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Published in:
Physics Letters B

DOI:
[10.1016/S0370-2693\(00\)00592-X](https://doi.org/10.1016/S0370-2693(00)00592-X)

IMPORTANT NOTE: You are advised to consult the publisher's version (publisher's PDF) if you wish to cite from it. Please check the document version below.

Document Version
Publisher's PDF, also known as Version of record

Publication date:
2000

[Link to publication in University of Groningen/UMCG research database](#)

Citation for published version (APA):

Berman, DS., & Parikh, MK. (2000). Confinement and the AdS/CFT correspondence. *Physics Letters B*, 483(1-3), 271-276. [https://doi.org/10.1016/S0370-2693\(00\)00592-X](https://doi.org/10.1016/S0370-2693(00)00592-X)

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15 June 2000

PHYSICS LETTERS B

Physics Letters B 483 (2000) 271–276

Confinement and the AdS/CFT correspondence

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Received 7 March 2000; received in revised form 2 May 2000; accepted 8 May 2000

Editor: M. Cvetič

Abstract

We study the thermodynamics of the confined and unconfined phases of $\mathcal{N} = 4$ Yang-Mills in finite volume and at large N using the AdS/CFT correspondence. We discuss the necessary conditions for a smooth phase crossover and obtain an N -dependent curve for the phase boundary. © 2000 Elsevier Science B.V. All rights reserved.

1. Introduction

It was realized many years ago [1] that in the large N limit of Yang-Mills theory a remarkable simplification takes place: the physics is dominated by “planar” graphs, Feynman diagrams with no line-crossings. In this limit, the gauge theory ought to be described by a “QCD string,” and it was a hope that such a simplification might shed light on some of the mysteries of nonabelian gauge theories, notably the puzzle of confinement.

The AdS/CFT correspondence [2] makes explicit this relation between gauge fields and strings. Specifically, the correspondence says that IIB string theory in a background of five-dimensional anti-de Sitter space times a five-sphere is dual to the large N limit of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory in four dimensions. This is a conformal theory with no

confinement; however, the thermal theory in finite volume could still have a confined phase.

An important ingredient in the correspondence is the principle of holography [3,4], the notion that the physics of a gravitational theory is dual to a different theory in one lower dimension. Conversely, given the dual theory on a boundary, we must consider all the possible bulk manifolds whose boundaries have the same intrinsic geometry as the background of the dual theory [5,6]. For thermal super Yang-Mills on S^3 , there are at least two distinct Einstein manifolds with the requisite boundary geometry: thermal anti-de Sitter space and a Schwarzschild-like black hole in AdS [7]. These classical bulk solutions are a sort of master field of the gauge theory. The distinct geometries are interpreted in the gauge theory as different phases in the strong 't Hooft coupling limit; the thermodynamics of the black hole corresponds to the thermodynamics of strong-coupling SYM in the unconfined phase while thermal AdS is seen as dual to the confined phase of the gauge theory.

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In this paper, we investigate the thermodynamics of the different phases of super Yang-Mills at finite volume on S^3 using the AdS/CFT correspondence. In particular, we examine the conditions for phase change in a microcanonical framework. Formally, a phase transition cannot occur in finite volume at finite N . However, a crossover between these qualitatively different phases can still occur when their weights in the partition sum are the same. In the dual picture, the black hole dominates the path integral when the horizon is large compared to the inverse AdS curvature, and the thermal AdS geometry dominates for sufficiently low temperatures. The crossover between these two geometries is known as the Hawking-Page transition and corresponds in the field theory to a transition between the confining and unconfined phases. Since a microcanonical framework requires that energy be conserved during any transition, we shall consider not empty AdS with thermal identifications, but rather thermally-identified AdS with a thermal gas in it. The energy of the system is then measured with respect to the thermal AdS background.

This paper is structured as follows. We begin, in Section 2, by reviewing the thermal properties of AdS black holes and their interpretation in light of the AdS/CFT correspondence. Then, in Section 3, we consider a thermal bath in AdS. In Section 4 we obtain our main result: an N -dependent phase boundary between the confined and unconfined phases of the CFT.

