Sufficient Conditions for Minimality of a Nonlinear Realization via Controllability and Observability Functions

Scherpen, Jacquelien M.A.; Gray, W. Steven

Published in:
Proceedings of the 1998 American Control Conference

Proceedings of the American Control Conference
Philadelphia, Pennsylvania • June 1998

Sufficient Conditions for Minimality of a Nonlinear Realization via Controllability and Observability Functions

Jacquelien M.A. Scherpen* and **W. Steven Gray****
* Delft University of Technology, Fac. ITS, Dept. Electrical Engineering
P.O. Box 5031, 2600 GA Delft, The Netherlands
** Dept. Electrical & Computer Engineering, Old Dominion University
Norfolk, Virginia 23529-0246, U.S.A.
* J.M.A.Scherpen@et.tudelft.nl, ** gray@ece.odu.edu

Abstract

In this paper we develop a set of sufficient conditions in terms of controllability and observability functions under which a given state space realization of a formal power series is minimal. Specifically, it will be shown that positivity of these functions, plus a few technical conditions, implies minimality. In doing so, connections are established between Hamilton-Jacobi type optimal control theory and the well known necessary and sufficient conditions for minimality in terms of Kalman type rank conditions on the accessibility and observability distributions.

1 Introduction

The problem of determining when the dimension of a state space realization for a given input-output map is minimal is a fundamental problem in systems. Its connects to many other topics in realization theory like similarity invariance, controllability and observability properties, model reduction and balanced realizations. The theory is quite complete in the case of linear systems. For example, it is well known that minimality is equivalent to joint controllability and observability, and for stable systems, this is further equivalent to the positive definiteness of the controllability and observability Gramians. These Gramian matrices naturally appear in balanced realization theory, and are related to optimal control problems. In the nonlinear case, minimality theory is not nearly as well developed. For example, there are several existing theories for minimality depending on the exact nature in which the input-output mapping is described, i.e., in terms of a set of input-output differential equations (see [18] and the references therein), a Volterra series [6, 7, 8] or a formal power series/Chen-Fliess functional expansion [6]. At present, the exact connections between these different approaches are not completely understood. Furthermore, motivated by the linear case, we might expect that minimality should have connections to the nonlinear extensions of the Gramians, which have been developed for nonlinear balancing [1]-[4],[10]-[13]. But these connections are also largely unknown at present.

The specific purpose of this paper is to develop a set of sufficient conditions in terms of controllability and observability functions under which a given state space realization of a formal power series is minimal. Specifically, it will be shown that positivity of these functions, plus a few technical conditions, implies minimality. Of course there exists well known necessary and sufficient conditions for minimality in terms of Kalman type rank conditions on the accessibility and observability distributions. So the novelty of the approach taken here is in establishing a connection between these differential geometric type minimality conditions and properties of functions that are connected with Hamilton-Jacobi type optimal control theory. As an added benefit, it also seems possible to use this new minimality characterization to further develop the nonlinear notions of similarity invariance [13], the Kalman decomposition [12], and Hankel operators.

The paper is organized as follows. In Sections 2, the background material pertaining to all the relevant subjects is briefly reviewed, specifically: the main definitions and rank conditions associated with reachability and observability, the definitions and known properties of controllability and observability functions, and minimality theory for state space realizations of formal power series. In Section 3.1 we then develop a relationship between positivity of the controllability function and the accessibility rank condition. The analogous connections between positivity of the observability function and the observability rank condition are covered in Section 3.2. The main result of the paper involving minimality, plus some concluding remarks, are presented in the final section.

The mathematical notation used throughout is fairly standard. Vector norms are represented by $\|x\| = \sqrt{x^T x}$ for $x \in \mathbb{R}^n$. $L_2(a,b)$ represents the set of Lebesgue measurable functions, possibly vector-valued, with finite L_2 norm $\|x\|_{L_2} = \sqrt{\int_a^b \|x(t)\|^2\, dt}$. If $L : \mathbb{R}^n \mapsto \mathbb{R}$ is a differentiable function, then its partial derivative $\frac{\partial L}{\partial x}$ will be the row vector of partial derivatives $\frac{\partial L}{\partial x_i}$ where $i = 1,\ldots,n$.

2 Background

2.1 Controllability and observability functions

Controllability and observability functions play an important role in balancing and model reduction for stable nonlinear systems [10, 12]. In this section we give a brief review of the results that are important for the minimality theory presented in Section 3.

