Convergence of the state of a passive nonlinear plant with an L2 input

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Convergence of the state of a passive nonlinear plant with an L^2 input

Bayu Jayawardhana and George Weiss

***Abstract*— In this paper, we consider a strictly output passive nonlinear plant P with storage function H. We assume that P is zero-state detectable. Under some mild conditions on H, we show that the state x of the plant converges to zero for any L^2 input. This implies the solvability for all $t \geq 0$ of the system equations, for every input in L^2_{loc}.**

We define a stability notion called L^2 system-stable, a variant to the L^2-stability concept, which has a nice interconnection properties.

I. INTRODUCTION

Passive systems have a $\mathscr{C}^1$ storage function H (defined on the state space $\mathbb{R}^n$) which has the intuitive meaning of stored energy. The input signal u and the output signal y take values in the same inner product space. We denote the state of the system at time t by $x(t)$. The defining property of a passive system is that if a state trajectory exists then

$$\dot{H} \leq \langle y, u \rangle, \quad \text{where} \quad \dot{H} = \frac{\partial H(x)}{\partial x}\dot{x}. \tag{1}$$

The dynamics of many physical systems such as electrical circuits or mechanical systems can be described as passive systems, if one chooses properly the input u and the output y. The product of y and u should correspond to the power flow into the system.

It is known that passive systems have inherent stability properties. The Lyapunov stability of the equilibrium points corresponding to $u = 0$ can be shown by using H as a Lyapunov function (see, for example, Willems [19]). A stability property that some passive systems have is L^2-*stability*, i.e., if the input u is in L^2 (for $t \geq 0$), then (for any initial state) the equations of the system have a unique solution (for all $t \geq 0$) and the output y is also in L^2 (see van der Schaft [14] for details).

It is shown in [14] that *a strictly output passive system*, i.e., a passive system where the storage function H satisfies

$$\dot{H} \leq \langle y, u \rangle - k\|y\|^2, \qquad k > 0, \tag{2}$$

has an L^2 gain $\leq \frac{1}{k}$. Such a system is locally asymptotically stable at 0 if it is zero-state detectable [14]. Moreover, if $\lim_{\|x\|\to\infty} H(x) = \infty$ (i.e., H is proper) then the system is globally asymptotically stable at 0.

Many references study the conditions under which a nonlinear system is passive, and when a nonlinear system can be made passive by state feedback. For affine nonlinear systems, Moylan [12] described necessary and sufficient conditions for the system to be passive. The conditions are analogous to the Kalman-Yakubovich-Popov conditions for linear time-invariant systems. In Byrnes *et al* [1], it is shown that if a nonlinear system has relative degree one and it is minimum phase, then the system can be rendered passive by state feedback.

Based on these results, passivity-based controller design exploits the stability properties of passive systems. Ortega *et al* [13] describes several passivity-based controller design methods for electrical and mechanical systems modeled by Euler-Lagrange equations. The book [14] introduced passivity-based control for port-controlled Hamiltonian systems. Lozano *et al* [11] describes control applications of dissipative systems theory. Jayawardhana [3] proposes a controller design to reject input disturbance signals generated by an exosystem and to track constant reference signals. For a fully actuated mechanical system, a passivity-based tracking controller has been proposed by Slotine and Li [15] (see also [13] for the passivity property of the closed-loop system using the *Slotine-Li* controller). The combination of the Slotine-Li controller with an internal model is explored in our paper [4].

In this paper, we study the behavior of a strictly output passive nonlinear system given an L^2 input signal. Under mild assumptions on the differential equation, we show that if the system is zero-state detectable and its storage function H is proper (these concepts are defined in Section II), then the state x converges to zero. This implies that a unique solution of the differential equation exists for all $t \geq 0$, for any input signal in L^2_{loc}.

The intuition behind our main result is the following: According to the global asymptotic stability result stated after (2), when $u = 0$ then $x(t) \to 0$. If $u \in L^2$, then for very large τ the energy left in u for $t \geq \tau$ becomes negligible, and the system behaves as it would for $u = 0$, i.e., we have $x(t) \to 0$. However, a rigorous proof of this result is not easy. Our proof uses techniques from infinite-dimensional system theory.

The main result of this paper has been used in our paper [5] to solve an input disturbance rejection problem, where the disturbance can be decomposed into a signal generated by an exosystem and an L^2 signal. The technique used in this paper can also be used for certain nonlinear systems to show the convergence of the state given an L^p input signal, where $p \in [1, \infty)$, see Jayawardhana [6].

The linear version of our main result is the following: For a linear time-invariant (LTI) system **P** which is detectable and strictly output passive we have $x(t) \to 0$, for every L^2

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input u. The proof of this is easy: suppose that $\mathbf{P}$ is described by

$$\dot{x} = Ax + Bu, \quad y = Cx + Du, \tag{3}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$. From the detectability and the strict output passivity of $\mathbf{P}$, it follows that $\mathbf{P}$ is stable. Thus, $u \in L^2$ implies that $x \in L^2$. From (3), we also have that $\dot{x} \in L^2$. Using Barbălat's lemma (see Logemann and Ryan [10]), it follows that $x(t) \to 0$.

