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Published in:
International Journal of Control

DOI:
10.1080/00207170110101766

Publication date:
2002

INT. J. CONTROL, 2002, VOL. 75, NO. 4, 242–251

Adaptive switching gain for a discrete-time sliding mode controller

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Sliding mode control is a well-known technique capable of making the closed loop system robust with respect to certain kinds of parameter variations and unmodelled dynamics. The sliding mode control law consists of a continuous component which is based on the model knowledge and discontinuous component which is based on the model uncertainty. This paper extends two known adaption laws for the switching gain for continuous-time sliding mode controllers to the multiple input case. Because these adaption laws have some fundamental problems in discrete-time, we introduce a new adaption law specifically designed for discrete-time sliding mode controllers.

1. Introduction

Sliding mode control is a well-known robust control algorithm for linear as well as non-linear systems (DeCarlo *et al.* 1988, Utkin 1992, Hung *et al.* 1993, Edwards and Spurgeon 1998, Utkin *et al.* 1999). Continuous-time sliding mode control has been extensively studied and has been used in various applications. Much less is known of discrete-time sliding mode controllers. In practice it is often assumed that the sampling frequency is sufficiently high to assume that the controller is continuous-time (Young and Özgüner 1999). Another possibility is to design the sliding mode controller in discrete-time, based on a discrete-time model, however stability has not yet been assured (Gao *et al.* 1995, Bartoszewicz 1998, van den Braembussche 1998).

The field of adaptive sliding mode controllers has received quite a bit of attention as well. In the case of continuous-time controllers, the field of adaptive sliding mode controllers can be divided into several groups. One group is formed by the model adaptive sliding mode controllers, for which we refer to, for example, Feng and Wu (1996) and Kwan (1995). Also the combination of adaptive backstepping and sliding mode control has recently been a topic of research as can be found in various publications (Sira-Ramirez and Llanes-Santiago 1993, Bartolini *et al.* 1997, Koshkouei and Zinober 1999, Sankaranarayanan *et al.* 1999). However, most attention from the research society, at least spoken in terms of the amount of publications, has been devoted to the adaptive switching gain sliding mode controllers. The major part of these publications focus on a very simple adaption procedure (see, for example, Leung *et al.* 1991, Su *et al.* 1991, Wang and Fan 1993, Jiang *et al.* 1994, Roh and Oh 2000. In this paper that simple procure is called the method I adaption law. A slightly more advanced method has been published in Lenz *et al.* (1998) and Wheeler *et al.* (1998), which we call the method II adaption law.

The field of discrete-time adaptive sliding mode controllers has, so far, mainly been focused on model adaptive controllers (see, for example, Bartolini *et al.* 1995, Park and Kim 1996, Chan 1997, Haskara *et al.* 1997, Utkin 1998). Here we focus on an adaptive switching gain sliding mode controller. We first study the effectiveness of the existing adaptive switching gains (method I and method II) when they are converted to the discrete-time domain. It is shown that the Method I procedure will, in general, lead to an unstable closed-loop system. Method II leads to much better results but still has the potential of leading to an unstable closed-loop system if the parameters are not chosen carefully. To overcome the drawbacks of these two adaption laws in discrete-time, a new adaption law specifically designed for discrete-time sliding mode controllers is introduced. Preliminary results of this adaption law were introduced by the authors in Monsees and Scherpen (2000) for the single input case.

The outline of this paper is as follows: §2 briefly introduces the continuous-time controllers employing the method I and method II adaption procedures. Section 3 introduces a discrete-time controller which uses the discretized version of method I and method II, and their applicability is studied. In §4 the rules defined by Gao *et al.* (1995) are introduced. According to these rules a discrete-time sliding mode controller is derived. These rules also form the basis on which the method III adaptive switching gain is defined. In §5 a discrete time multiple input simulation example is used to compare the discretized method II adaption law with the newly defined method III adaption law. Finally §6 presents the conclusions.

Received 9 October 2000. Revised 14 August 2001.
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International Journal of Control ISSN 0020–7179 print/ISSN 1366–5820 online © 2002 Taylor & Francis Ltd
http://www.tandf.co.uk/journals
DOI: 10.1080/00207170110101766

2. Continuous-time sliding mode control

2.1. *Introduction*

In this section we briefly introduce a continuous-time sliding mode controller for a multiple input system. We consider the system

$$\dot{x}(t) = Ax(t) + Bu(t) + f(t, x, u) \tag{1}$$

with the system matrices $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$, and the disturbance and modelling error vector $f \in \mathbb{R}^n$. Define $\sigma(t) \in \mathbb{R}^m$ by

$$\sigma(t) = Sx(t) \tag{2}$$

where matrix $S \in \mathbb{R}^{m\times n}$ should be chosen such that once the system reaches sliding mode (i.e. $\sigma(t) = 0$), the systems' dynamics are stable and consequently the system will 'slide' along the surface $\sigma(t) = 0$ towards the origin in state-space. The design procedure for S can be found in the literature (see, for example, Utkin 1992, Edwards and Spurgeon 1998, Utkin *et al.* 1999). A control-law which drives the system into sliding mode and subsequently keeps it in sliding mode can be found to be

$$u(t) = -(SB)^{-1}(SA - \Phi S)x(t) - \rho(t, x, u)(SB)^{-1}\frac{\sigma(t)}{\|\sigma(t)\|} \tag{3}$$

where $\Phi \in \mathbb{R}^{m\times m}$ is a design parameter and $\rho(t, x, u) \in \mathbb{R}^+$ is known as the switching gain. If the switching gain $\rho(t, x, u)$ is larger then a certain minimum value (which depends on the disturbance $f(t, x, u)$) then the closed-loop system is guaranteed to reach sliding mode in finite time and subsequently maintain sliding mode (Utkin 1992, Edwards and Spurgeon 1998, Utkin *et al.* 1999). A large discontinuous component may however excite unmodelled dynamics which could lead to chattering. For this reason the switching gain should not be excessively large. Finding an expression for $\rho(t, x, u)$ can be hard or even impossible so in practice one could be forced to determine the switching gain by experimentation. But in case of time-varying circumstances (for example different modes of operation or changing external conditions), the switching gain should be tuned on a regular basis to maintain the best possible performance. Another solution would be to have an adaptive switching gain which is the topic of this paper.