Consider a smooth, i.e., C^∞, nonlinear system of the form

$$\dot{x} = f(x) + g(x)u, \quad y = h(x) \tag{1}$$

where $u = (u_1,\ldots,u_m) \in \mathbb{R}^m$, $y = (y_1,\ldots,y_p) \in \mathbb{R}^p$ and $x = (x_1,\ldots,x_n)$ are local coordinates for a smooth state space manifold denoted by M. Throughout we assume that the system has an equilibrium. Without loss of generality we take this equilibrium to be at 0, i.e. $f(0) = 0$, and we also take $h(0) = 0$.

Definition 2.1 [10] The *controllability* and *observability function* of a system (1) are defined as

$$L_c(x_0) = \min_{\substack{u \in L_2(-\infty,0) \\ x(-\infty)=0,x(0)=x_0}} \frac{1}{2}\int_{-\infty}^{0} \| u(t) \|^2 \, dt \tag{2}$$

$$L_o(x_0) = \frac{1}{2}\int_{0}^{\infty} \| y(t) \|^2 \, dt, \tag{3}$$

for $x(0) = x_0,\ u(t) \equiv 0,\ 0 \leq t < \infty$, respectively. □

The value of the controllability function at x_0 is the minimum amount of control energy required to reach the state x_0, and the value of the observability function at x_0 is the amount of output energy generated by x_0. We assume throughout that L_c and L_o are *finite*. Also, for the rest of this paper we assume that L_c and L_o are *smooth* functions of x.

Theorem 2.2 [10] *If $f(x)$ is asymptotically stable on a neighborhood W of 0, then for all $x \in W$, $L_o(x)$ is the unique smooth solution of the following Lyapunov type equation:*

$$\frac{\partial L_o}{\partial x}(x)f(x) + \frac{1}{2}h^T(x)h(x) = 0, \qquad L_o(0) = 0. \tag{4}$$

Furthermore for all $x \in W$, $L_c(x)$ is the unique smooth solution of the following Hamilton-Jacobi equation:

$$\frac{\partial L_c}{\partial x}(x)f(x) + \frac{1}{2}\frac{\partial L_c}{\partial x}(x)g(x)g^T(x)\frac{\partial^T L_c}{\partial x}(x) = 0, \qquad L_c(0) = 0 \tag{5}$$

with $-(f(x)+ g(x)g^T(x)\frac{\partial^T L_c}{\partial x}(x))$ *asymptotically stable on* W. □

Remark 2.3 [12] If we assume that $f(x)$ is asymptotically stable and that (4) has a smooth solution, it then follows that L_o, as in (3), exists, i.e., is finite. Furthermore, if we assume that (5) has a smooth solution L_c that is anti-stabilizing (i.e., $-(f(x)+g(x)g(x)^T\frac{\partial^T L_c}{\partial x}(x))$ is asymptotically stable), it follows that L_c, as in (2), exists. □

Theorem 2.4 [10] *Assume $f(x)$ is asymptotically stable on a neighborhood W of 0 and (5) has a smooth solution $\bar{L}_c$ on W. Then $\bar{L}_c(x) > 0$ for $x \in W,\ x \neq 0$, if and only if $-(f(x)+g(x)g^T(x)\frac{\partial^T \bar{L}_c}{\partial x}(x))$ is asymptotically stable on W.* □

For the definitions of (strong) accessibility and observability, see, e.g.,[5], [6], [7], or [9].

Definition 2.5 Consider the system (1).
• The system is *zero-state observable* if any trajectory where $u(t) \equiv 0, y(t) \equiv 0$ implies $x(t) \equiv 0$, i.e., for all $x \in M$, $h(\varphi(t,0,x,0)) = 0, t \geq 0 \Rightarrow \varphi(t,0,x,0) = 0, t \geq 0$.
• The system (1) is *locally zero-state observable*, if there exists a neighborhood W of 0 such that for all $x \in W$, $h(\varphi(t,0,x,0)) = 0$, for all $t \geq 0 \Rightarrow \varphi(t,0,x,0) = 0$ for all $t \geq 0$. □

Remark 2.6 The definitions are standard, but usually given in the context where only piecewise constant inputs are admissible. However, the effects of approximations of more general inputs by piecewise constant inputs has been considered in earlier work [16], and statements about these properties holding for larger classes of inputs can be found in [15, 17]. □

The following definition is an addition to the well-known definitions of the (strong) accessibility distribution and observability codistribution [7].

