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Douven, I.; Romeyn, J.-W.

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The Discursive Dilemma as a Lottery Paradox

Igor Douven
Institute of Philosophy, University of Leuven
igor.douven@hiw.kuleuven.be

Jan-Willem Romeijn
Department of Psychology, University of Amsterdam
j.w.romeijn@uva.nl

Abstract

List and Pettit have stated an impossibility theorem about the aggregation of individual opinion states. Building on recent work on the lottery paradox, this paper offers a variation on that result. The present result places different constraints on the voting agenda and the domain of profiles, but it covers a larger class of voting rules, which need not satisfy the proposition-wise independence of votes.

The discursive dilemma concerns the question of how to determine the opinion state of a collective on the basis of the opinion states of its members. List and Pettit [2002] have stated an impossibility theorem about voting rules, that is, rules which are meant to answer the aforementioned question. Building on recent work on the lottery paradox, we show that their result persists if certain assumptions are added while the arguably most problematic condition of their theorem is relaxed. Specifically, we employ a voting agenda with richer logical structure and focus only on certain voting profiles, but in exchange for that we need not assume that votes on separate propositions are independent, or that the collective opinion profile is complete.

We start by rehearsing the discursive dilemma, List and Pettit's impossibility theorem, and the ways in which the present result deviates from it. Then we report a generalisation of the lottery paradox and exhibit the salient structural similarity between the discursive dilemma and the generalised version of the lottery paradox. Finally, we use this similarity to produce a new impossibility result, and we review its conditions in relation to those of List and Pettit's theorem. We also explain briefly how our result relates to another impossibility theorem by Pauly and van Hees [2006].

1. The Discursive Dilemma. Consider a parliament whose members each have individual opinions on some designated set of propositions, and imagine that the parliament must come to a collective opinion on this set. To this aim the parliament may employ some voting rule, which transforms the individual opinions regarding the propositions into an opinion for the parliament as a whole. A standard rule is majority voting, but many other voting rules are possible. Now, if the members of the parliament all have consistent opinion states, one would expect that there exist voting rules that guarantee that the parliament has a consistent collective opinion state, too. However, as List and Pettit [2002] have shown, if voting rules are required to satisfy certain minimal and prima facie plausible conditions, this is not so.

To make their result precise, we first need to settle some logical and notational issues. Let the voting agenda Φ be a set containing at least two propositions that are contingent and logically independent of each other, and be closed under the relation of standard logical consequence, meaning that any proposition logically entailed by Φ is also an element of it. A valuation $v\colon \Phi \to \{0,1\}$ is said to be consistent iff there is no $\Psi \subseteq \Phi$ such that $v(\psi) = 1$, for all $\psi \in \Psi$, and Ψ entails $\bot$, the inconsistent proposition; it is said to be complete iff $v(\varphi) = 1$ or $v(\neg\varphi) = 1$ for all $\varphi \in \Phi$; and it is said to be closed under logical consequence iff for all $\Psi \subseteq \Phi$ and all $\varphi \in \Phi$, if $v(\psi) = 1$ for all $\psi \in \Psi$ and Ψ logically entails φ, then $v(\varphi) = 1$. Let V be the set of all valuations on Φ, and $V_\star$ the set of consistent and complete valuations. Note that it follows from the definitions of consistency and completeness and the closure conditions on Φ that each $v \in V_\star$ is closed under logical consequence.[1]

Further, let $M = \{m_1, \ldots, m_n\}$ be a parliament with members m_i and $n \geqslant 2$. Each member m_i is associated with a consistent and complete valuation $v_i \in V_M$, where v_i can be thought of as the member's individual opinion state (at least with respect to Φ; we take this relativization to be implied from now on) and $V_M \subseteq V_\star$ is the set of valuations the members of M are allowed to adopt as individual opinion states.[2] Let $V_0 \subseteq V$ be the set of allowed collective valuations; note that these valuations are not by definition consistent or complete. Finally, a voting rule for the parliament is defined to be a function $r\colon (V_M)^n \to V_0$. Recall that the valuations v_i with $i \geqslant 0$ are themselves functions over a set of propositions, $v_i\colon \Phi \to \{0,1\}$. Thus, a voting rule can be decomposed into—possibly partial—functions r_φ for all propositions $\varphi \in \Phi$ separately, according to $r_\varphi(v_1, \ldots, v_n) = \big(r(v_1, \ldots, v_n)\big)(\varphi)$ for all $\langle v_1, \ldots, v_n\rangle \in (V_M)^n$. Note also that, since a voting rule is a function, rules that render the collective opinion empty do not qualify.

With these preliminaries in place, we can state List and Pettit's [2002] impossibility result, as follows:

Proposition 1.1 *There is no voting rule that satisfies all of the following requirements:*

- Universal Domain. *Members of the parliament are allowed to adopt* any *consistent and complete valuation of Φ as their individual opinion state, that is, $V_M = V_\star$.*

- Consistent and Complete Range. *The range of the voting rule r is restricted to the set of* consistent *and* complete *valuations, that is, $V_0 = V_\star$.*

[1] For suppose there is some $\Psi \subset \Phi$, a $\varphi \in \Phi$, and a $v \in V_\star$ such that $v(\psi) = 1$ for all $\psi \in \Psi$, and Ψ entails φ, but $v(\varphi) = 0$. Then, because Φ is supposed to be closed under logical consequence, $\varphi \in \Phi$. Because v is complete and, by supposition, $v(\varphi) = 0$, it must be that $v(\neg\varphi) = 1$. Thus, for all $\chi \in \Psi \cup \{\neg\varphi\}$, $v(\chi) = 1$. But because Ψ entails φ, the union set $\Psi \cup \{\neg\varphi\}$ entails $\bot$, and this contradicts the consistency of v.

