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Superconformal $N = 2$, $D = 5$ matter with and without actions

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Abstract: We investigate $N = 2$, $D = 5$ supersymmetry and matter-coupled supergravity theories in a superconformal context. In a first stage we do not require the existence of a lagrangian. Under this assumption, we already find at the level of rigid supersymmetry, i.e. *before* coupling to conformal supergravity, more general matter couplings than have been considered in the literature. For instance, we construct new vector-tensor multiplet couplings, theories with an *odd* number of tensor multiplets, and hypermultiplets whose scalar manifold geometry is *not* hyperkähler. Next, we construct rigid superconformal lagrangians. This requires some extra ingredients that are not available for all dynamical systems. However, for the generalizations with tensor multiplets mentioned above, we find corresponding new actions and scalar potentials. Finally, we extend the supersymmetry to local superconformal symmetry, making use of the Weyl multiplet. Throughout the paper, we will indicate the various geometrical concepts that arise, and as an application we compute the non-vanishing components of the Ricci tensor of hypercomplex group manifolds. Our results can be used as a starting point to obtain more general matter-couplings to Poincaré supergravity.

Keywords: Conformal and W Symmetry, Extended Supersymmetry, Supergravity Models.

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JHEP10(2002)045

Contents

JHEP10(2002)045

1. Introduction

Recently, much attention has been given to $D = 5$ matter-coupled supergravity theories [1, 2], thereby generalizing the earlier results of [3, 4]. This is mainly due to the fact that matter couplings in five dimensions play an important role in theories with large extra dimensions [5]–[8]. In particular, the properties of the scalar potential determine whether or not a supersymmetric Randall-Sundrum (RS) scenario [7, 8] is possible. The possibility of such a supersymmetric RS scenario relies on the existence of a domain-wall solution containing a warp factor with the correct asymptotic behaviour such that gravity is suppressed in the transverse direction. It turns out that constructing such a domain-wall solution is nontrivial.

With only vector multiplets and no singular source insertions, a no-go theorem was established for smooth domain-wall solutions [9, 10]. It has been shown that solutions acceptable for a supersymmetric RS scenario can be found provided one allows for branes as singular insertions [11]. Another approach is to include hypermultiplets [12]–[14]. The general mixing of vector and hypermultiplets was considered in [15], and its possibilities were further analysed in [16]. It seems that with such general matter couplings there is no *a priori* obstruction for a supersymmetric RS scenario, although an acceptable smooth solution has not yet been found. Improvements in the last year involve curved branes [17]–[20] and the use of non-homogeneous quaternionic spaces [21].

Matter-coupled $D = 5$ supergravity theories also play an important role in AdS_6/ CFT_5 [22] and AdS_5/CFT_4 [23] correspondences. In particular, the $D = 5$ domain-wall solutions describe the renormalization group flow of the corresponding four-dimensional field theory. The geometrical warp factor now plays the role of an energy scale. The structure of the domain wall is determined by the properties of the scalar potential. Finally, domain wall solutions have been applied to cosmology in the context of e.g. inflation [24] and quintessence [25]. In this context, it is important to find out what the detailed properties of the scalar potential are, and which kind of domain walls they give rise to.

The reasons given above motivated us to reconsider matter couplings in five dimensions, to independently derive the most general $D = 5$ matter couplings of [2] and, perhaps, to find more general matter couplings. Our strategy was to use the so-called conformal approach [26]–[29]. An advantage of the conformal construction is that, by past experience, it leads to insights into the structure of the matter couplings. A recent example is the insight in relations between hyperkähler cones and quaternionic manifolds, based on the study of superconformal matter couplings with hypermultiplets [30, 31].

In [32, 33], the first step in the conformal programme has been performed by constructing the Weyl multiplets of $N = 2$ conformal supergravity in five dimensions. The purpose of this paper is to take the next step in the conformal programme and introduce the different $D = 5$ matter multiplets with 8 conformal supersymmetries together with the corresponding actions (when they exist). Similar steps, have been performed in [33]–[35]. These authors also constructed off-shell superconformal multiplets. We will be able to generalize their results by not restricting ourselves to off-shell multiplets. Especially for the hypermultiplets this is important, as general quaternionic manifolds are not obtained from an off-shell calculus. In this context we should also mention earlier work on (non-conformal) on-shell multiplets by Zucker [36, 37]. In a next paper, we will take the last step in the conformal programme and impose different gauge-fixings. This will give us the $D = 5$ matter couplings we are aiming at. It was recently [38] shown how this method can be applied in the context of the RS scenario, for coupling the $D = 5$ bulk supergravity to $D = 4$ brane matter multiplets in a superconformal invariant way. We hope that our more general results may also be helpful in these investigations.

There is a rather different, more general, motivation of why the $D = 5$ matter-coupled supergravities are interesting to study. The reason is that they belong to the class of theories with eight supersymmetries [39]. Such theories are especially interesting since the geometry, determined by the kinetic terms of the scalars, contains undetermined functions. Theories with 32 supersymmetries have no matter multiplets while the geometry of those with 16 supersymmetries is completely determined by the number of matter multiplets. Of course, theories with 4 supersymmetries allow for more general geometries. The restricted class of geometries, in the case of 8 supersymmetries, makes these theories especially interesting and manageable. For instance, the work of Seiberg and Witten [40, 41] heavily relies on the presence of 8 supersymmetries. Theories with 8 supersymmetries are thus the maximally supersymmetric theories that, on the one hand, are not completely determined by the number of matter multiplets in the model and, on the other hand, allow arbitrary functions in their definition, i.e. continuous deformations of the metric of the manifolds.

The geometry related to supersymmetric theories with 8 real supercharges is called 'special geometry'. Special geometry was first found in [42, 43] for local supersymmetry and in [44, 45] for rigid supersymmetry. It occurs in Calabi-Yau compactifications of type II superstrings as the moduli space of these manifolds [46]–[51]. Special geometry was a very useful tool in the investigation of supersymmetric black holes [52, 53]. The work of Seiberg and Witten [40, 41] was based on the use of (rigid) special geometry. Later, the AdS/CFT correspondence [54] gave new applications of special geometry. So far, special geometry had been mainly investigated in the context of four dimensions. In the context of M theory compactifications on a Calabi-Yau [55], and with the advent of the brane-world scenarios [7, 8], also the $D = 5$ variant of special geometry [3], called 'very special geometry', received a lot of attention. The connection to special geometry was made in [56]. Last but not least, mathematicians got interested in special geometry due to its relation with quaternionic geometry [47], which lead to new results on the classification of homogeneous quaternionic spaces [57, 58].

We mentioned already that a conformal tensor calculus for $D = 5$ matter multiplets with 8 supersymmetries has already been introduced in [34, 35, 33]. However, there are still some ingredients missing: in particular the geometrical features have not been discussed at the most general level. In this paper, we use superconformal methods to fill this gap. We start with listing the basic superconformal matter multiplets: vector/tensor multiplets, linear multiplets and hypermultiplets. Some of these multiplets are off-shell, others imply equations of motion that define dynamical models. The closure of the algebra leads to equations that determine the evolution of the fields. In fact, by now we are used to handle

JHEP10(2002)045

theories without starting from a bona-fide action. Indeed, this is the way in which we often work with IIB supergravity, or theories with self-dual antisymmetric tensor fields. Therefore, rather than starting to analyse the most general matter couplings from looking for invariant actions, we first can start the analysis of the multiplets, which in some cases already gives dynamical systems. The latter allow more general matter couplings than those constructed from a lagrangian.

In particular, we will not only introduce vector multiplets in the adjoint representation but, more generally, so-called 'vector-tensor' multiplets in arbitrary representations. This includes couplings with an *odd* number of tensor multiplets. This may generalize the analysis made e.g. recently in [59]. Furthermore, as far as the hypermultiplets are concerned, we will introduce more general geometries than hyperkähler for rigid supersymmetry, or quaternionic-Kähler for supergravity. We can find dynamical theories also without the need of an action, i.e. in hypercomplex geometry, which is hyperkähler geometry where there may not be a metric. Also in $N = 8$ theories in 5 dimensions, more general possibilities were found in [60] by considering theories where the dynamical equations are considered without the necessity of an action.

In a second step, we construct rigid superconformal lagrangians. This will require an extra ingredient, namely the existence of a certain covariant tensor, that is not available for all dynamical systems and leads to a restriction on the possible geometries. In a last step, we will extend the supersymmetry to a local conformal supersymmetry, making use of the Weyl multiplet constructed in [32, 33].

The first two steps discussed above only deal with the case of rigid conformal supersymmetry. This case is sufficient to explain most of the subtleties concerning the possible geometrical structures. It is only at the last step that we introduce the full complications of coupling the matter multiplets to conformal supergravity.

The paper is organised as follows. First, in section 2, we perform step one and list the basic superconformal matter multiplets. We construct and discuss the possible matter couplings in the absence of a lagrangian. Next, in section 3, we perform the second step and construct rigid superconformal lagrangians. We discuss the restrictions on the possible geometries that follow from the requirement of a lagrangian. Finally, in section 4, we perform the last step and extend the supersymmetry to local superconformal symmetry, making use of the Weyl multiplet constructed in [32, 33]. Our aim is twofold: we want to determine and deduce the various restrictions from supersymmetry, and we want to determine the independent geometrical quantities that are needed for constructing superconformal matter theories. Our results can be used as a starting point to obtain more general matter couplings to Poincaré supergravity.

In a first appendix, we mention the linear multiplet, which does not play a big role in our paper. Appendix B gives a summary of the properties of hypercomplex manifolds and their place in the family of quaternionic-like manifolds. Explicit examples of hypercomplex manifolds that are not hyperkähler are given in appendix C. In that last appendix we calculate explicitly the non-vanishing antisymmetric Ricci tensor for these manifolds, which is also a new result.

The conventions that we use are given in [32, appendix A].

JHEP10(2002)045

2. Multiplets of rigid conformal supersymmetry

In this section, we will introduce the basic superconformal matter multiplets. We start with giving a short review of rigid conformal supersymmetry in the first subsection. For a more extended discussion, see e.g. [61]. In the remaining subsections, we will discuss the various multiplets: the vector-tensor multiplet, the linear multiplet and the hypermultiplet.

2.1 Definition of rigid conformal (super-)symmetry

We first introduce conformal symmetry and in a second step extend this to conformal supersymmetry. Given a spacetime with a metric tensor $g_{\mu\nu}(x)$, the conformal transformations are defined as the general coordinate transformations that leave "angles" invariant. The parameters of these special coordinate transformations define a conformal Killing vector $k^\mu(x)$. The defining equation for this conformal Killing vector is given by

$$\delta_{\text{g.c.t.}}(k)g_{\mu\nu}(x) \equiv \nabla_\mu k_\nu(x) + \nabla_\nu k_\mu(x) = \omega(x) g_{\mu\nu}(x)\,, \tag{2.1}$$

where $\omega(x)$ is an arbitrary function, $k_\mu = g_{\mu\nu}k^\nu$ and the covariant derivative is given by $\nabla_\mu k_\nu = \partial_\mu k_\nu - \Gamma^\rho_{\mu\nu}k_\rho$. In flat D-dimensional Minkowski spacetime, (2.1) implies

$$\partial_{(\mu}k_{\nu)}(x) - \frac{1}{D}\eta_{\mu\nu}\partial_\rho k^\rho(x) = 0\,. \tag{2.2}$$

In dimensions $D > 2$, the conformal algebra is finite-dimensional. The solutions of (2.2) are given by

$$k^\mu(x) = \xi^\mu + \lambda_M^{\mu\nu}x_\nu + \lambda_D x^\mu + \left(x^2\Lambda_K^\mu - 2x^\mu x\cdot\Lambda_K\right). \tag{2.3}$$

Corresponding to the parameters ξ^μ are the translations P_μ, the parameters $\lambda_M^{\mu\nu}$ correspond to Lorentz rotations $M_{\mu\nu}$, to λ_D are associated the dilatations D, and Λ_K^μ are the parameters of 'special conformal transformations' K_μ. Thus, the full set of conformal transformations δ_C can be expressed as follows:

$$\delta_C = \xi^\mu P_\mu + \lambda_M^{\mu\nu}M_{\mu\nu} + \lambda_D D + \Lambda_K^\mu K_\mu\,. \tag{2.4}$$

The commutators between different generators define the conformal algebra which is isomorphic to the algebra of $\mathrm{SO}(D,2)$.

We wish to consider representations of the conformal algebra on fields $\phi^\alpha(x)$ where α stands for a collection of internal indices referring to the stability subalgebra of $x^\mu = 0$. From the expression (2.3) for the conformal Killing vector, we deduce that this algebra is isomorphic to the algebra generated by $M_{\mu\nu}, D$ and K_μ. We denote the generators of this stability subalgebra by $\Sigma_{\mu\nu}, \Delta$ and κ_μ. Applying the theory of induced representations, it follows that any representation (Σ, Δ, κ) of the stability subalgebra induces a representation of the full conformal algebra with the following transformation rules (we suppress any internal indices):

$$
\begin{aligned}
\delta_P\phi(x) &= \xi^\mu\partial_\mu\phi(x)\,,\\
\delta_M\phi(x) &= \frac{1}{2}\lambda_M^{\mu\nu}(x_\nu\partial_\mu - x_\mu\partial_\nu)\phi(x) + \delta_\Sigma(\lambda_M)\phi(x)\,,\\
\delta_D\phi(x) &= \lambda_D x^\lambda\partial_\lambda\phi(x) + \delta_\Delta(\lambda_D)\phi(x)\,,\\
\delta_K\phi(x) &= \lambda_K^\mu(x^2\partial_\mu - 2x_\mu x^\lambda\partial_\lambda)\phi(x)\ +\\
&\quad + \Big(\delta_\Delta(-2x\cdot\Lambda_K) + \delta_\Sigma(-4x_{[\mu}\lambda_{K\nu]}) + \delta_\kappa(\lambda_K)\Big)\phi(x)\,. \qquad (2.5)
\end{aligned}
$$

We now look at the non-trivial representation (Σ, Δ, κ) that we use in this paper. First, concerning the Lorentz representations, in this paper we will encounter anti-symmetric tensors $\phi_{a_1\cdots a_n}(x)$ $(n = 0, 1, 2, \ldots)$ and spinors $\psi_\alpha(x)$:

$$
\begin{aligned}
\delta_\Sigma(\lambda_M)\phi_{a_1\cdots a_n}(x) &= -n(\lambda_M)_{[a_1}{}^b\phi_{|b|a_2\cdots a_n]}(x)\,,\\
\delta_\Sigma(\lambda_M)\psi(x) &= -\frac{1}{4}\lambda_M^{ab}\gamma_{ab}\psi(x)\,. \qquad (2.6)
\end{aligned}
$$

Second, we consider the dilatations. For most fields, the Δ transformation is just determined by a number w, which is called the Weyl weight of ϕ^α:

$$\delta_\Delta(\lambda_D)\phi^\alpha(x) = w\lambda_D\phi^\alpha(x)\,. \qquad (2.7)$$

For scalar fields, it is often convenient to consider the set of fields ϕ^α as the coordinates of a scalar manifold with affine connection $\Gamma_{\alpha\beta}{}^\gamma$. With this understanding, the transformation of ϕ^α under dilatations can be characterized by:

$$\delta_\Delta(\lambda_D)\phi^\alpha = \lambda_D k^\alpha(\phi)\,. \qquad (2.8)$$

Requiring dilatational invariance of kinetic terms determined by a metric $g_{\alpha\beta}$, the vector k^α should be a homothetic Killing vector, i.e. it should satisfy the conformal Killing equation (2.1) for *constant* $\omega(x)$:

$$\mathfrak{D}_\alpha k_\beta + \mathfrak{D}_\beta k_\alpha = (D-2)g_{\alpha\beta}\,, \qquad (2.9)$$

where D denotes the spacetime dimension and $\mathfrak{D}_\alpha k_\beta = \partial_\alpha k_\beta - \Gamma_{\alpha\beta}{}^\gamma k_\gamma$. However, (2.5) shows that the Δ-transformation also enters in the special conformal transformation. It turns out that invariance of the kinetic terms under these special conformal transformations restricts $k^\alpha(\phi)$ further to a so-called *exact* homothetic Killing vector, i.e.,

$$k_\alpha = \partial_\alpha\chi\,, \qquad (2.10)$$

for some function $\chi(\phi)$. One can show that the restrictions (2.9) and (2.10) are equivalent to

$$\mathfrak{D}_\alpha k^\beta \equiv \partial_\alpha k^\beta + \Gamma_{\alpha\gamma}{}^\beta k^\gamma = w\delta_\alpha{}^\beta\,. \qquad (2.11)$$

The constant w is identified with the Weyl weight of ϕ^α and is in general $w = (D-2)/2$, i.e. 3/2 in our case. The proof of the necessity of (2.11) can be extracted from [62], see also [63, 64]. In these papers the conditions for conformal invariance of a sigma model with either gravity or supersymmetry are investigated. By restricting the proof to rigid conformal symmetry (without supersymmetry) we find the same conditions.

Note that the condition (2.11) can be formulated *independent* of a metric. Only an affine connection is necessary. Indeed, we will find the same condition from the closure of the superconformal algebra before any metric and/or action has been introduced. In four spacetime dimensions, this was done in [63].

For the special case of a zero affine connection, the homothetic Killing vector is given by $k^\alpha = w\phi^\alpha$ and the transformation rule (2.8) reduces to $\delta_\Delta(\lambda_D)\phi^\alpha = w\lambda_D\phi^\alpha$. Note that the homothetic Killing vector $k^\alpha = w\phi^\alpha$ is indeed exact with χ given by

$$\chi = \frac{1}{(D-2)} k^\alpha g_{\alpha\beta} k^\beta \,. \tag{2.12}$$

Finally, all fields that we will discuss in this paper are invariant under the internal special conformal transformations, i.e. $\delta_\kappa\phi^\alpha = 0$.

We next consider the extension to conformal supersymmetry. The parameters of these supersymmetries define a conformal Killing spinor $\epsilon^i(x)$ whose defining equation is given by

$$\nabla_\mu\epsilon^i(x) - \frac{1}{D}\gamma_\mu\gamma^\nu\nabla_\nu\epsilon^i(x) = 0\,. \tag{2.13}$$

In D-dimensional Minkowski spacetime this equation implies

$$\partial_\mu\epsilon^i(x) - \frac{1}{D}\gamma_\mu\partial\!\!\!/\epsilon^i(x) = 0\,. \tag{2.14}$$

The solution to this equation is given by

$$\epsilon^i(x) = \epsilon^i + \mathrm{i}\, x^\mu\gamma_\mu\eta^i\,, \tag{2.15}$$

where the (constant) parameters ϵ^i correspond to "ordinary" supersymmetry transformations Q^i_α and the parameters η^i define special conformal supersymmetries generated by S^i_α. The conformal transformation (2.3) and the supersymmetries (2.15) do not form a closed algebra. To obtain closure, one must introduce additional R-symmetry generators. In particular, in the case of 8 supercharges Q^i_α in $D = 5$, there is an additional SU(2) R-symmetry with generators $U_{ij} = U_{ji}$ $(i = 1, 2)$. Thus, the full set of superconformal transformations δ_C is given by:

$$\delta_C = \xi^\mu P_\mu + \lambda_M^{\mu\nu}M_{\mu\nu} + \lambda_D D + \Lambda_K^\mu K_\mu + \Lambda^{ij}U_{ij} + \mathrm{i}\,\bar\epsilon Q + \mathrm{i}\,\bar\eta S\,. \tag{2.16}$$

We refer to [32] for the full superconformal algebra $F^2(4)$ formed by (anti-)commutators between the (bosonic and fermionic) generators.

To construct field representations of the superconformal algebra, one can again apply the method of induced representations. In this case one must use superfields $\Phi^a(x^\mu, \theta^i_\alpha)$, where a stands for a collection of internal indices referring to the stability subalgebra of $x^\mu = \theta^i_\alpha = 0$. This algebra is isomorphic to the algebra generated by $M_{\mu\nu}, D, K_\mu, U_{ij}$ and S^i_α.

An additional complication, not encountered in the bosonic case, is that the representation one obtains is reducible. To obtain an irreducible representation, one must impose constraints on the superfield. It is at this point that the transformation rules become nonlinear in the fields. In this paper, we will follow a different approach. Instead of working with superfields we will work with the component "ordinary" fields. The different nonlinear transformation rules are obtained by imposing the superconformal algebra.

In the supersymmetric case, we must specify the SU(2)-properties of the different fields as well as the behaviour under S-supersymmetry. Concerning the SU(2), we will only encounter scalars ϕ, doublets ψ^i and triplets $\phi^{(ij)}$ whose transformations are given by

$$\begin{aligned} \delta_{\mathrm{SU}(2)}(\Lambda^{ij})\phi &= 0\,, \\ \delta_{\mathrm{SU}(2)}(\Lambda^{ij})\psi^i(x) &= -\Lambda^i{}_j\psi^j(x)\,, \\ \delta_{\mathrm{SU}(2)}(\Lambda^{ij})\phi^{ij}(x) &= -2\Lambda^{(i}{}_k\phi^{j)k}(x)\,. \end{aligned} \tag{2.17}$$

The scalars of the hypermultiplet will also have an SU(2) transformation despite the absence of an i index. We refer for that to section 2.3.2.

This leaves us with specifying how a given field transforms under the special supersymmetries generated by S^i_α. In superfield language the full S-transformation is given by a combination of an x-dependent translation in superspace, with parameter $\epsilon^i(x) = \mathrm{i}\,x^\mu\gamma_\mu\eta^i$, and an internal S-transformation. This is in perfect analogy to the bosonic case. In terms of component fields, the same is true. The x-dependent contribution is obtained by making the substitution

$$\epsilon^i \to \mathrm{i}\,\not{x}\eta^i \tag{2.18}$$

in the Q-supersymmetry rules. The internal S-transformations can be deduced by imposing the superconformal algebra. In the next three subsections, we will give the explicit form of these internal S-transformations for different matter multiplets.

Finally, we give below some of the commutators of the (rigid) superconformal algebra expressed in terms of commutators of variations of the fields. These commutators are realized on *all* matter multiplets discussed in the next subsections. The commutators between Q- and S-supersymmetry are given by

$$[\delta_Q(\epsilon_1), \delta_Q(\epsilon_2)] = \delta_P\left(\frac{1}{2}\bar\epsilon_2\gamma_\mu\epsilon_1\right), \tag{2.19}$$

$$[\delta_S(\eta), \delta_Q(\epsilon)] = \delta_D\left(\frac{1}{2}\,\mathrm{i}\,\bar\epsilon\eta\right) + \delta_M\left(\frac{1}{2}\,\mathrm{i}\,\bar\epsilon\gamma^{ab}\eta\right) + \delta_U\left(-\frac{3}{2}\,\mathrm{i}\,\bar\epsilon^{(i}\eta^{j)}\right), \tag{2.20}$$

$$[\delta_S(\eta_1), \delta_S(\eta_2)] = \delta_K\left(\frac{1}{2}\bar\eta_2\gamma^a\eta_1\right). \tag{2.21}$$

For later use we list a few more commutators:

$$\left[\delta_D(\Lambda_D), \delta_Q(\epsilon^i)\right] = \delta_Q\left(\frac{1}{2}\epsilon^i\Lambda_D\right), \tag{2.22}$$

$$\left[\delta_{SU(2)}(\Lambda^{ij}), \delta_Q(\epsilon^k)\right] = \delta_Q\left(\epsilon^j\Lambda_j{}^i\right), \tag{2.23}$$

$$\left[\delta_K(\Lambda_K), \delta_Q(\epsilon^i)\right] = \delta_S\left(\mathrm{i}\,\not{\Lambda}_K\epsilon^i\right). \tag{2.24}$$

Field	SU(2)	w	# d.o.f.
	off-shell vector multiplet		
A_μ^I	1	0	$4n$
Y^{ijI}	3	2	$3n$
σ^I	1	1	$1n$
ψ^{iI}	2	3/2	$8n$
	on-shell tensor multiplet		
$B_{\mu\nu}^M$	1	0	$3m$
ϕ^M	1	1	$1m$
λ^{iM}	2	3/2	$4m$
	on-shell hypermultiplet		
q^X	2	3/2	$4r$
ζ^A	1	2	$4r$
	off-shell linear multiplet		
L^{ij}	3	3	3
E_a	1	4	4
N	1	4	1
φ^i	2	7/2	8

Table 1: The $D = 5$ matter multiplets. We introduce n vector multiplets, m tensor multiplets and r hypermultiplets. Indicated are their degrees of freedom, the Weyl weights and the SU(2) representations, including the linear multiplet for completeness.

Note that to verify these commutators one should use not only the internal but the *full* superconformal transformation rules including the x-dependent translations (see (2.5)) and Q-supersymmetries (see (2.18)).

Now it's clear how generic fields transform under the superconformal group, we briefly give the field content and properties of the basic superconformal multiplets in five dimensions. They will be used for studying matter couplings in the remainder of this article. The linear multiplet will only be used as the multiplet of the equations of motion for the vector multiplet.

2.2 The vector-tensor multiplet

In this section, we will discuss superconformal vector multiplets that transform in arbitrary representations of the gauge group. From work on $N = 2$, $D = 5$ Poincaré matter couplings [1] it is known that vector multiplets transforming in representations other than the adjoint have to be dualized to tensor fields. We define a vector-tensor multiplet to be a vector multiplet transforming in a reducible representation that contains the adjoint representation as well as another, arbitrary representation.

We will show that the analysis of [1] can be extended to superconformal vector multiplets. In doing this we will generalize the gauge transformations for the tensor fields [1] by allowing them to transform into the field-strengths for the adjoint gauge fields. These more general gauge transformations are consistent with supersymmetry, even after breaking the conformal symmetry.

The vector-tensor multiplet contains *a priori* an arbitrary number of tensor fields. The restriction to an even number of tensor fields is not imposed by the closure of the algebra. If one demands that the field equations do not contain tachyonic modes, an even number is required [65]. Closely related to this is the fact that one can only construct an action for an even number of tensor multiplets. But supersymmetry without an action allows the more general possibility. Note that these main results are independent of the use of superconformal or super-Poincaré algebras.

To make contact with other results in the literature we will break the rigid conformal symmetry by using a vector multiplet as a compensating multiplet for the superconformal symmetry. The adjoint fields of the vector-tensor multiplet are given constant expectation values, and the scalar expectation values will play the role of a mass parameter. This will reduce the superconformal vector-tensor multiplet, for the case of two tensor multiplets, to the massive self-dual complex tensor multiplet of [65].

2.2.1 Adjoint representation

We will start with giving the transformation rules for a vector multiplet in the adjoint representation [33]. An off-shell vector multiplet has $8+8$ real degrees of freedom whose SU(2) labels and Weyl weights we have indicated in table 1.