Definition 2.7 Consider the system (1).
• The *zero-observation space* $\mathcal{O}_0$ is the linear space of functions on M containing $h_1,\ldots,h_p$ and all repeated Lie derivatives $L_f^k h_j$, $j \in 1,\ldots,p$, $k = 1,2,\ldots$.
• The *zero-observability codistribution* $d\mathcal{O}_0$ is given by $d\mathcal{O}_0(q) = \text{span}\{dH(q)|\ H \in \mathcal{O}_0\}$, where $q \in M$. □

Theorem 2.8 [7] *Consider the system (1). Let C and C_0 denote the accessibility and strong accessibility distributions, respectively, and $d\mathcal{O}$ and $d\mathcal{O}_0$ be the observability and zero-observability co-distributions, respectively.*
• *If $\dim C(x_0) = n$, then the system is locally accessible from x_0.*
• *If $\dim C_0(x_0) = n$, then the system is locally strongly accessible from x_0.*
• *If $\dim d\mathcal{O}(x_0) = n$, then the system is locally observable at x_0.*
• *If $\dim d\mathcal{O}_0(0) = n$, then the system is locally zero-state observable.* □

Remark 2.9 A consequence of this theorem and Definition 2.7 is that local zero-state observability implies local observability at 0. Furthermore, it follows that local strong accessibility at x_0 implies local accessibility at x_0. □

The following theorems are closely related to results that appear in [5] and [9]. They reveal important properties of the system (1) in terms of the relationships between zero-state observability, positive definiteness of the observability function, and asymptotic stability of the equilibrium at 0.

Theorem 2.10 [10] *Assume $f(x)$ is asymptotically stable on a neighborhood W of 0. If the system (1) is zero-state observable on W, then $L_o(x) > 0$, $\forall x \in W, x \neq 0$.* □

Theorem 2.11 [10] *If the system (1) is zero-state observable and L_o is well-defined, smooth and positive definite, then the system $\dot{x} = f(x)$ is locally asymptotically stable. If L_o is proper, then $\dot{x} = f(x)$ is globally asymptotically stable.* □

2.2 A more general observability function

Local zero-state observability is certainly more restrictive than local observability. In order to extend the previous results to a more general setting, a more general observability function needs to be considered in which the input plays a direct role. Given the system (f,g,h), the corresponding homogeneous system is denoted by $(f,\tilde{g},h)$, where $\tilde{g}(x) = g(x) - g(0)$. It is easily shown that $(f,\tilde{g},h)$ and it homogeneous counterpart always have the same observability spaces, and thus have basically the same observability properties. Consider the following definition.

Definition 2.12 [1]-[4] The *natural observability function* for the system (1) is defined as

$$L_o^N(x_0) = \max_{\substack{u \in L_2(0,\infty),\ \|u\|_{L_2} \leq \alpha \\ x(0)=x_0,\ x(\infty)=0}} \frac{1}{2}\int_{0}^{\infty} \|\tilde{y}(t)\|^2 \, dt, \tag{6}$$

where $\alpha \geq 0$ is a fixed real number, and $\tilde{y}$ is the output response of the corresponding homogeneous system. □

Clearly $L_o^N(x_0)$ is the maximum output energy one could expect from initializing the homogeneous system at $x(0) = x_0$ and applying any input with energy bounded by α. When $\alpha = 0$, we have the observability function given in Definition 2.1. The following theorem provides a defining equation for L_o^N analogous to equation (4).

Theorem 2.13 [1]-[4] *For any fixed x_0 in a neighborhood W of 0, $L_o^N(x_0)$ is uniquely determined by evaluating the smooth solution of*

$$\frac{\partial \bar{L}_o}{\partial x} f + \frac{1}{2} h^T h - \frac{1}{2}\mu^{-1}(x_0)\frac{\partial \bar{L}_o}{\partial x}\bar{g}\bar{g}^T\frac{\partial \bar{L}_o}{\partial x}^T = 0, \tag{7}$$

with $\bar{L}_o(0,x_0) = 0$ at $x = x_0$ and under the assumptions that a smooth solution $\bar{L}_o(x,x_0)$ exists on W, and 0 is an asymptotically stable equilibrium on W of $\bar{f} := (f - \mu^{-1}(x_0)\bar{g}\bar{g}^T\frac{\partial \bar{L}_o}{\partial x}^T(\cdot,x_0))$ with $\mu(x_0) := -\|\bar{g}^T(\phi)\frac{\partial \bar{L}_o}{\partial x}^T(\phi,x_0)\|_{L_2}$ a negative real number when $\dot{\phi} = \bar{f}(\phi)$, $\phi(0) = x_0$. □

