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Boundary Relations and Generalized Resolvents of Symmetric Operators

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Abstract. The Kreĭn–Naĭmark formula provides a parametrization of all selfadjoint exit space extensions of a (not necessarily densely defined) symmetric operator in terms of maximal dissipative (in $\mathbb{C}_+$) holomorphic linear relations on the parameter space (the so-called Nevanlinna families). The new notion of boundary relation makes it possible to interpret these parameter families as Weyl families of boundary relations and to establish a simple coupling method to construct generalized resolvents from given parameter families. A general version of the coupling method is introduced and the role of the boundary relations and their Weyl families in the Kreĭn–Naĭmark formula is investigated and explained. These notions lead to several new results and new types of solutions to problems involving generalized resolvents and their applications, e.g., in boundary-value problems for (ordinary and partial) differential operators. For instance, an old problem going back to M. A. Naĭmark and concerning the analytic characterization of the so-called Naĭmark extensions is solved.

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1. INTRODUCTION

Let $\mathfrak{H}$ be a separable Hilbert space, let A be a not necessarily densely defined closed symmetric operator or a relation on $\mathfrak{H}$ with equal defect numbers $n_+(A) = n_-(A) \leqslant \infty$, and let $\widetilde{A}$ be a selfadjoint (canonical or exit space) extension of A acting on a Hilbert space $\widetilde{\mathfrak{H}}$. The compressed resolvent $P_{\mathfrak{H}}(\widetilde{A} - \lambda)^{-1}|_{\mathfrak{H}}$ is referred to as the *generalized resolvent* of A. The set of all generalized resolvents of A can be described by the Kreĭn–Naĭmark formula, which was first established in [43, 36] for densely defined symmetric operators with defect numbers $(1, 1)$ and then extended to the general case in a number of papers [37, 38, 40]. The Kreĭn–Naĭmark formula plays an important role in the extension theory of the operator A (see [1–3, 14, 19, 21, 26] and the references therein) and its numerous applications to classical interpolation problems (see [9, 35–37, 43, 19, 21]), to boundary value problems (see [22, 23, 27, 28] and the recent publications [30, 31, 42]), as well as to different types of physical problems (see [2, 3, 11, 12, 45–47] and the references therein).

In the present paper, a new approach to the extension theory of symmetric operators is developed; it uses the concept of boundary triplet and the notion of coupling. Recall the basic definitions for the case in which A is a densely defined symmetric operator. Denote by A^* the adjoint of A.

Definition 1.1 [27, 28]. A collection $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ formed by a Hilbert space $\mathcal{H}$ with $\dim \mathcal{H} = n_\pm(A)$ and two linear mappings Γ_0 and Γ_1 from $\operatorname{dom} A^*$ to $\mathcal{H}$ is called a *boundary triplet* for A^* if

(BT1) abstract Green's identity holds,

$$(A^*f, g) - (f, A^*g) = (\Gamma_1 f, \Gamma_0 g)_{\mathcal{H}} - (\Gamma_0 f, \Gamma_1 g)_{\mathcal{H}}, \qquad f, g \in \operatorname{dom} A^*; \tag{1.1}$$

(BT2) the linear mapping $\Gamma : f \in \operatorname{dom} A^* \mapsto \{\Gamma_0 f, \Gamma_1 f\} \in \mathcal{H}^2$ is surjective.

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A wide class of selfadjoint (exit space) extensions $\widetilde{A}$ of A can be defined via the following coupling construction (see [14]) in terms of boundary triplets. Let $S_1 = A$ and S_2 be two densely defined symmetric operators acting on Hilbert spaces $\mathfrak{H}_1 := \mathfrak{H}$ and $\mathfrak{H}_2$, $n_\pm(A) = n_\pm(S_2) = n \leqslant \infty$, and let $\Pi_A = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ and $\Pi_{S_2} = \{\mathcal{H}, \chi_0, \chi_1\}$ be two boundary triplets for $S_1^* = A^*$ and S_2^*, respectively. Then the restriction $\widetilde{A}$ of the operator $A^* \oplus S_2^*$ to the domain

$$\operatorname{dom} \widetilde{A} = \{\, f_1 \oplus f_2 \in \operatorname{dom} A^* \oplus \operatorname{dom} S_2^* : \Gamma_0 f_1 - \chi_0 f_2 = \Gamma_1 f_1 + \chi_1 f_2 = 0 \,\} \tag{1.2}$$

is called a *coupling* of $S_1 = A$ and S_2 corresponding to the coupling of boundary triplets Π_A and Π_{S_2}. It is a selfadjoint extension of $S_1 \oplus S_2$ acting on $\mathfrak{H}_1 \oplus \mathfrak{H}_2$.

Now consider an arbitrary selfadjoint (exit space) extension $\widetilde{A}$ of A in the direct sum $\widetilde{\mathfrak{H}} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$, where $\mathfrak{H}_1 = \mathfrak{H}$ and $\mathfrak{H}_2 = \widetilde{\mathfrak{H}} \ominus \mathfrak{H}$. Assume for simplicity that $\widetilde{A}$ is an operator and write

$$S_j = \widetilde{A} \cap \mathfrak{H}_j^2, \qquad T_j = \{\, \{P_j\varphi, P_j\varphi'\} : \{\varphi, \varphi'\} \in \widetilde{A} \,\}, \qquad j = 1, 2, \tag{1.3}$$

where $P_j := P_{\mathfrak{H}_j}$ are orthogonal projections of $\widetilde{\mathfrak{H}}$ onto $\mathfrak{H}_j$ $(j = 1, 2)$. Moreover, choose a boundary triplet $\Pi_A = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for $A^* = S_1^*$. As was shown in [14], every selfadjoint extension $\widetilde{A}$ of A such that

$$S_1 = A \quad \text{and} \quad T_2 \quad \text{is closed} \tag{1.4}$$

can be obtained as a coupling (1.2) of A and the operator S_2 in (1.3). More precisely, as was shown in [14], there is a unique boundary triplet $\Pi_{S_2} = \{\mathcal{H}, \chi_0, \chi_1\}$ for S_2^* which is defined by

$$\chi_0 f_2 = \Gamma_0 f_1, \qquad \chi_1 f_2 = -\Gamma_1 f_1, \qquad f_1 \oplus f_2 \in \operatorname{dom} \widetilde{A}, \tag{1.5}$$

from which $\widetilde{A}$ can (uniquely) be recovered using (1.2) and the relation $\widetilde{A} = (A^* \oplus S_2^*)|_{\operatorname{dom} \widetilde{A}}$.

To show the relationship between the coupling construction (1.2) and the theory of generalized resolvents, the notion of Weyl function of the operator A is needed.

Definition 1.2 [18, 19]. Let $\Pi_A = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^*, and let A_0 be a selfadjoint extension of A with the domain $\operatorname{dom} A_0 := \ker \Gamma_0$. The operator-valued function $M(\lambda)$ defined on the resolvent set $\rho(A_0)$ by

$$\Gamma_1 f_\lambda = M(\lambda)\Gamma_0 f_\lambda, \qquad f_\lambda \in \mathfrak{N}_\lambda := \ker(A^* - \lambda), \qquad \lambda \in \rho(A_0), \tag{1.6}$$

is called the *Weyl function* of A corresponding to the triplet Π_A.

The role of the Weyl function M in extension theory of the operator A is similar to that of the classical Weyl–Titchmarsh coefficient in the spectral theory of Sturm–Liouville operators. Moreover, in [41], Definitions 1.1 and 1.2 were extended to the case of nondensely defined symmetric operators, and the m-function of the Jacobi matrix was interpreted as the Weyl function of a nondensely defined symmetric operator. As was shown in [19, 21], the Weyl function M is a Q-function of the pair $\{A, A_0\}$ in the sense of [38], and consequently, it belongs to the class $R[\mathfrak{H}]$ of Nevanlinna functions, i.e., $M(\lambda)$ is holomorphic, has a nonnegative imaginary part $\operatorname{Im} M(\lambda)$ for $\lambda \in \mathbb{C}_+$, and satisfies the symmetry condition $M(\bar{\lambda}) = M(\lambda)^*$. Moreover, every Weyl function is uniformly strict, i.e., $0 \in \rho(\operatorname{Im} M(\lambda))$ for $\lambda \in \mathbb{C}_+$, and A is a densely defined symmetric operator if and only if its Weyl function satisfies the additional conditions (see, e.g., [40, 41])

$$\lim_{y\uparrow\infty} y^{-1} M(iy) = 0 \quad \text{and} \quad \lim_{y\uparrow\infty} y \cdot \operatorname{Im}(M(iy)h, h) = \infty, \qquad h \in \mathcal{H} \setminus \{0\}. \tag{1.7}$$

If a boundary triplet $\Pi_A = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for A^* is chosen, then the mapping $\Gamma = \{\Gamma_0, \Gamma_1\}$ establishes a bijective correspondence $\widetilde{A} \to \Theta := \Gamma(\widetilde{A}_\Theta)$ between the set of symmetric (selfadjoint) extensions $\widetilde{A}_\Theta := \widetilde{A}$, $A \subset \widetilde{A}_\Theta \subset A^*$, of A and the set of symmetric (selfadjoint) linear relations Θ in the parameter space $\mathcal{H}$. Moreover, as was proved in [18, 19], the following resolvent formula holds:

$$(\widetilde{A}_\Theta - \lambda)^{-1} = (A_0 - \lambda)^{-1} + \gamma(\lambda)(\Theta - M(\lambda))^{-1}\gamma(\bar{\lambda})^*, \qquad \lambda \in \rho(\widetilde{A}_\Theta) \cap \rho(A_0). \tag{1.8}$$

Here $\gamma(\lambda) := (\Gamma_0 \restriction \mathfrak{N}_\lambda)^{-1}$ and $M(\lambda)$ are the γ-field and the Weyl function of A corresponding to Π_A, respectively. Recall ([40]) that the Kreĭn–Naĭmark formula

$$\mathbf{R}_\lambda = (A_0 - \lambda)^{-1} - \gamma(\lambda)(M(\lambda) + \tau(\lambda))^{-1}\gamma(\bar{\lambda})^*, \qquad \lambda \in \rho(A_0) \cap \rho(\widetilde{A}), \tag{1.9}$$

gives a bijective correspondence between the set of all generalized resolvents $\mathbf{R}_\lambda := P_{\mathfrak{H}}(\widetilde{A}-\lambda)^{-1}\upharpoonright\mathfrak{H}$ of A and the set of all Nevanlinna families $\tau(\lambda)$, which are holomorphic families of maximal dissipative linear relations on $\mathbb{C}_+$ continued to $\mathbb{C}_-$ by the symmetry condition $\tau(\bar\lambda)=\tau(\lambda)^*$.

The proof of the Kreĭn–Naĭmark formula using the coupling construction (1.2) is sketched below. The key idea of this approach is the following realization result establishing a relationship between the underlying geometric and analytic objects: every uniformly strict Nevanlinna function is a Weyl function in the sense of Definitions 1.1 and 1.2 ([40, 21]). Thus, for every Nevanlinna function τ which is uniformly strict and satisfies (1.7), there exists a densely defined symmetric operator S_2 in $\mathfrak{H}_2$ and a boundary triplet $\Pi_{S_2}=\{\mathcal{H},\chi_0,\chi_1\}$ for S_2^* such that the corresponding Weyl function is τ. Let H^* be the restriction of $A^*\oplus S_2^*$ to the domain

$$\operatorname{dom} H^* = \{f=f_1\oplus f_2\in \operatorname{dom} A^*\oplus \operatorname{dom} S_2^* : \Gamma_0 f_1-\chi_0 f_2=0\}.$$

Then $H:=(H^*)^*$ is a symmetric extension of $A\oplus S_2$ and $\tau(\lambda)+M(\lambda)$ is the Weyl function of the pair $\{H,\widetilde{A}\}$ corresponding to the boundary triplet

$$\{\mathcal{H},\Gamma_0^{(1)},\Gamma_1^{(1)}\},\quad \text{where}\quad \Gamma_0^{(1)}f=\Gamma_0 f_1\quad\text{and}\quad \Gamma_1^{(1)}f=\Gamma_1 f_1+\chi f_2.$$

The selfadjoint extension $\widetilde{A}$ of H with the domain $\operatorname{dom}\widetilde{A}=\ker\Gamma_1^{(1)}$ coincides with the coupling construction in (1.2). Due to (1.8), the resolvent $(\widetilde{A}-\lambda)^{-1}$ becomes

$$(\widetilde{A}-\lambda)^{-1}=(\widetilde{A}_0-\lambda)^{-1}-\begin{pmatrix}\gamma(\lambda)\\ \gamma^{(2)}(\lambda)\end{pmatrix}(\tau(\lambda)+M(\lambda))^{-1}\begin{pmatrix}\gamma(\bar\lambda)^* & \gamma^{(2)}(\bar\lambda)^*\end{pmatrix}, \tag{1.10}$$

where $\widetilde{A}_0=\operatorname{diag}(A_0,A_0^{(2)})$, $A_0^{(2)}=\ker\chi_0$, and $\gamma^{(2)}(\lambda)=(\chi_0\upharpoonright\mathfrak{N}_\lambda(S_2))^{-1}$. The compression of (1.10) to $\mathfrak{H}$ gives (1.9).

This construction can be generalized to the case in which $\tau(\lambda)$ does not satisfy (1.7) by invoking the nondensely defined operator S_2. However, this coupling construction is essentially restricted to uniformly strict Nevanlinna functions $\tau(\lambda)$ or, equivalently, to the case in which (1.4) is satisfied. To explain the main difficulties arising (even in the case $n_\pm(A)=2$) when extending the coupling method to arbitrary Nevanlinna families, consider two examples.

Example 1.3. Let A be a minimal symmetric operator generated in $L_2(0,1)$ by the differential expression $-D^2=-d^2/dx^2$ on $\operatorname{dom} A=\overset{\circ}{W}{}_2^2(0,1)$. If $\widetilde{A}=\widetilde{A}^*$ is its exit space extension on

$$\widetilde{\mathfrak{H}}=L_2(-\infty,1)=L_2(-\infty,0)\oplus L_2(0,1)=:\widetilde{\mathfrak{H}}_2\oplus\widetilde{\mathfrak{H}}_1,$$

then $S_2=-D^2$ with $\operatorname{dom} S_2=\overset{\circ}{W}{}_2^2(-\infty,0)$. Note that $n_\pm(A)=2$, whereas $n_\pm(S_2)=1$. The boundary triplets $\Pi_A=\{\mathbb{C}^2,\Gamma_0,\Gamma_1\}$ and $\Pi_{S_2}=\{\mathbb{C},\chi_0,\chi_1\}$ for A^* and S_2^* can be defined in a usual way by setting

$$\Gamma_0\widehat{f}_1=\begin{pmatrix}f_1(0+)\\ f_1(1)\end{pmatrix},\quad \Gamma_1\widehat{f}_1=\begin{pmatrix}f_1'(0+)\\ -f_1'(1)\end{pmatrix},\qquad \begin{matrix}\chi_0\widehat{f}_2=f_2(0-), & \widehat{f}_1\in A^*,\\ \chi_1\widehat{f}_2=-f_2'(0-), & \widehat{f}_2\in S_2^*.\end{matrix} \tag{1.11}$$

The coupling (1.2) cannot be defined because $n_\pm(A)\neq n_\pm(S_2)$. Nevertheless, by setting

$$\chi=\bigl\{\{\widehat{f}_2,\operatorname{col}\,(f_2(0-),c,-f_2'(0-),hc)\}:\ \widehat{f}_2\in S_2^*,\ c\in\mathbb{C}\bigr\},$$

one can still define a (multivalued) boundary mapping $\chi\colon S_2^*\to\mathcal{H}^2$, and then formula (1.2) (written in an appropriate way) remains valid.

In this example, the first of the conditions (1.3) is violated. Typically, one meets this situation when considering higher-order ordinary differential operators on the entire line or on a half-line. On the other hand, partial differential operators usually lead to situations in which the second condition in (1.3) is violated.

Example 1.4. Let (a_{ij}) be a positive definite selfadjoint matrix with complex entries. Consider the second-order elliptic differential operator

$$\widetilde{A} := -\sum_{i,j=1}^{n} (\partial/\partial x_i)\, a_{ij}(\partial/\partial x_j) \tag{1.12}$$

on $\widetilde{\mathfrak{H}} = L_2(\mathbb{R}^n)$ with constant coefficients. Then $\widetilde{A}$ is selfadjoint on the natural domain $\operatorname{dom}\widetilde{A} = W_2^2(\mathbb{R}^n)$. Let now Ω be a bounded domain with smooth boundary $\partial\Omega$. Denote by $A = A_{\min}$ the minimal elliptic operator generated in $L_2(\Omega)$ by the differential expression (1.12). As is known (see, for instance, [9]), if $\partial\Omega$ is smooth, then $\operatorname{dom} A = \overset{\circ}{W}{}_2^2(\Omega)$. Using the decomposition

$$\widetilde{\mathfrak{H}} = L_2(\mathbb{R}^n) = L_2(\Omega_1) \oplus L_2(\Omega_2) =: \mathfrak{H}_1 \oplus \mathfrak{H}_2, \qquad \Omega_1 := \Omega, \quad \Omega_2 := \mathbb{R}^n \setminus \Omega,$$

we define the operators S_j and T_j, $j = 1, 2$, by (1.3). Then it can be shown (see Example 7.12) that $S_1 = A$ and $\operatorname{dom} S_j = \overset{\circ}{W}{}_2^2(\Omega_j)$, $j = 1, 2$. Further, $P_j W_2^2(\mathbb{R}^n) = W_2^2(\Omega_j)$. Therefore, in this case, T_j is a closable operator which is not closed and is defined by

$$T_j P_j f = P_j \widetilde{A} f, \quad f \in W_2^2(\mathbb{R}^n), \quad \operatorname{dom} T_j = W_2^2(\Omega_j), \quad j = 1, 2.$$

In other words, T_j is the elliptic operator generated in $L_2(\Omega_j)$ by the differential expression (1.12) on the domain $\operatorname{dom} T_j = W_2^2(\Omega_j)$. At the same time, as is known (see [9]), $W_2^2(\Omega_j)$ is dense in $\operatorname{dom}(S_j)_{\max} = \operatorname{dom} S_j^*$, and the closure $\overline{T}_j = S_j^*$ coincides with the maximal elliptic operator $(S_j)_{\max}$, i.e., $\overline{T}_j = S_j^* = (S_j)_{\max}$.

It is possible to overcome both geometrical difficulties explained above by applying the new concepts of boundary relations and their Weyl families introduced by the authors in [15]. These concepts generalize the notions of boundary triplet and the corresponding Weyl functions. Recall (see [15]) that a (possibly multivalued) mapping $\chi\colon T_2 \to \mathcal{H}^2$ is called a *boundary relation for S_2^** if the Green identity (1.1) and a certain maximality condition (similar to (BT2)) hold and the linear manifold T_2 is dense in S_2^*. The Weyl family τ of S_2 corresponding to the boundary relation χ is defined by

$$\tau(\lambda) = \Gamma(\widehat{\mathfrak{N}}_\lambda), \quad \text{where} \quad \widehat{\mathfrak{N}}_\lambda = \{\{f, \lambda f\} :\, f \in \ker(T_2 - \lambda)\}, \tag{1.13}$$

and now it belongs to the class of Nevanlinna families. If $\widetilde{A}$ is an arbitrary selfadjoint extension of A not satisfying (1.4), then the induced boundary relation χ defined by the formula

$$\chi = \left\{ \left\{ \widehat{f}_2, \begin{pmatrix} \Gamma_0 \widehat{f}_1 \\ -\Gamma_1 \widehat{f}_1 \end{pmatrix} \right\} :\, \widehat{f}_1 \oplus \widehat{f}_2 \in \widetilde{A},\, \widehat{f}_1 \in A^*,\, \widehat{f}_2 \in T_2 \right\} \tag{1.14}$$

is either multivalued or unbounded. For instance, the boundary relation χ in Example 1.3 is multivalued and the corresponding Weyl function $\tau(\lambda) = \operatorname{diag}\,(i\sqrt{\lambda}, h)$ is not strict. However, $\widetilde{A}$ can still be recovered from A^* and T_2 by using precisely the same coupling conditions as in (1.2).

To include an arbitrary Nevanlinna family in the framework of the coupling method, the main realization theorem from [15] is needed, which shows that every Nevanlinna family can be realized as the Weyl family of a boundary relation. One of the key results of the present paper is that, due to this new inverse result, the coupling construction in (1.2) can also be extended to the case of arbitrary Nevanlinna families. In this approach, the Nevanlinna family $\tau(\lambda)$ arising in the Kreĭn–Naĭmark formula (1.9) is treated as the Weyl family of a boundary relation $\chi : S_2^* \to \mathcal{H}^2$ as defined in (1.14). This geometric object contains the information concerning the exit space, whereas the ordinary boundary triplet $\Pi_A = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ contains the information concerning the original space. The exit space extensions $\widetilde{A}$ of A are then described by using the coupling conditions in (1.2), and this leads to new results and new types of solutions of problems involving generalized resolvents.

In the development of the coupling method in this general form, it is shown that one of the Weyl functions of the pair $\{A \oplus S_2, \widetilde{A}\}$ corresponding to a certain boundary relation of $A^* \oplus S_2^*$ is given by

$$\mathcal{M}(\lambda) = \begin{pmatrix} -(\tau + M)^{-1} & I - (\tau + M)^{-1} \cdot M \\ I - M \cdot (\tau + M)^{-1} & (\tau^{-1} + M^{-1})^{-1} \end{pmatrix}. \tag{1.15}$$

Further, the Nevanlinna functions $-(\tau+M)^{-1}$ and $(\tau^{-1}+M^{-1})^{-1}$ occurring on the diagonal of $\mathcal{M}(\lambda)$ in (1.15) can be treated as the Weyl functions of the pairs $\{H_1,\widetilde{A}\}$ and $\{H_2,\widetilde{A}\}$, where H_j are some intermediate symmetric extensions of A, $A\subset H_j\subset\widetilde{A}$, $j=1,2$. The Weyl function $\mathcal{M}(\lambda)$ turns out to be a very important and useful object in extension theory. For instance, it immediately leads to a general analytic criterion for Π-admissibility (see formulas (1.18) below). Recall that a parameter function τ in (1.9) is said to be Π-*admissible* if the corresponding minimal selfadjoint extension $\widetilde{A}$ of A is an operator. Moreover, a geometric treatment of the functions $-(\tau+m)^{-1}$ and $(\tau^{-1}+m^{-1})^{-1}$ as Weyl functions leads to other simple criteria for Π-admissibility. Note that functions of the form (1.15) appeared in several papers devoted to the spectral analysis of differential operators. For instance, 2×2-matrix functions $\mathcal{M}(\lambda)$ of the form (1.15) were used by Kac [32] in connection with the Sturm–Liouville operator $L=-D^2+q$ on $\mathbb{R}$, where M and τ are the Weyl functions corresponding to the minimal operator L on $L_2(\mathbb{R}_+)$ and $L_2(\mathbb{R}_-)$, respectively (see also [22, 23]).

The advantage of the approach developed in the present paper can also be shown by a simple geometric characterization of Π-admissibility by means of the so-called forbidden lineals $\mathcal{F}_\Gamma$ and $\mathcal{F}_\chi$ associated with a boundary mapping Γ and the induced boundary relation χ, respectively. Further, geometric and analytic characterizations of selfadjoint extensions of the second kind in the sense of Naĭmark are established. Recall that $\widetilde{A}$ is a selfadjoint extension of the second kind of a densely defined symmetric operator A (we write $\widetilde{A}\in\mathrm{Nai}_2(A)$) if and only if $\operatorname{dom}\widetilde{A}\cap\mathfrak{H}=\operatorname{dom}A$. For instance, if $n_\pm(A)=n<\infty$, then $\widetilde{A}\in\mathrm{Nai}_2(A)$ if and only if $n_\pm(S_2)=n$ and $\overline{\operatorname{dom}}S_2=\mathcal{H}_2$. On the other hand, as is known, the latter conditions can be expressed by means of the Weyl function τ of S_2 as follows:

$$\lim_{y\uparrow\infty} y^{-1}\tau(iy)=0 \quad\text{and}\quad \lim_{y\uparrow\infty} y\cdot\operatorname{Im}(\tau(iy)h,h)=\infty,\qquad h\in\mathcal{H}\setminus\{0\}. \tag{1.16}$$

Note that a criterion for the inclusion $\widetilde{A}\in\mathrm{Nai}_2(A)$ was obtained earlier by Štraus [52] in another form; however, the present approach is substantially simpler.

