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INSTABILITY OF A HIERARCHICAL WEDDING CAKE IN A RANDOM MEDIUM: A MEAN FIELD RESULT*

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ABSTRACT

We consider a hierarchical interface model imbedded in a disordered medium in interface dimensions $d = 2$. We rigorously investigate the renormalization group flow for the stochastic variables describing the disorder in the mean field limit of infinite blocklength at zero temperature. The renormalization of these processes can be described on the level of an infinite vector of covariances. It is shown that, although the strength of the disorder renormalizes to zero, the interface exhibits unbounded fluctuations when the system size goes to infinity.

1. Introduction

In the past years there has been considerable progress in the understanding of disordered spin systems. An essential step in this direction was made by the rigorous renormalization group (RG) analysis of Bricmont and Kupiainen [3,4] of the three dimension random field Ising model (RFIM). The authors were able to show that, for small disorder and at sufficiently low temperatures, there exists a ferromagnetically ordered phase. An alternative approach to this problem was given by Zahradnik [16].

To problems of interfaces in a random environment which we are interested in here the RG-technique could also be applied. In the framework of the solid on solid model, it could be shown that there exist Gibbs measures describing flat interfaces, for small disorder and at sufficiently low temperatures, in *interface* dimensions $d > 2$. The type of randomness that could be treated include a random bond and a random field environment ([7], for the latter see [14]). For this analysis the previous investigation of hierarchical models [5,6,8] has been very instructive.

However, the renormalization group analysis was used here to show the irrelevance of the randomness, i.e. in a situation where the randomness does not essentially modify the behaviour of the system. In contrast to that for the RFIM Aizenman and Wehr [1] have shown (by Martingale techniques) the uniqueness of the Gibbs measure, for arbitrarily small randomness, at all temperatures in space dimensions $D = 2$. Briefly, [1] and [3] thus confirmed precisely the prediction of the simple (but in the case $D = 3$ discussed) Imry-Ma argument. In an analogous

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way, for the interface model in $d = 2$ the Imry-Ma argument predicts an interface undergoing unbounded fluctuations when the volume of the system is allowed to go to infinity. But, still there exists no rigorous analysis on this.

The present note is an attempt to tackle this issue from the point of view of the renormalization group in a simplified situation. It continues the investigation of the hierarchical interface model, (as analyzed before in [5,6,8]) to treat the case of the marginal dimension $d = 2$. In this hierarchical model only contours (i.e. lines separating regions of different heights) are allowed whose bases are nested squares. For our present analysis we even simplify this model, further restricting the class of allowed interfaces by imposing the height of the interface to increase by one when crossing a contour from the outside. Our surfaces become thus 'hierarchical wedding cakes' [9], imbedded in a random medium. The hierarchical model allows to obtain explicitely RG equations for the stochastic processes describing the disorder. However, the resulting equations seem to be difficult to analyze still; therefor we restrict ourselves to treat the mean field limit of infinite blocklength. In this way we have finally arrived at the simplest possible approximation of the interface model from the point of view of the RG. Here, the reason for the simplicity of infinite blocksize limit lies in the fact that it allows us to describe the renormalization of distributions of stochastic processes on the level of an infinite vector of covariances. This resulting recursion relation we get then will be seen to be reminiscent of a discrete nonlinear diffusion equation in the space of covariances.

In the above framework we show be elementary arguments as the result of this paper that, although the strength of the randomness renormalizes to zero, the interface undergoes unbounded fluctuations when the system size goes to infinity (see the theorem in chapter 4).

The organisation of the paper is as follows. The formal definition of the model is given in chapter 2. In chapter 3 we derive the recursion relation. In chapter 4 we prove our results on the recursion relation.

