When is a Linear Complementarity System Controllable?

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Abstract

This paper deals with the controllability problem of a class of piecewise linear systems, known as linear complementarity systems. It is well-known that checking certain controllability properties of very simple piecewise linear systems are undecidable problems. In an earlier paper, however, a complete characterization of the controllability of the so-called conewise linear systems has been achieved. By employing this characterization and exploiting the special structure of linear complementarity systems, we present a set of inequality-type conditions as necessary and sufficient conditions for their controllability. Our treatment is based on the ideas and the techniques from geometric control theory together with mathematical programming.

Introduction

Ever since Kalman's seminal work [10] introduced the notion of controllability in the state space framework, it has been one of the central notions in systems and control theory. In the early 1960s, Kalman [11] himself and many others (see e.g. [9] for historical details) studied controllability of finite-dimensional linear systems extensively and established algebraic tests for controllability. Soon after, constrained controllability problems, i.e. problems for which the inputs are constrained to assume values from a subset of the entire input space, became popular (see for instance [12]). Early work in this direction consider only constraint sets which contain the origin in their interior [12, Thm. 8, p. 92]. However, the constraint set does not contain the origin in its interior in many interesting cases, for instance, when only nonnegative controls are allowed. Saperstone and Yorke [14] were the first to consider constraint sets that do not have the origin in their interior. In particular, they considered the case for which the inputs are constrained to the set [0,1]. More general constraint sets were studied by Brammer [2]. He showed that the usual controllability condition together with a condition on the real eigenvalues of the system matrix is necessary and sufficient for controllability of linear systems with nonnegative inputs [2, Thm. 1.4].

While the algebraic characterization of controllability of finite dimensional linear systems is among the classical results of systems theory, global controllability results for nonlinear systems have been hard to come by. When it comes to hybrid systems, the situation gets even more hopeless. In fact, Blondel and Tsitsiklis [1] proved that the reachability problem of a bimodal piecewise linear discrete-time system is an undecidable problem. However, our recent work [3–5] shows that one can come up with algebraic conditions for controllability of conewise linear. In this paper, our aim is to extend the ideas of [3–5] to a class of hybrid systems called *linear complementarity systems* (LCSs).

The following notational conventions will be in force throughout the paper. The symbol $\mathfrak{R}$ denotes the set of real numbers, $\mathfrak{R}^n$ n-tuples of real numbers, and $\mathfrak{R}^{n\times m}$ $n \times m$ real matrices. The set of complex numbers is denoted by C. For a matrix $A\in \mathfrak{R}^{n\times m}$, A^{T} stands for its transpose, A^{-1} for its inverse (if exists), im A for its image, i.e. the set $\{y\in \mathfrak{R}^n \mid y = Ax$ for some $x\in \mathfrak{R}^m\}$. We write A_{ij} for the (i, j)th element of A. For $\alpha \subseteq \{1, 2,\ldots, n\}$, and $\beta \subseteq \{1, 2,\ldots, m\}$, $A_{\alpha\beta}$ denotes the submatrix $\{A_{jk}\}_{j\in\alpha,k\in\beta}$. If $\alpha = \{1,2, \ldots,n\}$ ($\beta = \{1,2,\ldots ,m\}$), we also write $A_{*\beta}(A_{\alpha *})$. Inequalities for vectors must be understood componentwise. Similarly, max operator acts on the vectors componentwise. We write $x \perp y$ if $x^{\mathrm{T}}y = 0$.

Linear Complementarity Problem/System

The problem of finding a vector $z \in \mathfrak{R}^m$ such that

$$z \geq 0, \tag{1a}$$

$$q + Mz \geq 0, \tag{1b}$$

$$z^{\mathrm{T}}(q + Mz) = 0 \tag{1c}$$

for a given vector $q \in \mathfrak{R}^m$ and a matrix $M \in \mathfrak{R}^{m\times m}$ is known as the linear complementarity problem. We denote (1) by $LCP(q, M)$. It is well-known [7, Thm. 3.3.7] that the $LCP(q,M)$ admits a unique solution for each q if, and only if, M is a P-matrix. It is also known that z depends on q in a Lipschitz continuous way in this case.

Linear complementarity systems consist of nonsmooth dynamical systems that are obtained in the following way. Take a standard linear input/output system. Select a number of input/output pairs (z_i, w_i), and impose for each of these pairs complementarity relation of the type (1) at each time t, i.e. both $z_i(t)$ and $w_i(t)$ must be non-negative, and at least one of them should be zero for each time instant $t \geq 0$. This results in a dynamical system of the form

$$\dot{x}(t) = Ax(t) + Bu(t) + Ez(t), \quad \text{(2a)}$$

$$w(t) = Cx(t) + Du(t) + Fz(t), \quad \text{(2b)}$$

$$0 \leq z(t) \perp w(t) \geq 0, \quad \text{(2c)}$$

where $u \in \Re^m$, $x \in \Re^n$, and $z, w \in \Re^k$. A wealth of examples and application areas of LCSs can be found in [6,8,15,16].

