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Geodesic Flows in Cosmology

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Abstract. In this brief note we discuss geodesic flows that correspond to cosmological solutions in higher-dimensional supergravity. On the one hand, we explain that S-brane solutions are in one-to-one correspondence with geodesic curves on the moduli space through dimensional reduction over the brane worldvolume. On the other hand, reduction over the transversal space gives rise to a scalar potential for the moduli and the geodesic motion is deformed. Nonetheless, in most cases, the scalar flow becomes geodesic asymptotically in which case the solution is described by a multi-field scaling cosmology.

1. Cosmology and higher-dimensional supergravity

To study cosmology in string theory it is instructive, as a first approximation, to consider time-dependent solutions in 10- and 11-dimensional supergravity. In a simple truncation this comes down to studying time-dependent solutions of the following d-dimensional action

$$S = \int \mathrm{d}^d x \sqrt{-g} \Big\{ \mathcal{R} - \tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{2n!} \mathrm{e}^{a\phi} F_n^2 \Big\} , \tag{1}$$

where $\mathcal{R}$ is the usual Einstein–Hilbert term, ϕ is a dilaton and $F_n = \mathrm{d}A_{n-1}$ is the field strength of a $(n-1)$-form gauge potential.

In order for a time-dependent solution to have an interpretation in terms of a four-dimensional FLRW-universe, the space-time metric should at least contain two blocks, one block describing a four-dimensional FLRW-universe and another block that can be considered as an internal space. To interpret one block as an internal space requires compactness. Non-compact directions are allowed if these directions are Killing directions such that they can be mod out by a discrete, non-compact symmetry. As an example, consider the maximally symmetric spaces endowed with the metric $\mathrm{d}\Sigma_k^2$. When $k = +1$ the space is a compact sphere, for $k = 0$ the space is flat space which can be turned into a torus via discrete identifications and for $k = -1$ the space

describes a hyperboloid which can also be made compact by identification of the non-compact directions. The Ansatz for a two-block solution is then

$$\mathrm{d}s^2 = \mathrm{e}^{2\alpha\varphi(t)}\eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + \mathrm{e}^{2\beta\varphi(t)}\mathrm{d}\Sigma_k^2\,, \tag{2}$$

where α, β are specific real numbers [1], η is diag$(-+\ldots+)$. For the moment we keep the dimensions of the two blocks arbitrary. Several solutions of this kind have been constructed, see for instance [1].

In a cosmological context we take the first part of the metric four-dimensional. For such solutions it was noticed that, when a non-zero p-form flux is turned on, a period of four-dimensional cosmic acceleration can be present for all values of k [2]. When there is no flux this is only possible for $k = -1$ [3]. This observation sparked off lots of effort in the construction of general time-dependent solutions of the above form and generalizations thereof.

The phenomenological value of these solutions is rather low due to several shortcomings [4]. Nonetheless these solutions can serve as interesting and simple toy models with partial phenomenological successes (such as the cosmic coincidence problem [5]).

Yet, another insight increased the relevance of these solutions. The above form of the metric can be rewritten more suggestively by re-parametrizing time

$$\mathrm{d}s^2 = \mathrm{e}^{2A(t)}\delta_{ab}\mathrm{d}x^a \mathrm{d}x^a + \mathrm{e}^{2B(t)}[-\mathrm{d}t^2 + t^2\Sigma_k^2]\,. \tag{3}$$

Now one block is flat Euclidean space and, when $k = -1$, the other block is a Minkowski space in Milne coordinates. As suggested by Gutperle and Strominger such supergravity solutions could correspond to specific time-dependent processes in string theory [6]. Because of the similarity with stationary p-brane solutions the above time-dependent solutions are called *spacelike* branes (S-branes). In this language the usual stationary p-branes are called *timelike* p-branes and are characterized by a two-block solution where one block is the Lorentzian $(p+1)$-dimensional worldvolume whereas the other block is sliced with spheres and the functions in the metric only depend on the radial coordinate r. Spacelike branes have an Euclidean $(p+1)$-dimensional worldvolume whereas the transversal space is sliced with hyperboloids and the functions in the metric depend only on time t.

2. Lower-dimensional descriptions of Sp-branes

The complexity of a time-dependent solution carried by non-trivial form fields is somewhat smaller when the form fields are just scalars. This can be achieved in two different ways. Either by reducing the solution over its transversal space (not including the time) or by reducing over the Euclidean worlvolume. Below we present both cases.