2. The model

Our surface will be lying over the finite box $\{0, \ldots, L^N - 1\}^2 = \Lambda_{L,N} \subset \mathbb{Z}^2$. It will be build of towers of height zero or one whose bases are plaquettes in $L^n \mathbb{Z}^2$, $n \in \{0, \ldots, N\}$. $N + 1$ is the total number of hierarchies and L is the blocklength. We will label a plaquette in $L^n \mathbb{Z}^2$ by the L-adic expansion of its lower left edgepoint y. More precisely, we will write $(0, \ldots, 0, y_n, \ldots, y_{N-1})$ with $y_i \in S \equiv \{0, \ldots, L-1\}^2$ to denote the plaquette whose lower left edgepoint is $y = \sum_{i=n}^{N-1} y_i L^i$. To each such label is then associated a partial height variable $h^{(n)}_{(0,\ldots,0,x_n,\ldots,x_{N-1})} \in \{0, 1\}$. It is the latter condition that makes the model a 'wedding cake', which is the only difference to [5,6,8], where the partial height variables were allowed to take values in $\mathbb{Z}$. Thus, the state space of the model is $\Omega_{L,N} = \{(h^{(n)}_{(0,\ldots,0,x_n,\ldots,x_{N-1})})_{n=0,\ldots N; x_n,\ldots,x_{N-1} \in S} \,|h^{(n)}_{(0,\ldots,0,x_n,\ldots,x_{N-1})} \in \{0, 1\}\}$ and we will write $\underline{h}, \underline{h}', \ldots$ to denote its elements. We have employed here a notation slightly different from [5,6,8] which will be convenient for the mean field limit.

Next we must specify the disorder entering into the model. We treat here disorder of random bond type (see [8]). We assume thus that we are given centered i.i.d.

Gaussian random variables $J_{(x_0,x_1,\ldots)}(h)$, for different $h \in \mathbb{Z}$ and $x_n \in \{0,1,2,\ldots,\}^2 \equiv S_\infty$, $n \in \{0,1,2,\ldots\}$, with variance $\mathbb{E}\left[J^2_{(x_0,x_1,\ldots)}(h)\right] = \sigma^2$. x_n runs in the infinite set S_∞, since we want to take the mean field limit $L \uparrow \infty$ later.

As in [5,6,8], the energy of a surface described by $\underline{h}$ is defined by

$$E_{L,N,J}(\underline{h}) = \sum_{n=0}^{N} \sum_{x_n,\ldots,x_{N-1}\in S} h^{(n)}_{(0,\ldots,0,x_n,\ldots,x_{N-1})} L^n + \sum_{x_0,\ldots,x_{N-1}\in S} J_{(x_0,\ldots,x_{N-1})}(H_{(x_0,\ldots,x_{N-1})}) \tag{1}$$

Here $H_{(x_0,\ldots,x_{N-1})} = \sum_{n=0}^{N} h^{(n)}_{(0,\ldots,0,x_n,\ldots,x_{N-1})}$ is the total height over a base point $(x_0,\ldots,x_{N-1})$.

For given realisation of the J's, we will write $\underline{h}^*(L,N,J)$ to denote the (a.s. unique) ground state height configuration. Thus

$$E_{L,N,J}(\underline{h}^*(L,N,J)) = \inf_{\underline{h}\in\Omega_{L,N}} E_{L,N,J}(\underline{h}) \tag{2}$$

The *mean field model for the ground state* with $N+1$ hierarchies is now obtained by taking the limit $L \uparrow \infty$. Its state space is thus $\Omega_{\infty,N} = \{(h^{(n)}_{(0,\ldots,0,x_n,\ldots,x_{N-1})})_{n=0,\ldots N;x_n,\ldots,x_{N-1}\in S_\infty} \mid h^{(n)}_{(0,\ldots,0,x_n,\ldots,x_{N-1})} \in \{0,1\}\}$.