A set of standing assumptions throughout this paper are the following.

Assumption 1. The following conditions are satisfied for the LCS (2)

1. *The matrix F is a P-matrix*
2. $k = m$
3. *The transfer matrix* $D + C(sI - A)^{-1}B$ *is invertible as a rational matrix*

These assumptions are technical in nature and most of the subsequent results can be generalized in cases for which these assumptions do not hold. However, we focus on LCSs that satisfy Assumption 1 in order not to blur the main message of the paper.

It follows from Assumption 1 that $z(t)$ is a piecewise linear function of $Cx(t) + Du(t)$. This means that for each initial state x_0 and locally-integrable input u there exist a unique absolutely continuous state trajectory $x^{x_0,u}$ and locally-integrable trajectories $(z^{x_0,u}, w^{x_0,u})$ such that $x^{x_0,u}(0)=x_0$ and the triple $(x^{x_0,u}, z^{x_0,u}, w^{x_0,u})$ satisfies the relations (2) for almost all $t \geq 0$.

We say that the LCS (2) is (completely) controllable if for any pair of states $(x_o, x_f) \in \Re^{n+n}$ there exists a locally integrable input u such that the trajectory $x^{x_0,u}$ of (2) satisfies $x^{x_0,u}(T) = x_f$ for some $T > 0$.

In two particular cases, one can employ the available results for the linear systems to determine whether (2) is controllable.

Linear systems

Consider the LCS

$$\dot{x}(t) = Ax(t) + Bu(t), \quad \text{(3a)}$$

$$w(t) = u(t) + z(t), \quad \text{(3b)}$$

$$0 \leq z(t) \perp w(t) \geq 0. \quad \text{(3c)}$$

It can be verified that Assumption 1 holds. Note that this system is controllable if, and only if, the linear system (3a) is controllable. In turn, this is equivalent to the implication

$$\lambda \in C,\ z \in C^n,\ z^* A = \lambda z^*,\ B^T z = 0 \Rightarrow z = 0. \quad \text{(4)}$$

In this case, we say that the pair (A,B) is controllable.

Linear systems with nonnegative inputs

Consider the LCS

$$\dot{x}(t) = Ax(t) + Bu(t)+Bz(t), \tag{5a}$$

$$w(t) = u(t) + z(t), \tag{5b}$$

$$0 \leq z(t) \perp w(t) \geq 0. \tag{5c}$$

Note that the solution to the LCP (5b) and (5c) can be given as $z(t) = u^{-}(t)$ and $w(t) = u^{+}(t)$ where $\xi^{+} := \max(\xi, 0)$ and $\xi^{-} := \max(-\xi, 0)$ denote the positive and negative part of the real vector $\xi = \xi^{+} - \xi^{-}$ respectively.

Therefore, this LCS is controllable if, and only if, the linear system

$$\dot{x}(t) = Ax(t) + Bv(t)$$

with the input constraint $v(t) \geq 0$ is controllable. It follows from [2, Cor. 3.3] that this system is controllable if, and only if, the following two conditions hold:

1. the pair (A,B) is controllable
2. the implication

$$\lambda \in \Re,\ z \in \Re^{n},\ z^{\mathrm{T}} A = \lambda z^{\mathrm{T}}, B^{\mathrm{T}} z \geq 0 \Rightarrow z = 0 \tag{6}$$

holds.

Main results

To formulate the main results we need some nomenclature. Consider the linear system $S(A, B, C, D)$

$$\dot{x} = Ax + Bu, \tag{7a}$$

$$y = Cx + Du, \tag{7b}$$

where $x \in \Re^{n}$ is the state, $u \in \Re^{m}$ is the input, $y \in \Re^{p}$ is the output, and the matrices A, B, C, D are of appropriate sizes. We define the *invariant zeros* of the system (7) to be the zeros of the nonzero polynomials on the diagonal of the Smith form of

$$P_{\Sigma(s)} = \begin{bmatrix} A - SI & B \\ C & D \end{bmatrix}. \tag{8}$$

The matrix $P_{\Sigma}(s)$ is sometimes called *the system matrix.* It is known, for instance from [17, Cor. 8.14], that the transfer matrix $D + C(sI - A)^{-1}B$ is invertible as a rational matrix if, and only if, the system matrix $P_{\Sigma}(\lambda)$ is of rank $n + m$

for all but finitely many $\lambda \in C$. In this case, the values of $\lambda \in C$ such that rank $P_{\sum}(\lambda) < n+m$ coincide with the invariant zeros. Let Λ (*A*, *B*, *C*, *D*) denote the set of all invariant zeros of the system (7).

