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Published in:
Systems & Control Letters

DOI:
10.1016/S0167-6911(01)00144-X

Systems & Control Letters 44 (2001) 219–232
www.elsevier.com/locate/sysconle

On the nonuniqueness of singular value functions and balanced nonlinear realizations

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Received 14 March 2000; received in revised form 9 May 2001; accepted 5 June 2001

Abstract

The notion of balanced realizations for nonlinear state space model reduction problems was first introduced by Scherpen in 1993. Analogous to the linear case, the so-called *singular value functions* of a system describe the relative importance of each state component from an input–output point of view. In this paper it is shown that the procedure for nonlinear balancing has some interesting ambiguities that do not occur in the linear case. Specifically, distinct sets of singular value functions and balanced realizations are possible.

Keywords: State dependent matrices; Singular value functions; Balanced realizations; Model reduction; Nonlinear systems

1. Introduction

Balanced realizations for nonlinear state space model reduction problems was first introduced by Scherpen in [9]. Analogous to the Gramians matrices used in the linear case, controllability and observability (energy) functions are used to determine how important each state component is in influencing the input–output map of the system. These functions are then transformed, through a change of coordinates, into a simultaneous diagonal form in order to identify the so-called *singular value functions* of the system. In the linear case, these functions are equivalent to the square of the (constant) Hankel singular values of the system. State truncation is finally accomplished by examining the singular value functions in a neighborhood of 0 and deleting states that correspond to the smallest singular value functions in a local sense.

The procedure for nonlinear balancing, however, has two interesting ambiguities that do not occur in the linear case. First, it appears that the singular value functions defined in [9] are dependent on a particular factorization of the observability function. It will be shown by example that in a fixed coordinate frame this factorization is not unique, and thus other distinct definitions for the singular value functions are possible. Of course, this is of great concern in model reduction applications since decisions about state deletion should only depend on the coordinate frame of the state space and on intrinsic qualities of the input–output map [7,8,10]. (Analogous issues arise in other applications where state dependent matrices are introduced, e.g. state dependent Riccati equations and linear parameter varying (LPV) systems.) Next, given a fixed factorization, there

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0167-6911/01/$ - see front matter
PII: S0167-6911(01)00144-X

is a rich source of nonuniqueness for singular value functions via *norm preserving* coordinate transformations. However, the particular subclass of *orthogonal transformations* has a natural kind of invariance property.

The paper is organized as follows. In Section 2, the background for the problem is provided by outlining some standard results on nonlinear balanced realizations from [9]. Then a simple example is given to illustrate the nonuniqueness phenomena. In Section 3, each nonuniqueness source is examined independently, and the notion of consistency conditions is introduced. The final section summarizes the conclusions of the paper.

The mathematical notation used throughout is fairly standard. Vector norms are represented by $\|x\| = \sqrt{x^{\mathrm{T}}x}$ for $x \in \mathbb{R}^n$. $L_2(a,b)$ represents the set of Lebesgue measurable functions, possibly vector-valued, with finite L_2 norm $\|x\|_{L_2} = \sqrt{\int_a^b \|x(t)\|^2 \, \mathrm{d}t}$. If $L : \mathbb{R}^n \mapsto \mathbb{R}$ is a differentiable function, then its partial derivative $\partial L/\partial x$ will be the row vector of partial derivatives $\partial L/\partial x_i$ where $i = 1, \ldots, n$.

2. The nature of the problem

Let $\mathscr{M}$ be an n-dimensional smooth manifold, and let

$$\dot{x} = f(x) + g(x)u,$$
$$y = h(x)$$

be a system defined in terms of local coordinates on $\mathscr{M}$ with $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^p$. It is assumed that f, g and h are smooth on $\mathscr{M}$, $f(0) = 0$ and $h(0) = 0$. The corresponding controllability and observability functions (or energy functions, collectively) for such a system are defined below.

Definition 2.1. The *controllability* and *observability functions* for the system (f, g, h) are defined, respectively, as

$$L_c(x) = \min_{\substack{u \in L_2(-\infty,0) \\ x(-\infty)=0,\ x(0)=x}} \frac{1}{2} \int_{-\infty}^{0} \|u(t)\|^2 \, \mathrm{d}t$$

and

$$L_o(x) = \frac{1}{2} \int_0^{\infty} \|y(t)\|^2 \, \mathrm{d}t,$$

when $x(0) = x$, and $u(t) = 0$ for $0 \leqslant t < \infty$.

In order for a balanced realization to exist, the following system properties are assumed:

1. f is asymptotically stable on some neighborhood Y of 0.
2. The system (f, g, h) is zero-state observable on Y.
3. L_c and L_o exist and are smooth on Y.

The next collection of results form the core of the standard nonlinear balancing procedure.

Lemma 2.1 (Milnor [5]). *Let L be a smooth real-valued function on a convex neighborhood $V \subset \mathbb{R}^n$ of* 0 *with $L(0) = 0$. Then L exhibits the factorization*

$$L(x) = a^{\mathrm{T}}(x)x,$$

where a is the smooth vector field on V with component functions

$$a_i(x) = \int_0^1 \frac{\partial L}{\partial x_i}(tx_1, \ldots, tx_n) \, \mathrm{d}t.$$

Observe that $a^{\mathrm{T}}(0) = \partial L/\partial x(0)$, and in fact *any* factorization of the form $L(x) = \tilde{a}^{\mathrm{T}}(x)x$ necessarily has the property that $\tilde{a}^{\mathrm{T}}(0) = \partial L/\partial x(0)$. The following lemma comes from applying Morse's Lemma to L_c [5], and the above lemma twice to L_o.

Lemma 2.2. *For a system* (f,g,h) *with corresponding energy functions* (L_c,L_o), *there exists a coordinate transformation* $x=\phi(\bar{x})$, $\phi(0)=0$, *defined on a neighborhood* V *of* 0 *which converts the system into an* input-normal realization, *where*

$$\bar{L}_c(\bar{x}) := L_c(\phi(\bar{x})) = \tfrac{1}{2}\bar{x}^{\mathrm{T}}\bar{x},$$
$$\bar{L}_o(\bar{x}) := L_o(\phi(\bar{x})) = \tfrac{1}{2}\bar{x}^{\mathrm{T}}M(\bar{x})\bar{x}$$

with M *an* $n\times n$ *symmetric matrix-valued function having smooth component functions on* $\bar{V} := \phi^{-1}(V)$ *and* $M(0)=\partial^2\bar{L}_o/\partial\bar{x}^2(0)$.