[2] We throughout speak of parliaments. However, this is no more than a stylistic choice. Everything to be said about parliaments applies equally well to any other kind of voting body whose members have complete, consistent, and deductively closed individual opinion states.

- Anonymity. *All members of the parliament have an equal say in the collective opinion, that is, for any permutation* $u\colon M \to M$ *of members we have* $r(v_1, \ldots, v_n) = r\big(u(v_1), \ldots, u(v_n)\big)$.

- Neutrality. *All propositions on the agenda are voted for in the same way, that is, for any permutation* $f\colon \Phi \to \Phi$ *of propositions and any pair of* n*-tuples of valuations* $\langle v_1, \ldots, v_n\rangle$ *and* $\langle v'_1, \ldots, v'_n\rangle$*, if for all* $\varphi \in \Phi$ *and all* $i \in \{1, \ldots, n\}$ *we have* $v_i(\varphi) = v'_i(f(\varphi))$*, then* $r_\varphi = r_{f(\varphi)}$.

- Independence. *The collective opinion on a proposition is a function strictly of the individual opinions on it, that is, for all* $\varphi \in \Phi$*, if* $v_i(\varphi) = v'_i(\varphi)$ *for all* $i \in \{1, \ldots, n\}$*, then* $r_\varphi(v_1, \ldots, v_n) = r_\varphi(v'_1, \ldots, v'_n)$.

List and Pettit [2002] specify the last two conditions as a conjunction under one label, *Systematicity*, but following Pauly and van Hees [2006] we have stated the conjuncts separately; this facilitates a comparison of Proposition 1.1 with our result to be presented later.

Pauly and van Hees generalize Proposition 1.1 partly in ways other than we intend to pursue. One of their generalizations is that they allow valuations which can take on more than two values, so that members can for example abstain from voting. A further generalization is that they weaken Anonymity. They replace this condition with *Responsiveness* and *Non-Dictatorship*. Responsiveness says that, for at least two propositions, the collective opinion on them is not the same given any possible collection of individual opinion states, that is, there exist distinct propositions φ and ψ such that $r_\varphi(v_1, \ldots, v_n) \neq r_\varphi(v'_1, \ldots, v'_n)$ and $r_\psi(v_1, \ldots, v_n) \neq r_\psi(v'_1, \ldots, v'_n)$, for some $\langle v_1, \ldots, v_n\rangle, \langle v'_1, \ldots, v'_n\rangle \in (V_M)^n$. Non-Dictatorship says that the parliament must not be a dictatorship, meaning that the collective opinion state must not, as a rule, coincide with the opinion state of some designated individual. [* Note that Non-Dictatorship is entailed by the conditions of List and Pettit. To see this, consider the condition of *Unanimity*, which a voting rule is said to satisfy iff it includes in the collective opinion state only propositions on which the votes are unanimous. List and Pettit rule out Unanimity because it violates the completeness of the collective opinion. But under the assumption of Anonymity, Dictatorship comes down to assuming Unanimity, because if one individual determines the collective profile and if individuals are interchangeable, then all individuals do. So for List and Pettit, assuming Completeness, and therefore ruling out Unanimity, automatically rules out Dictatorship.

In this paper, we focus primarily on List and Pettit's condition of Systematicity. List and Pettit [2002:99] seem right that the other conditions mentioned in Proposition 1.1 are hardly contestable, but that Systematicity may be more controversial. In section 4 of their paper, they briefly consider the possibility of relaxing Systematicity, more in particular the component of Neutrality, which requires that for all propositions, inclusion (or otherwise) in the collective opinion state depends on the individual opinions in the same way. Pauly and van Hees are able to eliminate the condition of Neutrality by making some strong assumptions about the logical properties of the agenda. Dietrich and List [2006a] considerably weaken Neutrality to the condition of Unbiasedness, which is the requirement that only the voting rules for a proposition and its negation must be identical. It will be seen that, although our result does not permit a complete elimination of Neutrality, it does permit a significant weakening of this condition.

The main focus of the present result, however, is on the other component of Systematicity, namely Independence. All results to date rely on this condition, *] according to which inclusion of a proposition in the collective opinion state should depend exclusively on the individual opinions on *that* proposition. In our view this is an unreasonably strong requirement. Imagine a voting rule that accepts a proposition in the collective opinion state if a majority agrees with it, *provided* there do not exist majorities for other propositions that jointly undermine the former proposition, where "undermine" could be cashed out in various ways, for instance in terms of forming a coherent set of propositions on their own, but an incoherent one when conjoined with the proposition voted on.[3] While that rule may prove to be untenable on close scrutiny, one certainly would not want to reject it offhand. However, the prospects for saying anything informative about voting rules might seem bleak once Independence is dropped. For surely there are indefinitely many ways already to amend the proviso of the previous example; and of course a voting rule need not even make majority agreement a requirement for acceptance. Nevertheless, a remarkably general result concerning voting rules can be obtained that also applies to ones that violate Independence, and it can be obtained almost for free. For it follows immediately from a recent result concerning the lottery paradox, once we have exhibited the structural similarity between that paradox and the discursive dilemma. What the result shows is that a voting rule may let the collective verdict depend on the opinions on as many propositions as one likes, and in ways as complex as one likes; as long as this dependence is definable in formal terms (in a sense to be made precise), there still is no guarantee that application of the rule to consistent individual opinion states results in a consistent collective opinion state.