2.2. *Adaption method I*

The most straightforward adaption mechanism for the switching gain can be found in Leung *et al.* (1991), Su *et al.* (1991), Wang and Fang (1993), Jiang *et al.* (1994) and Roh and Oh (2000)

$$\hat{\rho}(t) = \int_{\upsilon=t_o}^{t} \|\sigma(\upsilon)\|\, d\upsilon \tag{4}$$

This adaption law is based on the fact that once the switching gain is sufficiently large, the system will be forced to the switching surface $\sigma(t) = 0$. However, this adaption law has three major drawbacks:

(1) In case of a large initial error, the switching gain $\hat{\rho}(t)$ will increase quickly due to this error and not because of a model-mismatch. This may result in a switching gain which is significantly larger then necessary.

(2) Noise on the measurements will prevent $\rho(t)$ to be exactly zero so the adaptive gain will continue to increase.

(3) The adaption law can only increase the gain but never decrease it. So if the circumstances change such that a smaller switching gain is permitted the adaption law is not able to adapt to these new circumstances.

To overcome these drawbacks, the next section introduces another adaption law which does not have these disadvantages.

2.3. *Adaptation method II*

Another way of determining the switching gain is by the adaption law as introduced by Lenz *et al.* (1998) (in Wheeler *et al.* (1998) a similar, but more advanced adaption procedure is used) for the single input case. We give here the straightforward extension to the multiple input case.

For this adaption law we change the discontinuous control component in equation (3) to

$$u_d(t) = \begin{cases} -\rho(t, x)(SB)^{-1}\|\sigma(t)\|^{-1}\sigma(t) & \text{if } \|\sigma(t)\| > \delta \\ -\rho(t, x)(SB)^{-1}\delta^{-1}\sigma(t) & \text{if } \|\sigma(t)\| \le \delta \end{cases}$$

(with $\delta \in \mathbb{R}$ some small positive constant scalar) which is the straightforward vector extension of the scalar saturation function. Now, $u_d(t)$ steers the system within the boundary region $\|\sigma(t)\| < \delta$. Once the system enters the boundary region and stays in it, the system is said to be in *pseudo sliding mode* (Slotine and Li 1991). The effect of this modification is that the discontinuous control part is softened (in fact it is no longer discontinuous) which prevents the chattering effect (Edwards and Spurgeon 1998).

The switching gain $\hat{\rho}(t)$ can now be adapted according to

$$\hat{\rho}(t) = \int_{\upsilon=t_o}^{t} (\|\sigma(\upsilon)\| - \psi)\, d\upsilon \tag{5}$$

where $\psi \in \mathbb{R}$ is a positive constant satisfying $\psi < \delta$. Intuitively, equation (5) is simple to explain: increase the switching gain $\hat{\rho}$ while you are outside the region $\|\sigma(t)\| < \psi$ and decrease $\hat{\rho}$ if $\|\sigma(t)\| < \psi$.

If we compare this adaption law with the drawbacks of the first method then we see that:

(1) In case of a large initial error, the switching gain $\hat{\rho}$ will increase fast due to this error, but once the system has reached the boundary region $\|\sigma(t)\| < \psi$ the switching gain will be decreased again.
(2) Noise on the measurements does not disturb the adaption procedure if the boundary region is chosen sufficiently large.
(3) The method II adaption law seeks the lowest possible switching gain which keeps the system within the boundary region $\|\sigma(t)\| < \delta$. So when the circumstances permit a lower switching gain, the adaption law will automatically adjust the switching gain to the new circumstances.

3. Discrete-time sliding mode control

3.1. *Introduction*

The switching part in a continuous-time sliding mode controller brings the system to the switching surface and keeps the system on the surface despite any modelling errors and disturbances with known bound. The underlying motivation is given by the fact that the switching part can instantaneously react to an error such that it is cancelled out directly. This is obviously no longer possible in discrete-time. The switching function can only change its value at specific time-instances dictated by the sampling frequency. Because of this limitation of the switching time, the system will no longer stay on the switching surface and no 'true' sliding mode will be possible.

We will now define a sliding mode controller for the discrete-time system defined by

$$x[k+1] = Ax[k] + Bu[k] + f[k, x, u] \tag{6}$$

The switching function is defined by

$$\sigma[k] = Sx[k] \tag{7}$$

The design procedure for S can be found in §4. We now search for a controller which fulfills the discrete-time reaching condition (Gao *et al.* 1995)

$$\sigma[k+1] - \sigma[k] = -\rho\frac{\sigma[k]}{\|\sigma[k]\|} - \Phi\sigma[k] \tag{8}$$

($\Phi \in \mathbb{R}^{m\times m}$ being a stable design matrix) from which, together with equation (6), we can determine in a similar way as for the continuous time sliding mode controller the required input to be

$$u[k] = -(SB)^{-1}(SAx[k] - (I_m - \Phi)\sigma[k]) - \rho[k, x, u](SB)^{-1}\frac{\sigma[k]}{\|\sigma[k]\|} \tag{9}$$

In contrast to the continuous-time case where there is only a lower bound on the switching gain, discrete-time sliding mode controllers have an upper bound on the switching gain as well (van den Braembussche 1998). As in the continuous-time case, one has to find an expression for $\rho[k, x, u]$ which is again depending on the disturbance $f[k, x, u]$. In order to do so, we design an adaptive switching gain, as is done in the previous section for the continuous-time case.

3.2. *Adaption method I*

The method I adaption law defined for the continuous-time sliding mode controller (§2.2) can be directly translated to the discrete-time domain by discretizing the integral function in equation (4) by the summation

$$\hat{\rho}[k] = \hat{\rho}[k-1] + \gamma\|\sigma[k]\| \tag{10}$$

where γ is a small positive constant. However, as was already pointed out, 'true' sliding mode is no longer possible in discrete-time. For this reason, the term $\|\sigma[k]\|$ will never converge to zero and consequently the adaptive switching gain will grow unbounded.