The paper is organized as follows. In Section 2, basic notions are introduced and various preliminary results are established. In particular, some new and useful facts on unitary relations in Kreĭn spaces are presented (for instance, on the composition of unitary relations; see Theorem 2.10). In Section 3, the notion of boundary relations for S^*, the corresponding Weyl families, orthogonal couplings, and unitary transformations (in the sense of Kreĭn spaces) of boundary relations are discussed. In that section, generalized boundary triplets and boundary triplets whose Weyl functions take values in $[\mathcal{H}]$ are also investigated. In Section 4, there are some general transformation results concerning boundary relations $\Gamma\colon\mathfrak{H}^2\to\mathcal{H}^2$ for S^* whose Weyl family $M(\lambda)$ belongs to the class $R[\mathcal{H}]$, i.e., $M(\cdot)$ is a Weyl function with values in $[\mathcal{H}]$. In this case, an arbitrary orthogonal decomposition $\mathcal{H}=\mathcal{H}_1\oplus\mathcal{H}_2$ of $\mathcal{H}$ induces the corresponding block operator representation

$$M(\lambda)=(M_{ij}(\lambda))_{i,j=1}^2 \tag{1.17}$$

of $M(\cdot)$. It is shown how one can identify intermediate closed symmetric extensions H of A and associated boundary relations for H^* such that the corresponding Weyl function is a given transform of blocks of $(M_{ij}(\lambda))$ including, for instance, linear combinations of $M_{ij}(\lambda)$ and Schur complements. In particular, induced boundary relations $\widetilde{\Gamma}$ arise for H^*, whose Weyl function $\widetilde{M}(\cdot)$ is equal to either $M_{11}+M_{22}$ or $-(M_{11}+M_{22})^{-1}$. Similar results for ordinary boundary triplets $\Pi=\{\mathcal{H},\Gamma_0,\Gamma_1\}$ for A^* have been published in [14]. However, the present generalizations are needed here for applications involving generalized resolvents. In Section 5, the coupling method of [14], as was briefly described above, is extended to the case of arbitrary Nevanlinna families $\tau(\cdot)$. This approach leads to new results and further geometric insight into various questions in this area. In the coupling method, the selfadjoint exit space extension $\widetilde{A}$ in $\widetilde{\mathfrak{H}}\supset\mathfrak{H}$ is constructed by means of a boundary triplet of A^* whose Weyl function is $M(\cdot)$, together with a boundary relation corresponding to the family $\tau(\cdot)\in\widetilde{R}(\mathcal{H})$. The coupling method makes it possible to treat the families $\tau(\cdot)$ and $-(\tau(\cdot)+M(\cdot))^{-1}$ arising in (1.9) as the Weyl families of $S_2:=\widetilde{A}\cap(\widetilde{\mathfrak{H}}\ominus\mathfrak{H})^2$ (see formula (5.1)) and of a certain intermediate extension of $A\oplus S_2$ (see formula (5.34)), respectively. In Section 6, the coupling method is applied to a new proof of Kreĭn–Naĭmark formula for generalized resolvents. A complete solution to the problem of Π-admissibility is also given (cf. [14]). Recall that, if A is nondensely defined, then exit space extensions $\widetilde{A}$ of A need not be single-valued operators. Using a coupling construction, the following simple analytic criterion for $\tau(\cdot)$ to generate a (minimal) selfadjoint operator extension $A^{(\tau)}$ of A is established: the Nevanlinna family $\tau(\cdot)$ in (1.9) corresponds to an operator $A^{(\tau)}$ (i.e., $\tau(\cdot)$ is Π-admissible) if and only if the following two conditions are satisfied:

$$\text{s-}\lim_{y\uparrow\infty} (\tau(iy)+M(iy))^{-1}/y = 0, \qquad \text{s-}\lim_{y\uparrow\infty} (\tau(iy)^{-1}+M(iy)^{-1})^{-1}/y = 0. \tag{1.18}$$

Moreover, the results on intermediate extensions given in Section 5 (a geometric treatment of $(\tau+M)^{-1}$ as a Weyl function) make it possible to show that, if, in addition, A_0 (A_1) is an operator, then $A^{(\tau)}$ is an operator if and only if the first (second) condition in (1.18) is satisfied, respectively. Further, in Section 6, an answer to a problem posed by Langer and Textorius in [40] is given. Finally, Section 7 contains both geometric and analytic characterizations for selfadjoint extensions of the second kind (in the sense of Naĭmark) of a densely defined symmetric operator.

A preliminary version of the results presented in this paper was published as a preprint [17]; a part of the results was announced in [16]. Later on, the coupling method (as it was introduced in [14, 17] and now further developed in its full generality in the present paper) was also successfully applied by other authors in diverse general settings; see, for instance, [5–7] and the references therein.

2. PRELIMINARIES

2.1. *Linear Relations in Hilbert Spaces*

Let $\mathfrak{H}$ and $\mathfrak{H}'$ be Hilbert spaces. A linear relation T from $\mathfrak{H}$ to $\mathfrak{H}'$ is a linear subspace of $\mathfrak{H}\times\mathfrak{H}'$. We systematically identify a linear operator T with its graph. It is convenient to write $T\colon \mathfrak{H}\to\mathfrak{H}'$ and treat the linear relation T as a multivalued linear mapping from $\mathfrak{H}$ into $\mathfrak{H}'$. If $\mathfrak{H}'=\mathfrak{H}$, one speaks of a linear relation T on $\mathfrak{H}$.

For a linear relation $T\colon \mathfrak{H}\to\mathfrak{H}'$, the inverse T^{-1} is the linear relation $\{\,\{f',f\}\,:\,\{f,f'\}\in T\,\}$ from $\mathfrak{H}'$ to $\mathfrak{H}$. The symbols $\operatorname{dom} T$, $\ker T$, $\operatorname{ran} T$, $\operatorname{mul} T$ $(=\ker T^{-1})$, and T^* stand for the domain, the kernel, the range, the multivalued part, and the adjoint of T, respectively (see [8] or [13]). The sum T_1+T_2 and the componentwise sum $T_1\widehat{+}T_2$ of linear relations T_1 and T_2 are defined by the rules $T_1+T_2=\{\{f,g+h\}:\{f,g\}\in T_1,\{f,h\}\in T_2\}$ and $T_1\widehat{+}T_2=\{\{f+h,g+k\}:\{f,g\}\in T_1,$ $\{h,k\}\in T_2\}$. If the componentwise sum is orthogonal, we denote it by $T_1\oplus T_2$. The null spaces of $T-\lambda$, $\lambda\in\mathbb{C}$, are defined by

$$\mathfrak{N}_\lambda(T)=\ker(T-\lambda), \qquad \widehat{\mathfrak{N}}_\lambda(T)=\{\,\{f,\lambda f\}\in T:\, f\in\mathfrak{N}_\lambda(T)\,\}. \tag{2.1}$$

The symbol $\rho(T)$ ($\hat{\rho}(T)$) stands for the set of regular (regular type) points of T. The closure of a linear relation T is denoted by $\operatorname{clos} T$, and a linear relation T_2 is an *extension* of T_1 if $T_1\subset T_2$.

The product of linear relations is defined in the standard way. Some basic properties of operator product remain valid for the product of relations. For instance, the following assertion holds.

Lemma 2.1. *Let $\mathfrak{H}_1$, $\mathfrak{H}_2$, and $\mathfrak{H}_3$ be Hilbert spaces and let $B:\mathfrak{H}_1\to\mathfrak{H}_2$ and $A:\mathfrak{H}_2\to\mathfrak{H}_3$ be linear relations. Then*

(i) $(AB)^{-1}=B^{-1}A^{-1}$ *and* $(A^*)^{-1}=(A^{-1})^*$;
(ii) $(AB)^*\supset B^*A^*$;
(iii) *if* $A\in[\mathfrak{H}_2,\mathfrak{H}_3]$ *or* $B^{-1}\in[\mathfrak{H}_2,\mathfrak{H}_1]$, *then* $(AB)^*=B^*A^*$.

Recall that a linear relation T on $\mathfrak{H}$ is said to be *symmetric* (*dissipative* or *accumulative*) if $\operatorname{Im}(h',h)=0$ ($\geqslant 0$ or $\leqslant 0$, respectively) for all $\{h,h'\}\in T$. These properties are preserved under closure. By polarization, it follows that a linear relation T on $\mathfrak{H}$ is symmetric if and only if $T\subset T^*$. A linear relation T on $\mathfrak{H}$ is said to be *selfadjoint* if $T=T^*$ and *essentially selfadjoint* if $\operatorname{clos} T=T^*$. A dissipative (accumulative) linear relation T on $\mathfrak{H}$ is said to be *maximal dissipative* (*maximal accumulative*) if it has no proper dissipative (accumulative) extensions.

Assume that T is closed. If T is dissipative or accumulative, then the orthogonal decomposition $\mathfrak{H}=(\operatorname{mul} T)^\perp\oplus\operatorname{mul} T$ induces an orthogonal decomposition of T,

$$T=T_s\oplus T_\infty, \qquad T_\infty=\{0\}\times\operatorname{mul} T, \qquad T_s=\{\,\{f,g\}\in T:\, g\perp\operatorname{mul} T\,\}, \tag{2.2}$$

where T_∞ is a selfadjoint relation on $\operatorname{mul} T$ and T_s is an operator on $\mathfrak{H}\ominus\operatorname{mul} T$ such that $\overline{\operatorname{dom}}\, T_s=\overline{\operatorname{dom}}\, T=(\operatorname{mul} T^*)^\perp$, which is dissipative or accumulative, respectively.

Definition 2.2. A symmetric linear relation S on $\mathfrak{H}$ is said to be *simple* if there is no nontrivial orthogonal decomposition of $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$ such that, in the corresponding orthogonal decomposition $S = S_1 \oplus S_2$, the relation S_1 is symmetric on $\mathfrak{H}_1$ and S_2 is selfadjoint on $\mathfrak{H}_2$.

The decomposition (2.2) for $S = S_s \oplus S_\infty$ shows that a simple closed symmetric relation is necessarily an operator. Recall (cf., e.g., [40]) that a closed symmetric linear relation S on a Hilbert space $\mathfrak{H}$ is simple if and only if $\mathfrak{H} = \overline{\operatorname{span}}\{\, \mathfrak{N}_\lambda(S^*) : \lambda \in \mathbb{C} \setminus \mathbb{R}\,\}$.

2.2. *Unitary Relations on Kreĭn Spaces*

Recall that a signature operator j on a Hilbert space is a bounded linear operator such that $j = j^* = j^{-1}$. A signature operator provides the Hilbert space with a Kreĭn space structure with the inner product $(j\,\cdot,\cdot)$. Let $\mathfrak{H}$ and $\mathcal{H}$ be Hilbert spaces with signature operators $j_\mathfrak{H}$ and $j_\mathcal{H}$, respectively, and denote the corresponding Kreĭn spaces by $(\mathfrak{H}, j_\mathfrak{H})$ and $(\mathcal{H}, j_\mathcal{H})$. Then the adjoint $T^{[*]}$ of a linear relation T from the Kreĭn space $(\mathfrak{H}, j_\mathfrak{H})$ to the Kreĭn space $(\mathcal{H}, j_\mathcal{H})$ is given by $T^{[*]} = j_\mathfrak{H} T^* j_\mathcal{H}$. In what follows, linear relations $T\colon \mathfrak{H} \to \mathfrak{H}$ are often regarded as subspaces of the space $\mathfrak{H} \times \mathfrak{H}$ interpreted as the Kreĭn space $(\mathfrak{H}^2, J_\mathfrak{H})$ with the fundamental symmetry

$$J_\mathfrak{H} := \begin{pmatrix} 0 & -iI_\mathfrak{H} \\ iI_\mathfrak{H} & 0 \end{pmatrix}. \tag{2.3}$$

Definition 2.3 [48]. A linear relation T from the Kreĭn space $(\mathfrak{H}, j_\mathfrak{H})$ to the Kreĭn space $(\mathcal{H}, j_\mathcal{H})$ is said to be *isometric* if $T^{-1} \subset T^{[*]}$ and *unitary* if $T^{-1} = T^{[*]}$.

The following statements are due to Yu. L. Shmul'jan [48]. They can also be obtained directly from the relation $T^{[*]} = T^{-1}$ and from [15, Prop. 2.2].

Proposition 2.4. *Let T be a unitary relation from the Kreĭn space $(\mathfrak{H}, j_\mathfrak{H})$ to the Kreĭn space $(\mathcal{H}, j_\mathcal{H})$. Then*

(i) *T is closed and the inverse T^{-1} and the adjoint $T^{[*]}$ are unitary, too*;
(ii) *$\operatorname{dom} T$ is closed if and only if $\operatorname{ran} T$ is closed*;
(iii) *the following equalities hold*:

$$\ker T = (\operatorname{dom} T)^{[\perp]}, \qquad \operatorname{mul} T = (\operatorname{ran} T)^{[\perp]}. \tag{2.4}$$

A unitary relation $T : (\mathfrak{H}, j_\mathfrak{H}) \to (\mathcal{H}, j_\mathcal{H})$ can be multivalued, nondensely defined, and unbounded. It is the graph of an operator if and only if its range is dense, and the operator need not be densely defined or bounded; if it is bounded, it need not be densely defined. To distinguish classical unitary operators from those in Definition 2.3 (see also [15, p. 10]), we use the following definition.

Definition 2.5. A unitary relation T from a Kreĭn space $\mathfrak{H}$ to a Kreĭn space $\mathcal{H}$ is called

(i) a *standard unitary operator* if T belongs to $[\mathfrak{H}, \mathcal{H}]$;
(ii) a *nonstandard unitary operator* if T is a (single-valued) operator not belonging to $[\mathfrak{H}, \mathcal{H}]$.

Here is an example of a nonstandard bounded unitary operator.

Example 2.6. Let $\mathfrak{H}$ be a Hilbert space, let $\mathfrak{L}$ be a subspace of $\mathfrak{H}$, and let P be the orthogonal projections of $\mathfrak{H}$ onto $\mathfrak{L}$. Consider $\mathfrak{H} \times \mathfrak{H}$ and its subspace $\mathfrak{L} \times \mathfrak{L}$ as the Kreĭn spaces equipped with the fundamental symmetries $J_\mathfrak{H}$ and $J_\mathfrak{L}$, respectively, as in (2.3). Define the operator T from $\mathfrak{H} \times \mathfrak{H}$ to $\mathfrak{L} \times \mathfrak{L}$ by the rule

$$T = \left\{ \left\{ \left(\begin{smallmatrix} f \\ h \end{smallmatrix}\right), \left(\begin{smallmatrix} Pf \\ h \end{smallmatrix}\right) \right\} : \, f \in \mathfrak{H},\, h \in \mathfrak{L} \right\}.$$

Then, clearly, $\operatorname{dom} T = \mathfrak{H} \times \mathfrak{L}$, $\ker T = \ker P \times \{0\}$, and T is bounded. It is easy to see that the Kreĭn space adjoint $T^{[*]}\colon \mathfrak{L} \times \mathfrak{L} \to \mathfrak{H} \times \mathfrak{H}$ coincides with T^{-1}. Thus, T is a bounded unitary operator which is not standard.

Corollary 2.7. *Let T be a unitary relation from the Kreĭn space $(\mathfrak{H}, j_\mathfrak{H})$ to the Kreĭn space $(\mathcal{H}, j_\mathcal{H})$. Then $T \in [\mathfrak{H}, \mathcal{H}]$ if and only if $T^{-1} \in [\mathcal{H}, \mathfrak{H}]$.*

Proof. Let $T \in [\mathfrak{H}, \mathcal{H}]$. Then $\operatorname{dom} T = \mathfrak{H}$ and $\operatorname{mul} T = \{0\}$, i.e., $\operatorname{ran} T^{-1} = \mathfrak{H}$ and $\ker T^{-1} = \{0\}$. Proposition 2.4 implies now that $T^{-1} \in [\mathcal{H}, \mathfrak{H}]$.

Note that, for a unitary relation T from $(\mathfrak{H}, j_{\mathfrak{H}})$ to $(\mathcal{H}, j_{\mathcal{H}})$, both T and T^{-1} are operators if and only if $\overline{\operatorname{dom}} T = \mathfrak{H}$ and $\overline{\operatorname{ran}} T = \mathcal{H}$. Moreover, in this case, $\operatorname{dom} T = \mathfrak{H}$ if and only if $\operatorname{ran} T = \mathcal{H}$.

Remark 2.8. An operator T from the Kreĭn space $(\mathfrak{H}, j_{\mathfrak{H}})$ to the Kreĭn space $(\mathcal{H}, j_{\mathcal{H}})$ is unitary in the sense of M. G. Kreĭn (*Kreĭn unitary*) if $\operatorname{dom} T = \mathfrak{H}$, $\operatorname{ran} T = \mathcal{H}$, and $[Tf, Tf]_{\mathcal{H}} = [f, f]_{\mathfrak{H}}$, $f \in \mathfrak{H}$ (see [4, Chap. 2, Def. 5.1 and Cor. 5.8]). In this case, $[Tf, Tg]_{\mathcal{H}} = [f, g]_{\mathfrak{H}}$, $f, g \in \mathfrak{H}$, i.e., $T^{-1} \subset T^{[*]}$, and now it follows from $\operatorname{dom} T = \mathfrak{H}$ and $\operatorname{ran} T = \mathcal{H}$ that $T^{-1} = T^{[*]}$; cf. [15, Prop. 2.8]. Moreover, $T \in [\mathfrak{H}, \mathcal{H}]$, and then $T^{-1} \in [\mathcal{H}, \mathfrak{H}]$ by Corollary 2.7. Hence, a Kreĭn unitary operator is standard in the sense of Definition 2.5. Conversely, if T is standard unitary in the sense of Definition 2.5, i.e., $T^{-1} = T^{[*]}$ and $T \in [\mathfrak{H}, \mathcal{H}]$ (or, equivalently, $T^{-1} \in [\mathcal{H}, \mathfrak{H}]$), then T is Kreĭn unitary.

Unitary relations between Kreĭn spaces admit useful composition properties. We first present a result concerning the adjoint of the composition (product) of linear relations if the domain or the range of one of the relations is closed; note that Lemma 2.1 remains valid in the Kreĭn space situation.

Lemma 2.9. *Let $\mathfrak{K}_j$, $j = 0, 1, 2, 3$, be Kreĭn spaces, and let $S\colon \mathfrak{K}_1 \to \mathfrak{K}_2$ be a closed relation. In this case,*

(i) *if $\operatorname{dom} S$ is closed, then $(SX)^{[*]} = X^{[*]} S^{[*]}$ for every linear relation $X\colon \mathfrak{K}_0 \to \mathfrak{K}_1$ such that $\operatorname{ran} X \subset \operatorname{dom} S$;*

(ii) *if $\operatorname{ran} S$ is closed, then $(YS)^{[*]} = S^{[*]} Y^{[*]}$ for every linear relation $Y\colon \mathfrak{K}_2 \to \mathfrak{K}_3$ such that $\operatorname{dom} Y \subset \operatorname{ran} S$.*

Proof. (i) The inclusion $(SX)^{[*]} \supset X^{[*]} S^{[*]}$ is always satisfied, cf. (ii) in Lemma 2.1. To prove the reverse inclusion, take $\{f, g\} \in (SX)^{[*]}$; then

$$[g, h]_{\mathfrak{K}_1} = [f, k]_{\mathfrak{K}_3} \quad \text{for all} \quad \{h, k\} \in SX. \tag{2.5}$$

Since the linear relation SX contains the set $\{\{0, f_0\} : f_0 \in \operatorname{mul} S\}$, it follows from (2.5) that $[f, f_0] = 0$ for all $f_0 \in \operatorname{mul} S$, and thus $f \in (\operatorname{mul} S)^{[\perp]} = \overline{\operatorname{dom}}\, S^{[*]}$. Since S is closed and $\operatorname{dom} S$ is closed, the domain $\operatorname{dom} S^{[*]}$ is also closed by [15, Prop. 2.2]. Hence, $f \in \operatorname{dom} S^{[*]}$ and $\{f, f'\} \in S^{[*]}$ for some $f' \in \mathfrak{K}_1$. Now it suffices to show that $\{f', g\} \in X^{[*]}$, because this yields $\{f, g\} \in X^{[*]} S^{[*]}$. Indeed, for each $\{h, u\} \in X$, there is a $u' \in \mathfrak{K}_2$ such that $\{u, u'\} \in S$, due to the condition $\operatorname{ran} X \subset \operatorname{dom} S$. Then, for all $\{f, f'\} \in S^{[*]}$,

$$[g, h] - [f', u] = [g, h] - [f, u']. \tag{2.6}$$

Clearly, $\{h, u'\} \in SX$, and therefore (2.5) and (2.6) yield $[g, h] = [f', u]$ for any $\{h, u\} \in X$. This means that $\{f', g\} \in X^{[*]}$. Thus, $(SX)^{[*]} \subset X^{[*]} S^{[*]}$.

(ii) This follows by applying (i) to the inverse $(YS)^{-1} = S^{-1} Y^{-1}$; see also Lemma 2.1.

The following theorem treats the composition of two unitary relations; the result is needed below.

Theorem 2.10. *Let $\mathfrak{K}_1$, $\mathfrak{K}_2$, and $\mathfrak{K}_3$ be Kreĭn spaces, and let the linear relations $T\colon \mathfrak{K}_1 \to \mathfrak{K}_2$ and $S\colon \mathfrak{K}_2 \to \mathfrak{K}_3$ be unitary. In this case,*

(i) *if*
$$\operatorname{ran} T \subset \operatorname{dom} S \quad \textit{and} \quad \operatorname{dom} S \quad \textit{is closed}, \tag{2.7}$$
then $ST : \mathfrak{K}_1 \to \mathfrak{K}_3$ is unitary and $\operatorname{dom} ST = \operatorname{dom} T$;

(ii) *if*
$$\operatorname{ran} T \supset \operatorname{dom} S \quad \textit{and} \quad \operatorname{dom} T \quad \textit{is closed}, \tag{2.8}$$
then $ST : \mathfrak{K}_1 \to \mathfrak{K}_3$ is unitary and $\operatorname{ran} ST = \operatorname{ran} S$;

(iii) *if $\operatorname{ran} T = \operatorname{dom} S$ and $\operatorname{ran} S = \mathfrak{K}_3$, then the unitary relation $ST\colon \mathfrak{K}_1 \to \mathfrak{K}_3$ is bounded and single-valued* (*not necessarily densely defined*);

(iv) *if $T \in [\mathfrak{K}_1, \mathfrak{K}_2]$ or $S \in [\mathfrak{K}_2, \mathfrak{K}_3]$, then $ST\colon \mathfrak{K}_1 \to \mathfrak{K}_3$ is unitary*;

(v) *if $T \in [\mathfrak{K}_1, \mathfrak{K}_2]$ and $S \in [\mathfrak{K}_2, \mathfrak{K}_3]$, then ST is a unitary operator belonging to $[\mathfrak{K}_1, \mathfrak{K}_3]$.*

Proof. (i) By the assumptions in formula (2.7) and by part (i) of Lemma 2.9, we have

$$(ST)^{[*]} = T^{[*]}S^{[*]} = T^{-1}S^{-1} = (ST)^{-1}$$

because S and T are unitary. Therefore, ST is unitary. The relation $\operatorname{dom} ST = \operatorname{dom} T$ follows from the assumption $\operatorname{ran} T \subset \operatorname{dom} S$.

(ii) This is an immediate consequence of Lemma 2.9; it can also be obtained from (i) by using inverses.

(iii) If $\operatorname{ran} T = \operatorname{dom} S$ and $\operatorname{ran} S = \mathfrak{K}_3$, then $\operatorname{dom} S$ and $\operatorname{dom} T$ are closed by Proposition 2.4 (ii). Therefore, by part (ii), the relation $ST\colon \mathfrak{K}_1 \to \mathfrak{K}_3$ is unitary and $\operatorname{ran} ST = \operatorname{ran} S = \mathfrak{K}_3$. Therefore, ST is bounded and single-valued; cf. [15, Cor. 2.4].

(iv) If $S \in [\mathfrak{K}_2, \mathfrak{K}_3]$ then $\operatorname{dom} S = \mathfrak{K}_2$ by definition. Hence, the relation ST is unitary by part (i). On the other hand, if $T \in [\mathfrak{K}_1, \mathfrak{K}_2]$, then $\operatorname{dom} T = \mathfrak{K}_1$ and $\operatorname{ran} T = \mathfrak{K}_2$, and now part (ii) shows that ST is unitary.

(v) This is an obvious and well-known fact (see [4]).

The following examples show that, for infinite-dimensional spaces, unitary operators can be unbounded and can form a family which is not a semigroup, i.e., the product of two unitary operators need not be a unitary operator.

Example 2.11. Let K be a densely defined operator on a Hilbert space $\mathfrak{H}$. Define the block operator matrix T by

$$T = \begin{pmatrix} I_{\mathfrak{H}} & K \\ 0 & I_{\mathfrak{H}} \end{pmatrix}. \tag{2.9}$$

Then T is an injective operator, i.e., $\ker T = \{0\}$ and $\operatorname{mul} T = \{0\}$. It is easy to see that T is closed if and only if K is closed. The inverse of T is given by

$$T^{-1} = \begin{pmatrix} I_{\mathfrak{H}} & -K \\ 0 & I_{\mathfrak{H}} \end{pmatrix} \tag{2.10}$$

and hence T is densely defined with dense range; in fact, $\operatorname{dom} T = \operatorname{ran} T = \mathfrak{H} \oplus \operatorname{dom} K$. Now we regard $\mathfrak{H} \oplus \mathfrak{H}$ as the Kreĭn space $(\mathfrak{H}^2, J_{\mathfrak{H}})$ with the fundamental symmetry $J_{\mathfrak{H}}$, as in (2.3). Then

$$T^{[*]} = \begin{pmatrix} I_{\mathfrak{H}} & -K^* \\ 0 & I_{\mathfrak{H}} \end{pmatrix}. \tag{2.11}$$

Identities (2.10) and (2.11) show that T is isometric (unitary) if and only if K is symmetric (selfadjoint, respectively). Therefore, if K_1 and K_2 are two unbounded selfadjoint operators on $\mathfrak{H}$ such that $K_1 + K_2$ is not selfadjoint, then the composition T_1T_2 of the unitary operators T_1 and T_2,

$$T_1T_2 = \begin{pmatrix} I_{\mathfrak{H}} & K_1 \\ 0 & I_{\mathfrak{H}} \end{pmatrix} \begin{pmatrix} I_{\mathfrak{H}} & K_2 \\ 0 & I_{\mathfrak{H}} \end{pmatrix} = \begin{pmatrix} I_{\mathfrak{H}} & K_1 + K_2 \\ 0 & I_{\mathfrak{H}} \end{pmatrix},$$

is not a unitary operator on $(\mathfrak{H}^2, J_{\mathfrak{H}})$. Here both assumptions in (2.7) can fail to hold. This is the case if, for instance, K_1 and K_2 are selfadjoint operators on $\mathfrak{H}$ such that $\operatorname{dom} K_1 \cap \operatorname{dom} K_2 = \{0\}$.