$\bar{L}_c$ and $\bar{L}_o$ are the energy functions for the transformed system $(\bar{f},\bar{g},\bar{h})$ since each satisfies the corresponding Hamilton–Jacobi–Bellman equation. Analogous to the above observation, *any* factorization of the form $\bar{L}_o(\bar{x})=\frac{1}{2}\bar{x}^{\mathrm{T}}M'(\bar{x})\bar{x}$ necessarily has the property that $M'(0)=\partial^2\bar{L}_o/\partial\bar{x}^2(0)$. In order to diagonalize M, the following technical lemma is needed.

Lemma 2.3 (Kato [3]). *If there exists a neighborhood* $\bar{V}$ *of* 0, *where the number of distinct eigenvalues of* M *is constant everywhere* $\bar{V}$, *then the eigenvalues and orthonormalized eigenvectors* (λ_i, p_i), $i=1,\dots,n$ *of* M *are smooth functions of* $\bar{x}\in\bar{V}$.

Theorem 2.1. *For a system* (f,g,h) *satisfying the condition in Lemma* 2.3, *there exists a coordinate transformation* $x=\psi(z)$, $\psi(0)=0$, *defined on a neighborhood* U *of* 0 *which converts the system into a* input-normal/output-diagonal realization, *where*

$$\tilde{L}_c(z) := L_c(\psi(z)) = \tfrac{1}{2}z^{\mathrm{T}}z,$$
$$\tilde{L}_o(z) := L_o(\psi(z)) = \tfrac{1}{2}z^{\mathrm{T}}\,\mathrm{diag}(\tau_1(z),\dots,\tau_n(z))z$$

with $\tau_1(z)\geqslant\cdots\geqslant\tau_n(z)$ *being smooth functions on* $W := \psi^{-1}(U)$.

The set of functions τ_i, $i=1,\dots,n$ are called the *singular value functions* of (f,g,h). The final step of this balancing procedure is given below.

Theorem 2.2. *For the system in Theorem* 2.1, *there exists a coordinate transformation* $z=\eta(\bar{z})$, $\eta(0)=0$, *defined on the neighborhood* W *of* 0 *which converts the system into a* balanced realization, *where*

$$\begin{aligned}\check{L}_c(\bar{z}) &:= \tilde{L}_c(\eta(\bar{z}))\\ &= \tfrac{1}{2}\bar{z}^{\mathrm{T}}\,\mathrm{diag}(\sigma(\bar{z}_1)^{-1},\dots,\sigma(\bar{z}_n)^{-1})\bar{z}\\ \check{L}_o(\bar{z}) &:= \tilde{L}_o(\eta(\bar{z}))\\ &= \tfrac{1}{2}\bar{z}^{\mathrm{T}}\,\mathrm{diag}(\sigma_1(\bar{z}_1)^{-1}\tau_1(\eta^{-1}(\bar{z})),\dots,\sigma_n(\bar{z}_n)^{-1}\tau_n(\eta^{-1}(\bar{z})))\bar{z},\end{aligned}$$

with $\sigma_i(\bar{z}_i) := \tau_i(0,\dots,0,\eta_i^{-1}(\bar{z}_i),0,\dots,0)^{1/2}$ *for* $i=1,\dots,n$.

Note that along coordinate axes it is easily verified for $i=1,\dots,n$ that

$$\check{L}_c(0,\dots,0,\bar{z}_i,0,\dots,0) = \tfrac{1}{2}\bar{z}_i^2\sigma_i(\bar{z}_i)^{-1}$$
$$\check{L}_o(0,\dots,0,\bar{z}_i,0,\dots,0) = \tfrac{1}{2}\bar{z}_i^2\sigma_i(\bar{z}_i).$$

To illustrate the nonuniqueness features in the above balancing procedure, consider the following example.

Example 2.1. Consider the system well defined on an open neighborhood of 0 in $\mathbb{R}^2$:

$$f(x) = -\begin{bmatrix} \alpha^2 x_1 + 2\alpha x_2 + (\alpha^2 - 2)x_2^2 \\ x_2 \end{bmatrix},$$

$$g(x) = \sqrt{2}\begin{bmatrix} \alpha - 2x_2 \\ 1 \end{bmatrix},$$

$$h(x) = \frac{1}{\sqrt{3}}(3\alpha(x_1 + x_2^2) + (\alpha - 2\sqrt{2})x_2),$$

where $\alpha = (\sqrt{3} + \sqrt{2})(\sqrt{3} + 2)$. The corresponding energy functions can be shown to be

$$L_c(x) = \tfrac{1}{2}(x_1^2 + 2x_1x_2^2 + x_2^2 + x_2^4),$$

$$L_o(x) = \tfrac{1}{4}(3x_1^2 + 2x_1x_2 + 6x_1x_2^2 + 3x_2^2 + 2x_2^3 + 3x_2^4)$$

for all $x \in \mathbb{R}^2$. Now applying the coordinate transformation

$$x = \phi(\bar{x}) = \begin{bmatrix} \bar{x}_1 + \bar{x}_1^2 \\ \bar{x}_2 \end{bmatrix}$$

yields an input-normal form with energy functions:

$$L_c(\bar{x}) = \tfrac{1}{2}\bar{x}^{\mathrm{T}}\bar{x},$$

$$L_o(\bar{x}) = \tfrac{1}{2}\bar{x}^T M(\bar{x})\bar{x} = \tfrac{1}{2}\bar{x}^{\mathrm{T}}\begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix}\bar{x}.$$

Since M is constant in this representation, the singular value functions appear to be the constant functions: $\tau_1(z) = 2$, $\tau_2(z) = 1$ in the diagonalized coordinate frame $x = \psi(z)$. The situation is more complex, however, than it first appears. For example, consider the smooth symmetric matrix-valued function

$$A(\bar{x}) = c_1(\bar{x})\begin{bmatrix} -2\bar{x}_2 & \bar{x}_1 \\ \bar{x}_1 & 0 \end{bmatrix} + c_2(\bar{x})\begin{bmatrix} 0 & \bar{x}_2 \\ \bar{x}_2 & -2\bar{x}_1 \end{bmatrix},$$

where $c_1, c_2 \in C^\infty(\mathbb{R}^2)$, the ring of smooth real-valued functions defined on $\mathbb{R}^2$. Since $\bar{x}^{\mathrm{T}}A(\bar{x})\bar{x} = 0$ everywhere on $\mathbb{R}^2$ and $A(0) = 0$, another input-normal form in the *same* coordinate system is:

$$\bar{L}_c(\bar{x}) = \tfrac{1}{2}\bar{x}^{\mathrm{T}}\bar{x},$$

$$\begin{aligned} \bar{L}_o(\bar{x}) &= \tfrac{1}{2}\bar{x}^{\mathrm{T}}(M(\bar{x}) + A(\bar{x}))\bar{x} \\ &:= \tfrac{1}{2}\bar{x}^{\mathrm{T}}M'(\bar{x})\bar{x} \\ &= \tfrac{1}{2}\bar{x}^{\mathrm{T}}\begin{bmatrix} \frac{3}{2} - 2c_1(\bar{x})\bar{x}_2 & \frac{1}{2} + c_1(\bar{x})\bar{x}_1 + c_2(\bar{x})\bar{x}_2 \\ \frac{1}{2} + c_1(\bar{x})\bar{x}_1 + c_2(\bar{x})\bar{x}_2 & \frac{3}{2} - 2c_2(\bar{x})\bar{x}_1 \end{bmatrix}\bar{x}. \end{aligned} \tag{1}$$

For most choices of c_1, c_2, the condition in Lemma 2.3 is satisfied, and thus M' is smoothly diagonalizable. Consider, for example, the case: $c_1(\bar{x}) = \bar{x}_1$ and $c_2(\bar{x}) = \bar{x}_2$. Then it follows that the eigenvalues of M' are $\lambda_1'(\bar{x}) = 2 + (\bar{x}_1 - \bar{x}_2)^2$ and $\lambda_2'(\bar{x}) = 1 - (\bar{x}_1 + \bar{x}_2)^2$, which are distinct everywhere on $\mathbb{R}^2$. The diagonalizing transformation

$$x = \psi'(z') = \begin{bmatrix} \frac{1}{\sqrt{2}} + \frac{1}{2}z_1' - z_2' & \frac{1}{\sqrt{2}} + \frac{1}{2}z_2' \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} z'$$

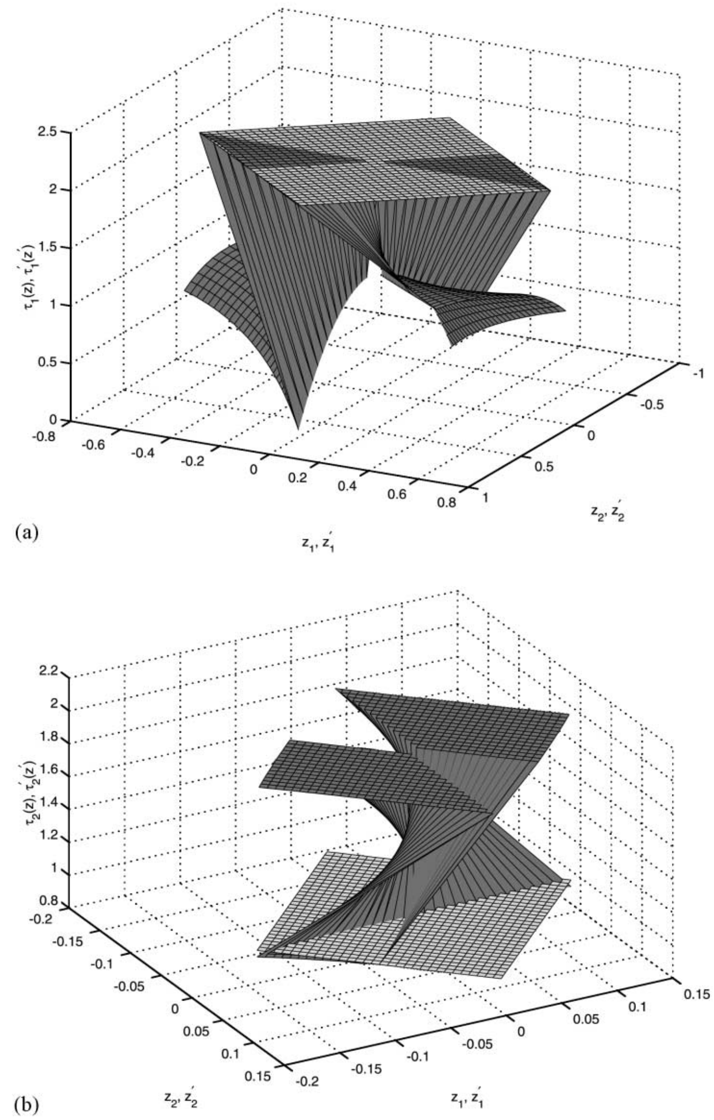

Fig. 1. The singular value functions for Example 2.1 when $c_1(x) = c_2(x) = 0$ (light gray), and when $c_1(x) = x_1$ and $c_2(x) = x_2$ (dark gray).

yields the corresponding input-normal/output-diagonal form:

$$\tilde{L}_c'(z') := L_c(\psi'(z')) = \tfrac{1}{2}(z')^{\mathrm{T}} z',$$

$$\begin{aligned}\tilde{L}_o'(z') &:= L_o(\psi'(z')) \\ &= \tfrac{1}{2}(z')^{\mathrm{T}}\,\mathrm{diag}(\tau_1'(z'), \tau_2'(z'))z' \\ &= \tfrac{1}{2}(z')^{\mathrm{T}}\,\mathrm{diag}(2 + 2(z_2')^2, 1 - 2(z_1')^2)z'.\end{aligned}$$

Thus, it is clear that a different factorization of L_o, via the introduction of the matrix-valued function A, leads to a different set of singular value functions. Note, however, that they *are* identical along respective coordinate directions, i.e., $\tau_1'(z_1', 0) = \tau_1(z_1, 0)$ and $\tau_2'(0, z_2') = \tau_2(0, z_2)$. This is illustrated in Fig. 1. However, notice in Fig. 2 that this relation does not hold for every set of c_i functions. Furthermore, observe that any