The gauge transformations that we consider satisfy the commutation relations ($I = 1,\ldots,n$)

$$\left[\delta_G(\Lambda_1^I),\delta_G(\Lambda_2^J)\right] = \delta_G(\Lambda_3^K)\,,\qquad \Lambda_3^K = g\Lambda_1^I\Lambda_2^J f_{IJ}{}^K\,. \tag{2.25}$$

The gauge fields A_μ^I ($\mu = 0,1,\ldots,4$) and general matter fields of the vector multiplet as e.g. X^I transform under gauge transformations with parameters Λ^I according to

$$\delta_G(\Lambda^J)A_\mu^I = \partial_\mu\Lambda^I + gA_\mu^J f_{JK}{}^I\Lambda^K\,,\qquad \delta_G(\Lambda^J)X^I = -g\Lambda^J f_{JK}{}^I X^K\,, \tag{2.26}$$

where g is the coupling constant of the group G. The expression for the gauge-covariant derivative of X^I and the field-strengths are given by

$$\mathcal{D}_\mu X^I = \partial_\mu X^I + gA_\mu^J f_{JK}{}^I X^K\,,\qquad F_{\mu\nu}^I = 2\partial_{[\mu}A_{\nu]}^I + gf_{JK}{}^I A_\mu^J A_\nu^K\,. \tag{2.27}$$

The field-strength satisfies the Bianchi identity

$$\mathcal{D}_{[\mu}F_{\nu\lambda]}^I = 0\,. \tag{2.28}$$

The rigid Q- and S-supersymmetry transformation rules for the off-shell Yang-Mills multiplet are given by [33]

$$\begin{aligned}
\delta A_\mu^I &= \frac{1}{2}\bar\epsilon\gamma_\mu\psi^I\,,\\
\delta Y^{ijI} &= -\frac{1}{2}\bar\epsilon^{(i}\not{\mathcal{D}}\psi^{j)I} - \frac{1}{2}\,\mathrm{i}\,g\bar\epsilon^{(i}f_{JK}{}^I\sigma^J\psi^{j)K} + \frac{1}{2}\,\mathrm{i}\,\bar\eta^{(i}\psi^{j)I}\,,\\
\delta\psi^{iI} &= -\frac{1}{4}\gamma\cdot F^I\epsilon^i - \frac{1}{2}\,\mathrm{i}\not{\mathcal{D}}\sigma^I\epsilon^i - Y^{ijI}\epsilon_j + \sigma^I\eta^i\,,\\
\delta\sigma^I &= \frac{1}{2}\,\mathrm{i}\,\bar\epsilon\psi^I\,.
\end{aligned} \tag{2.29}$$

The commutator of two Q-supersymmetry transformations yields a translation with an extra G-transformation

$$[\delta(\epsilon_1), \delta(\epsilon_2)] = \delta_P\left(\frac{1}{2}\bar{\epsilon}_2\gamma_\mu\epsilon_1\right) + \delta_G\left(-\frac{1}{2}\,\mathrm{i}\,\sigma\bar{\epsilon}_2\epsilon_1\right). \tag{2.30}$$

Note that even though we are considering rigid superconformal symmetry, the algebra (2.30) contains a field-dependent term on the righthand side. Such soft terms are commonplace in local superconformal symmetry but here they already appear at the rigid level. In hamiltonian language, it means that the algebra is satisfied modulo constraints.

2.2.2 Reducible representation

Starting from n vector multiplets we now wish to consider a more general set of fields $\mathcal{H}^{\widetilde{I}}_{\mu\nu}$ $(\widetilde{I} = 1,\ldots,n+m)$. We write $\mathcal{H}^{\widetilde{I}}_{\mu\nu} = \{F^I_{\mu\nu}, B^M_{\mu\nu}\}$ with $\widetilde{I} = (I, M)$ $(I = 1,\ldots,n; M = n+1,\ldots n+m)$. The first part of these fields corresponds to the generators in the adjoint representation. These are the fields that we used in subsection 2.2.1. The other fields may belong to an arbitrary, possibly reducible, representation:

$$(t_I)_{\widetilde{J}}{}^{\widetilde{K}} = \begin{pmatrix} (t_I)_J{}^K & (t_I)_J{}^N \\ (t_I)_M{}^K & (t_I)_M{}^N \end{pmatrix}, \qquad \begin{cases} I, J, K = 1,\ldots,n \\ M, N = n+1,\ldots,n+m\,. \end{cases} \tag{2.31}$$

It is understood that the $(t_I)_J{}^K$ are in the adjoint representation, i.e.

$$(t_I)_J{}^K = f_{IJ}{}^K\,. \tag{2.32}$$

If $m \neq 0$, then the representation $(t_I)_{\widetilde{J}}{}^{\widetilde{K}}$ is reducible. We will see that this representation can be more general than assumed so far in treatments of vector-tensor multiplet couplings. The requirement that m is even will only appear when we demand the existence of an action in section 3.2, or if we require absence of tachyonic modes. The matrices t_I satisfy commutation relations

$$[t_I, t_J] = -f_{IJ}{}^K t_K\,, \qquad \text{or} \qquad t_{I\widetilde{N}}{}^{\widetilde{M}} t_{J\widetilde{M}}{}^{\widetilde{L}} - t_{J\widetilde{N}}{}^{\widetilde{M}} t_{I\widetilde{M}}{}^{\widetilde{L}} = -f_{IJ}{}^K t_{K\widetilde{N}}{}^{\widetilde{L}}\,. \tag{2.33}$$

If the index $\tilde{L}$ is a vector index, then this relation is satisfied using the matrices as in (2.32).

Requiring the closure of the superconformal algebra, we find Q- and S-supersymmetry transformation rules for the vector-tensor multiplet and a set of constraints. The transformations are

$$\begin{aligned}
\delta\mathcal{H}^{\widetilde{I}}_{\mu\nu} &= -\bar{\epsilon}\gamma_{[\mu}\mathcal{D}_{\nu]}\psi^{\widetilde{I}} + \mathrm{i}\,g\bar{\epsilon}\gamma_{\mu\nu}t_{(\widetilde{J}\widetilde{K})}{}^{\widetilde{I}}\sigma^{\widetilde{J}}\psi^{\widetilde{K}} + \mathrm{i}\,\bar{\eta}\gamma_{\mu\nu}\psi^{\widetilde{I}}\,,\\
\delta Y^{ij\widetilde{I}} &= -\frac{1}{2}\bar{\epsilon}^{(i}\not{\mathcal{D}}\psi^{j)\widetilde{I}} - \frac{1}{2}\,\mathrm{i}\,g\bar{\epsilon}^{(i}\left(t_{[\widetilde{J}\widetilde{K}]}{}^{\widetilde{I}} - 3t_{(\widetilde{J}\widetilde{K})}{}^{\widetilde{I}}\right)\sigma^{\widetilde{J}}\psi^{j)\widetilde{K}} + \frac{1}{2}\,\mathrm{i}\,\bar{\eta}^{(i}\psi^{j)\widetilde{I}}\,,\\
\delta\psi^{i\widetilde{I}} &= -\frac{1}{4}\gamma\cdot\mathcal{H}^{\widetilde{I}}\epsilon^i - \frac{1}{2}\,\mathrm{i}\not{\mathcal{D}}\sigma^{\widetilde{I}}\epsilon^i - Y^{ij\widetilde{I}}\epsilon_j + \frac{1}{2}g t_{(\widetilde{J}\widetilde{K})}{}^{\widetilde{I}}\sigma^{\widetilde{J}}\sigma^{\widetilde{K}}\epsilon^i + \sigma^{\widetilde{I}}\eta^i\,,\\
\delta\sigma^{\widetilde{I}} &= \frac{1}{2}\,\mathrm{i}\,\bar{\epsilon}\psi^{\widetilde{I}}\,.
\end{aligned} \tag{2.34}$$

The curly derivatives denote gauge-covariant derivatives as in (2.27) with the replacement of structure constants by general matrices t_I according to (2.32). We have extended the range of the generators from I to $\widetilde{I}$ in order to simplify the transformation rules with the understanding that

$$(t_M)_{\widetilde{J}}{}^{\widetilde{K}} = 0\,. \tag{2.35}$$

We use a convention where (anti)symmetrizations are done with total weight 1. We find that the supersymmetry algebra (2.30) is satisfied provided the representation matrices are restricted to

$$t_{(\widetilde{J}\widetilde{K})}{}^{I} = 0\,, \tag{2.36}$$

and provided the following two constraints on the fields are imposed:

$$L^{ij\widetilde{I}} \equiv t_{(\widetilde{J}\widetilde{K})}{}^{\widetilde{I}} \left(2\sigma^{\widetilde{J}} Y^{ij\widetilde{K}} - \frac{1}{2}\,\mathrm{i}\,\bar{\psi}^{i\widetilde{J}} \psi^{j\widetilde{K}} \right) = 0\,, \tag{2.37}$$

$$E^{\widetilde{I}}_{\mu\nu\lambda} \equiv \frac{3}{g}\mathcal{D}_{[\mu}\mathcal{H}_{\nu\lambda]}{}^{\widetilde{I}} - \varepsilon_{\mu\nu\lambda\rho\sigma} t_{(\widetilde{J}\widetilde{K})}{}^{\widetilde{I}} \left(\sigma^{\widetilde{J}} \mathcal{H}^{\rho\sigma\widetilde{K}} + \frac{1}{4}\,\mathrm{i}\,\bar{\psi}^{\widetilde{J}} \gamma^{\rho\sigma} \psi^{\widetilde{K}} \right) = 0\,. \tag{2.38}$$

For $\widetilde{I} = I$, the constraint (2.38) reduces to the Bianchi identity (2.28). The tensor $F^I_{\mu\nu}$ can therefore be seen as the curl of a gauge vector A^I_μ. Moreover, the constraint (2.37) is trivially satisfied for $\widetilde{I} = I$. We conclude that the fields with indices $\widetilde{I} = I$ form an off-shell vector multiplet in the adjoint representation of the gauge group.

On the other hand, when $\widetilde{I} = M$, the constraint (2.38) does not permit the fields $B^M_{\mu\nu}$ to be written as the curl of a gauge field and they should be seen as independent tensor fields. Instead, the constraint (2.38) is a massive self-duality condition that puts the tensors $B^M_{\mu\nu}$ *on-shell*. The constraint (2.37) implicitly allows us to eliminate the fields Y^{ijM} altogether. The general vector-tensor multiplet can then be interpreted as a set of m on-shell tensor multiplets in the background of n off-shell vector multiplets.

Using (2.36) we have reduced the representation matrices t_I to the following block-upper-triangular form:

$$(t_I)_{\widetilde{J}}{}^{\widetilde{K}} = \begin{pmatrix} f_{IJ}{}^K & (t_I)_J{}^N \\ 0 & (t_I)_M{}^N \end{pmatrix}. \tag{2.39}$$

In [1] it is mentioned that, "since terms of the form $B^M \wedge F^I \wedge A^J$ appear to be impossible to supersymmetrize in a gauge invariant way (except possibly in very special cases) we shall also assume that $C_{MIJ} = 0$". This corresponds, as we will see below, to the assumption that the representation is completely reducible, i.e. $t_{IJ}{}^N = 0$, meaning that gauge transformations do not mix the pure Yang-Mills field-strengths and the tensor fields. However, we find that off-diagonal generators *are* allowed, both when requiring closure of the superconformal algebra and when writing down an action. We thus allow reducible, but not necessarily completely reducible representations.

Recall that every *unitary* reducible representation of a Lie group is also completely reducible, and that every representation of a *compact* Lie group is equivalent to a unitary representation. Hence, every reducible representation of a compact Lie group is also completely reducible. Non-compact Lie groups, on the other hand, have no non-trivial and finite-dimensional unitary representations. However, every reducible representation of a *connected*, *semi-simple*, non-compact Lie group or a semi-simple, non-compact Lie *algebra* is also completely reducible. See [66] for an exposition of these theorems.

This leaves us with the class of non-compact Lie algebras that contain an abelian invariant subalgebra. Examples of non-diagonal terms can thus be given for t_I of the form

$$(t_I)_{\widetilde{J}}{}^{\widetilde{K}} = \begin{pmatrix} 0 & (t_I)_J{}^M \\ 0 & 0 \end{pmatrix} . \qquad (2.40)$$

The simplest one is thus with one gauge multiplet and a number of tensor multiplets, with only the off-diagonal parts $t_{11}{}^M$ non-vanishing. But more examples are possible, e.g. the lower right corner does not have to be zero.

The constraints (2.37) and (2.38), with $\widetilde{I} = M$, do not form a supersymmetric set: they are invariant under S-supersymmetry but under Q-supersymmetry they lead to a constraint on the spinors ψ^{iM} which we will call φ^{iM}:

$$\delta L^{ijM} = \mathrm{i}\,\bar{\epsilon}^{(i}\varphi^{j)M} , \qquad \delta E^M_{\mu\nu\rho} = \bar{\epsilon}\gamma_{\mu\nu\rho}\varphi^M . \qquad (2.41)$$

The expression for this constraint is given by

$$\begin{aligned} \varphi^{iM} &\equiv t_{(\widetilde{J}\widetilde{K})}{}^M \left[\mathrm{i}\,\sigma^{\widetilde{J}} \not{D}\psi^{i\widetilde{K}} + \frac{1}{2}\mathrm{i}\left(\not{D}\sigma^{\widetilde{J}}\right)\psi^{i\widetilde{K}} + Y^{ik\widetilde{J}}\psi_k^{\widetilde{K}} - \frac{1}{4}\gamma\cdot\mathcal{H}^{\widetilde{J}}\psi^{i\widetilde{K}} \right] - \\ &\quad - g\left(\left[t_{[\widetilde{J}\widetilde{K}]}{}^{\widetilde{L}} - 3t_{(\widetilde{J}\widetilde{K})}{}^{\widetilde{L}} \right] t_{(\widetilde{I}\widetilde{L})}{}^M + \frac{1}{2} t_{\widetilde{I}\widetilde{J}}{}^{\widetilde{L}} t_{(\widetilde{K}\widetilde{L})}{}^M \right) \sigma^{\widetilde{I}}\sigma^{\widetilde{J}}\psi^{i\widetilde{K}} \\ &= 0 . \end{aligned} \qquad (2.42)$$

The second line can be rewritten, by splitting the indices in tensor versus vector parts, as

$$+\frac{1}{2}g\sigma^I\sigma^J\psi^{\widetilde{K}}(t_It_J)_{\widetilde{K}}{}^M + \frac{1}{4}g\sigma^I\sigma^{\widetilde{K}}\psi^J\,(t_It_J + 2t_Jt_I)_{\widetilde{K}}{}^M . \qquad (2.43)$$

Varying the new constraint $\varphi^{i\,M}$ under Q-and S-supersymmetry, one finds at first sight two more constraints, E_a^M and N^M, of which the first one turns out to be dependent (see below):

$$\begin{aligned} \delta\varphi^{i\,M} &= -\frac{1}{2}\mathrm{i}\,\not{D}L^{ij\,M}\epsilon_j - \frac{1}{2}\mathrm{i}\,\gamma^a E_a^M\epsilon^i + \frac{1}{2}N^M\epsilon^i - \frac{1}{2}gt_{\widetilde{J}\widetilde{K}}{}^M\sigma^{\widetilde{J}}L^{ij\widetilde{K}}\epsilon_j - \\ &\quad - \frac{1}{12}\mathrm{i}\,gt_{(\widetilde{J}\widetilde{K})}{}^M\gamma^{abc}\sigma^{\widetilde{J}}E^{\widetilde{K}}_{abc}\epsilon^i + 3L^{ijM}\eta_j . \end{aligned} \qquad (2.44)$$

The constraint N^M is given by

$$\begin{aligned} N^M &\equiv t_{(\widetilde{J}\widetilde{K})}{}^M \left(\sigma^{\widetilde{J}}\Box\sigma^{\widetilde{K}} + \frac{1}{2}\mathcal{D}^a\sigma^{\widetilde{J}}\mathcal{D}_a\sigma^{\widetilde{K}} - \frac{1}{4}\mathcal{H}^{\widetilde{J}}_{ab}\mathcal{H}^{ab\widetilde{K}} - \frac{1}{2}\bar{\psi}^{\widetilde{J}}\not{D}\psi^{\widetilde{K}} + Y^{ij\widetilde{J}}Y_{ij}{}^{\widetilde{K}} \right) - \\ &\quad - \mathrm{i}\,g\left[-\frac{1}{2}\,t_{[\widetilde{J}\widetilde{K}]}{}^{\widetilde{L}}t_{(\widetilde{I}\widetilde{L})}{}^M + 2\,t_{(\widetilde{I}\widetilde{J})}{}^{\widetilde{L}}t_{(\widetilde{K}\widetilde{L})}{}^M \right] \sigma^{\widetilde{I}}\bar{\psi}^{\widetilde{J}}\psi^{\widetilde{K}} + \\ &\quad + \frac{1}{2}g^2\,(t_It_Jt_K)_{\widetilde{L}}{}^M\sigma^I\sigma^J\sigma^K\sigma^{\widetilde{L}} \\ &= 0 , \end{aligned} \qquad (2.45)$$

and for E_a^M we find

$$E_a^M \equiv t_{(\widetilde{J}\widetilde{K})}{}^M \left(\mathcal{D}^b\left(\sigma^{\widetilde{J}}\mathcal{H}_{ba}{}^{\widetilde{K}} + \frac{1}{4}\mathrm{i}\,\bar{\psi}^{\widetilde{J}}\gamma_{ba}\psi^K \right) - \frac{1}{8}\varepsilon_{abcde}\mathcal{H}^{bc\widetilde{J}}\mathcal{H}^{de\widetilde{K}} \right) = 0 . \qquad (2.46)$$

We made use of identities as

$$t_{K\widetilde{I}}{}^{\widetilde{L}}t_{(\widetilde{J}\widetilde{L})}{}^{M}+t_{K\widetilde{J}}{}^{\widetilde{L}}t_{(\widetilde{I}\widetilde{L})}{}^{M}-t_{(\widetilde{I}\widetilde{J})}{}^{\widetilde{L}}t_{K\widetilde{L}}{}^{M}=0\,, \tag{2.47}$$

which follow from the commutator relation (2.33), and the restrictions (2.35) and (2.36).

We find that the expression for E_a is related to the one corresponding to E^M_{abc} as follows:

$$E_a^M=-\frac{1}{12}\varepsilon_{abcde}\mathcal{D}^bE^{cdeM}\,. \tag{2.48}$$

By now we have found a set of constraints that under Q- and S-supersymmetry transform to each other. These constraints do not seem to form a multiplet by themselves.

2.2.3 The massive self-dual tensor multiplet

To obtain the massive self-dual tensor multiplet of [65], we consider a vector-tensor multiplet for general n and m. Our purpose is to use the vector multiplet as a compensating multiplet for the superconformal symmetry. Thus, we impose conditions on the fields that break the conformal symmetry, and preserve Q-supersymmetry. We give the fields of the vector multiplets the following vacuum expectation values

$$F^I_{\mu\nu}=Y^{ijI}=\psi^{iI}=0\,,\qquad \sigma^I=\frac{2m^I}{g}\,, \tag{2.49}$$

where m^I are constants. Note that these conditions break the conformal group to the Poincaré group, and break S-supersymmetry ($\eta=0$). This is an example of a compensating multiplet in rigid supersymmetry. The breaking of conformal symmetry is characterized by the mass parameters m^I in (2.49). If we substitute (2.49) into the expression (2.37) for L^{ijM}, then we find that we can eliminate the field Y^{ijM}

$$Y^{ijM}=0\,. \tag{2.50}$$

Moreover, we can also substitute (2.49) into the constraints $E^M_{\mu\nu\lambda}$, φ^{iM} and N^M obtaining

$$\begin{aligned}3\partial_{[\mu}B^M_{\nu\lambda]}&=\frac{1}{2}\varepsilon_{\mu\nu\lambda\rho\sigma}\mathcal{M}_N{}^MB^{\rho\sigma N}\,,\\ \partial\!\!\!/\psi^{iM}&=\mathrm{i}\,\mathcal{M}_N{}^M\psi^{iN}\,,\\ \Box\sigma^M&=-\left(\mathcal{M}^2\right)_N{}^M\sigma^N-\frac{4}{g}t_{IJ}{}^Nm^Im^J\mathcal{M}_N{}^M\,.\end{aligned} \tag{2.51}$$

The mass-matrix $\mathcal{M}_N{}^M$ is defined as

$$\mathcal{M}_N{}^M\equiv g\sigma^I(t_I)_N{}^M=2m^I(t_I)_N{}^M\,, \tag{2.52}$$

and has been assumed to be invertible. The last term of (2.51) can be eliminated by redefining σ^M with a constant shift. In order for the tensor fields to have no tachyonic modes, the mass-matrix needs to satisfy a symplectic condition which can only be satisfied if the number of tensor fields is even [65]. We denote the number of tensor multiplets by $m=2k$.

In the particular gauge (2.49) and representation (2.40) the mass matrix $\mathcal{M}$ is zero. The last two equations in (2.51) are not present and the first one becomes the usual Bianchi identity for a set of m abelian vectors. Thus, we are dealing with $n+m$ off-shell gauge vectors.

To obtain the massive self-dual tensor multiplet of [65] we consider the case $n=1$, $m=2$, i.e. two (real) tensor multiplets $\{B^M_{\mu\nu}, \lambda^{iM}, \phi^M\}$ $(M,N=2,3)$ in the background of one vector multiplet $\{F_{\mu\nu}\,\psi^i, \sigma\}$, which has been given the vacuum expectation value (2.49). In what follows we will use a complex notation:

$$B_{\mu\nu} = B^2_{\mu\nu} + \mathrm{i}\,B^3_{\mu\nu}\,, \qquad \overline{B}_{\mu\nu} = B^2_{\mu\nu} - \mathrm{i}\,B^3_{\mu\nu}\,. \tag{2.53}$$

The generators $(t_1)_{\widetilde{I}}{}^{\widetilde{J}}$ must form a representation of $\mathrm{U}(1) \simeq \mathrm{SO}(2)$. Under a U(1) transformation the field-strength $F_{\mu\nu}$ is invariant and the tensor field gets a phase

$$B'_{\mu\nu} = e^{\,\mathrm{i}\,\theta} B_{\mu\nu} \to \begin{pmatrix} B^2_{\mu\nu} \\ B^3_{\mu\nu} \end{pmatrix}' = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \cdot \begin{pmatrix} B^2_{\mu\nu} \\ B^3_{\mu\nu} \end{pmatrix}. \tag{2.54}$$

From this we obtain the generator

$$(t_1)_{\widetilde{I}}{}^{\widetilde{J}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{2.55}$$

After substituting the conditions (2.49) into the transformation rules we obtain

$$\begin{aligned} \delta B_{\mu\nu} &= -\bar{\epsilon}\gamma_{[\mu}\partial_{\nu]}\lambda - m\bar{\epsilon}\gamma_{\mu\nu}\lambda\,, \\ \delta\lambda^i &= -\frac{1}{4}\gamma\cdot B\epsilon^i - \frac{1}{2}\,\mathrm{i}\,\partial\!\!\!/\phi\epsilon^i - \mathrm{i}\,m\phi\epsilon^i\,, \\ \delta\phi &= \frac{1}{2}\,\mathrm{i}\,\bar{\epsilon}\lambda\,, \end{aligned} \tag{2.56}$$

and

$$3\partial_{[\mu}B_{\nu\lambda]} - \mathrm{i}\,m\varepsilon_{\mu\nu\lambda\rho\sigma}B^{\rho\sigma} = 0\,. \tag{2.57}$$

This reproduces the massive self-dual tensor multiplet of [65]. Note that the commutator of two Q-supersymmetries yields a translation plus a (rigid) U(1)-transformation whose parameter can be obtained from the general G-transformation in the superconformal algebra, see (2.30), by making the substitution (2.49).

From a six-dimensional point of view the interpretation of the mass parameter m is that it is the label of the m-th Kaluza-Klein mode in the reduction of the $D=6$ self-dual tensor multiplet. The zero-mode of the reduced tensor multiplet corresponds to a vector multiplet as can be seen from (2.57) which becomes a Bianchi identity for a field-strength when $m=0$.

2.3 The hypermultiplet

In this subsection, we discuss hypermultiplets in five dimensions. As for the tensor multiplets, there is in general no known off-shell formulation with a finite number of auxiliary fields. Therefore, the supersymmetry algebra already leads to the equations of motion.

A single hypermultiplet contains four real scalars and two spinors subject to the symplectic Majorana reality condition. For r hypermultiplets, we introduce real scalars $q^X(x)$, with $X = 1, \ldots, 4r$, and spinors $\zeta^A(x)$ with $A = 1, \ldots, 2r$. To formulate the symplectic Majorana condition, we introduce two matrices $\rho_A{}^B$ and $E_i{}^j$, with

$$\rho\rho^* = -\mathbb{1}_{2r}\,, \qquad EE^* = -\mathbb{1}_2\,. \tag{2.58}$$

This defines symplectic Majorana conditions for the fermions and supersymmetry transformation parameters [67]:

$$\alpha\mathcal{C}\gamma_0\zeta^B\rho_B{}^A = \left(\zeta^A\right)^*\,, \qquad \alpha\mathcal{C}\gamma_0\epsilon^j E_j{}^i = \left(\epsilon^i\right)^*\,, \tag{2.59}$$

where $\mathcal{C}$ is the charge conjugation matrix, and α is an irrelevant number of modulus 1. We can always adopt the basis where $E_i{}^j = \varepsilon_{ij}$, and will further restrict to that.

The scalar fields are interpreted as coordinates of some target space, and requiring the on-shell closure of the superconformal algebra imposes certain conditions on the target space, which we derive below. Superconformal hypermultiplets in four spacetime dimensions were discussed in [30]; our discussion is somehow similar, but we extend it to the case where an action is not needed, in the spirit explained in [39].

2.3.1 Rigid supersymmetry

We will show how the closure of the supersymmetry transformation laws leads to a 'hypercomplex manifold'. The closure of the algebra on the bosons leads to the defining equations for this geometry, whereas the closure of the algebra on the fermions and its further consistency leads to equations of motion in this geometry, independent of an action.

The supersymmetry transformations (with ϵ^i constant parameters) of the bosons $q^X(x)$, are parametrized by arbitrary functions $f^X_{iA}(q)$. Also for the transformation rules of the fermions we write the general form compatible with the supersymmetry algebra. This introduces other general functions $f^{iA}_X(q)$ and $\omega_{XB}{}^A(q)$:[1]

$$\begin{aligned} \delta(\epsilon)q^X &= -\,\mathrm{i}\,\bar\epsilon^i\zeta^A f^X_{iA}\,,\\ \delta(\epsilon)\zeta^A &= \frac{1}{2}\,\mathrm{i}\,\partial\!\!\!/ q^X f^{iA}_X\epsilon_i - \zeta^B\omega_{XB}{}^A\big(\delta(\epsilon)q^X\big)\,. \end{aligned} \tag{2.60}$$

The functions satisfy reality properties consistent with reality of q^X and the symplectic Majorana conditions, e.g.:

$$\left(f^{iA}_X\right)^* = f^{jB}_X E_j{}^i\rho_B{}^A\,, \qquad \left(\omega_{XA}{}^B\right)^* = \left(\rho^{-1}\omega_X\rho\right)_A{}^B\,. \tag{2.61}$$

[1]In fact, one can write down a more general supersymmetry transformation rule for the fermions than in (2.60), but using Fierz relations and simple considerations about the supersymmetry algebra, one can bring its form into the one written above.