The following theorem describes a sufficient condition for having $L_o^N(x) > 0$ on $\forall x \in W, x \neq 0$, in terms of an observability condition on the set of inputs $B_\alpha := \{u \in L_2[0,\infty) : \|u\|_{L_2} \leq \alpha\}$.

Theorem 2.14 [1]-[4] *Suppose 0 is an asymptotically stable equilibrium of the system (f,g,h) on a neighborhood W of 0 and $h(0) = 0$. If the system (f,g,h) is observable with respect to B_α then $L_o^N(x) > 0$ when $x \in W$, $x \neq 0$.* □

Remark 2.15 It is clear that the property of zero-state observability in the previous section is playing the same role as observability with respect to the trivial input class B_0. □

Remark 2.16 [2, 4] It is also known that when the system (f,g,h) is observable with respect to B_α then L_o^N has a *strong* local minimum equal to zero at $x = 0$, i.e., $L_o^N(0) = 0$, $\frac{\partial L_o^N}{\partial x}(0) = 0$ and the Hessian $\frac{\partial^2 L_o^N}{\partial x^2}(0) > 0$. This last property is critical in the balancing transformation theory presented in [2]. □

2.3 Minimality via formal power series

In this section we briefly review a theory of minimal state space realizations for input-output systems that can be represented by a formal power series (Chen-Fliess functional expansion). A detailed treatment may be found in [6]. Ultimately this leads to rank conditions as in Theorem 2.8, which are necessary and sufficient conditions for a given realization to be minimal.

Let S be a given input-output map represented by a convergent generating series

$$S : u \rightarrow y(t) = \sum_{\eta \in I^*} c(\eta)\, E_\eta[u](t,t_0), \tag{8}$$

where I^* is the set of multi-indices for the index set $I = \{0, 1, \ldots, m\}$, $c(\eta) \in \mathbb{R}^p$, and

$$E_{i_k \ldots i_0}[u](t,t_0) = \int_{t_0}^{t} u_{i_k}(\tau) E_{i_{k-1}\ldots i_0}[u](\tau,t_0)\, d\tau \tag{9}$$

for $t \in [t_0, T]$ with $E_\emptyset(t,t_0)[u] := 1$ and $u_0(t) := 1$. The mapping S can then also be represented by a formal power series in noncommuting monomials $\mathcal{Z} = \{z_0, z_1, \ldots, z_m\}$ via

$$c = \sum_{\eta \in I^*} c(\eta) z_\eta, \tag{10}$$

where $z_\eta = z_{i_k} \cdots z_{i_0}$ when $\eta = (i_k \ldots i_0)$. Now define the sets:

$\mathbb{R} < \mathcal{Z} >$: the set of polynomials in $\mathcal{Z}$ over $\mathbb{R}$;

$\mathbb{R}^p \ll \mathcal{Z} \gg$: the set of formal power series in $\mathcal{Z}$ over $\mathbb{R}^p$.

Then the (block) Hankel mapping associated with c is defined to be the $\mathbb{R}$-vector space morphism $\mathcal{H} : \mathbb{R} < \mathcal{Z} > \rightarrow \mathbb{R}^p \ll \mathcal{Z} \gg$, uniquely specified by the generalized shifting property $[\mathcal{H}(z_\zeta)](\eta) = c(\eta\, \zeta)$, where $\eta, \zeta \in I^*$. In this context we have the following definition.

Definition 2.17 The *Lie rank* of a formal power series c is defined as $\rho_L(c) := \dim(\mathcal{H}(\mathcal{L}(\mathcal{Z})))$, where $\mathcal{L}(\mathcal{Z})$ denotes the smallest Lie algebra containing $\mathcal{Z}$. □

An analytic state space realization (f,g,h) defined locally about x_0 is said to *realize* a formal power series c if

$$c(i_k \ldots i_0) = L_{X_{i_0}} L_{X_{i_1}} \cdots L_{X_{i_k}} h(x_0) \tag{11}$$

for every $(i_k \ldots i_0) \in I^*$, where X_i, $i \in \underline{k}$, in the set $\{f, g_1, \ldots, g_m\}$. It is well known that if a certain growth condition on the coefficients $\{c(\eta)\}_{\eta \in I^*}$ is satisfied, then there exists a realization of c if and only if the Lie rank of c is finite. The following results characterize minimality.