[* Finally, the present result addresses the condition of Consistent Complete Range. Gärdenfors [2006] and Dietrich and List [2006b] have recently proved impossibility results for an incomplete range of the voting rule. As will be seen, our result also allows for an incomplete range. Specifically, it only employs that the collective profile is closed under conjunction. In sum, by making different agenda assumptions than in the above, the present paper arrives at an impossibility result in which Neutrality is weakened, and in which Independence and Complete Range can be dropped. *]

2. The Lottery Paradox. It has seemed plausible to many that high but nonperfect probability is sufficient for rational acceptability. However, Kyburg's [1961] so-called lottery paradox shows that, its plausibility notwithstanding, this idea cannot be maintained, at least not if we also want to maintain that rational acceptability is closed under conjunction (meaning that if two propositions are rationally acceptable then so is their conjunction). The argument goes as follows: Suppose you own a ticket in a large and fair lottery with exactly one winner. Then although it is highly unlikely that your ticket is the winner, this cannot make it rational to accept that your ticket won't win. If it did, then by the same token it should be rational to

[3]And where in turn the notion of coherence could be understood along the lines of one of the probabilistic theories of coherence that have been proposed of late.

accept of each of the other tickets that they won't win, for all tickets have the same high probability of losing. And by conjunctive closure that would make it rational to accept that no ticket will win, contradicting our knowledge that the lottery has a winner.

In response to this, some philosophers have proposed to abandon the idea that rational acceptability is closed under conjunction. Arguably, however, this proposal has some quite unpalatable consequences (see Douven [2002, Sect. 2] for an overview; see also Douven and Williamson [2006]). On a more popular approach, high probability *defeasibly* warrants rational acceptance, meaning that a proposition is rationally acceptable if it is highly probable, *unless* it satisfies some defeating condition D. Proposals of this type invariably aim to define a defeater that applies selectively, or at least as selectively as possible, to the kind of propositions from which the lottery paradox seems to emanate, that is, highly probable propositions stating or entailing that a given ticket is a loser; most, and preferably even all, other propositions that are highly probable are supposed still to qualify as rationally acceptable on account of their high probability. However, so far attempts to specify a satisfactory defeater have been unsuccessful in this respect; they have been shown to reduce the above proposal to the trivial claim that probability 1 is sufficient for rational acceptability.[4] More importantly, a result by Douven and Williamson [2006, Sect. 2] exhibits that what to many prima facie had seemed the most attractive type of conditions—namely, those that are definable in formal terms—are unavailing, because they too would trivialize the proposal.[5]

The following makes this precise. Let W be a set of worlds, and think of propositions as subsets of W. Further assume a probability distribution Pr on $\wp(W)$. Then a function f is said to be an *automorphism* of $\langle W, \wp(W), \Pr\rangle$ iff f is a $1:1$ function from $\wp(W)$ onto itself that satisfies these conditions:

1. $f(\varphi \wedge \psi) = f(\varphi) \wedge f(\psi)$,
2. $f(\neg\varphi) = \neg f(\varphi)$,
3. $\Pr(\varphi) = \Pr\big(f(\varphi)\big)$,

for all propositions $\varphi, \psi \in \wp(W)$. A *structural property* of propositions is any property P such that for any proposition φ and any automorphism f of propositions, φ has P iff $f(\varphi)$ has P. This definition can be extended to cover relations in the

[4]See Douven and Williamson [2006, Sect. 1] for an argument to this effect.

[5]Another response to the lottery paradox, made by Harman [1986:71], is that if we always conditionalize our probabilities after accepting a proposition to the effect that a given ticket will lose, no contradiction will arise. For by repeating such conditionalization for "enough" tickets, we will come to the point were it will no longer be rational to accept of any of the remaining tickets that it will lose (because conditional on what we already accept, it will no longer be highly probable for any of the remaining ones that it will lose). A similar proposal in the case of the discursive dilemma would be this: vote sequentially on the propositions on the agenda, and include a proposition in the collective opinion state only if it is consistent with the deductive closure of the propositions that have already been accepted in the collective opinion state at that stage. However, Harman's proposal has been criticized for making what it is rational to accept dependent on the order in which we accept propositions (cf. Nelkin [2000], but also Douven [2007] for another view on the matter); it is obvious that a parallel critique would apply to the suggestion of sequential voting. One could try to prioritize the propositions on the agenda in some way, aiming thereby to avoid the arbitrariness, but, as List and Pettit [2002:104 f] point out, that strategy is hopeless.

obvious way. A predicate is structural iff it denotes either a structural property or a structural relation. An *aggregative property* of propositions is any property such that whenever two propositions have it, their conjunction has it too. Call a probability distribution Pr on a set W of worlds *equiprobable* iff $\Pr(\{w\}) = \Pr(\{w'\})$ for all $w, w' \in W$. Finally, a proposition φ is defined to be inconsistent iff $\varphi = \emptyset = \bot$.

Then Douven and Williamson prove the following:

Proposition 2.1 *Let W be finite and let* Pr *be an equiprobable distribution on $\wp(W)$. Further, let P be structural, Q aggregative, and P sufficient for Q. Then if some proposition φ such that* $\Pr(\varphi) < 1$ *has P, then $\bot$ has Q.*

It may be useful briefly to sketch the proof. Assume there is some proposition φ that has the property P and such that $\Pr(\varphi) < 1$. Because of the latter fact and the fact that Pr is equiprobable, there must be some $w \in W$ such that $w \notin \varphi$. Then consider all permutations on W that map some world in φ onto w and all other worlds onto themselves; it is easy to show that each such permutation defines an automorphism of propositions. So, since φ has P and P is structural, each image of φ under any of the thus-defined automorphisms has P, too, and since P is sufficient for Q, the proposition φ and its said images all have Q. Because of how the permutations were defined, there is no one world that is an element of all of these propositions, so their conjunction is inconsistent. But since Q is aggregative, that conjunction has Q. So the inconsistent proposition has Q.