A priori the functions f^X_{iA} and f^{iA}_X are independent, but the commutator of two supersymmetries on the scalars only gives a translation if one imposes

$$\begin{aligned} &f^{iA}_Y f^X_{iA} = \delta^X_Y \,, \qquad f^{iA}_X f^X_{jB} = \delta^i_j \delta^A_B \,, \\ &\mathfrak{D}_Y f^X_{iB} \equiv \partial_Y f^X_{iB} - \omega_{YB}{}^A f^X_{iA} + \Gamma_{ZY}{}^X f^Z_{iB} = 0 \,, \end{aligned} \tag{2.62}$$

where $\Gamma_{XY}{}^Z$ is some object, symmetric in the lower indices. This means that f^X_{iA} and f^{iA}_X are each others inverse and are covariantly constant with connections Γ and ω. It also implies that ρ is covariantly constant. The conditions (2.62) encode all the constraints on the target space that follow from imposing the supersymmetry algebra. Below, we show that there are no further geometrical constraints coming from the fermion commutator; instead this commutator defines the equations of motion for the on-shell hypermultiplet.

The supersymmetry transformation rules are covariant with respect to two kinds of reparametrizations. The first ones are the target space diffeomorphisms, $q^X \to \widetilde{q}^X(q)$, under which f^X_{iA} transforms as a vector, $\omega_{XA}{}^B$ as a one-form, and $\Gamma_{XY}{}^Z$ as a connection. The second set are the reparametrizations which act on the tangent space indices $A, B, \ldots$ On the fermions, they act as

$$\zeta^A \to \widetilde{\zeta}^A(q) = \zeta^B U_B{}^A(q) \,, \tag{2.63}$$

where $U(q)_A{}^B$ is any invertible matrix. In general, such a transformation brings us into a basis where the fermions depend on the scalars q^X. In this sense, the hypermultiplet is written in a special basis where q^X and ζ^A are independent fields. The supersymmetry transformation rules (2.60) are covariant under (2.63) if we transform $f^{iA}_X(q)$ as a vector and $\omega_{XA}{}^B$ as a connection,

$$\omega_{XA}{}^B \to \widetilde{\omega}_{XA}{}^B = [(\partial_X U^{-1})U + U^{-1}\omega_X U]_A{}^B \,. \tag{2.64}$$

These considerations lead us to define the covariant variation of the fermions:

$$\widehat{\delta}\zeta^A \equiv \delta\zeta^A + \zeta^B \omega_{XB}{}^A \delta q^X \,, \tag{2.65}$$

for any transformation δ (supersymmetry, conformal transformations,...). Two models related by either target space diffeomorphisms or fermion reparametrizations of the form (2.63) are equivalent; they are different coordinate descriptions of the same system. Thus, in a covariant formalism, the fermions can be functions of the scalars. However, the expression $\partial_X \zeta^A$ makes only sense if one compares different bases. But in the same way also the expression $\zeta^B \omega_{XB}{}^A$ makes only sense if one compares different bases, as the connection has no absolute value. The only covariant object is the covariant derivative

$$\mathfrak{D}_X \zeta^A \equiv \partial_X \zeta^A + \zeta^B \omega_{XB}{}^A \,. \tag{2.66}$$

The covariant transformations are also a useful tool to calculate any transformation on e.g. a quantity $W_A(q)\zeta^A$:

$$\begin{aligned} \delta\left(W_A(q)\zeta^A\right) &= \partial_X\left(W_A\zeta^A\right)\delta q^X + W_A\, \delta\zeta^A\big|_q \\ &= \mathfrak{D}_X\left(W_A\zeta^A\right)\delta q^X + W_A\left(\widehat{\delta}\zeta^A - \mathfrak{D}_X\zeta^A\delta q^X\right) \\ &= \left(\mathfrak{D}_X W_A\right)\delta q^X\zeta^A + W_A\widehat{\delta}\zeta^A \,. \end{aligned} \tag{2.67}$$

We will frequently use the covariant transformations (2.65). It can similarly be used on target-space vectors or tensors. E.g. for a quantity Δ^X:

$$\widehat{\delta}\Delta^X = \delta\Delta^X + \Delta^Y \Gamma_{ZY}{}^X \delta q^Z \,. \tag{2.68}$$

The geometry of the target space is that of a *hypercomplex* manifold. It is a weakened version of hyperkähler geometry where no hermitian covariantly constant metric is defined. We refer the reader to appendix B for an introduction to these manifolds, references and the mathematical context in which they can be situated.

The crucial ingredient is a triplet of complex structures, the hypercomplex structure, defined as

$$J^\alpha{}_X{}^Y \equiv -\,\mathrm{i}\, f_X^{iA} (\sigma^\alpha)_i{}^j f_{jA}^Y \,. \tag{2.69}$$

Using (2.62), they are covariantly constant and satisfy the quaternion algebra

$$J^\alpha J^\beta = -\mathbb{1}_{4r} \delta^{\alpha\beta} + \varepsilon^{\alpha\beta\gamma} J^\gamma \,. \tag{2.70}$$

At some places we also use a doublet notation, for which

$$J_X{}^Y{}_i{}^j \equiv \mathrm{i}\, J^\alpha{}_X{}^Y \left(\sigma^\alpha\right)_i{}^j = 2 f_X^{jA} f_{iA}^Y - \delta_i^j \delta_X^Y \,. \tag{2.71}$$

The same transition between doublet and triplet notation is used also for other SU(2)-valued quantities.

The holonomy group of such a space is contained in $\mathrm{G}\ell(r,\mathbb{H}) = \mathrm{SU}^*(2r)\times\mathrm{U}(1)$, the group of transformations acting on the A,B-indices. This follows from the integrability conditions on the covariantly constant vielbeins f_X^{iA}, which relates the curvatures of the $\omega_{XA}{}^B$ and $\Gamma_{XY}{}^Z$ connections (see appendix B.2 for conventions on the curvatures),

$$R_{XYZ}{}^W = f_{iA}^W f_Z^{iB} \mathcal{R}_{XYB}{}^A \,, \qquad \delta_j^i \mathcal{R}_{XYB}{}^A = f_W^{iA} f_{jB}^Z R_{XYZ}{}^W \,, \tag{2.72}$$

such that the Riemann curvature only lies in $\mathrm{G}\ell(r,\mathbb{H})$. Moreover, from the cyclicity properties of the Riemann tensor, it follows that

$$\begin{aligned} f_{Ci}^X f_{jD}^Y \mathcal{R}_{XYB}{}^A &= -\frac{1}{2}\varepsilon_{ij} W_{CDB}{}^A \,, \\ W_{CDB}{}^A \equiv f_C^{iX} f_{iD}^Y \mathcal{R}_{XYB}{}^A &= \frac{1}{2} f_C^{iX} f_{iD}^Y f_{jB}^Z f_W^{Aj} R_{XYZ}{}^W \,, \end{aligned} \tag{2.73}$$

where W is symmetric in all its three lower indices. For a more detailed discussion on hypercomplex manifolds and their curvature relations, we refer to appendix B. We show there that, in contrast with hyperkähler manifolds, hypercomplex manifolds are not necessarily Ricci flat; instead, the Ricci tensor is antisymmetric and defines a closed two-form.

We have so far only used the commutator of supersymmetry on the hyperscalars, and this lead us to the geometry of hypercomplex manifolds. Before continuing, we want to see what are the independent objects that determine the theory, and what are the independent constraints. We start in the supersymmetric theory from the vielbeins f_X^{iA}. They have to be real in the sense of (2.61) and invertible. With these vielbeins, we can construct the complex structures as in (2.69). In the developments above, the only remaining independent equation is the covariant constancy of the vielbein in (2.62). This equation contains the affine connection $\Gamma_{XY}{}^{Z}$ and the $\mathrm{G}\ell(r,\mathbb{H})$-connection $\omega_{XA}{}^{B}$. These two objects can be determined from the vielbeins if and only if the ('diagonal') Nijenhuis tensor (B.24) vanishes. Indeed, for vanishing Nijenhuis tensor, the 'Obata'-connection [68]

$$\Gamma_{XY}{}^{Z} = -\frac{1}{6}\left(2\partial_{(X}J^{\alpha}{}_{Y)}{}^{W} + \varepsilon^{\alpha\beta\gamma}J^{\beta}{}_{(X}{}^{U}\partial_{|U|}J^{\gamma}{}_{Y)}{}^{W}\right)J^{\alpha}{}_{W}{}^{Z}\,, \tag{2.74}$$

leads to covariantly constant complex structures. Moreover, one can show that any torsionless connection that leaves the complex structures invariant is equal to this Obata connection (similar to the fact that a connection that leaves a metric invariant is the Levi-Civita connection). With this connection one can then construct the $\mathrm{G}\ell(r,\mathbb{H})$-connection

$$\omega_{XA}{}^{B} = \frac{1}{2}f^{iB}_{Y}\left(\partial_X f^{Y}_{iA} + \Gamma^{Y}_{XZ}f^{Z}_{iA}\right)\,, \tag{2.75}$$

such that the vielbeins are covariantly constant.

Dynamics. Now we consider the commutator of supersymmetry on the fermions, which will determine the equations of motion for the hypermultiplets.

Using (2.62), (2.72) and (2.73), we compute this commutator on the fermions, and find[2]

$$[\delta(\epsilon_1),\delta(\epsilon_2)]\zeta^{A} = \frac{1}{2}\partial_a\zeta^{A}\bar{\epsilon}_2\gamma^{a}\epsilon_1 + \frac{1}{4}\Gamma^{A}\bar{\epsilon}_2\epsilon_1 - \frac{1}{4}\gamma_a\Gamma^{A}\bar{\epsilon}_2\gamma^{a}\epsilon_1\,. \tag{2.76}$$

The Γ^{A} are the non-closure functions, and define the equations of motion for the fermions,

$$\Gamma^{A} = \not{\mathfrak{D}}\zeta^{A} + \frac{1}{2}W_{CDB}{}^{A}\zeta^{B}\bar{\zeta}^{D}\zeta^{C}\,, \tag{2.77}$$

where we have introduced the covariant derivative with respect to the transformations (2.65)

$$\mathfrak{D}_\mu\zeta^{A} \equiv \partial_\mu\zeta^{A} + (\partial_\mu q^{X})\zeta^{B}\omega_{XB}{}^{A}\,. \tag{2.78}$$

By varying the equations of motion under supersymmetry, we derive the corresponding equations of motion for the scalar fields:

$$\widehat{\delta}(\epsilon)\Gamma^{A} = \frac{1}{2}\,\mathrm{i}\,f^{iA}_{X}\epsilon_i\Delta^{X}\,, \tag{2.79}$$

where

$$\Delta^{X} = \Box q^{X} - \frac{1}{2}\bar{\zeta}^{B}\gamma_a\zeta^{D}\partial^{a}q^{Y}f^{iC}_{Y}f^{X}_{iA}W_{BCD}{}^{A} - \frac{1}{4}\mathfrak{D}_Y W_{BCD}{}^{A}\bar{\zeta}^{E}\zeta^{D}\bar{\zeta}^{C}\zeta^{B}f^{iY}_{E}f^{X}_{iA}\,, \tag{2.80}$$

and the covariant laplacian is given by

$$\Box q^{X} = \partial_a\partial^{a}q^{X} + \left(\partial_a q^{Y}\right)\left(\partial^{a}q^{Z}\right)\Gamma_{YZ}{}^{X}\,. \tag{2.81}$$

[2]To obtain this result, we use Fierz identities expressing that only the cubic fermion combinations of [32, (A.11)] are independent:

$$\zeta^{(B}\bar{\zeta}^{C}\gamma_a\zeta^{D)} = -\gamma_a\zeta^{(B}\bar{\zeta}^{C}\zeta^{D)}\,.$$

In conclusion, the supersymmetry algebra imposes the hypercomplex constraints (2.62) and the equations of motion (2.77) and (2.80). These form a multiplet, as (2.79) has the counterpart

$$\widehat{\delta}(\epsilon)\Delta^X = -\,\mathrm{i}\,\bar{\epsilon}^i \not{\mathfrak{D}}\Gamma^A f_{iA}^X + 2\,\mathrm{i}\,\bar{\epsilon}^i \Gamma^B \bar{\zeta}^C \zeta^D f_{Bi}^Y \mathcal{R}^X{}_{YCD}\,, \tag{2.82}$$

where the covariant derivative of Γ^A is defined similar to (2.78). In the following, we will derive further constraints on the target space geometry from requiring the presence of conformal symmetry.

2.3.2 Superconformal symmetry

Now we define transformation rules for the hypermultiplet under the full (rigid) superconformal group. The scalars do not transform under special conformal transformations and special supersymmetry, but under dilatations and SU(2) transformations, we parametrize

$$\begin{aligned} \delta_D(\Lambda_D) q^X &= \Lambda_D k^X(q)\,, \\ \delta_{\mathrm{SU}(2)}(\Lambda^{ij}) q^X &= \Lambda^{ij} k_{ij}^X(q)\,, \end{aligned} \tag{2.83}$$

for some unknown functions $k^X(q)$ and $k_{ij}^X(q)$.

To derive the appropriate transformation rules for the fermions, we first note that the hyperinos should be invariant under special conformal symmetry. This is due to the fact that this symmetry changes the Weyl weight with one. If we realize the commutator (2.24) on the fermions ζ^A, we read off the special supersymmetry transformation

$$\delta_S(\eta^i)\zeta^A = -k^X f_X^{iA} \eta_i\,. \tag{2.84}$$

To proceed, we consider the commutator of regular and special supersymmetry (2.20). Realizing this on the scalars, we determine the expression for the generator of SU(2) transformations in terms of the dilatations and complex structures,

$$k_{ij}^X = \frac{1}{3} k^Y J_Y{}^X{}_{ij} \qquad \text{or} \qquad k^{\alpha X} = \frac{1}{3} k^Y J^\alpha{}_Y{}^X\,. \tag{2.85}$$

Realizing (2.20) on the hyperinos, we determine the covariant variations

$$\widehat{\delta}_D \zeta^A = 2\Lambda_D \zeta^A\,, \qquad \widehat{\delta}_{\mathrm{SU}(2)}\zeta^A = 0\,, \tag{2.86}$$

and furthermore the commutator (2.20) only closes if we impose

$$\mathfrak{D}_Y k^X = \frac{3}{2}\delta_Y{}^X\,, \tag{2.87}$$

which also implies

$$\mathfrak{D}_Y k^{\alpha X} = \frac{1}{2} J^\alpha{}_Y{}^X\,. \tag{2.88}$$

Note that (2.87) is imposed by supersymmetry. In a more usual derivation, where one considers symmetries of the lagrangian, we would find this constraint by imposing dilatation invariance of the action, see (2.11). Our result, though, doesn't require the existence of an action. The relations (2.87) and (2.85) further restrict the geometry of the target space, and it is easy to derive that the Riemann tensor has four zero eigenvectors,

$$k^X R_{XYZ}{}^W = 0\,, \qquad k^{\alpha X} R_{XYZ}{}^W = 0\,. \tag{2.89}$$

Also, under dilatations and SU(2) transformations, the hypercomplex structure is scale invariant and rotated into itself,

$$\begin{aligned} \Lambda_D \left(k^Z \partial_Z J^\alpha{}_X{}^Y - \partial_Z k^Y J^\alpha{}_X{}^Z + \partial_X k^Z J^\alpha{}_Z{}^Y\right) &= 0\,, \\ \Lambda^\beta \left(k^{\beta Z} \partial_Z J^\alpha{}_X{}^Y - \partial_Z k^{\beta Y} J^\alpha{}_X{}^Z + \partial_X k^{\beta Z} J^\alpha{}_Z{}^Y\right) &= -\epsilon^{\alpha\beta\gamma}\Lambda^\beta J^\gamma{}_X{}^Y\,. \end{aligned} \tag{2.90}$$

All these properties are similar to those derived from superconformal hypermultiplets in four spacetime dimensions [69, 30]. There, the $\mathrm{Sp}(1)\times \mathrm{G}\,\ell(r,\mathbb{H})$ sections, or simply, hypercomplex sections, were introduced

$$A^{iB}(q) \equiv k^X f_X^{iB}\,, \qquad (A^{iB})^* = A^{jC} E_j{}^i \rho_C{}^B\,, \tag{2.91}$$

which allow for a coordinate independent description of the target space. This means that all equations and transformation rules for the sections can be written without the occurrence of the q^X fields. For example, the hypercomplex sections are zero eigenvectors of the $\mathrm{G}\,\ell(r,\mathbb{H})$ curvature,

$$A^{iB} W_{BCD}{}^E = 0\,, \tag{2.92}$$

and have supersymmetry, dilatation and SU(2) transformation laws.

$$\widehat{\delta} A^{iB} = \frac{3}{2} f_X^{iB} \delta q^X = -\frac{3}{2}\,\mathrm{i}\,\bar{\epsilon}^i \zeta^B + \frac{3}{2}\Lambda_D A_i{}^B - \Lambda^i{}_j A^{jB}\,, \tag{2.93}$$

where $\widehat{\delta}$ is understood as a covariant variation, in the sense of (2.65).

2.3.3 Symmetries

We now assume the action of a symmetry group on the hypermultiplet. We have no action, but the 'symmetry' operation should leave invariant the set of equations of motion. The symmetry algebra must commute with the supersymmetry algebra (and later with the full superconformal algebra). This leads to hypermultiplet couplings to a non-abelian gauge group G. The symmetries are parametrized by

$$\begin{aligned} \delta_G q^X &= -g\Lambda_G^I k_I^X(q)\,, \\ \widehat{\delta}_G \zeta^A &= -g\Lambda_G^I t_{IB}{}^A(q)\zeta^B\,. \end{aligned} \tag{2.94}$$

The vectors k_I^X depend on the scalars and generate the algebra of G with structure constants $f_{IJ}{}^K$,

$$k_{[I|}^Y \partial_Y k_{|J]}^X = -\frac{1}{2} f_{IJ}{}^K k_K^X\,. \tag{2.95}$$

The commutator of two gauge transformations (2.25) on the fermions requires the following constraint on the field-dependent matrices $t_I(q)$,

$$[t_I, t_J]_B{}^A = -f_{IJ}{}^K t_{KB}{}^A - 2k_{[I|}^X \mathfrak{D}_X t_{|J]B}{}^A + k_I^X k_J^Y \mathcal{R}_{XYB}{}^A\,. \tag{2.96}$$

Requiring the gauge transformations to commute with supersymmetry leads to further relations between the quantities k_I^X and $t_{IB}{}^A$. Vanishing of the commutator on the scalars yields

$$t_{IB}{}^A f_{iA}^X = \mathfrak{D}_Y k_I^X f_{iB}^Y\,. \tag{2.97}$$

These constraints determine $t_I(q)$ in terms of the vielbeins f_X^{iA} and the vectors k_I^X,

$$t_{IA}{}^B = \frac{1}{2} f_{iA}^Y \mathfrak{D}_Y k_I^X f_X^{iB}\,, \tag{2.98}$$

and furthermore

$$f_A^{Y(i} f_X^{j)B} \mathfrak{D}_Y k_I^X = 0\,. \tag{2.99}$$

The relations (2.99) and (2.98) are equivalent to (2.97). We interpret (2.98) as the definition for $t_{IA}{}^B$. The vanishing of an (ij)-symmetric part in an equation as (2.99) can be expressed as the vanishing of the commutator of $\mathfrak{D}_Y k_I^X$ with the complex structures:[3]

$$\left(\mathfrak{D}_X k_I^Y\right) J^\alpha{}_Y{}^Z = J^\alpha{}_X{}^Y \left(\mathfrak{D}_Y k_I^Z\right). \tag{2.100}$$

Extracting affine connections from this equation, it can be written as

$$\left(\mathcal{L}_{k_I} J^\alpha\right)_X{}^Y \equiv k_I^Z \partial_Z J^\alpha{}_X{}^Y - \partial_Z k_I^Y J^\alpha{}_X{}^Z + \partial_X k_I^Z J^\alpha{}_Z{}^Y = 0\,. \tag{2.101}$$

The left-hand side is the Lie derivative of the complex structure in the direction of the vector k_I. In part B.5 of the appendix, it is mentioned that (2.101) is a special case of the statement that the vector k_I normalizes the hypercomplex structures. The latter would allow that this Lie derivative is proportional to a complex structure. Killing vectors which normalize the hypercomplex structure can be decomposed in an SU(2) part and a $\mathrm{G}\ell(r,\mathbb{H})$ part. The vanishing of this Lie derivative, or (2.99), is expressed by saying that the gauge transformations act *triholomorphic*. Thus, it says that all the symmetries are embedded in $\mathrm{G}\ell(r,\mathbb{H})$.

Vanishing of the gauge-supersymmetry commutator on the fermions requires

$$\mathfrak{D}_Y t_{IA}{}^B = k_I^X \mathcal{R}_{YXA}{}^B\,. \tag{2.102}$$

Using (2.97) this implies a new constraint,

$$\mathfrak{D}_X \mathfrak{D}_Y k_I^Z = R_{XWY}{}^Z k_I^W\,. \tag{2.103}$$

Note that this equation is in general true for any Killing vector of a metric. As we have no metric here, we could not rely on this fact, but here the algebra imposes this equation. It turns out that (2.99) and (2.103) are sufficient for the full commutator algebra to hold. In particular, (2.102) follows from (2.103), using the definition of t as in (2.98), and (2.72).

A further identity can be derived: substituting (2.102) into (2.96) one gets

$$[t_I, t_J]_B{}^A = -f_{IJ}{}^K t_{KB}{}^A - k_I^X k_J^Y \mathcal{R}_{XYB}{}^A\,. \tag{2.104}$$

[3]This can be seen directly from lemma 2 in appendix B.

This identity can also be obtained from substituting (2.98) in the commutator on the left hand side, and then using (2.95), (2.99), (2.103) and (2.72).

The group of gauge symmetries should also commute with the superconformal algebra, in particular with dilatations and SU(2) transformations. This leads to

$$k^Y \mathfrak{D}_Y k_I^X = \frac{3}{2} k_I^X , \qquad k^{\alpha Y} \mathfrak{D}_Y k_I^X = \frac{1}{2} k_I^Y J^\alpha{}_Y{}^X , \tag{2.105}$$

coming from the scalars, and there are no new constraints from the fermions or from other commutators. Since $\mathfrak{D}_Y k_I^X$ commutes with $J^\alpha{}_Y{}^X$, the second equation in (2.105) is a consequence of the first one.

In the above analysis, we have taken the parameters Λ^I to be constants. In the following, we also allow for local gauge transformations. The gauge coupling is done by introducing vector multiplets and defining the covariant derivatives

$$\begin{aligned} \mathfrak{D}_\mu q^X &\equiv \partial_\mu q^X + g A_\mu^I k_I^X , \\ \mathfrak{D}_\mu \zeta^A &\equiv \partial_\mu \zeta^A + \partial_\mu q^X \omega_{XB}{}^A \zeta^B + g A_\mu^I t_{IB}{}^A \zeta^B . \end{aligned} \tag{2.106}$$

The commutator of two supersymmetries should now also contain a local gauge transformation, in the same way as for the multiplets of the previous sections, see (2.30). This requires an extra term in the supersymmetry transformation law of the fermion,

$$\widehat{\delta}(\epsilon)\zeta^A = \frac{1}{2}\,\mathrm{i}\,\not{\mathfrak{D}} q^X f_X^{iA} \epsilon_i + \frac{1}{2} g \sigma^I k_I^X f_{iX}^A \epsilon^i . \tag{2.107}$$

With this additional term, the commutator on the scalars closes, whereas on the fermions, it determines the equations of motion

$$\Gamma^A \equiv \not{\mathfrak{D}}\zeta^A + \frac{1}{2} W_{BCD}{}^A \bar\zeta^C \zeta^D \zeta^B - g(\,\mathrm{i}\, k_I^X f_{iX}^A \psi^{iI} + \mathrm{i}\, \zeta^B \sigma^I t_{IB}{}^A) = 0 , \tag{2.108}$$

with the same conventions as in (2.76).

Acting on Γ^A with supersymmetry determines the equation of motion for the scalars

$$\begin{aligned} \Delta^X =& \Box q^X - \frac{1}{2}\bar\zeta^B \gamma_a \zeta^D \mathfrak{D}^a q^Y f_Y^{iC} f_{iA}^X W_{BCD}{}^A - \frac{1}{4}\mathfrak{D}_Y W_{BCD}{}^A \bar\zeta^E \zeta^D \bar\zeta^C \zeta^B f_E^{iY} f_{iA}^X - \\ & - g\left(2\,\mathrm{i}\,\bar\psi^{iI}\zeta^B t_{IB}{}^A f_{iA}^X - k_I^Y J_Y{}^X{}_{ij} Y^{ijI}\right) + g^2 \sigma^I \sigma^J \mathfrak{D}_Y k_I^X k_J^Y . \end{aligned} \tag{2.109}$$

The first line is the same as in (2.80), the second line contains the corrections due to the gauging. The gauge-covariant laplacian is here given by

$$\Box q^X = \partial_a \mathfrak{D}^a q^X + g \mathfrak{D}_a q^Y \partial_Y k_I^X A^{aI} + \mathfrak{D}_a q^Y \mathfrak{D}^a q^Z \Gamma_{YZ}^X . \tag{2.110}$$

The equations of motions Γ^A and Δ^X still satisfy the same algebra with (2.79) and (2.82).

3. Rigid superconformal actions

In this section, we will present rigid superconformal actions for the multiplets discussed in the previous section. We will see that demanding the existence of an action is more restrictive than only considering equations of motion. For the different multiplets, we find that new geometric objects have to be introduced.

3.1 Vector multiplet action

The coupling of Poincaré-supergravity to n vector multiplets (having n scalars φ^x) is completely determined by an $(n+1)$-dimensional constant symmetric tensor C_{IJK} [3]. The reason for the difference in the number of scalars and the rank of C_{IJK} is that the graviton multiplet also contains a vector field called the graviphoton.

The tensor C_{IJK} appears directly in the $A\wedge F\wedge F$ Chern–Simons couplings, and indirectly in all other terms of the action.

In particular, the manifold parametrized by the scalars φ^x of the vector multiplets can be viewed as an n-dimensional hypersurface in an $(n+1)$-dimensional space parametrized by $n+1$ coordinates $\sigma^I(\varphi^x)$:

$$C_{IJK}\sigma^I\sigma^J\sigma^K = 1\,. \tag{3.1}$$

The resulting geometry goes under the name of "very special geometry". For every value of n there are many different "very special real" manifolds: a classification of such spaces that are homogeneous was given in [56]. This classification includes the previously found symmetric spaces [3, 70].

From the viewpoint of superconformal symmetry, the equation (3.1) looks like a gauge-fixing condition for dilatation invariance. Indeed, it turns out that the coupling of n vector multiplets (with n scalars σ^I) in rigid supersymmetry (or in conformal supergravity as we will give the generalization in section 4) is also completely determined by the tensor C_{IJK}, but in contrast to the case of Poincaré supergravity, this tensor will multiply the complete action, not just the Chern–Simons term.

