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Topology design for fast convergence of network consensus algorithms

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Abstract— **The quantities of coefficient of ergodicity and algebraic connectivity have been used to estimate the convergence rates of discrete-time and continuous-time network consensus algorithms respectively. Both of these two quantities are defined with respect to network topologies without the symmetry assumption, and they are applicable to the case when network topologies change with time. We present results identifying deterministic network topologies that optimize these quantities. We will also propose heuristics that can accelerate convergence in random networks by redirecting a small portion of the links assuming that the network topology is controllable.**

I. INTRODUCTION

Network consensus algorithms have been studied extensively in the field of parallel and distributed computation for decades [1], [2]. Recently these algorithms attracted much attention because they can be exploited to distributively coordinate groups of autonomous agents, such as mobile robot teams, wireless sensor networks and UAV (Unmanned Aerial Vehicles) formations. Consensus algorithms are discussed using both discrete-time [3], [4] and continuous-time models [5]. One sufficient condition for the asymptotic convergence of consensus algorithms is that the directed graphs of the possibly changing network topologies are "connected" repeatedly as time goes to infinity.

Since autonomous agents are always constrained by limited power and capacities of computation and communication, there is much interest in studying the convergence rates of consensus algorithms [6], [7], [8], [9]. To estimate convergence rates of discrete-time consensus algorithms, eigenvalues of the system matrix was used in [10], a technique that is not applicable when network topologies are changing; and joint spectral radius is used in [11] which requires formidable computation. Following Hajnal's pioneering work on infinite products of nonhomogeneous matrices, we use coefficient of ergodicity [12] in this paper which can be computed easily. Similarly, we will use algebraic connectivity, first introduced in [13] and extended to directed graphs in [14], to estimate convergence rates of continuous-time consensus algorithms. Both the coefficient of ergodicity and the algebraic connectivity are suitable for estimating convergence rates in nonsymmetric and dynamic networks. Note that most of the existing works only consider symmetric network topologies, namely those

modelled by undirected graphs, when estimating convergence rates of continuous-time consensus algorithms.

With the rapid development of mobile network technology, the topology of more and more autonomous agent systems are controllable. Since the convergence rate will be largely determined by network topologies, it is of great importance to identify those topologies on which consensus algorithms converge fast. Using coefficient of ergodicity and algebraic connectivity as proxies for convergence rates, we show that the union of star graphs is the optimal topology. In some cases, the network topologies are initialized as random graphs and for the sake of maneuvering cost, it will be infeasible to change the topology into the optimal one which will inevitably involve constructing or breaking many links. For small-world random graphs, we propose a very simple heuristic that only redirects a small portion of existing links and give experiment results to show that the convergence rates can be improved substantially.

The rest of this paper is organized as follows. In section II, we introduce the discrete-time and the continuous-time models of network consensus algorithms. In section III, we define the coefficient of ergodicity and the algebraic connectivity for directed graphs. In section IV we prove that the union of star graphs is the optimal topology as evaluated by both the coefficient of ergodicity and the algebraic connectivity. And in section V, we present our heuristic to accelerate consensus algorithms on random graphs and simulation results.

II. NETWORK CONSENSUS ALGORITHMS

The network consensus problem is also called the "agreement problem" in which we consider a network consisting of n autonomous agents. All the agents are trying to agree on a specific value of some quantity, denoted by x_i for agent $i \in \{1, \ldots, n\}$. They can only communicate with their physical or logical neighbors and the neighbor relationship is not symmetric, namely that agent i is a neighbor of agent j does not necessarily imply that agent j is a neighbor of agent i. In the sequel, we will introduce two most commonly used consensus algorithms using a discrete-time model and a continuous-time model respectively. Suppose at time t, the network topology can be described as a directed graph $\mathbb{G}(t)$ with vertex set $\mathcal{V}_n = \{1, \ldots, n\}$ and edge set $\mathcal{E}(\mathbb{G}(t))$, where there is a directed edge $(i, j) \in \mathcal{E}(\mathbb{G}(t))$ if and only if agent i is a neighbor of agent j. Let $\mathcal{N}_i(t)$ denote the set of indices of agent i's neighbors at time t. Throughout this paper, we will

only consider the network topologies which can be described by simple graphs. A graph is a *simple graph* if there is no self-loop at each vertex and no multiple edges between any pair of vertices [15]. Note that here the definition of the orientation of directed edges is the opposite of that used in [14]; as a result the graph discussed in this paper is the converse of that used in [14].

