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Classical resolution of singularities in dilaton cosmologies

E A Bergshoeff¹, A Collinucci¹, D Roest², J G Russo³ and P K Townsend⁴

¹ Centre for Theoretical Physics, University of Groningen, Nijenborgh 4, 9747 AG Groningen, The Netherlands

² Department of Mathematics, King's College, London, Strand, London WC2R 2LS, UK

³ Institució Catalana de Recerca i Estudis Avançats, Departament ECM, Facultat de Física, Universitat de Barcelona, Spain

⁴ Department of Applied Mathematics and Theoretical Physics, Centre for Mathematical Sciences, University of Cambridge, Wilberforce Road, Cambridge CB3 0WA, UK

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Abstract

For models of dilaton gravity with a possible exponential potential, such as the tensor–scalar sector of IIA supergravity, we show how cosmological solutions correspond to trajectories in a 2D Milne space (parametrized by the dilaton and the scale factor). Cosmological singularities correspond to points at which a trajectory meets the Milne horizon, but the trajectories can be smoothly continued through the horizon to an instanton solution of the Euclidean theory. We find some exact cosmology/instanton solutions that lift to black holes in one higher dimension. For one such solution, the singularities of a big crunch to big bang transition mediated by an instanton phase lift to the black hole and cosmological horizons of de Sitter Schwarzschild spacetimes.

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1. Introduction

In models of gravity coupled to scalar fields, homogeneous and isotropic cosmologies correspond to trajectories in an ‘augmented target space’ or ‘superspace’, parametrized by the scalar fields and the FLRW scale factor. There is a natural conformal class of Lorentzian metrics on this space, the scale factor playing the role of ‘time’, and when the scalar field target space is a hyperboloid this Lorentzian metric can be chosen to be the metric of a Milne ‘universe’, which is just Minkowski space in polar-type coordinates. In this case, cosmological singularities correspond to points at which the trajectory reaches the Milne horizon [1]. For the models considered in [1], the trajectories were actually geodesics and hence straight lines in Minkowski space that typically cross the Milne horizon twice, corresponding to a big bang and a big crunch singularity. In particular, there are trajectories

that represent a universe undergoing a transition from a collapsing big crunch universe to an expanding big bang universe across a ‘forbidden’ region of ‘superspace’ behind the Milne horizon.

In the absence of a potential for the scalar fields, it turns out that FLRW cosmologies are geodesics in the Milne metric only for a particular radius of curvature of the target space hyperboloid. We may parametrize these models by a constant γ such that the motion is geodesic for $\gamma = 1$. In a previous paper, we generalized the construction of [1] to (generically) non-geodesic Milne motion in models for which the target space hyperboloid has an arbitrary finite radius, corresponding to arbitrary non-zero γ , and we showed that parts of a trajectory behind the Milne horizon may be interpreted as instanton solutions of the Euclidean theory [2]. We also showed that for $\gamma = 2$ models there exist trajectories that are smooth closed curves in the Minkowski extension of the Milne superspace, corresponding to cyclic universes in which cosmological ‘phases’ are separated by instanton ‘phases’, each of which mediates a big crunch to big bang transition across a region behind the Milne horizon⁵.

These ideas are applicable to IIB supergravity because the dilaton and axion of that theory parametrize the hyperbolic space $SI(2; \mathbb{R})/SO(2) \equiv H_2$, and they potentially imply a resolution of singularities of flat ($k = 0$) cosmologies in IIB superstring theory since these correspond to null lines in the Minkowski extension of the Milne superspace. However, the H_2 radius of curvature (which corresponds to $\gamma = 2/3$) is not in the range for which a generic trajectory has a smooth continuation through the Milne horizon for $k \neq 0$, and the same applies to the models in spacetime dimension $d < 10$ obtained by toroidal compactification.

One purpose of this paper is to apply these ideas to models of dilaton gravity such as the dilaton-gravity sector of IIA supergravity. As already pointed out in [2], cosmological solutions of IIB supergravity for which the axion is identically zero are also solutions of IIA supergravity. More generally, one could set the axion to zero after having redefined the fields by an $SI(2; \mathbb{R})$ transformation. This means that any *planar* cosmological trajectory in the three-dimensional (3D) Minkowski augmented target space of IIB supergravity must correspond to some solution of the scalar–tensor sector of IIA supergravity, so this already shows that the latter can be viewed as trajectories in a two-dimensional (2D) Minkowski space. The possible consistent truncations simply correspond to the possible choices of this 2D Minkowski subspace. A subtlety that arises in these truncations is that the parameter γ ceases to be physical because the truncated target space is now a ‘one-dimensional hyperboloid’ for which the intrinsic curvature is trivially zero. Thus, planar trajectories for various values of γ that previously corresponded to different solutions of different models must now be viewed as different solutions of the *same* model. This leads to ambiguities in the continuation of trajectories across Milne horizons. We investigate the consequences of this in some detail.

A further feature of IIA supergravity is that it allows an extension to a ‘massive’ IIA theory with an exponential dilaton potential. This motivates the investigation of cosmological trajectories in dilaton-gravity models in d spacetime dimensions with an exponential potential characterized by an arbitrary dilaton coupling constant a . Thus, our starting point will be the d -dimensional action

$$I = \int d^d x \sqrt{\epsilon \det g} \left[R - \frac{1}{2} (\partial\phi)^2 - \Lambda e^{-a\phi} \right] \quad (1.1)$$

⁵ Cyclic universes without cosmological singularities are possible in some models with both a hyperbolic target space and a scalar potential [3] but these have no instanton ‘phases’ and are therefore quite different from the cyclic universes under discussion here.

for d -metric g of signature $(\epsilon, 1, \dots, 1)$ and dilaton ϕ , and constants Λ and a . We may suppose that $a \geq 0$ without loss of generality, but allow Λ to be positive, negative or zero. As in [2], we have introduced a sign ϵ such that $\epsilon = 1$ yields the Euclidean Lagrangian⁶. For $\epsilon = -1$ and $d = 10$ we have a consistent truncation of IIA supergravity when $\Lambda = 0$, and when $\Lambda > 0$, we have a consistent truncation of massive IIA supergravity when $a = 5/2$. Other values of a , for positive and negative Λ , arise in M-theory compactifications to $d < 10$.

These models have been extensively studied in the past, at least for $\epsilon = -1$. The equations describing cosmological solutions are known to be equivalent to those of an autonomous dynamical system [4], such that cosmologies correspond to trajectories in a 2D phase plane⁷. This allows a determination of the qualitative nature of the full space of solutions, and in this sense the cosmological solutions are already well understood. In particular, all $k = 0$ cosmologies can be found exactly [5–7]. Here we provide a new ‘explanation’ for this fact: these solutions are geodesics in the Milne wedge of a 2D Minkowski space. This description allows us to explore their continuation through cosmological spacetime singularities as smooth trajectories through the Milne horizon.

A subset of the models studied here have the property that their action is the reduction of the $(d + 1)$ -dimensional Einstein–Hilbert action, with a possible cosmological constant. This allows the d -dimensional cosmological solutions to be lifted to $(d + 1)$ -dimensional Einstein metrics. This is true not only for the cosmological solutions but also for the instanton solutions because the d -dimensional Euclidean action is obtained in these cases by considering time-independent fields. Thus, both cosmological and instanton solutions in d dimensions lift to Lorentzian-signature $(d + 1)$ -metrics. For example, when $\Lambda = 0$ the action (1.1) is the dimensional reduction of the $(d + 1)$ -dimensional Einstein–Hilbert action. As already mentioned in [2] it is known that d -dimensional cosmologies lift to the interior of a $(d + 1)$ -dimensional Schwarzschild black hole, and that instanton solutions lift to the Schwarzschild exterior. These results are applicable to IIA supergravity and they show that cosmology or instanton solutions can be lifted to solutions of 11D supergravity.

Our interest here is the lift of the full cosmology/instanton solutions corresponding to complete smooth 2D Minkowski trajectories. We find, in particular, that the $k = 1$ Milne geodesics lift to a sequence of all the possible $(d + 1)$ -dimensional (anti-)de Sitter Schwarzschild spacetimes for a given absolute value of the black hole mass, the negative and positive mass black holes corresponding to pre- and post-big bang phases, respectively. Moreover, the big crunch singularity is resolved in the higher dimension in the same way that certain dilaton black hole singularities are resolved [8].