The rigidly superconformal invariant action describing n vector multiplets was obtained from tensor calculus using an intermediate linear multiplet in [35]. The abelian part can be obtained by just taking the (cubic) action of one vector multiplet as given in [32], adding indices I,J,K on the fields and multiplying with the symmetric tensor C_{IJK}. For the non-abelian case, we need conditions expressing the gauge invariance of this tensor:

$$f_{I(J}{}^{H}C_{KL)H} = 0\,. \tag{3.2}$$

Moreover one has to add a few more terms, e.g. to complete the Chern–Simons term to its non-abelian form. This leads to the action

$$\begin{aligned}\mathcal{L}_{\text{vector}} = \Big[\Big(&-\frac{1}{4}F^I_{\mu\nu}F^{\mu\nu J} - \frac{1}{2}\bar\psi^I\not{\mathcal{D}}\psi^J - \frac{1}{2}\mathcal{D}_a\sigma^I\mathcal{D}^a\sigma^J + Y^I_{ij}Y^{ijJ}\Big)\sigma^K - \\ &-\frac{1}{24}\varepsilon^{\mu\nu\lambda\rho\sigma}A^I_\mu\left(F^J_{\nu\lambda}F^K_{\rho\sigma} + \frac{1}{2}g[A_\nu,A_\lambda]^J F^K_{\rho\sigma} + \frac{1}{10}g^2[A_\nu,A_\lambda]^J[A_\rho,A_\sigma]^K\right) - \\ &-\frac{1}{8}\,\mathrm{i}\,\bar\psi^I\gamma\cdot F^J\psi^K - \frac{1}{2}\,\mathrm{i}\,\bar\psi^{iI}\psi^{jJ}Y^K_{ij} + \frac{1}{4}\,\mathrm{i}\,g\bar\psi^L\psi^H\sigma^I\sigma^J f_{LH}{}^K\Big]C_{IJK}\,. \end{aligned} \tag{3.3}$$

The equations of motion for the fields of the vector multiplet following from the action (3.3) are

$$0 = L_I^{ij} = \varphi_I^i = E_I^a = N_I\,, \tag{3.4}$$

where we have defined

$$
\begin{aligned}
L_I^{ij} &\equiv C_{IJK}\left(2\sigma^J Y^{ijK} - \frac{1}{2}\,\mathrm{i}\,\bar\psi^{iJ}\psi^{jK}\right),\\
\varphi_I^i &\equiv C_{IJK}\left(\mathrm{i}\,\sigma^J \not{\mathcal{D}}\psi^{iK} + \frac{1}{2}\,\mathrm{i}\,(\not{\mathcal{D}}\sigma^J)\psi^{iK} + Y^{ikJ}\psi_k^K - \frac{1}{4}\gamma\cdot F^J\psi^{iK}\right) -\\
&\quad - gC_{IJK}f_{LH}{}^K\sigma^J\sigma^L\psi^{iH}\,,\\
E_{aI} &\equiv C_{IJK}\left[\mathcal{D}^b\left(\sigma^J F_{ba}{}^K + \frac{1}{4}\,\mathrm{i}\,\bar\psi^J\gamma_{ba}\psi^K\right) - \frac{1}{8}\varepsilon_{abcde}F^{bcJ}F^{deK}\right] -\\
&\quad - \frac{1}{2}gC_{JKL}f_{IH}{}^J\sigma^K\bar\psi^L\gamma_a\psi^H - gC_{JKH}f_{IL}{}^J\sigma^K\sigma^L\mathcal{D}_a\sigma^H\,,\\
N_I &\equiv C_{IJK}\left(\sigma^J\Box\sigma^K + \frac{1}{2}\mathcal{D}^a\sigma^J\mathcal{D}_a\sigma^K - \frac{1}{4}F_{ab}^J F^{abK} - \frac{1}{2}\bar\psi^J\not{\mathcal{D}}\psi^K + Y^{ijJ}Y_{ij}{}^K\right) +\\
&\quad + \frac{1}{2}\,\mathrm{i}\,gC_{IJK}f_{LH}{}^K\sigma^J\bar\psi^L\psi^H\,. \qquad (3.5)
\end{aligned}
$$

We have given these equations of motion the names $L_I^{ij}, \phi_I^i, E_{aI}, N_I$ since they form a linear multiplet in the adjoint representation of the gauge group for which the transformation rules have been given in (A.1).

3.2 The vector-tensor multiplet action

We will now generalize the vector action (3.3) to an action for the vector-tensor multiplets (with n vector multiplets and m tensor multiplets) discussed in section 2.2.2.

The supersymmetry transformation rules for the vector-tensor multiplet (2.34) were obtained from those for the vector multiplet (2.29) by replacing all contracted indices by the extended range of tilde indices. In addition, extra terms of $\mathcal{O}(g)$ had to be added to the transformation rules. Similar considerations apply to the generalization of the action, as we will see below.

To obtain the generalization of the Chern-Simons (CS) term, it is convenient to rewrite this CS-term as an integral in six dimensions which has a boundary given by the five-dimensional Minkowski spacetime. The six-form appearing in the integral is given by

$$
I_{\text{vector}} = C_{IJK}F^I F^J F^K\,, \qquad (3.6)
$$

where we have used form notation. This six-form is both gauge-invariant and closed, by virtue of (3.2) and the Bianchi identities (2.28). It can therefore be written as the exterior derivative of a five-form which is gauge-invariant up to a total derivative. The spacetime integral over this five-form is the CS-term given in the second line of (3.3).

We now wish to generalize (3.6) to the case of vector-tensor multiplets. It turns out that the generalization of (3.6) is somewhat surprising. We find

$$
I_{\text{vec-tensor}} = C_{\tilde I\tilde J\tilde K}\mathcal{H}^{\tilde I}\mathcal{H}^{\tilde J}\mathcal{H}^{\tilde K} - \frac{3}{g}\Omega_{MN}\mathcal{D}B^M\mathcal{D}B^N\,. \qquad (3.7)
$$

The tensor Ω_{MN} is antisymmetric and invertible, and it restricts the number of tensor multiplets to be *even*

$$
\Omega_{MN} = -\Omega_{NM}\,, \qquad \Omega_{MP}\Omega^{MR} = \delta_P{}^R\,. \qquad (3.8)
$$

JHEP10(2002)045

The covariant derivative of the tensor field is

$$\begin{aligned}\mathcal{D}_\lambda B^N_{\rho\sigma} &= \partial_\lambda B^N_{\rho\sigma} + g\,A^I_\lambda t_{I\tilde{J}}{}^N \mathcal{H}^{\tilde{J}}_{\rho\sigma} \\ &= \partial_\lambda B^N_{\rho\sigma} + g\,A^I_\lambda t_{IJ}{}^N F^J_{\rho\sigma} + g\,A^I_\lambda t_{IP}{}^N B^P_{\rho\sigma}\,. \end{aligned} \tag{3.9}$$

When this is reduced to 5 dimensions, one of the $\mathcal{H}$ factors of the first term of (3.7) should correspond to a vector field strength F^I in order that it can be written as a 5-form $A^I\mathcal{H}^{\tilde{J}}\mathcal{H}^{\tilde{K}}$. Thus, the components of C can have only three different forms, namely C_{IJK}, C_{IJM} and C_{IMN} (and permutations).

To see why (3.7) is a closed six-form, we write out the first term of (3.7)

$$C_{\tilde{I}\tilde{J}\tilde{K}}\mathcal{H}^{\tilde{I}}\mathcal{H}^{\tilde{J}}\mathcal{H}^{\tilde{K}} = C_{IJK}F^IF^JF^K + 3C_{IJM}F^IF^JB^M + 3C_{IMN}F^IB^MB^N\,. \tag{3.10}$$

Since the B^M tensors in (3.10) do not satisfy a Bianchi identity, we also need the second term in (3.7) to render it a closed six-form. This requirement of closure leads to the following relations between the C and Ω tensors:

$$C_{IJM} = t_{(IJ)}{}^N\Omega_{NM}\,, \qquad C_{IMN} = \frac{1}{2}t_{IM}{}^P\Omega_{PN}\,. \tag{3.11}$$

We stress that the tensor $C_{\tilde{I}\tilde{J}\tilde{K}}$ is not a fundamental object: the essential data for the vector-tensor multiplet are the representation matrices $t_{I\tilde{J}}{}^{\tilde{K}}$, the Yang-Mills components C_{IJK}, and the symplectic matrix Ω_{MN}. The tensor components of the C tensor are derived quantities, and we can summarize (3.11) as

$$C_{M\tilde{J}\tilde{K}} = t_{(\tilde{J}\tilde{K})}{}^P\Omega_{PM}\,. \tag{3.12}$$

From (3.11), we deduce that the second term of (3.10) only depends on the off-diagonal (between vector and tensor multiplets) transformations. The first term of (3.10) will induce the usual five-dimensional CS-term. The generalized CS-term induced by the third term of (3.10) was given in [1]. With our extension to also allow for the off-diagonal term in (2.39), we also get CS-terms induced by the C_{IJM} components, which were not present in [1].

Gauge invariance of the first term of (3.7) requires that the tensor C satisfies a modified version of (3.2)

$$f_{I(J}{}^HC_{KL)H} = t_{I(J}{}^Mt_{KL)}{}^N\Omega_{MN}\,. \tag{3.13}$$

In addition to this, the second term of (3.7) is only gauge invariant if the tensor Ω satisfies

$$t_{I[M}{}^P\Omega_{N]P} = 0\,, \tag{3.14}$$

such that the last one of (3.11) is consistent with the symmetry (MN). The two conditions (3.13) and (3.14) combined with the definition (3.12) imply the following generalization of (3.2)

$$t_{I(\tilde{J}}{}^{\widetilde{M}}C_{\tilde{K}\tilde{L})\widetilde{M}} = 0\,. \tag{3.15}$$

The superconformal action for the combined system of $m = 2k$ tensor multiplets and n vector multiplets contains the CS-term induced by (3.7) and the generalization of the vector action (3.3) to the extended range of indices. Some extra terms are necessary to complete it to an invariant action: we need mass terms and/or Yukawa coupling for the fermions at $\mathcal{O}(g)$, and a scalar potential at $\mathcal{O}(g^2)$. We thus find the following action:

$$\begin{aligned}
\mathcal{L}_{\text{vec}-\text{tensor}} = & \left(-\frac{1}{4}\mathcal{H}^{\widetilde{I}}_{\mu\nu}\mathcal{H}^{\mu\nu\widetilde{J}} - \frac{1}{2}\bar{\psi}^{\widetilde{I}}\not{\mathcal{D}}\psi^{\widetilde{J}} - \frac{1}{2}\mathcal{D}_a\sigma^{\widetilde{I}}\mathcal{D}^a\sigma^{\widetilde{J}} + Y^{\widetilde{I}}_{ij}Y^{ij\widetilde{J}}\right)\sigma^{\widetilde{K}}C_{\widetilde{I}\widetilde{J}\widetilde{K}} + \\
& + \frac{1}{16g}\varepsilon^{\mu\nu\lambda\rho\sigma}\Omega_{MN}B^M_{\mu\nu}\left(\partial_\lambda B^N_{\rho\sigma} + 2gt_{IJ}{}^N A^I_\lambda F^J_{\rho\sigma} + gt_{IP}{}^N A^I_\lambda B^P_{\rho\sigma}\right) - \\
& - \frac{1}{24}\varepsilon^{\mu\nu\lambda\rho\sigma}C_{IJK}A^I_\mu\left(F^J_{\nu\lambda}F^K_{\rho\sigma} + f_{FG}{}^J A^F_\nu A^G_\lambda\left(-\frac{1}{2}gF^K_{\rho\sigma} + \frac{1}{10}g^2 f_{HL}{}^K A^H_\rho A^L_\sigma\right)\right) - \\
& - \frac{1}{8}\varepsilon^{\mu\nu\lambda\rho\sigma}\Omega_{MN}t_{IK}{}^M t_{FG}{}^N A^I_\mu A^F_\nu A^G_\lambda\left(-\frac{1}{2}gF^K_{\rho\sigma} + \frac{1}{10}g^2 f_{HL}{}^K A^H_\rho A^L_\sigma\right) + \\
& + \left(-\frac{1}{8}\,\mathrm{i}\,\bar{\psi}^{\widetilde{I}}\gamma\cdot\mathcal{H}^{\widetilde{J}}\psi^{\widetilde{K}} - \frac{1}{2}\,\mathrm{i}\,\bar{\psi}^{i\widetilde{I}}\psi^{j\widetilde{J}}Y^{\widetilde{K}}_{ij}\right)C_{\widetilde{I}\widetilde{J}\widetilde{K}} + \\
& + \frac{1}{4}\,\mathrm{i}\,g\bar{\psi}^{\widetilde{I}}\psi^{\widetilde{J}}\sigma^{\widetilde{K}}\sigma^{\widetilde{L}}\left(t_{[\widetilde{I}\widetilde{J}]}{}^{\widetilde{M}}C_{\widetilde{M}\widetilde{K}\widetilde{L}} - 4t_{(\widetilde{I}\widetilde{K})}{}^{\widetilde{M}}C_{\widetilde{M}\widetilde{J}\widetilde{L}}\right) - \\
& - \frac{1}{2}g^2\sigma^K C_{KMN}t_{I\widetilde{L}}{}^M\sigma^I\sigma^{\widetilde{L}}t_{J\widetilde{P}}{}^N\sigma^J\sigma^{\widetilde{P}}\,.
\end{aligned} \tag{3.16}$$

To check the supersymmetry of this action, one needs all the relations between the various tensors given above. Another useful identity implied by the previous definitions is

$$t_{(\widetilde{I}\widetilde{J})}{}^M C_{\widetilde{K}\widetilde{L}M} = -t_{(\widetilde{K}\widetilde{L})}{}^M C_{\widetilde{I}\widetilde{J}M}\,. \tag{3.17}$$

The terms in the action containing the fields of the tensor multiplets can also be obtained from the field equations (2.42). They are now related to the action as

$$\frac{\delta S_{\text{vec}-\text{tensor}}}{\delta\bar{\psi}^{iM}} = \mathrm{i}\,\varphi^N_i\Omega_{NM}\,, \tag{3.18}$$

and the remaining bosonic terms can be obtained from comparing with N^M in (2.45). One may then further check that also the field equations (2.37) and (2.38) follow from this action.

Note however that the equations of motion for the vector multiplet fields, obtained from this action, are similar to those given in (3.5), but with the contracted indices running over the extended range of vector and tensor components. Furthermore, the A^I_μ equation of motion gets corrected by a term proportional to the self-duality equation for $B^M_{\mu\nu}$:

$$\frac{\delta S_{\text{vec}-\text{tensor}}}{\delta A^I_a} = E^a_I + \frac{1}{12}g\varepsilon^{abcde}A^J_b E_{cde}{}^M t_{JI}{}^N\Omega_{MN}\,. \tag{3.19}$$

Finally, we remark that the action (3.16) is invariant under supersymmetry for the completely general form (2.39) of the representation matrices $(t_I)_{\widetilde{J}}{}^{\widetilde{K}}$.

We thus conclude that in order to write down a superconformal action for the vector-tensor multiplet, we need to introduce another geometrical object, namely a gauge-invariant anti-symmetric invertible tensor Ω_{MN}. This symplectic matrix will restrict the number of tensor multiplets to be even. We can still allow the transformations to have off-diagonal terms between vector and tensor multiplets, if we adapt (3.2) to (3.15). In this way, we have constructed more general matter couplings than were known so far. Terms of the form $A \wedge F \wedge B$ did not appear in previous papers. We see that such terms appear generically in our lagrangian by allowing for these off-diagonal gauge transformations for the tensor fields. In some cases these may disappear after field redefinitions.

3.3 The hypermultiplet

Until this point, the equations of motion we derived, found their origin in the fact that we had an open superconformal algebra. The non-closure functions Γ^A, together with their supersymmetric partners Δ^X yielded these equations of motion. We discovered a hypercomplex scalar manifold $\mathcal{M}$, where $\Gamma_{XY}{}^Z$ was interpreted as an affine connection. We also needed a $\mathrm{G}\,\ell(r,\mathbb{H})$-connection $\omega_{XA}{}^B$ on a vector bundle and discovered that the manifold also admitted a trivial SU(2)-vector bundle.

Now, we will introduce an action to derive the field equations of the hypermultiplet. An important point to note is that the necessary data for the scalar manifold we had in the previous section, are not sufficient any more. This is not specific to our setting, but is a general property of non-linear sigma models.

In such models, the kinetic term for the scalars is multiplied by a scalar-dependent symmetric tensor $g_{\alpha\beta}(\phi)$,

$$S = -\frac{1}{2}\int d^D x g_{\alpha\beta}(\phi)\partial_\mu\phi^\alpha\partial^\mu\phi^\beta\,, \tag{3.20}$$

in which α and β run over the dimensions of the scalar manifold. The tensor g is interpreted as the metric on the target space $\mathcal{M}$. As the field equations for the scalars should now be also covariant with respect to coordinate transformations on the target manifold, the connection on the tangent bundle $T\mathcal{M}$ should be the Levi-Civita connection. Only in that particular case, the field equations for the scalars will be covariant. In other words, in $\Box\phi^\alpha + \cdots = 0$ the Levi-Civita connection on $T\mathcal{M}$ will be used in the covariant box.

To conclude, we will need to introduce a metric on the scalar manifold, in order to be able to write down an action. This metric will also restrict the possible target spaces for the theory.

Observe that most steps in this section do not depend on the use of superconformal symmetry.[4] Only at the end of section 3.3.2, we make explicitly use of the this symmetry.

3.3.1 Without gauged isometries

To start with, we take the non-closure functions Γ^A to be proportional to the field equations for the fermions ζ^A. In other words, we ask

$$\frac{\delta S_{\text{hyper}}}{\delta\bar{\zeta}^A} = 2C_{AB}\Gamma^B\,. \tag{3.21}$$

[4]Of course, the form of the field equations does reflect the superconformal symmetry.

In general, the tensor C_{AB} could be a function of the scalars and bilinears of the fermions. If we try to construct an action with the above Ansatz, it turns out that the tensor has to be anti-symmetric in AB and

$$\frac{\delta C_{AB}}{\delta \zeta^C} = 0\,, \tag{3.22}$$

$$\mathfrak{D}_X C_{AB} = 0\,. \tag{3.23}$$

This means that the tensor does not depend on the fermions and is covariantly constant.[5]

This tensor C_{AB} will be used to raise and lower indices according to the NW–SE convention similar to ε_{ij}:

$$A_A = A^B C_{BA}\,, \qquad A^A = C^{AB} A_B\,, \tag{3.24}$$

where ε^{ij} and C^{AB} for consistency should satisfy

$$\varepsilon_{ik}\varepsilon^{jk} = \delta_i{}^j\,, \qquad C_{AC}C^{BC} = \delta_A{}^B\,. \tag{3.25}$$

We may choose C_{AB} to be constant. To prove this, we look at the integrability condition for (3.23)

$$[\mathfrak{D}_X, \mathfrak{D}_Y]\, C_{AB} = 0 = -2\mathcal{R}_{XY[A}{}^C C_{B]C}\,. \tag{3.26}$$

This implies that the anti-symmetric part of the connection $\omega_{XAB} \equiv \omega_{XA}{}^C C_{CB}$ is pure gauge, and can be chosen to be zero. If we do so, the covariant constancy condition for C_{AB} reduces to the equation that C_{AB} is just constant. For this choice, the connection ω_{XAB} is symmetric, so the structure group $\mathrm{G}\,\ell(r,\mathbb{H})$ breaks to $\mathrm{USp}(2r-2p,2p)$. The signature is the signature of d_{CB}, which is defined as $C_{AB} = \rho_A{}^C d_{CB}$ where $\rho_A{}^C$ was given in (2.58). However, we will allow C_{AB} also to be non-constant, but covariantly constant.

We now construct the metric on the scalar manifold as

$$g_{XY} = f_X^{iA} C_{AB} \varepsilon_{ij} f_Y^{jB}\,. \tag{3.27}$$

The above-mentioned requirement that the Levi-Civita connection should be used (as $\Gamma_{XY}{}^Z$) is satisfied due to (3.23). Indeed, this guarantees that the metric is covariantly constant, such that the affine connection is the Levi-Civita one. On the other hand we have seen already that for covariantly constant complex structures we have to use the Obata connection. Hence, the Levi-Civita and Obata connection should coincide, and this is obtained from demanding (3.23) using the Obata connection. This makes us conclude that we can only write down an action for a hyperkähler scalar manifold.

We can now write down the action for the rigid hypermultiplets. It has the following form:

$$S_{\rm hyper} = \int d^5x \left(-\frac{1}{2} g_{XY}\partial_a q^X \partial^a q^Y + \bar\zeta_A \not{\mathfrak{D}} \zeta^A - \frac{1}{4} W_{ABCD} \bar\zeta^A \zeta^B \bar\zeta^C \zeta^D \right), \tag{3.28}$$

[5]This can also easily be seen by using the Batalin-Vilkovisky formalism.

where the tensor W_{ABCD} can be proven to be completely symmetric in all of its indices (see appendix B). The field equations derived from this action are

$$\begin{aligned} \frac{\delta S_{\text{hyper}}}{\delta \bar{\zeta}^A} &= 2C_{AB}\Gamma^B \,, \\ \frac{\delta S_{\text{hyper}}}{\delta q^X} &= g_{XY}\Delta^Y - 2\bar{\zeta}_A \Gamma^B \omega_{XB}{}^A \,. \end{aligned} \tag{3.29}$$

Also remark that due to the introduction of the metric, the expression of Δ^X simplifies to

$$\Delta^X = \Box q^X - \bar{\zeta}^A \not{\partial} q^Y \zeta^B \mathcal{R}^X{}_{YAB} - \frac{1}{4}\mathfrak{D}^X W_{ABCD}\bar{\zeta}^A\zeta^B\zeta^C\zeta^D \,. \tag{3.30}$$

Let us mention that we could also have followed a slightly different route. We could have introduced the metric g_{XY} first, and shown that the connection $\Gamma_{XY}{}^Z$ is the Levi-Civita connection with respect to this metric, as pointed out in the introduction of this section. Then, the identification of the vielbeins f^X_{iA} of the tangent bundle $T\mathcal{M}$ with the $\mathrm{G}\,\ell(r,\mathbb{H}) \otimes \mathrm{SU}(2)$ vector bundle would enable us to find a standard antisymmetric tensor $C \otimes \varepsilon$ on the latter bundle. As the metric is covariantly constant, this should be inherited by $C\otimes\varepsilon$, reflecting the possibility to choose it to be constant. The result of the introduction of a metric is that the scalar manifold should be hyperkähler.

Conformal invariance. Due to the presence of the metric, the condition for the homothetic Killing vector (2.87) implies that k_X is the derivative of a scalar function as in (2.10). This scalar function $\chi(q)$ is called the hyperkähler potential [71, 63, 30]. It determines the conformal structure, but should be restricted to

$$\mathfrak{D}_X\mathfrak{D}_Y\chi = \frac{3}{2}g_{XY} \,. \tag{3.31}$$

The relation with the homothetic Killing vector is

$$k_X = \partial_X\chi \,, \qquad \chi = \frac{1}{3}k_X k^X \,. \tag{3.32}$$

Note that this implies that, when χ and the complex structures are known, one can compute the metric with (3.31), using the formula for the Obata connection (2.74).

3.3.2 With gauged isometries

With a metric, the symmetries of section 2.3.3 should be isometries, i.e.

$$\mathfrak{D}_X k_{YI} + \mathfrak{D}_Y k_{XI} = 0 \,. \tag{3.33}$$

This makes the requirement (2.103) superfluous, but we still have to impose the triholomorphicity expressed by either (2.99) or (2.100) or (2.101).

In order to integrate the equations of motion to an action we have to define (locally) triples of 'moment maps', according to

$$\partial_X P^\alpha_I = -\frac{1}{2}J^\alpha{}_{XY}k^Y_I \,. \tag{3.34}$$

The integrability condition that makes this possible is the triholomorphic condition.

In the kinetic terms of the action, the derivatives should now be covariantized with respect to the new transformations. We are also forced to include some new terms proportional to g and g^2

$$S^g_{\text{hyper}} = \int d^5x \Big(-\frac{1}{2} g_{XY} \mathfrak{D}_a q^X \mathfrak{D}^a q^Y + \bar{\zeta}_A \not{\mathfrak{D}} \zeta^A - \frac{1}{4} W_{ABCD} \bar{\zeta}^A \zeta^B \bar{\zeta}^C \zeta^D - \\ - g \left(2\,\mathrm{i}\, k_I^X f^A_{iX} \bar{\zeta}_A \psi^{iI} + \mathrm{i}\, \sigma^I t_{IB}{}^A \bar{\zeta}_A \zeta^B - 2 P_{Iij} Y^{Iij} \right) - g^2 \frac{1}{2} \sigma^I \sigma^J k_I^X k_{JX} \Big)\,, \tag{3.35}$$

[where the covariant derivatives $\mathfrak{D}$ now also include gauge-covariantization proportional to g as in (2.106)], while the field equations have the same form as in (3.29). Supersymmetry of the action imposes

$$k_I^X J^\alpha{}_{XY} k_J^Y = 2 f_{IJ}{}^K P^\alpha_K\,. \tag{3.36}$$

As only the derivative of P appears in the defining equation (3.34), one may add an arbitrary constant to P. But that changes the right-hand side of (3.36). One should then consider whether there is a choice of these coefficients such that (3.36) is satisfied. This is the question about the center of the algebra, which is discussed in [72, 73]. For simple groups there is always a solution.[6] For abelian theories the constant remains undetermined. This free constant is the so-called Fayet–Iliopoulos term.

In a conformal invariant theory, the Fayet–Iliopoulos term is not possible. Indeed, dilatation invariance of the action needs

$$3 P^\alpha_I = k^X \partial_X P^\alpha_I\,. \tag{3.37}$$

Thus, P^α_I is completely determined [using (3.34) or (2.105)] as (see also [74])

$$-6 P^\alpha_I = k^X J^\alpha{}_{XY} k_I^Y = -\frac{2}{3} k^X k^Z J^\alpha{}_Z{}^Y \mathfrak{D}_Y k_{IX}\,. \tag{3.38}$$

The proof of the invariance of the action under the complete superconformal group, uses the equation obtained from (2.105) and (3.34):

$$k^{X\alpha} \mathfrak{D}_X k_I^Y = \partial^Y P^\alpha_I\,. \tag{3.39}$$

If the moment map P^α_I has the value that it takes in the conformal theory, then (3.36) is satisfied due to (2.95). Indeed, one can multiply that equation with $k_X k^Z J^\alpha{}_Z{}^W \mathfrak{D}_W$ and use (2.100), (2.103) and (2.89). Thus, in the superconformal theory, the moment maps are determined and there is no further relation to be obeyed, i.e. the Fayet–Iliopoulos terms of the rigid theories are absent in this case.

To conclude, isometries of the scalar manifold that commute with dilatations, see (2.105), can be gauged. The resulting theory has an extra symmetry group G, its algebra is generated by the corresponding Killing vectors.

[6] We thank Gary Gibbons for a discussion on this subject.

3.4 Potential

We complete this section with a discussion of the scalar potential for the general matter-coupled (rigid) superconformal theory. Gathering together our results (3.16) and (3.35) the total lagrangian describing the most general couplings of vector/tensor multiplets to hypermultiplets with rigid superconformal symmetry is

$$\mathcal{L}_{\text{total}} = \mathcal{L}_{\text{vec-tensor}} + \mathcal{L}^g_{\text{hyper}} \,. \tag{3.40}$$

From this expression the explicit form of the total scalar potential can be read off as

$$V(\sigma^{\widetilde{I}}, q^X) = \sigma^{\widetilde{K}} C_{\widetilde{I}\widetilde{J}\widetilde{K}} Y^{\widetilde{I}}_{ij} Y^{ij\,\widetilde{J}} + \frac{1}{2} g^2 \sigma^K C_{KMN} t_{I\widetilde{L}}{}^M \sigma^I \sigma^{\widetilde{L}} t_{J\widetilde{P}}{}^N \sigma^J \sigma^{\widetilde{P}} + \frac{1}{2} g^2 \sigma^I \sigma^J k_I^X k_{JX} \,, \tag{3.41}$$

where

$$Y^{ij\widetilde{J}} C_{I\widetilde{J}\widetilde{K}} \sigma^{\widetilde{K}} = -g P_I^{ij} \,, \qquad Y^{ij\widetilde{J}} C_{M\widetilde{J}\widetilde{K}} \sigma^{\widetilde{K}} = 0 \,. \tag{3.42}$$

Note that the auxiliary field Y has been eliminated here. Secondly, written as in (3.41), the potential does not contain the auxiliary field $Y^{\widetilde{I}}$ any more, but rather its solution of the field equations. This explains the apparent wrong sign in the $Y^I_{ij} Y^{ij\,J}$ term, and the field equation made use of the term $2gP_{Iij}Y^{Iij}$ in (3.35). In fact, the first term of (3.41) is equal to $-gP_{Iij}Y^{Iij}$.