Theorem 2.18 *An analytic realization (f,g,h) about x_0 of a formal power series c is minimal if and only if its dimension is equal to the Lie rank $\rho_L(c)$.* □

Theorem 2.19 *An analytic realization (f,g,h) about x_0 of a formal power series c is minimal if and only if* $\dim\, C(x_0) = n$ *and* $\dim\, d\mathcal{O}(x_0) = n$. □

Finally any two minimal realizations (f,g,h) about x_0 and $(\tilde{f},\tilde{g},\tilde{h})$ about $\tilde{x}_0$ are necessarily related by a diffeomorphism $T : V \rightarrow \tilde{V}$ where V and $\tilde{V}$ neighborhoods of x^o and $\tilde{x}^o$, respectively. Thus minimal realizations of formal power series are unique modulo a diffeomorphism.

3 Minimality and Energy Functions

3.1 Accessibility

In this section we study the controllability function and characterizations that are important for obtaining the accessibility rank condition in order to use Theorem 2.19. It can be easily deduced that we have the following relation (following the lines of the proof of Theorem 13 in [9])

$$L_c(x_0) = L_r(x_0) \quad := \inf_{\substack{u \in L_2(-\bar{t},0) \\ \bar{t} \geq 0 \\ x(-\infty)=0, x(0)=x_0}} \frac{1}{2}\int_{-\bar{t}}^{0} \| u(t) \|^2 dt, \tag{12}$$

and clearly reachability from x_0 implies L_r is well-defined for all $x \in M$, and thus, likewise for L_c. However, reachability is not implied from a well-defined and positive definite L_c. For our application it is sufficient (as observed from Theorem 2.4) to consider only the anti-stabilizability of the solution of the Hamilton-Jacobi equation (5), which is a condition that can be seen as reachability from 0 in infinite time (so called asymptotic reachability from 0). We formally define this notion below.

Definition 3.1 A system (1) is said to be *asymptotically reachable from* x_0 on a neighborhood W of x_0 if $\forall x \in W$ there exists a $u \in L_2(0,\infty)$ such that $\varphi(\tau,0,x_0,u) \in W$ for all $\tau \geq 0$, and $\lim_{t\to\infty} \varphi(t,0,x_0,u) = x$, i.e., $\forall x \in W$, $\forall \varepsilon > 0$, $\exists K < \infty$, s.t. $\varphi(\tau,0,x_0,u) \in W$ for all $\tau \leq K$, and s.t. $|\varphi(K,0,x_0,u) - x| < \varepsilon$.
A system (1) is said to be *locally asymptotically reachable from* x_0 if there exists a neighborhood W of x_0 such that the system is asymptotically reachable from x_0 on every neighborhood $V \subset W$ of x_0. □

In the following theorem, we obtain the relation between local asymptotic reachability from x_0 and local accessibility from x_0.

Theorem 3.2 *Assume that the accessibility distribution C has constant dimension about x_0. Then local asymptotic reachability from x_0 implies that the system is locally accessible from x_0.*

Proof Suppose that the system is not locally accessible from x_0, then we know from standard results in the literature (e.g., [7]) that $\dim C(x_0) = k < n$. Hence from Proposition 3.12 in [7] we know that there exists a neighborhood V of x_0 and local coordinates $x_1,\ldots,x_n$ such that the submanifold $S_{x_0} = \{q \in V | x_i(q) = x_i(x_0), i = k+1,\ldots,n\}$ contains $R_T^{\bar{V}}(x_0)$ (where $R^V(x_0,T)$ is the reachable set from x_0 at time $T > 0$, following the trajectories which remain in the neighborhood V of x_0 for $t \leq T$, and then $R_T^V(x_0) := \cup_{\tau\leq T} R^V(x_0,\tau)$) for any neighborhood $\bar{V} \subset V$ of x_0 and for all $T > 0$. This implies that all $q \in V$ such that $q \notin S_{x_0}$ are not asymptotically reachable from x_0 on V, and thus the local asymptotic reachability from x_0 is contradicted. ■

Our main aim is now to relate the positive definiteness and well-definedness of the controllability function to the accessibility rank condition.