2.1. $(p+2)$-dimensional cosmology

From the expression (2) we notice that we can reduce the solution over a maximally symmetric space $\mathrm{d}\Sigma_k^2$. One can generalize this to more complicated internal spaces. The reduced action will be of the following kind

$$S = \int \mathrm{d}^D x\sqrt{-g}\Big\{\mathcal{R} - \tfrac{1}{2}G_{ij}(\Phi)g^{\mu\nu}\partial_\mu\Phi^i\partial_\nu\Phi^j - V(\Phi)\Big\}\,. \tag{4}$$

The cosmological FLRW-solutions of this system lift up to higher-dimensional time-dependent solutions (S-brane type solutions). This dimensional reduction procedure maps an Sp-brane to

[1] For φ to have standard kinetic term we choose $\alpha^2 = \frac{n}{2(D+n-2)(D-2)}$, $\beta = -\frac{D-2}{n}\alpha$ where n is the dimension of the internal space and D is the number of non-compact dimensions.

a FLRW-cosmology in $p+2$ dimensions. To describe our four-dimensional world we need an S2 brane and then our three-dimensional universe is the worldvolume of the S2 brane.

We mentioned that spacelike branes are rather similar to the usual timelike branes. In fact, the above reasoning also applies to timelike branes. The dimensional reduction of a timelike brane over its transversal space (apart from the radial coordinate) maps the brane to a domain wall solution of the above action (4). Domain wall solutions and cosmologies are very similar [9, 25] as we notice from the expression of the metric

$$\mathrm{d}s^2 = \epsilon f(z)^2 \mathrm{d}z^2 + g(z)^2 (\eta_\epsilon)_{ab} \mathrm{d}x^a \mathrm{d}x^b \,, \tag{5}$$

where η_ϵ is $\mathrm{diag}(-\epsilon, +, \ldots, +)$. When $\epsilon = -1$ this is the metric for a cosmological solution and z is the time direction. When $\epsilon = +1$ this is the metric for a stationary domain wall solution and z is the distance from the wall.

An advantage with stationary solutions is that there exist supersymmetric ones which can be constructed from first-order BPS-equations by imposing vanishing fermion variations. Since time-dependent solutions are not supersymmetric in ordinary supergravity theories one would guess that there is no similar first-order framework for cosmologies [2]. But, interestingly, BPS-equations do not only arise for supersymmetric configurations and their existence is more general (for the case of time-dependent solutions, see [9, 10, 11]). Here we give the derivation as presented in [12].

Assume the scalar potential $V(\Phi)$ can be written in terms of another scalar function $W(\Phi)$ as follows

$$V = \epsilon \left\{ \tfrac{1}{2} G^{ij} \partial_i W \partial_j W - \tfrac{D-1}{4(D-2)} W^2 \right\}, \tag{6}$$

then the ϵ will be an overall factor of a one-dimensional action S which can be written as a sum of squares

$$S = \epsilon \int \mathrm{d}z \, f g^{D-1} \left\{ \tfrac{(D-1)}{4(D-2)} \left[W - 2(D-2) \frac{\dot{g}}{fg} \right]^2 - \tfrac{1}{2} || \frac{\dot{\Phi}^i}{f} + G^{ij} \partial_j W ||^2 \right\}, \tag{7}$$

where a dot denotes a derivative w.r.t. z and we ignored the boundary term. If the terms within brackets are zero, the action is stationary under variations, leading to the following *first-order* equations of motions

$$fgW = 2(D-2)\dot{g} \,, \qquad\qquad \dot{\Phi}^i + f G^{ij} \partial_j W = 0 \,. \tag{8}$$

For $\epsilon = +1$ these equations are the standard BPS equations for domain walls that arise from demanding the susy-variation of the fermions to vanish. The function W is then the superpotential that appears in the susy-variation rules and equation (6) with $\epsilon = +1$ is natural for supergravity theories. For every W that obeys (6) we can find a corresponding DW-solution, and if W is not related to the susy variations we call the solutions fake supersymmetric.

For $\epsilon = -1$ these first-order equations are named pseudo-BPS equations and W is named the *pseudo*-superpotential because of the immediate analogy with BPS domain walls in supergravity [9].