When Λ is non-zero there is a similar story for a particular (d -dependent) value of a but the $(d + 1)$ -dimensional action now has a cosmological constant. Exact cosmological and instanton solutions of the d -dimensional theory can be found for this value of a . It turns out that the $k = 1$ cosmology/instanton solutions corresponding to 2D Minkowski trajectories that are smooth for $\gamma = 1$ lift to $(d + 1)$ -dimensional (anti-)de Sitter Schwarzschild spacetimes. The $\Lambda > 0$ case is of particular interest because there are then trajectories on which an entire instanton mediated big bang to big crunch transition becomes non-singular in the higher dimension, in the sense that the big bang and big crunch curvature singularities become regular horizons of the $(d + 1)$ -metric.

⁶ To have a Euclidean action that is positive definite when $R = 0$ one would need an additional overall factor of $-\epsilon$, but this does not affect the field equations, which is all that is needed here.

⁷ The phase plane should not be confused with the 2D Minkowski superspace; it is essentially the space of velocities in this space for a particular time parametrization.

2. Cosmologies and instantons

We begin by explaining how the problem of finding cosmological or instanton solutions of the general model with action (1.1) can be reduced to solving the equations of motion of an effective action describing the motion of a particle in a 2D Minkowski space. This is achieved by the ansatz

$$ds_d^2 = \epsilon (e^{\alpha\varphi} f)^2 d\tau^2 + e^{2\alpha\varphi/(d-1)} d\Sigma_k^2, \quad \phi = \phi(\tau), \quad (2.1)$$

where f is an arbitrary function of τ , and

$$\alpha = \sqrt{\frac{d-1}{2(d-2)}}. \quad (2.2)$$

The $(d-1)$ -metric $d\Sigma_k^2$ is (at least locally) a maximally symmetric space of positive ($k=1$), negative ($k=-1$) or zero ($k=0$) curvature. One can choose coordinates such that

$$d\Sigma_k^2 = (1 - kr^2)^{-1} dr^2 + r^2 d\Omega_{d-2}^2, \quad (2.3)$$

where $d\Omega_{d-2}^2$ is an $SO(d-1)$ -invariant metric on the unit $(d-2)$ -sphere; note that $d\Sigma_k^2 = d\Omega_{d-1}^2$ when $k=1$. This ansatz constitutes a consistent reduction of the original degrees of freedom to a two-dimensional subspace, the ‘augmented target space’, with coordinates (φ, ϕ) . The full equations of motion reduce to a set of equations that can themselves be derived by variation (with respect to ϕ , φ and f) of the effective action

$$I = \int d\tau \left\{ \frac{1}{2} f^{-1} \epsilon (\dot{\phi}^2 - \phi^2) + f [k(d-1)(d-2) e^{\varphi/\alpha} - \Lambda e^{2\alpha\varphi - a\phi}] \right\}, \quad (2.4)$$

where the overdot indicates differentiation with respect to τ . For $\epsilon = -1$ we can interpret τ as a time coordinate related to the time t of FLRW cosmology in standard coordinates by

$$dt = e^{\alpha\varphi} f d\tau. \quad (2.5)$$

For $\epsilon = 1$ the metric has Euclidean signature and we can interpret τ as imaginary time.

To proceed we introduce new field variables (U, V) that we will shortly be able to interpret as null coordinates for a 2D Minkowski space. These variables are defined by

$$e^\varphi = (-\epsilon UV)^{\alpha\gamma}, \quad e^\phi = \left(-\frac{\epsilon V}{U} \right)^{\alpha\gamma}, \quad (2.6)$$

for some constant γ . Although different choices of γ give equivalent solutions in a given patch, say $U > 0, V > 0, \epsilon = -1$, the continuation of these solutions to the entire Minkowski plane (U, V) is sensitive to the value of γ , as will be explained in section 3. Note that we need both signs of ϵ to cover the entire Minkowski plane (U, V) with real scalars (φ, ϕ) : if the coordinates U and V have the same sign we have to choose $\epsilon = -1$ and if they have opposite signs $\epsilon = 1$. With this proviso one can always assume both $(-\epsilon UV)$ and $(-\epsilon V/U)$ to be positive. In terms of the new variables, the effective action is

$$I = \int d\tau \{ 2\alpha^2 \gamma^2 \epsilon (fUV)^{-1} \dot{U}\dot{V} + f [k(d-1)(d-2)(-\epsilon UV)^\gamma - \Lambda (-\epsilon UV)^{2\alpha^2\gamma} (-\epsilon U/V)^{\alpha\gamma a}] \}. \quad (2.7)$$

To obtain a canonical kinetic term we now fix the time-reparametrization invariance by choosing

$$f = -\frac{2}{(d-2)\epsilon UV}. \quad (2.8)$$

The resulting action is

$$I = -\frac{1}{2}(d-1) \int d\tau \left\{ \gamma^2 \dot{U} \dot{V} - 4k(-\epsilon UV)^{\gamma-1} + \frac{4\Lambda}{(d-1)(d-2)} (-\epsilon UV)^{2\alpha^2\gamma-1} (-\epsilon U/V)^{\alpha\gamma a} \right\}. \tag{2.9}$$

However, the equations of motion of this action must be supplemented by the f equation of motion of the original action, which is the constraint

$$\gamma^2 \dot{U} \dot{V} = -4k(-\epsilon UV)^{\gamma-1} + \frac{4\Lambda}{(d-1)(d-2)} (-\epsilon UV)^{2\alpha^2\gamma-1} (-\epsilon U/V)^{\alpha\gamma a}. \tag{2.10}$$

Note that this is just the condition that a particle in a 2D space with action (2.9) has zero energy. Given a zero-energy solution of the equations of motion for U and V we may read off the (Einstein-frame) metric and dilaton field from

$$\begin{aligned} ds_d^2 &= \frac{4\epsilon}{(d-2)^2} (-\epsilon UV)^{2\alpha^2\gamma-2} d\tau^2 + (-\epsilon UV)^{2\alpha^2\gamma/(d-1)} d\Sigma_k^2, \\ e^\phi &= \left(-\frac{\epsilon V}{U} \right)^{\alpha\gamma}. \end{aligned} \tag{2.11}$$

For $\epsilon = -1$, the FLRW time t is now related to the parameter τ by

$$dt \propto (UV)^{\alpha^2\gamma-1} d\tau. \tag{2.12}$$

We have now formulated the problem in such a way that cosmologies, given by (2.11), correspond to trajectories in a 2D Minkowski space with null coordinates U, V . More precisely, cosmological solutions correspond to trajectories in either the future Milne patch ($U > 0, V > 0$) or the past Milne patch ($U < 0, V < 0$) of the 2D Minkowski space. The null lines $U = 0$ and $V = 0$ are the Milne horizon. The 2D Minkowski space should therefore be thought of as the analytic extension of a Milne ‘universe’. Trajectories in this ‘universe’ correspond to cosmologies, and a cosmological singularity in spacetime corresponds to a point at which a trajectory crosses the Milne horizon. On crossing the horizon, into the Rindler patches of the 2D Minkowski space, the trajectory must be re-interpreted as an ‘instanton’, i.e., as a solution of the Euclidean action.

Despite the cosmological spacetime singularity, the continuation of the cosmological trajectory through the Milne horizon may be smooth. For example, if $a = 0$ and $k = 0$ we may choose γ so as to make the potential a constant, in which case the motion is geodesic motion and all trajectories are straight lines in the 2D Minkowski ‘superspace’. This is essentially the point made in [1] in the context of a model with N scalar fields, in which case the trajectories were straight lines in an $(N + 1)$ -dimensional Minkowski space. Here we are restricted to $N = 2$ but we allow for non-zero a and consider non-flat ($k \neq 0$) cosmologies. Clearly, in this more general context the motion in the 2D Minkowski space will not be geodesic but there may nevertheless be smooth trajectories that cross the Milne horizon. From [2] we know that this indeed happens when $\Lambda = 0$ and even that there can be elliptical trajectories in the analytic continuation of Milne to Minkowski space that correspond to a closed ($k = 1$) universe undergoing an endless cycle of big bang to big crunch transitions mediated by ($k = -1$) instanton ‘phases’.