Lemma 3.3 *Assume that f is asymptotically stable on a neighborhood W of 0. Then the controllability function $L_c(x)$ is smooth, finite and satisfies $L_c(x) > 0$ for $x \in W$, $x \neq 0$, if and only if the system (1) is asymptotically reachable from 0 on W.*

Proof The result follows immediately from the definition of asymptotic reachability and Theorem 2.4. ■

The previous two results, combined with Proposition 3.12 and Corollary 3.13 of [7], give rise to the following corollary:

Corollary 3.4 *Assume that the accessibility distribution C has constant dimension about 0, and assume that f is locally asymptotically stable. If there exists a neighborhood W of 0 such that the controllability function $L_c(x)$ is smooth, finite and satisfies $L_c(x) > 0$ for $x \in V$, $x \neq 0$, for all $V \subset W$, then $\dim C(0) = n$.* □

Remark 3.5 The above corollary is restricted by local requirements on L_c, since we need local asymptotic reachability from 0 in order to use Theorem 3.2. Only asymptotic reachability on a neighborhood W of 0 does not suffice. An example of a smooth system that is asymptotically reachable on a neighborhood W of 0 and that is not locally accessible is easy to construct. However, if we assume that the system (1) is *analytic*, then we can relax the local requirements on L_c to requirements on a neighborhood W of 0. This is due to the fact that asymptotic reachability from x_0 implies local accessibility from x_0 for analytic systems, e.g., [15]. Analyticity is actually not a strong restriction in our setting, i.e., it is also an assumption for the realization theory presented in the previous section. □

Corollary 3.6 *Let the system (1) be analytic. Assume that the accessibility distribution C has constant dimension about 0, and assume that f is asymptotically stable on a neighborhood W of 0. If the controllability function $L_c(x)$ is smooth, finite and satisfies $L_c(x) > 0$ for $x \in W$, $x \neq 0$, then $\dim C(0) = n$.* □

So far, we have been concentrating on the concept of local accessibility. However, for state space analysis of similarity invariants, which is closely related to the analysis of this paper, see e.g [14], the nonlinear counterpart of the Kalman decomposition is considered. Therefore, it is also of interest to consider the concept of local strong accessibility. The local strong accessibility version of Theorem 3.2 is given below.

Theorem 3.7 *Assume that the strong accessibility distribution C_0 has constant dimension about x_0. Then local asymptotic reachability from x_0 implies that the system is locally strongly accessible from x_0.*

Proof Suppose that the system is not locally strongly accessible from x_0, then we know from standard results in the literature (e.g., [7]) that $\dim C_0(x_0) = k < n$. Hence from Proposition 3.22 in [7] we know that there are two possibilities:
(i) If $f(x_0) \in C_0(x_0)$, then it follows similar to the proof of Theorem 3.2.
(ii) If $f(x_0) \notin C_0(x_0)$, then by continuity $f(q) \notin C_0(q)$ for all $q \in \tilde{U}$, $\tilde{U} \subset U$ neighborhood of x_0, and $\dim C(q) = \dim C_0(q) + 1$ for all $q \in \tilde{U}$. In this case, we can select the coordinates $\tilde{x}_{k+1},\ldots,\tilde{x}_n$ in such a way that $S_{x_0}^T = \{q \in \tilde{U} | \tilde{x}_{k+1}(q) = T, \tilde{x}_{k+2}(q) = \cdots = \tilde{x}_n(q) = 0\}$ contains $R^{\tilde{U}}(x_0,T)$ for any $T > 0$. Again, we have two cases: (a) If $\dim C_0(x_0) < n-1$, then this implies that all $q \in \tilde{U}$ such that $q \notin S_{x_0}^T$ are not asymptotically reachable from x_0 on V, and thus the local asymptotic reachability from x_0 is contradicted. (b) If $\dim C_0(x_0) = n-1$, then all $q \in \tilde{U}$ such that $\tilde{x}_n = -K$, $K > 0$, are not asymptotically reachable from x_0 on $\tilde{U}$. This concludes the proof. ■

This theorem gives rise to corollaries similar to Corollary 3.4 and 3.6, except with accessibility replaced by strong accessibility.