The following theorem presents algebraic necessary and sufficient conditions for the controllability of an LCS.

Theorem 2. Consider an LCS (2) satisfying Assumption 1. It is controllable if, and only if, the following two conditions hold:

1. The pair (*A*, [*B E*]) is controllable
2. *For all* $\lambda \in \Lambda$ $(A, B, C, D) \cap \Re$, *the system of inequalities*

$$\eta \geq 0, \tag{9a}$$

$$\begin{bmatrix}\xi^{\mathrm{T}} & \eta^{\mathrm{T}}\end{bmatrix}\begin{bmatrix}A-SI & B \\ C & D\end{bmatrix}=0, \tag{9b}$$

$$\begin{bmatrix}\xi^{\mathrm{T}} & \eta^{\mathrm{T}}\end{bmatrix}\begin{bmatrix}E \\ F\end{bmatrix}\leq 0, \tag{9c}$$

admits no nonzero solution (ξ,η).

A quick sketch of the proof

The main ingredients of the proof are conewise linear systems. A conewise linear system (CLS) is a dynamical system of the form

$$\dot{x}(t) = Ax\{t) + Bu\{t) + f(Cx(t) + Du\{t)), \tag{10}$$

where x $\in \Re^n$ is the state, u $\in \Re^m$ is the input, $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, $C \in \Re^{p\times n}$, D $\in \Re^{p\times m}$ and the function *f* is a conewise linear function, i.e., there exist an integer r, solid polyhedral cones y_i and matrices $M^i \in \Re^{n\times p}$ for *i* = 1, 2,...,r such that $U^r_{i=1} y_i$ = $\Re^P$ and $f(y) = M^i y$ if $y \in Y_i$.

Note that the function *f* is necessarily continuous since the cones y_i are closed due to polyhedrality. In turn, continuity implies Lipschitz continuity in this case. A somewhat more explicit representation for CLSs can be given by

$$\dot{x}(t) = (A + M^iC)x(t) + (B + M^iD)u(t) \text{ if } Cx(t) + Du(t) \in Y_i . \tag{11}$$

By using the fact that the solutions of an LCP with a *P*-matrix depend on the data in a Lipschitz continuous way, we can reformulate the LCS (2) as a CLS. This results in a CLS of the form

$$x = P^{\alpha}x + Q^{\alpha}u, \text{ whenever } R^{\alpha}x + S^{\alpha}u \geq 0. \tag{12}$$

where

$$P^{\alpha} := A - E_{*\alpha}F^{-1}_{\alpha\alpha}C_{\alpha *} \quad Q^{\alpha} := B - E_{*\alpha}F^{-1}_{\alpha\alpha}D_{\alpha *}, \tag{13a}$$

$$R^{\alpha} := \begin{bmatrix} -F_{\alpha\alpha}^{-1}C_{\alpha\bullet} \\ C_{\alpha^c\bullet} - F_{\alpha^c\alpha}F_{\alpha\alpha}^{-1}C_{\alpha\bullet} \end{bmatrix} \quad S^{\alpha} := \begin{bmatrix} -F_{\alpha\alpha}^{-1}D_{\alpha\bullet} \\ D_{\alpha^c\bullet} - F_{\alpha^c\alpha}F_{\alpha\alpha}^{-1}D_{\alpha\bullet} \end{bmatrix}. \quad (13b)$$

At this point, we invoke the following theorem on the controllability of LCS.

Theorem 3. *Consider the CLS* (10) *such that p = m and the transfer matrix* $D + C(sl - A)^{-1}B$ is invertible as a rational matrix. It is completely controllable if, and only if,

1. the relation

$$\sum_{i=1}^{r} \left\langle A + M^i C \mid im(B + M^i D) \right\rangle = \Re^n \quad (14)$$

is satisfied and

2. the implication $\lambda \in \Re, z \in \Re^n, w_i \in \Re^m$

$$\begin{bmatrix} z^T & w_i^T \end{bmatrix} \begin{bmatrix} A + M^i C - \lambda I & B + M^i D \\ C & D \end{bmatrix} = 0, w_i \in Y_i \text{ for all } i = 1,2,..,r \Rightarrow z = 0$$

holds.