As this all looks fairly abstract, it may be helpful to illustrate the result by means of a simple example.[6]

Example 2.1 Let $W = \{w_1, w_2, w_3\}$ and let $\Pr(w_i) = 1/3$, with $i \in \{1, 2, 3\}$. The powerset $\wp(W)$ of W is this:

0	$\emptyset$	4	$\{w_1, w_2\}$
1	$\{w_1\}$	5	$\{w_1, w_3\}$
2	$\{w_2\}$	6	$\{w_2, w_3\}$
3	$\{w_3\}$	7	$\{w_1, w_2, w_3\}$

One readily checks that $\langle W, \wp(W), \Pr\rangle$ has the following automorphisms:

	0	1	2	3	4	5	6	7
f_0	0	1	2	3	4	5	6	7
f_1	0	1	3	2	5	4	6	7
f_2	0	2	1	3	4	6	5	7
f_3	0	2	3	1	6	4	5	7
f_4	0	3	1	2	5	6	4	7
f_5	0	3	2	1	6	5	4	7

[6]Thanks to an anonymous referee for suggesting to include this example.

Now assume that some proposition $\varphi \in \wp(W)$ such that $\Pr(\varphi) < 1$ has a structural property P, where P is sufficient for an aggregative property Q. We do not have to consider proposition $\{w_1, w_2, w_3\}$, for, by finite additivity, $\Pr(\{w_1, w_2, w_3\}) = \Pr(\{w_1\}) + \Pr(\{w_2\}) + \Pr(\{w_3\}) = 1$. If the inconsistent proposition $\emptyset$ has P, then, because P is sufficient for Q, the inconsistent proposition also has Q. If $\{w_1\}$ has P, then, as P is structural, so have all propositions $f_i(\{w_1\})$, for $i \in \{0, 1, \ldots, 5\}$. Consultation of the above table shows that, apart from $\{w_1\}$ itself, these are the propositions $\{w_2\}$ and $\{w_3\}$ (they are the propositions occurring in the second column of the table). Again because P is sufficient for Q, under the given supposition, $\{w_1\}$, $\{w_2\}$, and $\{w_3\}$ all have Q. But because the intersection of these propositions is empty, and Q is aggregative—meaning, in set-theoretic terms, that it is closed under the operation of taking intersections—this entails that the inconsistent proposition has Q. Similarly if $\{w_2\}$ has P or if $\{w_3\}$ has P. If $\{w_1, w_2\}$ has P, then again all propositions $f_i(\{w_1, w_2\})$ ($i \in \{0, 1, \ldots, 5\}$) have P too. Apart from $\{w_1, w_2\}$, these are propositions $\{w_1, w_3\}$ and $\{w_2, w_3\}$, as the above table shows. So in that case $\{w_1, w_2\}$, $\{w_1, w_3\}$, and $\{w_2, w_3\}$ all have Q. But as the intersection of these propositions is empty, that must again mean that the inconsistent proposition has Q. Similarly if $\{w_1, w_3\}$ has P or if $\{w_2, w_3\}$ has P. These are all the cases to be considered. Hence, from our assumption it follows that the inconsistent proposition has Q.

As Douven and Williamson point out, interpreted in the context of conditions for rational acceptability, Proposition 2.1 means that if rational acceptability is to be closed under conjunction, and thus an aggregative property, then if there is a sufficient condition for rational acceptability that is structural as well as non-trivial—in the sense that some proposition with probability less than 1 has it—then the inconsistent proposition is rationally acceptable: just let Q be the property of being rationally acceptable and P some structural and non-trivial condition sufficient for rational acceptability. Note that, when stated in this form, the lottery paradox really has nothing essentially to do with propositions about lotteries or lottery tickets. Provided sufficient conditions for rational acceptability are to be structural, one faces inconsistency as soon as there is *any* proposition with non-perfect probability which qualifies as rationally acceptable, whether this proposition is about lottery tickets or about wholly different things.

To appreciate the generality of this result, it suffices to check that what can reasonably be regarded as the primitive predicates from (meta-)logic, set theory and probability theory (and more generally measure theory) all define structural properties or relations. For instance, given an automorphism f, φ is inconsistent iff $\varphi = \emptyset$ iff $f(\varphi) = \emptyset$ iff $f(\varphi)$ is inconsistent; similar procedures show the other (meta-)logical and set-theoretic predicates to be structural. And from the fact that automorphisms were defined as mappings which are, among others, probability-preserving, it follows immediately that "probability" and related predicates (such as "conditional probability" and "high probability") are all structural ones too.[7] Proposition 2.3 of Douven and Williamson [2006] then does the rest, for it says that any predicate defined strictly in terms of structural predicates by means of the Boolean operators and quantification (of any order) is itself structural. Thus, the above result applies

[7] See on this also Tarski [1986].

not only to the "simple" proposal that a proposition is rationally acceptable if it is highly probable, but also to all proposals that add a defeating condition to the foregoing one, at least if that condition is definable in structural terms (note that such a definition may be as complicated as one likes). In fact, the result applies to all proposals according to which a proposition is rationally acceptable if it satisfies some condition definable in structural terms, whether or not the definition makes reference to the notion of probability. There is thus no hope to define even a sufficient condition for rational acceptability—let alone rational acceptability itself—in logical and/or mathematical terms, unless one is willing to grant—which few are—that rational acceptability requires probability 1.

A last thing that merits remark before we return to the discursive dilemma is that the above result crucially hinges on the fact that the model that is assumed is a *finite* probability space. But surely there are infinitely many propositions expressible in our language, and thus also infinitely many propositions that might be (or fail to be) rationally acceptable. Douven and Williamson [2006, Sect. 5] offer various responses to this objection, but for present concerns the most relevant one is that we need not think of the worlds in W as being maximally specific. We can simply assume that W is a set of mutually exclusive and jointly exhaustive worlds that determine answers to all the questions that are relevant in some given context; the subsets of W then represent the contextually relevant propositions.