3.3. *Adaption method II*

The method II adaption law introduced in §2.3 can be discretized by

$$\hat{\rho}[k] = \hat{\rho}[k-1] + \gamma(\|\sigma[k]\| - \psi)$$

where γ and ψ are small positive constants similar to the continuous time case described in §2.3.

The above adaption law still increases the gain $\hat{\rho}$ until the system remains within in the boundary $\|\sigma\| < \psi$. However, since in discrete-time 'true' sliding mode is no longer achievable, the boundary region $\|\sigma\| < \psi$ cannot be chosen arbitrarily small. If the boundary ψ is chosen smaller then achievable there will not exist a switching gain which is able to keep the system within the selected boundary region, and consequently the adaptive gain will grow unbounded. Hence, ψ has to be chosen carefully. The simulation example in §5 demonstrates this.

4. New adaption method (III)

4.1. *Discrete-time sliding mode definition*

The method II adaption law works in continuous-time rather well but as described in the previous section and demonstrated in §5, it is not always suitable in the discrete-time case. To overcome this problem we introduce a new adaption method which is based on the following definition of discrete-time sliding mode for single input systems, introduced by (Gao *et al.* (1995):

- R_I Starting from any initial state, the trajectory will move monotonically towards the switching plane and cross it in finite time.
- R_{II} Once the trajectory has crossed the switching plane the first time, it will cross the plane again in every successive sampling period, resulting in a zigzag motion about the switching plane.
- R_{III} The size of each successive zigzagging step is non-increasing and the trajectory stays within a specified band.

The above definitions can be extended to multiple input problems by applying the rules to the m entries of the switching function $\sigma_i[k]$ independently (Gao *et al.* 1995). To study the implications of the above rules on the switching gain we first bring the system (6) in the so-called regular form by the orthogonal transformation $T_r \in \mathbb{R}^{n\times n}$

$$\begin{bmatrix} x_1[k] \\ x_2[k] \end{bmatrix} = T_r x[k] \tag{11}$$

where $x_1[k] \in \mathbb{R}^{n-m}$ and $x_2[k] \in \mathbb{R}^m$, resulting in

$$x_1[k+1] = A_{11}x_1[k] + A_{12}x_2[k] + f_u[k,x,u] \tag{12}$$

$$x_2[k+1] = A_{21}x_1[k] + A_{22}x_2[k] + B_2u[k] + f_m[k,x,u] \tag{13}$$

where the matrices in the above equations can be found from

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = T_r A T_r^T \quad \begin{bmatrix} 0 \\ B_2 \end{bmatrix} = T_r B$$

$$[S_1 \;\; S_2] = S T_r^T \quad \begin{bmatrix} f_u[k,x,u] \\ f_m[k,x,u] \end{bmatrix} = T_r f[k,x,u]$$

where $B_2 \in \mathbb{R}^{m\times m}$ has full rank. The term $f_u[k,x,u]$ is called the unmatched uncertainty since it is in the null-space of B, the term $f_m[k,x,u]$ is called the matched uncertainty since it is in the range of B. The switching function in the new coordinate-system is given by

$$\sigma[k] = S_1x_1[k] + S_2x_2[k] \tag{14}$$

If we assume that the closed-loop system is in 'true', or continuous-time, sliding mode (i.e. $\sigma[k] = 0$) then we can write with equations (12) and (14) (setting the unknown term $f_u[k,x,u]$ to zero)

$$x_1[k+1] = \left(A_{11} - A_{12}S_2^{-1}S_1\right)x_1[k] \tag{15}$$

The matrices S_1 and S_2 are design parameters and should be chosen such that the matrix $(A_{11} - A_{12}S_2^{-1}S_1)$ is stable. We choose S_2 such that $S_2B_2 = I_m$ (where I_m is the identity matrix of size m). In this way the ith input only affects the ith component of the switching function. The matrix S_1 can be found by the use of, for example, pole-placement or LQR design.

By applying the invertible coordinate transformation $T_\sigma \in \mathbb{R}^{n\times n}$

$$\begin{bmatrix} x_1[k] \\ \sigma[k] \end{bmatrix} = T_\sigma \begin{bmatrix} x_1[k] \\ x_2[k] \end{bmatrix} \tag{16}$$

the system representation (12) and (13) becomes

$$x_1[k+1] = \bar{A}_{11}x_1[k] + \bar{A}_{12}\sigma[k] + f_u[k,x,u] \tag{17}$$

$$\sigma[k+1] = \bar{A}_{21}x_1[k] + \bar{A}_{22}\sigma[k] + u[k] + S_1f_u[k,x,u] + S_2f_m[k,x,u] \tag{18}$$

where $\bar{A}_{11} = (A_{11} - A_{12}S_2^{-1}S_1)$, $\bar{A}_{12} = (A_{12}S_2^{-1})$, $\bar{A}_{21} = (S_1\bar{A}_{11} + S_2(A_{21} - A_{22}S_2^{-1}S_1))$, and $\bar{A}_{22} = (S_1A_{12}S_2^{-1} + S_2A_{22}S_2^{-1})$. We now propose the control-law

$$u[k] = -\bar{A}_{21}x_1[k] - (\bar{A}_{22} - \Phi)\sigma[k] + u_d[k] \tag{19}$$

where $\Phi \in \mathbb{R}^{m\times m}$ is a diagonal design matrix with diagonal entries $0 \le \phi_i < 1$ and

$$u_d[k] = -\begin{bmatrix} \rho_1 \operatorname{sign}(\sigma_1[k]) \\ \vdots \\ \rho_m \operatorname{sign}(\sigma_m[k]) \end{bmatrix} \tag{20}$$

Note that control law (19) is equal to control-law (9) written in the new coordinates, where the discontinuous control part is changed to the above definition (20). Substituting the control-law (19) into equation (18) leads to

$$\sigma_i[k+1] = \phi_i\sigma_i[k] - \rho_i \operatorname{sign}(\sigma_i[k]) + f_i[k,x,u] \tag{21}$$

where the subscript $i = 1\dots m$ denotes the ith entry of a vector, and the vector $f_i[k,x,u]$ is the shorthand notation for the term $P_iS_2f_m[k,x,u] + P_iS_1f_u[k,x,u]$, where $P_i \in \mathbb{R}^{m\times m}$ is a matrix with the ith diagonal entry is equal to 1 and all other entries equal 0. We assume that the disturbance $f_i[k,x,u]$ is bounded by $\|f_i[k,x,u]\| < F_i$.