Note that, if K_1 is an unbounded selfadjoint operator on $\mathfrak{H}$ and $K_2 = -K_1$, then $\operatorname{ran} T_2 = \operatorname{dom} T_1$, cf. (2.10), and $\operatorname{dom} T_1T_2 = \operatorname{dom} T_2$. Now the composition T_1T_2 is not closed, and hence it cannot be unitary. In this case, the first assumption in (2.7) and (2.8) is satisfied, whereas the other assumption in (2.7) and (2.8) fails to hold.

2.3. *Main Transform*

It is convenient to interpret the Hilbert space $\mathfrak{H}^2 = \mathfrak{H} \oplus \mathfrak{H}$ as a Kreĭn space $(\mathfrak{H}^2, J_{\mathfrak{H}})$ whose inner product is determined by the fundamental symmetry $J_{\mathfrak{H}}$ of the form (2.3). There is a useful and important transform which gives a relationship between the subspaces of the Hilbert space $(\mathfrak{H} \oplus \mathcal{H})^2$ and linear relations from the Kreĭn space $(\mathfrak{H}^2, J_{\mathfrak{H}})$ to $(\mathcal{H}^2, J_{\mathcal{H}})$ [15]. To recall this relationship, define the linear mapping $\mathcal{J}$ from $\mathfrak{H}^2 \times \mathcal{H}^2$ to $(\mathfrak{H} \oplus \mathcal{H})^2$ by

$$\mathcal{J} : \left\{ \begin{pmatrix} f \\ f' \end{pmatrix}, \begin{pmatrix} h \\ h' \end{pmatrix} \right\} \mapsto \left\{ \begin{pmatrix} f \\ h \end{pmatrix}, \begin{pmatrix} f' \\ -h' \end{pmatrix} \right\}, \qquad f, f' \in \mathfrak{H}, \quad h, h' \in \mathcal{H}.$$

This mapping establishes a one-to-one correspondence between the (closed) linear relations $\Gamma\colon \mathfrak{H}^2 \to \mathcal{H}^2$ and the (closed) linear relations $\widetilde{A}$ on $\widetilde{\mathfrak{H}} = \mathfrak{H} \oplus \mathcal{H}$ by the rule

$$\Gamma \mapsto \widetilde{A} := \mathcal{J}(\Gamma) = \left\{ \left\{ \begin{pmatrix} f \\ h \end{pmatrix}, \begin{pmatrix} f' \\ -h' \end{pmatrix} \right\} : \left\{ \begin{pmatrix} f \\ f' \end{pmatrix}, \begin{pmatrix} h \\ h' \end{pmatrix} \right\} \in \Gamma \right\}. \tag{2.12}$$

The mapping $\mathcal{J}$ plays a principal role and is referred to as the *main transform*. According to [15, Prop. 2.10], the main transform $\mathcal{J}$ establishes a one-to-one correspondence between the contractive, isometric, and unitary relations Γ from $(\mathfrak{H}^2, J_{\mathfrak{H}})$ to $(\mathcal{H}^2, J_{\mathcal{H}})$ and the dissipative, symmetric, and selfadjoint relations $\widetilde{A}$ on $\mathfrak{H} \oplus \mathcal{H}$, respectively.

2.4. *Nevanlinna Families*

A family of linear relations $M(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, on a Hilbert space $\mathcal{H}$ is called a *Nevanlinna family* if

(i) for every $\lambda \in \mathbb{C}_+(\mathbb{C}_-)$, the relation $M(\lambda)$ is maximal dissipative (accumulative, respectively);

(ii) $M(\lambda)^* = M(\bar{\lambda})$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$;

(iii) for some (and hence for all) $\mu \in \mathbb{C}_+(\mathbb{C}_-)$, the operator family $(M(\lambda) + \mu)^{-1} (\in [\mathcal{H}])$ is holomorphic for all $\lambda \in \mathbb{C}_+(\mathbb{C}_-)$.

By the maximality condition, each relation $M(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, is necessarily closed. The *class of all Nevanlinna families* in a Hilbert space is denoted by $\widetilde{R}(\mathcal{H})$. If the multivalued part $\operatorname{mul} M(\lambda)$ of $M(\cdot) \in \widetilde{R}(\mathcal{H})$ is nontrivial, then it does not depend on $\lambda \in \mathbb{C} \setminus \mathbb{R}$, and hence

$$M(\lambda) = M_s(\lambda) \oplus M_\infty, \qquad M_\infty = \{0\} \times \operatorname{mul} M(\lambda), \qquad \lambda \in \mathbb{C} \setminus \mathbb{R}, \tag{2.13}$$

where $M_s(\lambda)$ is a Nevanlinna family of densely defined operators on $\mathcal{H} \ominus \operatorname{mul} M(\lambda)$ (see [38]).

Clearly, if $M(\cdot) \in \widetilde{R}(\mathcal{H})$, then $M_\infty \subset M(\lambda) \cap M(\lambda)^*$ for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$. The following *subclasses* of the class $\widetilde{R}(\mathcal{H})$ are useful:

$$\begin{aligned}
R(\mathcal{H}) &= \bigl\{M(\cdot) \in \widetilde{R}(\mathcal{H}) :\ \operatorname{mul} M(\lambda) = \{0\}\bigr\};\\
R^s(\mathcal{H}) &= \bigl\{M(\cdot) \in \widetilde{R}(\mathcal{H}) :\ M(\lambda) \cap M(\lambda)^* = \{0\} \quad \text{for all} \quad \lambda \in \mathbb{C} \setminus \mathbb{R}\bigr\};\\
R^u(\mathcal{H}) &= \bigl\{M(\cdot) \in \widetilde{R}(\mathcal{H}) :\ M(\lambda) \widehat{+} M(\lambda)^* = \mathcal{H}^2 \quad \text{for all} \quad \lambda \in \mathbb{C} \setminus \mathbb{R}\bigr\};\\
R[\mathcal{H}] &= \bigl\{M(\cdot) \in \widetilde{R}(\mathcal{H}) :\ \operatorname{dom} M(\lambda) = \mathcal{H} \quad \text{for all} \quad \lambda \in \mathbb{C} \setminus \mathbb{R}\bigr\};\\
R^s[\mathcal{H}] &= \bigl\{M(\cdot) \in R[\mathcal{H}] :\ \ker \operatorname{Im} M(\lambda) = \{0\} \quad \text{for all} \quad \lambda \in \mathbb{C} \setminus \mathbb{R}\bigr\};\\
R^u[\mathcal{H}] &= \bigl\{M(\cdot) \in R^s[\mathcal{H}] :\ 0 \in \rho(\operatorname{Im} M(\lambda)) \quad \text{for all} \quad \lambda \in \mathbb{C} \setminus \mathbb{R}\bigr\}.
\end{aligned}$$

The subclasses of $\widetilde{R}(\mathcal{H})$ can be equivalently defined by assuming the corresponding property of $M(\lambda)$ at a single point $\lambda \in \mathbb{C} \setminus \mathbb{R}$ only. Moreover, it is easy to show that $R^u[\mathcal{H}] = R^u(\mathcal{H})$; see [15] for further details. The Nevanlinna functions in $R^s(\mathcal{H})$ and $R^u[\mathcal{H}]$ are said to be *strict* and *uniformly strict*, respectively.

A pair $\{\Phi, \Psi\}$ of holomorphic $[\mathcal{H}]$-valued functions on $\mathbb{C}_+ \cup \mathbb{C}_-$ is said to be a *Nevanlinna pair* if

(N1) $\operatorname{Im} \Phi(\lambda)^* \Psi(\lambda) / \operatorname{Im} \lambda \geqslant 0$, $\lambda \in \mathbb{C}_+ \cup \mathbb{C}_-$;

(N2) $\Psi(\bar{\lambda})^* \Phi(\lambda) - \Phi(\bar{\lambda})^* \Psi(\lambda) = 0$, $\lambda \in \mathbb{C}_+ \cup \mathbb{C}_-$;

(N3) $0 \in \rho(\Psi(\lambda) \pm i\Phi(\lambda))$, $\lambda \in \mathbb{C}_\pm$.

Two Nevanlinna pairs $\{\Phi_1, \Psi_1\}$ and $\{\Phi_2, \Psi_2\}$ are said to be *equivalent* if $\Phi_2(\lambda) = \Phi_1(\lambda)\chi(\lambda)$ and $\Psi_2(\lambda) = \Psi_1(\lambda)\chi(\lambda)$ for some operator function $\chi(\lambda) \in [\mathcal{H}]$ which is holomorphic and invertible on $\mathbb{C}_+ \cup \mathbb{C}_-$. If $\{\Phi, \Psi\}$ is a Nevanlinna pair, then the following kernel is nonnegative on $\mathbb{C}_+ \cup \mathbb{C}_-$:

$$\mathsf{N}_{\Phi\Psi}(\lambda, \mu) = \frac{\Phi(\mu)^* \Psi(\lambda) - \Psi(\mu)^* \Phi(\lambda)}{\lambda - \bar{\mu}}, \qquad \lambda, \mu \in \mathbb{C}_+ \cup \mathbb{C}_-. \tag{2.14}$$

The set of Nevanlinna families $\tau(\lambda)$ and the set of equivalence classes of Nevanlinna pairs $\{\Phi, \Psi\}$ are in a one-to-one correspondence via the formula

$$\tau(\lambda) = \{\Phi(\lambda), \Psi(\lambda)\} := \bigl\{\,\{\Phi(\lambda)h, \Psi(\lambda)h\} :\ h \in \mathcal{H}\,\bigr\}. \tag{2.15}$$

Moreover, the strict and uniformly strict Nevanlinna families are characterized by the conditions $0 \notin \sigma_p(\mathsf{N}_{\Phi\Psi}(\lambda, \lambda))$ and $0 \in \rho(\mathsf{N}_{\Phi\Psi}(\lambda, \lambda))$ for some $\lambda \in \mathbb{C} \setminus \mathbb{R}$, respectively.

3. BOUNDARY RELATIONS AND WEYL FAMILIES

3.1. *Definitions and Basic Properties*

Let S be a closed symmetric linear relation on the Hilbert space $\mathfrak{H}$. It is not assumed that the defect numbers of S are equal or finite. A boundary relation for S^* is defined as follows (cf. [15]).

Definition 3.1. Let $\mathcal{H}$ be a Hilbert space. A linear relation $\Gamma\colon \mathfrak{H}^2 \mapsto \mathcal{H}^2$ is called a *boundary relation* for S^* if

(G1) $\operatorname{dom}\Gamma$ is dense in S^* and the identity

$$(f',g)_{\mathfrak{H}} - (f,g')_{\mathfrak{H}} = (h',k)_{\mathcal{H}} - (h,k')_{\mathcal{H}} \tag{3.1}$$

holds for every $\{\widehat{f},\widehat{h}\}$, $\{\widehat{g},\widehat{k}\} \in \Gamma$;

(G2) Γ is maximal, which means that, if $\{\widehat{g},\widehat{k}\} \in \mathfrak{H}^2 \times \mathcal{H}^2$ satisfies (3.1) for every $\{\widehat{f},\widehat{h}\} \in \Gamma$, then $\{\widehat{g},\widehat{k}\} \in \Gamma$.

Here $\widehat{f} = \{f,f'\}$, $\widehat{g} = \{g,g'\} \in \operatorname{dom}\Gamma$ and $\widehat{h} = \{h,h'\}$, $\widehat{k} = \{k,k'\} \in \operatorname{ran}\Gamma$.

Condition (3.1) in (G1) can be interpreted as an abstract Green's identity. In the terminology of Kreĭn spaces, identity (3.1) means that Γ is an isometric relation from the Kreĭn space $(\mathfrak{H}^2, J_{\mathfrak{H}})$ to the Kreĭn space $(\mathcal{H}^2, J_{\mathcal{H}})$ because

$$(J_{\mathfrak{H}}\widehat{f},\widehat{g})_{\mathfrak{H}^2} = (J_{\mathcal{H}}\widehat{h},\widehat{k})_{\mathcal{H}^2}, \qquad \{\widehat{f},\widehat{h}\},\{\widehat{g},\widehat{k}\} \in \Gamma. \tag{3.2}$$

The maximality condition (G2) ensures that a boundary relation Γ is in fact a unitary relation from the Kreĭn space $(\mathfrak{H}^2, J_{\mathfrak{H}})$ to the Kreĭn space $(\mathcal{H}^2, J_{\mathcal{H}})$; in particular, it is closed and linear. Conversely, if $\Gamma\colon (\mathfrak{H}^2, J_{\mathfrak{H}}) \to (\mathcal{H}^2, J_{\mathcal{H}})$ is a unitary relation whose domain is dense in S^*, then Γ is a boundary relation for S^* (cf. [15]).

Note that the inverse relation $\Gamma^{-1}\colon (\mathcal{H}^2, J_{\mathcal{H}}) \to (\mathfrak{H}^2, J_{\mathfrak{H}})$ is also unitary; see Proposition 2.4. Therefore, in this case, Γ^{-1} can be interpreted as a boundary relation for $\widetilde{S}^* \subset \mathcal{H}^2$ adjoint to the closed symmetric relation

$$\widetilde{S} := \ker\Gamma^{-1} = \operatorname{mul}\Gamma\,(\subset \mathcal{H}^2). \tag{3.3}$$

Let Γ be a boundary relation for S^* and $T = \operatorname{dom}\Gamma$. According to [15, Prop. 2.12], the linear relation T on $\mathfrak{H}$ satisfies the condition

$$S \subset T \subset S^*, \qquad \operatorname{clos} T = S^*. \tag{3.4}$$

Recall that the eigenspaces $\mathfrak{N}_\lambda(T)$ and $\widehat{\mathfrak{N}}_\lambda(T)$ for T are defined by (2.1). For any $\{\widehat{f}_\lambda,\widehat{h}\},\{\widehat{g}_\mu,\widehat{k}\} \in \Gamma$ with $\widehat{f}_\lambda \in \widehat{\mathfrak{N}}_\lambda(T)$ and $\widehat{g}_\mu \in \widehat{\mathfrak{N}}_\mu(T)$, one has

$$(\lambda - \bar{\mu})(f_\lambda, g_\mu)_{\mathfrak{H}} = (h',k)_{\mathcal{H}} - (h,k')_{\mathcal{H}}, \qquad \lambda,\mu \in \mathbb{C}\setminus\mathbb{R}, \tag{3.5}$$

which follows from identity (3.1). Hence, the subspace $\widehat{\mathfrak{N}}_\lambda(T)$ is positive in the Kreĭn space $(\mathfrak{H}^2, J_{\mathfrak{H}})$ for $\lambda \in \mathbb{C}_+$ and negative for $\lambda \in \mathbb{C}_-$.

Proposition 3.2. *Let $\Gamma\colon \mathfrak{H}^2 \mapsto \mathcal{H}^2$ be a boundary relation for S^*. In this case,*

(i) $n_\pm(S) \leqslant \dim\mathcal{H}$;

(ii) *if* $n_\pm(S) < \infty$, *then* $\dim\mathcal{H} - n_\pm(S) = \dim\operatorname{mul}\Gamma$;

(iii) *if* $\dim\mathcal{H} < \infty$, *then* $n_+(S) = n_-(S)$.

Proof. (i) Let $\widetilde{A} = \mathcal{J}(\Gamma)$ be the main transform of Γ. Then $n_\pm(S) = n_\pm(\widetilde{S})$ by [15, Lemma 2.14], where $\widetilde{S} = \operatorname{mul}\Gamma \subset \mathcal{H}^2$; cf. (3.3). This implies (i).

(ii) If $n_\pm(S) < \infty$, or, equivalently, $n_\pm(\widetilde{S}) < \infty$, then $\dim\operatorname{mul}\Gamma = \dim\widetilde{S} = \dim\mathcal{H} - \dim n_\pm(\widetilde{S})$, where the latter equality holds because $\dim\widetilde{S}$ in $\mathcal{H}^2$ is equal to $\dim\operatorname{ran}(\widetilde{S} - \lambda)$ in $\mathcal{H}$ for any $\lambda \in \mathbb{C}\setminus\mathbb{R}$.

(iii) If $\dim\mathcal{H} < \infty$, then clearly $n_+(\widetilde{S}) = n_-(\widetilde{S})$, and thus $n_+(S) = n_-(S)$.

Definition 3.3. A boundary relation $\Gamma\colon \mathfrak{H}^2 \mapsto \mathcal{H}^2$ of S^* is said to be *minimal* if

$$\mathfrak{H} = \mathfrak{H}_{\min} := \overline{\operatorname{span}}\{\, \mathfrak{N}_\lambda(T) : \lambda \in \mathbb{C}_+ \cup \mathbb{C}_- \,\}.$$

Definition 3.4. The *Weyl family* $M(\cdot)$ of S corresponding to a boundary relation $\Gamma\colon \mathfrak{H}^2 \mapsto \mathcal{H}^2$ is defined by $M(\lambda) := \Gamma(\widehat{\mathfrak{N}}_\lambda(T))$, i.e.,

$$M(\lambda) := \left\{ \widehat{h} \in \mathcal{H}^2 : \{\widehat{f}_\lambda, \widehat{h}\} \in \Gamma \quad \text{for some} \quad \widehat{f}_\lambda = \{f_\lambda, \lambda f_\lambda\} \in \mathfrak{H}^2 \right\}, \tag{3.6}$$

where $\lambda \in \mathbb{C} \setminus \mathbb{R}$. If $M(\cdot)$ is operator-valued, it is called the *Weyl function* of S corresponding to the boundary relation Γ.

Definition 3.5. The *γ-field* $\gamma(\cdot)$ of S corresponding to a boundary relation $\Gamma\colon \mathfrak{H}^2 \to \mathcal{H}^2$ is defined by

$$\gamma(\lambda) := \left\{ \{h, f_\lambda\} \in \mathcal{H} \times \mathfrak{H} : \{\widehat{f}_\lambda, \widehat{h}\} \in \Gamma \quad \text{for some} \quad \widehat{f}_\lambda = \{f_\lambda, \lambda f_\lambda\} \in \mathfrak{H}^2 \right\}, \tag{3.7}$$

where $\lambda \in \mathbb{C} \setminus \mathbb{R}$ and the symbol $\widehat{\gamma}(\lambda)$ ($\lambda \in \mathbb{C} \setminus \mathbb{R}$) stands for

$$\widehat{\gamma}(\lambda) := \left\{ \{h, \widehat{f}_\lambda\} \in \mathcal{H} \times \mathfrak{H}^2 : \{h, f_\lambda\} \in \gamma(\lambda),\ \widehat{f}_\lambda = \{f_\lambda, \lambda f_\lambda\} \in \mathfrak{H}^2 \right\}. \tag{3.8}$$

Assign to Γ the following linear relations which are not necessarily closed:

$$\Gamma_0 = \left\{ \{\widehat{f}, h\} : \{\widehat{f}, \widehat{h}\} \in \Gamma,\ \widehat{h} = \{h, h'\} \right\}, \qquad \Gamma_1 = \left\{ \{\widehat{f}, h'\} : \{\widehat{f}, \widehat{h}\} \in \Gamma,\ \widehat{h} = \{h, h'\} \right\}. \tag{3.9}$$

It is clear that $\operatorname{dom} M(\lambda) = \Gamma_0(\widehat{\mathfrak{N}}_\lambda(T)) \subset \operatorname{ran}\Gamma_0$ and $\operatorname{ran} M(\lambda) = \Gamma_1(\widehat{\mathfrak{N}}_\lambda(T)) \subset \operatorname{ran}\Gamma_1$. If the boundary relation Γ is single-valued, then $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is referred to as a *boundary triplet* associated with the boundary relation $\Gamma\colon \mathfrak{H}^2 \mapsto \mathcal{H}^2$. In this case, the Weyl family corresponding to the boundary triplet $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ can be also defined by the relation

$$\Gamma_1(\{f_\lambda, \lambda f_\lambda\}) = M(\lambda)\Gamma_0(\{f_\lambda, \lambda f_\lambda\}), \qquad \{f_\lambda, \lambda f_\lambda\} \in T. \tag{3.10}$$

The γ-field $\gamma(\cdot)$ associated with the boundary relation $\Gamma\colon \mathfrak{H}^2 \to \mathcal{H}^2$ is the first component of the mapping $\widehat{\gamma}(\lambda)$ in (3.8). Note that

$$\widehat{\gamma}(\lambda) := (\Gamma_0 \upharpoonright \widehat{\mathfrak{N}}_\lambda(T))^{-1}, \qquad \lambda \in \mathbb{C} \setminus \mathbb{R},$$

is a linear mapping from $\Gamma_0(\widehat{\mathfrak{N}}_\lambda(T)) = \operatorname{dom} M(\lambda)$ onto $\widehat{\mathfrak{N}}_\lambda(T)$; $\widehat{\gamma}(\lambda)$ is single-valued by (3.5). Hence, the γ-field is a single-valued mapping from $\operatorname{dom} M(\lambda)$ onto $\mathfrak{N}_\lambda(T)$ and satisfies the relation $\gamma(\lambda)\Gamma_0\widehat{f}_\lambda = f_\lambda$ for any $\widehat{f}_\lambda \in \widehat{\mathfrak{N}}_\lambda(T)$.

3.2. *Realization Theorem*

Identity (3.5) implies that every Weyl family is a Nevanlinna family. In [15], the converse assertion was also proved: every Nevanlinna family can be realized as the Weyl family of a minimal boundary relation.

Theorem 3.6 [15]. *Let $\Gamma\colon \mathfrak{H}^2 \to \mathcal{H}^2$ be a boundary relation for S^*. Then the corresponding Weyl family $M(\cdot)$ belongs to the class $\widetilde{R}(\mathcal{H})$.*

Conversely, if $M(\cdot)$ belongs to the class $\widetilde{R}(\mathcal{H})$, then there exists a minimal boundary relation whose Weyl function coincides with $M(\cdot)$.

Due to Theorem 3.6, the subclasses of $\widetilde{R}(\mathcal{H})$ defined in Section 2 can be characterized in geometric terms by using boundary relations.

Proposition 3.7 [15]. *Let $\Gamma\colon \mathfrak{H}^2 \to \mathcal{H}^2$ be a boundary relation for S^* with the Weyl family $M(\lambda) = \Gamma(\widehat{\mathfrak{N}}_\lambda(T))$. In this case,*

(i) *$M(\cdot) \in R(\mathcal{H})$ if and only if $\operatorname{mul}\Gamma \cap (\{0\} \times \mathcal{H}) = \{0\}$;*
(ii) *$M(\cdot) \in R^s(\mathcal{H})$ if and only if $\operatorname{ran}\Gamma$ is dense in $\mathcal{H}^2$;*
(iii) *$M(\cdot) \in R[\mathcal{H}]$ if and only if $\Gamma_0(\widehat{\mathfrak{N}}_\lambda(T)) = \mathcal{H}$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$;*
(iv) *$M(\cdot) \in R^s[\mathcal{H}]$ if and only if $\operatorname{mul}\Gamma_0 = \{0\}$ and $\Gamma_0(\widehat{\mathfrak{N}}_\lambda(T)) = \mathcal{H}$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$;*
(v) *$M(\cdot) \in R^u[\mathcal{H}]$ if and only if $\operatorname{ran}\Gamma = \mathcal{H}^2$.*

The case of $\operatorname{mul}\Gamma \neq \{0\}$ can now be specified in more detail.

Proposition 3.8. *Let $\Gamma\colon \mathfrak{H}^2 \to \mathcal{H}^2$ be a boundary relation for S^*, and let $M(\cdot) = \{\Phi(\cdot), \Psi(\cdot)\}$ be the corresponding Weyl family. Then*

$$\dim \operatorname{mul}\Gamma = \dim M(\lambda) \cap M(\lambda)^* = \dim \ker \mathsf{N}_{\Phi,\Psi}(\lambda, \lambda), \qquad \lambda \in \mathbb{C} \setminus \mathbb{R}. \tag{3.11}$$

In particular, $\operatorname{mul}\Gamma = \{0\} \Longleftrightarrow M \in R^s(\mathcal{H}) \Longleftrightarrow \ker \mathsf{N}_{\Phi,\Psi}(\lambda, \lambda) = \{0\}$.

Proof. Define the linear mapping $T(\lambda)$ from $\mathcal{H}$ to $\mathcal{H}^2$ by

$$T(\lambda) = \begin{pmatrix} \Phi(\lambda) \\ \Psi(\lambda) \end{pmatrix}, \qquad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

Then $M(\lambda) = T(\lambda)\mathcal{H}$. If $\mathcal{H}_0 := \ker \mathsf{N}_{\Phi,\Psi}(\lambda_0, \lambda_0) \neq 0$ with a chosen $\lambda_0 \in \mathbb{C} \setminus \mathbb{R}$, then $T(\lambda_0)\mathcal{H}_0$ is the isotropic subspace of the space $T(\lambda_0)\mathcal{H}$ regarded as a subspace of the Kreĭn space $(\mathcal{H}^2, J_{\mathcal{H}})$. Therefore, $T(\lambda_0) \ker \mathsf{N}_{\Phi,\Psi}(\lambda_0, \lambda_0) = M(\lambda_0) \cap M(\lambda_0)^*$. According to [15, Lemma 4.1], we have $M(\lambda_0) \cap M(\lambda_0)^* = \operatorname{mul}\Gamma$, and this yields the equalities in (3.11).

3.3. *Linear Transformations of Boundary Relations*

Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces and let W be a linear relation from the Hilbert space $\mathcal{H}^2 = \mathcal{H} \oplus \mathcal{H}$ to the Hilbert space $\mathcal{K}^2 = \mathcal{K} \oplus \mathcal{K}$. For any linear relation Θ on $\mathcal{H}$, the formula

$$W[\Theta] = \left\{ \widehat{k} \in \mathcal{K}^2 : \{\widehat{h}, \widehat{k}\} \in W, \ \widehat{h} \in \Theta \right\} \tag{3.12}$$

defines a linear relation $W[\Theta]$ on $\mathcal{K}$.

