18}$$

which is consistent with the time-independence of Ξ. Thus, fixing it to any specific value (as zero) breaks the symmetry, and in this sense supergravity may at best be viewed as a broken phase of E_{10}. After setting $\Xi = 0$ and contracting with $\Theta_{\mathcal{MN}}$, (4.17) implies

$$g\Theta_{\mathcal{MN}}\varepsilon^{mn}\partial_t A_n{}^{\mathcal{N}} + g^2 e N\mathcal{G}^{\mathcal{KL}}g^{mn}\Theta_{\mathcal{MK}}\Theta_{\mathcal{NL}}A_n{}^{\mathcal{N}} \;=\; 0\;, \tag{4.19}$$

which coincides with the duality relation (4.13) from supergravity up to the replacement $\mathcal{G}^{\mathcal{MN}} \to \frac{1}{2}(\mathcal{G}^{\mathcal{MN}} + \eta^{\mathcal{MN}})$.

Finally, insertion of (4.15) and (4.19) into the equations of motion (4.14) for g_{mn} and $G^{\mathcal{MN}}$ as obtained from E_{10} yields

$$\begin{aligned}
&\frac{\delta L_0}{\delta g^{mn}} + \frac{1}{8}g^2\mathtt{e}G^{\mathcal{MN}}\Theta_{\mathcal{MK}}\Theta_{\mathcal{NL}}\left(g_{mn}g^{kl}A_k{}^{\mathcal{K}}A_l{}^{\mathcal{L}} - A_m{}^{\mathcal{K}}A_n{}^{\mathcal{L}}\right)\\
&\qquad\qquad + \frac{1}{2}g^2\mathtt{e}g_{mn}\left(\frac{1}{56}\mathcal{G}^{\mathcal{MK}}\mathcal{G}^{\mathcal{NL}}\tilde{\Theta}_{\mathcal{MN}}\tilde{\Theta}_{\mathcal{KL}} + \theta^2\right) \;=\; 0\;,\\
&\frac{\delta L_0}{\delta G^{\mathcal{MN}}} - \frac{1}{8}g^2\mathtt{e}g^{mn}\Theta_{\mathcal{MK}}\Theta_{\mathcal{NL}}A_m{}^{\mathcal{K}}A_n{}^{\mathcal{L}} - \frac{1}{56}g^2\mathtt{e}\mathcal{G}^{\mathcal{KL}}\tilde{\Theta}_{\mathcal{MK}}\tilde{\Theta}_{\mathcal{NL}} \;=\; 0\;.
\end{aligned} \tag{4.20}$$

Here, we have used (3.36) and (4.3). By comparing (4.20) with (4.11) and (4.12) we observe that the equations are structure-wise the same, but differ in the details. For one thing, on the E_{10} side we generically have just $\mathcal{G}^{\mathcal{MN}}$ instead of $\frac{1}{2}(\mathcal{G}^{\mathcal{MN}} + \eta^{\mathcal{MN}})$. Apart from that, the indefinite contributions to the supergravity potential are not reproduced, but only the leading term quadratic in $\mathcal{G}^{\mathcal{MN}}$.

Let us now inspect the simultaneous inclusion of gauge couplings and spatial gradients. As before this requires an analysis at the level of $\mathcal{P}$ that carries flat indices. Specifically, we can supplement the dictionary (4.3) and (4.5) with

$$P^{\mathcal{AB}}(t) \equiv \frac{1}{28}Ng\tilde{T}_{\mathcal{AB}}(t,\mathbf{x}_0), \qquad\qquad P(t) \equiv Ng\theta(t,\mathbf{x}_0) \tag{4.21}$$

on level $\ell = 2$. This dictionary is derived from the integrability conditions (3.44)–(3.45) such that they match exactly the common terms in the equations (2.55d)–(2.55c) for E_{10} (the terms involving Q_a^{IJ} do not match just as in the Einstein equation (4.6)). Moreover,

we have 'covariantized' the dictionary since it only fixes P^{AB} and $P^{A\,IJ}$, but not $P^{IJ\,KL}$. However, using the dictionary (4.21) in the Einstein equation one finds that the scalar potential is not reproduced correctly. The terms coming from the positive definite $(A_2^{I\dot{A}})^2$ contribution in (3.11), however, appear precisely in the E_{10} Einstein equation. If we only consider the terms in the scalar equation of motion arising from the positive definite part, then the dictionary (including also P^{IJKL}) gives the correct relative coefficients, but the overall coefficient is wrong. This can be seen by comparing (3.20) and (2.55b). For the full potential we find disagreement since the potential is not positive-definite, unlike the Cartan-Killing form used on the E_{10} side, and one can see that there is no choice for the dictionary such that all equations match. In addition, it is not the case that E_{10} predicts a different potential. Rather, the scalar dependence in the E_{10} equations is such that it cannot be integrated to a corresponding single scalar potential in a $D = 3$ field theory. To summarize, while there is no precise agreement between the corresponding equations, the E_{10} model predicts and provides an embedding tensor in the correct E_8 representation, which in the present truncation is forced to be constant by the geodesic equations. It is noteworthy that the E_{10} model naturally contains both the constant embedding tensor and the scalar field dependent T-tensor via dressing with the level zero vielbein.