A. Discrete-time model

At time $t = 1, 2, \ldots$, each agent $i \in \mathcal{V}_n$ will update its value $x_i(t)$ to the average of its current value and the values of its neighbors:

$$x_i(t+1) = \frac{1}{n_i(t)+1}\left(x_i(t) + \sum_{j\in\mathcal{N}_i(t)} x_j(t)\right) \qquad (1)$$

where $n_i(t)$ is the number of elements in $\mathcal{N}_i(t)$. The system equation of the n-agent system can be concisely written into its state form. Towards this end, define the system state to be $x = [\begin{matrix} x_1 & x_2 & \cdots & x_n \end{matrix}]^T$. Then

$$x(t+1) = P(\mathbb{G}(t))x(t), \qquad t = 1, 2, \ldots \qquad (2)$$

Here $P(\mathbb{G})$ is the *consensus matrix* of graph $\mathbb{G}$ and is defined as

$$P(\mathbb{G}) \triangleq (I+D)^{-1}(I+A^T) \qquad (3)$$

where I is the identity matrix, D is the diagonal matrix each entry of which is the in-degree of the corresponding vertex, and A is the adjacency matrix of $\mathbb{G}$. It is easy to check that $P(\mathbb{G})$ is always a stochastic matrix [16]. One sufficient condition for the convergence of consensus algorithm (2) is that each $\mathbb{G}(t)$ contains a spanning tree. For detailed discussion about more relaxed sufficient conditions, we refer readers to [4].

B. Continuous-time model

At time $t > 0$, each agent $i \in \mathcal{V}_n$ will change its value $x_i(t)$ with respect to the differences between its current value and the values of its neighbors:

$$\dot{x}_i(t) = \sum_{j\in\mathcal{N}_i(t)} (x_j(t) - x_i(t)) \qquad (4)$$

As before, define the system state of the n-agent system to be $x = [\begin{matrix} x_1 & x_2 & \cdots & x_n \end{matrix}]^T$. Then

$$\dot{x}(t) = -L(\mathbb{G}(t))x(t), \qquad t > 0 \qquad (5)$$

where $L(\mathbb{G})$ is the *Laplacian matrix* of graph $\mathbb{G}$ and is defined as

$$L(\mathbb{G}) \triangleq D - A^T \qquad (6)$$

One sufficient condition for the convergence of consensus algorithm (5) is that for all $t > 0$, the second smallest eigenvalue in magnitude of $L(\mathbb{G}(t))$ is always strictly greater than zero. We refer readers to [5], [9] for a detailed discussion about various sufficient conditions.

III. COEFFICIENT OF ERGODICITY AND ALGEBRAIC CONNECTIVITY

In this section, we will define the coefficient of ergodicity $\tau(\mathbb{G})$ and the algebraic connectivity $\alpha(\mathbb{G})$ for a directed graph $\mathbb{G}$. They will be used in the next section as the approximations of convergence rates to identify those graphs on which consensus algorithms converge quickly. Let $\mathcal{G}_n$ denote the set of directed simple graphs with vertex set $\mathcal{V}_n$.

Definition 1: [12], [17] For a directed graph $\mathbb{G} \in \mathcal{G}_n$, its *coefficient of ergodicity* $\tau(\mathbb{G})$ is defined as

$$\tau(\mathbb{G}) \triangleq \frac{1}{2} \max_{i,j\in\mathcal{V}_n} \sum_{s=1}^{n} |p_{is} - p_{js}| \qquad (7)$$

where $P = \{p_{ij}\}$ is the consensus matrix of $\mathbb{G}$. Since P is a stochastic matrix, $\tau(\mathbb{G})$ can be written in a different form as

$$\tau(\mathbb{G}) = 1 - \min_{i,j\in\mathcal{V}_n, i\neq j} \sum_{s=1}^{n} \min\{p_{is}, p_{js}\} \qquad (8)$$

Definition 2: [14] For a directed graph $\mathbb{G} \in \mathcal{G}_n$, its *algebraic connectivity* $\alpha(\mathbb{G})$ is defined as

$$\alpha(\mathbb{G}) \triangleq \min_{x\in\mathbb{R}^n, x\neq 0, x\perp\mathbf{1}} \frac{x^T L x}{x^T x} \qquad (9)$$

where L is the Laplacian matrix of $\mathbb{G}$ and $\mathbf{1}$ is the all ones n-vector. The significance of τ and α is that $\max_t \tau(G(t))$ and $\min_t \alpha(G(t))$ can serve as lower bounds on the convergence rates of Eq. (2) and Eq. (5) respectively.