2. AdS black holes

The five-dimensional Einstein-Hilbert action with a cosmological constant is given by

$$I_{\text{BH}} = -\frac{1}{16\pi G_5} \int d^5x \sqrt{-g} (R + 12l^2), \quad (1)$$

where G_5 is the five-dimensional Newton constant, R is the Ricci scalar, the cosmological constant is $\Lambda = -6l^2$, and we have neglected a surface term at infinity. Anti-de Sitter solutions derived from this action can be embedded in ten-dimensional IIB supergravity such that the supergravity background is of the form $\text{AdS}_5 \times S^5$.

The line element of a ‘‘Schwarzschild’’ black hole in anti-de Sitter space [7] in five spacetime dimensions can be written as

$$ds^2 = -\left(1 - \frac{2MG_5}{r^2} + r^2l^2\right) dt^2 + \left(1 - \frac{2MG_5}{r^2} + r^2l^2\right)^{-1} dr^2 + r^2 d\Omega_3^2, \quad (2)$$

where l is the inverse radius of AdS space. This solution has a horizon at $r = r_+$ where

$$r_+^2 = \frac{1}{2l^2} \left(-1 + \sqrt{1 + 8MG_5 l^2}\right). \quad (3)$$

When $r_+ l \ll 1$, the black hole could become unstable to localization on the S^5 by an analog of the Gregory-Laflamme mechanism [8]. As a rule, one may determine a necessary (though not sufficient) condition for instability from entropic considerations. A straightforward computation then shows that localization instability could occur for very small black holes with $r_+ l \ll 1$ [9]. Here we shall work with black holes with $r_+ l > 1$ for which we do not expect such an instability.

To study the black hole’s thermodynamics, we Euclideanize the metric. The substitution $\tau = it$ makes the metric positive definite and, by the usual removal of the conical singularity at r_+ , yields a periodicity in τ of

$$\beta_{\text{BH}} = \frac{2\pi r_+}{1 + 2r_+^2 l^2}, \quad (4)$$

which is identified with the inverse temperature of the black hole. The entropy is given by

$$S_{\text{BH}} = \frac{A}{4G_5} = \frac{\pi^2 r_+^3}{2G_5}, \quad (5)$$

where A is the ‘‘area’’ (that is, three-volume) of the horizon. The mass above the anti-de Sitter background is

$$U_{\text{BH}} = \frac{3\pi}{4} M = \frac{3\pi}{8G_5} r_+^2 (1 + r_+^2 l^2). \quad (6)$$

This is the AdS equivalent of the ADM mass, or energy at infinity. (Actually if the black hole is to be

considered at thermal equilibrium it should properly be regarded as being surrounded by a thermal envelope of Hawking particles. Because of the infinite blueshift at the horizon, the envelope contributes a formally infinite energy. Here we shall neglect this infinite energy as unphysical, absorbed perhaps by a renormalization of the Newton constant.) We can now also write down the free energy:

$$F_{\text{BH}} = \frac{\pi r_+^2}{8G_5} (1 - r_+^2 l^2). \tag{7}$$

Eqs. (5)–(7) then satisfy the first law of thermodynamics. To express them in terms of the gauge theory parameters N , T_{CFT} , and V_{CFT} , we substitute physical data taken from the boundary of the black hole spacetime. At fixed $r \equiv r_0 \gg r_+$, the boundary line element tends to

$$ds^2 \rightarrow r_0^2 [-l^2 dt^2 + d\Omega_3^2], \tag{8}$$

giving a volume of

$$V_{\text{CFT}} = 2\pi^2 r_0^3. \tag{9}$$

The field theory temperature is the physical temperature at the boundary:

$$T_{\text{CFT}} = \frac{T_{\text{BH}}}{\sqrt{-g_{tt}}} \approx \frac{T_{\text{BH}}}{lr_0}. \tag{10}$$

To obtain an expression for N , we invoke the AdS/CFT correspondence. This relates N to the radius of S^5 and the cosmological constant:

$$R_{S^5}^2 = \sqrt{4\pi g_s \alpha'^2 N} = \frac{1}{l^2}. \tag{11}$$

Then, since

$$(2\pi)^7 g_s^2 \alpha'^4 = 16\pi G_{10} = 16 \frac{\pi^4}{l^5} G_5, \tag{12}$$

we have

$$N^2 = \frac{\pi}{2l^3 G_5}. \tag{13}$$

With these substitutions, we see that in the limit $r_+ l \gg 1$, the black hole entropy can be expressed in terms of conformal field theory parameters as

$$S_{\text{BH}} = \frac{1}{2} \pi^2 N^2 V_{\text{CFT}} T_{\text{CFT}}^3. \tag{14}$$

The dimensionful terms in this expression are in accord with expectations for a conformal theory. The matching – but for a numerical discrepancy presumably arising from a 't Hooft coupling dependence – has been extended to rotating black holes and their field theory dual, Yang-Mills with angular momentum [10–18]. The dependence on N^2 indicates that the conformal field theory is in its unconfined phase; the N^2 species of free gluons make independent contributions to the free energy. We shall see in the next section that the thermodynamics of the confined phase is rather different.