Definition 3.9. The linear relation $W[\Theta]$ on $\mathcal{K}$ defined by (3.12) is called the *Shmul'yan transform* of Θ on $\mathcal{H}$ induced by the linear relation $W\colon \mathcal{H}^2 \to \mathcal{K}^2$.

If W is a standard unitary operator, then some known properties can readily be recovered, cf. [39, 49].

Proposition 3.10. *Let W be a standard unitary operator from the Kreĭn space $(\mathcal{H}^2, J_{\mathcal{H}})$ onto the Kreĭn space $(\mathcal{K}^2, J_{\mathcal{K}})$. Let Θ be a linear relation on $\mathcal{H}$. Then*
(i) $W[\Theta^*] = W[\Theta]^*$;
(ii) Θ *is maximal dissipative if and only if* $W[\Theta]$ *is maximal dissipative*;
(iii) Θ *is maximal symmetric if and only if* $W[\Theta]$ *is maximal symmetric*;
(iv) Θ *is selfadjoint if and only if* $W[\Theta]$ *is selfadjoint.*

If W is a standard unitary operator from $\mathcal{H}^2$ onto $\mathcal{K}^2$, then the Shmul'yan transform is usually written out componentwise. In this case, W is bounded with bounded inverse, and it can be represented in the block form

$$W = \begin{pmatrix} W_{00} & W_{01} \\ W_{10} & W_{11} \end{pmatrix}, \qquad W_{ij} \in [\mathcal{H}, \mathcal{K}], \qquad i, j = 0, 1. \tag{3.13}$$

If Θ is a linear relation on $\mathcal{H}$, then $W[\Theta]$ in (3.12) becomes

$$W[\Theta] = \{ \{W_{00}h + W_{01}h', W_{10}h + W_{11}h'\} : \{h, h'\} \in \Theta \}. \tag{3.14}$$

Clearly, the Shmul'yan transform $W[\Theta]$ is contained in the linear-fractional transformation of Θ given by

$$\Lambda = (W_{10} + W_{11}\Theta)(W_{00} + W_{01}\Theta)^{-1} := \{ \{W_{00}h + W_{01}h', W_{10}h + W_{11}h''\} : \{h, h'\}, \{h, h''\} \in \Theta \}. \tag{3.15}$$

In fact, the following equality holds:

$$(W_{10} + W_{11}\Theta)(W_{00} + W_{01}\Theta)^{-1} = W[\Theta] \mathbin{\widehat{+}} \{0, W_{11}(\operatorname{mul}\Theta)\}. \tag{3.16}$$

Hence, if Θ is a linear relation with $W_{11}(\operatorname{mul}\Theta) = \{0\}$ and, in particular, if Θ is an operator, then the linear relations in (3.14) and in (3.15) coincide.

Proposition 3.11. *Let* $\Gamma\colon \mathfrak{H}^2 \to \mathcal{H}^2$ *be a boundary relation for* S^* *with* γ*-field* $\gamma(\lambda)$ *and Weyl family* $M(\lambda)$*. Let* $W = (W_{ij})_{i,j=0}^1$ *be a standard unitary operator on the Kreĭn space* $(\mathcal{H}^2, J_{\mathcal{H}})$*. Then*

(i) *the composition* $W\Gamma\colon \mathfrak{H}^2 \to \mathcal{H}^2$ *is a boundary relation for* S^*;

(ii) *the* γ*-field* $\gamma_W(\lambda)$ *associated with* $W\Gamma$ *is given by*

$$\gamma_W(\lambda) = \left\{ \{W_{00}h + W_{01}h', \gamma(\lambda)h\} : \{h, h'\} \in M(\lambda) \right\}, \qquad \lambda \in \mathbb{C} \setminus \mathbb{R}; \tag{3.17}$$

(iii) *the corresponding Weyl family* $M_W(\lambda)$ *is given by the Shmul'yan transform*

$$M_W(\lambda) = \left\{ \{W_{00}h + W_{01}h', W_{10}h + W_{11}h'\} : \{h, h'\} \in M(\lambda) \right\}, \qquad \lambda \in \mathbb{C} \setminus \mathbb{R}. \tag{3.18}$$

Proof. (i) This statement is immediate from Theorem 2.10 and [15, Prop. 3.5] since $\ker W\Gamma = \ker \Gamma = S$.

(ii) According to (3.7), the (graph of) the γ-field $\gamma_W(\lambda)$ corresponding to $W\Gamma$ is the set of elements given by

$$\left\{\{k, f_\lambda\} \in \mathcal{H} \times \mathfrak{H} : \{\widehat{f}_\lambda, \widehat{k}\} \in W\Gamma\right\} = \left\{\{k, f_\lambda\} \in \mathcal{H} \times \mathfrak{H} : \widehat{k} = W\widehat{h}, \{\widehat{f}_\lambda, \widehat{h}\} \in \Gamma\right\},$$

where $\lambda \in \mathbb{C} \setminus \mathbb{R}$, which leads to (3.17).

(iii) By Definition 3.4, the Weyl family $M_W(\cdot)$ of S corresponding to the boundary relation $\Gamma_W : \mathfrak{H}^2 \mapsto \mathcal{H}^2$ is

$$M_W(\lambda) = \left\{ \widehat{k} \in \mathcal{H}^2 : \{\widehat{f}_\lambda, \widehat{k}\} \in W\Gamma \right\} = \left\{ \widehat{k} \in \mathcal{H}^2 : \widehat{k} = W\widehat{h},\ \{\widehat{f}_\lambda, \widehat{h}\} \in \Gamma \right\},$$

where $\lambda \in \mathbb{C} \setminus \mathbb{R}$, and this leads to (3.18).

Remark 3.12. If $W = J_{\mathcal{H}}$, then the boundary relation $W\Gamma$ becomes

$$\Gamma^\top := \Gamma_{J_{\mathcal{H}}} = \left\{ \{\widehat{f}, J_{\mathcal{H}}\widehat{h}\} : \{\widehat{f}, \widehat{h}\} \in \Gamma \right\}, \tag{3.19}$$

and is called the *transposed boundary relation*. As follows from (3.18), the corresponding Weyl family $M^\top(\cdot)$ for $\Gamma^\top$ coincides with $-M(\cdot)^{-1}$.

3.4. *Ordinary Boundary Triplets*

The notion of boundary triplet (or a *boundary value space*) was introduced for a densely defined symmetric operator by A. N. Kochubei and V. M. Bruk (see [27, 28] and the references therein). We present the corresponding definition for a nondensely defined operator.

Definition 3.13. Let S be a symmetric operator on a Hilbert space $\mathfrak{H}$ with defect indices $n_\pm(S)$. A triple $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ formed by a Hilbert space $\mathcal{H}$ and two linear mappings Γ_0 and Γ_1 from S^* to $\mathcal{H}$ is referred to as an *ordinary boundary triplet* for S^* if

(BT1) the abstract Green's identity holds,

$$(f', g) - (f, g') = (\Gamma_1\widehat{f}, \Gamma_0\widehat{g})_{\mathcal{H}} - (\Gamma_0\widehat{f}, \Gamma_1\widehat{g})_{\mathcal{H}}, \tag{3.20}$$

for all $\widehat{f} = \{f, f'\},\ \widehat{g} = \{g, g'\} \in S^*$;

(BT2) the linear mapping $\Gamma := \{\Gamma_0, \Gamma_1\} : S^* \to \mathcal{H}^2$ is surjective.

Simple observations (see [15, Prop. 5.3]) imply the following statement.

Proposition 3.14. *The following statements are equivalent*:

(i) *a triple* $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ *is an ordinary boundary triplet for* S^*;

(ii) $\Gamma = \{\Gamma_0, \Gamma_1\} :\ \mathfrak{H}^2 \mapsto \mathcal{H}^2$ *is a boundary relation for* S^* *with* $\operatorname{ran}\Gamma = \mathcal{H}^2$;

(iii) *the corresponding Weyl family* $M(\cdot)$ *belongs to* $R^u[\mathcal{H}]$.

The term *ordinary boundary triplet* is used to distinguish boundary triplets occurring in Definition 3.13 from those corresponding to single-valued boundary relations (see Subsection 3.1).

A linear relation $\widetilde{A}$ is said to be an *intermediate* extension of S if $S \subset \widetilde{A} \subset S^*$. Ordinary boundary triplets provide a tool for describing all intermediate extensions of S. As is well known [19, 41], the set of all intermediate extensions of A in $\mathfrak{H}$ admits the parametrization

$$\widetilde{A}_\Theta := \{\widehat{f} \in A^* : \Gamma\widehat{f} \in \Theta\} = \ker(\Gamma_1 - \Theta\Gamma_0), \tag{3.21}$$

where Θ ranges over the set of all linear relations on $\mathcal{H}$. Moreover, in this case, the linear relation $\widetilde{A}_\Theta$ is closed (symmetric, selfadjoint) if and only if the linear relation Θ is closed (symmetric, selfadjoint, respectively).

The definitions of the Weyl function $M(\cdot)$ and the γ-field $\gamma(\cdot)$ corresponding to the ordinary boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ can be rewritten in a simpler form,

$$\widehat{\gamma}(\lambda) := (\Gamma_0 \upharpoonright \widehat{\mathfrak{N}}_\lambda)^{-1}, \qquad \gamma(\lambda) := \pi_1 (\Gamma_0 \upharpoonright \widehat{\mathfrak{N}}_\lambda)^{-1}, \qquad M(\lambda) = \Gamma_1 \widehat{\gamma}(\lambda), \tag{3.22}$$

with $\lambda \in \rho(A_0)$. Here $\widehat{\mathfrak{N}}_\lambda := \widehat{\mathfrak{N}}_\lambda(S^*)$, and the symbol π_1 stands for the projection to the first component of $\mathcal{H} \oplus \mathcal{H}$. The Weyl function $M(\cdot)$ and the γ-field $\gamma(\cdot)$ satisfy the following identities:

$$\gamma(\lambda) = [I + (\lambda - \mu)(A_0 - \lambda)^{-1}]\gamma(\mu), \qquad \lambda, \mu \in \mathbb{C} \setminus \mathbb{R}, \tag{3.23}$$

$$M(\lambda) = M(\mu)^* + (\lambda - \bar{\mu})\gamma(\mu)^*[I + (\lambda - \mu)(A_0 - \lambda)^{-1}]\gamma(\mu), \qquad \lambda, \mu \in \mathbb{C} \setminus \mathbb{R}. \tag{3.24}$$

For an ordinary boundary triplet, the resolvent of an intermediate extension $\widetilde{A}$ of A can be expressed in terms of the corresponding Weyl function.

Proposition 3.15 [21]. *Let $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be an ordinary boundary triplet for S^*, let $M(\cdot)$ be the corresponding Weyl function, let Θ be a linear relation on $\mathcal{H}$, and let $\lambda \in \rho(A_0)$. In this case, $\lambda \in \rho(\widetilde{A}_\Theta)$ if and only if $0 \in \rho(\Theta - M(\lambda))$, and the resolvent of $\widetilde{A}_\Theta$ is given by*

$$(\widetilde{A}_\Theta - \lambda)^{-1} = (A_0 - \lambda)^{-1} + \gamma(\lambda)(\Theta - M(\lambda))^{-1}\gamma(\bar{\lambda})^*. \tag{3.25}$$

The fact that analytic properties of uniformly strict Nevanlinna functions, i.e., $M(\cdot) \in R^u[\mathcal{H}]$, are much simpler than those of general Nevanlinna families $M(\cdot) \in \widetilde{R}(\mathcal{H})$ reflects the fact that general boundary relations are much more complicated objects than ordinary boundary triplets; cf. Proposition 3.14. Many standard geometric and operator-theoretic properties known for ordinary boundary triplets are not shared by general boundary relations.

3.5. *Boundary Relations Whose Weyl Functions Belong to the Class $R[\mathcal{H}]$*

In this subsection, a special attention is paid to the boundary relations whose Weyl families belong to the class $R[\mathcal{H}]$. A purely geometric characterization of this class of boundary relations is given in the next proposition.

Proposition 3.16. *Let S be a closed relation on a Hilbert space $\mathfrak{H}$. Let $\mathcal{H}$ be a Hilbert space, and let $\Gamma\colon \mathfrak{H}^2 \to \mathcal{H}^2$ be a* (*possibly multivalued*) *linear relation with $\overline{\operatorname{dom}}\,\Gamma = S^*$ such that*

(B1) *Green's identity* (3.1) *holds*;
(B2) $\operatorname{ran}\Gamma_0 = \mathcal{H}$;
(B3) $A_0 := \ker \Gamma_0$ *is a selfadjoint relation on* $\mathfrak{H}$.

In this case, $\ker \Gamma = S$ *and* $\Gamma\colon \mathfrak{H}^2 \to \mathcal{H}^2$ *is a boundary relation for S^* for which*

$$\Gamma_0(\widehat{\mathfrak{N}}_\lambda(T)) = \mathcal{H}, \qquad \lambda \in \mathbb{C} \setminus \mathbb{R}. \tag{3.26}$$

Further, every closed isometric linear relation $\Gamma\colon \mathfrak{H}^2 \to \mathcal{H}^2$ satisfying $\overline{\operatorname{dom}}\,\Gamma = S^*$ and (3.26) *is a boundary relation for S^* which satisfies conditions* (B1)–(B3).

If conditions (B1)–(B3) *are satisfied, then S has equal defect numbers, and the corresponding Weyl function belongs to the class $R[\mathcal{H}]$. Moreover, every $R[\mathcal{H}]$-function is the Weyl function of some boundary relation $\Gamma\colon \mathfrak{H}^2 \to \mathcal{H}^2$ with properties* (B1)–(B3).

Proof. The proof of the first statement was given in [15, Prop. 5.9].

Now assume that $\Gamma\colon \mathfrak{H}^2 \to \mathcal{H}^2$ with $\overline{\operatorname{dom}}\,\Gamma = S^*$ is a closed isometric linear relation satisfying (3.26). Let $\{\widehat{f}_\lambda, \widehat{h}\}$, $\{\widehat{g}_{\bar{\lambda}}, \widehat{k}\} \in \Gamma$ with

$$\widehat{f}_\lambda = \begin{pmatrix} f_\lambda \\ \lambda f_\lambda \end{pmatrix} \in \widehat{\mathfrak{N}}_\lambda(T), \qquad \widehat{g}_{\bar{\lambda}} = \begin{pmatrix} g_{\bar{\lambda}} \\ \bar{\lambda} g_{\bar{\lambda}} \end{pmatrix} \in \widehat{\mathfrak{N}}_{\bar{\lambda}}(T), \qquad \widehat{h} = \begin{pmatrix} h \\ h' \end{pmatrix}, \ \widehat{k} = \begin{pmatrix} k \\ k' \end{pmatrix} \in \mathcal{H}^2,$$

let $M(\lambda) = \Gamma(\widehat{\mathfrak{N}}_\lambda(T))$, and let Γ_0, Γ_1 be as in (3.9). Then, in view of (3.5), we have

$$0 = (\lambda f_\lambda, g_{\bar{\lambda}})_{\mathfrak{H}} - (f_\lambda, \bar{\lambda} g_{\bar{\lambda}})_{\mathfrak{H}} = (h', k)_{\mathcal{H}} - (h, k')_{\mathcal{H}}, \qquad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

Since $\widehat{h} \in M(\lambda)$ and $\widehat{k} \in M(\bar{\lambda})$ are arbitrary, we can conclude that $M(\lambda) \subset M(\bar{\lambda})^*$. Assumption (3.26) implies that $\operatorname{dom} M(\lambda) = \operatorname{dom} M(\bar{\lambda}) = \mathcal{H}$. Hence, $M(\bar{\lambda})^*$ (and thus $M(\lambda)$ as well) is a bounded operator, and the relation $M(\lambda) = M(\bar{\lambda})^*$ holds for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Since the operator $M(\lambda)$ is dissipative (accumulative) for $\lambda \in \mathbb{C}_+$ ($\lambda \in \mathbb{C}_-$), this implies that

$$\operatorname{ran}(\Gamma(\widehat{\mathfrak{N}}_\lambda) + \lambda) = \operatorname{ran}(M(\lambda) + \lambda) = \mathcal{H}, \qquad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

Therefore, by [15, Prop. 3.6], $\Gamma\colon \mathfrak{H}^2 \to \mathcal{H}^2$ is a boundary relation for S^*.

Moreover, $\mathcal{H} = \Gamma_0(\widehat{\mathfrak{N}}_\lambda(T)) \subset \operatorname{ran}\Gamma_0$, and thus $\operatorname{ran}\Gamma_0 = \mathcal{H}$, i.e., (B2) holds. Property (B3) is also obtained from $\Gamma_0(\mathfrak{N}_\lambda(T)) = \mathcal{H}$ by using [15, Prop. 4.15]. Condition (B1) is clearly valid.

The fact that every $R[\mathcal{H}]$-function is the Weyl function of some boundary relation $\Gamma\colon \mathfrak{H}^2 \to \mathcal{H}^2$ satisfying the conditions (B1)–(B3) follows from Theorem 3.6 and Proposition 3.7. Finally, property (B3) implies that the defect numbers of $S = \ker\Gamma \subset A_0 = A_0^*$ are equal.

Recall that, for a boundary relation $\Gamma\colon \mathfrak{H}^2 \to \mathcal{H}^2$ satisfying conditions (B1)–(B3), the operator function

$$\gamma(\lambda) = \pi_1(\Gamma_0{\restriction}\,\widehat{\mathfrak{N}}_\lambda(T))^{-1} : \mathcal{H} \to \mathfrak{N}_\lambda(T)$$

is bounded and single-valued for every $\lambda \in \mathbb{C} \setminus \mathbb{R}$, see [15]. Further, the Weyl function $M(\cdot)$ and the γ-field $\gamma(\cdot)$ satisfy identities (3.23) and (3.24). Let $E(t)$ be the spectral family of A_0, and let $P = E(\infty)$ be the orthogonal projection onto $\overline{\operatorname{dom} A_0}$. Then (3.24) leads to the following integral representation of $M(\lambda)$:

$$(M(\lambda)h,h) = a_h + b_h\lambda + \int_{\mathbb{R}} \Big(\frac{1}{t-\lambda} - \frac{t}{t^2+1}\Big)\, d\sigma_h(t), \qquad h \in \mathcal{H}, \tag{3.27}$$

where $a_h = (\operatorname{Re} M(i)h,h)_{\mathcal{H}}$, $b_h = ((I-P)\gamma(i)h, \gamma(i)h)$, and $d\sigma_h(t) = (t^2+1)d(E(t)P\gamma(i)h, P\gamma(i)h)_{\mathfrak{H}}$. Representation (3.27) leads to the following characterization (which is one of the basic tools below). It can be derived from results in [15]; for completeness, a more immediate proof is presented here.

Proposition 3.17. *Let S be a symmetric operator on $\mathfrak{H}$. Let $\Gamma\colon \mathfrak{H}^2 \to \mathcal{H}^2$ be a boundary relation for S^* satisfying conditions* (B1)–(B3)*, and let $M(\lambda)$ be the corresponding Weyl function. Moreover, let $\mathcal{H}_0 = \operatorname{mul}\Gamma_0$, $A_0 = \ker\Gamma_0$, and $T = \operatorname{dom}\Gamma$. Then*

(i) $\operatorname{mul} A_0 = \{0\}$ *if and only if*

$$\lim_{y\to\infty} (M(iy)h,h)_{\mathcal{H}}/y = 0, \qquad h \in \mathcal{H}; \tag{3.28}$$

(ii) $\operatorname{mul} T = \{0\}$ *if and only if M satisfies condition* (3.28) *and*

$$\lim_{y\uparrow\infty} y\operatorname{Im}\big(M(iy)h,h\big) = \infty, \qquad h \in \mathcal{H} \ominus \mathcal{H}_0. \tag{3.29}$$

Proof. (i) It follows from representation (3.27) that

$$\lim_{y\to\infty} (M(iy)h,h)_{\mathcal{H}}(iy) = \|(I-P)\gamma(i)h\|^2_{\mathfrak{H}} = \|(I-P)\gamma(\mu)h\|^2_{\mathfrak{H}}, \qquad \mu \in \mathbb{C} \setminus \mathbb{R}. \tag{3.30}$$

Since A_0 is a selfadjoint extension of S and $\operatorname{ran}\gamma(\mu) = \mathfrak{N}_\mu(T)$ is dense in $\mathfrak{N}_\mu(S^*)$ (see Lemma 5.1, [15, Lemma 2.14]), it follows that $S = \big\{\, \{f,g\} \in A_0 : (g - \bar{\mu}f, \gamma(\mu)) = 0 \,\big\}$ and, in particular,

$$\operatorname{mul} S = \big\{\, g \in \operatorname{mul} A_0 : (g, \gamma(\mu)) = (g, (I-P)\gamma(\mu)) = 0 \,\big\}. \tag{3.31}$$

Hence, $\operatorname{mul} A_0 = \operatorname{mul} S$ if and only if $(I-P)\gamma(\mu) = 0$, which is equivalent to (3.28) by (3.30). Thus, (i) follows from the assumption $\operatorname{mul} S = \{0\}$.

(ii) Under the assumption (3.28), the limit in (3.29) acquires the form

$$\lim_{y\uparrow\infty} y\operatorname{Im}\big(M(iy)h,h\big) = \int_{\mathbb{R}} (t^2+1)\, d\|E_t\gamma(i)h\|^2_{\mathfrak{H}}. \tag{3.32}$$

Since $\ker\gamma(\mu) = \operatorname{mul}\Gamma_0$ by (3.7), the restriction of the mapping $\gamma(i)$ to $\mathcal{H} \ominus \mathcal{H}_0$ is injective. Hence, the limit in (3.32) is finite for some $h \in \mathcal{H} \ominus \mathcal{H}_0$, $h \neq 0$, if and only if

$$\mathfrak{N}_i(T) \cap \operatorname{dom} A_0 = (A_0 - \lambda)^{-1}(\operatorname{mul} T)$$

is nontrivial. For the proof of the last relation, see [15, Prop. 4.20].

The boundary relations with additional properties (B1)–(B3) are invariant under a special class of transforms, cf. Proposition 3.11. Let $B \in [\mathcal{H}]$. Assume that $G \in [\mathcal{H}]$ is invertible and $BG = (BG)^*$. Define the block operator $\widetilde{W}$ by

$$\widetilde{W} = \begin{pmatrix} G^{-1} & 0 \\ B & G^* \end{pmatrix} \quad \text{with} \quad BG = (BG)^*. \tag{3.33}$$

It can readily be seen that $\widetilde{W}$ is a standard unitary operator in $\mathcal{H}^2$.

Proposition 3.18. *Let $\Gamma\colon \mathfrak{H}^2 \to \mathcal{H}^2$ be a boundary relation for S^* which satisfies conditions* (B1)–(B3), *let $\gamma(\lambda)$ and $M(\lambda)$ be the corresponding γ-field and the Weyl function, respectively, and let $\widetilde{W} \in [\mathcal{H} \oplus \mathcal{H}]$ be given by* (3.33). *Then*

(i) *the transform $\widetilde{\Gamma} = \widetilde{W}\Gamma$ of Γ given by*

$$\widetilde{\Gamma} = \left\{ \left\{ \widehat{f}, \begin{pmatrix} G^{-1}h \\ Bh + G^*h' \end{pmatrix} \right\} : \{\widehat{f}, \widehat{h}\} \in \Gamma \right\}, \tag{3.34}$$

is a boundary relation for S^* with $\operatorname{dom}\widetilde{\Gamma} = \operatorname{dom}\Gamma$ *satisfying conditions* (B1)–(B3);

(ii) *the γ-field and the Weyl function of $\widetilde{\Gamma}$ are given by*

$$\widetilde{\gamma}(\lambda) = \gamma(\lambda)G, \qquad \widetilde{M}(\lambda) = BG + G^*M(\lambda)G \;\; (\in [\mathcal{H}]), \qquad \lambda \in \mathbb{C} \setminus \mathbb{R}. \tag{3.35}$$

Proof. (i) By Proposition 3.11 (i), $\widetilde{\Gamma} = \widetilde{W}\Gamma\colon \mathfrak{H}^2 \to \mathcal{H}^2$ is a boundary relation for S^* with $\operatorname{dom}\widetilde{\Gamma} = \operatorname{dom}\Gamma$ and $\ker\widetilde{\Gamma} = \ker\Gamma$. Clearly, $\widetilde{\Gamma}$ admits a representation of the form (3.34). Since $\operatorname{ran}\Gamma_0 = \mathcal{H}$ and $G \in [\mathcal{H}]$ is invertible, the relation $\operatorname{ran}\widetilde{\Gamma}_0 = \mathcal{H}$ holds and $\ker\widetilde{\Gamma}_0 = \ker\Gamma_0$ is selfadjoint. Hence, $\widetilde{\Gamma}$ satisfies conditions (B1)–(B3).

(ii) By Proposition 3.11 and (3.33), the Weyl function $\widetilde{M}(\lambda)$ of $\widetilde{\Gamma}$ is given by

$$\begin{aligned} \widetilde{M}(\lambda) &= \left\{ \{G^{-1}h, Bh + G^*h'\} : \{h, h'\} \in M(\lambda) \right\} \\ &= \left\{ \{k, BGk + G^*M(\lambda)Gk\} : h = Gk \in \operatorname{dom} M(\lambda) = \mathcal{H} \right\} = BG + G^*M(\lambda)G, \end{aligned} \tag{3.36}$$

where $BG = (BG)^*$. Similarly, for the γ-field $\widetilde{\gamma}(\lambda)$ of $\widetilde{\Gamma}$, relations (3.17) and (3.33) yield

$$\widetilde{\gamma}(\lambda) = \left\{ \{G^{-1}h, \gamma(\lambda)h\} : \{h, h'\} \in M(\lambda) \right\} = \left\{ \{k, \gamma(\lambda)Gk\} : h = Gk \in \operatorname{dom} M(\lambda) = \mathcal{H} \right\}.$$

Hence, $\widetilde{\gamma}(\lambda)$ acquires the form appearing in (3.35).