It is well known that the rules I, II, and III are satisfied if (Bartoszewicz 1996)

$$\rho_i > \frac{1+\phi_i}{1-\phi_i}F_i \tag{22}$$

If the above condition is met then the system will converge in finite time to the quasi-sliding mode band Δ given by

$$\Delta_i = \rho_i + F_i \tag{23}$$

The above expression clearly demonstrates that the quasi-sliding mode band is a function of the switching gain. It is desired to make the quasi-sliding mode band as small as possible, therefore the switching gain should be chosen as small as possible. Taking this into account, the definition of sliding mode (Rules I, II, and III) can be used to formulate an adaption law for the switching gain. This is introduced in the next section.

4.2. *Introducing the new adaption law (method III)*

The three rules for discrete-time sliding mode introduced in the previous section can be used as a guideline for an adaptive switching gain. The following adaption law exploits rule R_{II}

$$\hat{\rho}_i[k] = |\hat{\rho}_i[k-1] + \gamma_i \operatorname{sign}(\sigma_i[k]) \operatorname{sign}(\sigma_i[k-1])| \quad (24)$$

where $\gamma_i > 0 \in \mathbb{R}^+$ is the adaption constant which determines the speed of adaption of the ith switching gain. According to rule R_{II}, the switching surface should be crossed in every successive time step. The above adaption law increases the switching gain if the surface is not crossed and decreases the switching gain if the surface was crossed.

We study the adaption law by use of equation (21). The first theorem states that, regardless of the initial conditions (though bounded), the switching surface is crossed in finite time.

Theorem 1: *For any bounded* $\sigma_i[0]$ *and* $\hat{\rho}_i[0] > 0$ *there exists a finite time p such that* $\operatorname{sign}(\sigma_i[p]) = -\operatorname{sign}(\sigma_i[0])$.

Proof: If $\sigma_i[0] < 0$. Then if $\sigma_i[j] < 0, \forall 0 < j < p$, we can write for $\sigma_i[p]$

$$\sigma_i[p] = \phi_i^p \sigma_i[0] + \hat{\rho}_i[0] \sum_{j=0}^{p-1} \phi_i^j + \sum_{j=1}^{p-1} (p-j)\phi_i^{j-1}\gamma_i - \sum_{j=0}^{p-1} \phi_i^{p-1-j} f_i[j]$$

Using the fact that $|f_i[k]| < F_i$ and $\sum_{j=0}^{l} r^j < 1/(1-r)$ we obtain the expression

$$\sigma_i[p] > \phi_i^p \sigma_i[0] + \hat{\rho}_i[0] \sum_{j=0}^{p-1} \phi_i^j + \sum_{j=1}^{p-1} (p-j)\phi_i^{j-1}\gamma_i - \frac{1}{1-\phi_i} F_i$$

Looking at the right-hand terms, we see that the negative terms $\phi_i^p \sigma_i[0]$ and $-1/(1-\phi_i x)F_i$ are bounded. The positive term $\hat{\rho}_i[0] \sum_{j=0}^{p-1} \phi_i^j$ is bounded as well, but the positive term $\sum_{j=1}^{p-1}(p-j)\phi_i^{j-1}\gamma_i$ is growing with p. Therefore, there exists some time instant p for which the right-hand side, and consequently $\sigma_i[p]$, is positive. For the case where $\sigma_i[0] > 0$ the proof follows along the same lines. □

The above theorem states that the sliding surface is crossed in finite time. As the next theorem states, this also implies that, under some additional conditions, the switching gain has become large enough to cross the sliding surface in finite time.

Theorem 2: *If* $\operatorname{sign}(\sigma_i[p+1]) = -\operatorname{sign}(\sigma_i[p])$ *and* $\operatorname{sign}(f_i[p]) = \operatorname{sign}(\sigma_i[p])$ *then* $\hat{\rho}_i[p] > |f_i[p]|$.

Proof: We can write for $\sigma_i[p+1]$

$$\sigma_i[p+1] = \phi_i \sigma_i[p] - \hat{\rho}_i[p] \operatorname{sign}(\sigma_i[p]) + f_i[p]$$

Making $\hat{\rho}_i[p]$ explicit results in

$$\hat{\rho}_i[p] = \frac{\phi_i \sigma_i[p]}{\operatorname{sign}(\sigma_i[p])} - \frac{\sigma_i[p+1]}{\operatorname{sign}(\sigma_i[p])} + \frac{f_i[p]}{\operatorname{sign}(\sigma_i[p])}$$

Since $\operatorname{sign}(\sigma_i[p+1]) = -\operatorname{sign}(\sigma_i[p])$ it follows that

$$\hat{\rho}_i[p] = |\phi_i \sigma_i[p]| + |\sigma_i[p+1]| + \frac{f_i[p]}{\operatorname{sign}(\sigma_i[p])}$$

By assumption $\operatorname{sign}(f_i[p]) = \operatorname{sign}(\sigma_i[p])$ hence $f_i[p]/\operatorname{sign}(\sigma_i[p]) = |f_i[p]|$. Furthermore, $|\sigma_i[p]| > 0$ and $|\sigma_i[p+1]| > 0$, leading to

$$\hat{\rho}_i[p] > |f_i[p]|$$

Which proves the theorem. □

Theorem 1 states that the system will cross the switching surface in finite time, starting from any initial condition. Consequently, the system will cross the switching surface over and over again. Then Theorem 2 states that under the condition that at the moment of crossing the switching surface the sign of the disturbance is the same as the sign of the switching function, the switching gain is larger than the absolute value of the disturbance. Therefore it may be concluded, especially for slowly varying disturbances, that the switching gain will pass some lower bound.