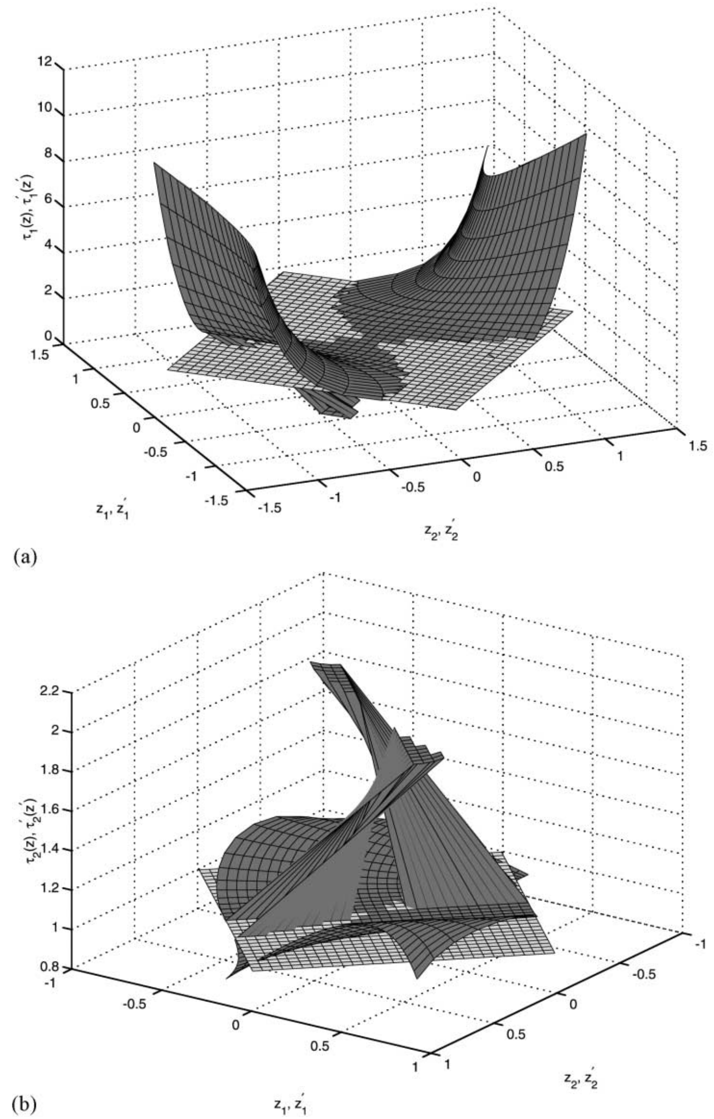

Fig. 2. The singular value functions for Example 2.1 when $c_1(x)=c_2(x)=0$ (light gray), and when $c_1(x)=x_2^3$ and $c_2(x)=-3x_1^3$ (dark gray).

coordinate transformation of the form $x=v(w)=T(w)w$ with $T^{\mathrm{T}}(w)T(w)=I$ transforms the energy functions in (1) to yet another input-normal/output-diagonal form after applying the diagonalizing transformation $w=\hat{\psi}(y)$:

$$\hat{L}_c(y):=L_c((v\circ\hat{\psi})(y))=\tfrac{1}{2}y^{\mathrm{T}}y,$$

$$\hat{L}_o(y):=L_o((v\circ\hat{\psi})(y))=\tfrac{1}{2}y^{\mathrm{T}}\,\mathrm{diag}(\hat{\tau}_1(y),\hat{\tau}_2(y))y,$$

where $\hat{\tau}_i(y)=\lambda_i((v\circ\hat{\psi})(y))$, $i=1,2$. Such *orthogonal transformations* thus represent a second source of nonuniqueness that has immediate consequences in nonlinear balancing and model reduction. In the next section these issues are considered in detail.

3. Sources of nonuniqueness

In this section two sources of nonuniqueness in computing the singular value functions of a system are examined: the addition of a null matrix function and a norm preserving coordinate transformation.

3.1. Null matrix functions

Let V be an open neighborhood of 0, and let $C^\infty(V)$ denote the abelian ring of smooth real-valued functions defined on V. (Addition and multiplication are defined in the obvious pointwise fashion on V, see for example [4].) Let $M_n(C^\infty(V))$ denote the set of $n \times n$ matrices with components from $C^\infty(V)$. Using the usual notions of matrix addition and multiplication, $M_n(C^\infty(V))$ is an associative ring with identity [2]. The subset $S_n(C^\infty(V))$ consists of all symmetric matrices in $M_n(C^\infty(V))$. The following subset of $S_n(C^\infty(V))$ is most relevant in this paper.

Definition 3.1. $\mathscr{A}(V) \subset S_n(C^\infty(V))$ is the set of matrix-valued functions on V where $A \in \mathscr{A}(V)$ if

i. $A(0) = 0$,
ii. $x^{\mathrm{T}} A(x) x = 0, \quad \forall x \in V$.

Any $A \in \mathscr{A}(V)$ is called a *null matrix function* on V. Properties of $\mathscr{A}(V)$ are considered in the following lemma, and then an application is given in the subsequent lemma.

Lemma 3.1. *For any neighborhood V of* 0, *the following statements are true*:

i. *$\mathscr{A}(V)$ is a vector space over R.*
ii. *$\mathscr{A}(V)$ is a module over $C^\infty(V)$.*
iii. *The matrix $A \equiv 0$ is the only constant matrix in $\mathscr{A}(V)$.*
iv. *The relation $M \sim M' \Leftrightarrow M - M' \in \mathscr{A}(V)$ is an equivalence relation on $S_n(C^\infty(V))$.*

Proof. Proofs of these statements are elementary. □

Lemma 3.2. *On any neighborhood V of* 0 *and for any $M, M' \in S_n(C^\infty(V))$*

$$x^{\mathrm{T}} M(x) x = x^{\mathrm{T}} M'(x) x, \quad x \in V \Leftrightarrow M \sim M'.$$

Proof. The proof is trivial using the fact that the equivalence on the left-hand side also implies $M(0) = M'(0)$. □

An interesting observation about the set $\mathscr{A}(V)$ is its relationship to an isotropy subgroup of the matrix group

$$GL_n(C^\infty(V)) := \{E \in M_n(C^\infty(V)) \colon \exists F \in M_n(C^\infty(V)) \text{ with } EF = I\},$$

where I denotes the identity matrix [6]. Viewing $GL_n(C^\infty(V))$ as a transformation group on V with the usual group action

$$\begin{aligned} \psi &: GL_n(C^\infty(V)) \times V \mapsto V \\ &: (E, x) \mapsto E(x)x, \end{aligned}$$

the isotropy subgroup for any $x \in V$ is

$$I_x := \{E \in GL_n(C^\infty(V)) \colon E(x)x = x\}.$$

The corresponding isotropy subgroup for V is

$$I_V := \bigcap_{x\in V} I_x.$$

Now given any symmetric element $\tilde{E}\in I_V$ such that $\tilde{E}(0)=I$, it is immediate that $\tilde{E}-I\in\mathscr{A}(V)$, that is,

$$x^{\mathrm{T}}(\tilde{E}(x)-I)x = x^{\mathrm{T}}(\tilde{E}(x)x - x) = 0.$$

However, it is easy to find examples of null matrices with no corresponding element in I_V. Specifically, it is possible for $x^{\mathrm{T}}A(x)x=0$ everywhere on V without $A(x)x=0$. Hence, the usual methods associated with matrix groups do not completely describe the nature of $\mathscr{A}(V)$.