We now plan to investigate to what extent these features carry over to the general model summarized by the effective action (2.9). We will concentrate on those special cases for which an exact solution can be found. These cases are as follows:

- $\Lambda = 0$. This case was partly dealt with in [2], as the special case of planar trajectories in a IIB-type model. In the IIA-type case considered here, the Lagrangian (1.1) from which we started is the dimensional reduction of the Einstein–Hilbert action in $(d + 1)$ dimensions. The dimensional-reduction formula is

$$\begin{aligned}
 ds_{d+1}^2 &= -\epsilon e^{\phi/\alpha} dz^2 + e^{-2\alpha\phi/(d-1)} ds_d^2 \\
 &= -\epsilon \left(-\frac{\epsilon V}{U} \right)^\gamma dz^2 + \frac{4\epsilon}{(d-2)^2} (-\epsilon UV)^{2\alpha^2\gamma-2} \left(-\frac{\epsilon U}{V} \right)^{2\alpha^2\gamma/(d-1)} d\tau^2 \\
 &\quad + (U^2)^{2\alpha^2\gamma/(d-1)} d\Sigma_k^2.
 \end{aligned}
 \tag{2.13}$$

Note that this metric has Lorentzian signature for either sign of ϵ . This allows any cosmology/instanton solution of the d -dimensional theory to be re-interpreted as a Lorentzian-signature solution of $(d + 1)$ -dimensional general relativity.

- $a = \sqrt{2}/\sqrt{(d-2)(d-1)}$. For this value of a the Lagrangian (1.1) is the dimensional reduction of $(d + 1)$ -dimensional gravity with a cosmological constant, so we can use the $(d + 1)$ -dimensional Schwarzschild (anti-)de Sitter solution to obtain a solution in d dimensions.
- $a = 0$. In this case, the motion in 2D Minkowski space is analogous to motion in a central potential.
- $k = 0$. In this case all cosmological solutions are already known explicitly. Here we give an alternative description of them as straight-line geodesics in a 2D Milne ‘superspace’.

3. Zero cosmological constant ($\Lambda = 0$)

Setting $\Lambda = 0$ in (2.9) we get the action

$$I = -\frac{1}{2}(d-1) \int d\tau \{ \gamma^2 \dot{U} \dot{V} - 4k(-\epsilon UV)^{\gamma-1} \}.
 \tag{3.1}$$

We have still to choose γ . The choices $\gamma = 1$ and $\gamma = 2$ are special because the equations of motion are then linear, so we shall consider only these two possibilities. From (2.6) it is obvious that the variables (U, V) for the choice $\gamma = 1$ are not the same as these variables for the choice $\gamma = 2$, so we rename the latter (\tilde{U}, \tilde{V}) . Also, it is clear from (2.12) that the independent parameter τ for the choice $\gamma = 1$ is not the same as the independent parameter for the choice $\gamma = 2$, so we call the latter λ . These considerations lead us to consider the following cases:

- $\gamma = 1$. In this case, we take the independent variable to be τ and the dependent variables to be (U, V) , so we seek solutions of the equations of motion for $U(\tau)$ and $V(\tau)$.
- $\gamma = 2$. In this case, we take the independent variable to be λ and the dependent variables to be (\tilde{U}, \tilde{V}) . In other words, we seek solutions of the equations of motion for $\tilde{U}(\lambda)$ and $\tilde{V}(\lambda)$. Note that the formula (2.11) for the metric and dilaton still applies but with (U, V) replaced by (\tilde{U}, \tilde{V}) and τ replaced by λ .

For either choice, the cosmological solutions correspond to trajectories in a Milne wedge of 2D Minkowski space with $UV > 0$, or $\tilde{U}\tilde{V} > 0$, since this condition is required for real φ . The null lines $UV = 0$ (or $\tilde{U}\tilde{V} = 0$) are the Milne horizon, and cosmological singularities correspond to points at which this horizon is crossed. This can be seen from the fact that the scale factor e^φ goes to zero as $UV \rightarrow 0$. Inasmuch as we are concerned only with cosmological solutions away from their big bang and big crunch singularities, the two choices of γ yield equivalent solutions; the correspondence between them is given by

$$|U(\tau)| = \tilde{U}^2(\lambda), \quad |V(\tau)| = \tilde{V}^2(\lambda),
 \tag{3.2}$$

and the relation

$$d\tau = -\epsilon \tilde{U}(\lambda) \tilde{V}(\lambda) d\lambda. \tag{3.3}$$

We shall solve the equations of motion for both values of γ and verify that the solutions are related in this way.

Given this relation, it might seem pointless to consider both values of γ . However, the two choices of γ need not yield equivalent trajectories in the analytic continuation of 2D Milne to 2D Minkowski space. The reason for this is that the Minkowski space with null coordinates (U, V) is *not the same space* as the Minkowski space with null coordinates (\tilde{U}, \tilde{V}) ; the two differ by a conformal transformation that is singular on the Milne horizon. This possibility is a special feature of 2D Minkowski space: the conformal transformation can be removed by a time reparametrization, so that what was initially motion in a 2D Minkowski space again becomes motion in a 2D Minkowski space, albeit with a different potential energy function. Thus, there is more than one possible continuation of a cosmological trajectory to a smooth trajectory in the 2D Minkowski continuation of 2D Milne. Moreover, a smooth trajectory that crosses the Milne horizon in the coordinates (U, V) will not be smooth in the coordinates (\tilde{U}, \tilde{V}) because of the singularity of the conformal rescaling that relates the Minkowski metrics in these two sets of variables.

3.1. Milne geodesics ($\gamma = 1$)

We now set $\gamma = 1$ in (3.1) to get the action

$$I = -\frac{1}{2}(d-1) \int d\tau \{ \dot{U}\dot{V} - 4k \}. \tag{3.4}$$

The potential is now a constant, so the equations of motion are

$$\ddot{U} = 0, \quad \ddot{V} = 0. \tag{3.5}$$

The solutions, subject to the zero-energy constraint

$$\dot{U}\dot{V} = -4k, \tag{3.6}$$

are the straight lines

$$U = c\tau, \quad V = -\frac{4k}{c}\tau + m, \tag{3.7}$$

for constants c and m . Each of these straight lines corresponds to some cosmology/instanton solution of the equations of motion of our original action (1.1). It will be a cosmology in a region of the (U, V) -plane with $UV > 0$ and an instanton in a region with $UV < 0$. Each of these solutions must descend from a Lorentzian-signature $(d+1)$ -dimensional metric, and we now wish to determine what these higher dimensional metrics are.

We may restrict ourselves here to $k \neq 0$ as the general $k = 0$ case will be discussed later in a separate section, so this leaves $k = \pm 1$. The $m = 0$ case is clearly special, as the straight line trajectory passes through the origin of the (U, V) space, and $k\epsilon = 1$ everywhere; in these cases, the dilaton is constant and the d -metric is flat (for $k = -1$ it is the d -dimensional Milne metric). In addition, the parameter c can be adjusted by rescaling $U \rightarrow c_0 U, V \rightarrow V/c_0$, so we choose $c = 2$. One then finds that the $(d+1)$ -metric (2.13) is