Let $\mathcal{G}_{n,m}$ denote the subset of $\mathcal{G}_n$ which consists of those simple graphs with $m > 0$ edges. Obviously, $\mathcal{G}_{n,m}$ is a finite set, so we can define:

$$\tau(\mathcal{G}_{n,m}) = \min_{\mathbb{G}\in\mathcal{G}_{n,m}} \tau(\mathbb{G}) \qquad (10)$$

and

$$\alpha(\mathcal{G}_{n,m}) = \max_{\mathbb{G}\in\mathcal{G}_{n,m}} \alpha(\mathbb{G}) \qquad (11)$$

IV. OPTIMAL NETWORK TOPOLOGIES

We call a graph in $\mathcal{G}_n$ with $n-1$ edges a *star graph* if we can relabel its vertices in such a way so that vertex 1 has out-degree $n-1$ and every other vertex has in-degree 1. In such a graph, vertex 1 is the root of the graph and the only vertex having positive out-degree. We say $\mathbb{G} \in \mathcal{G}_n$ is *the union of graphs* $\mathbb{G}_1, \mathbb{G}_2 \in \mathcal{G}_n$ if $\mathbb{G}$'s edge set is the union of the edge sets of $\mathbb{G}_1$ and $\mathbb{G}_2$.

Now we will prove several results identifying those graphs in $\mathcal{G}_{n,m}$ which achieve $\tau(\mathcal{G}_{n,m})$ and $\alpha(\mathcal{G}_{n,m})$.

Theorem 1: In the set $\mathcal{G}_{n,k(n-1)}$, where $1 \leq k < n$, the union of k stars achieves $\tau(\mathcal{G}_{n,k(n-1)}) = \frac{1}{k+1}$.

Theorem 2: In the set $\mathcal{G}_{n,k(n-1)}$, where $1 \leq k < n$, the union of k stars achieves $\alpha(\mathcal{G}_{n,k(n-1)}) = k$.

In the sequel, for $x \in \mathbb{R}$, let $\lfloor x \rfloor$ denote the largest integer less than or equal to x. We will first prove Theorem 1.

Proof of Theorem 1: Let $\mathbb{S} \in \mathcal{G}_{n,k(n-1)}$ denote the union of k stars. Since for a given graph, its coefficient of ergodicity is determined, we can compute according to the definition of τ that $\tau(\mathbb{S}) = \frac{1}{k+1}$. Now we will prove by contradiction that $\tau(\mathcal{G}_{n,k(n-1)}) = \tau(\mathbb{S})$. Suppose there exists a graph $\mathbb{H} \in \mathcal{G}_{n,k(n-1)}$ such that $\tau(\mathbb{H}) < \frac{1}{k+1}$. We will consider the definition of τ in the form of equation (8) where rows u and v are the pair of rows in $P(\mathbb{H})$ achieving $\tau(\mathbb{H})$. Without loss of generality, we assume $d = d_i(u) \geq d_i(v)$, where for a vertex $j \in \mathcal{V}_n$, $d_i(j)$ denotes the in-degree of vertex j. Then each nonzero element in row u is $\frac{1}{d+1}$. Let $q(i,j)$, $i, j \in \mathcal{V}_n$ and $i \neq j$, denote the number of columns of $P(\mathbb{H})$ each of which has positive intersection elements with both row i and row j.

From the assumption $\tau(\mathbb{H}) < \frac{1}{k+1}$ and the selection of u and v, we know that for any $w \in \mathcal{V}_n$

$$\sum_{s=1}^{n} \min\{p_{us}, p_{ws}\} \geq \sum_{s=1}^{n} \min\{p_{us}, p_{vs}\} > \frac{k}{k+1} \qquad (12)$$

On the other hand, we have

$$\sum_{s=1}^{n} \min\{p_{us}, p_{ws}\} \leq q(u,w)\frac{1}{d+1} \leq \frac{d_i(w)+1}{d+1} \qquad (13)$$

We choose w to be the vertex of the minimum in-degree in $\mathbb{H}$, then

$$d_i(w) \leq \left\lfloor \frac{k(n-1)}{n} \right\rfloor = k - 1 \qquad (14)$$

Combining inequalities (12), (13) and (14) together, we have $\frac{k}{d+1} > \frac{k}{k+1}$, which implies $d < k$. Then there are at most k non-zero elements in row u. Consequently, for each $l \in \mathcal{V}_n$ with $l \neq u$ and $l \neq v$, we have

$$\frac{k}{d_i(l)+1} \geq \sum_{s=1}^{n} \min\{p_{us}, p_{ls}\} > \frac{k}{k+1} \qquad (15)$$

where the second inequality sign holds because of (12).