The ground state is then the random field $(h^{(\infty,n)}_{(0,\ldots,0,x_n,\ldots,x_{N-1})})_{n=0,1,\ldots,N;x_n,\ldots,x_{N-1}\in S_\infty}$ whose finite dimensional marginals are obtained from the finite-L marginals by the weak limit $L \uparrow \infty$. The latter means of course that, for finite all finite collections of indices $n_1,\ldots,n_k$, and $\delta_1,\ldots,\delta_k \in \{0,1\}$

$$\begin{aligned} &\mathbb{P}\left[h^{(\infty,n_i)}_{(0,\ldots,0,x_{n_i},\ldots,x_{N-1})} = \delta_i,\ \forall i=1,\ldots,k\right] \\ &\qquad = \lim_{N\uparrow\infty} \mathbb{P}\left[h^{*,(n_i)}_{(0,\ldots,0,x_{n_i},\ldots,x_{N-1})}(L,N,J) = \delta_i,\ \forall i=1,\ldots,k\right] \end{aligned} \tag{3}$$

3. The renormalization group transformation

The renormalization group equations can of course be obtained from [5,6] by restricting the partial heights to the values zero and one. For the convenience of the reader we include here its derivation. For the ground state analysis, the RG transformation is nothing but the search for the energy minimum hierarchy after hierarchy. This means more precisely, that we introduce recursively *renormalized* energy functions by

$$\begin{aligned} &E^{(0)} = E_{L,N,J} \\ &E^{(n+1)}\left(\left(h^{(k)}_{(0,\ldots,0,x_k,\ldots,x_{N-1})}\right)_{k=n+1,\ldots,N;x_i\in S}\right) \\ &\quad = L \inf_{\left(h^{(n)}_{(x_n,\ldots,x_{N-1})}\right)_{x_i\in S}} E^{(n)}\left(\left(h^{(k)}_{(0,\ldots,0,x_k,\ldots,x_{N-1})}\right)_{k=n,\ldots,N;x_i\in S}\right) \end{aligned} \tag{4}$$

Due to the hierarchical nature of the model, these renormalized energy functions $E^{(k)}$ then have the same form of dependence on the height variables as the initial one, if one introduces new ('renormalized') random variables. With the recursive definition

$$
\begin{aligned}
J^{(0;L)}_{(x_0,\dots,x_{N-1})}(H) &= J_{(x_0,\dots,x_{N-1})}(H) \\
J^{(n+1;L)}_{(x_{n+1},\dots,x_{N-1})}(H) &= \frac{1}{L}\sum_{x_n\in S}\left(\inf_{h=0,1}\left(h+J^{(n;L)}_{(x_n,x_{n+1},\dots,x_{N-1})}(H+h)\right)\right. \\
&\qquad \left. - I\!E\left[\inf_{h=0,1}\left(h+J^{(n;L)}_{(x_n,\dots,x_{N-1})}(H+h)\right)\right]\right)
\end{aligned}
\tag{5}
$$

we have

$$
\begin{aligned}
&E^{(n)}\left(\left(h^{(k)}_{(0,\dots,0,x_k,\dots,x_{N-1})}\right)_{k=n,\dots,N;x_i\in S}\right) \\
&=\sum_{k=n}^{N}\sum_{x_k,\dots,x_{N-1}\in S} h^{(n)}_{(0,\dots,0,x_k,\dots,x_{N-1})}L^{k-n}+\sum_{x_k,\dots,x_{N-1}\in S} J^{(n;L)}_{(x_n,\dots,x_{N-1})}(H^{(n)}_{(x_n,\dots,x_{N-1})})+\text{Const}
\end{aligned}
\tag{6}
$$

where $H^{(n)}_{(x_n,\dots,x_{N-1})}=\sum_{k=n}^{N} h^{(k)}_{(0,\dots,0,x_k,\dots,x_{N-1})}$. 'Const' comes from the subtraction of the expectation in (5) and does not depend on the height variables. It is therefor of no importance for us. Note that we have made explicite the L-dependence in our notation.