II. PRELIMINARIES

Notation. Throughout this paper, the inner product on any Hilbert space is denoted by $\langle \cdot, \cdot \rangle$ and $\mathbb{R}_+ = [0, \infty)$. We refer to [8] and [14] for basic concepts on nonlinear systems and on passivity theory. For a finite-dimensional vector x, we use the norm $\|x\| = \left(\sum_n |x_n|^2\right)^{\frac{1}{2}}$ and for matrices, we use the operator norm induced by $\|\cdot\|$ (the largest singular value). For any $\varepsilon \geq 0$, we denote $\mathbf{B}_\varepsilon = \{x \in \mathbb{R}^n \mid \|x\| \leq \varepsilon\}$. For any finite-dimensional vector space $\mathscr{V}$ endowed with a norm $\|\cdot\|_{\mathscr{V}}$, the space $L^2(\mathbb{R}_+, \mathscr{V})$ consists of all the measurable functions $f : \mathbb{R}_+ \to \mathscr{V}$ such that $\int_0^\infty \|f(t)\|_{\mathscr{V}}^2 \mathrm{d}t < \infty$. The square-root of the last integral is denoted by $\|f\|_{L^2}$. For $f \in L^2(\mathbb{R}_+, \mathscr{V})$ and $T > 0$, we denote by f_T the truncation of f to $[0, T]$. The space $L^2_{loc}(\mathbb{R}_+, \mathscr{V})$ consists of all the measurable functions $f : \mathbb{R}_+ \to \mathscr{V}$ such that $f_T \in L^2(\mathbb{R}_+, \mathscr{V})$, for all $T > 0$. The space $\mathscr{H}^1(\mathbb{R}_+, \mathscr{V})$ consists of all the functions $f : \mathbb{R}_+ \to \mathscr{V}$ such that $f, \frac{\mathrm{d}f}{\mathrm{d}t} \in L^2(\mathbb{R}_+, \mathscr{V})$ (where $\frac{\mathrm{d}f}{\mathrm{d}t}$ is understood in the sense of distributions). The space $\mathscr{C}(\mathbb{R}^l, \mathbb{R}^p)$ (respectively $\mathscr{C}^1(\mathbb{R}^l, \mathbb{R}^p)$) consists of all the continuous (respectively continuously differentiable) functions $f : \mathbb{R}^l \to \mathbb{R}^p$.

Consider the time-invariant plant $\mathbf{P}$ described by

$$\dot{x} = f(x, u), \tag{4}$$

$$y = h(x), \tag{5}$$

where the state x, the input u and the output y are functions of $t \geq 0$, such that $x(t) \in \mathbb{R}^n$, $u(t), y(t) \in \mathbb{R}^m, m \leq n$. We assume that $f \in \mathscr{C}^1(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^n)$ with $f(0, u) = 0 \Leftrightarrow u = 0$ and $h \in \mathscr{C}^1(\mathbb{R}^n, \mathbb{R}^m)$ with $h(0) = 0$. We assume that there exists a storage function $H \in \mathscr{C}^1(\mathbb{R}^n, \mathbb{R}_+)$ such that for some $k > 0$,

$$\frac{\partial H(x)}{\partial x} f(x, u) \leq \langle y, u \rangle - k\|y\|^2. \tag{6}$$

The plant $\mathbf{P}$ as in (4)–(5) with the storage function H satisfying (6) is strictly output passive, which means that it satisfies (2) (this is easy to verify). H is called *proper* if $H(x) \to \infty$ whenever $\|x\| \to \infty$.

We recall a result on the existence and uniqueness of the solution of the differential equation (4) (see also Sontag [16, Appendix C] for details).

Definition 2.1: A solution of (4) *with a measurable input* u *on an interval* $\mathscr{I}$ *containing* 0 is an absolutely continuous function $x : \mathscr{I} \to \mathbb{R}^n$ such that

$$x(t) - x(0) = \int_0^t f(x(\tau), u(\tau)) \mathrm{d}\tau \qquad \forall t \in \mathscr{I}.$$

Theorem 2.2: Assume that $u : \mathbb{R}_+ \to \mathbb{R}^m$ is measurable, $f \in \mathscr{C}(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^n)$ and the following two conditions hold for every $a \in \mathbb{R}^n$:

(S1) There exists a constant $c > 0$ and a locally integrable function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\|f(x, u(t)) - f(y, u(t))\| \leq \alpha(t)\|x - y\|$$

for almost every $t \in \mathbb{R}_+$ and for all $x, y \in a + \mathbf{B}_c$.

(S2) There exists a locally integrable function $\beta : \mathbb{R}_+ \to \mathbb{R}_+$ such that for almost every $t \in \mathbb{R}_+$,

$$\|f(a, u(t))\| \leq \beta(t).$$

Then for every $x(0) \in \mathbb{R}^n$ there exists $\delta > 0$ and a unique solution of (4) with input u on $[0, \delta)$.

This theorem is an immediate consequence of Theorem 36 in [16]. We need this result in Section 3 when dealing with an L^2 input signal.

Corollary 2.3: Suppose that u and f are as in Theorem 2.2 and for some $x(0) \in \mathbb{R}^n$, $[0, \delta)$ (where $\delta > 0$) is the maximal interval of existence of the solution of (4). If $\delta < \infty$ then for every compact set $\mathscr{K} \subset \mathbb{R}^n$, there exists $T \in [0, \delta)$ such that $x(T) \notin \mathscr{K}$.

Proof: The property **(S1)** and **(S2)** in Theorem (2.2) implies also that for any compact $K \subset \mathbb{R}^n$, there is a locally integrable function γ such that

$$\|f(x, u(t))\| \leq \gamma(t), \tag{7}$$

for almost every $t \in \mathbb{R}_+$ and for all $x \in \mathscr{K}$. Indeed, given any $a \in K$, there exists $c > 0$ and function α and β as in the Theorem 2.2. Thus,

$$\|f(x,t)\| \leq \|f(a,t)\| + \|f(x,t) - f(a,t)\| \leq \beta(t) + c\alpha(t),$$

for all $x \in a + \mathbf{B}_c$ and almost every $t \in \mathbb{R}_+$. Denote the last inequality above by $\gamma_a(t) = \beta(t) + c\alpha(t)$ which is locally integrable. Consider the open covering of K by the sets of the form $\mathbf{B}_{c_j} + a_j$, $a_j \in K$, $j = \{1, 2, \ldots\}$. By compactness, the open covering has a finite subcovering, i.e., j is finite. Choose $\gamma(t) = \max_j \{\gamma_{a_j}(t)\}$, then γ satisfies (7) since γ_j is locally integrable for each j.

We prove the corollary by using contradiction. Suppose that there exists a compact set $K \subset \mathbb{R}^n$ such that $x(t) \in K$ for all $t \in [0, I(x(0)))$. First, we show that $\lim_{t \to I(x(0))} x(t)$ exists. For the compact set K, we know that there exists a locally integrable function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ such that (7) holds. Then we have

$$\|x(t_k) - x(t_j)\| \leq \int_{t_j}^{t_k} \|f(x(\tau), u(\tau)\| \mathrm{d}\tau \leq \int_{t_j}^{t_k} \gamma(t),$$

where $t_k, t_j \in [0, I(x(0)))$. Since $\|x(t_k) - x(t_j)\| \to 0$ as $t_k, t_j \to I(x(0))$. Since K is a complete metric space, $\lim_{t \to I(x(0))} x(t)$ exists and $x(I(x(0))) \in K$. However, we could use again Theorem 2.2 with $I(x(0))$ as the initial time and $x(I(x(0)))$ as the initial state to show the existence of solution of (4) on the interval $[I(x(0)), \eta)$, $\eta > I(x(0))$. This shows that $I(x(0))$

is not the maximal interval of existence of the solution of (4). □

If $I(x(0)) < \infty$ is as in Corollary 2.3, then it is called *the finite escape time*.