Remark 3.19. If the transposed boundary relation $\Gamma^\top$ satisfies (B1)–(B3), then the corresponding Weyl family $M^\top(\cdot) = -M(\cdot)^{-1}$ is single-valued and belongs to the class $R[\mathcal{H}]$.

Up to this point, the boundary relations $\Gamma\colon (\mathfrak{H}^2, J_{\mathfrak{H}}) \to (\mathcal{H}^2, J_{\mathcal{H}})$ satisfying the conditions (B1)–(B3) are multivalued in general. Let us briefly recall the single-valued case; cf. [15].

Definition 3.20 [21]. If a boundary relation $\Gamma\colon (\mathfrak{H}^2, J_{\mathfrak{H}}) \to (\mathcal{H}^2, J_{\mathcal{H}})$ is single-valued and satisfies conditions (B1)–(B3), then the triplet $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is called a *generalized boundary triplet*.

The following assertion results from Proposition 3.7 and Proposition 3.16.

Corollary 3.21 [21]. *A single-valued boundary relation $\Gamma = \{\Gamma_0, \Gamma_1\}\colon \mathfrak{H}^2 \mapsto \mathcal{H}^2$ corresponds to a generalized boundary triplet $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ if and only if the corresponding Weyl function $M(\cdot)$ belongs to the class $R^s[\mathcal{H}]$.*

In the case of a generalized boundary triplet, the condition (3.29) in Proposition 3.17 can be simplified as follows:

$$\lim_{y \uparrow \infty} y \operatorname{Im}\bigl(M(iy)h, h\bigr) = \infty, \qquad h \in \mathcal{H}. \tag{3.37}$$

4. WEYL FUNCTIONS FOR INTERMEDIATE EXTENSIONS

Some results of [14] on intermediate extensions for ordinary boundary triplets are extended in this section to the class of boundary relations satisfying properties (B1)–(B3) introduced in Subsection 3.5 (see Proposition 3.16). Although many formulas are similar, a more careful treatment is needed, since the boundary mapping Γ is now unbounded in general and can be multivalued. For this reason, the proofs are different here and use compositions, where a bounded *nonstandard* unitary operator is applied to a boundary relation Γ satisfying (B1)–(B3).

Let S be a closed symmetric operator on a separable Hilbert space $\mathfrak{H}$, and let $\Gamma : \mathfrak{H}^2 \to \mathcal{H}^2$ be a boundary relation for S^* satisfying conditions (B1)–(B3), and thus the corresponding Weyl family $M(\lambda)$ belongs to the class $R[\mathcal{H}]$. The purpose is to associate intermediate symmetric extensions H of S to different types of Nevanlinna functions (say, to linear combinations of M_{ij}, to Schur complements, and to compressions of linear-fractional transformations of $M(\lambda)$), which are obtained as block transforms of the operator matrix representation of $M(\lambda)$ in

$$\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2, \qquad M(\lambda) = (M_{ij}(\lambda))_{i,j=1}^2. \tag{4.1}$$

Let π_j be the orthogonal projection taking $\mathcal{H}$ onto $\mathcal{H}_j$, $j = 1, 2$. Consider the linear relations

$$\mathcal{P}^{(j)} = \left\{ \left\{ \begin{pmatrix} h \\ h' \end{pmatrix}, \begin{pmatrix} h \\ \pi_j h' \end{pmatrix} \right\} : h \in \mathcal{H}_j,\, h' \in \mathcal{H} \right\}, \qquad j = 1, 2, \tag{4.2}$$

which are nonstandard unitary operators from $(\mathcal{H}^2, J_{\mathcal{H}})$ to $(\mathcal{H}_j^2, J_{\mathcal{H}_j})$ (cf. Example 2.6). In general, it is unclear whether or not $\mathcal{P}^{(j)} \circ \Gamma$ is a unitary relation if $\operatorname{ran}\Gamma \not\subseteq \operatorname{dom}\mathcal{P}^{(j)} = \mathcal{H}_j \times \mathcal{H}$ (cf. Theorem 2.10). However, if $\Gamma\colon \mathfrak{H}^2 \to \mathcal{H}^2$ satisfies conditions (B1)–(B3), it turns out that $\mathcal{P}^{(j)} \circ \Gamma$ is a unitary relation from $(\mathcal{H}^2, J_{\mathcal{H}})$ to $(\mathcal{H}_j^2, J_{\mathcal{H}_j})$, $j = 1, 2$.

Proposition 4.1. *Let $\Gamma\colon \mathfrak{H}^2 \to \mathcal{H}^2$ be a boundary relation for S^* satisfying conditions* (B1)–(B3), *let $\gamma(\lambda)$ be the corresponding γ-field, and let the corresponding Weyl function $M(\lambda)$ be decomposed as in* (4.1). *Then*

(i) *the linear relation $H^{(1)}$ given by*

$$H^{(1)} = \left\{ \widehat{f} \in S^* : \left\{ \widehat{f}, \begin{pmatrix} 0 \\ h' \end{pmatrix} \right\} \in \Gamma,\ \pi_1 h' = 0 \right\} \tag{4.3}$$

is closed and symmetric on $\mathfrak{H}$ and has equal defect numbers;

(ii) *the linear relation $\Gamma^{(1)}\colon \mathfrak{H}^2 \to \mathcal{H}_1^2$ given by*

$$\Gamma^{(1)} := \mathcal{P}^{(1)} \circ \Gamma = \left\{ \left\{ \widehat{f}, \begin{pmatrix} h \\ \pi_1 h' \end{pmatrix} \right\} : \left\{ \widehat{f}, \begin{pmatrix} h \\ h' \end{pmatrix} \right\} \in \Gamma,\ \pi_2 h = 0 \right\} \tag{4.4}$$

is a boundary relation for $H^{(1)*}$ satisfying conditions (B1)–(B3);

(iii) *the domain $T_1 := \operatorname{dom}\Gamma^{(1)}$ is dense in $H^{(1)*}$ and*

$$T^{(1)} = \{ \widehat{f} \in S^* : \{\widehat{f}, \widehat{h}\} \in \Gamma,\, \pi_2 h = 0 \}; \tag{4.5}$$

(iv) *the corresponding γ-field $\gamma_1(\lambda)\colon \mathcal{H}_1 \to \mathfrak{H}$ and the Weyl function $M_1(\lambda) \in [\mathcal{H}_1]$ are given by*

$$\gamma_1(\lambda) = \gamma(\lambda) {\restriction} \mathcal{H}_1, \qquad M_1(\lambda) = M_{11}(\lambda), \qquad \lambda \in \mathbb{C} \setminus \mathbb{R}. \tag{4.6}$$

Proof. (i) & (ii) By definition, $\Gamma^{(1)}$ is a multivalued mapping from $\mathfrak{H}^2$ to $\mathcal{H}_1^2$. It satisfies Green's identity (3.1) since $(f', g)_{\mathfrak{H}} - (f, g')_{\mathfrak{H}} = (h', k)_{\mathcal{H}} - (h, k')_{\mathcal{H}} = (\pi_1 h', k)_{\mathcal{H}} - (h, \pi_1 k')_{\mathcal{H}}$ for any $\{\widehat{f}, \widehat{h}\}$, $\{\widehat{g}, \widehat{k}\} \in \Gamma$ with $h, k \in \mathcal{H}_1$. Property (B2) of Γ implies that $\operatorname{ran}\Gamma_0^{(1)} = \mathcal{H}_1$. Moreover, it follows from property (B3) of Γ that $\ker\Gamma_0^{(1)} = \ker\Gamma_0 = A_0$ is selfadjoint. Due to Proposition 3.16, $\Gamma^{(1)}$ is a boundary relation for $H^{(1)*}$ having properties (B1)–(B3).

Since $\Gamma^{(1)}$ is unitary, $H^{(1)} = \ker\Gamma^{(1)}$ is closed and symmetric. The description of $H^{(1)}$ in (4.3) is immediate from the definition of $\Gamma^{(1)}$ in (ii).

(iii) The description of $T_1 = \operatorname{dom}\Gamma^{(1)}$ in (4.5) is clear from the definition of $\Gamma^{(1)}$ in (i) because $T^{(1)}$ is dense in $H^{(1)*}$; equivalently, the identity $T^{(1)*} = \ker\Gamma^{(1)} = H^{(1)}$ holds by the definition of boundary relations.

(iv) By Proposition 3.16 (cf. [15, Prop. 5.9]), conditions (B1)–(B3) imply that

$$\Gamma_0(\widehat{\mathfrak{N}}_\lambda(T)) = \mathcal{H}, \qquad \Gamma_0^{(1)}(\widehat{\mathfrak{N}}_\lambda(T^{(1)}) = \mathcal{H}_1 \quad \text{for all} \quad \lambda \in \mathbb{C} \setminus \mathbb{R}. \tag{4.7}$$

Hence, $\operatorname{dom} \widehat{\gamma}(\lambda) = \mathcal{H}$ and $\operatorname{dom} \widehat{\gamma}_1(\lambda) = \mathcal{H}_1$, and the formulas

$$\gamma(\lambda) = \left\{ \{h, \widehat{f}_\lambda\} : \{\widehat{f}_\lambda, \widehat{h}\} \in \Gamma \right\}, \qquad \gamma_1(\lambda) = \left\{ \{h, \widehat{f}_\lambda\} : \{\widehat{f}_\lambda, \widehat{h}\} \in \Gamma, \, h \in \mathcal{H}_1 \right\}$$

show that these single-valued mappings satisfy the relation $\gamma_1(\lambda) = \gamma(\lambda) \upharpoonright \mathcal{H}_1$. Moreover, (4.7) yields $M_1(\lambda) \in [\mathcal{H}_1]$ and $M(\lambda) \in [\mathcal{H}]$, and thus $M_1(\lambda) = \left\{ \widehat{h} \in \mathcal{H}_1^2 : \{\widehat{f}_\lambda, \widehat{h}\} \in \Gamma^{(1)} \right\} = \left\{ \{h, \pi_1 h'\} \in \mathcal{H}^2 : \{\widehat{f}_\lambda, \widehat{h}\} \in \Gamma, \, h \in \mathcal{H}_1 \right\} = \pi_1 M(\lambda) \upharpoonright \mathcal{H}_1$. This completes the proof.

Replacing $\mathcal{H}_1$ by $\mathcal{H}_2$ in Proposition 4.1, we obtain the following corollary.

Corollary 4.2. *Let $\Gamma\colon \mathfrak{H}^2 \to \mathcal{H}^2$, $\gamma(\lambda)$, and $M(\lambda)$ be as in Proposition* 4.1. *Then*

(i) *the linear relation $H^{(2)}$ given by*

$$H^{(2)} = \left\{ \widehat{f} \in S^* : \left\{ \widehat{f}, \begin{pmatrix} 0 \\ h' \end{pmatrix} \right\} \in \Gamma, \; \pi_2 h' = 0 \right\} \tag{4.8}$$

is closed and symmetric on $\mathfrak{H}$ and has equal defect numbers;

(ii) *the linear relation $\Gamma^{(2)}\colon \mathfrak{H}^2 \to \mathcal{H}_2^2$ given by*

$$\Gamma^{(2)} := \mathcal{P}^{(2)} \circ \Gamma = \left\{ \left\{ \widehat{f}, \begin{pmatrix} h \\ \pi_2 h' \end{pmatrix} \right\} : \left\{ \widehat{f}, \begin{pmatrix} h \\ h' \end{pmatrix} \right\} \in \Gamma, \; \pi_1 h = 0 \right\} \tag{4.9}$$

is a boundary relation for $H^{(2)*}$ satisfying conditions (B1)–(B3);

(iii) *the domain $T^{(2)} := \operatorname{dom} \Gamma^{(2)}$ is dense in $H^{(2)*}$ and*

$$T^{(2)} = \left\{ \widehat{f} \in S^* : \{\widehat{f}, \widehat{h}\} \in \Gamma, \, \pi_1 h = 0 \right\}; \tag{4.10}$$

(iv) *the corresponding γ-field $\gamma_2(\lambda) : \mathcal{H}_2 \to \mathfrak{H}$ and the Weyl function $M_2(\lambda) \in [\mathcal{H}_2]$ are given by*

$$\gamma_2(\lambda) = \gamma(\lambda) \upharpoonright \mathcal{H}_2, \qquad M_2(\lambda) = M_{22}(\lambda), \qquad \lambda \in \mathbb{C} \setminus \mathbb{R}. \tag{4.11}$$

The next corollary concerning Schur complements of the Weyl function is of independent interest.

Corollary 4.3. *Let $\Gamma\colon \mathfrak{H}^2 \to \mathcal{H}^2$ be a boundary relation for S^* such that Γ, $\Gamma^\top$, and $\left(\Gamma^{(2)}\right)^\top$ (see* (3.19)) *satisfy* (B1)–(B3). *Decompose the corresponding Weyl function $M(\lambda)$ as in* (4.1). *Then*

(i) *the linear relation $S^{(1)}$ given by*

$$S^{(1)} = \left\{ \widehat{f} \in S^* : \left\{ \widehat{f}, \begin{pmatrix} h \\ 0 \end{pmatrix} \right\} \in \Gamma, \; \pi_1 h = 0 \right\} \tag{4.12}$$

is closed and symmetric on $\mathfrak{H}$ and has equal defect numbers;

(ii) *the linear relation $\Gamma' := \left(\mathcal{P}^{(1)} \circ \Gamma^\top\right)^\top = \left\{ \left\{ \widehat{f}, \begin{pmatrix} \pi_1 h \\ h' \end{pmatrix} \right\} : \left\{ \widehat{f}, \begin{pmatrix} h \\ h' \end{pmatrix} \right\} \in \Gamma, \; \pi_2 h' = 0 \right\}$ is a boundary relation for $S^{(1)*}$ satisfying conditions* (B1)–(B3);

(iii) *the corresponding Weyl function $M^{(1)}(\lambda) \in [\mathcal{H}_1]$ is given by*

$$M^{(1)}(\lambda) = M_{11}(\lambda) - M_{12}(\lambda) M_{22}(\lambda)^{-1} M_{21}(\lambda), \qquad \lambda \in \mathbb{C} \setminus \mathbb{R}. \tag{4.13}$$

Proof. By assumption and by Proposition 3.16, $M(\cdot)$ and $M(\cdot)^{-1}$ belong to the class $R[\mathcal{H}]$, and $M(\cdot)^{-1}$ has the block representation $M(\cdot)^{-1} = \left((M(\cdot)^{-1})_{ij}\right)_{i,j=1}^2$. Since $\left(\Gamma^{(2)}\right)^\top$ satisfies (B1)–(B3), we see by Proposition 3.16 that $-M_{22}(\cdot)^{-1}$ belongs to the class $R[\mathcal{H}_2]$. By the Frobenius formula,

$$\left(M(\lambda)^{-1}\right)_{11} = \left(M_{11}(\lambda) - M_{12}(\lambda) M_{22}(\lambda)^{-1} M_{21}(\lambda)\right)^{-1} \in [\mathcal{H}_1]. \tag{4.14}$$

Apply Proposition 4.1 to the transposed boundary relation $\Gamma^\top$. The composition $\mathcal{P}^{(1)} \circ \Gamma^\top$ is unitary and satisfies (B1)–(B3), and the corresponding Weyl function coincides with $-\left(M(\cdot)^{-1}\right)_{11}$; see (4.14). To complete the proof, it remains to show that the transposed boundary relation $\left(\mathcal{P}^{(1)} \circ \Gamma^\top\right)^\top$ satisfies (B1)–(B3). Since $\left(\mathcal{P}^{(1)} \circ \Gamma^\top\right)^\top$ is unitary, it is closed and isometric. Further, the corresponding Weyl function $M^{(1)}(\lambda) = \left(M(\cdot)^{-1}\right)_{11}^{-1} = M_{11}(\lambda) - M_{12}(\lambda) M_{22}(\lambda)^{-1} M_{21}(\lambda)$ belongs to the class $R[\mathcal{H}_1]$, which shows (iii) and proves that condition (3.26) is satisfied. Assertion (ii) follows now from Proposition 3.16, whereas (i) is immediate from part (i) of Proposition 4.1.

Proposition 4.4. *Let $\Gamma : \mathfrak{H}^2 \to \mathcal{H}^2$ be a boundary relation for S^* which satisfies* (B1)–(B3), *let $\gamma(\lambda)$ be the corresponding γ-field, and let the vectors $h = h_1 \oplus h_2$, $h' = h_1' \oplus h_2' \in \mathcal{H}_1 \oplus \mathcal{H}_2$ and the corresponding Weyl function $M(\lambda)$ be decomposed as in* (4.1)*. Then, for every $G \in [\mathcal{H}_2, \mathcal{H}_1]$,*

(i) *the linear relation H_G defined by*

$$H_G = \{ \widehat{f} \in S^* : \{\widehat{f}, \widehat{h}\} \in \Gamma,\ h = 0,\ h_2' = -G^* h_1' \} \tag{4.15}$$

is closed and symmetric on $\mathfrak{H}$ and has equal defect numbers;

(ii) *the linear relation $\Gamma_G : \mathfrak{H}^2 \to \mathcal{H}_2^2$ given by*

$$\Gamma_G = \left\{ \left\{ \widehat{f}, \begin{pmatrix} h_2 \\ G^* h_1' + h_2' \end{pmatrix} \right\} : \{\widehat{f}, \widehat{h}\} \in \Gamma,\ h_1 = G h_2 \right\} \tag{4.16}$$

is a boundary relation for H_G^* satisfying (B1)–(B3);

(iii) *the γ-field $\gamma_G(\lambda) : \mathcal{H}_2 \to \mathfrak{H}$ corresponding to Γ_G is given by*

$$\gamma_G(\lambda) = \gamma_1(\lambda) G + \gamma_2(\lambda), \tag{4.17}$$

where $\gamma(\lambda) = (\gamma_1(\lambda)\ \gamma_2(\lambda)) : \mathcal{H}_1 \oplus \mathcal{H}_2 \to \mathfrak{H}$ is decomposed according to $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$;

(iv) *the Weyl function $M_G(\lambda)$ associated with Γ_G is of the form*

$$M_G(\lambda) = G^* M_{11}(\lambda) G + G^* M_{12}(\lambda) + M_{21}(\lambda) G + M_{22}(\lambda). \tag{4.18}$$

Proof. Define the operator $\widetilde{G} \in [\mathcal{H}]$, where $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, and the operator $W \in [\mathcal{H} \oplus \mathcal{H}]$ by the block formulas $\widetilde{G} = \begin{pmatrix} I & G \\ 0 & I \end{pmatrix}$ and $W = \begin{pmatrix} \widetilde{G}^{-1} & 0 \\ 0 & \widetilde{G}^* \end{pmatrix}$, respectively. Then $\widetilde{G}$ is invertible, $\widetilde{G}^{-1} \in [\mathcal{H}]$, and W is a standard unitary operator on $\mathcal{H}^2 = \mathcal{H} \oplus \mathcal{H}$. According to Proposition 3.18, the composition $\widetilde{\Gamma} = W\Gamma : \mathfrak{H}^2 \to \mathcal{H}^2$ given by (3.34) is a unitary relation (in the Kreĭn space sense) satisfying properties (B1)–(B3). Moreover, according to (3.35), the γ-field and the Weyl function associated to $\widetilde{\Gamma}$ are given by

$$\widetilde{\gamma}(\lambda) h = \gamma(\lambda) \widetilde{G} h = \gamma_1(\lambda)(h_1 + G h_2) + \gamma_2(\lambda) h_2, \tag{4.19}$$

$$\widetilde{M}(\lambda) = \begin{pmatrix} M_{11}(\lambda) & M_{11}(\lambda) G + M_{12}(\lambda) \\ G^* M_{11}(\lambda) + M_{21}(\lambda) & G^* M_{11}(\lambda) G + G^* M_{12}(\lambda) + M_{21}(\lambda) G + M_{22}(\lambda) \end{pmatrix}, \tag{4.20}$$

respectively. Since $\widetilde{G}^{-1} h = \begin{pmatrix} h_1 - G h_2 \\ h_2 \end{pmatrix}$ and $\widetilde{G}^* h' = \begin{pmatrix} h_1' \\ G^* h_1' + h_2' \end{pmatrix}$, it follows from Corollary 4.2 that H_G in (4.15) is a closed symmetric relation on $\mathfrak{H}$ with equal defect numbers and the relation $\Gamma_G : \mathfrak{H}^2 \to \mathcal{H}_2^2$ defined by (4.16) is a boundary relation for H_G^* satisfying conditions (B1)–(B3). Moreover, the formulas for the γ-field and the Weyl function in (4.17) and (4.18) are obtained by applying (4.11) in Corollary 4.2 to (4.19) and (4.20). This completes the proof.

Corollary 4.5. *Let S_j be symmetric operators on Hilbert spaces $\mathfrak{H}_j$, let $\Gamma^{(j)} : \mathfrak{H}_j^2 \to \mathcal{H}^2$ be boundary relations for S_j^* satisfying conditions* (B1)–(B3), *and let $M_j(\lambda)$ be the corresponding Weyl functions of S_j, $j = 1, 2$. Then*

(i) *the linear relation*

$$H^{(3)} = \left\{ \widehat{f}_1 \oplus \widehat{f}_2 : \left\{ \widehat{f}_1, \begin{pmatrix} 0 \\ h_1 \end{pmatrix} \right\} \in \Gamma^{(1)}, \left\{ \widehat{f}_2, \begin{pmatrix} 0 \\ -h_1 \end{pmatrix} \right\} \in \Gamma^{(2)},\ h_1 \in \mathcal{H} \right\} \tag{4.21}$$

is closed and symmetric on $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$ and has equal defect numbers;

(ii) *the linear relation $\Gamma^{(3)} : \mathfrak{H}^2 \to \mathcal{H}^2$ given by*

$$\Gamma^{(3)} := \left\{ \left\{ \widehat{f}_1 \oplus \widehat{f}_2, \begin{pmatrix} h \\ h_1 + h_2 \end{pmatrix} \right\} : \left\{ \widehat{f}_i, \begin{pmatrix} h \\ h_i \end{pmatrix} \right\} \in \Gamma^{(i)},\ h, h_i \in \mathcal{H},\ i = 1, 2 \right\} \tag{4.22}$$

is a boundary relation for $H^{(3)*}$ satisfying conditions (B1)–(B3);

(iii) *the Weyl function $M(\lambda)$ corresponding to $\Gamma^{(3)}$ is given by*

$$M(\lambda) = M_1(\lambda) + M_2(\lambda), \qquad \lambda \in \mathbb{C} \setminus \mathbb{R}. \tag{4.23}$$

Proof. Let $\Gamma^{(1)} \oplus \Gamma^{(2)} = \{ \{ \widehat{f}_1 \oplus \widehat{f}_2, \widehat{h}_1 \oplus \widehat{h}_2 \} : \{ \widehat{f}_1, \widehat{h}_1 \} \in \Gamma^{(1)},\ \{ \widehat{f}_2, \widehat{h}_2 \} \in \Gamma^{(2)} \}$, be a boundary relation for $S_1^* \oplus S_2^*$; its Weyl function is $M(\lambda) = M_1(\lambda) \oplus M_2(\lambda)$. Now all assertions follow by applying Proposition 4.4 with $\Gamma = \Gamma^{(1)} \oplus \Gamma^{(2)}$ and $G = I_{\mathcal{H}}$.

5. ORTHOGONAL COUPLINGS

5.1. *Orthogonal Coupling and Boundary Relations*

Suppose $\mathfrak{H}_1$ and $\mathfrak{H}_2$ are Hilbert spaces, $\widetilde{A}$ is a selfadjoint relation on the orthogonal sum $\widetilde{\mathfrak{H}} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$, and P_j is the orthogonal projections onto $\mathfrak{H}_j$, $j = 1, 2$. Then the formulas

$$S_j = \widetilde{A} \cap \mathfrak{H}_j^2, \qquad T_j = \{\, \{P_j\varphi, P_j\varphi'\} : \{\varphi, \varphi'\} \in \widetilde{A}\,\}, \qquad j = 1, 2, \tag{5.1}$$

define closed symmetric linear relations S_1 and S_2 and not necessarily closed linear relations T_1 and T_2 (on $\mathfrak{H}_1$ and $\mathfrak{H}_2$, respectively). The relation $\widetilde{A}$ can be interpreted as a selfadjoint extension of the orthogonal sum $S_1 \oplus S_2$. It is called the *orthogonal coupling* of S_1 and T_2 (or of T_1 and S_2), see [52]. The selfadjoint relation $\widetilde{A}$ is said to be *minimal* with respect to the Hilbert space $\mathfrak{H}_j$ (where j is fixed, $j = 1, 2$) if

$$\mathfrak{H}_1 \oplus \mathfrak{H}_2 = \overline{\operatorname{span}}\,\{\, \mathfrak{H}_j + (\widetilde{A} - \lambda)^{-1}\mathfrak{H}_j : \lambda \in \rho(\widetilde{A})\,\}. \tag{5.2}$$

In the terminology used, e.g., in [40], $\widetilde{A}$ is called the *minimal selfadjoint extension of* S_j or of any of its restrictions $A \subset S_j$, $j = 1, 2$.

Assign to T_j the eigenspaces as in (2.1),

$$\mathfrak{N}_\lambda(T_j) = \ker(T_j - \lambda), \qquad \widehat{\mathfrak{N}}_\lambda(T_j) = \{\, \{f, \lambda f\} \in T_j : f \in \mathfrak{N}_\lambda(T_j)\,\}. \tag{5.3}$$

Note that S_2 is related to $\widetilde{S} = \operatorname{mul}\Gamma$ in (3.3) by the rule $S_2 = -\widetilde{S}$ (cf. (2.12)). Recall the following basic facts, see [15].