The inequality (15) implies that

$$d_i(l) < k, \qquad l \neq u \text{ and } l \neq v \qquad (16)$$

Combining with the fact that $d_i(v) \leq d_i(u) = d < k$, we have proved that all the vertices of the graph $\mathbb{H}$ have strictly less than k incoming edges. So there are at most $(k-1)n$ edges in $\mathbb{H}$ which contradicts the fact that $\mathbb{H}$ has $k(n-1)$ edges. Hence, the graph $\mathbb{H}$ satisfying $\tau(\mathbb{H}) < \frac{1}{k+1}$ does not exist. $\square$

To prove Theorem 2, we list a few useful Lemmas about $\alpha(\mathbb{G})$ that have been proved in [14].

Lemma 1: If $\mathbb{G} \in \mathcal{G}_n$ is a star graph, then $\alpha(\mathbb{G}) = 1$.

Lemma 2: (Lemma 1 in [14]) For $\mathbb{G}_1, \mathbb{G}_2 \in \mathcal{G}_n$,

$$\alpha(\mathbb{G}_1 \cup \mathbb{G}_2) \geq \alpha(\mathbb{G}_1) + \alpha(\mathbb{G}_2) \qquad (17)$$

Lemma 3: (Corollary 1 in [14]) Let $\mathcal{S}$ be a nontrivial subset of the vertex set $\mathcal{V}_n$ of $\mathbb{G} \in \mathcal{G}_n$. Then

$$\alpha(\mathbb{G}) \leq \frac{e(\overline{\mathcal{S}}, \mathcal{S})}{|\mathcal{S}|} + |\mathcal{S}| \qquad (18)$$

where $e(\overline{\mathcal{S}}, \mathcal{S})$ is the number of those edges which starts from a vertex in $\overline{\mathcal{S}}$ and ends at a vertex in $\mathcal{S}$.

Now we will develop some new results.

Lemma 4: For a graph $\mathbb{G} \in \mathcal{G}_{n,m}$, we have

$$\alpha(\mathbb{G}) \leq \left\lfloor \frac{m}{n} \right\rfloor + 1 \qquad (19)$$

Proof of Lemma 4: Let v be the vertex of graph $\mathbb{G}$ having the minimum in-degree $d_i(v)$ which implies that $d_i(v)$ is less than or equal to the average vertex in-degree of $\mathbb{G}$, namely $d_i(v) \leq \left\lfloor \frac{m}{n} \right\rfloor$. Let $\mathcal{S} = \{v\}$. Then

$$\begin{aligned} e(\overline{\mathcal{S}}, \mathcal{S}) &= d_i(v) \leq \left\lfloor \frac{m}{n} \right\rfloor \\ |\mathcal{S}| &= 1 \end{aligned}$$

In view of Lemma 3, we arrive at inequality (19). $\square$

Now we are in a position to prove Theorem 2.

Proof of Theorem 2: From Lemma 4, we know that

$$\alpha(\mathcal{G}_{n,k(n-1)}) \leq \left\lfloor \frac{k(n-1)}{n} \right\rfloor + 1 = k - 1 + 1 = k \qquad (20)$$

Now we consider the graph $\mathbb{S} \in \mathcal{G}_{n,k(n-1)}$ which is the union of k stars. Since $\mathbb{S} = \mathbb{T}_1 \cup \cdots \cup \mathbb{T}_k$ where $\mathbb{T}_i$, $1 \leq i \leq k$, are stars, using Lemma 2 and Lemma 1 we have

$$\alpha(\mathbb{S}) \geq \sum_{i=1}^{k} \alpha(\mathbb{T}_i) = k \qquad (21)$$

From inequalities (20), (21) and the fact $\mathbb{S} \in \mathcal{G}_{n,k(n-1)}$, we know that

$$\alpha(\mathcal{G}_{n,k(n-1)}) = k \qquad (22)$$

and the graph $\mathbb{S}$ achieves the maximum value $\alpha(\mathbb{S}) = k$. $\square$

V. ACCELERATED CONVERGENCE IN SMALL-WORLD NETWORKS

We have shown that for a deterministic graph in the set $\mathcal{G}_{n,k(n-1)}$, the graphs on which consensus algorithms converge the fastest are the union of star graphs with the following two properties: (1) As many as possible vertices have the maximum out-degree of $n - 1$; and (2) the in-degree of each vertex is around the average value of $\frac{m}{n}$. With the help of this insight into the characteristics of the optimal graphs in the set $\mathcal{G}_{n,k(n-1)}$, in this section we will define an operation on random graphs called "designed rewiring" which redirects a small portion of the edges of a given random graph and will show by experiment results that this operation can accelerate consensus algorithms effectively.