Finally, we comment on the meaning of the field $B_{mn}{}^{\mathcal{M}}$, which we truncated so far. One possible interpretation might be as a spatial gradient. Another attractive scenario is that it is related to a novel type of gauging, the so-called trombone gauging, which has recently appeared in the literature [36]. This gauging gives rise to embedding tensor components $\Theta_{\mathcal{M}}$, and it has been noted that they are in one-to-one correspondence with certain mixed Young tableaux representations within E_{11} and E_{10} [36]. Applied to $D = 3$ these degenerate to the symmetric $B_{mn}{}^{\mathcal{M}}$ and so one might hope to interpret this as a trombone gauging. However, given the ambiguity of the possible interpretations encountered so far, we postpone a detailed analysis of this proposal to future work.

4.4 Quadratic and gauge constraints

We now turn to a discussion of the constraint equations that supplement the dynamical equations discussed so far. From the E_{10} point of view these have to be considered as additional constraints on the geodesic. In [14] it has been shown that the constraint equations in maximal eleven-dimensional supergravity can be consistently imposed on the geodesic and are weakly conserved as the system evolves. Furthermore, the constraints there followed an intriguing pattern, displaying a certain grading property reminiscent of a Sugawara-type construction in terms of bilinear products of conserved currents. Here, we will encounter a similar phenomenon which extends up to the quadratic constraint, probing generators of E_{10} beyond the analysis carried out in [14].

Besides the Hamiltonian constraint, the constraint equations which have to be studied in the present context are

(i) the diffeomorphism constraint (the $(0a)$ component of the Einstein equation (3.17)),

(ii) the Gauss constraint (3.41) or (3.24),

(iii) the spatial constancy of $\Theta_{\mathcal{MN}}$ (3.42) and

(iv) the quadratic constraint (3.43) of standard gauging and possibly trombone gauging.

The first one arises from gauge fixing the shift vector $N^a = 0$, whereas the other three are all consequences of adopting the temporal gauges (3.37) for the tensors of gauged supergravity. There are no additional Bianchi type constraints as there were for $D = 11$ supergravity in [14] since these vanish identically in $D = 3$. For example, the equation $D_{[a}\mathtt{F}_{bc]}{}^{\mathcal{M}} = 0$ is fulfilled trivially since there are no three distinct spatial indices a, b, c.

Analyzing the four constraint equations with the use of the dictionaries derived in (4.1), (4.5) and (4.15), and using the duality relation (3.15), one finds that they have the schematic form

$$\begin{aligned}
\mathfrak{C}_a &= P_a{}^A P^A \,, \\
\mathfrak{C}^{\mathcal{A}} &= P^{\mathcal{A}B} P^B + f^{\mathcal{A}}{}_{\mathcal{BC}} \varepsilon^{ab} P_a{}^{\mathcal{B}} P_b{}^{\mathcal{C}} \,, \\
\mathfrak{C}_a{}^{\mathcal{AB}} &= f^{(\mathcal{A}}{}_{\mathcal{CD}} P^{\mathcal{B})\mathcal{C}} P_a{}^{\mathcal{D}} \,, \\
\mathfrak{C}^{\mathcal{AB},\mathcal{C}} &= P^{\mathcal{CD}} P^{\mathcal{E}(\mathcal{A}} f_{\mathcal{DE}}{}^{\mathcal{B})} \,,
\end{aligned} \tag{4.22}$$

in flat indices (where the SO(16) spinor indices A and B should not be confused with the adjoint E_8 indices $\mathcal{A}$ and $\mathcal{B}$). The important feature of these equations is the tensor structure and the fact that the levels of the P components occurring on the right hand side always add up to the same number in each constraint. In this way one can assign to the four equations the 'levels' $\ell = 1, 2, 3, 4$, respectively, since in the first one the combinations are $P^{(0)}P^{(1)}$ up to $P^{(2)}P^{(2)}$ in the last equation. Furthermore, they transform (after conversion to curved indices) in the $GL(2, \mathbb{R}) \times E_8$ representations indicated. As in [14] we can thus bring the above constraints into a Sugawara-like form by switching to curved indices $m, n, \ldots$ and $\mathcal{M}, \mathcal{N}, \ldots$, and by replacing the P's by the corresponding components of the conserved E_{10} Noether current.

In [14] it was also noted that the representation content of the graded constraints is very similar to that of a specific highest weight representation of E_{10}, sometimes called $L(\Lambda_1)$ as it is the highest weight module with highest weight corresponding to the fundamental weight of node 1 of the E_{10} Dynkin diagram in figure 1. We give the decomposition of this representation with respect to $\mathrm{SL}(2, \mathbb{R}) \times E_8$ at low levels in table 2. From this table we see that there is again agreement between the representations of the constraints at low levels and the tensors contained in the $L(\Lambda_1)$ representation. At higher levels there appear extra representations, some of which can probably be interpreted as recurrences (higher order gradients) of the constraints encountered before but this explanation seems incomplete and therefore we have partly left the interpretation open.