To recall the renormalization group algorithm for the determination of the groundstate of $E_{L,N,J}$, let us now briefly write $h^{(k)}$ for the collection of all height variables of hierarchy k. Similarly we write $h^{(k,*)}$ for the variables taken from the groundstate configuration. Now, the renormalization group strategy consists of the following steps: First 'compute' the infimum $h^{(N,*)}$ of the function $h^{(N)}\mapsto E^{(N)}\left(h^{(N)}\right)$. Next, *given* the knowledge of $h^{(N,*)}$, compute the minimal configuration $h^{(N-1,*)}$ of the function $h^{(N-1)}\mapsto E^{(N-1)}\left(h_0^{(N,*)},h^{(N-1)}\right)$. In this way proceed from coarser to finer hierarchies, until, in the last step, $h^{(0,*)}$ is computed.

This procedure is very similar in spirit to renormalization group algorithms used in image processing, see [12]. Here the word 'compute' indicates, that we have to be able in some sense to control the above recursion relations for the stochastic variables in some sense. So, let us focus on (5) now.

Due to the fact that we are in interface dimensions $d=2$, the normalisation $1/L$ in (5) is just the normalisation of the central limit theorem. Thus, if we replace the 'inf' by its value for $h=0$ the renormalized variables are just Gaussians of the same variance as the initial ones, showing that $d=2$ is the marginal dimension. We must therefor, *in contrast to earlier studies of this model,* carefully investigate the effects of the *inf*- expression. Despite the simplicity of the model, this seems to be difficult for general L, having no detailed knowledge about the distribution of the J-processes. Observe however, that, how larger L becomes, the more approaches their distribution a Gaussian and the latter is completely described by its covariance structure. This has been our motivation for passing to the mean field limit $L\uparrow\infty$. The reader may also note at this point, that the recursion relation (5) maps processes that are stationary w.r.t. the shift $h\to h+k$ to processes of the same type.

Thus, in the limit $L \uparrow \infty$, the recursion relations can be described on the level of a renormalization of an infinite vector of covariances. More precisely we have the following

PROPOSITION:

(i) With $L \uparrow \infty$ the random processes $\{J^{(n;L)}_{(x_n,\dots,x_{N-1})}(h)\}_{h\in\{0,1,2,\dots\}}$ converge in distribution to Gaussian processes $\{J^{(n)}_{(x_n,\dots,x_{N-1})}(h)\}_{h\in\{0,1,2,\dots\}}$ which are stationary w.r.t. the shift $h \to h+k$. Their vector of covariances,

$$a_h^{(n)} \equiv I\!E[J^{(n)}_{(x_n,\dots)}(h_0)J^{(n)}_{(x_n,\dots)}(h_0+h)], \qquad h\in\{0,1,\dots\} \tag{7}$$

is computed from the recursion relations

$$\begin{aligned} a_0^{(n+1)} &= a_0^{(n)} - 2\left(a_0^{(n)}-a_1^{(n)}\right) F\left(\frac{1}{\sqrt{2\left(a_0^{(n)}-a_1^{(n)}\right)}}, 1\right) \\ a_h^{(n+1)} &= a_h^{(n)} - 2\left(a_0^{(n)}-a_1^{(n)}\right) F\left(\frac{1}{\sqrt{2\left(a_0^{(n)}-a_1^{(n)}\right)}}, \frac{-a_{h-1}^{(n)}+2a_h^{(n)}-a_{h+1}^{(n)}}{2\left(a_0^{(n)}-a_1^{(n)}\right)}\right) \qquad h\geq 1 \end{aligned} \tag{8}$$

with initial condition $a_h^{(0)} = \sigma^2\delta_{h,0}$. Here the function F is given by

$$F(\alpha,q) \equiv qP(\alpha) - I\!E\left[(Q+\alpha)\,1_{Q<-\alpha}\,(Q_1+\alpha)\,1_{Q_1<-\alpha}\right] + \left(I\!E\left[(Q+\alpha)\,1_{Q<-\alpha}\right]\right)^2 \tag{9}$$

for a twodimensional Gaussian random vector (Q,Q_1) with $E[Q]=E[Q_1]=0$, $E[Q^2]=E[Q_1^2]=1$, $E[QQ_1]=q$. $P(\alpha)\equiv I\!P[Q<-\alpha]$ is the error function.