This potential reflects the general form in supersymmetry that it is the square of the transformations of the fermions, where the definition of 'square' uses the fermion kinetic terms. The first term is the square of the transformations of the gauginos, the second term depends on the transformations of the fermionic partners of the antisymmetric tensors, and the last one is the square of the transformation law of the hyperinos. Note that off-diagonal terms between the contributions of Y^{ij} and the $t_{I\widetilde{L}}{}^M \sigma^I \sigma^{\widetilde{L}}$ terms do not survive as these would be proportional to $\varepsilon^{ij} Y_{ij} = 0$.

The difference between our potential (3.41) and the one in a rigid limit of [2], is the generalization to off-diagonal couplings of vectors and tensor multiplets in the first two terms.

Summarizing, in this section the actions of rigid superconformal vector/tensor-hyper-multiplet couplings have been constructed. The full answer is (3.40). We found that the existence of an action requires the presence of additional tensorial objects. Table 2 gives an overview of what are the independent objects to know, either to determine the transformation laws, or to determine the action.

In the next section we generalize our results to the local case.

4. Local superconformal multiplets

We are now ready to perform the last step in our programme, i.e. extend the supersymmetry to a local conformal supersymmetry. We will make use here of the off-shell $32+32$ Weyl multiplet constructed in [32, 33], and in particular of the 'standard' Weyl multiplet. In fact, there exist two Weyl multiplets: the 'dilaton' Weyl multiplet and the 'standard' Weyl multiplet. They contain the same gauge fields but differ in their matter fields. We restrict

	ALGEBRA (no action)		ACTION	
multiplets	objects	Def/restriction	objects	Def/restriction
Vect.	$f_{[IJ]}{}^K$	Jacobi identities	$C_{(IJK)}$	$f_{I(J}{}^H C_{KL)H} = 0$▲
Vect./Tensor	$(t_I)_{\widetilde{J}}{}^{\widetilde{K}}$ $\widetilde{I}=(I,M)$	$[t_I, t_J] = -f_{IJ}{}^K t_K$ $t_{IJ}{}^K = f_{IJ}{}^K\,, \quad t_{IM}{}^J = 0$	$\Omega_{[MN]}$	invertible $f_{I(J}{}^H C_{KL)H}=t_{I(J}{}^M t_{KL)}{}^N \Omega_{MN}$ $t_{I[M}{}^P \Omega_{N]P}=0$
Hyper	$f_X{}^{iA}$	invertible and real using ρ Nijenhuis condition: $N_{XY}{}^Z = 0$	$C_{[AB]}$	$\mathfrak{D}_X C_{AB} = 0$
Hyper + gauging	k_I^X	$\mathfrak{D}_X \mathfrak{D}_Y k_I^Z = R_{XWY}{}^Z k_I^W$ ◀ $k^Y_{[I\|} \partial_Y k^X_{\|J]} = -\frac{1}{2} f_{IJ}{}^K k_K^X$ $\mathcal{L}_{k_I} J^\alpha = 0$ ◀	P_I^α▲	$\mathfrak{D}_X k_{YI} + \mathfrak{D}_Y k_{XI} = 0$ $\partial_X P_I^\alpha = -\frac{1}{2} J^\alpha{}_{XY} k_I^Y$▲ $k_I^X J^\alpha_{XY} k_J^Y = 2 f_{IJ}{}^K P_K^\alpha$▲
Hyper + conformal	k^X ◀	$\mathfrak{D}_Y k^X = \frac{3}{2} \delta_Y{}^X$ ◀	χ	$\mathfrak{D}_X \mathfrak{D}_Y \chi = \frac{3}{2} g_{XY}$
Hyper + conformal + gauged		$k^Y \mathfrak{D}_Y k_I^X = \frac{3}{2} k_I^X$		

Table 2: Various matter couplings with or without action. We indicate which are the geometrical objects that determine the theory and what are the independent constraints. The symmetries of the objects are already indicated when they appear first. In general, the equations should also be valid for the theories in the rows below (apart from the fact that 'hyper+gauging' and 'hyper+conformal' are independent, but both are used in the lowest row). However, the symbol ▲ indicates that these equations are not to be taken over below. E.g. the moment map P_I^α itself is completely determined in the conformal theory, and it should thus not any more be given as an independent quantity. For the rigid theory without conformal invariance, only constant pieces can be undetermined by the given equations, and are the Fayet–Iliopoulos terms. On the other hand, the equations indicated by ◀ have not to be taken over for the theories with an action, as they are then satisfied due to the Killing equation or are defined by χ.

ourselves here to the standard Weyl multiplet, due to two considerations. First, it turns out that with the standard Weyl multiplet we already find a local generalization for any rigid theory. Second, the experience in other similar situations has shown that two different sets of auxiliary fields for theories with the same rigid limit do not lead to physically different results. This has e.g. been investigated in full detail for the old minimal, new minimal and non-minimal set of auxiliary fields for $N = 1$, $D = 4$ in [75]. We therefore expect that the couplings to the dilaton Weyl multiplet are only those obtained from the replacement of the fields of the standard Weyl multiplet by their functions in terms of the dilaton Weyl multiplet given in [32, eq. (3.14)]. Whether the conformal gauge-fixing program will also be insensitive to the choice of Weyl multiplet, remains to be seen. For instance in [33], the connection between the dilaton Weyl multiplet and an inequivalent set of auxiliary fields for Poincaré supergravity [76] was discussed.

Field	#	Gauge	SU(2)	w
	Elementary gauge fields			
${e_\mu}^a$	9	P^a	1	-1
b_μ	0	D	1	0
$V_\mu^{(ij)}$	12	SU(2)	3	0
ψ_μ^i	24	Q_α^i	2	$-\frac{1}{2}$
	Dependent gauge fields			
${\omega_\mu}^{ab}$	–	$M^{[ab]}$	1	0
${f_\mu}^a$	–	K^a	1	1
ϕ_μ^i	–	S_α^i	2	$\frac{1}{2}$
	Matter fields			
$T_{[ab]}$	10		1	1
D	1		1	2
χ^i	8		2	$\frac{3}{2}$

Table 3: Fields of the standard Weyl multiplet. The symbol # indicates the off-shell degrees of freedom. The first block contains the (bosonic and fermionic) gauge fields of the superconformal algebra. The fields in the middle block are dependent gauge fields. In the lower block are the extra matter fields that appear in the standard Weyl multiplet.

We have listed all the gauge fields and matter fields of the standard Weyl multiplet in table 3. For the full details of the standard Weyl multiplet, we refer to [32].

The procedure for extending the rigid superconformal transformation rules for the various matter multiplets is to introduce covariant derivatives with respect to the superconformal symmetries. These derivatives contain the superconformal gauge fields which, in turn, will also transform to additional matter fields (this is explained in detail in [32]).

Since in the previous sections we have explained most of the subtleties concerning the possible geometrical structures, we can be brief here. We will obtain our results in two steps. First, we require that the local superconformal commutator algebra, as it is realized on the standard Weyl multiplet (see [32, eqs. (4.3)–(4.6)]) is also realized on the matter multiplets (with possible additional transformations under which the fields of the standard Weyl multiplet do not transform, and possibly field equations if the matter multiplet is on-shell). Note that this local superconformal algebra is a modification of the rigid superconformal algebra (2.21), (2.19) where all modifications involve the fields of the standard Weyl multiplet.

Now we will apply a standard Noether procedure to extend the rigid supersymmetric actions to a locally supersymmetric one. This will introduce the full complications of coupling the matter multiplets to conformal supergravity.

4.1 Vector-tensor multiplet

The local supersymmetry rules are given by

$$\begin{aligned}
\delta A_\mu^I &= \frac{1}{2}\bar\epsilon\gamma_\mu\psi^I - \frac{1}{2}\,\mathrm{i}\,\sigma^I\bar\epsilon\psi_\mu\,,\\
\delta B_{ab}^M &= -\bar\epsilon\gamma_{[a}D_{b]}\psi^M + \mathrm{i}\,g\bar\epsilon\gamma_{ab}t_{(\widetilde J\widetilde K)}{}^M\sigma^{\widetilde J}\psi^{\widetilde K} + \mathrm{i}\,\bar\eta\gamma_{ab}\psi^M\,,\\
\delta Y^{ij\widetilde I} &= -\frac{1}{2}\bar\epsilon^{(i}\not{D}\psi^{j)\widetilde I} + \frac{1}{2}\,\mathrm{i}\,\bar\epsilon^{(i}\gamma\cdot T\psi^{j)\widetilde I} - 4\,\mathrm{i}\,\sigma^{\widetilde I}\bar\epsilon^{(i}\chi^{j)} -\\
&\quad -\frac{1}{2}\,\mathrm{i}\,g\bar\epsilon^{(i}\left(t_{[\widetilde J\widetilde K]}{}^{\widetilde I} - 3t_{(\widetilde J\widetilde K)}{}^{\widetilde I}\right)\sigma^{\widetilde J}\psi^{j)\widetilde K} + \frac{1}{2}\,\mathrm{i}\,\bar\eta^{(i}\psi^{j)\widetilde I}\,,\\
\delta\psi^{i\widetilde I} &= -\frac{1}{4}\gamma\cdot\widehat{\mathcal H}^{\widetilde I}\epsilon^i - \frac{1}{2}\,\mathrm{i}\,\not{D}\sigma^{\widetilde I}\epsilon^i - Y^{ij\widetilde I}\epsilon_j + \sigma^{\widetilde I}\gamma\cdot T\epsilon^i + \frac{1}{2}gt_{(\widetilde J\widetilde K)}{}^{\widetilde I}\sigma^{\widetilde J}\sigma^{\widetilde K}\epsilon^i + \sigma^{\widetilde I}\eta^i\,,\\
\delta\sigma^{\widetilde I} &= \frac{1}{2}\,\mathrm{i}\,\bar\epsilon\psi^{\widetilde I}\,.
\end{aligned} \tag{4.1}$$

The covariant derivatives are given by

$$\begin{aligned}
D_\mu\sigma^{\widetilde I} &= \mathcal{D}_\mu\sigma^{\widetilde I} - \frac{1}{2}\,\mathrm{i}\,\bar\psi_\mu\psi^{\widetilde I}\,,\\
\mathcal{D}_\mu\sigma^{\widetilde I} &= (\partial_\mu - b_\mu)\sigma^{\widetilde I} + gt_{J\widetilde K}{}^{\widetilde I}A_\mu^J\sigma^{\widetilde K}\,,\\
D_\mu\psi^{i\widetilde I} &= \mathcal{D}_\mu\psi^{i\widetilde I} + \frac{1}{4}\gamma\cdot\widehat{\mathcal H}^{\widetilde I}\psi_\mu^i + \frac{1}{2}\,\mathrm{i}\,\not{D}\sigma^{\widetilde I}\psi_\mu^i + Y^{ij\widetilde I}\psi_{\mu j} - \sigma^{\widetilde I}\gamma\cdot T\psi_\mu^i -\\
&\quad -\frac{1}{2}gt_{(\widetilde J\widetilde K)}{}^{\widetilde I}\sigma^{\widetilde J}\sigma^{\widetilde K}\psi_\mu^i - \sigma^{\widetilde I}\phi_\mu^i\,,\\
\mathcal{D}_\mu\psi^{i\widetilde I} &= (\partial_\mu - \frac{3}{2}\,b_\mu + \frac{1}{4}\,\gamma_{ab}\omega_\mu{}^{ab})\psi^{i\widetilde I} - V_\mu^{ij}\psi_j^{\widetilde I} + gt_{J\widetilde K}{}^{\widetilde I}A_\mu^J\psi^{i\widetilde K}\,.
\end{aligned} \tag{4.2}$$

The covariant curvature $\widehat{\mathcal H}^{\widetilde I}_{\mu\nu}$ should be understood as having components $(\widehat F^I_{\mu\nu}, B_{\mu\nu})$ and

$$\widehat F^I_{\mu\nu} = 2\partial_{[\mu}A^I_{\nu]} + gf_{JK}{}^IA_\mu^JA_\nu^K - \bar\psi_{[\mu}\gamma_{\nu]}\psi^I + \frac{1}{2}\,\mathrm{i}\,\sigma^I\bar\psi_{[\mu}\psi_{\nu]}\,. \tag{4.3}$$

The locally superconformal constraints needed to close the algebra are given by the following extensions of (2.37) and (2.38) (which are non-zero only for $\widetilde I$ in the tensor multiplet range)

$$\begin{aligned}
L^{ijM} &\equiv t_{(\widetilde J\widetilde K)}{}^M\left(2\sigma^{\widetilde J}Y^{ij\widetilde K} - \frac{1}{2}\,\mathrm{i}\,\bar\psi^{i\widetilde J}\psi^{j\widetilde K}\right) = 0\,,\\
E^M_{\mu\nu\lambda} &\equiv \frac{3}{g}D_{[\mu}B_{\nu\lambda]}{}^M - \varepsilon_{\mu\nu\lambda\rho\sigma}t_{(\widetilde J\widetilde K)}{}^M\left(\sigma^{\widetilde J}\widehat{\mathcal H}^{\rho\sigma\widetilde K} - 8\sigma^{\widetilde J}\sigma^{\widetilde K}T^{\rho\sigma} + \frac{1}{4}\,\mathrm{i}\,\bar\psi^{\widetilde J}\gamma^{\rho\sigma}\psi^{\widetilde K}\right) -\\
&\quad -\frac{3}{2}\bar\psi^M\gamma_{[\mu}\widehat R_{\nu\lambda]}(Q)\\
&= 0\,.
\end{aligned} \tag{4.4}$$

Analogously to subsection 2.2.2, the full set of constraints could be obtained by varying these constraints under supersymmetry.

The action, invariant under local superconformal symmetry, can be obtained by replacing the rigid covariant derivatives in (3.16) by the local covariant derivatives (4.2) and adding extra terms proportional to gravitinos or matter fields of the Weyl multiplet, determined by supersymmetry:

$$
\begin{aligned}
e^{-1}\mathcal{L}^{\rm conf}_{\rm vec-ten}=\Bigg[&\left(-\frac{1}{4}\widehat{\mathcal{H}}^{\widetilde{I}}_{\mu\nu}\widehat{\mathcal{H}}^{\mu\nu\widetilde{J}}-\frac{1}{2}\bar{\psi}^{\widetilde{I}}\not{D}\psi^{\widetilde{J}}+\frac{1}{3}\sigma^{\widetilde{I}}\Box^c\sigma^{\widetilde{J}}+\frac{1}{6}D_a\sigma^{\widetilde{I}}D^a\sigma^{\widetilde{J}}+Y^{\widetilde{I}}_{ij}Y^{ij\widetilde{J}}\right)\sigma^{\widetilde{K}}-\\
&-\frac{4}{3}\sigma^{\widetilde{I}}\sigma^{\widetilde{J}}\sigma^{\widetilde{K}}\left(D+\frac{26}{3}T_{ab}T^{ab}\right)+4\sigma^{\widetilde{I}}\sigma^{\widetilde{J}}\widehat{\mathcal{H}}^{\widetilde{K}}_{ab}T^{ab}+\\
&+\left(-\frac{1}{8}\,\mathrm{i}\,\bar{\psi}^{\widetilde{I}}\gamma\cdot\widehat{\mathcal{H}}^{\widetilde{J}}\psi^{\widetilde{K}}-\frac{1}{2}\,\mathrm{i}\,\bar{\psi}^{i\widetilde{I}}\psi^{j\widetilde{J}}Y^{\widetilde{K}}_{ij}+\mathrm{i}\,\sigma^{\widetilde{I}}\bar{\psi}^{\widetilde{J}}\gamma\cdot T\psi^{\widetilde{K}}-8\,\mathrm{i}\,\sigma^{\widetilde{I}}\sigma^{\widetilde{J}}\bar{\psi}^{\widetilde{K}}\chi\right)+\\
&+\frac{1}{6}\sigma^{\widetilde{I}}\bar{\psi}_\mu\gamma^\mu\left(\mathrm{i}\,\sigma^{\widetilde{J}}\not{D}\psi^{\widetilde{K}}+\frac{1}{2}\,\mathrm{i}\,\not{D}\sigma^{\widetilde{J}}\psi^{\widetilde{K}}-\frac{1}{4}\gamma\cdot\widehat{\mathcal{H}}^{\widetilde{J}}\psi^{\widetilde{K}}+2\sigma^{\widetilde{J}}\gamma\cdot T\psi^{\widetilde{K}}-8\sigma^{\widetilde{J}}\sigma^{\widetilde{K}}\chi\right)-\\
&-\frac{1}{6}\bar{\psi}_a\gamma_b\psi^{\widetilde{I}}\left(\sigma^{\widetilde{J}}\widehat{\mathcal{H}}^{ab\widetilde{K}}-8\sigma^{\widetilde{J}}\sigma^{\widetilde{K}}T^{ab}\right)-\frac{1}{12}\sigma^{\widetilde{I}}\bar{\psi}_\lambda\gamma^{\mu\nu\lambda}\psi^{\widetilde{J}}\widehat{\mathcal{H}}^{\widetilde{K}}_{\mu\nu}+\\
&+\frac{1}{12}\,\mathrm{i}\,\sigma^{\widetilde{I}}\bar{\psi}_a\psi_b\left(\sigma^{\widetilde{J}}\widehat{\mathcal{H}}^{ab\widetilde{K}}-8\sigma^{\widetilde{J}}\sigma^{\widetilde{K}}T^{ab}\right)+\frac{1}{48}\,\mathrm{i}\,\sigma^{\widetilde{I}}\sigma^{\widetilde{J}}\bar{\psi}_\lambda\gamma^{\mu\nu\lambda\rho}\psi_\rho\widehat{\mathcal{H}}^{\widetilde{K}}_{\mu\nu}-\\
&-\frac{1}{2}\sigma^{\widetilde{I}}\bar{\psi}^i_\mu\gamma^\mu\psi^{j\widetilde{J}}Y^{\widetilde{K}}_{ij}+\frac{1}{6}\,\mathrm{i}\,\sigma^{\widetilde{I}}\sigma^{\widetilde{J}}\bar{\psi}^i_\mu\gamma^{\mu\nu}\psi^j_\nu Y^{\widetilde{K}}_{ij}-\frac{1}{24}\,\mathrm{i}\,\bar{\psi}_\mu\gamma_\nu\psi^{\widetilde{I}}\bar{\psi}^{\widetilde{J}}\gamma^{\mu\nu}\psi^{\widetilde{K}}+\\
&+\frac{1}{12}\,\mathrm{i}\,\bar{\psi}^i_\mu\gamma^\mu\psi^{j\widetilde{I}}\bar{\psi}^{\widetilde{J}}_i\psi^{\widetilde{K}}_j-\frac{1}{48}\sigma^{\widetilde{I}}\bar{\psi}_\mu\psi_\nu\bar{\psi}^{\widetilde{J}}\gamma^{\mu\nu}\psi^{\widetilde{K}}+\frac{1}{24}\sigma^{\widetilde{I}}\bar{\psi}^i_\mu\gamma^{\mu\nu}\psi^j_\nu\bar{\psi}^{\widetilde{J}}_i\psi^{\widetilde{K}}_j-\\
&-\frac{1}{12}\sigma^{\widetilde{I}}\bar{\psi}_\lambda\gamma^{\mu\nu\lambda}\psi^{\widetilde{J}}\bar{\psi}_\mu\gamma_\nu\psi^{\widetilde{K}}+\frac{1}{24}\,\mathrm{i}\,\sigma^{\widetilde{I}}\sigma^{\widetilde{J}}\bar{\psi}_\lambda\gamma^{\mu\nu\lambda}\psi^{\widetilde{K}}\bar{\psi}_\mu\psi_\nu\\
&+\frac{1}{48}\,\mathrm{i}\,\sigma^{\widetilde{I}}\sigma^{\widetilde{J}}\bar{\psi}_\lambda\gamma^{\mu\nu\lambda\rho}\psi_\rho\bar{\psi}_\mu\gamma_\nu\psi^{\widetilde{K}}+\frac{1}{96}\sigma^{\widetilde{I}}\sigma^{\widetilde{J}}\sigma^{\widetilde{K}}\bar{\psi}_\lambda\gamma^{\mu\nu\lambda\rho}\psi_\rho\bar{\psi}_\mu\psi_\nu\Bigg]C_{\widetilde{I}\widetilde{J}\widetilde{K}}+\\
&+\frac{1}{16g}e^{-1}\varepsilon^{\mu\nu\lambda\rho\sigma}\Omega_{MN}B^M_{\mu\nu}\left(\partial_\lambda B^N_{\rho\sigma}+2gt_{IJ}{}^NA^I_\lambda F^J_{\rho\sigma}+gt_{IP}{}^NA^I_\lambda B^P_{\rho\sigma}\right)-\\
&-\frac{1}{24}e^{-1}\varepsilon^{\mu\nu\lambda\rho\sigma}C_{IJK}A^I_\mu\left(F^J_{\nu\lambda}F^K_{\rho\sigma}+f_{FG}{}^JA^F_\nu A^G_\lambda\left(-\frac{1}{2}gF^K_{\rho\sigma}+\frac{1}{10}g^2f_{HL}{}^KA^H_\rho A^L_\sigma\right)\right)-\\
&-\frac{1}{8}e^{-1}\varepsilon^{\mu\nu\lambda\rho\sigma}\Omega_{MN}t_{IK}{}^Mt_{FG}{}^NA^I_\mu A^F_\nu A^G_\lambda\left(-\frac{1}{2}gF^K_{\rho\sigma}+\frac{1}{10}g^2f_{HL}{}^KA^H_\rho A^L_\sigma\right)+\\
&+\frac{1}{4}\,\mathrm{i}\,g\bar{\psi}^{\widetilde{I}}\psi^{\widetilde{J}}\sigma^{\widetilde{K}}\sigma^{\widetilde{L}}\left(t_{[\widetilde{I}\widetilde{J}]}{}^{\widetilde{M}}C_{\widetilde{M}\widetilde{K}\widetilde{L}}-4t_{(\widetilde{I}\widetilde{K})}{}^{\widetilde{M}}C_{\widetilde{M}\widetilde{J}\widetilde{L}}\right)+\\
&+\frac{1}{10}\,\mathrm{i}\,g\bar{\psi}_\mu\gamma^\mu\psi^{\widetilde{I}}\sigma^{\widetilde{J}}\sigma^{\widetilde{K}}\sigma^{\widetilde{L}}\left(\left[t_{[\widetilde{I}\widetilde{J}]}{}^{\widetilde{M}}-2t_{(\widetilde{I}\widetilde{J})}{}^{\widetilde{M}}\right]C_{\widetilde{M}\widetilde{K}\widetilde{L}}-\frac{1}{2}t_{(\widetilde{J}\widetilde{K})}{}^{\widetilde{M}}C_{\widetilde{M}\widetilde{I}\widetilde{L}}\right)-\\
&-\frac{1}{2}g^2\sigma^I\sigma^J\sigma^K\sigma^{\widetilde{M}}\sigma^{\widetilde{N}}t_{J\widetilde{M}}{}^Pt_{K\widetilde{N}}{}^QC_{IPQ}\,,
\end{aligned}
\tag{4.5}
$$

where the superconformal d'alembertian is defined as

$$
\begin{aligned}
\Box^c\sigma^{\widetilde{I}}&=D^aD_a\sigma^{\widetilde{I}}\\
&=\left(\partial^a-2b^a+\omega_b{}^{ba}\right)D_a\sigma^{\widetilde{I}}+gt_{J\widetilde{K}}{}^{\widetilde{I}}A^J_aD^a\sigma^{\widetilde{K}}-\frac{\mathrm{i}}{2}\bar{\psi}_\mu D^\mu\psi^{\widetilde{I}}-2\sigma^{\widetilde{I}}\bar{\psi}_\mu\gamma^\mu\chi+\\
&\quad+\frac{1}{2}\bar{\psi}_\mu\gamma^\mu\gamma\cdot T\psi^{\widetilde{I}}+\frac{1}{2}\bar{\phi}_\mu\gamma^\mu\psi^{\widetilde{I}}+2f_\mu{}^\mu\sigma^{\widetilde{I}}-\frac{1}{2}g\bar{\psi}_\mu\gamma^\mu t_{\widetilde{J}\widetilde{K}}{}^{\widetilde{I}}\psi^{\widetilde{J}}\sigma^{\widetilde{K}}\,.
\end{aligned}
\tag{4.6}
$$

4.2 Hypermultiplet

Imposing the local superconformal algebra we find the following supersymmetry rules:

$$\begin{aligned}\delta q^X &= -\,\mathrm{i}\,\bar\epsilon^i\zeta^A f^X_{iA}\,,\\ \widehat{\delta}\zeta^A &= \frac{1}{2}\,\mathrm{i}\,\not{D}q^X f^{iA}_X\epsilon_i - \frac{1}{3}\gamma\cdot T k^X f^A_{iX}\epsilon^i - \frac{1}{2}g\sigma^I k^X_I f^A_{iX}\epsilon^i + k^X f^A_{iX}\eta^i\,.\end{aligned} \tag{4.7}$$

The covariant derivatives are given by

$$\begin{aligned}D_\mu q^X &= \mathcal{D}_\mu q^X + \mathrm{i}\,\bar\psi^i_\mu\zeta^A f^X_{iA}\,,\\ \mathcal{D}_\mu q^X &= \partial_\mu q^X - b_\mu k^X - V^{jk}_\mu k^X_{jk} + gA^I_\mu k^X_I\,,\\ D_\mu\zeta^A &= \mathcal{D}_\mu\zeta^A - k^X f^A_{iX}\phi^i_\mu + \frac{1}{2}\,\mathrm{i}\,\not{D}q^X f^A_{iX}\psi^i_\mu + \frac{1}{3}\gamma\cdot T k^X f^A_{iX}\psi^i_\mu + g\frac{1}{2}\sigma^I k^X_I f^A_{iX}\psi^i_\mu\\ \mathcal{D}_\mu\zeta^A &= \partial_\mu\zeta^A + \partial_\mu q^X\omega_{XB}{}^A\zeta^B + \frac{1}{4}\omega_\mu{}^{bc}\gamma_{bc}\zeta^A - 2b_\mu\zeta^A + gA^I_\mu t_{IB}{}^A\zeta^B\,.\end{aligned} \tag{4.8}$$