Lemma 5.1 [15]. *Let $\widetilde{A}$ be a selfadjoint relation on $\widetilde{\mathfrak{H}} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$, and let the linear relations S_j and T_j, $j = 1, 2$, be defined by* (5.1). *Then*

- (i) $\operatorname{clos} T_j = S_j^*$, $j = 1, 2$;
- (ii) *T_1 is closed if and only if T_2 is closed*;
- (iii) $\mathfrak{N}_\lambda(T_1) = P_1(\widetilde{A} - \lambda)^{-1}\,\mathfrak{H}_2$, $\mathfrak{N}_\lambda(T_2) = P_2(\widetilde{A} - \lambda)^{-1}\,\mathfrak{H}_1$;
- (iv) *$\mathfrak{N}_\lambda(T_j)$ is dense in $\mathfrak{N}_\lambda(S_j^*)$ for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$, $j = 1, 2$;*
- (v) *the defect numbers of S_1 and $-S_2$ coincide*: $n_\pm(S_1) = n_\mp(S_2)$;
- (vi) *$\widetilde{A}$ is minimal with respect to $\mathfrak{H}_1$ ($\mathfrak{H}_2$) if and only if S_2 (S_1, respectively) is simple.*

Part (iv) of Lemma 5.1 and Definition 3.3 show that a boundary relation Γ for S^* is minimal if and only if $S = \ker\Gamma$ is simple (see [15]), which is equivalent (by part (vi) of Lemma 5.1) to the condition that $\widetilde{A} = \mathcal{J}(\Gamma)$ is minimal with respect to $\mathfrak{H}_2$.

5.2. *Shtraus [Štraus, Strauss] Families*

Consider the so-called Shtraus family of linear relations $T(\lambda)$ on $\mathfrak{H} = \mathfrak{H}_1$ defined for any $\lambda \in \rho(\widetilde{A})$ by the rule

$$T(\lambda) := \bigl\{\{P_1 f, P_1 f'\} : \{f, f'\} \in \widetilde{A},\ f' - \lambda f \in \mathfrak{H}\bigr\}, \qquad \lambda \in \rho(\widetilde{A}). \tag{5.4}$$

The following result gives a boundary relation whose Weyl family coincides with $T(-\lambda)$. Define the linear relation $\Delta\colon \mathfrak{H}_1^2 \to \mathfrak{H}_2^2$ as the main transform of $\widetilde{A}$, see (2.12), i.e., write

$$\Delta := \mathcal{J}^{-1}(\widetilde{A}) = \left\{\left\{\begin{pmatrix} f_1 \\ f_1' \end{pmatrix}, \begin{pmatrix} f_2 \\ -f_2' \end{pmatrix}\right\} : \widehat{f}_1 \oplus \widehat{f}_2 \in \widetilde{A},\ f_j, f_j' \in \mathfrak{H}_j,\ j = 1, 2\right\}. \tag{5.5}$$

Theorem 5.2. *Let A be a symmetric operator on $\mathfrak{H}_1$ with equal defect numbers, let $\widetilde{A} = \widetilde{A}^*$ be a selfadjoint exit space extension of A acting on $\widetilde{\mathfrak{H}} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$ and minimal with respect to $\mathfrak{H}_1$, and let S_j, T_j, $j = 1, 2$, be defined by* (5.1). *Then*

- (i) *the linear relation*

$$\Delta^{-1} = \left\{\left\{\begin{pmatrix} f_2 \\ -f_2' \end{pmatrix}, \widehat{f}_1\right\} : \widehat{f}_1 \oplus \widehat{f}_2 \in \widetilde{A},\ f_j, f_j' \in \mathfrak{H}_j,\ j = 1, 2\right\} \tag{5.6}$$

is a minimal boundary relation for $-S_2^*$;

- (ii) *the Weyl family $M^{\Delta^{-1}}(\lambda)$ corresponding to the boundary relation Δ^{-1} coincides with $T(-\lambda)$.*

Proof. (i) By [15, Prop. 2.10], $\Delta\colon (\mathfrak{H}_1^2, J_{\mathfrak{H}_1}) \to (\mathfrak{H}_2^2, J_{\mathfrak{H}_2})$ in (5.5) is unitary, and hence Δ^{-1} is a unitary linear relation from $(\mathfrak{H}_2^2, J_{\mathfrak{H}_2})$ to $(\mathfrak{H}_1^2, J_{\mathfrak{H}_1})$, see Proposition 2.4. Since $\ker \Delta^{-1} = -S_2$, Δ^{-1} is a boundary relation for $-S_2^*$ with $\operatorname{dom} \Delta^{-1} = -T_2$, and Δ^{-1} is minimal by Lemma 5.1.

(ii) Let $\{f_2, -f_2'\} = \{f_2, \lambda f_2\} \in \widehat{\mathfrak{N}}_\lambda(-T_2)$. Choose $f_1, f_1' \in \mathfrak{H}_1$ such that

$$\left\{\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \begin{pmatrix} f_1' \\ -\lambda f_2 \end{pmatrix}\right\} \in \widetilde{A}, \quad \text{or, equivalently,} \quad \left\{\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \begin{pmatrix} f_1' + \lambda f_1 \\ 0 \end{pmatrix}\right\} \in \widetilde{A} + \lambda.$$

The definition of Shtraus extension implies that $\{f_1, f_1'\} \in T(-\lambda)$. Therefore, $M^{\Delta^{-1}}(\lambda) \subset T(-\lambda)$. The proof of the reverse inclusion is analogous.

5.3. *Induced Boundary Relation and the Coupling of Symmetric Operators*

Let A be a symmetric operator on the Hilbert space $\mathfrak{H}_1$, let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be an ordinary boundary triplet for A^*, let $\widetilde{A}$ be a selfadjoint extension of A on $\mathfrak{H}_1 \oplus \mathfrak{H}_2$, and let S_2 and T_2 be defined as in (5.1). One can naturally assign a boundary relation on the exit space $\mathfrak{H}_2$ to S_2^*.

Theorem 5.3. *Let A be a symmetric operator on $\mathfrak{H}_1$ with equal defect numbers, and let $\Pi_A = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be an ordinary boundary triplet for A^*. Then,*

(i) *if $\widetilde{A} = \widetilde{A}^*$ is a minimal selfadjoint exit space extension of A on $\widetilde{\mathfrak{H}} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$ and S_2 is defined by* (5.1), *then the linear relation $\chi\colon \mathfrak{H}_2^2 \to \mathcal{H}^2$ defined by*

$$\chi = \left\{\left\{\widehat{f}_2, \begin{pmatrix} \Gamma_0 \widehat{f}_1 \\ -\Gamma_1 \widehat{f}_1 \end{pmatrix}\right\} : \widehat{f}_1 \oplus \widehat{f}_2 \in \widetilde{A},\ \widehat{f}_1 \in A^*,\ \widehat{f}_2 \in T_2\right\} \tag{5.7}$$

is a minimal boundary relation for S_2^*;

(ii) *if S_2 is a simple symmetric operator on $\mathfrak{H}_2$ and $\chi\colon \mathfrak{H}_2^2 \to \mathcal{H}^2$ is a minimal boundary relation for S_2^*, then the linear relation $\widetilde{A}$ defined by*

$$\widetilde{A} = \left\{ \widehat{f}_1 \oplus \widehat{f}_2 \in A^* \oplus S_2^* : \left\{\widehat{f}_2, \begin{pmatrix} \Gamma_0 \widehat{f}_1 \\ -\Gamma_1 \widehat{f}_1 \end{pmatrix}\right\} \in \chi \right\} \tag{5.8}$$

is a minimal selfadjoint extension of A such that $\widetilde{A} \cap \mathfrak{H}_2^2 = S_2$.

Proof. (i) Let $\widetilde{A}$ be a selfadjoint extension of A on the Hilbert space $\mathfrak{H}_1 \oplus \mathfrak{H}_2$. Introduce the linear relation Δ^{-1} by (5.6). According to Theorem 5.2, Δ^{-1} is a boundary relation for $-S_2^*$ with $\operatorname{dom} \Delta^{-1} = -T_2$ and $\operatorname{ran} \Delta^{-1} = T_1$. Since $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is an ordinary boundary triplet for A^* and $A \subset S_1$, it follows that $\operatorname{ran} \Delta^{-1} = T_1 \subset S_1^* \subset A^* = \operatorname{dom} \Gamma$ and $\operatorname{ran} \Gamma = \mathcal{H}^2$. By Theorem 2.10, the composition χ_- of the unitary relations Δ^{-1} and Γ,

$$\chi_- = \Gamma \circ \Delta^{-1} = \left\{\left\{\begin{pmatrix} f_2 \\ -f_2' \end{pmatrix}, \Gamma \widehat{f}_1\right\} : \widehat{f}_1 \oplus \widehat{f}_2 \in \widetilde{A},\ \widehat{f}_1 \in A^*,\ \widehat{f}_2 \in T_2\right\}, \tag{5.9}$$

is a unitary relation from $(\mathfrak{H}_2^2, J_{\mathfrak{H}_2})$ to $(\mathcal{H}^2, J_{\mathcal{H}})$ with $\operatorname{dom} \chi_- = -T_2$. Changing the signs of the second components in χ_- yields the linear relation χ of the form (5.7) with $\operatorname{dom} \chi = T_2$. Since $\operatorname{clos} T_2 = S_2^*$, $\chi\colon \mathfrak{H}_2^2 \to \mathcal{H}^2$ is a boundary relation for S_2^*.

To complete the proof of (i), it remains to prove that the boundary relation $\chi\colon \mathfrak{H}_2^2 \to \mathcal{H}^2$ for S_2^* is minimal. Since the sets $\mathfrak{N}_\lambda(T_2)$ are dense in $\mathfrak{N}_\lambda(S_2^*)$ (see Lemma 5.1 (iv)), the minimality of χ is equivalent to the simplicity of S_2. However, by Lemma 5.1 (vi), S_2 is simple if and only if $\widetilde{A}$ is minimal with respect to $\mathfrak{H}_1$, i.e., $\widetilde{A}$ is a minimal selfadjoint extension of S_1 or, equivalently, of $A \subset S_1$.

(ii) Let $\chi\colon \mathfrak{H}_2^2 \to \mathcal{H}^2$ be a boundary relation for S_2^*. Then

$$\chi_- = \left\{\left\{\begin{pmatrix} f_2 \\ -f_2' \end{pmatrix}, \begin{pmatrix} h \\ -h' \end{pmatrix}\right\} : \left\{\begin{pmatrix} f_2 \\ f_2' \end{pmatrix}, \begin{pmatrix} h \\ h' \end{pmatrix}\right\} \in \chi\right\} \tag{5.10}$$

is a boundary relation for $-S_2^*$. Since $\operatorname{ran} \chi_- \subset \mathcal{H}^2 = \operatorname{dom} \Gamma^{-1}$ and $\operatorname{ran} \Gamma^{-1} = A^* = A^{[\perp]} = (\operatorname{mul} \Gamma^{-1})^{[\perp]}$, it follows from Theorem 2.10 that the composition

$$\Delta^{-1} := \Gamma^{-1} \circ \chi_- = \left\{\left\{\begin{pmatrix} f_2 \\ -f_2' \end{pmatrix}, \widehat{f}_1\right\} : \left\{\begin{pmatrix} f_2 \\ f_2' \end{pmatrix}, \begin{pmatrix} \Gamma_0 \widehat{f}_1 \\ -\Gamma_1 \widehat{f}_1 \end{pmatrix}\right\} \in \chi,\ \widehat{f}_1 \in A^* \right\} \tag{5.11}$$

is a unitary relation from $(\mathfrak{H}_2^2, J_{\mathfrak{H}_2})$ to $(\mathfrak{H}_1^2, J_{\mathfrak{H}_1})$ with $\operatorname{dom} \Delta^{-1} = \operatorname{dom} \chi_-$.

Then
$$\Delta = \chi_-^{-1} \circ \Gamma = \left\{ \left\{ \widehat{f}_1, \begin{pmatrix} f_2 \\ -f_2' \end{pmatrix} \right\} : \left\{ \begin{pmatrix} f_2 \\ f_2' \end{pmatrix}, \begin{pmatrix} \Gamma_0 \widehat{f}_1 \\ -\Gamma_1 \widehat{f}_1 \end{pmatrix} \right\} \in \chi,\ \widehat{f}_1 \in A^* \right\} \tag{5.12}$$
is a unitary relation from $(\mathfrak{H}_1^2, J_{\mathfrak{H}_1})$ to $(\mathfrak{H}_2^2, J_{\mathfrak{H}_2})$ with $\operatorname{ran}\Delta = \operatorname{dom}\chi_-$ and $\operatorname{mul}\Delta = -S_2$. Applying the main transform $\mathcal{J}$ to the unitary relation Δ, we obtain a selfadjoint extension $\widetilde{A}$ of A given by (5.8) for which $\widetilde{A} \cap \mathfrak{H}_2^2 = \ker\chi = S_2$.

If $\chi\colon \mathfrak{H}_2^2 \to \mathcal{H}^2$ is a minimal boundary relation for S_2^*, then the minimality of $\widetilde{A}$ with respect to $\mathfrak{H}_1$ is implied by the same reasons as in (i).

Remark 5.4. Theorem 5.3 establishes a one-to-one correspondence between all minimal (with respect to $\mathfrak{H}_1$) selfadjoint extensions $\widetilde{A}$ of A and all minimal boundary relations $\chi\colon \mathfrak{H}_2^2 \to \mathcal{H}^2$ with a fixed parameter space $\mathcal{H}$. If $\widetilde{A}$ is a canonical selfadjoint extension of A, i.e., if $\mathfrak{H}_2 = \{0\}$, then χ becomes purely multivalued, i.e., $\operatorname{mul}\chi = \operatorname{ran}\chi$, and its Weyl family coincides with the canonical selfadjoint extension $-\Gamma(\widetilde{A})$; cf. also [15, Cor. 6.2]. Since the minimal boundary relations are uniquely determined (up to unitary equivalence) by their Weyl families [15], one can consider the correspondence established in Theorem 5.3 as a one-to-one correspondence between all minimal selfadjoint extensions of A and all Nevanlinna families $\tau(\cdot) \in \widetilde{R}(\mathcal{H})$. This correspondence for canonical selfadjoint extensions reduces to the parametrization stated in Proposition 3.15. For the minimal selfadjoint exit space extensions, the correspondence can be written explicitly in terms of generalized resolvents (see Section 6). Theorem 5.3 plays an important role in Section 7 as well.

Definition 5.5. The boundary relation χ defined by (5.7) is said to be the *induced boundary relation*; it is uniquely determined by the exit space extension $\widetilde{A} = \widetilde{A}^*$ of A on $\mathfrak{H}_1 \oplus \mathfrak{H}_2$ and an ordinary boundary triplet Π_A for A^*.

Proposition 5.6. *Let the assumptions of Theorem* 5.3 *be satisfied. Then the Weyl family* $\tau(\lambda)$ *of* S_2 *corresponding to the induced boundary relation* $\chi\colon \mathfrak{H}_2^2 \to \mathcal{H}^2$ *is given by*
$$\tau(\lambda) = \left\{ \{\Gamma_0 \widehat{f}_1, -\Gamma_1 \widehat{f}_1\} : \quad \widehat{f} = \{f, f'\} = \widehat{f}_1 \oplus \widehat{f}_2 \in \widetilde{A},\ f' - \lambda f \in \mathfrak{H}_1 \right\}. \tag{5.13}$$

Proof. Let $\widehat{f}_2 = \{f_2, f_2'\} \in \widehat{\mathfrak{N}}_\lambda(T_2)$. Then it follows from (5.1) that there are vectors $f_1, f_1' \in \mathfrak{H}_1$ for which $\widehat{f} = \{f, f'\} = \widehat{f}_1 \oplus \widehat{f}_2 \in \widetilde{A}$ and $f' - \lambda f \in \mathfrak{H}_1$. Hence, by (5.8),
$$\left\{ \widehat{f}_2, \begin{pmatrix} \Gamma_0 \widehat{f}_1 \\ -\Gamma_1 \widehat{f}_1 \end{pmatrix} \right\} \in \chi. \tag{5.14}$$
Since $\widehat{f}_2 \in \widehat{\mathfrak{N}}_\lambda(T_2)$, this shows that $\{\Gamma_0 \widehat{f}_1, -\Gamma_1 \widehat{f}_1\}$ belongs to the Weyl family $M_\chi(\lambda)$ of S_2 corresponding to the boundary relation χ. This proves the inclusion $M_\chi(\lambda) \subset \tau(\lambda)$, $\lambda \in \mathbb{C}_+ \cup \mathbb{C}_-$.

Conversely, if $\widehat{f}_1$, $\widehat{f}$ satisfy conditions (5.13), then $\widehat{f}_2 \in \widehat{\mathfrak{N}}_\lambda(T_2)$. Due to (5.14), we obtain $\{\Gamma_0 \widehat{f}_1, -\Gamma_1 \widehat{f}_1\} \in M_\chi(\lambda)$, which proves the inclusion $\tau(\lambda) \subset M_\chi(\lambda)$.

Definition 5.7. The selfadjoint extension $\widetilde{A}$ of A constructed by (5.8) is called the *coupling of the symmetric operators* A *and* S_2 *corresponding to the coupling of the boundary triplet* Π *and the boundary relation* χ.

Consider some examples of couplings of differential operators, both single-valued and multivalued. In what follows, $W_2^s(\Omega)$ stands for the Sobolev space on a domain Ω; see [9] or [53].

Example 5.8. Let A be the differential operator in $L_2(0,1)$ generated by the differential expression $-D^2$ on the domain $\operatorname{dom} A = \left\{ f \in W_2^2(0,1) : \ f(0+) = f'(0+) = f(1) = 0 \right\}$. It is clear that A is a symmetric operator, $n_\pm(A) = 1$, and
$$A^* = -D^2 \upharpoonright \operatorname{dom} A^*, \qquad \operatorname{dom} A^* = \left\{ f \in W_2^2(0,1) : \ f(1) = 0 \right\}. \tag{5.15}$$
A boundary triplet $\Pi_A = \{\mathbb{C}, \Gamma_0, \Gamma_1\}$ for the operator A^* is obtained by setting $\Gamma_0 f = f(0)$, $\Gamma_1 f = f'(0)$ for $f \in \operatorname{dom} A^*$. Next, consider a symmetric operator S_2 generated on $L_2(-1,0)$ by the same differential expression $-D^2$ on the domain
$$\operatorname{dom} S_2 = \left\{ f \in W_2^2(-1,0) : \ f(0-) = f'(0-) = f(-1) = 0 \right\}. \tag{5.16}$$

Again $n_\pm(S_2) = 1$ and $S_2^* = -D^2 \upharpoonright \operatorname{dom} S_2^*$, $\operatorname{dom} S_2^* = \{ f \in W_2^2(-1,0) : \ f(-1) = 0 \}$. Choose a boundary triplet $\Pi_{S_2} = \{\mathbb{C}, \chi_0, \chi_1\}$ for the operator S_2^* by setting $\chi_0 f = f(0)$, $\chi_1 f = -f'(0)$ for $f \in \operatorname{dom} S_2^*$. The boundary conditions (5.8) determining a coupling $\widetilde{A}$ of the symmetric operators A and S_2 corresponding to the coupling of the boundary triplets Π_A and Π_{S_2} now become

$$f(0+) = f(0-), \qquad f'(0+) = f'(0-),$$

and $f \in W_2^2(-1,1)$. Thus, the coupling $\widetilde{A}$ is a selfadjoint operator on $L_2(-1,1)$ associated with the Dirichlet boundary value problem for differential expression $-D^2$,

$$\widetilde{A} = -D^2 \upharpoonright \operatorname{dom} \widetilde{A}, \qquad \operatorname{dom} \widetilde{A} = \{ f \in W_2^2(-1,1) : \ f(1) = 0, \ f(-1) = 0 \}. \tag{5.17}$$

Example 5.9. Let A be a minimal differential operator in $L_2(0,1)$ associated with the differential expression $-D^2$. The domain of A is characterized by

$$\operatorname{dom} A = \overset{\circ}{W}{}_2^2(0,1) = \{ f \in W_2^2(0,1) : \ f(0+) = f'(0+) = f(1) = f'(1) = 0 \}.$$

Then A is a symmetric operator in $L_2(0,1)$ with the defect numbers $n_\pm(A) = 2$, and its adjoint A^* is generated by the same expression $-D^2$ on the domain $\operatorname{dom} A = W_2^2(0,1)$. Moreover, a boundary triplet $\Pi_A = \{\mathbb{C}^2, \Gamma_0, \Gamma_1\}$ for the operator A^* is obtained by setting

$$\Gamma_0 f = \begin{pmatrix} f(0+) \\ f(1) \end{pmatrix}, \qquad \Gamma_1 f = \begin{pmatrix} f'(0+) \\ -f'(1) \end{pmatrix}, \qquad f \in \operatorname{dom} A^*. \tag{5.18}$$

Consider now a minimal differential operator S_2 generated by the differential expression $-D^2$ on $L_2(-1,0)$ and introduce a boundary triplet $\Pi_{S_2} = \{\mathbb{C}, \chi_0, \chi_1\}$ for S_2^* by setting

$$\chi_0 f = \begin{pmatrix} f(0-) \\ f(-1) \end{pmatrix}, \qquad \chi_1 f = \begin{pmatrix} -f'(0-) \\ f'(-1) \end{pmatrix}, \qquad f \in \operatorname{dom} S_2^*. \tag{5.19}$$

Then the boundary conditions (5.8) determining the coupling $\widetilde{A}$ of the operators A and S_2 corresponding to the coupling of the boundary triplets Π_A and Π_{S_2} become

$$f(0+) = f(0-), \quad f'(0+) = f'(0-), \quad f(1) = f(-1), \quad f'(1) = f'(-1).$$

In other words, the coupling $\widetilde{A}$ is a selfadjoint operator on $L_2(-1,1)$ associated with the periodic boundary value problem for the differential expression $-D^2$, namely,

$$\widetilde{A} = -D^2 \upharpoonright \operatorname{dom} \widetilde{A}, \qquad \operatorname{dom} \widetilde{A} = \{ f \in W_2^2[-1,1] : \ f(1) = f(-1), \ f'(1) = f'(-1) \}.$$

Example 5.10. Let A be as in the previous example, and let S_2 be a minimal differential operator generated on $L_2(-\infty,0)$ by the differential expression $-D^2$, i.e., $S_2 = -D^2 \upharpoonright \operatorname{dom} S_2$, $\operatorname{dom} S_2 = \overset{\circ}{W}{}_2^2(-\infty,0)$. Here $n_\pm(A) = 2$, whereas $n_\pm(S_2) = 1$. Therefore, to obtain a coupling $\widetilde{A}$ of the operators A and S_2, a (multivalued) boundary relation $\chi \colon S_2^* \to \mathbb{C}^4$ for S_2^* is needed. Hence, for any fixed $h \in \mathbb{R}$, we write

$$\chi = \Big\{ \{ \widehat{f}, \ \operatorname{col}\,(f(0-), \ c, -f'(0-), \ hc) \} : \ \widehat{f} \in S_2^*, \ c \in \mathbb{C} \Big\}. \tag{5.20}$$

It follows that $\operatorname{mul} \chi = \{ \operatorname{col}\,(0, \ c, 0, \ hc) : \ c \in \mathbb{C} \}$ and $\dim \operatorname{mul} \chi = 1 = n_\pm(A) - n_\pm(S_2)$. This fact is in accordance with Proposition 3.2 (ii). For a fixed $\lambda \in \mathbb{C} \setminus \mathbb{R}$, the defect subspace $\mathfrak{N}_\lambda(S_2^*)$ is spanned by $f_\lambda = \exp(-i\sqrt{\lambda t})$, and $\{ \widehat{f}_\lambda, \ \operatorname{col}\,(1, \ c, \ i\sqrt{\lambda}, \ hc) \} \in \chi$ for every $c \in \mathbb{C}$. Hence, the corresponding Weyl function becomes

$$\tau(\lambda) = \begin{pmatrix} i\sqrt{\lambda} & 0 \\ 0 & h \end{pmatrix}. \tag{5.21}$$

Clearly, $\tau(\lambda)$ is not strict, reflecting the fact that it corresponds to the multivalued boundary relation χ (see Propositions 3.7 and 3.8). Relation (5.8) becomes

$$f(0+) = f(0-), \quad f'(0+) = f'(0-), \quad f(1) = c, \quad f'(1) = ch, \qquad c \in \mathbb{C},$$

and determines a selfadjoint operator $\widetilde{A}$ (a coupling) generated in $L_2(-\infty,1)$ by the differential expression $-D^2$ and the boundary condition $f'(1) = hf(1)$, namely,

$$\widetilde{A} = -D^2 \upharpoonright \operatorname{dom} \widetilde{A}, \qquad \operatorname{dom} \widetilde{A} = \{ f \in W_2^2(-\infty,1) : \ f'(1) = hf(1) \}.$$

Here the operator S_1 is a restriction of $-D^2$ to the domain

$$\operatorname{dom} S_1 = \{f \in W_2^2(0,1) : f(0+) = f'(0+) = f'(1) - hf(1) = 0\}.$$

It differs from A, and $\dim S_1/A = 1 = \dim\ker \operatorname{Im}\tau(\lambda)$. This illustrates the general situation stated in the next proposition.