$$ds_{d+1}^2 = k dr^2 + r^2 d\Sigma_k^2 - k dz^2, \tag{3.8}$$

where

$$r^{d-2} = 2\tau. \tag{3.9}$$

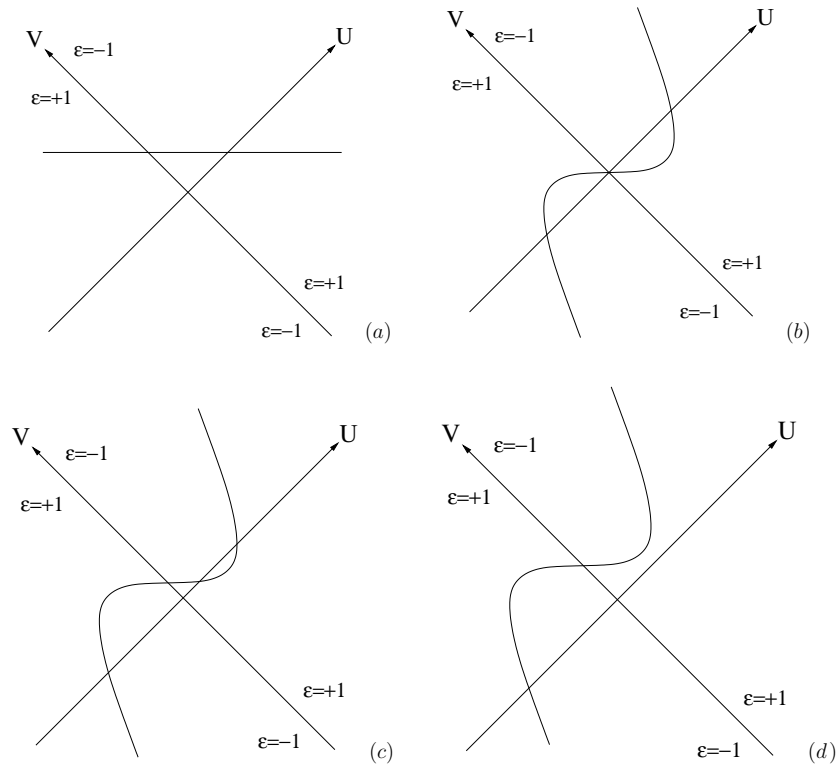


Figure 1. Generic trajectories for $k = 1$ and $m \geq 0$: (a) $\Lambda = 0$, (b) $\Lambda > 0$ and $m = 0$, (c) $0 < m\Lambda < 3$, (d) $m\Lambda > 3$. The behaviour near $U = 0$ is determined by k and m while the asymptotics depend on Λ .

For either sign of k , this is just a flat metric on $(d + 1)$ -dimensional Minkowski space. Thus the spacetime singularity at the Milne superspace horizon lifts in this case to a mere coordinate singularity at $r = 0$.

Let us now consider the less trivial case of $m \neq 0$. We shall assume that $m > 0$ and $c > 0$ since the other cases are entirely analogous. For $k = 1$ the straight-line trajectory originates at $\tau = -\infty$ in the Rindler patch $U < 0, V > 0$, where we must choose $\epsilon = 1$, so this part of the trajectory corresponds to an instanton solution. When the straight line crosses the $U = 0$ horizon it enters the Milne patch, where we must choose $\epsilon = -1$, so this part of the trajectory corresponds to a cosmological solution. Finally, the straight line passes through the $V = 0$ horizon to enter the other Rindler patch ($U > 0, V < 0$) where we must again choose $\epsilon = 1$. So this part of the trajectory describes another instanton solution. This is illustrated in figure 1(a). For $k = -1$ we again have three solutions corresponding to three pieces of the straight line but now two are cosmologies and one an instanton; the instanton mediates between a collapsing big crunch universe and an expanding big bang universe.

Thus a single straight-line trajectory comprises one cosmology and two instantons (for $k = 1$) or one instanton and two cosmologies (for $k = -1$), all three of which lift, via the formula (2.13), to the Lorentzian-signature $(d + 1)$ -metric:

$$ds_{d+1}^2 = -\frac{4}{(d-2)^2} \frac{U}{V} (U^2)^{-(d-3)/(d-2)} d\tau^2 + \frac{V}{U} dz^2 + (U^2)^{1/(d-2)} d\Sigma_k^2. \tag{3.10}$$

To see how this metric incorporates all three of the d -dimensional cosmology/instanton solutions, we must consider how it looks in the various sectors of the U, V -plane through which a straight-line trajectory passes. For sectors with $U > 0$ we may use the coordinate r defined in (3.9), and we again set $c = 2$, in which case the $(d + 1)$ -metric becomes

$$ds_{d+1}^2 = h(r)^{-1} dr^2 - h(r) dz^2 + r^2 d\Sigma_k^2, \quad (3.11)$$

with

$$h(r) \equiv -\frac{V}{U} = k - \frac{m}{r^{d-2}}. \quad (3.12)$$

For sectors with $U < 0$, and again for $c = 2$, we may introduce a new coordinate ρ by setting $2\tau = -\rho^{d-2}$, in which case the $(d + 1)$ -metric becomes

$$ds_{d+1}^2 = g(\rho)^{-1} d\rho^2 - g(\rho) dz^2 + \rho^2 d\Sigma_k^2, \quad (3.13)$$

with

$$g(\rho) = k + \frac{m}{\rho^{d-2}}. \quad (3.14)$$

Now consider the $k = 1$ case, for which $d\Sigma_k^2 = d\Omega_{d-1}^2$ (the $SO(d)$ -invariant metric on the unit $(d - 1)$ -sphere). In this case, the metric (3.11) is just the Schwarzschild black hole in $d + 1$ dimensions with mass proportional to m . This comprises the interior of the black hole ($V > 0$) and the exterior ($V < 0$). Therefore the interior of the black hole describes the cosmology part of the trajectory (similar to [9]) and the exterior describes one of the two instanton parts of the trajectory. The other instanton part of the trajectory lifts to the $k = 1$ case of the metric (3.13), which is just Schwarzschild $(d + 1)$ -metric with a negative mass (proportional to $-m$). Thus, the three pieces of a single $k = 1$ straight-line trajectory in the 2D Minkowski ‘superspace’ correspond to cosmological/instanton solutions that lift to the following three $(d + 1)$ -dimensional Lorentzian-signature spacetimes:

- negative mass Schwarzschild black hole;
- interior of positive mass Schwarzschild black hole;
- exterior of positive mass Schwarzschild black hole.

The situation for $k = -1$ is analogous, but now $d\Sigma_k = dH_{d-1}^2$ where dH_{d-1}^2 is the $SO(1, d - 1)$ -invariant metric on the unit radius $(d - 1)$ -hyperboloid. For example, the big bang cosmology lifts to the $(d + 1)$ -metric

$$ds_{d+1}^2 = -\left(1 + \frac{m}{r^{d-2}}\right)^{-1} dr^2 + \left(1 + \frac{m}{r^{d-2}}\right) dz^2 + r^2 dH_{d-1}^2. \quad (3.15)$$

Clearly, the coordinate r is now a time coordinate, and the metric is asymptotic at late times to the product of a d -dimensional Milne universe with a circle. There is a big bang singularity at $r = 0$, which is a curvature singularity of the metric. Thus, the big bang singularity of the d -dimensional metric is not resolved in the higher dimension (although the trajectory in ‘superspace’ passes smoothly through it). In contrast, the big crunch lifts to the horizon of the $(d + 1)$ -metric

$$ds_{d+1}^2 = -\left(1 - \frac{m}{\rho^{d-2}}\right)^{-1} d\rho^2 + \left(1 - \frac{m}{\rho^{d-2}}\right) dz^2 + \rho^2 dH_{d-1}^2. \quad (3.16)$$

The ‘exterior’ spacetime ($\rho^{d-2} > m$) corresponds to the d -dimensional big crunch cosmology while the interior ($\rho^{d-2} < m$) corresponds to the d -dimensional instanton.

It should be appreciated that the asymmetry between the big bang and big crunch singularities, for either sign of k , is due to an asymmetry in the formula (2.13) used to lift the cosmology/instanton solutions to one higher dimension, and is not an intrinsic feature of the

d -dimensional solutions. In fact, there are two formulae that yield the same d -dimensional action; one is related to the other by an interchange of U and V or, equivalently, by a flip of sign of the dilaton, which is a symmetry of the $\Lambda = 0$ action. If the other formula is used to lift to $d + 1$ dimensions then one finds that the black hole horizon and the curvature singularity behind the horizon are exchanged, so either singularity can be resolved by an appropriate lift⁸.