The topology of most of the networks running consensus algorithms, such as wireless sensor networks, are modeled as random graphs. The convergence of consensus algorithms have been studied on some classes of special random graphs, such as small-world and scale-free graphs [18], [19] and Ramanujan graphs [20]. In this paper we will focus on accelerating discrete-time consensus algorithms on small-world graphs achieved by the "random rewiring" process parameterized by $0 < p < 1$, where the *random rewiring* process [21] refers to the following operations: Consider a regular lattice on a ring in $\mathcal{G}_n$, where each vertex $i \in \mathcal{V}_n$ is connected to its k nearest neighbors in $\mathcal{N}_i$; with probability p, each edge is
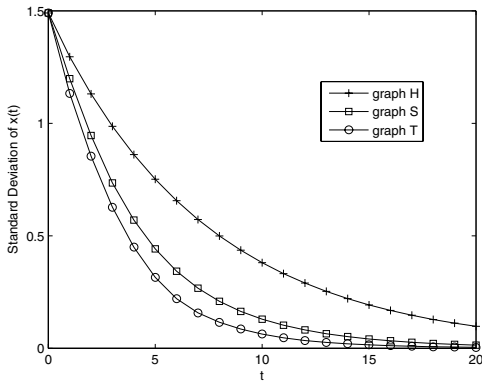
Fig. 1.   The evolution of standard deviation of $x(t)$ on graphs $\mathbb{H}$, $\mathbb{S}$ and $\mathbb{T}$
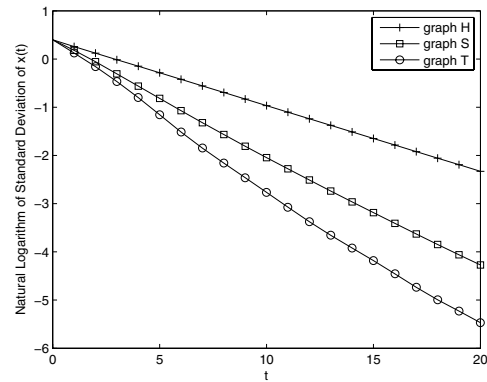

Fig. 2.   The logarithm of standard deviation of $x(t)$ on graphs $\mathbb{H}$, $\mathbb{S}$ and $\mathbb{T}$

randomly redirected to a vertex uniformly chosen from the set $\mathcal{V} - \{i\} \cup \mathcal{N}_i$.

We will consider the discrete-time consensus algorithm with fixed network topology $\mathbb{G}$ in the form

$$x(k+1) = P(\mathbb{G})x(k), \qquad k = 0, 1, 2, \ldots \qquad (23)$$

where $P(\mathbb{G})$ is the consensus matrix of graph $\mathbb{G}$. Let $\mathbb{H} \in \mathcal{G}_n$, $n = 10$, be the original lattice on the ring with $k = 2$. Let $\mathbb{S}$ be the random graph achieved after the random rewiring operation on $\mathbb{H}$ using $p = 0.15$. Now we will define the *designed rewiring* operation on $\mathbb{S}$. With probability one there are a pair of vertices u and v in $\mathbb{S}$ having the maximum out-degree and minimum in-degree respectively such that (u, v) is not an edge of $\mathbb{S}$. We randomly pick an edge ending at v and reattach its starting end to u. What results is a new graph $\mathbb{T}$ which only differs from $\mathbb{S}$ in one edge. We can compare the convergence rates for system (23) with $\mathbb{G}$ being $\mathbb{H}$, $\mathbb{S}$ and $\mathbb{T}$ respectively. The initial value of x is given as $x_i(i) = x(n - i + 1) = i$ for $1 \le i \le \frac{n}{2} = 5$. During each iteration, we compute the standard deviation of $x(t)$. We perform the experiments 100 times and plot the mean of the data in figure 1. Since all three processes converge exponentially, we plot the natural logarithm for the three processes in figure 2 in order to better compare their convergence rates. As one can see from figures 1 and 2, the consensus algorithm converges faster on the small-world graph $\mathbb{S}$ than on the lattice graph $\mathbb{H}$, which agrees with the analysis and experiments for the continuous case in [18], [19], [20]. What we show here is that we achieved a even faster convergence process on $\mathbb{T}$ by only redirecting one edge in $\mathbb{S}$ via designed rewiring.

VI. Conclusion

In this paper, we use the coefficient of ergodicity and the algebraic connectivity to study the convergence rates of network consensus algorithms. We found network topologies on which consensus algorithms converge quickly. We also presented a technique called designed rewiring for random graphs which accelerates convergence rates by redirecting only a small portion of edges.

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