Returning now to the main problem, it was observed in the example from the previous section that the equivalence $M\sim M'$ on $S_n(C^\infty(V))$ does not imply equivalence of their respective pointwise spectra. This is a fundamental source of nonuniqueness in the calculation of the singular value functions of a system. However, it is still possible to make some general statements relating their spectra. This is done using the following result.

Lemma 3.3. *If $A\in\mathscr{A}(V)$ then its (i,j)th component function can be factored as $a_{ij}(x)=\alpha_{ij}(x)x$, where α_{ij} is a smooth vector field on V. In particular, the kth component of α_{ij}, denoted α_{ijk}, satisfies*

i. $\alpha_{ijk}(0)=\partial a_{ij}/\partial x_k(0)$
ii. $\alpha_{ijk}(0)+\alpha_{kij}(0)+\alpha_{jki}(0)=0,\ \forall i,j,k$
iii. $\sum_{ijk}(\alpha_{ijk}(x)+\alpha_{kij}(x)+\alpha_{jki}(x))x_ix_jx_k=0$ *on* V.

Proof.

i. This result follows from the fact that $A(0)=0$ and applying Lemma 2.1 componentwise to A.
ii. Since $x^{\mathrm{T}}A(x)x=0$ everywhere on V then

$$\left.\frac{\partial^3}{\partial x_i\partial x_j\partial x_k}(x^{\mathrm{T}}A(x)x)\right|_{x=0} = \frac{\partial a_{ij}}{\partial x_k}(0)+\frac{\partial a_{ki}}{\partial x_j}(0)+\frac{\partial a_{jk}}{\partial x_i}(0)=0.$$

iii. Observe that

$$x^{\mathrm{T}}A(x)x=\sum_{ij}(\alpha_{ij}(x)x)x_ix_j=\sum_{ijk}\alpha_{ijk}(x)x_ix_jx_k=0.$$

Hence,

$$3\sum_{ijk}\alpha_{ijk}(x)x_ix_jx_k=0,$$

$$\sum_{ijk}(\alpha_{ijk}(x)+\alpha_{kij}(x)+\alpha_{jki}(x))x_ix_jx_k=0. \qquad \square$$

Next consider the following matrix perturbation theorem adapted from [1] (see p. 163).

Theorem 3.1. *Let $M_0\in\mathbb{R}^{n\times n}$ be a simple symmetric matrix with eigenvalues $\{\lambda_i\}_{i=1}^n$ and orthonormal eigenvectors $\{p_i\}_{i=1}^n$. For $\theta\in\mathbb{R}$ and symmetric matrices $M_1,M_2\in\mathbb{R}^{n\times n}$ define*

$$M(\theta)=M_0+M_1\theta+M_2\theta^2.$$

For sufficiently small $|\theta|$, the matrix $M(\theta)$ is also simple, and its corresponding eigenvalues $\{\lambda_i(\theta)\}_{i=1}^n$ and orthonormal eigenvectors $\{p_i(\theta)\}_{i=1}^n$ depend analytically on θ, i.e.,

$$\lambda_i(\theta)=\lambda_i^{(0)}+\lambda_i^{(1)}\theta+\lambda_i^{(2)}\theta^2+\cdots,$$

$$p_i(\theta)=p_i^{(0)}+p_i^{(1)}\theta+p_i^{(2)}\theta^2+\cdots,$$

for $i=1,2,\ldots,n$. *In particular*,

$$\lambda_i^{(0)}=\lambda_i,$$
$$\lambda_i^{(1)}=p_i^{\mathrm{T}}M_1 p_i,$$
$$\lambda_i^{(2)}=p_i^{\mathrm{T}}M_2 p_i+\sum_{\substack{j=1\\ i\neq j}}^{n}\frac{1}{\lambda_i-\lambda_j}|p_i^{\mathrm{T}}M_1 p_j|^2,$$
$$p_i^{(0)}=p_i,$$
$$p_i^{(1)}=\sum_{\substack{j=1\\ i\neq j}}^{n}\frac{p_i^{\mathrm{T}}M_1 p_j}{\lambda_i-\lambda_j}.$$

A main result of the paper is given below.

Theorem 3.2. *Suppose* $M\in S_n(C^\infty(V))$ *and* $M(0)$ *is simple. Let* $\{\lambda_i, p_i\}$ *denote the smoothly defined eigenvalue and orthonormal eigenvector pairs for* M *on a neighborhood* $\bar{V}\subset V$ *of* 0 (*cf.* *Lemma* 2.3). *Let* $A\in\mathscr{A}(V)$ *and define* $M'=M+A$ *with corresponding eigenvalues* $\{\lambda_i'\}_{i=1}^n$. *In the diagonalized coordinate frame* $z=\psi^{-1}(x)$ *for* M, *the eigenvalues of* M *and* M' *are equivalent to first order along their respective coordinate directions. That is*, *sufficiently close to* 0

$$\lambda_i'(\psi(0,\ldots,0,z_i,0,\ldots,0))=\lambda_i(\psi(0,\ldots,0,z_i,0,\ldots,0))+\mathcal{O}(z_i^2). \tag{2}$$

Proof. Let $M=P\Lambda P^{\mathrm{T}}$ be the spectral decomposition of M on $\bar{V}$. Then it follows directly that for any $x\in\bar{V}$

$$\begin{aligned}M'(x)&=M(x)+A(x)\\&=P(x)\Lambda(x)P^{\mathrm{T}}(x)+A(x),\end{aligned}$$

$$\underbrace{P^{\mathrm{T}}(x)M'(x)P(x)}_{N(x)}=\Lambda(x)+\underbrace{P^{\mathrm{T}}(x)A(x)P(x)}_{B(x)}. \tag{3}$$

Now set $z=P^{\mathrm{T}}(x)x=\psi^{-1}(x)$ or $x=\psi(z)$, then

$$N(\psi(z))=\Lambda(\psi(z))+B(\psi(z)),$$
$$\tilde{N}(z)=\tilde{\Lambda}(z)+\tilde{B}(z). \tag{4}$$