3.2. Cyclic cosmologies ($\gamma = 2$)

We now set $\gamma = 2$ in (3.1) to get the action

$$I = -2(d-1) \int d\lambda \{ \partial_\lambda \tilde{U} \partial_\lambda \tilde{V} + \epsilon k \tilde{U} \tilde{V} \}, \quad (3.17)$$

where, according to the earlier discussion, we have replaced the variables (U, V) by (\tilde{U}, \tilde{V}) and the independent variable τ by λ . The equations of motion are

$$\partial_\lambda^2 \tilde{U} = \epsilon k \tilde{U}, \quad \partial_\lambda^2 \tilde{V} = \epsilon k \tilde{V}, \quad (3.18)$$

and the zero-energy constraint is

$$\partial_\lambda \tilde{U} \partial_\lambda \tilde{V} = \epsilon k \tilde{U} \tilde{V}. \quad (3.19)$$

For $\epsilon k = -1$ we get cyclic universes, as described in [2]. The solutions are

$$\tilde{U} = \tilde{U}_0 \sin \lambda, \quad \tilde{V} = \tilde{V}_0 \cos \lambda. \quad (3.20)$$

for constants $(\tilde{U}_0, \tilde{V}_0)$. A single cycle comprises four segments, joined smoothly across the Milne horizon at $\tilde{U} \tilde{V} = 0$, one in each of the following four segments of the 2D Minkowski analytic continuation of the 2D Milne wedge:

$$\begin{aligned} \text{I: } & \tilde{U} > 0, \tilde{V} > 0 & \text{II: } & \tilde{U} > 0, \tilde{V} < 0 \\ \text{III: } & \tilde{U} < 0, \tilde{V} < 0 & \text{IV: } & \tilde{U} < 0, \tilde{V} > 0. \end{aligned} \quad (3.21)$$

Using (3.2), and relation (3.3), we can make contact with the straight-line solutions of the previous subsection. Firstly, for the cyclic universe solution (3.20) relation (3.3) can be integrated to give

$$\tau = -\frac{1}{2} \epsilon \tilde{U}_0 \tilde{V}_0 \sin^2(\lambda). \quad (3.22)$$

Recalling that $k\epsilon = -1$ for the cyclic universe, we then deduce from (3.2) that the parameters c, m of the corresponding $\gamma = 1$ straight-line solutions (3.7) are

$$m = \mp \tilde{V}_0^2, \quad c = \mp \frac{2\tilde{U}_0}{\tilde{V}_0}. \quad (3.23)$$

This yields

$$U = \pm \epsilon \tilde{U}_0^2 \sin^2(\lambda), \quad V = \mp \tilde{V}_0^2 \cos^2(\lambda). \quad (3.24)$$

Note that the slope of the straight line depends on ϵ : upon crossing the Milne horizon there is a ‘kink’ in the line. In this way, the elliptical trajectory in the (\tilde{U}, \tilde{V}) -plane is mapped to a parallelogram with equal sides, i.e., a rhombus, in the (U, V) -plane. The slope of the sides depends on the ratio $\tilde{U}_0^2/\tilde{V}_0^2$, which determines the eccentricity of the $\gamma = 2$ ellipse, with a circle mapping to a square. This is a concrete illustration of how a trajectory that is smooth for one choice of γ may have derivative discontinuities at the Milne horizon for some other value of γ .

⁸ Note, however, that the $\phi \rightarrow -\phi$ symmetry of the action that allows for this does not extend to the full IIA supergravity action, nor to the case of non-zero Λ .

Let us now investigate how the cyclic solutions look in $d + 1$ dimensions. Using the formula (2.13), we get

$$ds_{d+1}^2 = \epsilon \left(1 - \frac{\tilde{U}_0^2}{r^{d-2}} \right) dy^2 - \epsilon \left(1 - \frac{\tilde{U}_0^2}{r^{d-2}} \right)^{-1} dr^2 + r^2 d\Sigma_k^2, \tag{3.25}$$

where

$$y = \frac{\tilde{V}_0}{\tilde{U}_0} z, \quad r^{d-2} = \tilde{U}_0^2 \sin^2 \lambda. \tag{3.26}$$

Each cosmology phase of the cycle has $\epsilon = -1$ and therefore $k = 1$, so it lifts to the interior of a Schwarzschild black hole, as in the previous subsection. These are the segments of the ellipse lying on the Milne patches with $\tilde{U}\tilde{V} > 0$. Each instanton phase of the cycle has $\epsilon = 1$ and therefore $k = -1$. It lifts to

$$ds_{d+1}^2 = \left(1 - \frac{\tilde{U}_0^2}{r^{d-2}} \right) dy^2 - \left(1 - \frac{\tilde{U}_0^2}{r^{d-2}} \right)^{-1} dr^2 + r^2 dH_{d-1}^2. \tag{3.27}$$

This is equivalent to (3.16) with $m = \tilde{U}_0^2$.

Let us now follow a cycle as λ is varied from 0 to 2π . At $\lambda = 0$ ($\tilde{V} = 0$) we have a big bang singularity, which lifts to the curvature singularity at $r = 0$ of the $d + 1$ black hole. The universe expands and then collapses to a big crunch singularity at $\lambda = \pi/2$ ($\tilde{V} = 0$), which lifts to the horizon of the $d + 1$ black hole at $r^{d-2} = \tilde{U}_0^2$. For $\lambda \in (\frac{\pi}{2}, \pi)$ the solution lifts to (3.27). Note, however, that the $(d + 1)$ -metric changes *discontinuously* across the horizon; this can be seen from the fact that the near horizon geometry of the black hole differs from the near horizon geometry of the solution (3.27) because the $(d - 1)$ -sphere of the former becomes a $(d - 1)$ -hyperboloid of the latter. Thus, in contrast to the $\gamma = 1$ case, the trajectories that are smooth for $\gamma = 2$ do not lift to solutions in the higher dimension that are smooth across a horizon. This makes the higher dimension interpretation of cyclic cosmologies problematic, and indicates a ‘higher dimensional preference’ for the Milne geodesics that we found for $\gamma = 1$.

If we put this difficulty aside and continue with the cycle, then we approach the curvature singularity (at $r = 0$) of the solution (3.27) as λ increases to π . Passing through $\lambda = \pi$, we arrive at the second cosmological phase of the cycle, with $\lambda \in (\pi, 3\pi/2)$, described again in $d + 1$ dimensions by the interior of the black hole. Passing again through a big crunch singularity at $\lambda = 3\pi/2$ we come to the second instanton phase, in the interval $\lambda \in (\frac{3\pi}{2}, 2\pi)$, which again lifts to the $d + 1$ -metric (3.27).

4. Uplift to (anti-)de Sitter Schwarzschild

Although cosmological trajectories of the $\Lambda \neq 0$ models considered in this paper can be understood qualitatively for any value of the constant a , there is only one other value for which exact solutions can be readily found (leaving aside the $a = 0$ case, which we discuss in the following section). This is because there is only one value of a for which the equations of motion for U and V separate. From (2.9) it can be seen that the potential terms depend only on U iff

$$\gamma = 1, \quad 2\alpha^2\gamma - 1 = \alpha\gamma a, \tag{4.1}$$

which together require that

$$a = \sqrt{\frac{2}{(d-1)(d-2)}}. \tag{4.2}$$

The effective action (2.9) in this case is

$$I = -\frac{1}{2}(d-1) \int d\tau \left\{ \dot{U}\dot{V} - 4k + \frac{4\Lambda}{(d-1)(d-2)} (U^2)^{1/(d-2)} \right\}. \quad (4.3)$$

The field equations are now

$$\ddot{U} = 0, \quad \ddot{V} = \frac{8\Lambda(U^2)^{1/(d-2)}}{(d-1)(d-2)^2U}, \quad (4.4)$$

and the zero-energy constraint is

$$\dot{U}\dot{V} = -4k + \frac{4\Lambda(U^2)^{1/(d-2)}}{(d-1)(d-2)}. \quad (4.5)$$

The most general solution is the curve

$$U = c\tau, \quad V = -\frac{4k\tau}{c} + m + \frac{4\Lambda((c\tau)^2)^{1/(d-2)}\tau}{cd(d-1)}, \quad (4.6)$$

for constants c, m , as in the special case of $\Lambda = 0$ that we considered in the previous section. Following the discussion of the uplift to $(d+1)$ dimensions in that section, we set $c\tau = r^{d-2}$ (in regions with $U > 0$) to again get the metric

$$ds_{d+1}^2 = h(r)^{-1} dr^2 - h(r) dz^2 + r^2 d\Sigma_k^2, \quad (4.7)$$

but now with a different function h . Choosing $c = 2$, one finds that

$$h(r) = k - \frac{m}{r^{d-2}} - \frac{\Lambda}{d(d-1)} r^2. \quad (4.8)$$

For $k = 1$, this is the (anti-)de Sitter Schwarzschild metric.