3. From the Lottery Paradox to the Discursive Dilemma. [* We will now derive a variant of List and Pettit's impossibility theorem (Proposition 1.1) from the above result concerning the lottery paradox. Levi [2002] was the first to point out that the discursive dilemma is structurally similar to the lottery paradox. The present section elaborates and formalises this similarity, building on the idea that possible worlds may be thought of as voters. We construct a particular parliament and agenda, and show how these yield a model that is isomorphic to the one assumed in Proposition 2.1; that suffices to make Proposition 2.1 apply to our construction. The next section then presents Proposition 2.1 as an impossibility result in the context of voting rules. *]

Let $W = \{w_1, \ldots, w_n\}$ be a set of mutually exclusive and jointly exhaustive worlds and let Pr be an equiprobable distribution defined on $\wp(W)$. Furthermore, let $M_W = \{m_1, \ldots, m_n\}$ be a specific parliament, where the opinion states of the members of this parliament are defined as follows. For all $\varphi \in \wp(W)$ and $i \in \{1, \ldots, n\}$, $v_i(\varphi) = 1$ iff $w_i \in \varphi$. Note that it follows automatically that each individual opinion state is complete, consistent, and deductively closed. Let the parliament's agenda Φ consist of the elements of $\wp(W)$. It is obvious that this set is deductively closed too. Finally, define a function $g\colon \wp(M_W) \to [0, 1]$ as follows: $g(M') = |M'|/n$, for all $M' \in \wp(M_W)$. We may think of g as measuring the weight a subset of M_W has in determining the collective opinion state, but the interpretation of g need not be pinned down. It is simply intended to provide us with a formal equivalent of the equiprobable distribution Pr.

To prove that $\langle W, \wp(W), \Pr\rangle$ and $\langle M_W, \wp(M_W), g\rangle$ are isomorphic structures, it suffices to show, first, that there is a bijection h from W to M_W, and second, that

$\Pr(\{w \mid w \in \varphi\}) = g\big(\{h(w) \mid w \in \varphi\}\big)$ for all $\varphi \in \wp(W)$.[8] For the bijection, simply define $h(w_i) = m_i$ for all $i \in \{1, \ldots, n\}$. As to the second, note that since W is finite and Pr equiprobable, $\Pr(\varphi) = |\varphi| \,/\, |W|$ for all φ. We thus have for all φ, $\Pr(\{w \mid w \in \varphi\}) = |\{w \mid w \in \varphi\}| \,/ n = |\{h(w) \mid w \in \varphi\}| \,/ n = g\big(\{h(w) \mid w \in \varphi\}\big)$.

As a result, Proposition 2.1 of Douven and Williamson applies not only to $\langle W, \wp(W), \Pr\rangle$, but to $\langle M_W, \wp(M_W), g\rangle$ as well. To be maximally clear about what it says about the latter, it may be helpful to say a few words about what the crucial terms occurring in Proposition 2.1 come to when they are interpreted in $\langle M_W, \wp(M_W), g\rangle$ (insofar as this is not completely evident). We end by noting some peculiar features of the lottery setting, when interpreted in terms of a parliament, an agenda, and a voting rule.

Firstly, the term "proposition" now refers to elements of $\wp(M_W)$ instead of $\wp(W)$. But note that the above-defined bijection h yields a second bijection $h' \colon \wp(W) \to \wp(M_W)$ in the following obvious way: $h'(\varphi) = \{h(w) \mid w \in \varphi\}$, for all φ. Therefore, each proposition φ can be taken to be represented by the set of φ-voters in M_W as much as it can be taken to be represented by the set of φ-worlds in W. As suggested earlier, for the purposes of Douven and Williamson's paper the possible worlds may as well *be* the members of M_W as defined above. The set of propositions $\wp(M_W)$, or any subset of it that allows us to uniquely identify members of the parliament by their opinions on propositions in that subset, serves as the semantic equivalent of the voting agenda Φ referred to earlier. It will further be obvious that the voting agenda has the same logical properties whether we think of propositions as members of $\wp(W)$ or as members of $\wp(M_W)$.[9]

Secondly, when interpreted in $\langle M_W, \wp(M_W), g\rangle$ the term "Pr" is to be taken as referring to the function g, of course. From the isomorphism between the two models it follows that, formally speaking, g is a probability function on $\wp(M_W)$. Since, patently, $|\{m_i\}| \,/ n = |\{m_j\}|/n$ for all $i, j \in \{1, \ldots, n\}$, it is an equiprobable one. Note that, again in virtue of the correspondence between sets of worlds and sets of voters in the models, the function g can be thought of as measuring the fraction of the parliament that supports a given proposition. The function g may play a part in, or even fully determine, the voting rule, as is the case in majority voting. And if g completely determines the voting rule, the fact that it is equiprobable means, in the terminology of List and Pettit, that g assumes anonymity of the members of the parliament. Furthermore, whatever its precise role in the voting rule, the fact that $g(\{m_i \mid v_i(\varphi) = 1\}) < 1$ can be interpreted as meaning that φ is not unanimously supported by the parliament. This latter fact is central to the result to be presented in the next section.

Thirdly, concerning the property P from Proposition 2.1, let us say that a proposition φ satisfies the property R iff $r_\varphi(v_1, \ldots, v_n) = 1$. So, having property R is

[8] To state the following in a formally entirely precise fashion, one would have to make explicit that both our models also contain the rational interval $[0, 1] \cap \mathbb{Q}$, being the range of Pr and g, respectively. But that would only make the proof more cumbersome to read while not adding anything that is not obvious anyway.