Before it is shown that the switching gain does not grow unbounded, the notion of a p-cycle is introduced in the Definition 1.

Definition 3: With a p-cycle it is meant that $\operatorname{sign}(\sigma_i[k]) = \operatorname{sign}(\sigma_i[k+p])$, while $\operatorname{sign}(\sigma_i[k]) = -\operatorname{sign}(\sigma_i[k+i]) \forall i = \{1 ... p-1\}$.

The value of the switching value after a p-cycle can easily be determined, which is described in Lemma 1.

Lemma 4: *Given the adaption law* (24)*, the value of* $\hat{\rho}_i[k+p]$ *after a p-cycle will be*

$$\hat{\rho}_i[k+p] = \hat{\rho}_i[k] + (p-4)\gamma_i$$

Proof: Within every p-cycle, $\sigma_i[k]$ changes sign only twice. All other signs will be equal. This means that γ_i is subtracted twice from $\hat{\rho}_i[k]$ and added $p-2$ times to $\hat{\rho}_i[k]$, which adds effectively $(p-4)\gamma_i$ to $\hat{\rho}_i[k]$. □

Clearly, Lemma 1 states that the switching gain over one p-cycle is:

- decreasing for $p < 4$;
- constant for $p = 4$;
- increasing for $p > 4$;

If the switching gain fulfils (22), then it follows that the closed-loop will go into a 2-cycle. For 2-cycles, the switching gain is decreasing and therefore the switching gain is bounded from above.

In the case of a constant disturbance, the system will settle down in a 4-cycle instead. In this case, the switching gain remains constant in an average sense. The next theorem gives the upper and lower bound of the switching gain in a 4-cycle for constant disturbances.

Theorem 3: *Consider the system* (21) *with a constant disturbance* $-F_i$, *adaption law* (24) *and* $0.58 < \phi_i < 1$. *In steady-state the adaptive gain* $\hat{\rho}_i[k]$ *(where k is the time instant where* $\sigma_i[k] > 0$*) will be in the region*

$$\frac{1+\phi_i+\phi_i^2+\phi_i^3}{1+\phi_i+\phi_i^2-\phi_i^3}F_i - \frac{1-\phi_i^2}{1+\phi_i+\phi_i^2-\phi_i^3}\gamma_i < \hat{\rho}[k] < \frac{1+\phi_i+\phi_i^2+\phi_i^3}{1+\phi_i-\phi_i^2+\phi_i^3}F_i + \frac{\phi_i-\phi_i^3}{1+\phi_i-\phi_i^2+\phi_i^3}\gamma_i \quad (25)$$

Proof: In steady-state the system will be in a 4-cycle and, assuming $\sigma_i[k] > 0$

$$\sigma_i[k+1] = \phi_i\sigma_i[k] - \hat{\rho}_i[k] - F_i$$

$$\sigma_i[k+2] = \phi_i^2\sigma_i[k] + (1-\phi_i)\hat{\rho}_i[k] - (1+\phi_i)F_i - \gamma_i$$

$$\sigma_i[k+3] = \phi_i^3\sigma_i[k] + (1+\phi_i-\phi_i^2)\hat{\rho}_i[k] - (1+\phi_i+\phi_i^2)F_i - \phi_i\gamma_i$$

$$\sigma_i[k+4] = \phi_i^4\sigma_i[k] + (1+\phi_i+\phi_i^2-\phi_i^3)\hat{\rho}_i[k] - (1+\phi_i+\phi_i^2+\phi_i^3)F_i + (1-\phi_i^2)\gamma_i$$

As can be found in Lemma 2 (see the appendix), $\sigma_i[k+1] < 0$. Therefore, a 4-cycle also implies $\sigma_i[k+2] < 0$, $\sigma_i[k+3] < 0$, and $\sigma_i[k+4] > 0$, leading to the conditions

$$\hat{\rho}_i[k] < \frac{1+\phi_i+\phi_i^2+\phi_i^3}{-1+\phi_i+\phi_i^2+\phi_i^3}F_i - \frac{\phi_i-\phi_i^3}{-1+\phi_i+\phi_i^2+\phi_i^3}\gamma_i$$

$$\hat{\rho}_i[k] < \frac{1+\phi_i+\phi_i^2+\phi_i^3}{1-\phi_i+\phi_i^2+\phi_i^3}F_i + \frac{1-\phi_i^2}{1-\phi_i+\phi_i^2+\phi_i^3}\gamma_i$$

$$\hat{\rho}_i[k] < \frac{1+\phi_i+\phi_i^2+\phi_i^3}{1+\phi_i-\phi_i^2+\phi_i^3}F_i + \frac{\phi_i-\phi_i^3}{1+\phi_i-\phi_i^2+\phi_i^3}\gamma_i$$

$$\hat{\rho}_i[k] > \frac{1+\phi_i+\phi_i^2+\phi_i^3}{1+\phi_i+\phi_i^2-\phi_i^3}F_i - \frac{1-\phi_i^2}{1+\phi_i+\phi_i^2-\phi_i^3}\gamma_i$$

Taking the lowest upper bound and the lower bound results in (25). □

In the Appendix (Lemma 3) it is shown for the case of constant disturbances that shorter p-cycles are achieved by larger switching gains, and conversely, longer p-cycles for smaller gains. By Lemma 1, the switching will be increased for p-cycles with $p > 4$ and decreased for $p < 2$ decreased. Therefore, assuming that γ is chosen sufficiently small (ideally $\gamma << F_i$), the switching gain will always be driven into the region given by Theorem 3.

All analysis so far has focused on only one entry of the switching function. Because of the choice of S_2, each entry σ_i of σ is coupled to only one input u_i. Therefore, we can treat all entries of the switching function separately.