We note that it is to be expected that the constraints only form a representation of a Borel subgroup $E_{10}^{+} \subset E_{10}$ rather than of the whole E_{10} since explicit calculations of the transformation of the diffeomorphism constraint show that it is not annihilated by elements of the conjugate subgroup E_{10}^{-} [14].[9]

[9]Here, the $\pm$ superscripts on E_{10} should not be confused with further Kac-Moody extensions of E_{10} but refer to Borel subgroups generated by positive and negative level generators, respectively.

JHEP01(2009)020

Level ℓ	SL(2, $\mathbb{R}$) $\times$ E_8 representation	Interpretation
1	$(\mathbf{2},\mathbf{1})$	Diffeomorphism constraint
2	$(\mathbf{1},\mathbf{248})$	Gauss constraint
3	$(\mathbf{2},\mathbf{1})$	Spatial constancy of θ
	$(\mathbf{2},\mathbf{3875})$	Spatial constancy of $\tilde{\Theta}_{\mathcal{MN}}$
	$(\mathbf{2},\mathbf{248})$	Spatial constancy of $\Theta_{\mathcal{M}}$ (trombone)?
4	$(\mathbf{1},\mathbf{147250})$	Quadratic constraint
	$(\mathbf{3},\mathbf{30380})$	Quadratic constraint of trombone?
	$(\mathbf{1},\mathbf{30380})$	?
	$(\mathbf{3},\mathbf{3875})$	Quadratic constraint of trombone?
	$2\times(\mathbf{1},\mathbf{3875})$	Quadratic constraint
	$2\times(\mathbf{3},\mathbf{248})$	Quadratic constraint of trombone?
	$2\times(\mathbf{1},\mathbf{248})$	Recurrence of Gauss?
	$(\mathbf{3},\mathbf{1})$	Quadratic constraint of trombone?
	$(\mathbf{1},\mathbf{1})$	Recurrence of θ?

Table 2: SL(2, $\mathbb{R}$) $\times$ E_8 decomposition of $L(\Lambda_1)$ highest weight representation of E_{10}.

5. Discussion and outlook

In this paper we explored the E_{10}/supergravity correspondence for the case of gauged supergravity. Apart from the inclusion of spatial gradients and/or mass parameters discussed in the literature so far, this provides additional insights into the interpretation of part of the higher-level representations within E_{10}. As has been found before, in general dimensions D there are $(D-1)$-forms whose representations coincide with those of consistent gaugings in supergravity. Moreover, here we found that the quadratic constraint of gauged supergravity belongs to the same highest weight representation of E_{10} as the diffeomorphism constraint (but, we repeat, the constraints transform properly only under the Borel part E_{10}^{+} of that representation). In contrast, in the E_{11} approach the D-form Lagrangian multiplier for this constraint arises as one of the higher-level fields. While at a purely kinematical level the Kac-Moody algebras E_{10} and E_{11} therefore encode gauged supergravity, the sigma model theory discussed in this paper allows, in addition, to check the correspondence at the level of dynamics.

Most remarkably, we find that the equations of motion of gauged supergravity (here for the example of three space-time dimensions) adapted to a one-dimensional language can in part be matched to the E_{10} equations, even though the latter have a priori a rather different form. For one thing, the absence of gauge-covariant derivatives on the E_{10} side agrees with the supergravity expressions, once the gauge-fixing condition $\mathtt{A}_t{}^{\mathcal{M}} = 0$, which is inevitable for the comparison, has been imposed. Moreover, in spite of the fact that on the E_{10} side all fields appear with a 'kinetic' term, the (truncated) duality relation between vectors and scalars expected from supergravity naturally follows via integrating the one-dimensional equations of motion. Finally, the embedding tensor automatically appears as an integration constant in the right representation. In this sense, none of the

essential ingredients of gauged supergravity have to be introduced by hand, but rather they naturally follow from the E_{10} sigma model.

Irrespective of these promising observations, there remain mismatches at higher levels, which prohibit a full agreement between supergravity and the E_{10} model. One finds systematically that while in supergravity the combination $\frac{1}{2}(\mathtt{G}_{\mathcal{MN}} + \eta_{\mathcal{MN}})$ appears, the corresponding equations on the E_{10} side only contain $\mathcal{G}_{\mathcal{MN}}$. Similarly, the scalar potential is not fully reproduced by E_{10}. This is due to the fact that in supergravity the scalar potential is indefinite [33], while the corresponding 2-forms appearing in the E_{10} coset model necessarily enter with a positive-definite kinetic term. The latter is somewhat reminiscent to a discrepancy encountered in higher dimensions, once spatial gradients are introduced as the duals of higher-level fields.