[9] Douven and Williamson's response to the objection that their result requires a finite probability space in which only finitely many propositions can be represented applies, mutatis mutandis, here as well, or even with more right: voting bodies typically do not and, realistically speaking, cannot aim to decide about all propositions expressible in our language, but only on some subset of contextually relevant ones.

a sufficient condition for a proposition to end up being accepted in the collective opinion state, and thus takes on the role of the property P, the rational acceptability of a proposition. [* Recall that being structural is defined as invariance under automorphisms *of a given model*. Hence a property or relation (and, correspondingly, a predicate denoting that property or relation) which is structural with respect to one model need not be so with respect to another. However, again from the isomorphism between $\langle W, \wp(W), \Pr\rangle$ and $\langle M_W, \wp(M_W), g\rangle$ it follows that all properties and relations that are structural relative to the former are also structural relative to the latter. So a property such as P or R, when structural in the context of a lottery, is also a structural property in the context of voting. Finally note that, by the definition of R, demands placed on this property are in effect demands placed on the corresponding voting rule r. We call r structural, and say that it satisfies the condition of *Structuralness* iff R is a structural property.

Fourthly, the property Q in Proposition 2.1 comes down to the property of being accepted in the collective opinion state. Note that this property is only supposed to be aggregative. So next to consistency, we only need to assume that whenever two propositions are both in the collective opinion, so is their conjunction. Thus, if we call the set of valuations that satisfies this condition plus consistency $V_\wedge$, then the requirement of aggregativeness is that $V_0 = V_\wedge$.

We can now use the above notions in a first translation of Proposition 2.1. *] Given that the parliament M_W is finite and g is the weighting function on $\wp(M_W)$, and filling in property R for P and the property of being accepted in the collective opinion state for Q, this proposition says the following about $\langle M_W, \wp(M_W), g\rangle$: if R is a structural property and the property of being in the collective opinion state is aggregative, and if some proposition $\varphi \in \wp(M)$ such that $g(\{m_i \mid v_i(\varphi) = 1\}) < 1$ satisfies R, then $\bot$ is in the collective opinion state. In other words, given the parliament M_W, if r satisfies Structuralness and its range includes the collective opinion states that are aggregative, then r renders the collective opinion state inconsistent, unless it only includes propositions in that state that are unanimously supported by the members of M_W.

This translation bring us close to our impossibility theorem. But before stating this in a form similar to List and Pettit's theorem, it is worth noticing that all of the foregoing hinges on a highly specific construction, namely, a parliament M_W in which for every two members there is at least one proposition about which they disagree, so that every member can be individuated by her opinions on the agenda. [* Call such a parliament profile *disparate*. It is notable that the above result about the discursive dilemma does *not* mean that if a voting rule is structural and does not require unanimous support, then it will lead to inclusion of the inconsistent proposition in the collective opinion state, whatever the composition of the parliament. Whether it does will depend on whether the parliament is disparate. However, for the impossibility theorem to be stated below it is enough that disparate parliament profiles are *possible*.

The need for a disparate parliament leads us to consider the voting agenda, and specifically its relation to the parliament. *] In all impossibility results in the literature, the agenda is independent of the size and composition of the parliament. Unfortunately this is not so in the construction of the inconsistent parliament M_W. The agenda must be such that it allows for a disparate parliament, which provides a lower bound to the size of the agenda for any given parliament. Specifically, for a parliament of size n we need an agenda that has at least $k \geqslant \log_2 n$ logically independent propositions. And with an agenda of that size, the agenda must further contain all propositions that can be constructed with these k propositions by means of conjunction and negation operations. [* However, it can be noted immediately that if a parliament of n members can be divided into equally large parties of size d, $n = 0 \mod d$, then we may build a similar construction by taking the parties as single voters. This would require a smaller number of logically independent propositions, namely, $k \geqslant \log_2 n/d$. The requirement that the agenda be rich enough to make the parliament disparate can therefore be relaxed to the requirement that the agenda be rich enough to make the parliament *party-wise disparate*, that is, divide the parliament in equally large parties each two of which disagree about at least one proposition on the agenda.

We write down the set of party-wise disparate profiles as $D = \{\langle v_1, \ldots, v_n\rangle \in V^n : \exists d > 1 : (n = 0 \mod d) \wedge (\forall i, i' < d : v_i \neq v_{i'}) \wedge (\forall i \leq n/d, j < d : v_i = v_{i+jn/d})\}$. Note that for a parliament there may be many different values of d that yield elements of D. Ultimately, we can always choose $d = n$, but for smaller values of d the agenda will become progressively smaller. *] On the face of it, we do not find the resulting requirement on the size and richness of the agenda unnatural. Surely in real life it may happen that a parliament is disparate. It seems natural to require from a voting rule that it be capable of dealing with such eventualities.

4. A New Impossibility Result. [* With these translations between the lottery and the discursive setting in place, we can present our main result. Douven and Williamson [2006] prove Proposition 2.1 concerning the lottery paradox, and the preceding section proves that the lottery paradox is isomorphic to the discursive dilemma in case the parliament is party-wise disparate. So we have effectively proved:

Proposition 4.1 *Consider a parliament M and assume an agenda Φ which allows for the possibility that the parliament is party-wise disparate. Then there is no voting rule that satisfies all of the following requirements:*

- Disparaty *The domain of the voting rule includes party-wise disparate profiles, so $D \cap (V_M)^n \neq \emptyset$.*
- Consistent and Aggregative Range *The range of the voting rule is consistent and aggregative, $V_0 = V_\wedge$.*
- Structuralness *The voting rule is structural, meaning that for all automorphisms f, if $r_\varphi(v_1, \ldots, v_n) = 1$ then $r_{f(\varphi)}(v_1, \ldots, v_n) = 1$.*
- Non-Unanimity at Disparaty *The voting rule is not Unanimous at party-wise disparate profiles, meaning that $\exists \varphi \in \Phi, \langle v_1, \ldots, v_n\rangle \in D : r_\varphi(v_1, \ldots, v_n) = 1 \wedge g(\varphi) < 1$.*

In short, this says that structural voting does not allow for consistent, aggregative, and non-unanimous collective opinions in the domain of party-wise disparate opinions. Again, no direct proof for this Proposition is needed, since it follows from Proposition 2.1 and the isomorphism between $\langle W, \wp(W), \Pr\rangle$ and $\langle M_W, \wp(M_W), g\rangle$.