2.2. The Sp-brane/S(-1)-brane map

All directions of $\mathrm{d}\Sigma_k^2$ that appear in (2) are indeed Killing directions that can be reduced over. However, the same applies to the flat worldvolume directions x^a of the Sp-brane in the coordinate frame used in equation (3). The reduction of the Sp-brane solution does not generate

[2] There exist exotic supergravity theories with wrong signs kinetic terms for which time-dependent solutions are supersymmetric [7, 8]

a scalar potential for the moduli since a scalar potential originates from the curvature of the internal space and flux trough the internal space. Both of these are absent. The compactified Lagrangian is of the form (4) with $V = 0$. The metric of the lower-dimensional *S(-1)-brane* is again FLRW-type

$$\mathrm{d}s^2 = -f(t)^2\mathrm{d}t^2 + g(t)^2\mathrm{d}\Sigma_k^2\,, \tag{9}$$

and accordingly the moduli only depend on time.

From the scalar field Lagrangian we find that the scalars describe a geodesic flow on the moduli space with as affine parameter $h(t)$ defined by $\mathrm{d}h(t) = g^{1-D}f\mathrm{d}t$. The metric solution can be found easily

$$\mathrm{d}s_D^2 = -\frac{\mathrm{d}t^2}{a\,t^{-2(D-2)} - k} + t^2\mathrm{d}\Sigma_k^2\,, \tag{10}$$

where $a = ||v||^2/(2(D-1)(D-2))$ and $||v||^2$ is defined as the constant affine velocity along the geodesic curve.

The S(-1)-brane solution is then completely specified when the geodesic curve is specified [13]. This proves the one-to-one correspondence between geodesic curves and Sp-branes.

The same analysis can be carried out for timelike branes and the result is that one can map a Dp-brane to a D(-1)-brane in a Euclidean theory (aka instanton) [14]. An important difference with Sp-branes is that for Dp-branes the moduli space is non-Riemannian since the metric has indefinite signature. We hope to report soon on an analysis of the geodesic curves that correspond to Sp- and Dp-branes [15].

3. Asymptotic scaling behaviour

In this section we focus on the effective four-dimensional description of cosmological solutions of the action (4) with $V \neq 0$. Since explicit cosmological solutions are hard to find (for the interesting and often more complex situations) it is useful to first study the asymptotic behaviour of general solutions. Asymptotically many solutions become a simplified solution of the equations of motion, which we call attractor solutions. Especially in the context of late-time cosmology one is not interested in the details of the most general solutions and knowing the attractor solution is sufficient for many purposes.

A typical class of attractor solutions are so-called *scaling solutions* [17]. Such solutions are defined by the property that all terms contributing to the energy density maintain a fixed ratio with respect to each other during evolution. For instance the kinetic energy scales as the potential energy, $G_{ij}\dot{\Phi}^i\dot{\Phi}^j/V(\Phi) = \text{constant}$. As a consequence the scale factor is a power-law; $a(t) = t^p$. When the FLRW-space has non-zero spatial curvature ($k \neq 0$) the number p is restricted to be $p = 1$. Scaling solutions are the unique FLRW-universes that possess a timelike conformal Killing vector [3]. Scaling solutions have been shown to appear in supergravity theories and general Kaluza–Klein theories, see [18, 20, 19, 24] and references therein.

As we now show, an interesting relation emerges between scaling solutions and geodesic flows [16]. The conditions for the correspondence between geodesics and scaling solutions were found in [12] using the first-order formalism and the results of [23]. We briefly summarize this below.

The finite transformation associated with the conformal Killing vector leaves the equations of motion invariant if the action S scales with a constant factor, which is exactly what happens for scaling solutions since all terms in the Lagrangian scale like t^{-2}. Under the finite conformal transformation $g_{\mu\nu} \to \mathrm{e}^{2\lambda}g_{\mu\nu}$ the action scales as a whole if

$$V \to \mathrm{e}^{-2\lambda}V\,, \qquad g^{\tau\tau}G_{ij}\dot{\Phi}^i\dot{\Phi}^j \to \mathrm{e}^{-2\lambda}g^{\tau\tau}G_{ij}\dot{\Phi}^i\dot{\Phi}^j\,. \tag{11}$$

[3] This Killing vector comes from the following scaling: $t \to \lambda t$ and $x \to \lambda^{1-p}x$ where the x are the spacelike coordinates. This coordinate re-scaling results in an overall scaling of the metric.