In total we are led to conclude that further insights are required in order to understand the precise relation between supergravity theories and the E_{10} sigma model. It would be interesting to see whether modifications and/or extensions of the E_{10} model are possible to compensate for the present mismatches. We note that mismatches already occur before comparing to *gauged* supergravity and so an ultimate resolution of the present discrepancies must await a better understanding of the basic picture.

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A. Conventions for E_8

Our conventions for E_8 are as in [7]. The Lie algebra $\mathfrak{e}_8$ is generated by $t^{\mathcal{M}}$, with $\mathcal{M},\mathcal{N},\ldots = 1,\ldots,248$ denoting the adjoint indices, and bracket $[t^{\mathcal{M}}, t^{\mathcal{N}}] = f^{\mathcal{MN}}{}_{\mathcal{K}} t^{\mathcal{K}}$. Specifically, $\mathfrak{e}_8$ can be defined according to its $\mathfrak{so}(16)$ decomposition,

$$\begin{aligned} [t^{IJ}, t^{KL}] &= 4\delta^{JK} t^{IL} \,, \\ [t^{IJ}, t^{A}] &= -\frac{1}{2}\Gamma^{IJ}{}_{AB} t^{B} \,, \\ [t^{A}, t^{B}] &= \frac{1}{4}\Gamma^{IJ}{}_{AB} t^{IJ} \,. \end{aligned} \tag{A.1}$$

Here $I,J,\ldots = 1,\ldots,16$ are SO(16) vector indices, while $A,B,\ldots = 1,\ldots,128$ label spinor indices. The adjoint indices split according to $\mathcal{A} = ([IJ], A)$, where we employ the convention that summation over the antisymmetric $[IJ]$ is accompanied by a factor of $\frac{1}{2}$.

The spinor generators are defined by $\Gamma^I{}_{A\dot{A}}\Gamma^J{}_{B\dot{A}} = \delta^{IJ}\delta_{AB} + \Gamma^{IJ}{}_{AB}$. Like any other Kac-Moody algebra, $\mathfrak{e}_8$ admits an invariant Cartan-Killing form, which in the SO(16) decomposition (A.1) reads

$$\eta^{AB} \;=\; \delta^{AB}\,, \qquad \eta^{IJKL} \;=\; -2\delta^{IK}\delta^{JL}. \tag{A.2}$$

Accordingly, in the totally antisymmetric structure constants

$$\begin{aligned} f^{IJKLMN} \;&=\; -f^{IJKL}{}_{MN} \;=\; 8\delta^{IK}\delta^J{}_M\delta^L{}_N \\ f^{IJAB} \;&=\; -f_{IJ}{}^{AB} \;=\; -\frac{1}{2}\Gamma^{IJ}{}_{AB} \end{aligned} \tag{A.3}$$

we can freely raise and lower indices. We recall that we use the convention that the right hand side is always to be antisymmetrized in the same way as the left hand side. The E_8 structure constants and the Killing form are related by the identity $f^{\mathcal{AB}}{}_{\mathcal{C}}f_{\mathcal{ABD}} = -60\eta_{\mathcal{CD}}$, which implies $f^{\mathcal{ABC}}f_{\mathcal{ABC}} = -14880$. We also frequently use the relation

$$\mathtt{E}^{-1}t^{\mathcal{M}}\mathtt{E} \;=\; \mathtt{E}^{\mathcal{M}}{}_{\mathcal{A}}t^{\mathcal{A}} \tag{A.4}$$

for the adjoint matrix $\mathtt{E} \in E_8$, which can be easily checked by use of the Baker-Campbell-Hausdorff formula (2.44).

The tensor product of two adjoint representations decomposes as

$$\mathbf{248} \times \mathbf{248} = \mathbf{1} + \mathbf{248} + \mathbf{3875} + \mathbf{27000} + \mathbf{30380}, \tag{A.5}$$

and the corresponding projectors have the components [31]

$$\begin{aligned} (\mathbb{P}_{\mathbf{1}})_{\mathcal{AB}}{}^{\mathcal{CD}} &= \frac{1}{248}\eta_{\mathcal{AB}}\eta^{\mathcal{CD}}, \\ (\mathbb{P}_{\mathbf{248}})_{\mathcal{AB}}{}^{\mathcal{CD}} &= -\frac{1}{60}f^{\mathcal{E}}{}_{\mathcal{AB}}f_{\mathcal{E}}^{\mathcal{CD}}, \\ (\mathbb{P}_{\mathbf{3875}})_{\mathcal{AB}}{}^{\mathcal{CD}} &= \frac{1}{7}\delta_{(\mathcal{A}}{}^{\mathcal{C}}\delta_{\mathcal{B})}{}^{\mathcal{D}} - \frac{1}{56}\eta_{\mathcal{AB}}\eta^{\mathcal{CD}} - \frac{1}{14}f^{\mathcal{E}}{}_{\mathcal{A}}{}^{(\mathcal{C}}f_{\mathcal{EB}}{}^{\mathcal{D})}, \\ (\mathbb{P}_{\mathbf{27000}})_{\mathcal{AB}}{}^{\mathcal{CD}} &= \frac{6}{7}\delta_{(\mathcal{A}}{}^{\mathcal{C}}\delta_{\mathcal{B})}{}^{\mathcal{D}} + \frac{3}{217}\eta_{\mathcal{AB}}\eta^{\mathcal{CD}} + \frac{1}{14}f^{\mathcal{E}}{}_{\mathcal{A}}{}^{(\mathcal{C}}f_{\mathcal{EB}}{}^{\mathcal{D})}, \\ (\mathbb{P}_{\mathbf{30380}})_{\mathcal{AB}}{}^{\mathcal{CD}} &= \delta_{[\mathcal{A}}{}^{\mathcal{C}}\delta_{\mathcal{B}]}{}^{\mathcal{D}} + \frac{1}{60}f^{\mathcal{E}}{}_{\mathcal{AB}}f_{\mathcal{E}}^{\mathcal{CD}}. \end{aligned} \tag{A.6}$$