Let $\mathscr{X}$ be a metric space with distance μ. A set $G \subset \mathscr{X}$ is *relatively compact* if the closure of G is compact. Let $z : \mathbb{R}_+ \to \mathscr{X}$. A point $\xi \in \mathscr{X}$ is said to be an *ω-limit point of* z if there exists a sequence (t_n) in $\mathbb{R}_+$ such that $t_n \to \infty$ and $z(t_n) \to \xi$. The set of all the ω-limit points of z is denoted by $\Omega(z)$.

A map $\pi : \mathbb{R}_+ \times \mathscr{X} \to \mathscr{X}$ is said to be a *semiflow on* $\mathscr{X}$ if π is continuous, $\pi(0, x_0) = x_0$ for all $x_0 \in \mathscr{X}$ and

$$\pi(s+t, x_0) = \pi(s, \pi(t, x_0)) \qquad \forall s, t \in \mathbb{R}_+ \quad \forall x_0 \in \mathscr{X}.$$

A non-empty set $G \subset \mathscr{X}$ is *π-invariant* if $\pi(t, G) = G$ for all $t \in \mathbb{R}_+$.

Proposition 2.4: Let $\pi : \mathbb{R}_+ \times \mathscr{X} \to \mathscr{X}$ be a semiflow on a metric space $\mathscr{X}$. Let $x_0 \in \mathscr{X}$ and denote $z(t) = \pi(t, x_0)$. If $z(\mathbb{R}_+)$ is relatively compact, then $\Omega(z)$ is non-empty, compact, π-invariant and

$$\lim_{t\to\infty} \mu(z(t), \Omega(z)) = 0. \tag{8}$$

The proof is a straightforward extension from the result for finite-dimensional systems where $\mathscr{X} \subset \mathbb{R}^n$ (see, for example, La Salle [9] or Logemann and Ryan [10]). This result will be used for an infinite-dimensional system in Section 3. The proof is given below to make the paper self-contained. We mention that $\Omega(z)$ is also connected.

Proof: Since $z(\mathbb{R}_+)$ is relatively compact, $\Omega(z)$ is non-empty and compact.

To prove π-invariance, take $\xi \in \Omega(z)$, so that there exists a sequence (t_n) in $\mathbb{R}_+$ such that $t_n \to \infty$ and $z(t_n) \to \xi$. Take $t > 0$, then

$$\pi(t, \xi) = \lim_{n\to\infty} \pi(t, z(t_n)) = \lim_{n\to\infty} \pi(t + t_n, x_0) \in \Omega(z),$$

so that $\pi(t, \Omega(z)) \subset \Omega(z)$. To prove the opposite inclusion, take $\eta \in \Omega(z)$, so that $\eta = \lim_{n\to\infty} z(\tau_n)$ for some sequence (τ_n) with $\tau_n \to \infty$. The sequence $\pi(\tau_n - t, x_0)$ (defined for n large enough, so that $\tau_n - t > 0$) being contained in a compact set, has a convergent subsequence $\pi(\theta_n, x_0)$, where (θ_n) is a subsequence of $(\tau_n - t)$. If we put $\xi = \lim_{n\to\infty} \pi(\theta_n, x_0)$, then $\pi(t, \xi) = \eta$.

To prove (8), assume that (8) is false. Then there exists a sequence $(t_n) \in \mathbb{R}_+$ such that $t_n \to \infty$ and $\mu(z(t_n), \Omega(z)) \geq \varepsilon > 0$ for all n. This is a contradiction since for a subsequence (θ_n) of (t_n), we have $z(\theta_n) \to \xi \in \Omega(z)$. □

P is said to be *zero-state detectable* if the following is true: If $u(t) = 0$ and x is a solution of (4) on $[0, \infty)$ such that $y(t) = 0$ for all $t \geq 0$, then $\lim_{t\to\infty} x(t) = 0$.

III. MAIN RESULTS

We consider the system **P** described by (4) and (5), with the mild assumptions on f and g stated after (5).

We need additional assumptions on the function f:

(A1) For every compact set $\mathscr{K} \subset \mathbb{R}^n$, there exist constants $c_1, c_2 > 0$ such that

$$\|f(x_1, u) - f(x_2, u)\| \leq (c_1 + c_2\|u\|^2)\|x_1 - x_2\|, \tag{9}$$

for all $u \in \mathbb{R}^m$ and $x_1, x_2 \in \mathscr{K}$.

(A2) For each fixed $a \in \mathbb{R}^n$, there exist constants $c_3, c_4 > 0$ such that

$$\|f(a, u)\| \leq c_3 + c_4\|u\|^2 \qquad \forall u \in \mathbb{R}^m. \tag{10}$$

Remark 3.1: It can be shown that **(A1)** and **(A2)** are satisfied for affine nonlinear systems **P** described by

$$\dot{x} = \tilde{f}(x) + g(x)u, \tag{11}$$

$$y = h(x), \tag{12}$$

where $\tilde{f} \in \mathscr{C}^1(\mathbb{R}^n, \mathbb{R}^n)$, $g \in \mathscr{C}^1(\mathbb{R}^n, \mathbb{R}^{n\times m})$, $g(0)$ has rank m and h is as in (5). This class of systems includes also the port-controlled Hamiltonian systems [14].