A new feature with respect to the $\Lambda = 0$ case is that the de Sitter Schwarzschild metric that arises for $\Lambda > 0$ has two horizons for sufficiently small non-zero Λ . When this happens, the region between the horizons is the lift of an instanton solution which mediates a universe collapsing to a big crunch singularity and one expanding from a second big bang singularity. To illustrate this, we consider an example with $d = 4$ and $k = 1$. In this case, the $d+1$ solution is the five-dimensional de Sitter Schwarzschild black hole with

$$h(r) = -\frac{\Lambda}{12r^2} (r^2 - r_+^2)(r^2 - r_-^2), \quad (4.9)$$

where

$$r_{\pm}^2 = \frac{6}{\Lambda} \left(1 \pm \sqrt{1 - \frac{m\Lambda}{3}} \right). \quad (4.10)$$

There are two horizons for $m\Lambda < 3$. The horizon at $r = r_-$ is the black hole horizon, while $r = r_+$ is the de Sitter cosmological event horizon.

Let us consider the cosmological evolution for the cases: (a) $m = 0$; (b) $0 < m\Lambda < 3$; (c) $m\Lambda > 3$ (see figure 1).

- $\Lambda = 0$. This case was analysed in the previous section. The trajectory is just a straight line.
- $m = 0$. In this case, the trajectory passes through the origin $U = V = 0$ and is symmetric under $(U, V) \rightarrow -(U, V)$. Starting at negative U and V , we may interpret the trajectory as representing a contracting universe that collapses through a big crunch singularity into an instanton phase, which is connected to an isometric instanton phase through the origin $U = V = 0$. Then, as the trajectory passes again through the Milne horizon at $V = 0$ it becomes an expanding big bang universe, this being the time reverse of the original big crunch universe.

- $0 < m\Lambda < 3$. Starting from the big bang singularity at $\tau = 0$ ($U = 0$), the universe expands and then collapses to a big crunch singularity at $2\tau = r_-^2$ ($V = 0$). This evolution takes place in the future Milne sector ($U > 0, V > 0$). For $2\tau \in (r_-^2, r_+^2)$, the trajectory is in the Rindler region with $U > 0, V < 0$, and the corresponding solution is therefore an instanton. This instanton lifts to the $d + 1$ spacetime between the black hole and de Sitter horizons. As 2τ increases to r_+^2 , the trajectory returns to the $V = 0$ Milne horizon. Having passed through this horizon, the trajectory is once more in the future Milne sector and represents a second expanding big bang cosmology. This universe expands forever and at late times approaches the self-similar universe obtained by dimensional reduction of the five-dimensional de Sitter universe.

The full trajectory, which is shown in figure 1(c), includes another big crunch to big bang transition through an instanton that lifts to the interior spacetime of the negative mass Schwarzschild de Sitter spacetime. The exterior of this spacetime is the lift of a universe that is collapsing to a big crunch singularity, which lifts to the cosmological horizon of the negative mass de Sitter Schwarzschild spacetime.

- $m\Lambda = 3$. In this case, the black hole solution becomes the so-called Nariai solution with the topology $dS_2 \times S^{d-1}$. In the future Milne patch we get a bouncing cosmology where the singular bouncing point gets resolved into the dS_2 part of the Nariai solution.
- $m\Lambda > 3$. The part of the trajectory with $U < 0$ is similar to the previous case. Starting from the big bang singularity at $\tau = 0$ ($U = 0$), the universe first expands but then collapses to a minimum size, after which it re-expands, beginning a phase of eternal expansion.

The cases with $k = -1$ are related to those with $k = 1$ by flipping the sign of Λ, m and V : the evolution is similar with instantons and cosmologies interchanged (since V flips sign).

5. Cosmological constant models ($a = 0$)

We now allow for non-zero Λ but set $a = 0$. In other words, we now consider a cosmological constant term, of either sign. The effective action (2.9) becomes

$$I = -\frac{1}{2}(d-1) \int d\tau \left\{ \gamma^2 \dot{U} \dot{V} - 4k(-\epsilon UV)^{\gamma-1} + \frac{4\Lambda}{(d-1)(d-2)} (-\epsilon UV)^{2\alpha^2\gamma-1} \right\}. \quad (5.1)$$

We consider first the case of general d , showing that our prescription for passing through cosmological singularities leads fairly generically to cyclic universes. We then show how exact solutions may be found for $d = 3$ and $d = 4$.

5.1. Cyclic universes

For $a = 0$ we have a problem analogous to that of a particle in a central potential. To make this explicit, we set

$$\begin{aligned} U &= \eta e^{-\psi}, & V &= -\epsilon \eta e^{\psi}, & U > 0, \\ U &= -\eta e^{-\psi}, & V &= \epsilon \eta e^{\psi}, & U < 0, \end{aligned} \quad (5.2)$$

to put the action into the form

$$I = -\frac{1}{2}(d-1)\epsilon \int d\tau \left\{ \gamma^2 (\dot{\eta}^2 - \eta^2 \dot{\psi}^2) + 4k\epsilon \eta^{2(\gamma-1)} - \frac{4\epsilon\Lambda}{(d-1)(d-2)} \eta^{4\alpha^2\gamma-2} \right\}. \quad (5.3)$$

Let us note here that the relation between the new variables (η, ψ) and the original variables (φ, ϕ) is

$$e^{\varphi} = \eta^{2\alpha\gamma}, \quad \phi = 2\alpha\gamma\psi, \quad (5.4)$$

and that the spacetime metric in terms of η is

$$ds_d^2 = \frac{4\epsilon}{(d-2)^2} (\eta^2)^{2\alpha^2\gamma-2} d\eta^2 + (\eta^2)^{2\alpha^2\gamma/(d-1)} d\Sigma_k^2. \quad (5.5)$$

The potential term is greatly simplified by the choice $\gamma = 1$. Omitting an unimportant overall factor⁹, this choice yields the Lagrangian

$$L = \dot{\eta}^2 - \eta^2 \dot{\psi}^2 + 4k\epsilon - \frac{4\epsilon\Lambda}{(d-1)(d-2)} \eta^{2/(d-2)}. \quad (5.6)$$

The ψ equation of motion is trivially once-integrated to give

$$\dot{\psi} = \frac{j}{\eta^2}, \quad (5.7)$$

for some constant j . The η equation is

$$\eta\ddot{\eta} = -\frac{j^2}{\eta^2} - \frac{4\epsilon\Lambda}{(d-1)(d-2)^2} \eta^{2/(d-2)}. \quad (5.8)$$

This is trivially once-integrated and the integration constant is fixed by the zero-energy constraint. This yields the equation¹⁰

$$\dot{\eta}^2 + V_{\text{eff}}(\eta) = 4k\epsilon, \quad (5.9)$$

where

$$V_{\text{eff}}(\eta) = -\frac{j^2}{\eta^2} + \frac{4\epsilon\Lambda}{(d-1)(d-2)} \eta^{2/(d-2)}. \quad (5.10)$$

The cosmological ($\epsilon = -1$) solutions to this system are well known for $j = 0$; the dilaton is constant, and the metric is de Sitter or anti-de Sitter space. In what follows, we assume $j \neq 0$.

Our main interest at present is the possible continuation through a cosmological singularity to an instanton ($\epsilon = 1$) solution. As cosmological singularities occur at $\eta = 0$, the term containing the cosmological constant in equation (5.9) is irrelevant near the singularities. It follows that near a singularity the trajectories are straight lines in the (U, V) -plane, so the transition through the singularity is exactly the same as in the $\Lambda = 0$ case already discussed. Away from the singularity, the trajectories are bent by the cosmological constant term and the global behaviour can be quite different from the $\Lambda = 0$ case. Nevertheless, it can be understood by examining the behaviour of a particle in the potential $V_{\text{eff}}(\eta)$, noting that the potential changes across the Milne horizons because of the change in the sign of ϵ .