Note that $\tilde{N}(z)$ has the same eigenvalues as $M'(\psi(z))$ and $\tilde{B}(z)\in\mathscr{A}(\psi^{-1}(\bar{V}))$, that is,

$$\tilde{B}(0)=B(\psi(0))=B(0)=0,$$

$$\begin{aligned}z^{\mathrm{T}}\tilde{B}(z)z&=x^{\mathrm{T}}P(x)P^{\mathrm{T}}(x)A(x)P(x)P^{\mathrm{T}}(x)x\\&=x^{\mathrm{T}}A(x)x=0.\end{aligned}$$

Now evaluate Eq. (4) in the ith coordinate direction

$$\tilde{N}(0,\ldots,0,z_i,0,\ldots,0)=\tilde{\Lambda}(0,\ldots,0,z_i,0,\ldots,0)+\tilde{B}(0,\ldots,0,z_i,0,\ldots,0).$$

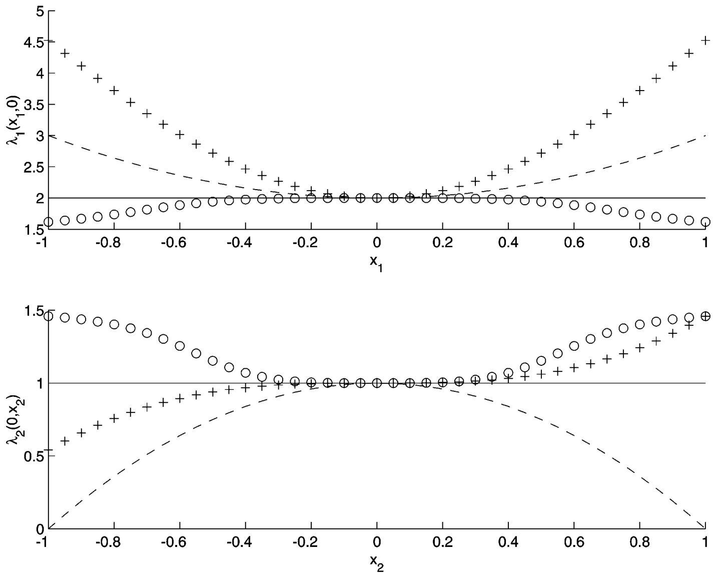

Fig. 3. Coordinate axis cross sections of the functions λ_1 and λ_2 in Example 2.1 when $c_1(x) = c_2(x) = 0$ (solid line), $c_1(x) = x_1$ and $c_2(x) = x_2$ (dashed line), $c_1(x) = x_2^3$ and $c_2(x) = -3x_1^3$ (marked 'o'), and $c_1(x) = \cos(x_1) - 1$ and $c_2(x) = 3\sin(x_2)$ (marked '+').

If $|z_i|$ is sufficiently small then

$$\tilde{N}(0,\ldots,0,z_i,0,\ldots,0) = \tilde{A}(0,\ldots,0,z_i,0,\ldots,0) + \underbrace{\left.\frac{\mathrm{d}\tilde{B}}{\mathrm{d}z_i}\right|_{z_i=0}}_{B_i} z_i + \mathcal{O}(z_i^2).$$

In light of Theorem 3.1 it follows that

$$\lambda_i'(\psi(0,\ldots,0,z_i,0,\ldots,0)) = \lambda_i(\psi(0,\ldots,0,z_i,0,\ldots,0)) + e_i^{\mathrm{T}} B_i e_i z_i + \mathcal{O}(z_i^2)$$

with $e_i = (\underbrace{0,\ldots,0,1}_{i\text{th position}},0,\ldots,0)^{\mathrm{T}}$. However, from Lemma 3.3, part ii, it is known that $e_i^{\mathrm{T}} B_i e_i = [B_i]_{ii} = 0$. Thus the theorem is proven. □

Remark. (a) In the context of the singular value functions, i.e., when $L_o(x) = \frac{1}{2}x^{\mathrm{T}}M(x)x$ and $L_o'(x) = \frac{1}{2}x^{\mathrm{T}}M'(x)x$, the identity (2) becomes

$$\lambda_i'(\psi(0,\ldots,0,z_i,0,\ldots,0)) = \tau_i(0,\ldots,0,z_i,0,\ldots,0) + \mathcal{O}(z_i^2).$$

The left-hand side of this identity is only equivalent to the true singular value functions for M' along the coordinates axes if the (orthogonal) diagonalizing transformation $z' = (\psi')^{-1}(x)$ for M' is identical to the diagonalizing transformation $z = \psi^{-1}(x)$ for M. This is the case in Example 2.1 from the previous section, M and M' are simultaneously diagonalized by the same coordinate transformation. In Fig. 3, the theorem is illustrated for this example using various sets of c_i functions.

(b) In general the eigenvalues of M and M' are not equivalent to second order or higher. However, if matrix $B_i = 0$ in the proof of Theorem 3.2 then equality up to second order follows from the expression for $\lambda_i^{(2)}$ in Theorem 3.1. This is exactly the case in Example 2.1 for the first choice of functions c_1 and c_2.

3.2. Norm preserving coordinate transformations

A smooth coordinate transformation $x=v(w)$ is said to be *norm preserving* on a convex neighborhood of the origin, W, if $\|x\|=\|w\|$ for all $w\in W$. Since all such maps satisfy $v(0)=0$, it follows directly from Lemma 2.1 that there exists at least one factorization of the form $v(w)=V(w)w$ where $V\in M_n(C^\infty(W))$, and $V(0)$ is nonsingular on at least an open neighborhood of 0. Thus, it is immediate that everywhere on W

$$\|v(w)\|^2=w^{\mathrm T}V^{\mathrm T}(w)V(w)w=w^{\mathrm T}w$$

or equivalently, $V^{\mathrm T}(w)V(w)-I=A_V(w)\in\mathscr{A}(W)$. In the context of energy functions, norm preserving transformations are interesting because they preserve input-normal forms, that is,

$$L_c(x)=\tfrac12 x^{\mathrm T}x=\tfrac12 w^{\mathrm T}w=\hat L_c(w).$$

A specific class of norm preserving transformations are the so-called *orthogonal transformations*, which are characterized by having a factorization $v(w)=T(w)w$ where $T^{\mathrm T}(w)T(w)=I$ for all $w\in W$. That is, $T^{\mathrm T}T$ is a symmetric, constant element of the isotropy subgroup I_W. The following theorem gives conditions under which an orthogonal transformation can be extracted from a given norm preserving transformation.