For any $\tau \geq 0$, we denote by $\mathbf{S}_\tau^*$ the left-shift operator by τ, acting on $X = L^2(\mathbb{R}_+, \mathbb{R}^m)$. The reason for this notation is that, traditionally, $\mathbf{S}_\tau$ denotes the right-shift by τ on X and $\mathbf{S}_\tau^*$ is the adjoint of $\mathbf{S}_\tau$. By denoting $d_0 = u$ and $d_t = \mathbf{S}_t^* d_0$, it follows that $d_t \in L^2(\mathbb{R}_+, \mathbb{R}^m)$ for all $t \geq 0$ and the following equation holds for almost every $t \geq 0$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|d_t\|_{L^2}^2 = \frac{\mathrm{d}}{\mathrm{d}t}\int_t^\infty ||d_0(\xi)||^2 \mathrm{d}\xi = -||d_0(t)||^2. \tag{13}$$

Theorem 3.2: Let the plant **P** defined by (4), (5) be zero-state detectable and assume **(A1)-(A2)**. Assume that **P** has a storage function H such that $H(x) > 0$ for $x \neq 0$, $H(0) = 0$, H is proper and (6) (strict output passivity) holds.

Then for every initial condition $x(0) \in \mathbb{R}^n$ and for every $u \in L^2(\mathbb{R}_+, \mathbb{R}^m)$, the state trajectory x of **P** is defined for all $t \geq 0$ and it satisfies $x(t) \to 0$ as $t \to \infty$ (and hence $y(t) \to 0$ as $t \to \infty$).

Proof: Using **(A1)**, we have that for every compact set $\mathscr{K} \subset \mathbb{R}^n$ there exist constants $c_1, c_2 > 0$ such that (9) holds. By denoting $\alpha(t) = c_1 + c_2\|u(t)\|^2$ and since $u \in L^2(\mathbb{R}_+, \mathbb{R}^m)$, it is easy to see that α is locally integrable and satisfies the condition **(S1)** in Theorem 2.2.

Using the assumption **(A2)**, we have that for each fixed $a \in \mathbb{R}^n \times \mathbb{R}^l$, there exist constants $c_3, c_4 > 0$ such that (10) holds. By denoting $\beta(t) = c_3 + c_4\|u(t)\|^2$ and since $u \in L^2(\mathbb{R}_+, \mathbb{R}^m)$, β is locally integrable and satisfies the condition **(S2)** in Theorem 2.2 for the state equation (4).

Then using α, β as above and using initial value $x(0) \in \mathbb{R}^n$, it follows from Theorem 2.2 that there exists $\delta > 0$ and a unique solution of (14) with input $u \in L^2(\mathbb{R}_+, \mathbb{R}^m)$ on $\mathscr{I} = [0, \delta)$. In particular, x is absolutely continuous as a function of t on $\mathscr{I}$.

We define an infinite-dimensional signal generator for the signal u. This signal generator has the state space $X = L^2(\mathbb{R}_+, \mathbb{R}^m)$ and the evolution of its state is governed by the operator semigroup $(\mathbf{S}_\tau^*)_{\tau\geq 0}$. Thus, the state of the signal generator at time t is $d_t = \mathbf{S}_t^* d_0$, where $d_0 \in X$ is the initial state. The generator of this semigroup is $\mathscr{A} = \frac{\mathrm{d}}{\mathrm{d}\xi}$ with domain $\mathscr{D}(\mathscr{A}) = \mathscr{H}^1(\mathbb{R}_+,\mathbb{R}^m)$. The observation operator of this signal generator is $\mathscr{C}$, defined for $\phi \in \mathscr{D}(\mathscr{A})$ by $\mathscr{C}\phi = \phi(0)$. It can be checked that $\mathscr{C}$ is admissible in the sense of Weiss [18]. We need the Lebesgue extension of $\mathscr{C}$, denoted by $\mathscr{C}_L$, defined by

$$\mathscr{C}_L\phi = \lim_{\varepsilon\to 0}\mathscr{C}\frac{1}{\varepsilon}\int_0^{\varepsilon}\mathbf{S}_t^*\phi\;\mathrm{d}t = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^{\varepsilon}\phi(\xi)\;\mathrm{d}\xi.$$

with $\mathscr{D}(\mathscr{C}_L)$ being the set of all $\phi \in X$ for which the above limit exists. We refer to [18] for more information on the concept of Lebesgue extension. The output function of the signal generator is $u(t) = \mathscr{C}_L d_t$, which is defined for almost every $t \geq 0$. It turns out that $u = d_0$ (the generated signal is the initial state).

We define an extended system $\mathbf{L}$ by connecting $\mathbf{P}$ to the generator for d_0 as shown in Figure 1. Then we have

$$\dot{x}(t) = f(x(t),u(t)), \quad (14)$$
$$d_t = \mathbf{S}_t^* d_0, \quad (15)$$
$$u(t) = \mathscr{C}_L d_t, \quad (16)$$
$$y(t) = h(x(t)). \quad (17)$$

Let $z(t) = \begin{bmatrix} x(t) \\ d_t \end{bmatrix}$ denote the state at time t of the above system, so that $z(t) \in Z = \mathbb{R}^n \times X$.

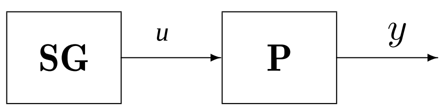

Fig. 1. The extended closed-loop system. The block **SG** is the infinite-dimensional linear signal generator for the L^2 signal u.

Consider the storage function $H_{cl} : Z \to \mathbb{R}_+$ defined for $z = \left[\begin{smallmatrix} x \\ d \end{smallmatrix}\right]$ by $H_{cl}(z) = H(x) + \gamma\|d\|^2$ where $\gamma > \frac{1}{4k}$, where $k > 0$ is the constant from (6). We show that $H_{cl}(z(t))$ is absolutely continuous as a function of t. Since $H \in \mathscr{C}^1(\mathbb{R}^n,\mathbb{R}_+)$ and the solution x of (14) is absolutely continuous as a function of t defined in $\mathscr{I}$, it follows that $H(x(t))$ is absolutely continuous on $\mathscr{I}$. From (13) and since $d_0 \in L^2(\mathbb{R}_+,\mathbb{R}^m)$, it follows that $\frac{\mathrm{d}}{\mathrm{d}t}\|d_t\|_{L^2}^2 \in L^1(\mathbb{R}_+,\mathbb{R}^m)$. This implies that $\|d_t\|_{L^2}^2$ is absolutely continuous on $\mathbb{R}_+$.