3. A hot bath in AdS

Now, following Hawking and Page, consider a gas of thermal radiation in anti-de Sitter space. Here we are taking the view that the gas is constituted of the massless fields of IIB supergravity. First we must determine the thermodynamically relevant number of spacetime dimensions. The energy eigenstates of AdS₅ are [19]

$$\begin{aligned} \Psi_{\omega j m n}(r, t, \theta, \phi, \psi) &= N_{\omega j} \exp(-i \omega l t) \sin^j \rho C_{\omega-j-1}^{j+1}(\cos \rho) \\ &\quad \times Y_j^{mn}(\theta, \phi, \psi), \end{aligned} \tag{15}$$

with the condition $\omega - 1 \geq j \geq |m|, |n|$ where $C_q^p(x)$ are Gegenbauer polynomials, $Y_j^{mn}(\theta, \phi, \psi)$ are the spherical harmonics in five-dimensional spacetime (with total angular momentum number j), and $\rho \equiv \arctan(r/l)$. Here ω is an integer and hence the spectrum is quantized in units of l , the inverse ‘‘radius’’ of AdS. Since this is also the quantum of excitations of the five-sphere, we should consider thermodynamics over the full ten-dimensional space. The appropriate line element is therefore

$$ds^2 = -(1 + r^2 l^2) dt^2 + (1 + r^2 l^2)^{-1} dr^2 + r^2 d\Omega_3^2 + l^{-2} d\Omega_5^2. \tag{16}$$

To obtain a thermal field theory, we again Euclideanize the metric. The periodicity of $\tau = it$ is then the inverse (asymptotic) temperature, T_{AdS}^{-1} , of the theory; the absence of a horizon means that T_{AdS} is an arbitrary parameter. However, the relevant

temperature for thermodynamics in the bulk is not T_{AdS} , but the local, redshifted, temperature:

$$T_{\text{local}} = \frac{T_{\text{AdS}}}{\sqrt{-g_{tt}}} = \frac{T_{\text{AdS}}}{\sqrt{1+r^2 l^2}}. \quad (17)$$

To calculate thermodynamic quantities we foliate spacetime into (timelike) slices of constant local temperature. Extensive thermodynamic quantities are then computed by adding the contribution of each such hypersurface.

The local five-dimensional energy density of the thermal gas of radiation can be written as

$$\rho_{\text{local}} = \sigma \frac{\pi^3}{l^5} T_{\text{local}}^{10}, \quad (18)$$

where we have neglected infrared effects due to curvature or nonconformality. Here σ is the ten-dimensional supersymmetric generalization of the Stefan-Boltzmann constant, which is approximated by its flat space value:

$$\sigma = \frac{62}{105} \pi^5, \quad (19)$$

where we have included a factor of 128, the number of massless bosonic physical degrees of freedom of IIB supergravity.

The total ‘‘ADM’’ energy-at-infinity of a gas contained in a ball of radius r_0 is then

$$\begin{aligned} U_{\text{gas}}^\infty &= \sigma \frac{\pi^3}{l^4} \int T_{\text{local}}^{10} \sqrt{-g_{tt}} \sqrt{g_{rr}} r^3 dr d\Omega_3 \\ &= \frac{2\pi^5}{l^5} \sigma T_{\text{AdS}}^{10} \int_0^{r_0} \frac{r^3 dr}{(1+r^2 l^2)^5} \equiv \sigma V_{\text{eff}}(r_0) T_{\text{AdS}}^{10}. \end{aligned} \quad (20)$$

Here the additional blueshift factor of $\sqrt{-g_{tt}}$ converts the local (fiducial) energy into an ADM-type energy, comparable to Eq. (6). We have also defined an effective volume,

$$V_{\text{eff}}(r_0) = \frac{2\pi^5}{l^9} \left(\frac{2}{3} - \frac{2+3(r_0 l)^2}{3(1+(r_0 l)^2)^{3/2}} \right), \quad (21)$$

which, as $r_0 \rightarrow \infty$, approaches

$$\frac{4\pi^5}{3l^9}. \quad (22)$$

Thermodynamically, anti-de Sitter space behaves as if it had a finite volume.