Elsewhere in the paper, we have dropped the subscript on $\mathbb{P}_{\mathbf{3875}}$. Splitting the indices, we get the following identities for a tensor $\tilde{T}^{\mathcal{AB}}$ that transforms in the **3875** representation:

$$\begin{aligned} \tilde{T}^{AIJ} &= -\frac{1}{6}\Gamma^{IK}{}_{AB}\tilde{T}^{BJK} = \frac{1}{26}\Gamma^{IJKL}{}_{AB}\tilde{T}^{BKL}, \\ \tilde{T}^{IJKL} &= \frac{3}{7}\delta^{IK}\tilde{T}^{JMLM} - \tilde{T}^{IKJL}, \\ \tilde{T}^{AB} &= \frac{1}{96}\Gamma^{IJKL}{}_{AB}\tilde{T}^{IJKL}. \end{aligned} \tag{A.7}$$

The two equations in the first line are equivalent. The last equation can be inverted to

$$\Gamma^{IJKL}{}_{AB}\tilde{T}^{AB} = 32\,\tilde{T}^{[IJKL]}. \tag{A.8}$$

We also note that $\tilde{T}^{IJIJ} = \tilde{T}^{AA} = 0$.

JHEP01(2009)020

B. Level decomposition of E_{10}

To determine the E_{10} commutation relation (2.18), we needed to identify the Chevalley generators, which are the 30 elements h_i, e_i, f_i ($i = 1, 2, \ldots, 10$) that satisy the Chevalley-Serre relations (2.2) and (2.3). We let any $x \in \mathfrak{e}_8$ have the components $x_{\mathcal{A}}$ in the $t^{\mathcal{A}}$ basis, $x = x_{\mathcal{A}} t^{\mathcal{A}}$. Then we get

$$\begin{aligned}
&e_1 = K^1{}_2, && e_2 = (-f_\theta)_{\mathcal{A}} E^2{}_{\mathcal{B}} \eta^{\mathcal{A}\mathcal{B}}, && e_i = (e_i)_{\mathcal{A}} t^{\mathcal{A}}, \\
&h_1 = K^1{}_1 - K^2{}_2, && h_2 = (-h_\theta)_{\mathcal{A}} t^{\mathcal{A}} - K^1{}_1, && h_i = (h_i)_{\mathcal{A}} t^{\mathcal{A}}, \\
&f_1 = K^2{}_1, && f_2 = (-e_\theta)_{\mathcal{A}} F_2{}^{\mathcal{A}}, && f_i = (f_i)_{\mathcal{A}} t^{\mathcal{A}},
\end{aligned} \tag{B.1}$$

for $i = 3, 4, \ldots, 10$. Here θ (not to be confused with the singlet embedding tensor) denotes the highest root of $\mathfrak{e}_8$, with the corresponding step operators e_θ, f_θ and Cartan element h_θ. We have

$$h_\theta = 2h_3 + 3h_4 + 4h_5 + 5h_6 + 6h_7 + 4h_8 + 2h_9 + 3h_{10} \tag{B.2}$$

and we get

$$K^1{}_1 = -h_\theta - h_2, \qquad K^2{}_2 = -h_\theta - h_2 - h_1, \qquad K = -2h_\theta - 2h_2 - h_1. \tag{B.3}$$

By inserting (B.1) into (2.2) and using (2.15), we see that the Chevalley relations $[h_i, e_j] = A_{ij} e_j$ and $[h_i, f_j] = -A_{ij} f_j$ are indeed satisfied. For the remaining relations to hold, $[e_i, f_j] = \delta_{ij} h_i$, we must have

$$[E^a{}_{\mathcal{A}},\, F_b{}^{\mathcal{B}}] = \delta^a{}_b f_{\mathcal{A}}{}^{\mathcal{B}}{}_{\mathcal{C}} t^{\mathcal{C}} + \delta_{\mathcal{A}}{}^{\mathcal{B}} K^a{}_b - \delta_{\mathcal{A}}{}^{\mathcal{B}} \delta^a{}_b K, \tag{B.4}$$