Using (6), (13), (14) – (16), we obtain that, for almost every $t \in \mathscr{I}$,

$$\begin{aligned} \dot{H}_{cl} &= \frac{\partial H(x)}{\partial x} f(x,d_0(t)) - \gamma\|d_0(t)\|^2 \\ &\leq \langle y, d_0(t)\rangle - k\|y\|^2 - \gamma\|d_0(t)\|^2, \\ &\leq \left(\frac{1}{2\theta} - \gamma\right)\|d_0(t)\|^2 + \left(\frac{\theta}{2} - k\right)\|y\|^2 \quad \forall\theta > 0. \end{aligned}$$

By choosing $\theta \in (1/2\gamma, 2k)$, we obtain

$$\dot{H}_{cl}(z(t)) \leq -c_5\|u(t)\|^2 - c_6\|y(t)\|^2 \leq 0, \quad (18)$$

where $c_5 = \gamma - \frac{1}{2\theta} > 0$ and $c_6 = k - \frac{\theta}{2} > 0$.

Let us prove that $\mathscr{I} = \mathbb{R}_+$. If the maximal interval of definition of a state trajectory is $\mathscr{I} = [0,\delta)$ with $\delta < \infty$, then it follows from Corollary 2.3 that $x(t)$ must leave any compact set $\mathscr{K} \subset \mathbb{R}^n$ at some finite time $T < \delta$. Since H_{cl} is absolutely continuous as a function of t and bounded from below, (18) implies that $H_{cl}(z(t))$ is bounded and non-increasing for all $t \in \mathscr{I}$. In particular, the state $x(t)$ never leaves the compact set $\{x \in \mathbb{R}^n \mid H(x) \leq H_{cl}(z(0))\}$ for all $t \in \mathscr{I}$. This contradiction shows that $\mathscr{I} = \mathbb{R}_+$ and $H_{cl}(z(t))$ has a limit h as $t \to \infty$.

We will prove the relative compactness of $z(\mathbb{R}_+)$. It has been shown that $x(t)$ is bounded for all $t \in \mathbb{R}_+$, hence $x(\mathbb{R}_+)$ is relatively compact in $\mathbb{R}^n$. Since $\lim_{t\to\infty}\|d_t\|_{L^2} = 0$, the mapping $t \mapsto d_t$ is a continuous mapping from the compact interval $[0,\infty]$ to $L^2(\mathbb{R}_+,\mathbb{R}^m)$. (Here, $[0,\infty]$ is the compactification of $\mathbb{R}_+$.) The image of a compact set through a continuous mapping is always compact. Thus, the state trajectory of the signal generator together with its limit point 0 is a compact set in $L^2(\mathbb{R}_+,\mathbb{R}^m)$, i.e., the set $\{d_t \mid t \geq 0\}$ is relatively compact in $L^2(\mathbb{R}_+,\mathbb{R}^m)$. Therefore $z(\mathbb{R}_+)$ is relatively compact in $\mathbb{R}^n \times X$.

Let π denote the semiflow of (14)–(15) so that $z(t) = \pi(t,z_0)$. According to Proposition 2.4 and the relative compactness of $z(\mathbb{R}_+)$, $\Omega(z)$ is non-empty, compact and π-invariant.

For any $\xi \in \Omega(z)$, there is a sequence (t_n) in $\mathbb{R}_+$ such that $t_n \to \infty$ and $z(t_n) \to \xi$. By the continuity of H_{cl}, $H_{cl}(\xi) = \lim_{n\to\infty} H_{cl}(z(t_n)) = h$. Therefore, $H_{cl}(z(t)) = h$ on $\Omega(z)$. Since $\Omega(z)$ is π-invariant, $\Omega(z) \subset E = \{z \mid \dot{H}_{cl}(z) = 0\}$.

Let M be the largest π-invariant set contained in E. Since $\Omega(z)$ is π-invariant and $\Omega(z) \subset E$, we have $\Omega(z) \subset M$.

In the invariant set M, H_{cl} is constant along state trajectories and $y = 0$ and $u = 0$ along such trajectories. By the assumptions of the theorem, $\mathbf{P}$ is zero-state detectable, i.e., if $u(t) = 0$ and $y(t) = 0$ for all $t \in \mathbb{R}_+$ then $x(t) \to 0$ as $t \to \infty$. Also, if $u(t) = 0$ for all $t \in \mathbb{R}_+$ then $d_0 = 0$, so that $d_t = 0$ for all $t \in \mathbb{R}_+$. Hence, in the invariant set M, $H_{cl}(z) = H_{cl}(0) = 0$ for all $z \in M$. Since $H_{cl}(z) > 0$ for all $z \neq 0$, we obtain that $M = \{0\}$, hence $\Omega(z) = \{0\}$. Using (8) it follows that $x(t) \to 0$ as $t \to \infty$.

The above argument is valid for any initial state $x(0) \in \mathbb{R}^n$ and for any $u \in L^2(\mathbb{R}_+,\mathbb{R}^m)$. □

Corollary 3.3: Let the plant $\mathbf{P}$ be as in Theorem *3.2*. Then for every $x(0) \in \mathbb{R}^n$ there exists a unique solution of (4) with $u \in L^2_{loc}(\mathbb{R}_+,\mathbb{R}^m)$ in $\mathbb{R}_+$.

Proof: To prove the result, we use a contradiction. Suppose that there exists an input $u \in L^2_{loc}(\mathbb{R}_+,\mathbb{R}^m)$ and a finite escape time $T > 0$ for the trajectory of x of the system with initial conditions $x(0) = x_0$. According to Corollary 2.3, $\|x(t)\| \to \infty$ as $t \to T$. Then using $\tilde{u}$ given by

$$\tilde{u}(t) = \begin{cases} u(t) & \forall t \in [0,T], \\ 0 & \forall t \in (T,\infty), \end{cases}$$

the trajectory $\tilde{x}$ of the system with $\tilde{x}(0) = x_0$ and input $\tilde{u}$ also has the same finite escape time T. This is a contradiction. Indeed, since $\tilde{u} \in L^2(\mathbb{R}_+,\mathbb{R}^m)$, it follows from Theorem 3.2

that the state trajectory $\tilde{x}$ corresponding to $\tilde{u}$ is bounded for $t \in [0,\infty)$, i.e., there is no finite escape time. □