Equations (11) imply that $G_{ij}\dot{\Phi}^i\dot{\Phi}^j$ remains invariant from which one deduces that $\mathrm{d}\Phi^i/\mathrm{d}\lambda = \xi^i$ must be a Killing vector on the targetspace or on any geodesic submanifold of the targetspace [23].

In terms of $t = \ln\tau$ the tangent vector $\dot{\Phi}$ itself is Killing since then $\mathrm{d}\Phi^i/\mathrm{d}\lambda = \mathrm{d}\Phi^i/\mathrm{d}\ln\tau$. Thus, a scaling solution is associated with an invariance of the equations of motion for a rescaling of cosmic time and is therefore associated with a conformal Killing vector on space-time and a Killing vector on the target space (or a totally geodesic subspace thereof).

Consider

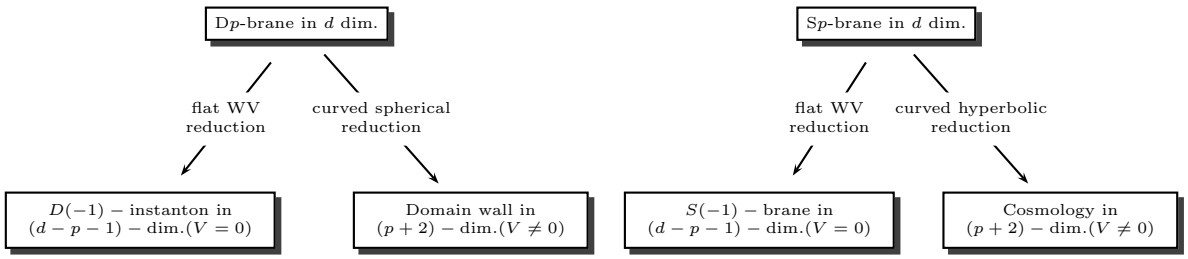
$$\nabla_{\dot{\Phi}}\dot{\Phi}_i = \dot{\Phi}^j\nabla_j\dot{\Phi}_i = \dot{\Phi}^j\Big\{\nabla_{(j}\dot{\Phi}_{i)} + \nabla_{[j}\dot{\Phi}_{i]}\Big\}, \tag{12}$$

where we denote $\dot{\Phi}_i = G_{ik}\dot{\Phi}^k$. The symmetric part is zero if we parametrize the curve with $t = \ln\tau$ since scaling makes $\dot{\Phi}$ a Killing vector. Also, $\nabla_{[j}\dot{\Phi}_{i]} = 0$ since the pseudo-BPS condition makes $\dot{\Phi}$ a curl-free flow $\dot{\Phi}_i = -f\partial_i W$. In the case $\dot{\Phi}$ is Killing on a totally geodesic submanifold the above shows that $\Phi^i(t)$ is a geodesic on the totally geodesic submanifold which implies that $\Phi^i(t)$ is geodesic on the whole target space. We believe that scaling solutions that are not geodesic are rather exceptional, see [21] for an example.

4. Discussion

In this note we reviewed the importance of geodesic motion on moduli spaces for the description of cosmological solutions inspired from higher-dimensional supergravity. Higher-dimensional time-dependent solutions correspond to spacelike p-branes which can be reduced over their worldvolume. The resulting spacelike (-1)-brane is identified with a geodesic motion on the moduli space that appears after the reduction. In another approach one reduces a spacelike brane over its transversal space. Consequently, the spacelike brane becomes a cosmological solution of lower-dimensional theory of gravity coupled to scalars with a non-zero scalar potential.

We pointed out that a similar description exists for timelike branes. The reduction of a timelike brane over the worldvolume give rise to geodesic instanton solutions and the reduction over the transversal space generates stationary domain wall solutions. This is summarized schematically in the picture below.



The similarity between timelike and spacelike p-branes translates into the similarity between domain walls and cosmologies after compactification over the transversal space. This similarity inspires the use of 'superpotential' techniques for studying cosmological solutions. We demonstrated that this can be particularly useful for understanding the asymptotic structure of a solution. More specifically we explained that scaling solutions (typically attractors) correspond to geodesic flows if a superpotential exist [4].

[4] We like to mention the results of [22] in this respect. There it was shown that cosmological solutions of the action (4) can always be seen as geodesic trajectories on a target space that extends the scalar manifold.

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