Similarly, the other thermodynamic quantities of the thermal bath are

$$F = -\frac{\sigma}{9} V_{\text{eff}} T_{\text{AdS}}^{10}, \quad S = \frac{10}{9} \sigma V_{\text{eff}} T_{\text{AdS}}^9, \quad (23)$$

consistent with the first law of thermodynamics. The absence of a G_5 in the free energy indicates, from the CFT point of view, that the free energy is of order N^0 . This is the confined phase of the theory – the free energy is of order N^0 because the N^2 species of gluons have condensed into hadronic color singlets. (Curiously, attempts to formulate confinement in terms of anti-de Sitter space date back at least to the 1970’s [20].)

The power of nine temperature dependence of the entropy is somewhat puzzling for a three-dimensional gauge theory. A similar puzzle is pointed out concerning the subleading N^0 corrections to the thermodynamics at large $r_+ l$ in [21]. Here we wish to point out that we are in the confined phase and the power of nine reflects the fact that the QCD (SYM) string is really a type IIB string which naturally lives in nine spatial dimensions. It has been suggested that the extra dimensions in which the open string worldsheet bounded by a Wilson loop can extend are akin to Liouville dimensions [2,22–25] and the effective dimensionality of the QCD string is greater than 3 + 1.

4. The crossover

A field theory living on a manifold, $S^3 \times S^1$ with three-sphere radius r_0 , is dual to those five-dimensional Einstein manifolds that have the same geometry at r_0 . In the microcanonical approach that we shall follow, the contributions to the partition function come from both the black hole and the gas in thermally-identified AdS where the energy of the gas and black hole are taken to be the same:

$$Z(U) = e^{-I_{\text{BH}}(U)} + e^{-I_{\text{gas}}(U)}. \quad (24)$$

Which of these two thermodynamic phases the system is found in is determined, in the saddle point approximation, by the relative values of the respective Euclidean classical actions. The action of the black hole is simply the Einstein-Hilbert action, Eq.

(1). This is proportional to the volume of the space-time and so needs to be regulated. A finite action is obtained by subtracting the (also infinite) action for thermally-identified anti-de Sitter space in which the hypersurface at a constant large radius has the same intrinsic geometry as a hypersurface at the same radius in the black hole background [7]. The regularized black hole action is

$$I_{\text{BH}} = \frac{\pi^2 r_+^3}{4G_5} \frac{1 - r_+^2 l^2}{1 + 2r_+^2 l^2}. \tag{25}$$

Subtracting the anti-de Sitter background is equivalent to choosing the ground state of the theory. Then the comparable value of the action for the gas in thermally-identified AdS should be just the action of the gas itself, namely F/T . Thus

$$I_{\text{gas}} = -\frac{1}{9} \sigma V_{\text{eff}} T_{\text{AdS}}^9. \tag{26}$$

The qualitative thermodynamic behavior of the system is determined by the action which dominates the partition function Eq. (24). At the crossover between the two phases, the action for the gas and the black hole are the same. Moreover, energy must

be conserved. Hence one may determine the conditions for a smooth crossover from the following equations:

$$U_{\text{gas}}^{\text{local}} = U_{\text{BH}}^{\text{local}}, \quad I_{\text{gas}} = I_{\text{BH}}. \tag{27}$$

Note that, since the two phases cannot be in physical contact, the physical temperature does not have to be the same for the two phases. (The physical temperatures are the same at $r_+ l = \sqrt{7}$.)