Some remarks on this are in order. First, the property of voting rules with which we avoid inconsistent collective opinions is a rather weak one: we need only make sure that at party-wise opinionated profiles D votes are unanimous. This sets apart the present result from many if not all other impossibility results. As discussed, the reason is simply that the parallel between the discursive dilemma and the lottery paradox can be drawn only at those specific elements of the domain of the aggregation function. One may argue that this limits the relevance of the result for the discussion on the discursive dilemma, but we think not. Note first that the impossibility result can also be derived by requiring Non-Unanimity over the whole domain, which implies Non-Unanimity over D. Moreover, it is a real life possibility that a parliament is disparate. And it seems rather awkward to adopt a voting rule that functions normally in case two or more members vote the same, but that reverts to Unanimity once members or equal-sized parties can be identified by their opinions. In our view, having to assume Unanimity at specific points in the domain is almost as bad as having to assume it over the whole domain. *]

It might further be said that the condition of Structuralness hardly has a natural interpretation in the context of voting rules, and thus that the above result is of limited interest at best. First, at the risk of repeating ourselves, there *is* a natural interpretation of Structuralness: A structural voting rule is a rule that is blind to the meaning, the order, or the name tags of the propositions involved, so that it is, in a sense, a completely impartial procedure. Its meaning is illustrated in the example of section 2. Now it may be objected that also under this interpretation, Structuralness is still an esoteric condition, and that there is no natural motivation for demanding it. But surely Structuralness is not an outlandish condition at all. For one thing, the rule of majority voting, which in practice is without any doubt more common than any other rule, satisfies Structuralness. It is not hard to think of more complicated but still intuitively reasonable rules that satisfy this condition too. One may think here of rules of the type hinted at towards the end of section 1, which brought in considerations on possible majorities undermining the proposition at issue. It is to such attempts at repairing voting rules that Proposition 4.1 applies. What our result shows, and what at least to our eyes came as a surprise, is that no matter how complicated we make such attempts at repairing the voting rule, as long as it is structural there is no guarantee that application of it will result in a consistent collective opinion state, even if all voters can be assumed to have consistent opinion states.[10]

Further, Proposition 4.1 invites a comparison with Proposition 1.1 of List and Pettit: [* Universal Domain, Consistent and Complete Range, Anonymity, Neutrality, and Independence. Before going through them, we want to emphasise again that Proposition 4.1 is based on the construction M_W involving party-wise disparate parliaments D. And to allow for those parliaments, we must make rather different

[10]Note that, while the condition of Structuralness is rather weak in that it includes all formal voting rules, it excludes voting rules that make the inclusion of a proposition in the collective opinion state depend on the propositions (if any) that have already been included, or more generally on the order of voting on the propositions in the agenda. Such rules violate the condition of Structuralness, because the position of propositions in the order of the voting agenda is not invariant under automorphisms. In other words, the Structuralness of the voting rule excludes Harman's response to the lottery paradox, as mentioned in note 5, when that response is translated for the discussion of the discursive dilemma.

assumptions on the agenda than List and Pettit. For some parliament profiles the agenda involved may be equally minimal, but the interdependence between agenda and parliament remains, and will in some cases lead to rather rich agendas. In sum, our result presents a different trade-off between the logical structure of the agenda and the generality of the voting rule: we can deduce an impossibility result under weaker conditions for the voting rule exactly because the assumptions on the agenda are in part stronger than those of List and Pettit.

Let us now turn to the condtions, starting with Universal Domain. To allow for a parliament and agenda structre that is isomorphic to the model used in the generalization of the lottery paradox, as described in section 2, we must suppose that there are profiles in the domain of the voting functions with regard to which the parliament is party-wise disparate. A domain that is universal in the sense of the condition of Universal Domain includes such party-wise disparate profiles, but smaller domains may also include them. *]

Secondly, the condition of Consistent and Complete Range may be weakened to the requirement of Consistent and Aggregative Range. In other words, we need not require the completeness of the collective opinion state. It can very well be that neither φ nor $\neg\varphi$ satisfies R, so that neither φ nor its negation need be an element of the collective opinion. Since the property of being accepted in the collective opinion state is only supposed to be aggregative, [* we only need to assume that if propositions φ and ψ are both in the collective opinion, then $\varphi \wedge \psi$ is as well. Most notably, we need not even assume that the range of the voting rule is closed under negation. *]

Thirdly, let us consider Anonimity. Recall that this condition requires that the voting rule be invariant under a permutation of voters, which means that it must have the same value at profiles in the domain that only differ in the order of voters. This requirement is defined by reference to the domain V_M of the voting rule. But notice that in the construction M_W, the behavior of the voting rule only matters at the party-wise disparate profiles in the domain. At these profiles the collective opinion is at danger of being inconsistent, and if at these profiles we allow the voting rule to give a deciding vote to some designated subset of its members, then the inconsistency can be avoided. Thus, for the present impossibility result, all that seems relevant is the invariance of the voting rule in the subdomain where the parliament is party-wise disparate.