Proposition 5.11. *Let the assumptions of Theorem* 5.3 *hold, and let $\tau = \{\Phi, \Psi\}$ be the Weyl family of S_2 in* (5.13) *corresponding to the induced boundary relation χ in* (5.7). *Then*

$$\dim S_1/A = \dim \operatorname{mul}\chi = \dim \tau(\lambda) \cap \tau(\lambda)^* = \dim\ker \mathsf{N}_{\Phi,\Psi}(\lambda,\lambda) \tag{5.22}$$

for $\lambda \in \mathbb{C}\setminus\mathbb{R}$. Further,

(i) *if $S_1 = A$, or, equivalently, if $\tau \in R^s(\mathcal{H})$, then T_1 is closed, i.e., $T_1 = T_1^{**} = S_1^*$ (nonclosed, i.e., $T_1 \neq S_1^*$), if and only if $\tau \in R^u[\mathcal{H}]$ ($\tau \in R^s(\mathcal{H}) \setminus R^u[\mathcal{H}]$, respectively), i.e., $0 \in \rho(\mathsf{N}_{\Phi,\Psi}(\lambda,\lambda))$ ($0 \in \sigma_c(\mathsf{N}_{\Phi,\Psi}(\lambda,\lambda))$, respectively) for each $\lambda \in \mathbb{C}_+$;*

(ii) *$S_1 = S_1^*$ if and only if $\operatorname{mul}\chi$ is selfadjoint, or, equivalently, the Weyl family $\tau(\lambda)$, $\lambda \in \mathbb{C}\setminus\mathbb{R}$, is constant; this also means that χ is purely multivalued, i.e., $\operatorname{mul}\chi = \operatorname{ran}\chi$.*

Proof. By setting $\widehat{f}_2 = 0$ in (5.7), one obtains a description of $\operatorname{mul}\chi$,

$$\begin{pmatrix} h \\ h' \end{pmatrix} \in \operatorname{mul}\chi \quad\Longleftrightarrow\quad \begin{pmatrix} h \\ -h' \end{pmatrix} = \Gamma\widehat{f}, \quad \text{where} \quad \widehat{f} \in \widetilde{A} \cap \mathfrak{H}_1^2 = S_1. \tag{5.23}$$

Since Γ is an isomorphism between the linear spaces A^*/A and $\mathcal{H}^2$, this implies that $\operatorname{mul}\chi$ and S_1/A are isomorphic. Therefore, $\dim S_1/A = \dim \operatorname{mul}\chi$. Now relations (3.11) in Proposition 3.8 yield (5.22).

(i) This follows from [15, Lemma 4.4] and from the description of the subclasses $R^s(\mathcal{H})$ and $R^u[\mathcal{H}]$ in Subsection 2.4.

(ii) The equivalence between the condition $S_1 = S_1^*$ and the selfadjointness of $\operatorname{mul}\chi$ is clear from (5.23) and the properties of Γ (cf. (3.21)). Since $\operatorname{mul}\chi = \tau(\lambda)\cap\tau(\lambda)^*$ and $\overline{\operatorname{ran}}\,\chi = \operatorname{clos}\left(\tau(\lambda) \widehat{+} \tau(\lambda)^*\right)$ by [15, Lemma 4.1], we see that $\operatorname{mul}\chi$ is selfadjoint if and only if $\tau(\lambda) = \operatorname{mul}\chi$ for any $\lambda \in \mathbb{C}\setminus\mathbb{R}$ (see also [15, Cor. 6.2]). On the other hand, if $\tau(\lambda) = B$ for any $\lambda \in \mathbb{C}\setminus\mathbb{R}$, then $B = B^*$ and $\operatorname{mul}\chi = \operatorname{ran}\chi = B$.

5.4. *Weyl Function of a Coupling*

It is shown here that, to every selfadjoint extension $\widetilde{A}$ of A, one can assign a special boundary relation (involving the linear relation $A^* \oplus T_2$) whose parameter space has the double dimension. The corresponding Weyl function of the operator $A \oplus S_2$ can be represented in block form, and it is frequently encountered in boundary-eigenvalue problems with boundary conditions depending on the eigenvalue parameter (see, e.g., [22, 23]).

Theorem 5.12. *Let A be a symmetric operator on $\mathfrak{H}_1$, let $\Pi = \{\mathcal{H}_1, \Gamma_0, \Gamma_1\}$ be an ordinary boundary triplet for A^* with the Weyl function $M(\lambda)$, let S_2 be a symmetric operator on a Hilbert space $\mathfrak{H}_2$, let $\chi\colon \mathfrak{H}_2^2 \mapsto \mathcal{H}^2$ be a boundary relation for S_2^* with the domain $\operatorname{dom}\chi = T_2$ and the Weyl family $\tau(\cdot) = \{\Phi(\cdot), \Psi(\cdot)\} \in \widetilde{R}(\mathcal{H})$, and let $\widetilde{\mathfrak{H}} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$. Then*

(i) *the linear relation $\Gamma^{\mathrm{coupl}} : \widetilde{\mathfrak{H}}^2 \mapsto \mathcal{H}^2 \oplus \mathcal{H}^2$ given by*

$$\Gamma^{\mathrm{coupl}} = \left\{ \left\{ \widehat{f}_1 \oplus \widehat{f}_2, \left\{ \begin{pmatrix} h' + \Gamma_1\widehat{f}_1 \\ h - \Gamma_0\widehat{f}_1 \end{pmatrix}, \begin{pmatrix} -\Gamma_0\widehat{f}_1 \\ h' \end{pmatrix} \right\} \right\} : \widehat{f}_1 \in A^*,\ \left\{ \widehat{f}_2, \widehat{h} \right\} \in \chi \right\} \tag{5.24}$$

is a boundary relation for $A^* \oplus S_2^*$ satisfying (B1)–(B3) (*see Proposition* 3.16);

(ii) *the corresponding Weyl function $\mathcal{M}(\cdot)$ belongs to the class $R[\mathcal{H}^2]$ and is given by*

$$\mathcal{M} = \begin{pmatrix} -\Phi(\Psi + M\Phi)^{-1} & I - \Phi(\Psi + M\Phi)^{-1}M \\ I - M\Phi(\Psi + M\Phi)^{-1} & \Psi(\Psi + M\Phi)^{-1}M \end{pmatrix} = \begin{pmatrix} -(\tau + M)^{-1} & I - (\tau + M)^{-1}M \\ I - M(\tau + M)^{-1} & (\tau^{-1} + M^{-1})^{-1} \end{pmatrix}. \tag{5.25}$$

Proof. (i) Clearly, the linear relation

$$\widetilde{\Gamma} = \left\{\left\{\widehat{f}_1 \oplus \widehat{f}_2, \left\{\begin{pmatrix}\Gamma_0 \widehat{f}_1\\ -h'\end{pmatrix}, \begin{pmatrix}\Gamma_1 \widehat{f}_1\\ h\end{pmatrix}\right\}\right\} : \ \widehat{f}_1 \in A^*, \ \left\{\widehat{f}_2, \begin{pmatrix}h\\ h'\end{pmatrix}\right\} \in \chi\right\} \tag{5.26}$$

forms a boundary relation for $A^* \oplus S_2^*$, and the corresponding Weyl family is

$$\widetilde{\tau}(\lambda) = M(\lambda) \oplus (-\tau(\lambda)^{-1}) = \{I \oplus (-\Psi(\lambda)), M(\lambda) \oplus \Phi(\lambda)\}. \tag{5.27}$$

Let W be the standard unitary operator defined by

$$W = \begin{pmatrix} W_{00} & I_{\mathcal{H}^2}\\ -I_{\mathcal{H}^2} & 0\end{pmatrix}, \quad \text{where} \quad W_{00} = \begin{pmatrix} 0 & -I_{\mathcal{H}}\\ -I_{\mathcal{H}} & 0\end{pmatrix}. \tag{5.28}$$

By Theorem 2.10, the composition $\Gamma^{\mathrm{coupl}} = W \circ \widetilde{\Gamma}$ is unitary, and hence, by Proposition 3.11, it defines a new boundary relation for $A^* \oplus S_2^*$ whose Weyl family is

$$\mathcal{M}(\lambda) = W[\widetilde{\tau}(\lambda)] = \left\{\Omega_0(\lambda), \begin{pmatrix} -I_{\mathcal{H}} & 0\\ 0 & \Psi(\lambda)\end{pmatrix}\right\}, \quad \text{where} \quad \Omega_0 = \begin{pmatrix} M(\lambda) & \Psi(\lambda)\\ -I_{\mathcal{H}} & \Phi(\lambda)\end{pmatrix}. \tag{5.29}$$

Since $\Omega_0(\lambda)$ is invertible (see [14, Remark 5.7]), $\Gamma_0^{\mathrm{coupl}}\upharpoonright \left(\widehat{\mathfrak{N}}_\lambda(A) \oplus \widehat{\mathfrak{N}}_\lambda(T)\right)$ is a surjective mapping, and thus, by Proposition 3.16, Γ^{coupl} is a boundary relation for $A^* \oplus S_2^*$ satisfying conditions (B1)–(B3). This proves (i).

(ii) Taking $\omega := (\Psi + M\Phi)^{-1}$, one easily derives from (5.29) the following formula for the corresponding Weyl function $\mathcal{M}(\cdot)$:

$$\begin{aligned}\mathcal{M}(\lambda) &= \begin{pmatrix} -I & 0\\ 0 & \Psi(\lambda)\end{pmatrix}\Omega_0(\lambda)^{-1} = \begin{pmatrix} -I & 0\\ 0 & \Psi(\lambda)\end{pmatrix}\begin{pmatrix} \Phi(\lambda)\omega(\lambda) & \Phi(\lambda)\omega(\lambda)M(\lambda) - I\\ \omega(\lambda) & \omega(\lambda)M(\lambda)\end{pmatrix}\\ &= \begin{pmatrix} -\Phi(\lambda)(\Psi(\lambda) + M(\lambda)\Phi(\lambda))^{-1} & I - \Phi(\lambda)(\Psi(\lambda) + M(\lambda)\Phi(\lambda))^{-1}M(\lambda)\\ \Psi(\lambda)(\Psi(\lambda) + M(\lambda)\Phi(\lambda))^{-1} & \Psi(\lambda)(\Psi(\lambda) + M(\lambda)\Phi(\lambda))^{-1}M(\lambda)\end{pmatrix},\end{aligned}$$

which coincides with the one in (5.25). Moreover, $\mathcal{M}(\cdot) \in R[\mathcal{H}^2]$.

Remark 5.13. (i) If the boundary relation χ in Theorem 5.3 is single-valued, then it can be decomposed into a boundary triplet $\Pi'' = \{\mathcal{H}, \chi_0, \chi_1\}$, where the boundary operators χ_j are given by $\chi_j = \pi_j\chi\colon\ T_2 \to \mathcal{H}$, $j = 0, 1$. In this case, the boundary relation $\widetilde{\Gamma}$ of the form (5.26) becomes a boundary triplet $\widetilde{\Pi} = \{\mathcal{H}^2, \widetilde{\Gamma}_0, \widetilde{\Gamma}_1\}$, where $\widetilde{\Gamma} = \begin{pmatrix}\widetilde{\Gamma}_0\\ \widetilde{\Gamma}_1\end{pmatrix}$ with $\widetilde{\Gamma}_0 = \begin{pmatrix}\Gamma_0\\ -\chi_1\end{pmatrix}$ and $\widetilde{\Gamma}_1 = \begin{pmatrix}\Gamma_1\\ \chi_0\end{pmatrix}$, and relation (5.8) becomes

$$\widetilde{A} = \ker(\widetilde{\Gamma}_1 - B\widetilde{\Gamma}_0) \quad \text{with} \quad B = \begin{pmatrix} 0 & I_{\mathcal{H}}\\ I_{\mathcal{H}} & 0\end{pmatrix}. \tag{5.30}$$

In other words, the coupling $\widetilde{A}$ is

$$\widetilde{A} = \{\, \widehat{f}_1 \oplus \widehat{f}_2 \in A^* \oplus T_2 : \Gamma_0\widehat{f}_1 - \chi_0\widehat{f}_2 = \Gamma_1\widehat{f}_1 + \chi_1\widehat{f}_2 = 0\,\}. \tag{5.31}$$

In this form, the construction of the coupling $\widetilde{A}$ of two boundary triplets has been introduced in [14] under the additional assumption that $\Pi'' = \{\mathcal{H}, \chi_0, \chi_1\}$ is an ordinary boundary triplet.

(ii) Suppose that the Nevanlinna family $\tau(\cdot)$ in Theorem 5.12 belongs to $R^s(\mathcal{H})$. Then the Weyl function corresponding to the triplet $\Gamma^{\mathrm{coupl}} = \{\,\mathcal{H}^2, \Gamma_0^{\mathrm{coupl}}, \Gamma_1^{\mathrm{coupl}}\,\}$ with

$$\Gamma_0^{\mathrm{coupl}} = \widetilde{\Gamma}_1 - B\widetilde{\Gamma}_0 = \begin{pmatrix}\chi_1 + \Gamma_1\\ \chi_0 - \Gamma_0\end{pmatrix}, \qquad \Gamma_1^{\mathrm{coupl}} = -\widetilde{\Gamma}_0 = \begin{pmatrix}-\Gamma_0\\ \chi_1\end{pmatrix}, \tag{5.32}$$

is $(B - \widetilde{\tau}(\cdot))^{-1}$. Using (5.26) and (5.27), we can readily see that

$$(B - \widetilde{\tau})^{-1} = \left(\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} - \begin{pmatrix}M & 0\\ 0 & -\tau^{-1}\end{pmatrix}\right)^{-1} = \begin{pmatrix} -(\tau + M)^{-1} & (\tau + M)^{-1}\tau\\ \tau(\tau + M)^{-1} & (\tau^{-1} + M^{-1})^{-1}\end{pmatrix}. \tag{5.33}$$

The matrix of the linear-fractional transformation $\widetilde{\tau}(\cdot) \to (B - \widetilde{\tau}(\cdot))^{-1}$ coincides with the block matrix W given by (5.28), i.e., $(B - \widetilde{\tau}(\cdot))^{-1} = W[\widetilde{\tau}(\cdot)]$. Comparing (5.25) with (5.33), we see that, in this case, the function $(B - \widetilde{\tau}(\cdot))^{-1}$ coincides with the Weyl function $\mathcal{M}(\cdot)$ corresponding to the boundary triplet Γ^{coupl}. Moreover, these arguments (borrowed from [14]) explain the appearance of the linear-fractional transformation W in formula (5.29). Under the additional assumption that $\chi = \{\mathcal{H}, \chi_0, \chi_1\}$ is an ordinary boundary triplet, Theorem 5.12 was proved in [14].

(iii) In the case of finite defect numbers, the function $\mathcal{M}(\cdot)$ arises, for instance, in connection with Sturm–Liouville operators and Hamiltonian systems with "λ-depending" boundary conditions expressed by means of a Nevanlinna pair $\{\Phi(\cdot), \Psi(\cdot)\}$ $(= \tau(\cdot))$. In this case, the function $\mathcal{M}(\cdot)$ is known as the *spectral matrix induced by the Nevanlinna pair* $\{\Phi(\cdot), \Psi(\cdot)\}$; cf. [32, 22, 23, 31].

Finally, the usefulness of the general results from Section 4 concerning intermediate extensions is demonstrated by applying them to the Weyl function $\mathcal{M}(\cdot)$ in Theorem 5.12. Namely, it is shown that the diagonal elements of the block operator function $\mathcal{M}(\lambda)$ are also Weyl families of some intermediate extensions of the operator $A \oplus S_2$. In particular, this result gives a geometric interpretation of the Nevanlinna function $(\tau(\cdot) + M(\cdot))^{-1}$ arising in the Kreĭn–Naĭmark formula for generalized resolvents (see (6.5)) as a Weyl function of some intermediate extension. The importance of this result will become clear in Section 7.

Theorem 5.14. *Let A be a symmetric operator in $\mathfrak{H}_1$, and let $\Pi = \{\mathcal{H}_1, \Gamma_0, \Gamma_1\}$ be an ordinary boundary triplet for A^* with the Weyl function $M(\lambda)$. Let S_2 be a symmetric operator in a Hilbert space $\mathfrak{H}_2$, let $\chi \colon \mathfrak{H}_2^2 \mapsto \mathcal{H}^2$ be a boundary relation for S_2^* with the domain $\operatorname{dom}\chi = T_2$ and the Weyl family $\tau(\cdot) = \{\Phi(\cdot), \Psi(\cdot)\} \in \widetilde{R}(\mathcal{H})$, and let $\widetilde{\mathfrak{H}} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$. Then*

(i) *the linear relation*

$$H^{(1)} = \left\{ \widehat{f}_1 \oplus \widehat{f}_2 \in A^* \oplus T_2 \colon \Gamma_0 \widehat{f}_1 = 0,\ \left\{ \widehat{f}_2, \begin{pmatrix} 0 \\ -\Gamma_1 \widehat{f}_1 \end{pmatrix} \right\} \in \chi \right\} \tag{5.34}$$

is closed and symmetric on $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$ and has equal defect numbers;

(ii) *the linear relation $\Gamma^{(1)} \colon \mathfrak{H}^2 \mapsto \mathcal{H}^2$ given by*

$$\Gamma^{(1)} = \left\{ \left\{ \widehat{f}_1 \oplus \widehat{f}_2, \begin{pmatrix} \Gamma_1 \widehat{f}_1 + h' \\ -\Gamma_0 \widehat{f}_1 \end{pmatrix} \right\} : \left\{ \widehat{f}_2, \begin{pmatrix} \Gamma_0 \widehat{f}_1 \\ h' \end{pmatrix} \right\} \in \chi,\ \widehat{f}_1 \in A^* \right\} \tag{5.35}$$

is a boundary relation for $H^{(1)*}$ satisfying (B1)–(B3);

(iii) *the Weyl function of $H^{(1)}$ corresponding to $\Gamma^{(1)}$ is given by*

$$M^{(1)}(\lambda) = -(\tau(\lambda) + M(\lambda))^{-1} = -\Phi(\lambda)(\Psi(\lambda) + M(\lambda)\Phi(\lambda))^{-1}; \tag{5.36}$$

(iv) *the linear relation*

$$H^{(2)} = \left\{ \widehat{f}_1 \oplus \widehat{f}_2 \in A^* \oplus T_2 : \Gamma_1 \widehat{f}_1 = 0,\ \left\{ \widehat{f}_2, \begin{pmatrix} \Gamma_0 \widehat{f}_1 \\ 0 \end{pmatrix} \right\} \in \chi \right\} \tag{5.37}$$

is closed and symmetric on $\widetilde{\mathfrak{H}} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$ and has equal defect numbers;

(v) *the linear relation $\Gamma^{(2)} \colon \widetilde{\mathfrak{H}}^2 \mapsto \mathcal{H}^2$ given by*

$$\Gamma^{(2)} = \left\{ \left\{ \widehat{f}_1 \oplus \widehat{f}_2, \begin{pmatrix} -\Gamma_0 \widehat{f}_1 + h \\ -\Gamma_1 \widehat{f}_1 \end{pmatrix} \right\} : \left\{ \widehat{f}_2, \begin{pmatrix} h \\ -\Gamma_1 \widehat{f}_1 \end{pmatrix} \right\} \in \chi,\ \widehat{f}_1 \in A^* \right\} \tag{5.38}$$

is a boundary relation for $H^{(2)*}$ satisfying (B1)–(B3);

(vi) *the Weyl function of $H^{(2)}$ corresponding to $\Gamma^{(2)}$ is given by*

$$M^{(2)}(\lambda) = (\tau(\lambda)^{-1} + M(\lambda)^{-1})^{-1} = \Psi(\lambda)(\Psi(\lambda) + M(\lambda)\Phi(\lambda))^{-1} M(\lambda). \tag{5.39}$$

Proof. To prove assertions (i)–(iii), we apply Proposition 4.1 to the boundary relation Γ^{coupl} in Theorem 5.12. Then the boundary conditions in (4.3) become $\Gamma_1 \widehat{f}_1 + h' = -\Gamma_0 \widehat{f}_1 + h = \Gamma_0 \widehat{f}_1 = 0$, or, equivalently, $\Gamma_1 \widehat{f}_1 = -h'$, $\Gamma_0 \widehat{f}_1 = h = 0$, which yields (5.34). Moreover, by part (i) of Proposition 4.1, $H^{(1)}$ is a closed symmetric linear relation on $\widetilde{\mathfrak{H}}$ with equal defect numbers. By applying (4.4) to the boundary relation Γ^{coupl} in (5.24), we can see that $h - \Gamma_0 \widehat{f}_1 = 0$, which then yields (5.35). Finally, part (iv) of Proposition 4.1 shows that the Weyl function corresponding to the boundary relation $\Gamma^{(1)}$ is the upper left corner of the block representation of $\mathcal{M}(\lambda)$ in (5.25).

Assertions (iv)–(vi) follow from Theorem 5.12 and Corollary 4.2.

The next result is used below to study the Naĭmark classification of exit space extensions in Section 7.

Proposition 5.15. *Let the boundary relation* $\chi\colon \mathfrak{H}_2^2 \to \mathcal{H}^2$ *in Theorem* 5.14 *be single-valued, let* A *be a closed nondensely defined symmetric operator on* $\mathfrak{H}_1$ *such that the linear relation*

$$A_\infty := A \,\widehat{+}\, (\{0\} \times \operatorname{mul} A^*) \tag{5.40}$$

is selfadjoint on $\mathfrak{H}_1$, *and let an* (*ordinary*) *boundary triplet* $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ *for* A^* *satisfy* $\ker \Gamma_1 = A_\infty$. *Then* $T^{(1)} = \operatorname{dom} \Gamma^{(1)}$ *and* $H^{(1)} = \ker \Gamma^{(1)}$ *satisfy the following conditions*:

(i) $\operatorname{mul} T^{(1)} = \{0\}$ *if and only if* $\operatorname{mul} T_2 = \{0\}$;

(ii) $\operatorname{mul} H^{(1)} = \{0\}$ *if and only if* $\operatorname{mul} S_2 = \{0\}$.

Proof. (i) Since χ is single-valued, (5.35) shows that $T^{(1)} = \operatorname{dom} \Gamma^{(1)}$ is given by

$$T^{(1)} := \left\{ \widehat{f}_1 \oplus \widehat{f}_2 \in A^* \oplus T_2 : \Gamma_0 \widehat{f}_1 = \chi_0 \widehat{f}_2 \right\} \tag{5.41}$$

and its multivalued part is

$$\operatorname{mul} T^{(1)} = \left\{ f_1' \oplus f_2' : \ \widehat{f}_1 = \{0, f_1'\} \in A^*, \ \widehat{f}_2 = \{0, f_2'\} \in T_2, \ \Gamma_0 \widehat{f}_1 = \chi_0 \widehat{f}_2 \right\}. \tag{5.42}$$

Assume that $\widehat{f}_2 = \{0, f_2'\} \in T_2$. Since the selfadjoint extension $A_\infty = \ker \Gamma_1$ of A is given by (5.40) and Γ is an ordinary boundary triplet for A^*, the selfadjoint extension $A_0 := \ker \Gamma_0$ is transversal to A_∞ and

$$\mathcal{H} = \Gamma_0(A^*) = \Gamma_0(A_0 \,\widehat{+}\, (\{0\} \oplus \operatorname{mul} A^*)) = \Gamma_0(\{0\} \oplus \operatorname{mul} A^*). \tag{5.43}$$

Thus, there exists an $\widehat{f}_1 = \{0, f_1'\} \in A^*$ such that $\Gamma_0 \widehat{f}_1 = \chi_0 \widehat{f}_2$, and hence $f_1' \oplus f_2' \in \operatorname{mul} T^{(1)}$ by (5.42). Therefore, if $\operatorname{mul} T_2$ is nonzero, the same holds for $\operatorname{mul} T^{(1)}$.

Assume now that $\operatorname{mul} T_2 = \{0\}$. Let $f_1' \oplus f_2' \in \operatorname{mul} T^{(1)}$. Then (5.42) shows that $f_2' = 0$ and $\Gamma_0 \widehat{f}_1 = \chi_0 \widehat{f}_2 = 0$ with $\widehat{f}_1 = \{0, f_1'\} \in A^*$. Thus, $\{0, f_1'\} \in A_0 \cap A_\infty = A$, see (5.40), and, since A is an operator, we obtain $f_1' = 0$. This shows that $\operatorname{mul} T^{(1)} = \{0\}$ and completes the proof of the first assertion.

(ii) Since χ is single-valued, the symmetric relation $H^{(1)}$ in (5.31) becomes

$$H^{(1)} = \left\{ \widehat{f}_1 \oplus \widehat{f}_2 \in A^* \oplus T_2 : \Gamma_0 \widehat{f}_1 = \chi_0 \widehat{f}_2 = \Gamma_1 \widehat{f}_1 + \chi_1 \widehat{f}_2 = 0 \right\}. \tag{5.44}$$

To prove the equivalence, let us show that

$$\operatorname{mul} H^{(1)} = \left\{ 0 \oplus f_2' : \ f_2' \in \operatorname{mul} S_2 \right\}. \tag{5.45}$$

If $f_1' \oplus f_2' \in \operatorname{mul} H^{(1)}$, then (5.44) yields $\Gamma_0 \widehat{f}_1 = 0$, where $\widehat{f}_1 = \{0, f_1'\} \in A^*$. Hence,

$$\widehat{f}_1 = \{0, f_1'\} \in A_0 \cap A_\infty = A,$$

and therefore, $f_1' = 0$ and $\Gamma_1 \widehat{f}_1 = 0$. Now (5.44) implies that $\chi \widehat{f}_2 = 0$, i.e., $\widehat{f}_2 = \{0, f_2'\} \in \ker \chi = S_2$. The reverse inclusion is clear, and this proves (5.45).

6. GENERALIZED RESOLVENTS AND ADMISSIBILITY

6.1. *Kreĭn's Formula for Generalized Resolvents*

Let A be a symmetric operator on a Hilbert space $\mathfrak{H}$ with equal defect numbers. Let $\widetilde{A}$ be a selfadjoint extension of A acting on a Hilbert space $\widetilde{\mathfrak{H}}$ which contains $\mathfrak{H}$ as a closed subspace. The compression $\mathbf{R}_\lambda = P_{\mathfrak{H}}(\widetilde{A} - \lambda)^{-1} \upharpoonright \mathfrak{H}$ of the resolvent of $\widetilde{A}$ to $\mathfrak{H}$ is said to be a *generalized resolvent* of A.