Solving these equations yields N^2 as a function of the dimensionless quantity $x \equiv r_+ l$ at the crossover:

$$N^2 = \frac{31}{2^5 \cdot 3^{13} \cdot 5 \cdot 7} \frac{(1 + x^2)^9 \cdot (1 + 2x^2)^{10}}{(x^2 - 1)^{10} \cdot x^{12}}. \tag{28}$$

This is the equation we want. Before discussing it, we comment on some of the underlying approximations. In using Eq. (17), we have omitted the back-reaction of the gas on the metric. One may estimate this. Consider a spherically symmetric sta-

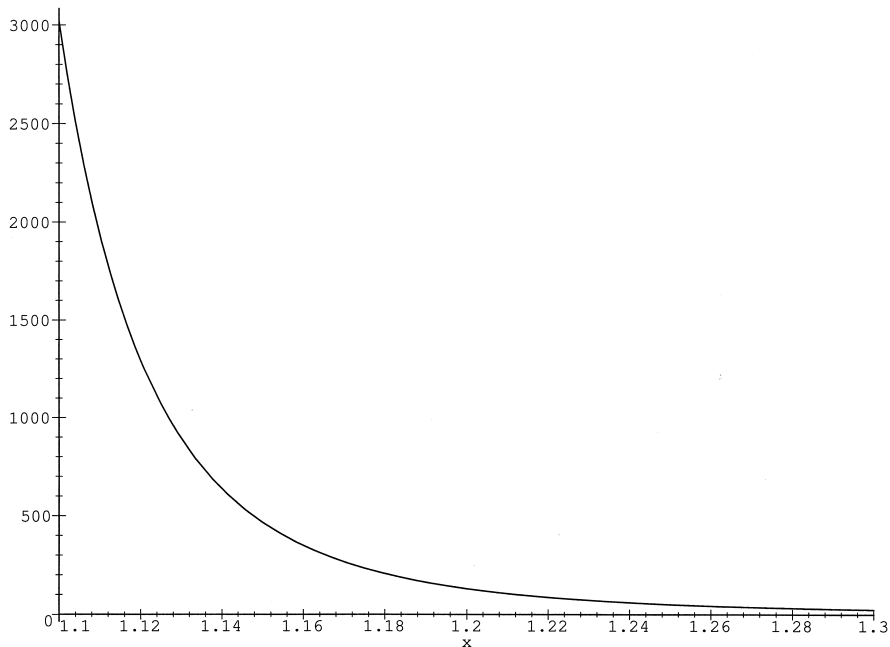


Fig. 1. N versus $x \equiv r_+ l$ for x near unity.

tionary spacetime with a cosmological constant and massive matter fields. The equation of hydrostatic equilibrium (equally, the $t-t$ Einstein equation) reads

$$\frac{d}{dr}m(r) = 8\pi^2 r^3 (G_5 \rho + \Lambda), \quad (29)$$

where $-g_{tt} \equiv 1 - m(r)/r^2$. Back-reaction can reliably be neglected when the matter term in the parenthesis is (much) smaller than the cosmological term. For an energy density given by Eq. (18) and a total energy matched to that of the black hole phase, Eq. (6), the condition $G_5 \rho < |\Lambda|$ amounts to

$$\frac{9\pi}{32} x^2 (1 + x^2) l^2 < 6 l^2, \quad (30)$$

and we see that the matter term becomes dominant at large x . So at high temperature Eq. (28) becomes unreliable; this limit has been studied elsewhere [21]. To accommodate the effect of matter, one might try to seek a solution to the linearized Einstein equations, perturbed around the AdS_5 background. The exact form of Eq. (28) would also be modified by including the correct Stefan-Boltzmann constant for anti-de Sitter space. And finally, when N is small, the supergravity approximation itself breaks down.

In Fig. 1, we plot N near the crossover for $x \sim 1$, near which backreaction is least important and N is large. The region below the crossover curve is dominated by the confined or AdS gas phase, whereas the region above is dominated by the unconfined or black hole phase. Note that x grows roughly like the dimensionless product $T_{\text{phys}} r_0$. As the temperature/volume increases, the graph confirms our expectation that the theory becomes conformal. As N goes to infinity, we recover the result [6] that the transition occurs at $r_+ l = 1$. This is in fact a very good approximation for finite but large N .

Acknowledgements

We would like to thank José Barbón, Gerard 't Hooft, Shiraz Minwalla, and Erik Verlinde for helpful discussions. D.B. is supported by European Commission TMR programme ERBFMRX-CT96-0045, and would like to acknowledge the hospitality of the Spinoza Institute, CERN and the support of ITF,

Utrecht. M.P. is by the Netherlands Organization for Scientific Research (NWO).

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