(ii) With $L\uparrow\infty$ the partial heights of the groundstate $\underline{h}^*(L,N,J)$ converge to the independent Bernoulli variables $h^{(\infty,n)}_{(0,\dots,0,x_{n_i},\dots,x_{N-1})}$ such that

$$I\!P\left[h^{(\infty,n)}_{(0,\dots,0,x_{n_i},\dots,x_{N-1})}=1\right] = P\left(\frac{1}{\sqrt{2\left(a_0^{(n)}-a_1^{(n)}\right)}}\right) \tag{10}$$

In particular, their distribution is independent of the total number of hierarchies $N+1$.

Proof: From the multidimensional central limit theorem follows that the limiting process is Gaussian and hence completely described by its covariances (see e.g. [15]). It is easy to see that covariances between J's of different hierarchies vanish with $L\uparrow\infty$. Hence it remains to compute the covariance of $\inf_{h=0,1}\left(h+J^{(n)}_{(x_n,\dots,x_{N-1})}(H+h)\right)$ between different values of H at the same site and the same hierarchie. Here we can assume that $\left(J^{(n)}_{(x_n,\dots,x_{N-1})}\right)_{h\in\{0,1,\dots\}}$ is Gaussian. This computation is given in the following lemma from which the 'lifted' recursion relations (8) and thus part (i) of the proposition follow iteratively by successive rescaling of the occuring variables by $\sigma^{(n)}=\frac{1}{c}$.

LEMMA: *Assume that* (K_0, K_1, K_2, K_3) *is a Gaussian random vector of mean zero and covariances* $\mathbb{E}K_0^2 = \mathbb{E}K_1^2 = \mathbb{E}K_2^2 = \mathbb{E}K_3^2 = 1$, $\mathbb{E}K_0K_1 = \mathbb{E}K_2K_3 = r$, $\mathbb{E}K_1K_2 = r_1$, $\mathbb{E}K_0K_2 = \mathbb{E}K_1K_3 = r_2$, $\mathbb{E}K_0K_3 = r_3$.

Define the variables

$$\begin{aligned} A &\equiv \inf\{K_0, K_1 + c\} \\ B &\equiv \inf\{K_2, K_3 + c\} \end{aligned} \tag{11}$$

Then

$$\mathbb{E}\left[AB\right] - \mathbb{E}\left[A\right]\mathbb{E}\left[B\right] = r_2 - w^2 F\left(\frac{1}{w}, q\right) \tag{12}$$

where

$$w \equiv \sqrt{\mathbb{E}\left[(K_1 - K_0)^2\right]} = \sqrt{2-2r} \tag{13}$$

and q *is the 'scalar product'*

$$q \equiv \frac{1}{w^2}\mathbb{E}\left[(K_1 - K_0)(K_3 - K_2)\right] = \frac{-r_1 + 2r_2 - r_3}{2(1-r)} \tag{14}$$

Proof (of Lemma): We first write A and B in the form

$$\begin{aligned} A &= K_0 + (K_1 - K_0 + c)\, 1_{K_1 - K_0 < -c} \\ B &= K_2 + (K_3 - K_2 + c)\, 1_{K_3 - K_2 < -c} \end{aligned} \tag{15}$$

It is useful to decompose the $\mathcal{L}^2$-random variables K_0, K_2 in the following way. We introduce

$$\begin{aligned} Q &\equiv \frac{1}{w}(K_1 - K_0) \\ Q_1 &\equiv \frac{1}{w}(K_3 - K_2) \end{aligned} \tag{16}$$