Theorem 3.3. *Suppose that $v(w)=V(w)w$ is a smooth, nonsingular, norm preserving coordinate transformation on an open neighborhood W of 0. Assume that V is a smooth $n\times n$ matrix-valued function on W and define $A=V^{\mathrm T}V-I\in\mathscr{A}(W)$. Consider a smooth $n\times n$ matrix-valued function Δ such that $\Delta(w)w=0$ for all $w\in W$. Then $v(w)=(V(w)+\Delta(w))w$ is an orthogonal transformation if $\Delta(w)$ satisfies the following state dependent Riccati equation*:

$$\Delta^{\mathrm T}(w)V(w)+V^{\mathrm T}(w)\Delta(w)+\Delta^{\mathrm T}(w)\Delta(w)+A(w)=0. \tag{5}$$

Proof. By definition $(V(w)+\Delta(w))w$ is an orthogonal factorization of $v(w)$ if

$$(V(w)+\Delta(w))^{\mathrm T}(V(w)+\Delta(w))=I.$$

Rewriting the latter equation, and using the expression for $A(w)$ yields (5). □

In the following theorem, it is observed that orthogonal coordinate transformations preserve the singular value functions in a natural sense.

Theorem 3.4. *Consider a system (f,g,h) with singular value functions τ_i, $i=1,\dots,n$ derived from a specific input-normal form*: $L_c(x)=\frac12 x^{\mathrm T}x$, $L_o(x)=\frac12 x^{\mathrm T}M(x)x$. *Any orthogonal coordinate transformation, $x=v(w)=T(w)w$, yields the corresponding singular value functions*

$$\hat\tau_i=\tau_i\circ\psi^{-1}\circ v\circ\hat\psi,\quad i=1,\dots,n, \tag{6}$$

where $x=\psi(z)$ and $w=\hat\psi(y)$ are diagonalizing transformations for $M(\cdot)$ and $M(v(\cdot))$, respectively.

Proof. After applying the coordinate transformation v and using the orthogonality condition, the new system has the input-normal form

$$\hat L_o(w)=\tfrac12 w^{\mathrm T}\underbrace{T^{-1}(w)M(v(w))T(w)}_{\hat M(w)}w. \tag{7}$$

Hence, it follows that the matrices $M(v(w))$ and $\hat M(w)$ have the same eigenvalues for each w. To compute the singular value functions starting from $\hat M(\cdot)$, use the fact that $x=\psi(z)=T_\psi(z)z$ diagonalizes $M(x)$ in the appropriate fashion, i.e.,

$$T_\psi^{\mathrm T}(z)M(\psi(z))T_\psi(z)=\operatorname{diag}(\tau_1(z),\dots,\tau_n(z)).$$

Consequently,

$$\begin{aligned} &T_\psi^{\mathrm{T}}(\psi^{-1}\circ v(w))T(w)\hat{M}(w)T^{-1}(w)T_\psi(\psi^{-1}\circ v(w)) \\ &\quad = T_\psi^{\mathrm{T}}(\psi^{-1}\circ v(w))M(\psi(\psi^{-1}\circ v(w)))T_\psi(\psi^{-1}\circ v(w)) \\ &\quad = \mathrm{diag}(\tau_1(\psi^{-1}\circ v(w)),\ldots,\tau_n(\psi^{-1}\circ v(w))) \\ &\quad = \mathrm{diag}(\hat{\tau}_1(y),\ldots,\hat{\tau}_n(y)), \end{aligned}$$

where $y=\hat{\psi}^{-1}(w)$ is the diagonalized coordinate frame for $\hat{M}(w)$. Equating the diagonal terms on the right-hand side of the last two equations gives

$$\hat{\tau}_i(y)=\tau_i(\psi^{-1}\circ v(\hat{\psi}(y))).$$

Hence, the theorem is proven. □

In Example 2.1, the first set of singular value functions were the constant values $\tau_1(z)=2$ and $\tau(z)=1$. Thus, in light of Eq. (6), they are invariant under all orthogonal coordinate transformations. This eliminates the possibility that such a transformation can relate $\{\tau_i\}$ to the second set of singular value transformations $\{\tau_i'\}$ derived from adding a null matrix. The next theorem is a combination of Theorems 3.2 and 3.4 and describes the effect of a general norm preserving coordinate transformation on a given set of singular value functions.

Theorem 3.5. *Consider a system (f,g,h) with singular value functions τ_i, $i=1,\ldots,n$ derived from a specific input-normal form: $L_c(x)=\frac{1}{2}x^{\mathrm{T}}x$, $L_o(x)=\frac{1}{2}x^{\mathrm{T}}M(x)x$ defined on a neighborhood W of 0 with $M\in S_n(C^\infty(V))$ and $M(0)$ simple. Any norm preserving coordinate transformation, $x=v(w)=V(w)w$, where $V(w)$ is a smooth function of w yields the following singular value functions expressed in the diagonalized coordinate frame for $M(v(\cdot))$:*

$$\lambda_i'(\hat{\psi}(0,\ldots,0,y_i,0,\ldots,0))=(\tau_i\circ\psi^{-1}\circ v\circ\hat{\psi})(0,\ldots,0,y_i,0,\ldots,0)+\mathcal{O}(y_i^2),\quad i=1,\ldots,n,$$

for y_i sufficiently close to 0, and where $x=\psi(z)$ and $w=\hat{\psi}(y)$ are the diagonalizing transformation for $M(\cdot)$ and $M(v(\cdot))$, respectively.