Similar to section 2.3, requiring closure of the commutator algebra on these transformation rules yields the equation of motion for the fermions

$$\begin{aligned}\Gamma^A_{\text{conf}} = {}& \not{D}\zeta^A + \frac{1}{2}W_{CDB}{}^A\zeta^B\bar\zeta^D\zeta^C - \frac{8}{3}\,\mathrm{i}\,k^X f^A_{iX}\chi^i + 2\,\mathrm{i}\,\gamma\cdot T\zeta^A -\\ &- g\left(\mathrm{i}\,k^X_I f^A_{iX}\psi^{iI} + \mathrm{i}\,\sigma^I t_{IB}{}^A\zeta^B\right).\end{aligned} \tag{4.9}$$

The scalar equation of motion can be obtained from varying (4.9):

$$\widehat{\delta}_Q\Gamma^A = \frac{1}{2}\,\mathrm{i}\,f^{iA}_X\Delta^X\epsilon_i + \frac{1}{4}\gamma^\mu\Gamma^A\bar\epsilon\psi_\mu - \frac{1}{4}\gamma^\mu\gamma^\nu\Gamma^A\bar\epsilon\gamma_\nu\psi_\mu\,, \tag{4.10}$$

where

$$\begin{aligned}\Delta^X_{\text{conf}} = {}& \Box^c q^X - \bar\zeta^B\gamma^a\zeta^C D_a q^Y\mathcal{R}^X{}_{YBC} + \frac{8}{9}T^2k^X +\\ &+ \frac{4}{3}Dk^X + 8\,\mathrm{i}\,\bar\chi^i\zeta^A f^X_{iA} - \frac{1}{4}\mathcal{D}^X W_{ABCD}\bar\zeta^A\zeta^B\bar\zeta^C\zeta^D -\\ &- g\big(2\,\mathrm{i}\,\bar\psi^{iI}\zeta^B t_{IB}{}^A f^X_{iA} - k^Y_I J_Y{}^X{}_{ij}Y^{Iij}\big) +\\ &+ g^2\sigma^I\sigma^J\mathfrak{D}_Y k^X_I k^Y_J\,,\end{aligned} \tag{4.11}$$

and the superconformal d'alembertian is given by

$$\begin{aligned}\Box^c q^X \equiv{}& D_a D^a q^X\\ ={}& \partial_a D^a q^X - \frac{5}{2}b_a D^a q^X - \frac{1}{2}V^{jk}_a J_Y{}^X{}_{jk}D^a q^Y + \mathrm{i}\,\bar\psi^i_a D^a\zeta^A f^X_{iA} +\\ &+ 2f_a{}^a k^X - 2\bar\psi_a\gamma^a\chi k^X + 4\bar\psi^{(j}_a\gamma^a\chi^{k)}k^X_{jk} - \bar\psi^i_a\gamma^a\gamma\cdot T\zeta^A f^X_{iA} -\\ &- \bar\phi^i_a\gamma^a\zeta^A f^X_{iA} + \omega_a{}^{ab}D_b q^X - \frac{1}{2}g\bar\psi^a\gamma_a\psi^I k^X_I - D_a q^Y\partial_Y k^X_I A^{aI} +\\ &+ D_a q^Y D^a q^Z\Gamma^X_{YZ}\,.\end{aligned} \tag{4.12}$$

Note that so far we didn't require the presence of an action. Introducing a metric, the locally conformal supersymmetric action is given by

$$\begin{aligned}
e^{-1}\mathcal{L}^{\rm conf}_{\rm hyper} = & -\frac{1}{2}g_{XY}\mathcal{D}_a q^X\mathcal{D}^a q^Y + \bar{\zeta}_A \not{\mathcal{D}}\zeta^A + \frac{4}{9}Dk^2 + \frac{8}{27}T^2k^2 - \\
& - \frac{16}{3}\,\mathrm{i}\,\bar{\zeta}_A\chi^i k^X f_{iX}^A + 2\,\mathrm{i}\,\bar{\zeta}_A\gamma\cdot T\zeta^A - \frac{1}{4}W_{ABCD}\bar{\zeta}^A\zeta^B\bar{\zeta}^C\zeta^D - \\
& - \frac{2}{9}\bar{\psi}_a\gamma^a\chi k^2 + \frac{1}{3}\bar{\zeta}_A\gamma^a\gamma\cdot T\psi_a^i k^X f_{iX}^A + \frac{1}{2}\,\mathrm{i}\,\bar{\zeta}_A\gamma^a\gamma^b\psi_a^i\mathcal{D}_b q^X f_{iX}^A + \\
& + \frac{2}{3}f_a{}^a k^2 - \frac{1}{6}\,\mathrm{i}\,\bar{\psi}_a\gamma^{ab}\phi_b k^2 - \bar{\zeta}_A\gamma^a\phi_a^i k^X f_{iX}^A + \\
& + \frac{1}{12}\bar{\psi}_a^i\gamma^{abc}\psi_b^j\mathcal{D}_c q^Y J_Y{}^X{}_{ij}k_X - \frac{1}{9}\,\mathrm{i}\,\bar{\psi}^a\psi^b T_{ab}k^2 + \frac{1}{18}\,\mathrm{i}\,\bar{\psi}_a\gamma^{abcd}\psi_b T_{cd}k^2 - \\
& - g\bigg(\,\mathrm{i}\,\sigma^I t_{IB}{}^A\bar{\zeta}_A\zeta^B + 2\,\mathrm{i}\,k_I^X f_{iX}^A\bar{\zeta}_A\psi^{iI} + \frac{1}{2}\sigma^I k_I^X f_{iX}^A\bar{\zeta}_A\gamma^a\psi_a^i + \\
& \qquad + \bar{\psi}_a^i\gamma^a\psi^{jI}P_{Iij} - \frac{1}{2}\,\mathrm{i}\,\bar{\psi}_a^i\gamma^{ab}\psi_b^j\sigma^I P_{Iij}\bigg) + \\
& + 2gY_I^{ij}P_{ij}^I - \frac{1}{2}g^2\sigma^I\sigma^J k_I^X k_{JX}\,.
\end{aligned} \tag{4.13}$$

No further constraints, other than those given in section 2.3 were necessary in this local case. In particular, the target space is still hypercomplex or, when an action exists, hyperkähler. This action leads to the following dynamical equations

$$\begin{aligned}
\frac{\delta\mathcal{S}^{\rm conf}_{\rm hyper}}{\delta\bar{\zeta}^A} &= 2\,C_{AB}\Gamma^B_{\rm conf}\,, \\
\frac{\delta\mathcal{S}^{\rm conf}_{\rm hyper}}{\delta q^X} &= g_{XY}\left(\Delta^Y_{\rm conf} - 2\bar{\zeta}_A\Gamma^B_{\rm conf}\omega^Y{}_B{}^A - \mathrm{i}\,\bar{\psi}_a^i\gamma^a\Gamma^A_{\rm conf}f^Y_{iA}\right).
\end{aligned} \tag{4.14}$$

The lagrangians (4.5) and (4.13) are the starting point for obtaining matter couplings to Poincaré supergravity. This involves a gauge fixing of the local scale and SU(2) symmetries, which will be studied in a forthcoming paper.

5. Conclusions and discussion

In this paper, we have analysed various multiplets in five spacetime dimensions with $N = 2$ supersymmetry in a superconformal context. Although we have so far only considered rigid supersymmetry and superconformal (both rigid and local) supersymmetry, we have found new couplings. The main emphasis was on the vector-tensor multiplet and on the hypermultiplet. Both these multiplets are on-shell and from the closure of the supersymmetry algebra, one can read off the equations of motion that determine the dynamics of the system. These equations of motion do not necessarily follow from an action. The existence of an action requires extra tensors which are needed to integrate the equations of motion into an action. In this way we have generalized the work of [33] where off-shell hypermultiplets were considered, leading e.g. to a restricted class of quaternionic-Kähler manifolds.

For vector-tensor multiplets, we have written down equations of motion with an *odd* number of tensor multiplets in the background of an arbitrary number of vector multiplets.

This is in contrast with formulations based on an action, where an even number of tensor multiplets is always needed. Even in the case when an action exists, we have found new couplings where vectors and tensors mix non-trivially due to the off-diagonal structure of the representation matrices for the gauge group. This introduces new terms in the scalar potential, such that we have a broader class of models than in the existing literature so far. We hope that these new potentials lead to interesting new physical applications.

For hypermultiplets, it has been known that the geometry of the scalars is hyperkähler for rigid supersymmetry [77] or quaternionic-Kähler for supergravity [78]. This was based on an analysis of the requirements imposed by the existence of an invariant action, and has been fully proved in [79]. We have written down equations of motion without the need of a target space metric (and thus a supersymmetric action), but which only involve a vielbein and a triplet of integrable complex structures. The resulting geometry is that of a *hypercomplex* manifold, which is a weakened version of hyperkähler geometry where the Ricci tensor is antisymmetric and not necessarily zero.

Since the appearance of hypercomplex geometry is somehow new in the physics literature, we have discussed their properties in appendix B. Group manifolds, e.g. SU(3), provide examples of hypercomplex geometries that are not hyperkähler, and we have computed the non-vanishing components of the Ricci tensor for hypercomplex group manifolds in appendix C. The main condition for a hypermultiplet action to exist, is the presence of a target space metric. In that case, the target space becomes hyperkähler. Our results then coincide with the literature.

The results of our analysis, both with and without actions, are summarized in table 2, where we indicate the various geometrical tensors and the restrictions they are subject to. The resulting scalar potential is displayed in section 3.4. After the analysis for rigid conformal supersymmetry, we have extended our results to local conformal supersymmetry. However, it turns out that the extra constraints that are necessary for allowing rigid conformal symmetry are also sufficient for the extension to local conformal supersymmetry. For this formulation, we have used the previously obtained results on the Weyl multiplet in five dimensions [32, 33].

Note that in constructing these superconformal theories, we have allowed kinetic terms for the scalars with arbitrary signature. This will be important for the conformal gauge-fixing programme, where the compensating scalars should have negative kinetic terms in order that the full theory has positive kinetic energy. The couplings of superconformal matter to the Weyl multiplet are gauge equivalent to matter-coupled Poincaré supergravities. This involves a partial gauge fixing, which we will investigate in a forthcoming paper, and which has been considered for some cases in [33, 80]. This should lead to actions that can be compared with those in [1, 2].

However, not all our results can fall in the theories of the present literature. We mentioned already above the extension to off-diagonal vector-tensor couplings. The other extension is due to not requiring the existence of an action.

From a string theory viewpoint, this is quite a natural thing to do. In fact, string theory does not lead to an action, but it leads to field equations, which in most cases can be integrated to an action. We should point out that there are also other techniques for constructing matter couplings that do not lead to an action. In many cases, the presence of self-dual antisymmetric tensor fields makes the construction of actions non-trivial. The gaugings of $N = 8$ supergravity in 5 dimensions require in some cases an odd number of antisymmetric tensors, which prohibits the construction of an action [60]. Its reduction to $N = 2$ theories should be in the class of the theories of this paper that are not based on an action.

This interesting aspect of our paper is not confined to five spacetime dimensions. A similar analysis can be done in other dimensions as well. The results were obtained by emphasizing the distinction between requirements from the algebra and requirements from action invariance, which is especially interesting for multiplets with an 'open' algebra, where equations of motions are generated from the anticommutator of two supersymmetries. e.g. the hypercomplex manifolds can be obtained in the same way for $D = 4$ and $D = 6$ theories with 8 supersymmetries.

We conclude by remarking that it is likely that our newly found matter-couplings will survive after gauge-fixing the local superconformal symmetry to $N = 2$ Poincaré supergravity. It will be of interest to see the consequences of our results for studying domain walls, renormalization group flows in the context of the AdS/CFT correspondence, and for finding a supersymmetric Randall-Sundrum scenario.

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A. The linear multiplet

The significance of the linear multiplet appears when we introduce an action for the vector multiplet, see (3.3) in section 3. This action contains a constant totally symmetric tensor C_{IJK}. In section 3 we saw that this tensor characterizes a special geometry. The linear multiplet is related to this vector multiplet action in the sense that the equations of motion (3.5) that follow from the action (3.3) transform precisely as a linear multiplet in the adjoint representation.

The degrees of freedom of the linear multiplet are given in table 1. We will consider a linear multiplet in the background of an off-shell (non-abelian) vector multiplet. We take the fields of the linear multiplet in an arbitrary representation of dimension m. The rigid

JHEP10(2002)045

conformal supersymmetry transformation rules for a linear multiplet in the background of a Yang-Mills multiplet are given by

$$\begin{aligned}
\delta L^{ijM} &= \mathrm{i}\,\bar\epsilon^{(i}\varphi^{j)M}\,,\\
\delta\varphi^{iM} &= -\frac{1}{2}\,\mathrm{i}\,\not{\mathcal{D}}L^{ijM}\epsilon_j - \frac{1}{2}\,\mathrm{i}\,\gamma^a E_a^M\epsilon^i + \frac{1}{2}N^M\epsilon^i + \frac{1}{2}g\sigma^I t_{IN}{}^M L^{ijN}\epsilon_j + 3L^{ijM}\eta_j\,,\\
\delta E_a^M &= -\frac{1}{2}\,\mathrm{i}\,\bar\epsilon\gamma_{ab}\mathcal{D}^b\varphi^M - \frac{1}{2}g\bar\epsilon\gamma_a t_{IN}{}^M\sigma^I\varphi^N + \frac{1}{2}g\bar\epsilon^{(i}t_{IN}{}^M\gamma_a\psi^{j)I}L_{ij}^N - 2\bar\eta\gamma_a\varphi^M\,,\\
\delta N^M &= \frac{1}{2}\bar\epsilon\not{\mathcal{D}}\varphi^M + \frac{1}{2}\,\mathrm{i}\,g\bar\epsilon^{(i}t_{IN}{}^M\psi^{j)I}L_{ij}^N + \frac{3}{2}\,\mathrm{i}\,\bar\eta\varphi^M\,.
\end{aligned} \tag{A.1}$$

The superconformal algebra closes provided the following constraint is satisfied

$$\mathcal{D}_a E^{aM} + g t_{IN}{}^M\left(Y^{ijI}L_{ij}^N + \mathrm{i}\,\bar\psi^I\varphi^N + \sigma^I N^N\right) = 0\,. \tag{A.2}$$

Note that the index I refers to the adjoint representation of the vector multiplet. To obtain the multiplet of equations of motion of the vector multiplet one should also take for M the adjoint representation in which case all t matrices become structure constants.

B. Hypercomplex manifolds

In this appendix we will present the essential properties of hypercomplex manifolds, and show the relation with hyperkähler and quaternionic (Kähler) manifolds. We show how properties of the Nijenhuis tensor determine whether suitable connections for these geometries can be defined. We give the curvature relations, and finally the properties of symmetry transformations of these manifolds.

Hypercomplex manifolds were introduced in [81]. A very thorough paper on the subject is [82]. Examples of homogeneous hypercomplex manifolds that are not hyperkähler, can be found in [83, 84], and are further discussed in section C. Non-compact homogeneous manifolds are dealt with in [85]. Various aspects have been treated in two workshops with mathematicians and physicists [86, 87]. To prepare this appendix, we used extensively [82], and some parts of this presentation use original methods.

B.1 The family of quaternionic-like manifolds

Let V be a real vector space of dimension $4r$, whose coordinates we indicate as q^X (with $X = 1,\ldots,4r$). We define a *hypercomplex structure* H on V to be a triple of complex structures J^α, (with $\alpha = 1,2,3$) which realize the algebra of quaternions,

$$J^\alpha J^\beta = -\delta^{\alpha\beta}\mathbb{1}_{4r} + \varepsilon^{\alpha\beta\gamma}J^\gamma\,. \tag{B.1}$$

A *quaternionic structure* is the space of linear combinations $a_\alpha J^\alpha$ with a_α real numbers. In this case the 3-dimensional space of complex structures is globally defined, but the individual complex structures do not have to be globally defined.

Let $\mathcal{M}$ be a $4r$ dimensional manifold. An *almost hypercomplex manifold* or *almost quaternionic manifold* is defined as a manifold $\mathcal{M}$ with a field of hypercomplex or quaternionic structures.

	no preserved metric	with a preserved metric
no SU(2) curvature	hypercomplex $G\ell(r,\mathbb{H})$	hyperkähler $USp(2r)$
non-zero SU(2) curvature	quaternionic $SU(2)\cdot G\ell(r,\mathbb{H})$	quaternionic-Kähler $SU(2)\cdot USp(2r)$

Table 4: Quaternionic-like manifolds. These are the manifolds that have a quaternionic structure satisfying (B.1) and (B.2). The holonomy group is indicated. For the right column the metric may give another real form as e.g. $USp(2,2(r-1))$.

The 'almost' disappears under one extra condition. Different terminologies are used to express this condition. Sometimes it is said that the structure should be 1-integrable. The same condition is also expressed as the statement that the structure should be covariantly constant using some connections, and it is also sometimes expressed as the 'preservation of the structure' using that connection. The connection[7] here should be a symmetric (i.e. 'torsionless') connection $\Gamma_{(XY)}{}^Z$ and possibly an SU(2) connection $\omega_X{}^\alpha$. The condition is

$$0=\mathfrak{D}_X J^\alpha{}_Y{}^Z \equiv \partial_X J^\alpha{}_Y{}^Z - \Gamma_{XY}{}^W J^\alpha{}_W{}^Z + \Gamma_{XW}{}^Z J^\alpha{}_Y{}^W + 2\varepsilon^{\alpha\beta\gamma}\omega_X{}^\beta J^\gamma{}_Y{}^Z\,. \tag{B.2}$$

If the SU(2) connection has non-vanishing curvature, the manifold is called *quaternionic.*[8] If the condition (B.2) holds with vanishing SU(2) connection, i.e.

$$0=\mathfrak{D}_X J^\alpha{}_Y{}^Z \equiv \partial_X J^\alpha{}_Y{}^Z - \Gamma_{XY}{}^W J^\alpha{}_W{}^Z + \Gamma_{XW}{}^Z J^\alpha{}_Y{}^W\,, \tag{B.3}$$

then the manifold is *hypercomplex.* If there is a hermitian metric, i.e. a metric such that

$$J^\alpha{}_X{}^Z g_{ZY} = -J^\alpha{}_Y{}^Z g_{ZX}\,, \tag{B.4}$$

and if this metric is preserved using the connection Γ (i.e. if Γ is the Levi-Civita connection of this metric) then the hypercomplex and quaternionic manifolds are respectively promoted to hyperkähler and quaternionic-Kähler manifolds. Hence this gives rise to the scheme[9] of table 4.

We will show in section B.4 that the spaces in the upper row have a Ricci tensor that is antisymmetric, and those in the right column have a Ricci tensor that is symmetric (and Einstein). It follows then that the hyperkähler manifolds are Ricci-flat. The restriction of holonomy group when one goes to the right column, just follows from the fact that the presence of a metric restricts the holonomy group further to a subgroup of $O(4r)$.[10]

[7]The word 'connection' is by mathematicians mostly used as the derivative including the 'connection coefficients'. We use here 'connection' as a word denoting these coefficients, i.e. gauge fields.

[8]For $r=1$ there are subtleties in the definition, to which we will return below.

[9]The table is essentially taken over from [82], where there is also the terminology unimodular hypercomplex or unimodular quaternionic if the $G\ell(r)$ is reduced to $S\ell(r)$.

[10]The dot notation means that it is the product up to a common factor in both groups that does not contribute. In fact, one considers e.g. SU(2) and $USp(2r)$ on coset elements as working one from the left, and the other from the right. Then if both are -1, they do not contribute. Thus: $SU(2)\cdot USp(2r) = \frac{SU(2)\times USp(2r)}{\mathbb{Z}_2}$.

JHEP10(2002)045

A theorem of Swann [71] shows that all quaternionic-Kähler manifolds have a corresponding hyperkähler manifold which admit a quaternionically extended homothety [a homothety extended to an SU(2) vector as in (2.85)] and which has three complex structures that rotate under an isometric SU(2) action. It has been shown in [30] that this can be implemented in superconformal tensor calculus to construct the actions of hypermultiplets in any quaternionic-Kähler manifold from a hyperkähler cone. Similarly, it has been proven in [88, 89] that any quaternionic manifold is related to a hypercomplex manifold.

Locally there is a vielbein f_X^{iA} (with $i = 1, 2$ and $A = 1, \ldots, r$) with reality conditions as in (2.61). In supersymmetry (and thus in this paper), we always start from these vielbeins and the integrability condition can be expressed as

$$\partial_X f_Y^{iA} - \Gamma_{XY}^Z f_Z^{iA} + f_Y^{jA} \omega_{Xj}{}^i + f_Y^{iB} \omega_{XB}{}^A = 0\,. \tag{B.5}$$

B.2 Conventions for curvatures and lemmas

We start with the notations for curvatures. The main conventions for target space curvature, fermion reparametrization curvature and SU(2) curvature are

$$\begin{aligned} R_{XYZ}{}^W &\equiv 2\partial_{[X}\Gamma_{Y]Z}{}^W + 2\Gamma_{V[X}{}^W\Gamma_{Y]Z}{}^V\,,\\ \mathcal{R}_{XYB}{}^A &\equiv 2\partial_{[X}\omega_{Y]B}{}^A + 2\omega_{[X|C|}{}^A\omega_{Y]B}{}^C\,,\\ \mathcal{R}_{XYi}{}^j &\equiv 2\partial_{[X}\omega_{Y]i}{}^j + 2\omega_{[X|k|}{}^j\omega_{Y]i}{}^k\,. \end{aligned} \tag{B.6}$$

The SU(2) curvature and connection $\omega_{Xi}{}^j$ are hermitian traceless,[11] and one can make the transition to triplet indices $\alpha = 1, 2, 3$ by using the sigma matrices

$$\begin{aligned} \mathcal{R}_{XYi}{}^j &= \mathrm{i}\,(\sigma^\alpha)_i{}^j\mathcal{R}_{XY}{}^\alpha\,,\\ \mathcal{R}_{XY}{}^\alpha &= -\frac{1}{2}\,\mathrm{i}\,(\sigma^\alpha)_i{}^j\mathcal{R}_{XYi}{}^j = 2\partial_{[X}\omega_{Y]}{}^\alpha + 2\varepsilon^{\alpha\beta\gamma}\omega_X{}^\beta\omega_Y{}^\gamma\,. \end{aligned} \tag{B.7}$$

This transition between doublet and triplet notation is valid for any triplet object as e.g. the complex structures. It is useful to know the translation of the inner product: $\mathcal{R}_i{}^j\mathcal{R}_j{}^i = -2\mathcal{R}^\alpha\mathcal{R}^\alpha$.

The curvatures by definition all satisfy the Bianchi identities that say that they are closed 2-forms, e.g.

$$\mathfrak{D}_{[X}R_{YZ]V}{}^W = 0\,. \tag{B.8}$$

Furthermore, due to the torsionless (symmetric) connection, also the cyclicity property holds.

$$R_{XYZ}{}^W + R_{ZXY}{}^W + R_{YZX}{}^W = 0\,. \tag{B.9}$$

The Ricci tensor is defined as

$$R_{XY} = R_{ZXY}{}^Z\,. \tag{B.10}$$

This is not necessarily symmetric. When Γ is the Levi-Civita connection of a metric, then one can raise and lower indices, $R_{WZXY} = R_{XYWZ}$ and the Ricci tensor is symmetric. Then one defines the scalar curvature as $R = g^{XY}R_{XY}$.

[11] This means symmetric if the indices are put at equal height using the raising or lowering tensor ε_{ij} (NW–SE convention).

We now present three lemmas that are useful in connecting scalar manifold indices with $G\ell(r,\mathbb{H})$ indices. These lemmas are used in section 2.3 and will simplify further derivations in this appendix.

Lemma 1. *If a matrix $M_X{}^Y$ satisfies*

$$[J^\alpha, M] = 2\varepsilon^{\alpha\beta\gamma} J^\beta m^\gamma\,, \tag{B.11}$$

for some numbers m^γ, then the latter are given by

$$4rm^\alpha = \mathrm{Tr}\,(J^\alpha M)\,, \tag{B.12}$$

and the matrix can be written as

$$M = -m^\alpha J^\alpha + N\,, \qquad [N, J^\alpha] = 0\,. \tag{B.13}$$

A matrix M of this type is said to 'normalize the hypercomplex structure'.

Proof. The first statement is proven by taking the trace of (B.11) with J^δ. Inserting this value of m^α in (B.13), it is obvious that the remainder N commutes with the complex structures. ■

Lemma 2. *If a matrix $M_X{}^Y$ commutes with the complex structures, then it can be written as*

$$M_X{}^Y = M_A{}^B f_X^{iA} f_{iB}^Y\,. \tag{B.14}$$

and vice-versa, any $M_A{}^B$ matrix can be transformed with (B.14) to a matrix commuting with the complex structures.

Proof. The vice-versa statement is easy. For the other direction, one replaces J^α with $J_i{}^j$ as in (2.71). Then multiply this equation with $f_{jA}^X f_Z^{kB}$ and consider the traceless part in AB. ■

Lemma 3. *If a tensor $R_{[XY]Z}{}^W$ satisfies the cyclicity condition (B.9) and commutes with the complex structures,*

$$R_{XYZ}{}^V J^\alpha{}_V{}^W - J^\alpha{}_Z{}^V R_{XYV}{}^W = 0\,, \tag{B.15}$$

it can be written in terms of a tensor $W_{ABC}{}^D$ that is symmetric in its lower indices. If $R_{XYZ}{}^Z = 0$, then also W is traceless.

Proof. By the previous theorem, we can write

$$R_{XYW}{}^Z = f_W^{iA} f_{iB}^Z \mathcal{R}_{XYA}{}^B\,, \qquad \mathcal{R}_{XYA}{}^B = \frac{1}{2} f_{iA}^W f_Z^{iB} R_{XYW}{}^Z\,. \tag{B.16}$$

We can change all indices to tangent indices, defining

$$R_{ij,CDB}{}^A \equiv f_{Ci}^X f_{jD}^Y \mathcal{R}_{XYB}{}^A = -R_{ji,DCB}{}^A\,. \tag{B.17}$$

The cyclicity property of R can be used to obtain

$$0 = f_Z^{iA} R_{[WXY]}{}^Z = f_{[Y}^{iB} \mathcal{R}_{WX]B}{}^A \,. \tag{B.18}$$

We multiply this with $f_{iC}^X f_{Dj}^Y f_{kE}^W$, leading to

$$R_{kj,ECD}{}^A + R_{kj,CDE}{}^A + 2R_{jk,DEC}{}^A = 0 \,. \tag{B.19}$$

The symmetric part in (jk) of this equation implies that $R_{(jk),ABC}{}^D = 0$ [multiply the equation by 3, and subtract both cyclicity rotated terms in (CDE)]. Thus we find

$$R_{ij,CDB}{}^A = -\frac{1}{2}\varepsilon_{ij} W_{CDB}{}^A \,, \tag{B.20}$$

with

$$W_{CDB}{}^A \equiv \varepsilon^{ij} f_{jC}^X f_{iD}^Y \mathcal{R}_{XYB}{}^A = \frac{1}{2}\varepsilon^{ij} f_{jC}^X f_{iD}^Y f_{kB}^Z f_W^{Ak} R_{XYZ}{}^W \,. \tag{B.21}$$

Now we prove that W is completely symmetric in the lower indices. The definition immediately implies symmetry in the first two. The $[jk]$ antisymmetric part of (B.19) gives

$$W_{ECD}{}^A + W_{DCE}{}^A - 2W_{EDC}{}^A = 0 \,. \tag{B.22}$$

Antisymmetrizing this in two of the indices gives the desired result.