3.2 Observability

For the observability counterpart of the previous section we consider the observability functions as defined in (3) and (6). We begin with the observability function in (3) for which zero-state observability plays an important role.

Lemma 3.8 *Assume that the observability function (3) is smooth and finite for system (1) on a neighborhood W of 0. Then $L_o(x) > 0$ for $x \in W$, $x \neq 0$ implies that the system (1) is zero-state observable on W.*

Proof Assume that the system is not zero-state observable on W. Then there exists a trajectory $x(t) = \phi(t,0,x_0,0) \in W$, $t \geq 0$ such that $x(\tau) \neq 0$ for $0 \leq \tau \leq \bar{t}$ for some $0 < \bar{t} < \infty$ and such that $h(x(t)) = 0$, $u(t) = 0$, $\forall t \geq 0$. Hence, by definition of L_o, $L_o(x(t)) = 0$ for all t, which gives the contradiction with the positivity of L_o, and thus proves the lemma. ■

In the following lemma, we present the relationship between local zero-state observability and local observability at 0.

Lemma 3.9 *Assume that the zero-state observability codistribution has constant dimension about 0. Then local zero-state observability implies local observability at 0.*

Proof Along the same lines as for local observability in [7], it can be proven that local zero-state observability implies that $\dim d\mathcal{O}_0(0) = n$. This implies that $\dim d\mathcal{O}(0) = n$, and by Theorem 2.8 it follows directly that the system is locally observable at 0. ■

Motivated by the minimality conditions of Theorem 2.19, we obtain the following corollary.

Corollary 3.10 *Assume that the zero-observability codistribution $d\mathcal{O}_0$ has constant dimension about 0. If the observability function (3) is smooth, finite and satisfies $L_o(x) > 0$, $x \in W$, $x \neq 0$, then* $\dim d\mathcal{O}_0(0) = n$. □

In the event that system (1) is not zero-state observable, it still may be locally observable at 0. In which case the natural observability function given in Definition 2.12 gives the following result analogous to Lemma 3.8.

Lemma 3.11 *Let $L_o^N(x)$ be the natural observability function (6) for some $\alpha > 0$. Assume that $L_o^N(x)$ is smooth and finite for system (1) on a neighborhood W of 0. Then $L_o^N(x) > 0$ for $x \in W$, $x \neq 0$, implies that the system (1) is locally observable at 0 with respect to B_α.*

Proof Assume that the system (1) is not locally observable at 0 with respect to B_α. Then the corresponding homogeneous system is also not locally observable at 0 with respect to B_α. Hence there exists an initial state $x_a \neq 0$ such that $h(\bar{\varphi}(t,0,0,u)) = h(\bar{\varphi}(t,0,x_a,u))$, $t \geq 0$, $\forall u \in B_\alpha$, where $\bar{\varphi}(\cdot)$ denotes the solution to homogeneous system. By definition of the natural observability function, we have that $L_o^N(0) = 0$ and by the positivity of L_o^N it follows that $L_o^N(x_a) > 0$. However, from equation (6) it follows immediately that the maximum over $u \in B_\alpha$ for both states 0 and x_a results in the same optimal input u. This implies that $L_o(0) = L_o(x_a)$, and yields the desired contradiction to prove the lemma. ■

This lemma gives the analogue of Corollary 3.10 in terms of the general observability function as follows.

Corollary 3.12 *Assume that the observability codistribution $d\mathcal{O}$ has constant dimension about 0. If the natural observability function (6) is smooth, finite and satisfies $L_o^N(x) > 0$ for $x \in W$, $x \neq 0$, then* $\dim d\mathcal{O}(0) = n$. □

Remark 3.13 We may now compare the results of this section to the previous section. It is clear that they do not completely follow along similar or "dual" lines. Specifically, the results related to the observability functions as given by (3) and (6) are given in terms of the zero-state observability and observability rank condition, respectively. Starting with the rank conditions the converse of these results also holds by the Theorems 2.2, 2.8, and 2.10. However, for the controllability function, we are considering asymptotic reachability which implies local accessibility, and which can be related to the accessibility rank condition. The reverse direction is far less clear in this case, mainly because accessibility from 0 is not sufficient for asymptotic reachability from 0. However, if asymptotic reachability can somehow be assumed for a given system, then the converse of these results also follows for the controllability function. □

4 Sufficient conditions for minimality

Briefly summarized we have obtained the following main result.