Note that the convergence of the state trajectory x to zero does not imply that $x \in L^2(\mathbb{R}_+,\mathbb{R}^n)$. We give an example where $u \in L^2(\mathbb{R}_+,\mathbb{R}^m) \Rightarrow y \in L^2(\mathbb{R}_+,\mathbb{R}^m)$ with a unique solution of the state $x(t)$ for all $t \in \mathbb{R}_+$, but $x \notin L^2(\mathbb{R}_+,\mathbb{R}^n)$. Let the strictly output passive plant $\mathbf{P}$ be described by

$$\dot{x} = -x^3 + u, \quad y = x^3, \tag{19}$$

where $x(t),u(t),y(t) \in \mathbb{R}$. Using the storage function $H(x) = \frac{1}{4}x^4$, it follows from Theorem 3.2 that for every $u \in L^2(\mathbb{R}_+,\mathbb{R})$ and every initial state $x(0) \in \mathbb{R}$, there exists a unique solution $x(t)$ of (19) in $\mathbb{R}_+$ and $\lim_{t\to\infty}|x(t)| = 0$. However, this does not imply that $x \in L^2(\mathbb{R}_+,\mathbb{R})$. Using $u = 0$ and initial state $x(0) = a$, the solution x of (19) is given by

$$x(t) = \left(2t + \frac{1}{a^2}\right)^{-0.5},$$

so that $x \notin L^2(\mathbb{R}_+,\mathbb{R})$.

IV. SYSTEM STABILITY

Consider the following single-input single-output plant $\mathbf{P}$

$$\dot{x} = -xu^{2p} - x + u, \quad y = x, \tag{20}$$

where p is a positive integer. This plant $\mathbf{P}$ is strictly output passive. Indeed, using the storage function $H(x) = \frac{1}{2}x^2$, we have

$$\begin{aligned} \dot{H} &= -x^2u^{2p} - x^2 + xu \\ &\leq \langle y,u\rangle - \|y\|^2. \end{aligned}$$

From this inequality, it can be shown that $\mathbf{P}$ has a finite L^2 gain of 1, i.e., $\|y_T\|_{L^2} \leq \|u_T\|_{L^2} + \sqrt{2H(x(0))}$ (see Lemma 6.5 in [8] for details).

However, this does not imply that for every $u \in L^2(\mathbb{R}_+,\mathbb{R})$ the solution $x(t)$ of (20) exists on some interval $t \in [0,\delta)$ with $\delta > 0$. Indeed, consider $p \geq 2$ and

$$u(t) = \begin{cases} t^{-\frac{1}{2p}} & \text{for } t \in [0,1), \\ 0 & \text{for } t \in [1,\infty), \end{cases} \tag{21}$$

so that $u \in L^2(\mathbb{R}_+)$. Now the state equation (20) can be written as follows:

$$\dot{x} = -xt^{-1} - x + t^{-\frac{1}{2p}} \quad \forall t \in [0,1). \tag{22}$$

It can be shown that if $x(0) \neq 0$ then a solution of (22) does not exist on any interval $[0,\delta)$, where $\delta > 0$. Without loss of generality, assume that $x(0) < 0$. Using contradiction, suppose that there exists a solution x of (22) on $[0,\delta)$ with $\delta > 0$. By the continuity of x on $[0,\delta)$, there exists $\varepsilon \in [0,\delta)$ such that $x_\varepsilon = \max_{t\in[0,\varepsilon)} x(t) < 0$. By Definition 2.1, the state trajectory x satisfies

$$\begin{aligned} x(t) &= x(0) + \int_0^t \left[-(\tau^{-1}+1)x(\tau) + \tau^{-\frac{1}{2p}}\right] \mathrm{d}\tau \\ &> x(0) + \int_0^t -(\tau^{-1}+1)x_\varepsilon \,\mathrm{d}\tau + \int_0^t \tau^{-\frac{1}{2p}} \,\mathrm{d}\tau \\ &= \infty \end{aligned}$$

for all $t \in (0,\varepsilon)$. This contradicts the existence of a solution x on $[0,\delta)$.

Note that if $x(0) = 0$, then the solution of (22) exists on $\mathbb{R}_+$ and for $t \in (0,1]$ it is given by

$$x(t) = \mathrm{e}^{(-\ln(t)-t)} \int_0^t \mathrm{e}^{(\ln(\tau)+\tau)} \tau^{-\frac{1}{2p}} \,\mathrm{d}\tau, \tag{23}$$

(this can be verified directly).

It has been shown that the plant $\mathbf{P}$ as in (20) with input u as in (21) does not have a solution on any interval of the type $[0,\delta)$ when $p \geq 2$. It has a unique solution when $p = 1$ which can also be concluded from Theorem 3.2 since it satisfies Assumption **(A1)**.

Now consider plant $\mathbf{P}$ described by

$$\dot{x} = \begin{cases} -x + \mathrm{sat}(u) & \forall x \in [-1,1], \\ x - 2 + \mathrm{sat}(u) & \forall x \in (1,\infty), \\ x + 2 + \mathrm{sat}(u) & \forall x \in (-\infty,-1), \end{cases} \tag{24}$$

$$y = x, \tag{25}$$

where $x(t),u(t),y(t) \in \mathbb{R}$, $\mathrm{sat}: \mathbb{R} \to \mathbb{R}$ is a saturation function defined by $\mathrm{sat}(u) = u$ for all $u \in (-1,1)$ and $\mathrm{sat}(u) = u/|u|$ otherwise. For the plant $\mathbf{P}$ as in (24), (25) and for every initial condition $x(0) \in \mathbf{B}_1$, it can be checked that every $u \in L^2(\mathbb{R}_+,\mathbb{R})$ implies the existence of a unique solution and the corresponding output $y \in L^2(\mathbb{R}_+,\mathbb{R})$. But when $x(0)$ is outside the ball $\mathbf{B}_3$, i.e., $x(0) \in \mathbb{R}\backslash\mathbf{B}_3$, $u \in L^2 \not\Rightarrow y \in L^2$ and $\lim_{t\to\infty} \|y(t)\| = \infty$.