However, this restricted form of Anonymity is of limited interest in the present context, since the condition of Anonymity, restricted or not, is covered by the requirement that the voting rule be structural. Recall that we call a voting rule r structural iff it is invariant under specific transformations of propositions, so-called automorphisms. With the further fact that in a party-wise disparate parliament propositions are represented by subsets of voters/parties, we can spell out automorphisms as transformations of propositions effected by a permutation of the voters/ parties. Now if a voting rule violates Anonymity at party-wise disparate profiles—so that it is not invariant under different labellings of voters at these profiles—then it is also not invariant over some set of propositions that is closed under automorphisms. In such a case it may happen that some proposition φ will be accepted in the collective opinion in virtue of the fact that a specific voter or party supports it, while the proposition ψ, the image of φ under the permutation of this voter, or party of voters, with a voter that does not support φ, will not be accepted in the collective opinion. In other words, a structural voting rule automatically satisfies Anonymity at all party-wise disparate profiles D. We may therefore subsume the condition of Anonymity at party-wise disparate profiles under the requirement of Structuralness.

The question may arise whether the condition that the voting rule satisfies Anonymity is equivalent to the condition that it is structural, because both concern permutations of voters. Indeed, the requirement that the voting rule be structural is equivalent to the requirement that it be invariant under all possible permutations of voters. But the permutation involved in Structuralness is not the permutation of voters simpliciter. In the present setting, sets of voters are propositions, so the permutation involved in Structuralness is a transformation over the language, whereas a permutation involved in Anonimity concerns the numbering, or the names, of the voters only. It is much less to require of a voting rule that its value for a specific proposition be invariant under different labellings of the voters simpliciter, without the transformation of the proposition induced by the permutation of voters.

Finally, there is the condition of Neutrality. Recall that the inclusion of a proposition in the collective opinion state by a voting rule r depends on whether a proposition satisfies the corresponding property R. This property is assumed to apply to all propositions, and in this sense our result assumes Neutrality. However, the only assumption we are making about the property is that it is structural. Because of this, it is possible to incorporate any structural difference between two propositions φ and ψ in the property R. In other words, our result is left intact under any violation of Neutrality that concerns types of propositions—in the sense that for propositions of one type one rule might be appropriate, for propositions of a second type a second rule might be appropriate, and so on—provided the types can be individuated in structural terms. So with the condition of Structuralness, we effectively replace the condition of Neutrality with the weaker condition of Neutrality for types of propositions of the aforementioned sort. In the formulation of Neutrality in Proposition 1.1, we replace "for any permutation of propositions" by "for any permutation of propositions that corresponds to an automorphism of those propositions."

[* Summing up, Proposition 4.1 only gets going if the agenda is rich enough to allow for party-wise disparate profiles, and it applies only to those profiles. In that sense, the conditions of Proposition 4.1 may be stronger than those of Proposition 1.1. On the other hand, the conditions of Proposition 4.1 are weaker in a number of respects: the former does not assume Consistent and Complete Range, but only Consistent and Aggregative Range, and the condition of Structuralness entails a restricted form of Neutrality. But above all, our result does not require Independence. The condition of Structuralness does not imply any restriction on relations between votes on different propositions.

Next to the discussion of List and Pettit's theorem, let us briefly discuss Pauly and van Hees's generalization of Proposition 1.1, without attempting to give a full translation between their conditions and the conditions of the present result. First, the conditions of our result are weaker than those of Pauly and van Hees in the sense that we drop Independence, and that we do not require the Completeness of the collective opinion. However, in the other conditions Pauly and van Hees seem more general, although the comparison is not entirely obvious since our result employs the fixed valuation of M_W. In the guise of Structuralness we assume Anonymity, while Pauly and van Hees only assume Responsiveness. Further, our result employs the requirement of Unanimity at party-wise disparate profiles, while Pauly and van Hees require Non-Dictatorship. *] Finally, Pauly and van Hees are also more general in that they drop the condition of Neutrality altogether, while the above result still assumes the weakened kind of Neutrality that is implicit in Structuralness. The complete absence of Neutrality in Pauly and van Hees's paper allows us to tell apart propositions on the basis of their non-formal (most likely, semantical) properties.

The latter remark relates to our next point, which is that our result may be less dramatic than the corresponding one about the lottery paradox. At least it is quite clear that many have hoped for a (non-trivial) formal solution to the lottery paradox, and even for a formal theory of rationality (which would seem to presuppose a formal solution to the lottery paradox). It is not so clear that something similar holds true for voting rules. Although, as we said above, the paradigmatic rule of majority voting *is* structural, and although many parliaments may very well be disparate, it may be argued that in general voting rules should be sensitive to the semantic content of the various propositions that are on the agenda, already for reasons independent of our result. A voting rule might then set higher standards for acceptance for (say) propositions whose acceptance would lead to tax benefits for farmers than for (say) propositions whose acceptance would have the effect of lowering the emission of pollutants. Be that as it may, it will still be good to know that already for purely logical reasons voting rules will have to be cast, at least partly, in non-formal terms.

Finally, we would like to point to a possible avenue for further research. We established an isomorphism between a structure relevant to the lottery paradox and one relevant to the discursive dilemma. This allowed us to employ a theorem concerning the lottery paradox in the context of judgement aggregation. But the bridge we built between the two discussions can also be crossed in the other direction, of course. And given the liveliness of the debate on judgement aggregation, and the many new results that keep coming out of that, it is not unrealistic to expect that at least some theorems originally derived, or still to be derived, within that context can be applied fruitfully to the context of the lottery paradox, and will teach us something new, and hopefully also important, about this paradox.

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