Then we define random variables $K_0^\perp, K_2^\perp$ by writing

$$\begin{aligned} K_0 &\equiv \langle K_0, Q_1\rangle Q_1 + K_0^\perp \\ K_2 &\equiv \langle K_2, Q\rangle Q + K_2^\perp \end{aligned} \tag{17}$$

Note that $K_0^\perp$ is a Gaussian orthogonal to Q_1 and $K_2^\perp$ is orthogonal to Q (w.r.t. the scalar product $\langle X, Y\rangle \equiv \mathbb{E}(X,Y)$). Introducing $\alpha \equiv \frac{c}{w}$ we may then write the variables A and B in the form

$$\begin{aligned} A &= K_0^\perp + \langle K_0, Q_1\rangle Q_1 + w\,(Q + \alpha)\, 1_{Q < -\alpha} \\ B &= K_2^\perp + \langle K_2, Q\rangle Q + w\,(Q_1 + \alpha)\, 1_{Q_1 < -\alpha} \end{aligned} \tag{18}$$

Using the centeredness of $Q, Q_1, K_0^\perp, K_2^\perp$ we thus obtain for the expectation

$$\mathbb{E}\left[A\right] = \mathbb{E}\left[B\right] = w\mathbb{E}\left[(Q+\alpha)\, 1_{Q<-\alpha}\right] \tag{19}$$

Now observe that from $\langle Q_1, K_0^\perp\rangle = 0$ follows that $Q_1, K_0^\perp$ are *independent* centered Gaussians. Using the independence follows e.g. $\mathbb{E}\left[K_0^\perp 1_{Q_1<-\alpha}\right] = \mathbb{E}\left[K_0^\perp\right]\mathbb{E}\left[1_{Q_1<-\alpha}\right] = 0$; in this way we obtain

$$\begin{aligned} \mathbb{E}\left[AB\right] &= \mathbb{E}\left[K_0K_2\right] + \left(\langle K_0, Q_1\rangle + \langle K_2, Q\rangle\right) w\mathbb{E}\left[Q\,(Q+\alpha)\, 1_{Q<-\alpha}\right] \\ &\quad + w^2\mathbb{E}\left[(Q+\alpha)\, 1_{Q<-\alpha}\,(Q_1+\alpha)\, 1_{Q_1<-\alpha}\right] \end{aligned} \tag{20}$$

Using $(\langle K_0, Q_1\rangle + \langle K_2, Q\rangle)\, w = r_1 - 2r_2 + r_3 = -qw^2$ and $\mathbb{E}\left[Q\,(Q+\alpha)\, 1_{Q<-\alpha}\right] = P(\alpha)$ we finally have

$$\mathbb{E}\,[AB] - \mathbb{E}\,[A]^2 = \mathbb{E}\,[K_0 K_2] - w^2 \Bigg(qP(\alpha) - \mathbb{E}\left[(Q+\alpha)\, 1_{Q<-\alpha}\, (Q_1+\alpha)\, 1_{Q_1<-\alpha}\right] + \left(\mathbb{E}\left[(Q+\alpha)\, 1_{Q<-\alpha}\right]\right)^2 \Bigg) \tag{21}$$

which proves the lemma. Informally speaking, we thus see that the $P(\alpha)$-term stems from the event that the inf is taken for partial height one in precisely one of the 'renormalized' variables. In fact, it gives the main part of the function F. The first term in the second line comes from the event that partial height one realizes both infs and the last term comes from the centering. ◊

To prove part (ii) of the proposition just note that from the independence of $J^{(n)}$, $J^{(m)}$ for $n \neq m$ follows that

$$\mathbb{P}\left[h^{(\infty,n)}_{(0,\ldots,0,x_{n_i},\ldots,x_{N-1})} = 1\right] = \mathbb{P}\left[J^{(n)}_{(0,\ldots)}(0) < 1 + J^{(n)}_{(0,\ldots)}(1)\right] = P\left(\frac{1}{\sqrt{2\left(a_0^{(n)} - a_1^{(n)}\right)}}\right) \tag{22}$$