4.3. *Extensions of method III*

In case of a constant disturbance, the adaptive gain converges to the region given by Theorem 3. Within this region, no further adaption takes place. However it is desirable to converge to the minimal switching gain which still results in a 4-cycle. To ensure this, the adaption procedure (24) could be changed to

$$\hat{\rho}_i[k] = |\hat{\rho}_i[k-1] + \gamma_i \operatorname{sign}(\sigma_i[k]) \operatorname{sign}(\sigma_i[k-1]) - \beta_i| \quad (26)$$

where $0 < \beta_i < \gamma_i$. The value of the switching gain after a p-cycle is now given by

$$\hat{\rho}_i[k+p] = \hat{\rho}_i[k] + (p-4)\gamma_i - p\beta_i$$

which can be obtained in a similar way as presented in Lemma 1.

Also, the proposed control strategy introduces a larger deviation from $\sigma_i[k] = 0$ then the disturbance itself. Especially for constant disturbances, the use of the proposed discontinuous control may lead to an excessive switching gain. Therefore, we could change the definition of the discontinuous control part (20) to

$$u_{i,d}[k] = \begin{cases} -\hat{\rho}_i^+[k] & \text{if} \, \sigma_i[k] \geq 0 \\ \hat{\rho}_i^-[k] & \text{if} \, \sigma_i[k] < 0 \end{cases} \quad (27)$$

where $u_{i,d}$ is the ith component of the discontinuous control vector. The adaptive gains $\hat{\rho}_i^+[k]$ and $\hat{\rho}_i^-[k]$ should be adapted according to the routine

$$\text{if } \sigma_i[k-1] \geq 0: \quad \begin{aligned} \hat{\rho}_i^+[k] &= \hat{\rho}_i^+[k-1] + \gamma_i \operatorname{sign}(\sigma_i[k]) \operatorname{sign}(\sigma_i[k-1]) \\ \hat{\rho}_i^-[k] &= \hat{\rho}_i^-[k-1] \end{aligned}$$

$$\text{if } \sigma_i[k-1] < 0: \quad \begin{aligned} \hat{\rho}_i^-[k] &= \hat{\rho}_i^-[k-1] + \gamma_i \operatorname{sign}(\sigma_i[k]) \operatorname{sign}(\sigma_i[k-1]) \\ \hat{\rho}_i^+[k] &= \hat{\rho}_i^+[k-1] \end{aligned}$$

Note that in this case the switching gains can become negative as well.

5. Simulation example

As a simulation example of the proposed controller set-up we have chosen the hover control of a Bell 205 helicopter. The simulation model can be obtained from

Pieper *et al.* (1996) or Trentini and Pieper (2001). The linear model is represented by the differential state equation

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{28}$$

with the matrices

$$A = \begin{bmatrix} 0 & 0.03 & 0.18 & -0.01 & -0.42 & 0.08 & -9.810 & \\ -0.10 & -0.39 & 0.09 & -0.10 & -0.72 & 0.68 & 0 & 0 \\ 0.01 & -0.01 & -0.19 & 0 & 0.23 & 0.04 & 0 & 0 \\ 0.02 & 0 & -0.41 & -0.05 & -0.27 & 0.27 & 0 & 9.81 \\ 0.03 & -0.02 & -0.88 & -0.04 & -0.57 & 0.14 & 0 & 0 \\ -0.01 & -0.02 & -0.06 & 0.07 & -0.32 & -0.71 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0.08 & 0.13 & 0 & 0 \\ -1.17 & 0.04 & 0 & 0.01 \\ 0 & -0.07 & 0 & 0.01 \\ -0.04 & 0 & 0.11 & 0.19 \\ -0.04 & 0 & 0.22 & 0.17 \\ 0.17 & 0 & 0.03 & -0.47 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The states and input variables are given by

$$x(t) = \begin{bmatrix} \text{forward velocity} \\ \text{vertical velocity} \\ \text{pitch rate} \\ \text{lateral velocity} \\ \text{roll rate} \\ \text{yaw rate} \\ \text{pitch attitude} \\ \text{roll attitude} \end{bmatrix} \quad u(t) = \begin{bmatrix} \text{collective} \\ \text{longitudinal cyclic} \\ \text{lateral cyclic} \\ \text{tail rotor collective} \end{bmatrix}$$

The model is discretized (zero order hold) with sampling frequency $T_s = 5\,\text{ms}$. The parameters of the sliding mode controller are given by Φ being a $m \times m$ zero matrix, and

$$S = \begin{bmatrix} 0.0428 & -0.1687 & -0.0167 & 0.0028 & -0.0013 & -0.0029 & 0.0002 & 0.0005 \\ 1.1753 & 0.0788 & -0.6302 & 0.0064 & -0.0001 & -0.0101 & 0.0132 & 0.0001 \\ -0.0084 & 0.0200 & -0.0022 & 0.3036 & 0.7072 & 0.3785 & -0.0003 & 0.0183 \\ 0.0196 & -0.0584 & 0.0031 & 0.0889 & 0.0081 & -0.3884 & -0.0012 & 0.0002 \end{bmatrix}$$

S has been obtained by LQR design. The non-zero components of the desired state, the vertical and lateral velocity (i.e. 2nd and 4th components of the state vector), are depicted in figure 1.

In simulation, the system matrix A is perturbed by the matrix δ_A (i.e. $A_s = A + \delta_A$) given by

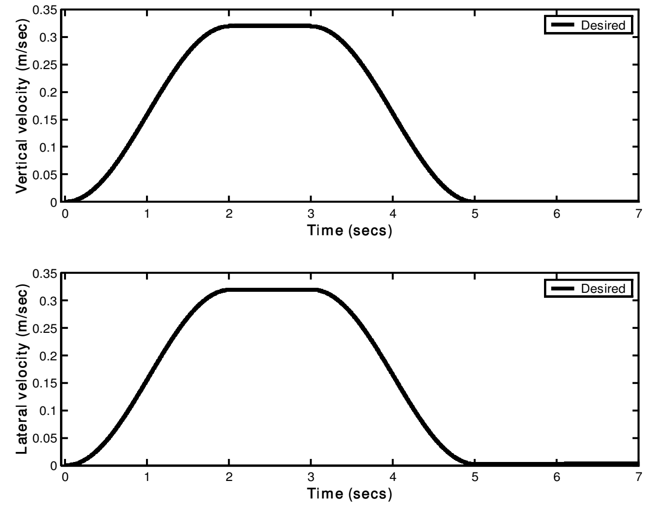

Figure 1. Non-zero desired states, all other desired state variables are zero.