Finally, it is obvious from (B.21) that the tracelessness of R and W are equivalent. ■

The full result for such a curvature tensor is thus

$$R_{XYW}{}^Z = -\frac{1}{2} f_X^{Ai} \varepsilon_{ij} f_Y^{jB} f_W^{kC} f_{kD}^Z W_{ABC}{}^D \,. \tag{B.23}$$

B.3 The connections

In the definition of hypercomplex and quaternionic manifolds, appear the affine connection $\Gamma_{XY}{}^Z$ and an SU(2) connection $\omega_X{}^\alpha$. In this subsection we will show how they can be obtained. The crucial ingredient is the Nijenhuis tensor.

Nijenhuis tensor. A Nijenhuis tensor $N_{XY}^{\alpha\beta\,Z}$ is defined for any combination of two complex structures, but we will use only the 'diagonal' Nijenhuis tensor (normalization for later convenience)

$$N_{XY}{}^Z \equiv \frac{1}{6} J^\alpha{}_X{}^W \partial_{[W} J^\alpha{}_{Y]}{}^Z - (X \leftrightarrow Y) = -N_{YX}{}^Z \,. \tag{B.24}$$

It satisfies a useful relation

$$N_{XY}{}^Z = J^\alpha{}_X{}^{X'} N_{X'Y}{}^{Z'} J^\alpha{}_{Z'}{}^Z \,, \tag{B.25}$$

from which one can deduce that it is traceless.

Obata connection and hypercomplex manifolds. The torsionless *Obata connection* [68] is defined as

$$\Gamma^{\mathrm{Ob}}{}_{XY}{}^{Z} = -\frac{1}{6}\left(2\partial_{(X}J^{\alpha}{}_{Y)}{}^{W} + \varepsilon^{\alpha\beta\gamma}J^{\beta}{}_{(X}{}^{U}\partial_{|U|}J^{\gamma}{}_{Y)}{}^{W}\right)J^{\alpha}{}_{W}{}^{Z}\,. \tag{B.26}$$

First, note that if a manifold is hypercomplex, i.e. if (B.3) is satisfied, then by inserting the expression for ∂J from that equation in the right hand side of (B.26), one finds that the affine connection of the hypercomplex manifold should be the Obata connection, $\Gamma = \Gamma^{\mathrm{Ob}}$. One may thus answer the question whether an almost hypercomplex manifold [i.e. with three matrices satisfying (B.1)], defines a hypercomplex manifold [i.e. satisfies (B.3)]. As we now know that the affine connection in (B.3) should be (B.26), this can just be checked. For that purpose, the following equation is useful:

$$\partial_X J^{\alpha}{}_{Y}{}^{Z} - \left(\Gamma^{\mathrm{Ob}}{}_{XY}{}^{W} + N_{XY}{}^{W}\right)J^{\alpha}{}_{W}{}^{Z} + \left(\Gamma^{\mathrm{Ob}}{}_{XW}{}^{Z} + N_{XW}{}^{Z}\right)J^{\alpha}{}_{Y}{}^{W} = 0\,. \tag{B.27}$$

It shows that any hypercomplex structure can be given a torsionful connection such that the complex structures are covariantly constant. The condition for a hypercomplex manifold is thus that this connection is torsionless, i.e. that the Nijenhuis tensor vanishes. In conclusion, *a hypercomplex manifold consists of the following data: a manifold $\mathcal{M}$, with a hypercomplex structure with vanishing Nijenhuis tensor.* In the main text, we only use the Obata connection, and we thus have $\Gamma = \Gamma^{\mathrm{Ob}}$.

Oproiu connection and quaternionic manifolds. For the quaternionic manifolds, the affine connection and SU(2) connection can not be uniquely defined. Indeed, one can easily check that (B.2) is left invariant when we change these two connections simultaneously using an arbitrary vector ξ_W as

$$\Gamma_{XY}{}^{Z} \to \Gamma_{XY}{}^{Z} + S^{WZ}_{XY}\xi_W\,, \qquad \omega_X{}^{\alpha} \to \omega_X{}^{\alpha} + J^{\alpha}{}_{X}{}^{W}\xi_W\,, \tag{B.28}$$

where S is the tensor

$$S^{XY}_{ZW} \equiv 2\delta^{X}_{(Z}\delta^{Y}_{W)} - 2J^{\alpha}{}_{Z}{}^{(X}J^{\alpha}{}_{W}{}^{Y)}\,, \tag{B.29}$$

which satisfies the relation

$$S^{XV}_{ZW}J^{\alpha}{}_{V}{}^{Y} - J^{\alpha}{}_{W}{}^{V}S^{XY}_{ZV} = 2\varepsilon^{\alpha\beta\gamma}J^{\beta}{}_{Z}{}^{X}J^{\gamma}{}_{W}{}^{Y}\,. \tag{B.30}$$

An invariant SU(2) connection is

$$\tilde{\omega}_X{}^{\alpha} = \omega_X{}^{\alpha} + \frac{1}{3}J^{\alpha}{}_{X}{}^{Y}J^{\beta}{}_{Y}{}^{Z}\omega_Z{}^{\beta} = \frac{2}{3}\omega_X{}^{\alpha} - \frac{1}{3}\varepsilon^{\alpha\beta\gamma}J^{\beta}{}_{X}{}^{Y}\omega_Y{}^{\gamma}\,. \tag{B.31}$$

If we use (B.2) in the expression for the Nijenhuis tensor, (B.24), we find that quaternionic manifolds do not have a vanishing Nijenhuis tensor, but the latter should satisfy

$$N_{XY}{}^{Z} = -J^{\alpha}{}_{[X}{}^{Z}\tilde{\omega}_{Y]}{}^{\alpha}\,. \tag{B.32}$$

This condition can be solved for $\tilde{\omega}$. We find

$$(1-2\,r)\,\tilde{\omega}_X{}^{\alpha} = N_{XY}{}^{Z}J^{\alpha}{}_{Z}{}^{Y}\,. \tag{B.33}$$

Thus the condition for an almost quaternionic manifold to be quaternionic is that the Nijenhuis tensor satisfies

$$(1-2\,r)\,N_{XY}{}^{Z} = -J^{\alpha}{}_{[X}{}^{Z} N_{Y]V}{}^{W} J^{\alpha}{}_{W}{}^{V}\,. \tag{B.34}$$

On the other hand, one may also use (B.2) in the expression for the Obata connection (B.26). Then we find that the affine connection for the quaternionic manifolds is given by

$$\Gamma_{XY}{}^{Z} = \Gamma^{\rm Ob}{}_{XY}{}^{Z} - J^{\alpha}{}_{(X}{}^{Z}\omega_{Y)}{}^{\alpha} - \frac{1}{3} S^{ZU}_{XY} J^{\alpha}{}_{U}{}^{V}\omega_{V}{}^{\alpha}\,, \tag{B.35}$$

which exhibits the transformation (B.28).

One can take a gauge choice for the invariance. A convenient choice is to impose

$$J^{\alpha}{}_{Y}{}^{Z}\omega_{Z}{}^{\alpha} = 0\,. \tag{B.36}$$

With this choice $\tilde\omega_X{}^{\alpha} = \omega_X{}^{\alpha}$. The affine connection in (B.35) simplifies, and this expression is called the Oproiu connection [90]

$$\begin{aligned}\Gamma^{\rm Op}{}_{XY}{}^{Z} &\equiv \Gamma^{\rm Ob}{}_{XY}{}^{Z} - J^{\alpha}{}_{(X}{}^{Z}\omega_{Y)}{}^{\alpha}\\ &= \Gamma^{\rm Ob}{}_{XY}{}^{Z} + N^{Z}_{XY} - J^{\alpha}{}_{Y}{}^{Z}\omega_{X}{}^{\alpha}\,.\end{aligned} \tag{B.37}$$

The last expression shows that the Oproiu connection, which up to here was only proven to be necessary for solving (B.2), gives indeed rise to covariantly constant complex structures under the condition (B.32). Indeed, the first two terms give already a (torsionful) connection that gives rise to a covariantly constant hypercomplex structure, see (B.27), and the last term cancels the SU(2) connection. The condition (B.32) is now the condition that the connection $\Gamma^{\rm Op}$ is torsionless.

In conclusion, *a quaternionic manifold consists of the following data: a manifold $\mathcal{M}$, with a hypercomplex structure with Nijenhuis tensor satisfying (B.34).*

Levi-Civita connection and hyperkähler or quaternionic-Kähler manifolds. For hyperkähler manifolds, the Obata connection should coincide with the Levi-Civita connection of a metric. For quaternionic-Kähler manifolds, the connection that preserves the metric can be one of the equivalence class defined from the Oproiu connection by a transformation (B.28).

Final note on connections. Note that for a given $\mathcal{M}$ and H, it is possible to find different connections which are all compatible with the hypercomplex structures. The resulting curvatures are then also different, which implies different (restricted) holonomy groups. An example on group manifolds, where we use a torsionful and a torsionless connection, will follow in section C. Other examples can be found in [91, 92, 93], which discuss 'HKT' manifolds, hypercomplex manifolds with torsion.

JHEP10(2002)045

B.4 Curvature relations

Splitting according to holonomy. There are two interesting possibilities of splitting the curvature on quaternionic-like manifolds. First of all, the integrability condition of (B.5) yields that the total curvature on the manifold is the sum of the SU(2) curvature and the $\mathrm{G}\,\ell(r,\mathbb{H})$ curvature which shows that the (restricted) holonomy splits in these two factors:

$$\begin{aligned} R_{XYW}{}^{Z} &= R^{\mathrm{SU}(2)}{}_{XYW}{}^{Z} + R^{\mathrm{G}\,\ell(r,\mathbb{H})}{}_{XYW}{}^{Z} \\ &= -J^{\alpha}{}_{W}{}^{Z}\,\mathcal{R}_{XY}{}^{\alpha} + L_{W}{}^{Z}{}_{A}{}^{B}\,\mathcal{R}_{XYB}{}^{A}\,, \qquad \text{with} \qquad L_{W}{}^{Z}{}_{A}{}^{B} \equiv f^{Z}_{iA} f^{iB}_{W}\,. \end{aligned} \tag{B.38}$$

The matrices $L_A{}^B$ and J^α commute and their mutual trace vanishes

$$J^{\alpha}{}_{X}{}^{Y} L_{Y}{}^{Z}{}_{A}{}^{B} = L_{X}{}^{Y}{}_{A}{}^{B} J^{\alpha}{}_{Y}{}^{Z}\,, \qquad J^{\alpha}{}_{Z}{}^{Y} L_{Y}{}^{Z}{}_{A}{}^{B} = 0\,. \tag{B.39}$$

For hypercomplex (or hyperkähler) manifolds, the SU(2) curvature vanishes. Then the Riemann tensor commutes with the complex structures and using the cyclicity, one may use lemmas 2 and 3 to write

$$R_{XYW}{}^{Z} = -\frac{1}{2} f^{Ai}_{X} \varepsilon_{ij} f^{jB}_{Y} f^{kC}_{W} f^{Z}_{kD} W_{ABC}{}^{D}\,. \tag{B.40}$$

This W is symmetric in its lower indices. The Ricci tensor is then

$$R_{XY} = \frac{1}{2}\varepsilon_{ij} f^{iB}_{X} f^{jC}_{Y} W_{ABC}{}^{A} = -R_{YX}\,. \tag{B.41}$$

Thus the Ricci tensor for hypercomplex manifolds is antisymmetric. In general, the antisymmetric part can be traced back to the curvature of the U(1) part in $\mathrm{G}\,\ell(r,\mathbb{H}) = \mathrm{S}\,\ell(r,\mathbb{H}) \times \mathrm{U}(1)$. Indeed, using the cyclicity condition:

$$R_{[XY]} = R_{Z[XY]}{}^{Z} = -\frac{1}{2} R_{XYZ}{}^{Z} = -\mathcal{R}^{\mathrm{U}(1)}_{XY}\,, \qquad \mathcal{R}^{\mathrm{U}(1)}_{XY} \equiv \mathcal{R}_{XYA}{}^{A}\,. \tag{B.42}$$

Splitting in Ricci and Weyl curvature. The separate terms in (B.38) for quaternionic manifolds do not satisfy the cyclicity condition, and thus are not bona-fide curvatures. We will now discuss another splitting

$$R = R^{\mathrm{Ric}}{}_{XYW}{}^{Z} + R^{(\mathrm{W})}{}_{XYW}{}^{Z}\,. \tag{B.43}$$

Both terms will separately satisfy the cyclicity condition. The first part only depends on the Ricci tensor of the full curvature, and is called the '*Ricci part*'. The Ricci tensor of the second part will be zero, and this part will be called the '*Weyl part*' [82]. We will prove that the second part commutes with the complex structures. The lemmas of section B.2 then imply that the second part can be written in terms of a tensor $\mathcal{W}_{ABC}{}^{D}$, symmetric in the lower indices and traceless. This tensor appears in supersymmetric theories, which is another reason for considering this construction. The case $r = 1$ needs a separate treatment which will be discussed afterwards.

To define the splitting (B.43), we define the first term as a function of the Ricci tensor, and $R^{(\mathrm{W})}$ is just defined as the remainder. The definition of R^{Ric} makes again use of the tensor S in (B.29):

$$
\begin{aligned}
R^{\mathrm{Ric}}{}_{XYZ}{}^{W} &\equiv 2S^{WV}_{Z[X}B_{Y]V}\,, \\
B_{XY} &\equiv \frac{1}{4r}R_{(XY)} - \frac{1}{2r(r+2)}\Pi_{(XY)}{}^{ZW}R_{ZW} + \frac{1}{4(r+1)}R_{[XY]}\,.
\end{aligned}
\tag{B.44}
$$

Here, Π projects bilinear forms onto hermitian ones, i.e.

$$
\Pi_{XY}{}^{ZW} \equiv \frac{1}{4}\left(\delta_X{}^Z\delta_Y{}^W + J^\alpha{}_X{}^Z J^\alpha{}_Y{}^W\right). \tag{B.45}
$$

The Ricci part satisfies several properties that can be checked by a straightforward calculation:

1. The Ricci tensor of R^{Ric} is just R_{XY}.
2. The cyclicity property (B.9).
3. Considered as a matrix in its last two indices, this matrix normalizes the hypercomplex structure (see lemma 1).

Especially to prove the last one, the property (B.30) can be used (multiplying it with B_{UX} and antisymmetrizing in $[ZU]$). The relation is explicitly

$$
\begin{aligned}
&J^\alpha{}_Z{}^W\, R^{\mathrm{Ric}}{}_{XYW}{}^V - R^{\mathrm{Ric}}{}_{XYZ}{}^W\, J^\alpha{}_W{}^V = 2\varepsilon^{\alpha\beta\gamma}J^\beta{}_Z{}^V\,\mathcal{R}^{\mathrm{Ric}}{}_{XY}{}^\gamma\,, \\
&\text{with} \qquad \mathcal{R}^{\mathrm{Ric}}{}_{XY}{}^\alpha = \frac{1}{4r}J^\alpha{}_W{}^Z R^{\mathrm{Ric}}{}_{XYZ}{}^W = 2J^\alpha{}_{[X}{}^Z B_{Y]Z}\,.
\end{aligned}
\tag{B.46}
$$

The important information is now that the full curvature also satisfies these 3 properties. The latter one is the integrability property of (B.2):

$$
0 = 2\mathfrak{D}_{[X}\mathfrak{D}_{Y]}J^\alpha{}_Z{}^V = R_{XYW}{}^V J^\alpha{}_Z{}^W - R_{XYZ}{}^W J^\alpha{}_W{}^V - 2\varepsilon^{\alpha\beta\gamma}\mathcal{R}_{XY}{}^\gamma J^\beta{}_Z{}^V\,. \tag{B.47}
$$

As in general for matrices normalizing the complex structure, we can also express $\mathcal{R}_{XY}{}^\alpha$ as

$$
R_{XYZ}{}^W J^\alpha{}_W{}^Z = 4r\,\mathcal{R}^\alpha_{XY}\,. \tag{B.48}
$$

This leads to properties of the Weyl part of the curvature. First of all, it implies that this part is Ricci-flat. Secondly it also satisfies the cyclicity property. Third, it also normalizes the hypercomplex structure, defining some $\mathcal{R}^{(\mathrm{W})\alpha}_{XY}$. We will now prove that the latter is zero for $r>1$.

The expression for this tensor satisfies a property that can be derived, starting from its definition, by first using the cyclicity of $R^{(\mathrm{W})}$, then the equation saying that it normalizes the hypercomplex structure, and finally that it is Ricci-flat

$$
\begin{aligned}
r\mathcal{R}^{(\mathrm{W})\alpha}_{XY} &= \frac{1}{4}J^\alpha{}_U{}^V R^{(\mathrm{W})}{}_{XYV}{}^U = -\frac{1}{2}J^\alpha{}_U{}^V R^{(\mathrm{W})}{}_{V[XY]}{}^U \\
&= -\epsilon^{\alpha\beta\gamma}\mathcal{R}^{(\mathrm{W})\beta}_{V[X}J^\gamma{}_{Y]}{}^V\,.
\end{aligned}
\tag{B.49}
$$

Multiplying with $J^{\alpha}{}_V{}^Y$ and antisymmetrizing leads to

$$J^{\alpha}{}_{[V}{}^Y \mathcal{R}^{(\mathrm{W})\alpha}_{X]Y} = 0\,. \tag{B.50}$$

Secondly, multiplying (B.49) with $J^{\delta}{}_Z{}^X J^{\delta}{}_W{}^Y$, and using (B.49) again for multiplying the complex structures at the right-hand side, leads to

$$J^{\beta}{}_X{}^Z J^{\beta}{}_Y{}^V \mathcal{R}^{(\mathrm{W})\alpha}_{ZV} = -\mathcal{R}^{(\mathrm{W})\alpha}_{XY} \qquad \text{or} \qquad \Pi_{XY}{}^{ZV}\mathcal{R}^{(\mathrm{W})\alpha}_{ZV} = 0\,. \tag{B.51}$$

Finally, multiplying (B.49) with $\varepsilon^{\alpha\delta\epsilon}J^{\delta}{}_Z{}^Y$ leads to

$$\mathcal{R}^{(\mathrm{W})\alpha}_{XY} = 0\,, \qquad \text{if} \quad r > 1\,. \tag{B.52}$$

Therefore $R^{(\mathrm{W})}{}_{XYZ}{}^V$ is a tensor that satisfies all conditions of lemma 3, and we can thus write

$$R_{XYZ}{}^W = R^{\mathrm{Ric}}{}_{XYZ}{}^W - \frac{1}{2} f_X^{Ai}\varepsilon_{ij} f_Y^{jB} f_W^{kC} f_{kD}^Z \mathcal{W}_{ABC}{}^D\,. \tag{B.53}$$

For hypercomplex manifolds, we found that the full curvature can be written in terms of a tensor $W_{ABC}{}^D$, see (B.40), which is symmetric in the lower indices, but not necessarily traceless. One can straightforwardly compute the corresponding $\mathcal{W}$, and find that this is its traceless part, the trace determining the Ricci tensor:

$$\mathcal{W}_{ABC}{}^D = W_{ABC}{}^D - \frac{3}{2(r+1)}\delta^D_{(A} W_{BC)E}{}^E\,, \qquad R_{XY} = -\mathcal{R}_{XYA}{}^A = \frac{1}{2}\varepsilon_{ij} f_X^{iA} f_Y^{jB} W_{ABC}{}^C\,. \tag{B.54}$$

The 1-dimensional case. As

$$\mathrm{G}\,\ell(1,\mathbb{H}) = \mathrm{S}\,\ell(1,\mathbb{H}) \times \mathrm{U}(1) = \mathrm{SU}(2)\times\mathrm{U}(1)\,, \tag{B.55}$$

we have now two SU(2) factors in the full holonomy group. This can be written explicitly by splitting L in (B.38) in a traceless and trace part:

$$L_X{}^Y{}_A{}^B = \frac{1}{2}\,\mathrm{i}\,(\sigma^{\alpha})_A{}^B J^{-\alpha}{}_X{}^Y + \frac{1}{2}\delta_X^Y\delta_A^B\,. \tag{B.56}$$

This leads to the $r=1$ form of (B.38):

$$R_{XYW}{}^Z = -J^{+\alpha}{}_W{}^Z\mathcal{R}^{+\alpha}_{XY} - J^{-\alpha}{}_W{}^Z\mathcal{R}^{-\alpha}_{XY} + \delta_W^Z\mathcal{R}^{\mathrm{U}(1)}_{XY}\,, \tag{B.57}$$

where for emphasizing the symmetry, we indicate the original complex structures as $J^{+\alpha}{}_X{}^Y$.

We saw that for $r=1$ we could not perform all steps to get to the decomposition (B.53). However, some authors define quaternionic and quaternionic-Kähler for $r=1$ as a more restricted class of manifolds such that this decomposition is still valid [94]. For quaternionic-Kähler manifolds, the definition that is taken in general leads for $r=1$ to the manifolds with holonomy $\mathrm{SU}(2)\times\mathrm{USp}(2)$, which is just SO(4). Thus with this definition all 4-dimensional riemannian manifolds would be quaternionic-Kähler. Therefore a further restriction is imposed. This further restriction is also natural in supergravity, as it is equivalent to a constraint that follows from requiring invariance of the supergravity action.

In general, as $\mathcal{R}^{(\mathrm{W})}$ normalizes the hypercomplex structure, we can by lemma 1 and lemma 2 write

$$R^{(\mathrm{W})}{}_{XYZ}{}^{W} = -\mathcal{R}^{(\mathrm{W})\alpha}_{XY} J^{\alpha}{}_{Z}{}^{W} + \mathcal{R}^{(\mathrm{W})}_{XYA}{}^{B} L_{Z}{}^{W}{}_{A}{}^{B} = R^{(\mathrm{W})+}{}_{XYZ}{}^{W} + R^{(\mathrm{W})-}{}_{XYZ}{}^{W} . \quad (\mathrm{B.58})$$

We impose

$$\mathcal{R}^{(\mathrm{W})\alpha}_{XY} = 0\,, \quad (\mathrm{B.59})$$

as part of the definition of quaternionic manifolds with $r = 1$. This is thus the equation that is automatically valid for $r > 1$. Using lemma 3, this implies that (B.53) is valid for all quaternionic manifolds.

In the 1-dimensional case, we can see that a possible metric is already fixed up to a multiplicative function. Indeed, the C_{AB} that is used in (3.27) can only be proportional to ε_{AB}. Therefore, it is said that there is a *conformal metric*, i.e. a metric determined up to a (local) scale function $\lambda(q)$:

$$g_{XY} \equiv \lambda(q) f^{iA}_{X} f^{jB}_{Y} \varepsilon_{ij} \varepsilon_{AB}\,. \quad (\mathrm{B.60})$$

One can check that this metric is hermitian for any $\lambda(q)$, i.e. $J^{\alpha}{}_{XY} = J^{\alpha}{}_{X}{}^{Z} g_{ZY}$ is anti-symmetric. The remaining question is whether this metric is covariantly constant, which boils down to the covariant constancy of C_{AB}. This condition can be simplified using the Schouten identity:

$$\mathfrak{D}_X C_{AB} = \partial_X C_{AB} + 2\omega_{X[A}{}^{C} C_{|C|B]} = \partial_X C_{AB} + \omega_{XC}{}^{C} C_{AB} = \varepsilon_{AB} \left(\partial_X \lambda(q) + \omega_{XC}{}^{C} \lambda(q) \right) . \quad (\mathrm{B.61})$$

We can choose a function $\lambda(q)$ such that C is covariantly constant iff $\omega_{XC}{}^{C}$ is a total derivative, i.e. if the U(1) curvature vanishes. Thus in the 1-dimensional case hypercomplex manifolds become hyperkähler, and quaternionic manifolds become quaternionic-Kähler if and only if the U(1) factor in the curvature part $G\ell(1,\mathbb{H})$ vanishes.

The curvature of Quaternionic-Kähler manifolds. In quaternionic-Kähler manifolds, the affine connection is the Levi-Civita connection of a metric. Therefore, the Ricci tensor is symmetric. As we have already proven that in the hypercomplex case the symmetric part vanishes, hyperkähler manifolds have vanishing Ricci tensor. Now we will prove that the quaternionic-Kähler spaces are Einstein, and that moreover the SU(2) curvatures are proportional to the complex structures with a proportionality factor that is dependent on the scalar curvature.

We start again from the integrability property (B.47). Multiplying with $J^{\delta}{}_{V}{}^{X}$ gives

$$\begin{aligned} &R_{YZ}\delta^{\alpha\delta} - \varepsilon^{\alpha\delta\beta} R_{XYZ}{}^{W} J^{\beta}{}_{W}{}^{X} + J^{\alpha}{}_{Z}{}^{W} R_{XYW}{}^{V} J^{\delta}{}_{V}{}^{X} - \\ &\qquad -2\varepsilon^{\alpha\beta\delta}\mathcal{R}^{\beta}_{ZY} + 2\delta^{\alpha\delta}\mathcal{R}^{\beta}_{XY} J^{\beta}{}_{Z}{}^{X} - 2\mathcal{R}^{\delta}_{XY} J^{\alpha}{}_{Z}{}^{X} = 0\,. \end{aligned} \quad (\mathrm{B.62})$$

The second and third term can be rewritten

$$\begin{aligned} R_{XYW}{}^{V} J^{\delta}{}_{V}{}^{X} &= -R_{YWX}{}^{V} J^{\delta}{}_{V}{}^{X} - R_{WXY}{}^{V} J^{\delta}{}_{V}{}^{X} \\ &= -R_{YWX}{}^{V} J^{\delta}{}_{V}{}^{X} + R_{YXW}{}^{V} J^{\delta}{}_{V}{}^{X}\,, \\ 2R_{XYW}{}^{V} J^{\delta}{}_{V}{}^{X} &= -4r\mathcal{R}^{\delta}_{YW}\,. \end{aligned} \quad (\mathrm{B.63})$$

In the first line, the cyclicity property of the Riemann tensor is used. Then, the symmetry in interchanging the first two and last two indices (here we use that the curvature originates from a Levi-Civita connection) and finally interchanging the indices on the last complex structure, using its antisymmetry (Hermiticity of the metric). This leads to

$$R_{YZ}\delta^{\alpha\delta} + \varepsilon^{\alpha\delta\beta}2(r-1)\mathcal{R}^{\beta}_{YZ} - 2(r-1)\mathcal{R}^{\delta}_{YX}J^{\alpha}{}_{Z}{}^{X} + 2\delta^{\alpha\delta}\mathcal{R}^{\beta}_{XY}J^{\beta}{}_{Z}{}^{X} = 0\,. \tag{B.64}$$

Multiplying with $\delta^{\alpha\delta}$ gives

$$R_{YZ} = -\frac{2}{3}(r+2)J^{\beta}{}_{Z}{}^{X}\mathcal{R}^{\beta}_{XY}\,. \tag{B.65}$$

On the other hand, multiplying (B.64) with $\varepsilon^{\alpha\delta\gamma}$ gives only a non-trivial result for $r \neq 1$, in which case we find

$$\text{for} \quad r > 1\ : \qquad 2\mathcal{R}^{\alpha}_{YZ} = \varepsilon^{\alpha\beta\gamma}J^{\beta}{}_{Y}{}^{X}\mathcal{R}^{\gamma}_{XZ}\,. \tag{B.66}$$

We impose the same equation for $r = 1$. We will connect this equation to another requirement below.