where we have set $K = K^a{}_a = K^1{}_1 + K^2{}_2$. The relations (2.17) and (2.18) can then be inverted to

$$\begin{aligned}
E &= \frac{1}{248} \varepsilon_{ab} \eta^{\mathcal{A}\mathcal{B}} [E^a{}_{\mathcal{A}},\, E^b{}_{\mathcal{B}}], \\
E_{\mathcal{A}\mathcal{B}} &= \frac{1}{2} \varepsilon_{ab} [E^a{}_{\mathcal{A}},\, E^b{}_{\mathcal{B}}] - \frac{1}{496} \varepsilon_{ab} \eta_{\mathcal{A}\mathcal{B}} \eta^{\mathcal{C}\mathcal{D}} [E^a{}_{\mathcal{C}},\, E^b{}_{\mathcal{D}}], \\
E^{ab}{}_{\mathcal{A}} &= \frac{1}{60} f_{\mathcal{A}}{}^{\mathcal{B}\mathcal{C}} [E^a{}_{\mathcal{B}},\, E^b{}_{\mathcal{C}}],
\end{aligned} \tag{B.5}$$

$$\begin{aligned}
F &= -\frac{1}{248} \varepsilon^{ab} \eta_{\mathcal{A}\mathcal{B}} [F_a{}^{\mathcal{A}},\, F_b{}^{\mathcal{B}}], \\
F^{\mathcal{A}\mathcal{B}} &= -\frac{1}{2} \varepsilon^{ab} [F_a{}^{\mathcal{A}},\, F_b{}^{\mathcal{B}}] + \frac{1}{496} \varepsilon^{ab} \eta^{\mathcal{A}\mathcal{B}} \eta_{\mathcal{C}\mathcal{D}} [F_a{}^{\mathcal{C}},\, F_b{}^{\mathcal{D}}], \\
F_{ab}{}^{\mathcal{A}} &= \frac{1}{60} f^{\mathcal{A}}{}_{\mathcal{B}\mathcal{C}} [F_a{}^{\mathcal{B}},\, F_b{}^{\mathcal{C}}],
\end{aligned} \tag{B.6}$$

$$\begin{aligned}
t^{\mathcal{A}} &= -\frac{1}{120} f^{\mathcal{A}\mathcal{B}}{}_{\mathcal{C}} [E^a{}_{\mathcal{B}},\, F_a{}^{\mathcal{C}}], \\
K^a{}_b &= \frac{1}{248} ([E^a{}_{\mathcal{A}},\, F_b{}^{\mathcal{A}}] - \delta^a{}_b [E^c{}_{\mathcal{A}},\, F_c{}^{\mathcal{A}}]).
\end{aligned} \tag{B.7}$$

JHEP01(2009)020

The remaining nonzero commutation relations follow from the Jacobi identity,

$$
\begin{aligned}
[E,\, F_a{}^{\mathcal{A}}] &= -\frac{1}{2}\varepsilon_{ab}\eta^{\mathcal{AB}}E^b{}_{\mathcal{B}}, & [F,\, E^a{}_{\mathcal{A}}] &= \frac{1}{2}\varepsilon^{ab}\eta_{\mathcal{AB}}F_b{}^{\mathcal{B}},\\
[E^{ab}{}_{\mathcal{A}},\, F_c{}^{\mathcal{B}}] &= -\delta^a{}_c f_{\mathcal{A}}{}^{\mathcal{BC}}E^b{}_{\mathcal{C}}, & [F_{ab}{}^{\mathcal{A}},\, E^c{}_{\mathcal{B}}] &= -\delta^c{}_a f^{\mathcal{A}}{}_{\mathcal{BC}}F_b{}^{\mathcal{C}},\\
[E_{\mathcal{AB}},\, F_a{}^{\mathcal{C}}] &= -14\varepsilon_{ab}\mathbb{P}_{\mathcal{AB}}{}^{\mathcal{CD}}E^b{}_{\mathcal{D}}, & [F^{\mathcal{AB}},\, E^a{}_{\mathcal{C}}] &= 14\varepsilon^{ab}\mathbb{P}^{\mathcal{AB}}{}_{\mathcal{CD}}F_b{}^{\mathcal{D}},\\
[E_{\mathcal{AB}},\, F^{\mathcal{CD}}] &= 2f^{\mathcal{C}}{}_{\mathcal{AE}}f^{\mathcal{ED}}{}_{\mathcal{F}}f^{\mathcal{F}}{}_{\mathcal{BG}}t^{\mathcal{G}} - 4\delta_{\mathcal{A}}{}^{\mathcal{C}}f^{\mathcal{D}}{}_{\mathcal{BE}}t^{\mathcal{E}} - 14\mathbb{P}_{\mathcal{AB}}{}^{\mathcal{CD}}K, & &\\
[E^{ab}{}_{\mathcal{A}},\, F_{cd}{}^{\mathcal{B}}] &= f_{\mathcal{A}}{}^{\mathcal{B}}{}_{\mathcal{C}}t^{\mathcal{C}} + 2\delta_{\mathcal{A}}{}^{\mathcal{B}}(\delta^a{}_c K^b{}_d - \delta^a{}_c\delta^b{}_d K), & [E,\, F] &= -K.
\end{aligned}
\tag{B.8}
$$