Here the notation $\langle M \mid imN \rangle$ denotes the so-called controllability subspace associated to the matrix pair (M, N), i.e. $\langle M \mid \mathrm{im}N \rangle = \mathrm{im}N + M\mathrm{im}N + \cdots + M^{P-1}\mathrm{im}N$ where $M \in \Re^{pxp}$ and F^* denotes the dual cone associated to the nonempty set F, i.e., $F = \{y \mid x^T y \geq \text{ for all } x \in F\}$.

By using (12) and Theorem 3, one can show that the two conditions of these theorems are equivalent.

Particular cases

We can recover the two particular cases that are mentioned earlier from Theorem 2 as follows.

Linear systems. If we take $C = 0$, $D = I$, $E = 0$, and $F = I$ as in (3), the two conditions of Theorem 2 boil down to:

1. The pair (A,B) is controllable
2. For all $\lambda \in \Lambda$ $(A, B, 0, I) \subset \Re$, the system of inequalities

$$\eta \geq 0, \quad (15a)$$

$$\begin{bmatrix}\xi^T & \eta^T\end{bmatrix}\begin{bmatrix}A-\lambda I & B \\ C & D\end{bmatrix}=0, \tag{15b}$$

$$\begin{bmatrix}\xi^T & \eta^T\end{bmatrix}\begin{bmatrix}0 \\ I\end{bmatrix}\leq 0, \tag{15c}$$

admits no nonzero solution (ξ,η).

Note that (15a) and (15c) imply that η = 0. This means that if (A,B) is controllable then (15b) the only solution (15b) is ξ = 0. Hence, we recover the case of linear systems.

Linear systems with nonnegative inputs. If we take $C = 0$, $D = I$, E = B, $F = I$ as in (5), the two conditions of Theorem 2 boil down to:

1. The pair (A,B) is controllable.
2. For all $\lambda \in \Lambda$ (A, B, 0, I)$\bigcap \Re$, the system of inequalities

$$\eta \geq 0, \tag{16a}$$

$$\begin{bmatrix}\xi^T & \eta^T\end{bmatrix}\begin{bmatrix}A-\lambda I & B \\ C & D\end{bmatrix}=0, \tag{16b}$$

$$\begin{bmatrix}\xi^T & \eta^T\end{bmatrix}\begin{bmatrix}B \\ I\end{bmatrix}\leq 0, \tag{16c}$$

admits no nonzero solution (ξ,η).

Note that (16c) is already satisfied for this case. Together with (16a), the equality (16b) implies that the second condition is equivalent to the second condition that is presented in (6).

Computational issues

Theorem 2 requires that one needs to check whether a set of inequalities of the form (9) admits only the trivial solution. However, it might be sometimes easier to check whether a given set of inequalities admits a nontrivial solution. To do so, one can employ the following *alternative theorem* which is originally due to Tucker [13, (1.6.10)].

Theorem 4. Let W $\in \Re^{p\times r}$, X $\in \Re^{p\times s}$, Y $\in \Re^{q\times r}$, and Z $\in \Re^{q\times s}$ be given matrices. Exactly one of the following statements hold:

1. There exists a nonzero $(\rho,\varsigma) \in \Re^{r+s}$ such that

$$\rho \geq 0,$$

$$W\rho + X\varsigma = 0,$$

$$Y\rho + Z\varsigma \geq 0.$$

2. There exists a nonzero $(\xi, \eta) \in \Re^{p+q}$ such that

$$\eta \geq 0,$$
$$W^{\mathrm{T}}\xi + Y^{\mathrm{T}}\varsigma \leq 0,$$
$$X^{\mathrm{T}}\xi + Z^{\mathrm{T}}\varsigma = 0.$$

A direct application of the theorem to (9) gives the following alternative formulation of the second condition in Theorem 2:

2' For all $\lambda \in \Lambda$ *(A, B, C, D)* $\bigcap \Re$, the system of inequalities

$$\rho \geq 0, \tag{17a}$$

$$E\rho + [A - \lambda IB]\varsigma = 0, \tag{17b}$$

$$F\rho + [CD]\varsigma \geq 0. \tag{17c}$$

admits a nonzero solution (ρ, ς).

Conclusions

In this paper, we studied the controllability problem for the linear complementarity class of hybrid systems. These systems are closely related to the so-called conewise linear systems. By exploiting this connection, together with the special structure of complementarity systems, we derived algebraic necessary and sufficient conditions for the controllability. We also showed that Kalman's and Bramer's results for linear systems can be recovered from our theorem. Our treatment employed a mixture of methods from both mathematical programming and geometric control theory. Obvious question is how one can utilize these techniques in order to establish necessary and/or sufficient conditions for the (feedback) stabilizability problem.

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