Proof. Since $V(0)$ is always nonsingular, then sufficiently close to 0 the matrix $V(w)$ is invertible. Applying the coordinate transformation v and using the identity

$$V^{\mathrm{T}}(w)=V^{-1}(w)[I+V(w)A_V(w)V^{-1}(w)],$$

where A_V a null matrix, gives a new input-normal form where

$$\check{L}_o(w)=\tfrac{1}{2}w^{\mathrm{T}}\{[I+A_V(w)]\underbrace{V^{-1}(w)M(v(w))V(w)}_{\hat{M}(w)}\}w$$

(cf. Eq. (7)). Letting

$$M'(w)=\hat{M}(w)+A_V(w)\hat{M}(w),$$

the proof proceeds similar to that of Theorem 3.2 (cf. Eq. (3)). That is, let $\hat{M}=\Lambda P^{\mathrm{T}}$ be the spectral decomposition near the origin. Then

$$\underbrace{P^{\mathrm{T}}(w)M'(w)P(w)}_{N(w)}=\Lambda(w)+\underbrace{P^{\mathrm{T}}(w)A_V(w)P(w)}_{B(w)}\Lambda(w).$$

After setting $y=P^{\mathrm{T}}(w)V^{-1}(w)w=\hat{\psi}^{-1}(w)$ it follows that

$$\begin{aligned} &N(\hat{\psi})(y)=\Lambda(\hat{\psi}(y))B(\hat{\psi}(y))\Lambda(\hat{\psi}(y)), \\ &\tilde{N}(y)=\tilde{\Lambda}(y)+\tilde{B}(y)\tilde{\Lambda}(y). \end{aligned}$$

As before, $\tilde{N}$ has the same eigenvalues as M', and $\tilde{B}$ is a null matrix. Along the ith coordinate direction and sufficiently close to the origin

$$\begin{aligned}\tilde{N}(0,\ldots,0,y_i,0,\ldots,0) &= \tilde{A}(0,\ldots,0,y_i,0,\ldots,0)\\ &\quad + \frac{\mathrm{d}}{\mathrm{d}y_i}[\tilde{B}(0,\ldots,0,y_i,0,\ldots,0)\tilde{A}(0,\ldots,0,y_i,0,\ldots,0)]|_{y_i=0}y_i + \mathcal{O}(y_i^2)\\ &= \tilde{A}(0,\ldots,0,y_i,0,\ldots,0) + \underbrace{\frac{\mathrm{d}}{\mathrm{d}y_i}[\tilde{B}(0,\ldots,0,y_i,0,\ldots,0)]|_{y_i=0}}_{B_i}\tilde{A}(0)y_i + \mathcal{O}(y_i^2).\end{aligned}$$

Applying Lemma 3.3 and Theorems 3.1 and 3.2,

$$\begin{aligned}\lambda_i'(\hat{\psi}(0,\ldots,0,y_i,0,\ldots,0)) &= \hat{\lambda}(\hat{\psi}(0,\ldots,0,y_i,0,\ldots,0)) + e_i^{\mathrm{T}}B_i\tilde{A}(0)e_i + \mathcal{O}(y_i^2)\\ &= (\tau_i \circ \psi^{-1} \circ v \circ \hat{\psi})(0,\ldots,0,y_i,0,\ldots,0) + e_i^{\mathrm{T}}B_ie_i\lambda_i(0) + \mathcal{O}(y_i^2)\\ &= (\tau_i \circ \psi^{-1} \circ v \circ \hat{\psi})(0,\ldots,0,y_i,0,\ldots,0) + \mathcal{O}(y_i^2). \qquad \square\end{aligned}$$

3.3. Consistency conditions

Given an input–output system and two distinct state space realizations related by a coordinate transformation, it is desirable to identify a coordinate balancing procedure which will result in the same singular value functions. At the heart of this problem are certain consistency conditions. First consider the smooth realization $(f(x), g(x), h(x))$ with a smooth energy function $L(x)$, which is related via a smooth coordinate transformation $z = v(x)$, $v(0) = 0$, to the realization $(\tilde{f}(z), \tilde{g}(z), \tilde{h}(z))$ with energy function $\tilde{L}(z) = L(v^{-1}(z))$. Assume that $L(0) = 0$ and $\partial L/\partial x(0) = 0$. It is well known then that on a convex neighborhood of 0 there exist $n \times n$ matrices $V(x)$, $M(x)$, and $\tilde{M}(z)$ such that

$$\begin{aligned}v(x) &= V(x)x,\\ L(x) &= x^{\mathrm{T}}M(x)x,\\ \tilde{L}(z) &= z^{\mathrm{T}}\tilde{M}(z)z,\end{aligned}$$

where the entries of $V(x)$, $M(x)$ and $\tilde{M}(z)$ are smooth functions of x, x and z, respectively, and where $M(x)$ and $\tilde{M}(z)$ are symmetric [9]. It follows directly that $L(x) = v(x)^{\mathrm{T}}\tilde{M}(v(x))v(x)$. Furthermore, for all such factorizations it is clear that $M(0) = V(0)^{\mathrm{T}}\tilde{M}(0)V(0)$. *A factorization procedure to produce* $V(x)$, $M(x)$, *and* $\tilde{M}(z)$ *would be consistent in this context if*

$$V(x)^{\mathrm{T}}\tilde{M}(v(x))V(x) = M(x). \tag{8}$$

The idea can be visualized in the diagram below:

$$\begin{array}{ccc}(\tilde{f}(z), \tilde{g}(z), \tilde{h}(z), \tilde{L}(z)) & \overset{\tilde{L}\text{-factorization}}{\longrightarrow} & \tilde{M}(z)\\ z = v(x)\downarrow & & V(x)\downarrow\\ (f(x), g(x), h(x), L(x)) & \overset{L\text{-factorization}}{\longrightarrow} & M(x)\end{array}$$

It is easily verified by the example that a procedure based on Lemma 2.1 does not exhibit property (8), except when restricted to linear systems. This suggests a second consistency condition. *Namely, if the coordinate transformation* $v(x)$ *is linear, then the factorization procedure should always produce a constant matrix, i.e.,* $V(x)$ *is a constant matrix. If the energy function* $L(x)$ *is a true quadratic form, then the factorization procedure must result in a constant matrix, i.e.,* $M(x)$ *is a constant matrix.* This latter property is exhibited by the factorization in Lemma 2.1. Finally, observe that any factorization with both consistency properties will always produce an orthogonal factorization of a norm preserving transformation. That is, since $M(x) = I$ and $\tilde{M}(z) = I$, then from (8), $V(x)^{\mathrm{T}}V(x) = I$.

4. Conclusions

It was shown that the current notion of singular value functions for nonlinear systems is not unique in two ways. In a fixed coordinate frame there are many ways to produce a state dependent quadratic form from a given energy function, all are related by the addition of a null matrix. Furthermore, norm preserving transformations can change the singular value functions at least to second order or higher. The special subclass of orthogonal transformations preserves the singular value functions modulo two diagonalizing transformations. These ideas then lead to the notion of consistency conditions in factoring an energy function.

Acknowledgements

This research was supported in part by a travel grant from the Netherlands Organization for Scientific Research (NWO), file number R62-554, and by the North Atlantic Treaty Organization through the NATO Collaborative Research Grant CRG-971135. The authors also wish to thank Prof. P.S. Krishnaprasad for insightful discussions on this research.

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