◊

We would like to conclude this chapter with the remark that the case $\beta < \infty$ is only nontrivial for $L < \infty$. In fact, if we start with a recursion relation for finite temperature this leads to a temperature renormalization of the form $\beta' = L\beta$ (compare [5,6,8]). Thus for $L \uparrow \infty$, the temperature becomes zero after the first step of the renormalization and, with no loss of generality, we are led back to the present analysis.

4. Analysis of renormalization group equations

Let us note first that all $J^{(n)}_{(x_n,\ldots,x_{N-1})}(h)$ are monotone functions of each of the initial i.i.d. random variables $J^{(0)}_{(x_n,\ldots,x_{N-1})}(h')$. Thus, from the FKG inequality follows that the covariances $a_h^{(n)}$ are nonnegative for all n and h. Note also that $F(\alpha, q=0) = 0$ and $\frac{\partial}{\partial q} F(\alpha, q) \geq 0$ (as is easily checked). Therefore, due to the presence of the lattice laplacian in the q-argument, the renormalization group equation (8) is reminiscent of a discrete nonlinear diffusion equation. A complete analysis of this equation including the computation of the asymptotics of its solution is difficult; at first sight it might not be clear that the solution exists for all n $(\uparrow \infty)$.

However, using explicite properties of the function F and the fact that the a_h's are covariances we can obtain by elementary arguments the qualitative information of interest, as stated in

THEOREM (MEAN FIELD MODEL): *Let the initial variance $\sigma^2 = \mathbb{E}[J_{(x_n,\ldots)}(h)^2]$ be positive. Denote by $H_0^\infty(N)$ the total height of the ground state at the base point 0 for the model with $N+1$ hierarchies. Then* *(i) The renormalized variances* $a_0^{(n)}$ *satisfy* $\lim_{n\uparrow\infty} a_0^{(n)} = 0$

(ii) $\lim_{N\uparrow\infty} H_0^\infty(N) = \infty$ *a.s.*

Remarks: Physically speakint it is the response of 'small' contours to the randomness leads to a screening of the randomness which causes the variance $a_0^{(n)}$ of the effective disorder variables $J^{(n)}$ to shrink to zero on large scales (i.e. when the index of the hierarchy n goes to infinity). Though, this effect is not strong enough to lead to a bounded interface in the infinite volume. The result is 'global' in the sense that we do not need the initial variance σ^2 to be small. Note that the theorem only gives a qualitative result. The determination of the asymptotics of the divergence of the average height at a given point as a function of the system size would require a much more elaborate analysis.

PROOF: Observe that the fixed points of the recursion relation (8) are precisely those given by the homogenous configurations $a_h \equiv a$ for all h. Clearly these are fixed points; in fact, there are no others: From the first equation of (8) follows that a fixed point $(a_h)_{h=0,1,\dots}$ must satisfy $a_0 = a_1$, since $F(\alpha, 1) > 0$. Since the a_h's are covariances for a stationary Gaussian process it follows from this that $a_h = a_0$.

Now, from the first equation of (8) and $a_0 \geq a_1$ (the latter holds since the a_h's are covariances) we see that $a_0^{(k)}$ is a monotonically decreasing function of the 'time' k. Hence the limit $\lim_{k\uparrow\infty} a_0^{(k)} \geq 0$ exists (and could possibly be larger than zero). We will now show that in fact it equals zero.