Here we have used that $\varepsilon^{ac}\varepsilon_{cb} = -\delta^a{}_b$ with our conventions. Using the invariance of the Cartan-Killing form, we have

$$
-\frac{1}{4}\langle [E^a{}_{\mathcal{A}},\, E^b{}_{\mathcal{B}}] | [F_c{}^{\mathcal{C}},\, F_d{}^{\mathcal{D}}] \rangle = (31\mathbb{P}_{(\mathbf{1},\mathbf{1})} + 15\mathbb{P}_{(\mathbf{3},\mathbf{248})} + 7\mathbb{P}_{(\mathbf{1},\mathbf{3875})})^{ab}{}_{cd}{}^{\mathcal{CD}}{}_{\mathcal{AB}}, \tag{B.9}
$$

where $\mathbb{P}_{(\mathbf{1},\mathbf{1})}$, $\mathbb{P}_{(\mathbf{3},\mathbf{248})}$ and $\mathbb{P}_{(\mathbf{1},\mathbf{3875})}$ are the projectors corresponding to the $\mathrm{SL}(2,\mathbb{R}) \times E_8$ representations at level $\ell = 2$ (cf. table 1). Explicitly,

$$
\begin{aligned}
\mathbb{P}_{(\mathbf{1},\mathbf{1})}{}^{ab}{}_{cd}{}^{\mathcal{CD}}{}_{\mathcal{AB}} &= \delta^a{}_{[c}\delta^b{}_{d]}\mathbb{P}_{\mathbf{1}}{}^{\mathcal{CD}}{}_{\mathcal{AB}},\\
\mathbb{P}_{(\mathbf{3},\mathbf{248})}{}^{ab}{}_{cd}{}^{\mathcal{CD}}{}_{\mathcal{AB}} &= \delta^a{}_{(c}\delta^b{}_{d)}\mathbb{P}_{\mathbf{248}}{}^{\mathcal{CD}}{}_{\mathcal{AB}},\\
\mathbb{P}_{(\mathbf{1},\mathbf{3875})}{}^{ab}{}_{cd}{}^{\mathcal{CD}}{}_{\mathcal{AB}} &= \delta^a{}_{[c}\delta^b{}_{d]}\mathbb{P}_{\mathbf{3875}}{}^{\mathcal{CD}}{}_{\mathcal{AB}},
\end{aligned}
\tag{B.10}
$$

where the E_8 projectors $\mathbb{P}_{\mathbf{1}}$, $\mathbb{P}_{\mathbf{248}}$ and $\mathbb{P}_{\mathbf{3875}}$ [31] were already given in (A.6).

References

[1] W. Nahm, *Supersymmetries and their representations*, *Nucl. Phys.* **B 135** (1978) 149.

[2] E. Cremmer, B. Julia and J. Scherk, *Supergravity theory in* 11 *dimensions*, *Phys. Lett.* **B 76** (1978) 409.

[3] E. Cremmer and B. Julia, *The* SO(8) *supergravity*, *Nucl. Phys.* **B 159** (1979) 141.

[4] B. Julia, *Application of supergravity to gravitation theory*, in *Unified field theories of* > 4 *dimensions*, World Scientific Singapore (1983), see p. 215.

[5] N. Marcus and J.H. Schwarz, *Three-dimensional supergravity theories*, *Nucl. Phys.* **B 228** (1983) 145.

[6] H. Nicolai and H. Samtleben, *Maximal gauged supergravity in three dimensions*, *Phys. Rev. Lett.* **86** (2001) 1686 [hep-th/0010076].

[7] H. Nicolai and H. Samtleben, *Compact and noncompact gauged maximal supergravities in three dimensions*, *JHEP* **04** (2001) 022 [hep-th/0103032].

[8] B. Julia, *Kac-Moody symmetry of gravitation and supergravity theories*, in *Applications of group theory in physics and mathematical physics*, M. Flato et al. eds., Lectures in Applied Mathematics, American Mathematical Society, Providece U.S.A. (1985), see p. 355.

[9] P.C. West, E_{11} *and M-theory*, *Class. and Quant. Grav.* **18** (2001) 4443 [hep-th/0104081].

[10] I. Schnakenburg and P.C. West, *Kac-Moody symmetries of IIB supergravity*, *Phys. Lett.* **B 517** (2001) 421 [hep-th/0107181].

[11] A. Kleinschmidt, I. Schnakenburg and P.C. West, *Very-extended Kac-Moody algebras and their interpretation at low levels*, *Class. and Quant. Grav.* **21** (2004) 2493 [hep-th/0309198].

[12] T. Damour, M. Henneaux and H. Nicolai, E_{10} *and a 'small tension expansion' of M-theory*, *Phys. Rev. Lett.* **89** (2002) 221601 [hep-th/0207267].