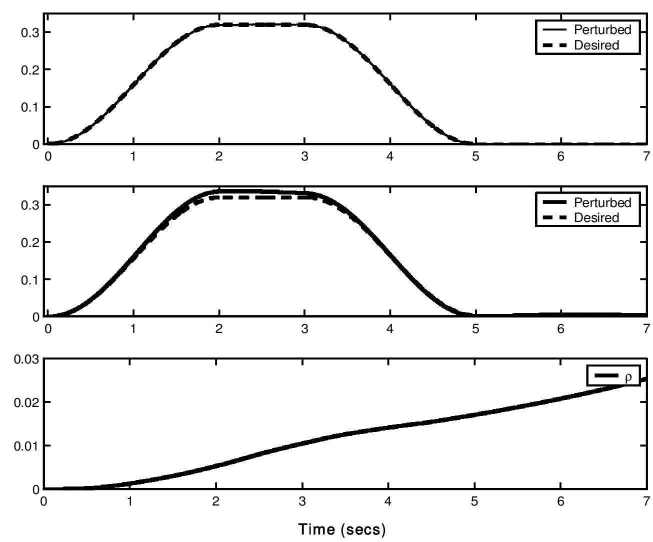

Figure 2. Simulation results for the method II adaption method with $\gamma = 5e^{-3}$, $\delta = 1e^{-2}$, and $\psi = \frac{1}{2}\delta$.

$$\delta_A = 1e^{-4} * \begin{bmatrix} 0.4564 & 0.1759 & 0.8124 & 0.0154 & 0.2044 & 0.7081 & 0.1285 & 0.8224 \\ 0.0185 & 0.4048 & 0.0089 & 0.7464 & 0.6704 & 0.4302 & 0.6970 & 0.6815 \\ 0.8211 & 0.9336 & 0.1370 & 0.4444 & 0.8339 & 0.3082 & 0.3381 & 0.8398 \\ 0.4443 & 0.9152 & 0.2056 & 0.9312 & 0.0174 & 0.1936 & 0.8382 & 0.7059 \\ 0.6154 & 0.4095 & 0.1995 & 0.4656 & 0.6776 & 0.1955 & 0.8235 & 0.3648 \\ 0.7916 & 0.8919 & 0.6045 & 0.4182 & 0.3741 & 0.6841 & 0.5547 & 0.3103 \\ 0.9220 & 0.0578 & 0.2697 & 0.8459 & 0.8278 & 0.3040 & 0.4513 & 0.3827 \\ 0.7381 & 0.3522 & 0.2004 & 0.5249 & 0.5000 & 0.5423 & 0.8636 & 0.5598 \end{bmatrix}$$

Simulation results for the discretized method II adaption method are shown in figure 2 for the parameters $\gamma = 5e^{-3}$, $\delta = 1e^{-2}$, and $\psi = \frac{1}{2}\delta$. Clearly, for these parameters the adaption method is unstable because the parameters δ and ψ have been chosen to small. Figure 3 presents the simulation results for the method III under the same circumstances. In this case, the single switching gain of the method II controller is

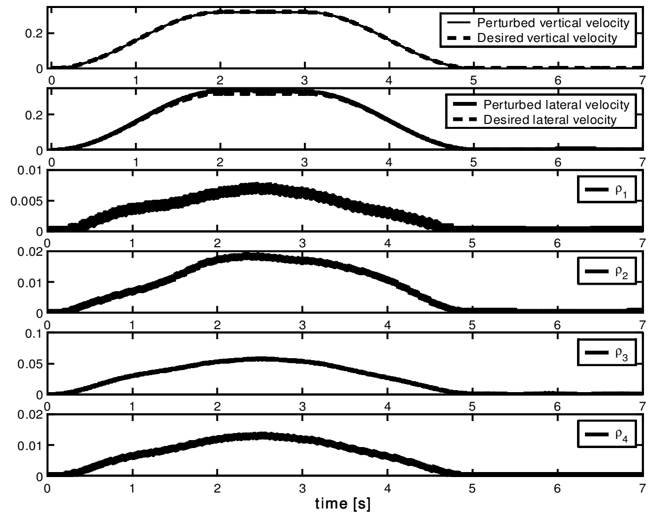

Figure 3. Simulation results for the method III adaption method with $\gamma = 5e^{-3}$ and $\beta = 0.1\gamma$.

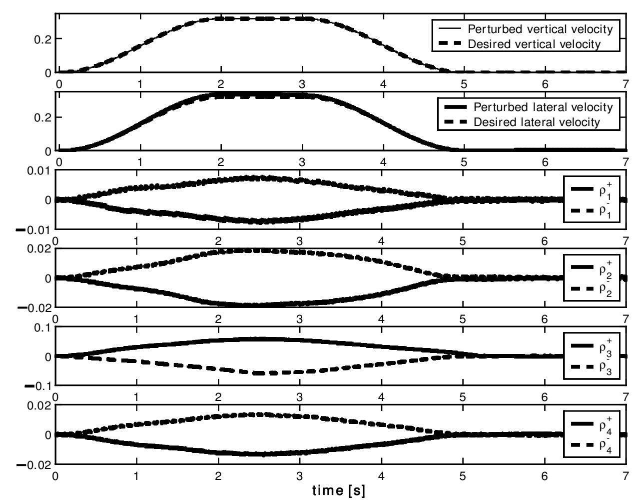

Figure 4. Simulation results for the extended method III adaption method ($\gamma = 5e^{-3}$ and $\beta = 0.1\gamma$).

replaced by four independent switching gains, each associated with one component of the switching function σ. As can be seen in figure 3, the switching gains remain bounded and the desired state is tracked with high accuracy. Finally, figure 4 presents the simulation results for the extended method III, where the discontinuous control part is changed as presented in §4.3, equation (27). In this case, the switching gains are again stable. The main difference between the (regular) method III and the extended method III can be seen by plotting the input signals, which is done in figure 5. The advantage of the extended method III is obvious, the high frequency component of the input signal for the regular method III has largely been suppressed.