The concept of L^2-stability is originally defined for mapping, see, for example, Vidyasagar [17, Chapter 6.3] or van der Schaft [14, Chapter 1.2]. Its generalization to state equations often overlooks the influence of the initial state on the output (for example in [17, Chapter 6.3]) or the existence of solution of the state equation (for example in [14, Remark 3.1.4] or in [8, Lemma 6.5]). Example (20) with $p \geq 2$ shows that the system having a finite L^2-gain (in the sense of [14, Definition 3.1.3]) does not imply L^2-stability. Example (24) shows that for every initial condition $x(0)$ in a compact set, every L^2 input u implies the existence of a unique solution to the system equations and the corresponding output y is in L^2, but this property does not hold anymore when the initial condition $x(0)$ is outside the set.

A good definition of L^2-stability for state equations is given in [14, Definition 1.2.11] but it omits the boundedness of the state trajectories. This omission allows an LTI system to be categorized as an L^2-stable system (in the sense of [14, Definition 1.2.11]) but the state grows unbounded for any L^2 input, for example, the plant $\mathbf{P}$ given by

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}1 & 1\\ 0 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u, \quad y = \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}. \tag{26}$$

In this section, we want to refine again the concept of L^2-stability for dynamical systems which combines the L^2-stability concept from van der Schaft [14] or Vidyasagar [17] with the concept of system stability for linear systems as defined in Curtain [2].

Definition 4.1: The plant $\mathbf{P}$ described by (4) is L^2 *system-stable* if for every $u \in L^2(\mathbb{R}_+,\mathbb{R}^m)$ and $x(0) \in \mathbb{R}^n$, there exists a unique solution x of (4) on $\mathbb{R}_+$, the state trajectory x is bounded and the output function y is in $L^2(\mathbb{R}_+,\mathbb{R}^m)$.

It follows that any plant $\mathbf{P}$ satisfying the assumptions in Theorem 3.2 is L^2 system-stable, while the plant $\mathbf{P}$ in (20) with $p \geq 2$, the plant $\mathbf{P}$ in (24),(25) and the plant $\mathbf{P}$ in (26) are not L^2 system-stable. Note that if a plant $\mathbf{P}$ is L^2 system-stable then it is also L^2-stable.

Proposition 4.2: Let the plant $\mathbf{P}$ be defined by (4) and assume **(A1)-(A2)**. Assume that $\mathbf{P}$ has a storage function H such that $H(x) > 0$ for $x \neq 0$, $H(0) = 0$, H is proper and $\mathbf{P}$ is *strictly output passive*, i.e.,

$$\dot{H} \leq \langle y,u\rangle - k\|y\|^2 \qquad (27)$$

holds with $k > 0$. Then $\mathbf{P}$ is L^2 system-stable.

Proof: Let $u \in L^2(\mathbb{R}_+,\mathbb{R}^m)$. It follows from the first part of the proof in Theorem 3.2 that for any initial conditions $x(0) \in \mathbb{R}^n$ that there exists a global solution x of (4) and the state trajectory x is bounded.

By the strict output passivity of $\mathbf{P}$, we have

$$\|y\|_{L^2} \leq \frac{1}{k}\|u\|_{L^2} + \sqrt{\frac{2}{k}H(x(0))}.$$

Thus $y \in L^2(\mathbb{R}_+,\mathbb{R}^m)$. □

Corollary 4.3: Let the plant $\mathbf{P}$ be as in Proposition 4.2. Then for every $x(0) \in \mathbb{R}^n$ and for every $u \in L^2_{loc}(\mathbb{R}_+,\mathbb{R}^m)$ there exists a global unique solution of (4).

Remark 4.4: A passive system with a proper storage function and satisfying **(S3)–(S4)**, does not necessarily have a global solution for every input $u \in L^2(\mathbb{R}_+)$. Indeed, let the plant $\mathbf{P}$ be given by

$$\dot{x} = (1-x)^2 u \qquad y = x(1-x)^2,$$

where $x(t),u(t),y(t) \in \mathbb{R}$, with the proper storage function $H = \frac{1}{2}x^2$. $\mathbf{P}$ is passive, since $\dot{H} = \langle y,u\rangle$. Note that $\mathbf{P}$ satisfies **(A1)–(A2)** but it is not strictly output passive. Suppose that the input u is given by

$$u(t) = \begin{cases} -2 & \forall t \in [0,1) \\ 0 & \text{elsewhere,} \end{cases}$$

so that $u \in L^2(\mathbb{R}_+)$ and consider the initial condition $x(0) = 0.5$. Then the solution of the differential equation is $x(t) = 1-(2-2t)^{-2}$, which is defined only on $[0,1)$ and $\lim_{t\to 1} x(t) = -\infty$. □

V. System-stable interconnections

The motivation to study L^2 system-stability is analogous to the study of Input-to-State Stability (ISS). By definition, for an ISS system with input u and state x, any input $u \in L^\infty$ implies that there exists a global solution x of the state equation and $x \in L^\infty$. If we define an output y which depends continuously on the state x, then it follows that $u \in L^\infty \Rightarrow y \in L^\infty$. In the same manner, an L^2 system-stable with input u, state x and output y has the property that any input $u \in L^2$ implies that there exists a global solution x of the state equation, $x \in L^\infty$ and $y \in L^2$.

A cascade connection of two ISS retains the ISS property of the interconnected systems. The same consequence also applies to the cascade connection of two plants which are L^2 system-stable. Let the plants $\mathbf{P}_i$, $i = 1,2$, be given by

$$\dot{x}_i = f_i(x_i,u_i), \qquad y_i = h_i(x_i) \qquad (28)$$

where $x_i(t) \in \mathbb{R}^{n_i}$ and $u_i(t),y_i(t) \in \mathbb{R}^{m_i}$. Consider $m_1 = m_2$ and $\mathbf{P}_1$, $\mathbf{P}_2$ are L^2 system-stable and are cascade connected by $u_2 = y_1$. Then the whole system with input u_1, state $\left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right]$ and output y_2 is L^2 system-stable. Indeed, by L^2 system-stability of $\mathbf{P}_1$, any $u_1 \in L^2$ implies the global solution of x_1, and we have $x_1 \in L^\infty$ and $y_1 \in L^2$. Since $u_2 = y_1 \in L^2$, by L^2 system-stability of $\mathbf{P}_2$, there exists global solution of x_2, and we have $x_2 \in L^\infty$ and $y_2 \in L^2$.

The feedback interconnection of ISS systems preserves the ISS property of the closed loop system provided that a small-gain type condition is satisfied (see Jiang *et al* [7] for details). The feedback interconnection version for L^2 system-stable is given in the following proposition.