Consider first $\Lambda > 0$. For an instanton, the potential is monotonically increasing with increasing η , going to $-\infty$ at $\eta = 0$ and $+\infty$ at $\eta = \infty$. So any trajectory will flow to $\eta = 0$ and cross the Milne horizon, to yield a big bang cosmology. What happens next depends on k and Λ . On the cosmology side of the Milne horizon the potential is everywhere negative and has a maximum. If this maximum is less than the energy $4k\epsilon$ then $\dot{\eta}$ is never zero and the universe expands forever. This happens when $k = -1$ for any $\Lambda > 0$ and also for $k = 1$ provided that

$$\Lambda > \Lambda_+ = (d-2)^2 \left(\frac{4}{(d-1)j^2} \right)^{1/(d-2)}. \quad (5.11)$$

If, instead, the maximum is greater than the energy $4k\epsilon$, which happens when $k = 1$ and $\Lambda < \Lambda_+$, then there will be a recollapse to a big crunch singularity. However, the trajectory

⁹ This includes a factor of ϵ , which ensures that there is no factor of ϵ in the kinetic term in the U, V variables that cover the full 2D Minkowski space.

¹⁰ This equation implies the equation of motion (5.8) except when $\dot{\eta} \equiv 0$, in which case both the constraint and the η equation of motion are needed.

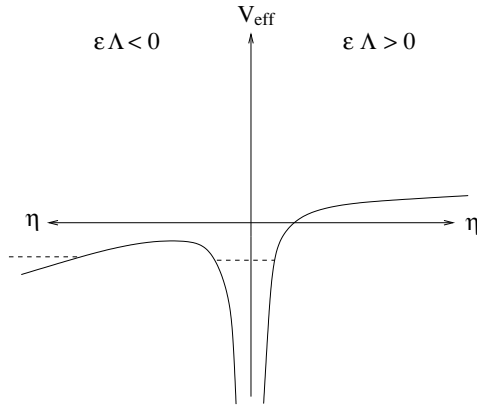


Figure 2. The potential $V_{\text{eff}}(\eta)$ with $j \neq 0$ for both signs of $\epsilon\Lambda$. The variable η is positive on both sides. A cyclic universe corresponds to a particle in the potential well around $\eta = 0$ (with dashed energy level). Classically it is trapped in the well; quantum mechanically it could tunnel through the energy barrier to the left region.

will pass straight through the singularity to the region behind the Milne horizon and we will have a second instanton phase. This instanton trajectory will be bent back to the Milne horizon at $\eta = 0$, yielding a second cosmological phase identical to the first one. Thus we have a cyclic universe, as is illustrated in figure 2.

A similar analysis applies for $\Lambda < 0$. If $k = 1$ or if $k = -1$ with $\Lambda < \Lambda_- = -\Lambda_+$, then we have an instanton phase that yields a big bang cosmology, which recollapses through a big crunch to another instanton phase. If $k = -1$ but $\Lambda > \Lambda_-$, then this process continues *ad infinitum* and we have a cyclic universe. Thus cyclic universes are generic in this model.

5.2. Explicit examples

The constraint (5.9) can be written as

$$d\tau = \frac{d\eta}{\sqrt{4k\epsilon - V_{\text{eff}}(\eta)}}. \tag{5.12}$$

Using this in (5.5) we have¹¹

$$ds_d^2 = \frac{4\epsilon}{(d-2)^2} \frac{\eta^{-\frac{2(d-3)}{d-2}}}{[4k\epsilon - V_{\text{eff}}(\eta)]} d\eta^2 + \eta^{2/(d-2)} d\Sigma_k^2, \tag{5.13}$$

so that η is now the time variable. The dilaton is given as a function of η by the formula

$$\phi(\eta) = 2\alpha j \int \frac{dx}{x^2 \sqrt{4k\epsilon - V_{\text{eff}}(x)}}. \tag{5.14}$$

More explicit formulae with τ as the time variable can be obtained for $d = 3$ and $d = 4$. For $d = 3$ equation (5.9) becomes

$$\dot{\eta}^2 = 4k\epsilon + \frac{j^2}{\eta^2} - 2\epsilon\Lambda\eta^2. \tag{5.15}$$

¹¹ Related solutions have appeared in the literature (see, e.g., [10]). Here our main interest is the fact that there exist smooth transitions through the Milne horizon connecting instanton and cosmology solutions.

Setting

$$y = \eta^2 - \frac{k}{\Lambda} \quad \text{and} \quad b = \frac{k^2}{\Lambda^2} + \frac{j^2}{2\epsilon\Lambda}, \quad (5.16)$$

this equation becomes

$$\dot{y}^2 = -8\epsilon\Lambda(y^2 - b). \quad (5.17)$$

This can be easily solved, and the solutions for the different cases are as follows,

- $\Lambda\epsilon < 0$

$$\begin{aligned} \eta^2 &= \sqrt{b} \cosh(\sqrt{-8\Lambda\epsilon}(\tau - \tau_0)) + \frac{k}{\Lambda} & \text{for } b > 0, \\ \eta^2 &= \sqrt{-b} \sinh(\sqrt{-8\Lambda\epsilon}(\tau - \tau_0)) + \frac{k}{\Lambda} & \text{for } b < 0, \end{aligned} \quad (5.18)$$

where τ_0 is an integration constant. For $b > 0$ with $k/\Lambda > 0$, the domain of validity for τ is $(-\infty, \infty)$. For the rest of the cases, τ_0 are chosen such that the domain of validity is $(0, \infty)$.

- $\Lambda\epsilon > 0$

$$\eta^2 = \sqrt{b} \sin(\sqrt{8\Lambda\epsilon}(\tau - \tau_0)) + \frac{k}{\Lambda}. \quad (5.19)$$

Since, by definition, $\sqrt{b} > k/\Lambda$ for $j \neq 0$, there is always a non-zero domain of validity for this solution.

Note that in this case of $d = 3$ the time τ coincides with the FLRW cosmic time t , as can be seen from equation (2.12) using $\alpha = 1$ and $\gamma = 1$, and $\eta(\tau)$ is the FLRW scale factor.

In the case of $d = 4$, equation (5.8) becomes

$$\eta\dot{\eta} = \sqrt{j^2 + 4k\epsilon\eta^2 - \frac{2}{3}\epsilon\Lambda\eta^3}. \quad (5.20)$$

The solution $\eta(\tau)$ can be expressed in terms of elliptic functions but the explicit formula is not illuminating and we omit it.

6. Flat universes ($k = 0$)

In this section, we consider flat universes ($k = 0$) for which the solutions are known exactly for any value of a [5–7]. Here we present these solutions as straight lines in a 2D Minkowski space, and consider the continuation through the cosmological singularities at the Milne horizon.

Going back to our original scalars ϕ and φ , the $k = 0$ effective action reads

$$I = \int d\tau \left\{ \frac{1}{2} f^{-1} \epsilon (\dot{\phi}^2 - \dot{\varphi}^2) - f \Lambda e^{2\alpha\varphi - a\phi} \right\}. \quad (6.1)$$

It is convenient to introduce the quantity

$$\Delta = a^2 - 4\alpha^2, \quad (6.2)$$

because this is invariant under $SO(1, 1)$ rotations of (φ, ϕ) . We will assume initially that $\Delta \neq 0$, in which case we may define s by $s = \text{sign}(\Delta)$. We now define new variables (ζ, ξ) by means of an $SO(1, 1)$ rotation:

$$\begin{pmatrix} \zeta \\ \xi \end{pmatrix} = \frac{1}{\sqrt{|\Delta|}} \begin{pmatrix} 2\alpha & -a \\ -a & 2\alpha \end{pmatrix} \begin{pmatrix} \varphi \\ \phi \end{pmatrix}. \quad (6.3)$$

This brings the action (6.1) to the form

$$I = \int d\tau \left\{ \frac{1}{2} f^{-1} s \epsilon (\dot{\xi}^2 - \dot{\xi}^2) - f \Lambda e^{\sqrt{|\Delta|} \xi} \right\}. \tag{6.4}$$