[13] T. Damour and H. Nicolai, *Eleven dimensional supergravity and the* $E_{10}/K(E_{10})$ σ*-model at low* A_9 *levels*, in *Theoretical methods in physics*, G.S. Pogoyan et al. eds, IoP Publishing, U.K. (2005), hep-th/0410245.

[14] T. Damour, A. Kleinschmidt and H. Nicolai, *Constraints and the* E_{10} *coset model*, *Class. and Quant. Grav.* **24** (2007) 6097 [arXiv:0709.2691].

[15] T. Damour, A. Kleinschmidt and H. Nicolai, *Hidden symmetries and the fermionic sector of eleven-dimensional supergravity*, *Phys. Lett.* **B 634** (2006) 319 [hep-th/0512163].

[16] S. de Buyl, M. Henneaux and L. Paulot, *Extended* E_8 *invariance of* 11*-dimensional supergravity*, *JHEP* **02** (2006) 056 [hep-th/0512292].

[17] T. Damour, A. Kleinschmidt and H. Nicolai, $K(E_{10})$*, supergravity and fermions*, *JHEP* **08** (2006) 046 [hep-th/0606105].

[18] F. Englert and L. Houart, $\mathcal{G}^{+++}$ *invariant formulation of gravity and M-theories: exact BPS solutions*, *JHEP* **01** (2004) 002 [hep-th/0311255].

[19] F. Englert and L. Houart, $\mathcal{G}^{+++}$ *invariant formulation of gravity and M-theories: Exact intersecting brane solutions*, *JHEP* **05** (2004) 059 [hep-th/0405082].

[20] H. Nicolai and T. Fischbacher, *Low level representations for* E_{10} *and* E_{11}, in the proceedings of the *Ramanujan International Symposium on Kac-Moody Algebras and Applications* ($ISKMAA - 2002$), Chennai, India (2002), hep-th/0301017.

[21] F. Riccioni and P.C. West, *Dual fields and* E_{11}, *Phys. Lett.* **B 645** (2007) 286 [hep-th/0612001].

[22] A. Kleinschmidt and H. Nicolai, E_{10} *and* $\mathrm{SO}(9,9)$ *invariant supergravity*, *JHEP* **07** (2004) 041 [hep-th/0407101].

[23] L.J. Romans, *Massive* $N = 2a$ *supergravity in ten-dimensions*, *Phys. Lett.* **B 169** (1986) 374.

[24] M. Henneaux, E. Jamsin, A. Kleinschmidt and D. Persson, *On the* E_{10}*/massive IIA correspondence*, arXiv:0811.4358.

[25] I. Schnakenburg and P.C. West, *Massive IIA supergravity as a non-linear realisation*, *Phys. Lett.* **B 540** (2002) 137 [hep-th/0204207].

[26] F. Riccioni and P.C. West, *The* E_{11} *origin of all maximal supergravities*, *JHEP* **07** (2007) 063 [arXiv:0705.0752].

[27] E.A. Bergshoeff, I. De Baetselier and T.A. Nutma, E_{11} *and the embedding tensor*, *JHEP* **09** (2007) 047 [arXiv:0705.1304].

[28] E.A. Bergshoeff, J. Gomis, T.A. Nutma and D. Roest, *Kac-Moody spectrum of (half-)maximal supergravities*, *JHEP* **02** (2008) 069 [arXiv:0711.2035].

[29] F. Riccioni and P.C. West, E_{11}*-extended spacetime and gauged supergravities*, *JHEP* **02** (2008) 039 [arXiv:0712.1795].

[30] *SimpLie: a simple program for Lie algebras*, http://strings.fmns.rug.nl/SimpLie/.

JHEP01(2009)020

[31] K. Koepsell, H. Nicolai and H. Samtleben, *On the Yangian* $[Y(\mathfrak{e}_8)]$ *quantum symmetry of maximal supergravity in two dimensions*, *JHEP* **04** (1999) 023 [hep-th/9903111].

[32] E.A. Bergshoeff, O. Hohm and T.A. Nutma, *A note on* E_{11} *and three-dimensional gauged supergravity*, *JHEP* **05** (2008) 081 [arXiv:0803.2989].

[33] B. de Wit, H. Nicolai and H. Samtleben, *Gauged supergravities, tensor hierarchies and M-theory*, *JHEP* **02** (2008) 044 [arXiv:0801.1294].

[34] B. de Wit and H. Samtleben, *Gauged maximal supergravities and hierarchies of nonabelian vector-tensor systems*, *Fortschr. Phys.* **53** (2005) 442 [hep-th/0501243].

[35] A. Kleinschmidt and H. Nicolai, *Gradient representations and affine structures in* AE_n, *Class. and Quant. Grav.* **22** (2005) 4457 [hep-th/0506238].

[36] A. Le Diffon and H. Samtleben, *Supergravities without an action: gauging the trombone*, arXiv:0809.5180.

JHEP01(2009)020