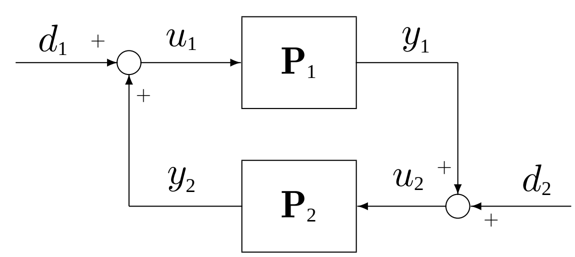

Fig. 2. The feedback interconnection of systems stable $\mathbf{P}_1$ and $\mathbf{P}_2$.

Proposition 5.1: Let the plants $\mathbf{P}_i$, $i = 1,2$, be given by (28) with $m_1 = m_2$. Suppose that for each $i = 1,2$, f_i assumes **(A1)-(A2)** and $\mathbf{P}_i$ is L^2 system-stable. Assume that for each $i = 1,2$, $\mathbf{P}_i$ has a finite L^2-gain denoted by γ_i. Suppose that $\mathbf{P}_1$ and $\mathbf{P}_2$ are feedback interconnected as in Figure 2 such that $u_1 = d_1 + y_2$ and $u_2 = d_2 + y_1$ where d_1, d_2 are external signals. If $\gamma_1\gamma_2 < 1$ then the closed-loop system with input $\left[\begin{smallmatrix} d_1 \\ d_2 \end{smallmatrix}\right]$ and output $\left[\begin{smallmatrix} y_1 \\ y_2 \end{smallmatrix}\right]$ is L^2 system-stable.

Proof: Let $\left[\begin{smallmatrix} d_1 \\ d_2 \end{smallmatrix}\right] \in L^2(\mathbb{R}_+,\mathbb{R}^{m_1+m_2})$. The closed-loop system $\mathbf{L}$ is given by the following state equation

$$\begin{array}{rcl} \dot{x}_1 & = & f_1(x_1,h_2(x_2)+d_1) \\ \dot{x}_2 & = & f_2(x_2,h_1(x_1)+d_2). \end{array} \qquad (29)$$

Using **(A1)**, we have that for every compact set $\mathscr{B} \subset \mathbb{R}^{n_1+n_2}$ there exist constants $c_1,c_2 > 0$ such that (9) holds for the closed-loop system $\mathbf{L}$. By denoting $\alpha(t) = c_1 + c_2\left\|\left[\begin{smallmatrix} d_1(t) \\ d_2(t) \end{smallmatrix}\right]\right\|^2$ and since $\left[\begin{smallmatrix} d_1 \\ d_2 \end{smallmatrix}\right] \in L^2(\mathbb{R}_+,\mathbb{R}^{m_1+m_2})$, it is easy to see that α is locally integrable and satisfies the condition **(S1)** in Theorem 2.2.

Using the assumption **(A2)**, we have that for each fixed $a \in \mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$, there exist constants $c_3,c_4 > 0$ such that (10) holds for $\mathbf{L}$. By denoting $\beta(t) = c_3 + c_4\left\|\left[\begin{smallmatrix} d_1(t) \\ d_2(t) \end{smallmatrix}\right]\right\|^2$ and since

$\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \in L^2(\mathbb{R}_+, \mathbb{R}^{m_1+m_2})$, β is locally integrable and satisfies the condition **(S2)** in Theorem 2.2 for the state equation (29).

It follows from Theorem 2.2 (with α, β as above) that for any initial value $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, there exists a maximal interval time of definition $\delta > 0$ and a unique solution of (29) with input $\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \in L^2(\mathbb{R}_+, \mathbb{R}^m)$ on $\mathscr{I} = [0, \delta)$. In particular, $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is absolutely continuous on $\mathscr{I}$.

For any measurable function f defined on $\mathscr{I}$, we denote by $\|f\|_{L^2(\mathscr{I})} = (\int_0^\delta \|f(t)\|^2 \mathrm{d}t)^{\frac{1}{2}}$. Using the finite L^2 gain of $\mathbf{P}_1$ and $\mathbf{P}_2$ in the interval time of definition $\mathscr{I}$, we have

$$\begin{aligned} \|y_1\|_{L^2(\mathscr{I})} &\leq \gamma_1 \|d_1 + y_2\|_{L^2(\mathscr{I})} + \beta_1 \\ \|y_2\|_{L^2(\mathscr{I})} &\leq \gamma_2 \|d_2 + y_1\|_{L^2(\mathscr{I})} + \beta_2 \end{aligned}$$

where $\beta_1, \beta_2 \in \mathbb{R}$. By simple algebraic manipulation, it can be shown that

$$\begin{aligned} \|y_1\|_{L^2(\mathscr{I})} &\leq \frac{1}{1-\gamma_1\gamma_2} \Big(\gamma_1 \|d_1\|_{L^2(\mathscr{I})} + \gamma_1\gamma_2 \|d_2\|_{L^2(\mathscr{I})} \\ &\qquad + \beta_1 + \gamma_1 \beta_2 \Big) \\ \|y_2\|_{L^2(\mathscr{I})} &\leq \frac{1}{1-\gamma_1\gamma_2} \Big(\gamma_1\gamma_2 \|d_1\|_{L^2(\mathscr{I})} + \gamma_2 \|d_2\|_{L^2(\mathscr{I})} \\ &\qquad + \gamma_2 \beta_1 + \beta_2 \Big). \end{aligned}$$

Since $\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \in L^2(\mathbb{R}_+, \mathbb{R}^{m_1+m_2})$, it implies that $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \in L^2(\mathscr{I}, \mathbb{R}^{m_1+m_2})$. It follows also that $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \in L^2(\mathscr{I}, \mathbb{R}^{m_1+m_2})$. By the L^2 system-stability of $\mathbf{P}_1$ and $\mathbf{P}_2$, x_1 and x_2 is bounded on $\mathscr{I}$. Using Corollary 2.3 we conclude that the maximal interval of definition of $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is $\mathbb{R}_+$. Hence the state trajectory $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is bounded on $\mathbb{R}_+$ and $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \in L^2(\mathbb{R}_+, \mathbb{R}^{m_1+m_2})$. □

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