By analogy with (2.6) we choose new coordinates $(\mathcal{U}, \mathcal{V})$

$$e^{\xi} = (-\epsilon \mathcal{U} \mathcal{V})^{\alpha \gamma}, \quad e^{\xi} = \left(-\frac{\epsilon \mathcal{V}}{\mathcal{U}} \right)^{\alpha \gamma}, \tag{6.5}$$

and we fix the time-reparametrization invariance by the choice

$$f = \frac{2\alpha^2 \gamma^2}{s \epsilon \mathcal{U} \mathcal{V}}. \tag{6.6}$$

The action becomes

$$I = \int d\tau \{ \dot{\mathcal{U}} \dot{\mathcal{V}} + 2\alpha^2 \gamma^2 s \Lambda (-\epsilon \mathcal{U} \mathcal{V})^{\alpha \gamma \sqrt{|\Delta|} - 1} \}, \tag{6.7}$$

and the f equation of motion of (6.4) becomes the constraint

$$\dot{\mathcal{U}} \dot{\mathcal{V}} = 2\alpha^2 \gamma^2 s \Lambda (-\epsilon \mathcal{U} \mathcal{V})^{\alpha \gamma \sqrt{|\Delta|} - 1}. \tag{6.8}$$

With the choice $\gamma = 1/(\alpha \sqrt{|\Delta|})$, the solutions to the field equations satisfying the constraint are the straight lines¹²

$$\mathcal{U} = c\tau, \quad \mathcal{V} = \frac{2s\Lambda}{c|\Delta|} \tau + m. \tag{6.9}$$

In order to find the general metric and dilaton solution explicitly, we need to express U, V appearing in equation (2.11) in terms of \mathcal{U}, \mathcal{V} . Using equations (2.6), (6.3) and (6.5), we find that

$$\mathcal{U}^2 = (U^2)^{\frac{2\alpha+a}{\sqrt{|\Delta|}}}, \quad \mathcal{V}^2 = (V^2)^{\frac{2\alpha-a}{\sqrt{|\Delta|}}}. \tag{6.10}$$

The analysis of the continuation through the Milne horizon in the $(\mathcal{U}, \mathcal{V})$ space is similar to the discussion of the $\Lambda = 0$ case in section 3. The role of k in that section is played here by $-\alpha^2 s \Lambda / 2$ (which, however, need not be ± 1). Consequently, the full straight line trajectory consists of cosmological and instanton phases.

The case $\Delta = 0$, i.e., $a = 2\alpha$, must be treated separately. We choose $\gamma = 1/\alpha^2$ and we define new coordinates (U, v) by

$$e^{\varphi} = (\epsilon U)^{1/\alpha} e^v, \quad e^{\phi} = (\epsilon U)^{-1/\alpha} e^v. \tag{6.11}$$

In the (U, v) -plane, the regions with $U < 0$ represent cosmologies, while the regions with $U > 0$ represent instantons.

The gauge choice

$$f = \frac{2}{\alpha \epsilon U}, \tag{6.12}$$

yields the following action,

$$I = \epsilon \int d\tau \left\{ \dot{U} \dot{v} - \frac{2\Lambda}{\alpha} \right\}, \tag{6.13}$$

and the additional constraint

$$\dot{U} \dot{v} = -\frac{2\Lambda}{\alpha}. \tag{6.14}$$

Again we have straight lines in the (U, v) -plane, but now cosmological singularities occur at points at which the trajectory crosses $U = 0$. As every trajectory crosses $U = 0$ once, every trajectory has one instanton and one cosmology phase.

¹² Note that the choice $\gamma = 2/(\alpha \sqrt{|\Delta|})$ also gives rise to linear field equations. In this case, for $s\epsilon\Lambda > 0$, the solution to the field equations will be a segment of an ellipse, but a smooth continuation through the Milne horizon would require changing the sign of $s\Lambda$, which is not possible because s and Λ are parameters of the model.

7. Discussion

In this paper, we have investigated the possibility of a classical resolution of cosmological singularities in a class of d -dimensional models with a dilaton field ϕ and a possible exponential potential of the form $\Lambda e^{-a\phi}$. In these models, a FLRW cosmology corresponds to a trajectory in a 2D Minkowski ‘superspace’ but with a Milne metric in the parametrization provided by the dilaton ϕ and the cosmological scale factor e^ϕ . In terms of null Minkowski coordinates (U, V) these fields cover only the Milne patches, for which $UV > 0$. Cosmological singularities correspond to points on trajectories that cross the Milne horizon at $UV = 0$. Although the d -dimensional spacetime metric is singular, the cosmological trajectory can cross the Milne horizon smoothly, and the continuation of the trajectory into the Rindler region behind the Milne horizon represents an instanton solution of the Euclidean action. Thus, a trajectory that starts in a Milne region, smoothly crosses to a Rindler region and then returns to a Milne region, represents a big crunch–big bang transition mediated by an instanton phase. Globally, the trajectories may be open curves that represent a solution with a finite number of cosmology/instanton phases, or closed curves that represent cyclic universes.

These results complement our earlier study of cosmology/instanton transitions in models of gravity coupled to scalar fields with a hyperbolic target space [2]. Indeed, for $\Lambda = 0$, the model we considered is a consistent truncation of the model of [2] to purely Einstein-dilaton system. However, an important aspect of the truncated theory considered here is that its action is the dimensional reduction of the Einstein–Hilbert action in one higher dimension. This is true not only for the standard metric/dilaton action, for a Lorentzian-signature metric, but also for the Euclidean action, which is obtained by considering a time-independent metric in the higher dimension. Thus, the cosmology/instanton solutions lift to solutions of Einstein’s equations in one higher dimension. For $k = 1$ a cosmology/instanton solution corresponding to a straight line trajectory in the 2D Minkowski space passes through both Rindler patches and either the future or past Milne patch of this space; the full trajectory therefore comprises three segments. The cosmological solution corresponding to the segment in the Milne patch lifts to the interior of a (positive mass) Schwarzschild black hole, while the pre-big bang and post-big crunch instanton solutions corresponding to the segments in the Rindler patches lift either to the exterior of the same black hole solution or to the negative mass Schwarzschild spacetime found by reversing the sign of the black hole mass. All three of these higher dimensional spacetimes (for given mass) become part of the *same* cosmology/instanton solution in one dimension lower. In particular, the black hole horizon becomes either the big bang or the big crunch singularity of the lower dimensional cosmology, depending on which of two possible dimensional-reduction ansätze is used, the two ansätze being related by a change of sign of the dilaton¹³.

Motivated in part by massive IIA supergravity, for which the dilaton has a positive exponential potential, we also considered the effect of an exponential dilaton potential parametrized by a constant a , such that $a = 0$ corresponds to a constant potential (i.e., a cosmological constant). For given d there is one value of a for which exact cosmology/instanton solutions can be found for any k ; namely, the value for which the action is the reduction of gravity with a cosmological constant in one higher dimension. These solutions lift to de Sitter Schwarzschild (for positive cosmological constant) or anti-de Sitter Schwarzschild (for negative cosmological constant). The de Sitter Schwarzschild case is particularly

¹³ An interesting corollary is a means of resolving the curvature singularity of the Schwarzschild solution: reduction by one dimension followed by a change of sign of the dilaton and a lift back to the higher dimension interchanges the positive and negative mass Schwarzschild solutions and also interchanges the black hole horizon with the curvature singularity behind the horizon.

interesting because of the occurrence (for sufficiently small positive black hole mass) of two horizons, the black hole horizon and the cosmological event horizon. These horizons are the lift, respectively, of a big crunch and big bang singularity on a trajectory that crosses and then recrosses the Milne horizon of the 2D Minkowski ‘superspace’. The region behind the Milne horizon is described by a d -dimensional instanton solution mediating the big crunch to big bang transition, and this lifts to the $(d + 1)$ -dimensional spacetime between the black hole and cosmological horizons. Thus, the entire big crunch to big bang transition via a cosmological instanton lifts to a non-singular spacetime in one higher dimension! The uniqueness of the analytic continuation of the $(d + 1)$ -dimensional spacetime through its horizons is evidence for the correctness of our prescription for the continuation of cosmological trajectories through their singularities. Moreover, since the horizons remain regular when small perturbations about spherical symmetry are included, this suggests that it may be possible to extend our prescription for continuation through cosmological singularities to more general cosmologies which are not homogeneous and isotropic.

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