By replacing $\varepsilon^{\alpha\beta\gamma}J^{\beta}{}_{Y}{}^{X}$ by $-(J^{\alpha}J^{\gamma})_{Y}{}^{X} - \delta^{X}_{Y}\delta^{\alpha\gamma}$ we get

$$\mathcal{R}^{\alpha}_{XY} = -\frac{1}{3}J^{\alpha}{}_{X}{}^{Z}J^{\beta}{}_{Z}{}^{V}\mathcal{R}^{\beta}_{VY} = \frac{1}{2(r+2)}J^{\alpha}{}_{X}{}^{Z}R_{ZY}\,. \tag{B.67}$$

We also have

$$J^{\alpha}{}_{X}{}^{Z}\mathcal{R}_{ZY}{}^{\beta} = \varepsilon^{\alpha\beta\gamma}\mathcal{R}_{XY}{}^{\gamma} - \frac{1}{2(r+2)}\delta^{\alpha\beta}R_{XY}\,. \tag{B.68}$$

The final step is obtained by using (B.47) once more. Now multiply this equation with $\varepsilon^{\alpha\beta\gamma}J^{\beta YX}J^{\gamma}{}_{V}{}^{U}$, and use for the contraction of the Riemann curvature tensor with $J^{\beta YX}$ that we may interchange pairs of indices such that (B.48) can be used. Then everywhere appears $J^{\alpha}\mathcal{R}^{\beta}$, for which we can use (B.68). This leads to the equation expressing that the manifold is Einstein:

$$R_{XY} = \frac{1}{4r}g_{XY}R\,. \tag{B.69}$$

With (B.67), the SU(2) curvature is proportional to the complex structure:

$$\mathcal{R}^{\alpha}_{XY} = \frac{1}{2}\nu J^{\alpha}_{XY}\,, \qquad \nu \equiv \frac{1}{4r(r+2)}R\,. \tag{B.70}$$

The Einstein property drastically simplifies the expression for B in (B.44) to

$$B_{XY} = \frac{1}{4}\nu g_{XY}\,. \tag{B.71}$$

The Ricci part of the curvature then becomes proportional to the curvature of a quaternionic projective space of the same dimension:

$$\left(R^{\mathbb{H}P^n}\right)_{XYWZ} = \frac{1}{2}g_{Z[X}g_{Y]W} + \frac{1}{2}J^{\alpha}_{XY}J^{\alpha}_{ZW} - \frac{1}{2}J^{\alpha}_{Z[X}J^{\alpha}_{Y]W} = \frac{1}{2}J^{\alpha}_{XY}J^{\alpha}_{ZW} + L_{[ZW]}{}^{AB}L_{[XY]AB}\,. \tag{B.72}$$

The full curvature decomposition is then

$$R_{XYWZ} = \nu(R^{\mathbb{H}P^n})_{XYWZ} + \frac{1}{2} L_{ZW}{}^{AB} \mathcal{W}_{ABCD} L_{XY}{}^{CD} \,, \qquad \text{(B.73)}$$

with $\mathcal{W}_{ABCD}$ completely symmetric. The constraint appearing in supergravity fixes the value of ν to -1. The quaternionic-Kähler manifolds appearing in supergravity thus have negative scalar curvature, and this implies that all such manifolds that have at least one isometry are non-compact.

Finally, we should still comment on the extra constraint (B.66) for $r = 1$. In the mathematics literature [94] the extra constraint is that the quaternionic structure annihilates the curvature tensor, which is the vanishing of

$$\begin{aligned}(J^\alpha \cdot R)_{XYWZ} &\equiv J^\alpha{}_X{}^V R_{VYWZ} + J^\alpha{}_Y{}^V R_{XVWZ} + J^\alpha{}_Z{}^V R_{XYWV} + J^\alpha{}_W{}^V R_{XYVZ} \\ &= \varepsilon^{\alpha\beta\gamma} \left(\mathcal{R}^\beta_{XY} J^\gamma_{ZW} + \mathcal{R}^\beta_{ZW} J^\gamma_{XY} \right) , \end{aligned} \qquad \text{(B.74)}$$

where the second expression is obtained using once more (B.47). We have proven that (B.66) was sufficient extra input to have $\mathcal{R}^\alpha_{XY}$ proportional to J^α_{XY} implying $J^\alpha \cdot R = 0$. Vice versa: multiplying (B.74) with $\varepsilon^{\alpha\delta\epsilon} J^\epsilon{}_{YZ}$ leads to (B.66) if $J^\alpha \cdot R = 0$. Thus indeed the vanishing of (B.74) is an equivalent condition that can be imposed for $r = 1$ and that is automatically satisfied for $r > 1$.

B.5 Symmetries

Symmetries of manifolds are most known as isometries for riemannian manifolds (i.e. when there is a metric). They are transformations $\delta q^X = k^X_I(q)\Lambda^I$, where Λ^I are infinitesimal parameters. They are determined by the Killing equation[12]

$$\mathfrak{D}_{(X} k_{Y)I} = 0 \,, \qquad k_{XI} \equiv g_{XY} k^Y{}_I \,. \qquad \text{(B.75)}$$

This definition can only be used when there is a metric. However, there is a weaker equation that can be used for defining symmetries also in the absence of a metric, but when parallel transport is defined. Indeed, the Killing equation implies that

$$-R_{YZX}{}^W k_{WI} = \mathfrak{D}_Y \mathfrak{D}_Z k_{XI} - \mathfrak{D}_Z \mathfrak{D}_Y k_{XI} = \mathfrak{D}_Y \mathfrak{D}_Z k_{XI} + \mathfrak{D}_Z \mathfrak{D}_X k_{YI} \,. \qquad \text{(B.76)}$$

Using the cyclicity condition on the left hand side to write

$$R_{YZX}{}^W = \frac{1}{2} \left(R_{YZX}{}^W - R_{ZXY}{}^W - R_{XYZ}{}^W \right) , \qquad \text{(B.77)}$$

we obtain

$$\mathfrak{D}_X \mathfrak{D}_Y k^Z_I = R_{XWY}{}^Z k^W_I \,. \qquad \text{(B.78)}$$

This equation does not need a metric any more. We will use it as definition of symmetries when there is no metric available. We will see that it leads to the group structure that is known from the riemannian case.

[12] See also 'conformal Killing vectors' in section 2.1.

Of course, we will require also that the symmetries respect the quaternionic structure. This is the statement that the vector k_I^X normalizes the quaternionic structure:

$$\mathcal{L}_{k_I} J^\alpha{}_X{}^Y \equiv k_I^Z \partial_Z J^\alpha{}_X{}^Y - \left(\partial_X k_I^Z\right) J^\alpha{}_Z{}^Y + J^\alpha{}_X{}^Z \left(\partial_Z k_I^Y\right) = b_I^{\alpha\beta} J^\beta{}_X{}^Y , \tag{B.79}$$

for some functions $b_I^{\alpha\beta}(q)$. This b_I is antisymmetric, as can be seen by multiplying the equation with $J^\gamma{}_Y{}^X$.

Thus we define symmetries in quaternionic-like manifolds as those $\delta q^X = k_I^X(q)\Lambda^I$, *such that the vectors* k_I^X *satisfy (B.78) and (B.79).*

We first consider (B.79). One can add an affine torsionless connection to the derivatives, because they cancel. As a total covariant derivative on J vanishes, we add in case of quaternionic manifolds the SU(2) connection to the first derivative. This addition is of the form of the right-hand side. Thus defining P_I^γ by $b_I^{\alpha\beta} - 2\varepsilon^{\alpha\beta\gamma}\omega_X{}^\gamma k_I^X = -2\varepsilon^{\alpha\beta\gamma}\nu P_I^\gamma$, the remaining statement is that there is a $P_I^\alpha(q)$ (possibly zero) such that[13]

$$J^\alpha{}_X{}^Z \left(\mathfrak{D}_Z k_I^Y\right) - \left(\mathfrak{D}_X k_I^Z\right) J^\alpha{}_Z{}^Y = -2\varepsilon^{\alpha\beta\gamma} J^\beta{}_X{}^Y \nu P_I^\gamma . \tag{B.80}$$

The equation now takes on the form of (B.11) in lemma 1. Thus, using this lemma, as well as lemma 2, we have

$$\mathfrak{D}_X k_I^Y = \nu J^\alpha{}_X{}^Y P_I^\alpha + L_X{}^Y{}_A{}^B t_{IB}{}^A . \tag{B.81}$$

$t_{IB}{}^A$ is the matrix that we saw in the fermion gauge transformation law (2.94). The rule (B.12) gives an expression for P_I^α, which is called the *moment map*:

$$4r\,\nu\, P_I^\alpha = -J^\alpha{}_X{}^Y \left(\mathfrak{D}_Y k_I^X\right) . \tag{B.82}$$

Using the second equation, (B.78) we now find

$$R_{ZWX}{}^Y k_I^W = \mathfrak{D}_Z \mathfrak{D}_X k_I^Y = \nu J^\alpha{}_X{}^Y (\mathfrak{D}_Z P_I^\alpha) + L_X{}^Y{}_A{}^B \left(\mathfrak{D}_Z t_{IB}{}^A\right) . \tag{B.83}$$

Using the curvature decomposition (B.38) and projecting onto the complex structures and L, we find two equations

$$\mathcal{R}_{ZW}{}^\alpha k_I^W = -\nu \mathfrak{D}_Z P_I^\alpha , \qquad \mathcal{R}_{ZWB}{}^A k_I^W = \mathfrak{D}_Z t_{IB}{}^A . \tag{B.84}$$

The algebra that the vectors k_I^X define is

$$2k_{[I}^Y \mathfrak{D}_Y k_{J]}^X + f_{IJ}{}^K k_K^X = 0 , \tag{B.85}$$

where $f_{IJ}{}^K$ are structure constants. Multiplying this relation with $J^\alpha{}_X{}^Z \mathfrak{D}_Z$, and using (B.78), and (B.82) gives

$$2J^\alpha{}_X{}^Z (\mathfrak{D}_Z k_{[I}^Y)(\mathfrak{D}_Y k_{J]}^X) + 2J^\alpha{}_X{}^Z R_{ZWY}{}^X k_{[I}^Y k_{J]}^W - 4r\nu f_{IJ}{}^K P_K^\alpha = 0 . \tag{B.86}$$

[13]Here we introduce in fact νP. The factor ν is included for agreement with other papers and allows a smooth limit $\nu = 0$ to the hypercomplex or hyperkähler case. In fact, we have seen in (2.101) that supersymmetry in the setting of hypercomplex manifolds demands that the right-hand side of (B.79) is zero. We will see below that this is unavoidable for hypercomplex manifolds even outside the context of supersymmetry.

The trace that appears in the first term can be evaluated by using (B.80) and once more (B.82), while in the second term we can use the cyclicity condition of the curvature and (B.48) to obtain

$$-2\nu^2\varepsilon^{\alpha\beta\gamma}P_I^\beta P_J^\gamma + \mathcal{R}_{YW}{}^\alpha k_I^Y k_J^W - \nu f_{IJ}{}^K P_K^\alpha = 0\,. \tag{B.87}$$

We thus found that the moment maps, defined in (B.82) satisfy (B.84) and (B.87). The first of these shows that we can take $\nu = 0$ for the hypercomplex or hyperkähler manifolds. Both these two relations vanish identically in this case. However, for quaternionic-Kähler and hyperkähler manifolds, we can use (B.70), and dividing by ν leads to

$$J^\alpha{}_{ZW}k_I^W = -2\mathfrak{D}_Z P_I^\alpha\,, \tag{B.88}$$

$$-2\nu\varepsilon^{\alpha\beta\gamma}P_I^\beta P_J^\gamma + \frac{1}{2}J^\alpha{}_{YW}k_I^Y k_J^W - f_{IJ}{}^K P_K^\alpha = 0\,. \tag{B.89}$$

These equations are thus equivalent to the previous ones for $\nu \neq 0$ if there is a metric. This is thus the quaternionic-Kähler case, for which these relations appear already in [95]. But we did not *derive* these equations for the $\nu = 0$ (hyperkähler) case. Rather, the first one is taken as the definition of P for this case. This equation also follows from supersymmetry requirements, where the moment map P_I^α is an object that is needed to define the action, see (3.34). The moment map is then determined up to constants. As we saw in section 3.3.2, the constants are fixed when conformal symmetry is imposed. Similarly, the second equation appears in supersymmetry as a requirement, see (3.36). For a conformal invariant theory, the constants in P_I^α are determined and the moment map again satisfies (B.89) automatically due to a similar calculation as the one that we did above for $\nu \neq 0$. Note, however, that for the quaternionic manifolds that are not quaternionic-Kähler, we can only use (B.84) and (B.87), as (B.88) and (B.89) need a metric. For hypercomplex manifolds, on the other hand, the moment maps are not defined.

C. Examples: hypercomplex group manifolds

In this appendix we illustrate explicit examples of hypercomplex manifolds. Specifically, we demonstrate the non-vanishing of the antisymmetric Ricci tensor for some of these manifolds. The examples that we have in mind are group manifolds, or cosets thereof. These have two connections preserving the complex structures, one with and one without torsion. The torsionful connection preserves a metric, which is on the group manifolds the Cartan-Killing metric. First we consider the generic setup which has such two connections.

C.1 Hypercomplex manifolds with metric and torsionful connection

We consider a space with a metric g_{XY} and torsionful connection coefficients

$$\Gamma_{\pm YZ}{}^X = \gamma_{YZ}{}^X \pm T_{YZ}{}^X\,, \tag{C.1}$$

where $\gamma_{YZ}{}^X$ are the Levi-Civita connection coefficients with respect to this metric, and where $T_{YZ}{}^X = -T_{ZY}{}^X$ is the torsion.

We assume that there are hypercomplex structures that are covariantly constant with respect to the connection (C.1). We also assume the Nijenhuis condition and therefore have an Obata connection $\Gamma_{XY}{}^Z$. Taking the plus sign in (C.1) we have

$$\begin{aligned} 0 = \mathfrak{D}_X J^\alpha{}_Y{}^Z &= \partial_X J^\alpha{}_Y{}^Z - (\gamma + T)_{XY}{}^W J^\alpha{}_W{}^Z + (\gamma + T)_{XW}{}^Z J^\alpha{}_Y{}^W \,, \\ &= \partial_X J^\alpha{}_Y{}^Z - \Gamma_{XY}{}^W J^\alpha{}_W{}^Z + \Gamma_{XW}{}^Z J^\alpha{}_Y{}^W \,. \end{aligned} \tag{C.2}$$

Then the Obata connection can be related to the Levi-Civita connection and torsion by

$$\Gamma_{XY}{}^Z = \gamma_{XY}{}^Z + \frac{1}{6}\varepsilon^{\alpha\beta\gamma} J^\alpha{}_X{}^U J^\beta{}_Y{}^V T_{UV}{}^W J^\gamma{}_W{}^Z + \frac{2}{3} J^\alpha{}_{(X}{}^V T_{Y)V}{}^W J^\alpha{}_W{}^Z \,. \tag{C.3}$$

The antisymmetric part of the Ricci tensor of the Obata connection is

$$\begin{aligned} R_{[XY]} &= \partial_{[Y}\Gamma_{X]Z}{}^Z \\ &= \frac{2}{3} J^\alpha{}_W{}^Z J^\alpha{}_{[X}{}^V \mathfrak{D}_{Y]} T_{ZV}{}^W + \frac{2}{3} T_{YX}{}^U J^\alpha{}_W{}^Z J^\alpha{}_U{}^V T_{ZV}{}^W + \\ &\quad + \mathfrak{D}_{[X} T_{Y]Z}{}^Z + T_{YX}{}^U T_{WU}{}^W \,, \end{aligned} \tag{C.4}$$

where $\mathfrak{D}_X$ is the torsionful connection. If the torsion is covariant constant and traceless, as it is in group manifolds, then

$$R_{[XY]} = \frac{2}{3} T_{XY}{}^Z V_Z \qquad \text{with} \qquad V_Z = J^\alpha{}_Z{}^W T_{WV}{}^U J^\alpha{}_U{}^V \,. \tag{C.5}$$

This is the only surviving part of the Ricci tensor in hypercomplex manifolds, and will be used below.

The Nijenhuis condition can be written as a condition on the torsion (using the metric to lower indices) as [83]

$$3 J^{(\alpha}{}_{[X}{}^U J^{\beta)}{}_Y{}^V T_{Z]UV} = \delta^{\alpha\beta} T_{XYZ} \,. \tag{C.6}$$

Using the quaternionic algebra $J^1 J^2 = J^3 = -J^2 J^1$ and the Nijenhuis condition for one of the complex structures, one can show that the contributions from $\alpha = 1, 2$ and 3 in (C.5) are all equal.

C.2 Group manifolds

In [83], 2-dimensional sigma models with extended supersymmetry on group manifolds were studied. In the case of $N = 4$, it was shown to be possible to construct three globally defined, covariantly constant complex structures, on certain groups. Using cohomology, one argues[14] that these manifolds are in fact hypercomplex. For these arguments one makes use the fact that the second de Rham cohomology vanishes for all simple groups, whereas Kähler manifolds have a non-trivial Kähler 2-form.

We will explicitly construct the Ricci tensor on the group manifolds considered in [83], and show that there are cases with non-vanishing Ricci tensor. As this is an antisymmetric tensor, there is no invariant metric for the Obata connection.

[14] We thank George Papadopoulos for pointing this out to us.

In [83], the complex structures were first constructed in one fibre, and then used to form a field of complex structures with the help of the left- or right-invariant vector fields, giving rise to ($J_\pm^\alpha$). As the sigma models included an antisymmetric tensor field in their action, the connection used in the equations of motion had torsion, which could be written in terms of the structure constants of the groups. The connections $\Gamma_\pm$ corresponding to $J_\pm^\alpha$ differed in a sign, in the sense of (C.1). The torsion T_{XYZ} is completely anti-symmetric, and defined as (denoting the flat indices on the group manifold with $\hat\Lambda, \hat\Sigma, \ldots$)

$$g^{ZV}T_{VXY} \equiv T_{XY}{}^Z = \frac{1}{2} e_X^{\hat\Lambda} e_Y^{\hat\Sigma} f_{\hat\Lambda\hat\Sigma}^{\hat\Gamma} e_{\hat\Gamma}^Z \tag{C.7}$$

where the $e_X^{\hat\Lambda}$ are vielbeins, and dual to the left- or right invariant vector fields. The vielbeins (and the torsion) are covariantly constant with respect to the connection $\Gamma_{\pm YZ}{}^X$.

We will now construct the vector V of (C.5) explicitly, using the connection Γ_+. This means that the complex structures, defined in one fibre, define a field of complex structures using the *left*-invariant vector fields.

A key concept in the construction of hypercomplex group manifolds, are the so-called stages. This is because the 3 complex structures in fact act within any such 'stage'. A stage consists of a subset of the generators of a group on which a hypercomplex structure is defined. One can start from any simple group G to define a stage. One starts by picking out a highest root θ. One adds $-\theta$, all the roots that are *not* orthogonal to θ and two more generators. One of these is the generator in the Cartan subalgebra (CSA) in the direction of θ and $-\theta$. If the subspace of roots orthogonal to θ form a root space of dimension ($\mathrm{rank}\, G - 2$), then the second one is the other element in the CSA that does not belong to the simple group defined with the roots orthogonal to θ. This happens only for $G = \mathrm{SU}(n)$ with $n \geq 3$. In all other cases one has to consider $G \times \mathrm{U}(1)$ in order to be able to define a hypercomplex structure. The roots θ and $-\theta$ and the two generators of the CSA define an algebra $\mathrm{SU}(2) \oplus \mathrm{U}(1)$. The stage can thus be written as

$$\mathrm{SU}(2) \oplus \mathrm{U}(1) \oplus W\,, \tag{C.8}$$

where W are all the roots not orthogonal to θ. These form a 'Wolf space'. The Wolf spaces

$$\begin{array}{lll} W = \frac{G}{H\times \mathrm{SU}(2)}\,, & G \neq \mathrm{SU}(n)\,, & \\ W = \frac{\mathrm{SU}(n)}{\mathrm{SU}(n-2)\times\mathrm{SU}(2)\times\mathrm{U}(1)}\,, & n \geq 3\,, & \end{array} \quad \dim W = 4(\tilde h_g - 2)\,, \tag{C.9}$$

where $\tilde h_g$ is the dual Coxeter number[15] of the group G, are the quaternionic symmetric spaces. So far, we considered compact groups. The only non-compact groups that are allowed are those real forms where just the generators in W are non-compact, and all the others are compact. Hereafter, the group generated by the roots orthogonal to θ, together with the remaining elements in the Cartan subalgebra [being H or $\mathrm{SU}(n-2)$ in (C.9)], is used to construct a new stage in the same way. By this procedure, one constructs the complex structures in one fibre of the group. For more details we refer to [83] or [84].

[15] Tables are given in [83], e.g. $\tilde h_g = n$ for $\mathrm{SU}(n)$.

We will now give explicitly the hypercomplex structures (in one stage) in a language adapted to this paper. As we use flat space indices on a Lie group, these take values in the Lie algebra. The base for our Lie algebra is taken to be Cartan-Weyl. We will use hatted Greek capitals to denote all Lie algebra elements. θ and $-\theta$ are the chosen highest root and its negative. Greek capital letters denote the positive generators in W. The full set in W consists thus of those indicated by Δ and those by $-\Delta$. Small Roman letters k, ℓ indicate elements of the Cartan subalgebra. The full set of generators is thus

$$\hat{\Delta} = \{-\theta, -\Delta, k, \Delta, \theta\} \oplus \text{other stages}\,, \tag{C.10}$$

where Δ runs over $2(\tilde{h}_g - 2)$ values and $k = 1, 2$.

First, it is useful to give some more information about the structure of the algebra in a stage. The root vectors are indicated as $\vec{\theta}$ or $\vec{\Delta}$ and particular components as θ_k or Δ_k. The following properties of structure constants, Cartan-Killing metric and root vectors are useful:

$$\begin{aligned}
&f_{k,\pm\Delta}^{\pm\Delta} = \pm\Delta_k\,, \qquad f_{k,\pm\theta}^{\pm\theta} = \pm\theta_k\,, \quad f_{\Delta,-\Delta}^{k} = \Delta_k\,, \quad f_{\theta,-\theta}^{k} = \theta_k\,, \quad f_{\pm\Delta,\mp\theta}^{\pm\Delta\mp\theta} = \pm\frac{1}{2x}\alpha_\Delta\,,\\
&2x^2 \equiv \tilde{h}_g = \frac{1}{\vec{\theta}^2}\,, \qquad \vec{\Delta}\cdot\vec{\theta} = \frac{1}{2}\vec{\theta}^2\,, \qquad \alpha_\Delta = -\alpha_{\theta-\Delta} = \pm 1\,,\\
&g_{k,\ell} = -\delta_{k\ell}\vec{\theta}^2\,, \qquad g_{\theta,-\theta} = g_{\Delta,-\Delta} = -\vec{\theta}^2\,.
\end{aligned} \tag{C.11}$$

These relations fix a normalization for the generators.

We can now write the non-zero elements of the complex structures as

$$\begin{aligned}
{J^1}_k{}^\ell &= \varepsilon_{k\ell}\,, & {J^1}_{\pm\theta}{}^{\pm\theta} &= \pm\mathrm{i}\,, & {J^1}_{\pm\Delta}{}^{\pm\Delta} &= \pm\mathrm{i}\\
{J^2}_{\pm\theta\mp\Delta}{}^{\mp\Delta} &= \mp\mathrm{i}\,\alpha_\Delta\,, & {J^2}_k{}^{\pm\theta} &= x\,(\pm\mathrm{i}\,\theta_k - \varepsilon_{k\ell}\theta_\ell)\,, & {J^2}_{\pm\theta}{}^k &= x\,(\pm\mathrm{i}\,\theta_k + \varepsilon_{k\ell}\theta_\ell)\,,\\
{J^3}_{\pm\theta\mp\Delta}{}^{\mp\Delta} &= \alpha_\Delta\,, & {J^3}_k{}^{\pm\theta} &= x\,(\theta_k \pm \mathrm{i}\,\varepsilon_{k\ell}\theta_\ell)\,, & {J^3}_{\pm\theta}{}^k &= x\,(-\theta_k \pm \mathrm{i}\,\varepsilon_{k\ell}\theta_\ell)\,.
\end{aligned} \tag{C.12}$$

These satisfy the Nijenhuis conditions (C.6).

As written at the end of section C.1, we can limit the calculation of V to the contribution of one of the complex structures. The torsion is proportional to the structure constants, and as J^1 is diagonal in the roots, the vector $V_{\hat{\Sigma}}$ has only non-zero components along the Cartan subalgebra:

$$V_k = \frac{3}{2}{J^1}_k{}^\ell f_{\ell,\hat{\Delta}}{}^{\hat{\Gamma}} {J^1}_{\hat{\Gamma}}{}^{\hat{\Delta}} = 3\varepsilon_{k\ell}\,\mathrm{i}\left(\theta_\ell + \sum_\Delta \Delta_\ell\right) = 3\,\mathrm{i}\,\varepsilon_{k\ell}\theta_\ell(\tilde{h}_g - 1)\,. \tag{C.13}$$

Though this is non-zero for all the groups under consideration, the Ricci tensor is only non-vanishing for $G = \mathrm{SU}(n)$ with $n \geq 3$. Indeed, in all other cases, the generator corresponding to the index k in (C.13) corresponds to the extra U(1) factor that was added to G, and there are thus no non-vanishing $R_{XY} = \frac{2}{3}{T_{XY}}^k V_k$.

The only case in which we find a non-vanishing Ricci tensor, is when the Wolf space is

$$W = \frac{\mathrm{SU}(n)}{\mathrm{SU}(n-2)\times\mathrm{SU}(2)\times\mathrm{U}(1)}\,, \qquad n \geq 3\,, \tag{C.14}$$

Then the non-vanishing components of the Ricci tensor are of the form

$$R_{\Delta,-\Delta} = -R_{-\Delta,\Delta} = \mathrm{i}\,\Delta_k \varepsilon_{k\ell}\theta_\ell(\tilde{h}_g - 1)\,, \tag{C.15}$$

and one can see again that $\Delta_k \varepsilon_{k\ell}\theta_\ell$ vanishes for all other cases than $G = \mathrm{SU}(n)$. In this case, it is simply a function of n.

The group manifolds that have a non-zero Ricci tensor are those that have a stage with the Wolf spaces (C.14). Checking the list in [83], these are $\mathrm{SU}(2n-1)$, $\mathrm{SU}(2n) \times \mathrm{U}(1)$ (both for $n \geq 2$) and $E_6 \times \mathrm{U}(1)^2$. The other cases are Ricci flat, and one may wonder whether there is a metric whose Levi-Civita tensor is the Obata connection. This can not be the Cartan-Killing metric as its Levi-Civita tensor has a non-vanishing Ricci tensor and we just proved that the Obata connection has vanishing Ricci tensor. One may try to use cohomological arguments to exclude also any other metric.

After obtaining this result, we can understand it from the geometrical structure of the stages. We see that the origin of a non-zero Ricci tensor sits in the fact that there are non-zero roots in the direction of the U(1) factor in the decomposition (C.8). Thus, we see that we obtain a non-zero Ricci tensor if this U(1) is already present in the structure of the Wolf space, i.e. the origin sits in the U(1) factor in the structure of the coset (C.14).

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