From the estimate $|F(\alpha, q)| \leq F(\alpha)$ follows that the time variation at any fixed 'site' h is bounded by the time variation at the site $h = 0$, i.e.

$$\sum_{k=n}^{\infty} \left| a_h^{(k+1)} - a_h^{(k)} \right| \leq \sum_{k=n}^{\infty} 2v^{(k)} F\left(\frac{1}{\sqrt{2v^{(k)}}}, 1 \right) = a_0^{(n)} - \lim_{k\uparrow\infty} a_0^{(k)} \tag{23}$$

with $v^{(k)} \equiv a_0^{(k)} - a_1^{(k)}$. Next we note that

$$\lim_{h\uparrow\infty} \sum_{k=0}^{\infty} \left| a_h^{(k+1)} - a_h^{(k)} \right| = 0 \tag{24}$$

This follows from

$$\sum_{k=0}^{\infty} \left| a_h^{(k+1)} - a_h^{(k)} \right| = \sum_{k=h-1}^{\infty} \left| a_h^{(k+1)} - a_h^{(k)} \right| \leq a_0^{(h-1)} - \lim_{k\uparrow\infty} a_0^{(k)} \tag{25}$$

The equality in (25) stems from the fact that $F(\alpha, q = 0) = 0$ and hence $a_h^{(k)} = 0$ for $h > k$.

Now, for all $h \geq 1$,

$$\left| a_h^{(k)} - a_{h+1}^{(k)} \right| \leq \left| a_{h-1}^{(k)} - 2a_h^{(k)} + a_{h+1}^{(k)} \right| + \left| a_{h-1}^{(k)} - a_h^{(k)} \right| \leq 2v^{(k)} + \left| a_{h-1}^{(k)} - a_h^{(k)} \right| \tag{26}$$

Here the second inequality stems from the fact that the a_h's are covariances. (It is equivalent with $|q| \leq 1$ for q as defined in (14).) Observe that $v^{(k)} \to 0$ (this follows from the convergence of the sum $\sum_{k=n}^{\infty} 2v^{(k)} F\left(1/\sqrt{2v^{(k)}}, 1\right)$. Hence we conclude from (26) by induction that $\left|a_h^{(k)} - a_{h+1}^{(k)}\right| \to 0$ with $k \uparrow \infty$ for all h. Subsequently, for any fixed h, we have

$$\lim_{k\uparrow\infty} \left|a_0^{(k)} - a_h^{(k)}\right| = 0 \tag{27}$$

Thus

$$\lim_{k\uparrow\infty} a_0^{(k)} = \lim_{k\uparrow\infty} a_h^{(k)} \leq \sum_{k=0}^{\infty} \left|a_h^{(k+1)} - a_h^{(k)}\right| \tag{28}$$

Taking the limit $h \uparrow \infty$ and using (24) now proves part (i) of the theorem.

To prove the nonsummability of the expectation of the partial height functions and thus part (ii) of the theorem recall that

$$\mathbb{E}\left[H_0^{\infty}(N)\right] = \sum_{k=0}^{N} P\left(\frac{1}{\sqrt{2v^{(k)}}}\right) \tag{29}$$

Now, it is easy to verify the bound $F(\alpha, q = 1) \leq Const\, P(\alpha)$ with some numerical Const. Hence, using part (i) of the theorem, we may write

$$a_0^{(n)} = \sum_{k=n}^{\infty} 2v^{(k)} F\left(\frac{1}{\sqrt{2v^{(k)}}}, 1\right) \leq 2Const \sum_{k=n}^{\infty} v^{(k)} P\left(\frac{1}{\sqrt{2v^{(k)}}}\right) \leq 2Const\, a_0^{(n)} \sum_{k=n}^{\infty} P\left(\frac{1}{\sqrt{2v^{(k)}}}\right) \tag{30}$$

In the last inequality we have used the fact that $a_0^{(n)}$ is monotonically decreasing. From (30) thus follows the uniform bound $\sum_{k=n}^{\infty} P\left(1/\sqrt{2v^{(k)}}, 1\right) \geq \frac{1}{2Const}$ for all n which proves (ii).◇

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6. References

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