Theorem 4.1 *Assume that the observability codistribution $d\mathcal{O}$ (or the zero-observability codistribution $d\mathcal{O}_0$, respectively) and the accessibility distribution C of a system (f,g,h) each have constant dimension about 0. Furthermore, assume that the analytic system (f,g,h) is a realization of the formal power series c. Then, if $0 < L_c(x) < \infty$ and $0 < L_o^N(x) < \infty$ (or $0 < L_o(x) < \infty$) for $x \in W$, $x \neq 0$, then (f,g,h) is a minimal realization of c.* □

The necessity of these conditions is not obtained due to the fact that, contrary to the linear case, accessibility and controllability are not equivalent in general. Only under additional assumptions can a converse result be obtained.

The use for and relation with similarity invariants for balanced realizations of nonlinear systems may be found in [14].

Acknowledgement

This research was supported in part by the North Atlantic Treaty Organization through the NATO Collaborative Research Grant CRG-97113

References

[1] Gray, W.S. and J. P. Mesko (1996). Controllability and observability functions for model reduction of nonlinear systems. *Proc. 1996 Conf. Inf. Sc. & Syst.*, Princeton, New Jersey, pp. 1244-1249.

[2] Gray, W.S. and J. P. Mesko (1997). General input balancing transformations for nonlinear systems. *Proc. 1997 Conf. Inf. Sc. & Syst.*, Baltimore, Maryland, pp. 264–269.

[3] Gray, W.S. and J. P. Mesko (1997). General input balancing and model reduction for linear and nonlinear systems. *Proc. 1997 Eur. Contr. Conf.*, Brussels, Belgium.

[4] Gray, W.S. and J. P. Mesko (1997). Observability functions for linear and nonlinear systems. *Syst. & Contr. Letters*, submitted.

[5] Hill, D. and P. Moylan (1976). The stability of nonlinear dissipative systems. *IEEE Trans. Aut. Contr.* AC-21, pp. 708–711.

[6] Isidori, A. (1995). *Nonlinear Control Systems*. Third Edition, Springer-Verlag, London.

[7] Nijmeijer, H. and A.J. van der Schaft (1990). *Nonlinear Dynamical Control Systems*. Springer-Verlag, New York.

[8] Rugh, W.J. (1981). *Nonlinear System Theory: the Volterra-Wiener Approach*. John Hopkins University Press.

[9] van der Schaft, A.J. (1992). L_2-gain analysis of nonlinear systems and nonlinear state feedback $\mathcal{H}_\infty$ control. *IEEE Trans. Aut. Contr.* AC-37, pp. 770–784.

[10] Scherpen, J.M.A. (1993). Balancing for nonlinear systems. *Syst. & Contr. Letters* 21, pp. 143–153.

[11] Scherpen, J.M.A. (1993). Balancing for nonlinear systems. *Proc. 1993 Eur. Contr. Conf.* Groningen, The Netherlands. Vol. 4. pp. 1838–1843.

[12] Scherpen, J.M.A. (1994). Balancing for nonlinear systems. PhD thesis. University of Twente.

[13] Scherpen, J.M.A. (1995). On the similarity invariance of balancing for nonlinear systems, *Preprints IFAC Nonl. Contr. Syst. Des. (NOLCOS)*. Tahoe City, CA. Vol. 2. pp. 783-788.

[14] Scherpen, J.M.A., W.S. Gray, Minimality and Similarity Invariants of a Nonlinear State Space Realization, submitted.

[15] Sontag, E.D. (1990). Integrability of certain distributions associated to actions on manifolds and an introduction to Lie-algebraic control. Rutgers Center SYCON Report 88-04, partly appeared in: *Nonlinear Controllability and Optimal Control*. H.J. Sussmann, ed. pp. 1–41.

[16] Sussman, H.J. (1977). Existence and uniqueness of minimal realizations of nonlinear systems. *Math. Syst. Theory*. 10. pp. 263–284.

[17] Sussmann, H.J. (1987). A general theorem on local controllability. *SIAM J. Contr. Opt.* Vol. 25. pp. 158–194.

[18] Zheng, Y.F., P. Liu, A.S.I. Zinober and C.H. Moog (1995). What is the dimension of the minimal realization of a nonlinear system? *Proc. 34th IEEE CDC*. New Orleans, Louisiana. Vol. 4. pp. 4239–4244.