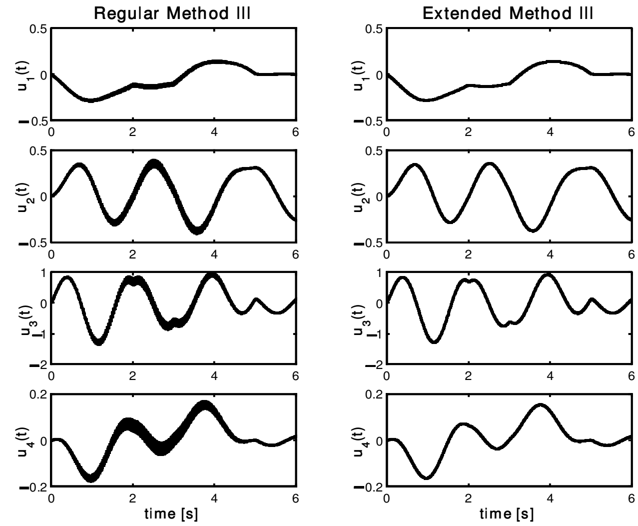

Figure 5. Input signals for the regular (left) and the extended (right) method III adaption law ($\gamma = 5e^{-3}$ and $\beta = 0.1\gamma$).

6. Conclusions

In this paper, a new adaption law for the switching gain was introduced. It has been specifically designed for discrete-time sliding mode controllers. This new method proved to have an important advantage over the discretized version of the adaptive gain introduced in Lentz *et al.* (1998), namely that there is no danger of instability of the adaption procedure because of a bad choice of adaption parameters. With the latter method a boundary region (quasi-sliding mode band) within which (discrete-time) sliding mode will take place has to be selected. This region can be chosen smaller than achievable in which case the adaptive gain can grow unbounded. With the new adaption law this can no longer happen. Simulation results visualize the above statements.

Acknowledgements

This work was supported by Brite-Euram under contract number BRPR-CT97-0508 and project number BE-97-4186.

Appendix

Lemma 2: *Consider the system* (21) *subject to the constant disturbance* $f_i[k] = -F_i$ *with adaption law* (24) *and* $0 \le \phi_i < 1$. *If* $\sigma_i[k-1] < 0$, $\sigma_i[k] > 0$ *and* $\gamma_i \ll F_i$, *then* $\sigma_i[k+1] < 0$.

Proof: By assumption $\sigma_i[k-1] < 0$, leading with (21) to

$$\sigma_i[k] = \phi_i \sigma_i[k-1] + \hat{\rho}_i[k-1] - F_i$$

Since $\sigma_i[k] > 0$ we know that $\hat{\rho}_i[k-1] > F_i - \phi_i\sigma_i[k-1]$. Using this lower bound for $\hat{\rho}_i[k-1]$ we can determine $\sigma_i[k+1]$ to be

$$\sigma_i[k+1] < \phi_i^2\sigma_i[k-1] - 2F_i + \gamma_i$$

from which it follows that $\sigma_i[k+1] < 0$ if $\gamma_i < 2F_i + |\phi_i^2\sigma_i[k-1]|$. Since $\gamma \ll F_i$ the latter condition is always met. □

The following lemma relates the value of the switching gain to length of the p-cycle.

Lemma 3: *Consider the system* (21) *subject to the constant disturbance* $f_i[k] = -F_i$ *with adaption law* (24), $0 \le \phi_i < 1$ *and* $\gamma_i \ll F_i$. *If* $\sigma_i[k-1] < 0$ *and* $\sigma_i[k] > 0$ *then* $\sigma_i[k+p^*] > 0$ *where the smallest* p^* *can be found from:*

- $p^* = 2$ if

$$\hat{\rho}_i[k] > \frac{-\phi_i^2\sigma_i[k] + (1+\phi_i)F_i + \gamma_i}{1-\phi_i} \qquad (29)$$

- p^* is the smallest p satisfying

$$\hat{\rho}_i[k] > \frac{-\phi_i^p\sigma_i[k] + \sum_{j=0}^{p-1}\phi_i^j F_i + (\phi_i^{p-2} - \sum_{j=1}^{p-2}\phi_i^{j-1}(p-j-1)\gamma_i}{\sum_{j=0}^{p-2}\phi_i^j - \phi_i^{p-1}} \qquad (30)$$

Proof: According to Lemma 2 $\sigma_i[k+1]$ is always negative, so $p \ge 2$. For $\sigma_i[k+2]$ we can write

$$\sigma_i[k+2] = \phi_i^2\sigma_i[k] - (1+\phi_i)F_i + (1-\phi_i)\hat{\rho}[k] - \gamma_i$$

which leads to condition (29). The subsequent values for $\sigma_i[k+p]$ (while $\sigma_i[k+j] < 0, \forall j = 1p-1$ and $p > 2$) can be found from

$$\sigma_i[k+p] = \phi_i^p\sigma_i[k] - \sum_{j=0}^{p-1}\phi_i^j F_i - \left(\phi_i^{p-1} - \sum_{j=0}^{p-2}\phi_i^j\right)\hat{\rho}_i[k] - \left(\phi_i^{p-2} - \sum_{j=1}^{p-2}\phi_i^{j-1}(p-j-1)\right)\gamma_i$$

which leads to condition (30). □

From the above lemma it can be concluded that for smaller switching gains $\hat{\rho}_i[k]$ the system will be in a longer p-cycle than for larger switching gains.

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