We claim that, using the coupling method, one can readily obtain a general version of the Kreĭn–Naĭmark formula which parametrizes all generalized resolvents of A by the maximal dissipative relations (Nevanlinna pairs) $\tau(\cdot)$. Indeed, at the first step, by combining Theorem 5.3, Proposition 5.6, and Theorem 3.6, one arrives at the following formula for generalized resolvents (in the Shtraus form).

Theorem 6.1. *Let A be a closed symmetric operator on a Hilbert space $\mathfrak{H}$ with equal defect numbers $n_+(A) = n_-(A)$. Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be an ordinary boundary triplet for A^*. Let $\widetilde{A}$ be a selfadjoint extension of A on a Hilbert space $\widetilde{\mathfrak{H}} \supset \mathfrak{H}$. In this case, there is a unique Nevanlinna family $\tau(\lambda) \in \widetilde{R}_{\mathcal{H}}$ such that*

$$P_{\mathfrak{H}}(\widetilde{A} - \lambda)^{-1} \upharpoonright \mathfrak{H} = (\widetilde{A}_{-\tau(\lambda)} - \lambda)^{-1}. \tag{6.1}$$

Moreover, for every $h \in \mathfrak{H}$, the vector $f_1 = P_{\mathfrak{H}}(\widetilde{A} - \lambda)^{-1}h$ is a solution of the following "boundary-value problem" with the spectral parameter $\tau(\lambda)$ in the boundary conditions,

$$f_1' - \lambda f_1 = h, \qquad \widehat{f}_1 = \{f_1, f_1'\} \in A^*, \qquad \{\Gamma_0 \widehat{f}_1, -\Gamma_1 \widehat{f}_1\} \in \tau(\lambda). \tag{6.2}$$

Conversely, for any given $\tau(\lambda) \in \widetilde{R}_{\mathcal{H}}$, there is a minimal selfadjoint extension $\widetilde{A}$ of A on a Hilbert space $\widetilde{\mathfrak{H}} \supset \mathfrak{H}$ such that (6.1) *holds.*

Proof. Let $\lambda \in \rho(\widetilde{A})$ and $h \in \mathfrak{H}$. Then there exists an $\widehat{f} = \{f, f'\} \in \widetilde{A}$ such that

$$f' - \lambda f = h. \tag{6.3}$$

Projecting (6.3) to $\mathfrak{H}_1 = \mathfrak{H}$ and $\mathfrak{H}_2 = \widetilde{\mathfrak{H}} \ominus \mathfrak{H}$, respectively, leads to

$$f_1' - \lambda f_1 = h, \qquad f_2' - \lambda f_2 = 0, \quad \text{where} \quad f_j = P_{\mathfrak{H}_j} f \quad \text{and} \quad f_j' = P_{\mathfrak{H}_j} f', \quad j = 1, 2. \tag{6.4}$$

It follows from (5.7) in Theorem 5.3 that $\left\{\widehat{f}_2, \begin{pmatrix} \Gamma_0 \widehat{f}_1 \\ -\Gamma_1 \widehat{f}_1 \end{pmatrix}\right\} \in \chi$, where $\widehat{f}_j = \{f_j, f_j'\}$. Since $\widehat{f}_2 \in \widehat{\mathfrak{N}}_\lambda(T_2)$ here, see (6.4), it follows that $\{\Gamma_0 \widehat{f}_1, -\Gamma_1 \widehat{f}_1\} \in \tau(\lambda)$, where $\tau(\cdot)$ is the Weyl family of S_2 corresponding to the boundary relation χ. This proves (6.2). In view of (3.21), the second condition in (6.2) means that $\widehat{f}_1 \in \widetilde{A}_{-\tau(\lambda)}$, and therefore, relation (6.1) holds as well. The uniqueness of $\widetilde{A}_{-\tau(\lambda)}$ and $\tau(\lambda)$ is clear from (6.1) and (6.2).

Conversely, starting from $\tau(\cdot) \in \widetilde{R}(\mathcal{H})$ and applying Theorem 3.6, one obtains a simple symmetric operator S_2 in $\mathfrak{H}_2$ and a minimal boundary relation $\chi \colon \mathfrak{H}_2^2 \to \mathcal{H}^2$ for S_2^* such that the corresponding Weyl family is $\tau(\cdot)$. Then, by Theorem 5.3, the linear relation $\widetilde{A}$ on $\widetilde{\mathfrak{H}} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$ (a coupling of T_1 and S_2) defined by (5.8) is a minimal selfadjoint extension of A satisfying (5.1) and hence, by the first part of the proof, (6.1) holds for some $\tau_1(\cdot) \in \widetilde{R}(\mathcal{H})$. The relation $\tau_1(\cdot) = \tau(\cdot)$ is clear from Proposition 5.6.

Combining Theorem 6.1 with formula (3.25) for canonical resolvents, we arrive at the following assertion.

Theorem 6.2 [38]. *Let A be a symmetric operator on $\mathfrak{H}$ with $n_+(A) = n_-(A)$, let $\Pi = \{\mathcal{H}_1, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^*, and let $M(\cdot)$ and $\gamma(\cdot)$ be the corresponding Weyl function and the γ-field, respectively. Then the formula*

$$\mathbf{R}_\lambda = (A_0 - \lambda)^{-1} - \gamma(\lambda)(M(\lambda) + \tau(\lambda))^{-1}\gamma(\bar{\lambda})^*, \qquad \lambda \in \rho(A_0) \cap \rho(\widetilde{A}), \tag{6.5}$$

with $A_0 = \ker \Gamma_0$ establishes a bijective correspondence between the generalized resolvents $\mathbf{R}_\lambda$ of A and the Nevanlinna families $\tau(\cdot) \in \widetilde{R}_{\mathcal{H}}$.

Proof. Let $\lambda \in \rho(A_0)$. According to Proposition 3.15, $\lambda \in \rho(A_{-\tau(\lambda)})$ if and only if $0 \in \rho(M(\lambda) + \tau(\lambda))$. In this case (see (3.25)),

$$(\widetilde{A}_{-\tau(\lambda)} - \lambda)^{-1} = (A_0 - \lambda)^{-1} - \gamma(\lambda)(M(\lambda) + \tau(\lambda))^{-1}\gamma(\bar{\lambda})^*. \tag{6.6}$$

Now the assertion follows from Theorem 6.1.

Remark 6.3. (i) For "good" families $\tau(\cdot)$, Theorem 6.2 can readily be derived from Theorem 5.12 and from formula (3.25) for canonical resolvents with the Weyl function $\mathcal{M}(\cdot)$ of the coupling (see (5.25)). In particular, if $\tau(\cdot) \in R^u[\mathcal{H}]$, the coupling becomes "symmetric" with respect to the decomposition $\widetilde{\mathfrak{H}} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$, $\mathfrak{H}_1 = \mathfrak{H}$. Indeed, by Theorem 5.14 and Proposition 3.14, there is an ordinary boundary triplet $\{\mathcal{H}, \chi_0, \chi_1\}$ for S_2^* whose Weyl function is $\tau(\cdot)$. The boundary triplet $\{\mathcal{H}^2, \Gamma_0^{\text{coupl}}, \Gamma_1^{\text{coupl}}\}$ for $A^* \oplus S_2^*$ defined by (5.24) has the Weyl function $\mathcal{M}(\cdot)$ of the form (5.25).

In view of (5.32), $\widetilde{A}$ and $A_0 \oplus A_1^{(2)}$ with $A_1^{(2)} = \ker \chi_1$ are canonical selfadjoint extensions of $A \oplus S_2$, and formula (3.25) gives

$$(\widetilde{A} - \lambda)^{-1} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} (A_0 - \lambda)^{-1} h_1 \\ (A_1^{(2)} - \lambda)^{-1} h_2 \end{pmatrix} - \begin{pmatrix} \gamma(\lambda) & 0 \\ 0 & \gamma^{(2)}(\lambda) \end{pmatrix} \mathcal{M}(\lambda) \begin{pmatrix} \gamma(\bar{\lambda})^* h_1 \\ \gamma^{(2)}(\bar{\lambda})^* h_2 \end{pmatrix}, \tag{6.7}$$

where $\gamma^{(2)}(\lambda)$ is the γ-field corresponding to the boundary triplet $\{\mathcal{H}, -\chi_1, \chi_0\}$. Setting $h_2 = 0$ and projecting the formula (6.7) onto $\mathfrak{H}_1$ yields (6.5).

(ii) Note that, in fact, the formulas (6.1) and (6.5) are equivalent to each other and can easily be deduced from each other (cf. [41, 21]).

(iii) By Proposition 5.11, the constant Nevanlinna families $\tau(\cdot) \equiv \tau \in \widetilde{R}(\mathcal{H})$ correspond to the induced boundary relations $\chi \colon \mathfrak{H}_2 \to \mathcal{H}^2$ which are purely multivalued, i.e., $\operatorname{mul} \chi = \operatorname{ran} \chi$. The minimal property of χ yields $\mathfrak{H}_2 = \{0\}$; see [15, Cor. 6.2]. Hence, the coupling technique used to prove Theorems 6.1 and 6.2 covers the case of constant Nevanlinna families $\tau \in \widetilde{R}(\mathcal{H})$ and their one-to-one correspondence to the canonical selfadjoint extensions of A acting on $\mathfrak{H}$.

Remark 6.4. The description of all generalized resolvents was originally given in different forms by M.G. Kreĭn [36] and M.A. Naĭmark [43]. It has been extended to the case of infinite indices by Saakyan (see [38, 19] and the references therein). Another description (in a form close to (6.1)) was given by A. V. Štraus [51, 52]. A relationship between the Kreĭn–Naĭmark formula and the boundary triplets was discovered in [19, 21, 41]. Moreover, other proofs and generalizations of the Kreĭn–Naĭmark formula for nondensely defined symmetric operators can be found in [21, 41, 24, 40]; also see the references there.

6.2. *Admissibility*

In this section, some new admissibility criteria are given ensuring that a given generalized resolvent of a symmetric operator corresponds to a selfadjoint operator extension. The relationships between these criteria and some other conditions (found earlier in [21, 41, 40]) are also discussed.

Let A be a symmetric operator on $\mathfrak{H}$ with equal defect numbers $n_+(A) = n_-(A) \leqslant \infty$, and let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for A^*. According to Theorem 6.2, the generalized resolvents $\mathbf{R}_\lambda$ of A are in a one-to-one correspondence, given by the Kreĭn–Naĭmark formula (6.5), with the Nevanlinna families $\tau \in \widetilde{R}(\mathcal{H})$. Let $\widetilde{A}$ be a minimal selfadjoint extension of A whose compressed resolvent is $\mathbf{R}_\lambda$. Then the family $\tau(\lambda)$ associated to $\widetilde{A}$ by (6.5) is said to be Π-*admissible* if $\widetilde{A}$ is an *operator* extension of A, i.e., if $\operatorname{mul} \widetilde{A} = \{0\}$. Recall the following definition (see [15]).

Definition 6.5. Let $\Gamma \colon \mathfrak{H}^2 \to \mathcal{H}^2$ be a boundary relation and $T = \operatorname{dom} \Gamma$. The *forbidden lineal* of Γ is $\mathcal{F}_\Gamma = \Gamma(\{0\} \times \operatorname{mul} T)$.

The first result here is a geometric characterization of Π-admissibility of $\tau(\lambda) \in \widetilde{R}(\mathcal{H})$ using the induced boundary relation χ introduced in Theorem 5.3.

Proposition 6.6. *Let A be a* (*nondensely defined*) *closed symmetric operator on $\mathfrak{H}$ with equal defect numbers, let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be an ordinary boundary triplet for A^* with the Weyl function $M(\lambda)$, and let $\chi : \mathfrak{H}_2^2 \to \mathcal{H}^2$ be the induced boundary relation for S_2^* with the Weyl family $\tau(\lambda)$ as in Theorem* 5.3. *Then $\tau(\lambda) \in \widetilde{R}(\mathcal{H})$ is Π-admissible* (*i.e.* $\operatorname{mul} \widetilde{A} = \{0\}$) *if and only if*

$$\mathcal{F}_\Gamma \cap -\mathcal{F}_\chi = \{0\}, \tag{6.8}$$

where $\mathcal{F}_\Gamma$ and $\mathcal{F}_\chi$ are the forbidden lineals of $\Gamma = \{\Gamma_0, \Gamma_1\}$ and χ, respectively.

Proof. Let $\widehat{f}_1 = \{0, f_1'\}$ and $\widehat{f}_2 = \{0, f_2'\}$. Assume that $\widehat{f}_1 \oplus \widehat{f}_2 \in \widetilde{A}$. Then $\Gamma \widehat{f}_1 \in \mathcal{F}_\Gamma$, and (5.7) shows that $\{\Gamma_0 \widehat{f}_1, \Gamma_1 \widehat{f}_1\} \in -\mathcal{F}_\chi$. Thus, (6.8) implies that $\Gamma \widehat{f}_1 = 0$ and $\widehat{f}_2 \in \ker \chi$, i.e., $\widehat{f}_1 \in A$ and $\widehat{f}_2 \in S_2$. Therefore, $f_1' = f_2' = 0$ (since A and S_2 are operators); recall that, by Lemma 5.1, $\widetilde{A}$ is a minimal selfadjoint extension of A if and only if S_2 is simple, and thus, in particular, an operator. The reverse implication is proved in the same way.

If A is densely defined, then $\operatorname{mul} A^* = \{0\}$ and $\mathcal{F}_\Gamma = \{0\}$, and thus condition (6.8) holds for any $\tau(\lambda) \in \widetilde{R}(\mathcal{H})$. Hence, the well-known fact that every minimal selfadjoint extension of a densely defined symmetric operator A is an operator is an immediate consequence of Proposition 6.6.

Using the analytic description of the forbidden lineal $\mathcal{F}_\Gamma$ established in [21] for an ordinary boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ of A^*, one can express the criterion in Proposition 6.6 in terms of the Weyl function $M(\lambda)$.

Corollary 6.7. *Let the assumptions be as in Proposition* 6.6. *Then* $\tau(\lambda) \in \widetilde{R}(\mathcal{H})$ *is* Π*-admissible if and only if*

$$-\mathcal{F}_\chi \cap \left(M(i\infty) \widehat{+} (\{0\} \times \operatorname{ran} B_M^{1/2})\right) = \{0\}, \tag{6.9}$$

where $B_M = \text{s-}\lim_{y\uparrow\infty} M(iy)/y \,(\in [\mathcal{H}])$ *and* $M(i\infty)h = \lim_{y\uparrow\infty} M(iy)h$ *with* $h \in \operatorname{dom} M(i\infty) = \operatorname{dom} \mathcal{F}_\Gamma = \{\, h \in \mathcal{H} : \lim_{y\uparrow\infty} y(\operatorname{Im} M(iy)h, h) < \infty \,\}$.

Proof. This follows from Proposition 6.6 and from the description of $\mathcal{F}_\Gamma$ in [21, Th. 1.1, Cor. 2.6].

The next theorem gives a general Π-admissibility criterion for the family $\tau(\lambda) = \{\phi(\lambda), \psi(\lambda)\}$ in purely analytic terms.

Theorem 6.8. *Let A be a (nondensely defined) closed symmetric operator on $\mathfrak{H}$ with equal defect numbers $n_+(A) = n_-(A) \leqslant \infty$, let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be an ordinary boundary triplet for A^* with the Weyl function $M(\lambda)$, and let $\tau(\lambda) = \{\phi(\lambda), \psi(\lambda)\}$ be a Nevanlinna pair in $\mathcal{H}$. Then*

(i) *the pair $\{\phi(\lambda), \psi(\lambda)\}$ is Π-admissible if and only if the following two conditions are satisfied*:

$$\text{s-}\lim_{y\uparrow\infty} \phi(iy)(\psi(iy) + M(iy)\phi(iy))^{-1}/y = \text{s-}\lim_{y\uparrow\infty} (\tau(iy) + M(iy))^{-1}/y = 0, \tag{6.10}$$

$$\text{s-}\lim_{y\uparrow\infty} \psi(iy)(\psi(iy) + M(iy)\phi(iy))^{-1}M(iy)/y = \text{s-}\lim_{y\uparrow\infty} (\tau(iy)^{-1} + M(iy)^{-1})^{-1}/y = 0; \tag{6.11}$$

(ii) *if, in addition, $A_0 = \ker \Gamma_0$ is an operator, then the Π-admissibility of $\{\phi(\lambda), \psi(\lambda)\}$ is equivalent to condition* (6.10);

(iii) *if $A_1 = \ker \Gamma_1$ is an operator, then the Π-admissibility of $\{\phi(\lambda), \psi(\lambda)\}$ is equivalent to condition* (6.11).

Proof. (i) By Theorem 3.6, there are a Hilbert space $\mathfrak{H}_2$, a symmetric operator S_2 on $\mathfrak{H}_2$, and a minimal boundary relation $\chi \colon \mathfrak{H}_2^2 \to \mathcal{H}^2$ for S_2^* whose Weyl family is $\tau(\lambda) = \{\phi(\lambda), \psi(\lambda)\}$. Let the selfadjoint extension $\widetilde{A}$ of $A \oplus S_2$ be as in Theorem 5.3. Then, by Theorem 5.12, the function $\mathcal{M}(\lambda)$ given by (5.25) is the Weyl function of $A \oplus S_2$ corresponding to the boundary relation $\Gamma_{\mathcal{M}} \colon \widetilde{\mathfrak{H}}^2 \to \mathcal{H}_{\mathcal{M}}^2$ (of the form (5.24)) satisfying conditions (B1)–(B3). Therefore, according to Proposition 3.17, $\operatorname{mul} \widetilde{A} = \{0\}$ if and only if

$$\text{w-}\lim_{y\uparrow\infty} \mathcal{M}(iy)/y = 0. \tag{6.12}$$

It remains to note that (6.12) is equivalent to the pair of conditions (6.10), (6.11).

(ii) Assume that A_0 is an operator. Let $\Gamma^{(1)} : \widetilde{\mathfrak{H}}^2 \mapsto \mathcal{H}^2$ be the boundary relation (5.35) as defined in Theorem 5.14 for $H^{(1)*}$, where $H^{(1)} = \ker \Gamma^{(1)}$ is the closed symmetric relation in $\widetilde{\mathfrak{H}} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$ given by (5.34). According to Theorem 5.14, the boundary relation $\Gamma^{(1)}$ satisfies conditions (B1)–(B3), and the corresponding Weyl function of $H^{(1)}$ is $-(M(\lambda) + \tau(\lambda))^{-1}$. To show that $H^{(1)}$ is an operator, assume that $\widehat{f}_1 \oplus \widehat{f}_2 = \{0, f_1'\} \oplus \{0, f_2'\} \in H^{(1)}$. Then $\widehat{f}_1 = 0$, since $\Gamma_0 \widehat{f}_1 = 0$ and $\operatorname{mul} A_0 = \{0\}$. Using the last condition in the definition of $H^{(1)}$ in (5.34), we obtain $\{\widehat{f}_2, 0\} \in \chi$, and hence $\widehat{f}_2 \in \ker \chi = S_2$. Since S_2 is an operator, it follows that $\widehat{f}_2 = 0$, and hence $H^{(1)}$ is an operator as well. By Proposition 3.17, $\widetilde{A} = \ker \Gamma_0^{(1)}$ is an operator if and only if (6.10) holds.

(iii) If $A_1 = \ker \Gamma_1$ is an operator, then it is clear from (5.37) that $H^{(2)}$ is also an operator. Hence, again by Proposition 3.17 and by part (vi) of Theorem 5.14, $\widetilde{A} = \ker \Gamma_0^{(2)}$ is an operator if and only if (6.11) holds.

Remark 6.9. It is of interest to note the following analytical facts resulting from Theorem 6.8.

(i) If $M \in R^u[\mathcal{H}]$, then the condition $\text{s-}\lim_{y\to\infty} M(iy)/y = 0$ together with condition (6.10) imply condition (6.11), since the limit condition analytically characterizes the fact that A_0 is an operator; see Proposition 3.17.

(ii) If $M \in R^u[\mathcal{H}]$, then the condition $\text{s-}\lim_{y\to\infty}(yM(iy))^{-1} = 0$ together with condition (6.11) imply condition (6.10), since the limit condition analytically characterizes the fact that A_1 is an operator.

(iii) If $M \in R^u[\mathcal{H}]$, then the limit conditions (3.28) and (3.29) imply conditions (6.10) and (6.11), since the first two limit conditions analytically characterize the fact that A^* is an operator (see Proposition 3.17). However, even together, the limit conditions $\text{s-}\lim_{y\to\infty} M(iy)/y = 0$ and $\text{s-}\lim_{y\to\infty}(yM(iy))^{-1} = 0$ neither imply condition (6.10) nor condition (6.11), since both A_0 and A_1 can be operators such that A is not densely defined.

We know no direct analytic proof of these facts, even in the case of $(n \times n)$-matrix functions with $n > 1$.

Select the selfadjoint extension $A_0 = \ker\Gamma$ for the boundary triplet $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for A^* such that $\operatorname{mul} A_0 = \operatorname{mul} A^*$. Then we obtain a criterion for the Π-admissibility of τ which involves only the symmetric condition $S_0 = \ker\chi_0$ for $T = \operatorname{dom}\chi \subset S_2^*$ from the exit space and only the limit value B_M from $M(\lambda)$.

Proposition 6.10. *Let the assumptions be as in Theorem* 6.8*, let the boundary triplet* $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ *for* A^* *be fixed in such a way that* $A_0 = \ker\Gamma_0$ *satisfies* $\operatorname{mul} A_0 = \operatorname{mul} A^*$*, and let* $S_0 = \ker\chi_0$*. Then*

(i) $\tau(\lambda) \in \widetilde{R}_{\mathcal{H}}$ *is* Π*-admissible if and only if*

$$\mathcal{F}_{0,\chi} \cap (\{0\} \times \operatorname{ran} B_M^{1/2}) = \{0\}, \tag{6.13}$$

where $\mathcal{F}_{0,\chi} = \chi(\{0\} \times \operatorname{mul} S_0)$ *and* $B_M = \text{s-}\lim_{y\uparrow\infty} M(iy)/y$ $(\in [\mathcal{H}])$;

(ii) *if* $\tau(\lambda)$ *is* Π*-admissible, then*

$$\operatorname{mul}\tau(\lambda) \cap \operatorname{ran} B_M^{1/2} = \{0\}, \qquad \lambda \in \mathbb{C}\setminus\mathbb{R}; \tag{6.14}$$

in addition, if S_0 *is an operator, then condition* (6.14) *is also sufficient for the* Π*-admissibility of* $\tau(\lambda)$*; this holds, for instance, if* $\tau(\lambda)$ *is a selfadjoint constant*;

(iii) *a family* $\tau(\lambda) \in R^u[\mathcal{H}]$ *is* Π*-admissible if and only if* $\operatorname{ran} B_\tau^{1/2} \cap \operatorname{ran} B_M^{1/2} = \{0\}$*, where* $B_\tau = \text{s-}\lim_{y\uparrow\infty} \tau(iy)/y\, (\in [\mathcal{H}])$.

Proof. (i) Since $\operatorname{mul} A_0 = \operatorname{mul} A^*$, we have $\Gamma(\{0\} \times \operatorname{mul} A^*) = \{0\} \times \Gamma_1(\operatorname{mul} A^*)$, and hence $\mathcal{F}_\Gamma = \{0\} \times \operatorname{mul}\mathcal{F}_\Gamma = \{0\} \times \operatorname{ran} B_M^{1/2}$, see Corollary 6.7. Assume now that $\widehat{h} = \{h, h'\} \in -\mathcal{F}_\chi \cap \mathcal{F}_\Gamma$. Then $h = 0$ and $\{\widehat{f}_2, \{0, -h'\}\} \in \chi$ for some $\widehat{f}_2 = \{0, f_2'\} \in \operatorname{dom}\chi$, so that $\widehat{f}_2 \in \ker\chi_0 = S_0$. Hence, $\widehat{h} \in -\mathcal{F}_{0,\chi}$, and this shows that $-\mathcal{F}_\chi \cap \mathcal{F}_\Gamma \subset -\mathcal{F}_{0,\chi} \cap \mathcal{F}_\Gamma$. Since $\mathcal{F}_{0,\chi} \subset \mathcal{F}_\chi$, the converse inclusion is clear. Thus, (6.13) is equivalent to (6.8), and therefore (i) follows from Proposition 6.6.

(ii) According to [15, Lemma 4.1],

$$\{0\} \times \operatorname{mul}\tau(\lambda) = \operatorname{mul}\chi \cap (\{0\} \times \mathcal{H}) \subset \mathcal{F}_{0,\chi} \tag{6.15}$$

for any $\lambda \in \mathbb{C}\setminus\mathbb{R}$, and thus the necessity of condition (6.14) follows from (6.13). If S_0 is an operator, then $\mathcal{F}_{0,\chi} = \operatorname{mul}\chi$, and (6.15) shows that relation (6.14) is equivalent to (6.13). If $\tau(\lambda)$ is constant, then the minimality of the boundary relation yields $\mathfrak{H}_2 = \{0\}$, and thus $S_2 = S_0 = S_2^* = \{0, 0\}$ is an operator; cf. [15, Cor. 6.2].

(iii) If $\tau(\lambda) \in R^u[\mathcal{H}]$, and hence χ is an ordinary boundary triplet for S_2^*, then we have $\mathcal{F}_{0,\chi} = \{0\} \times \operatorname{ran} B_\tau^{1/2}$ (see [21, Prop. 2.6]), and thus (iii) follows from (i).

Corollary 6.11. *Let* A_0 *be a selfadjoint extension of* A *such that* $A_0 = A \,\widehat{+}\, (\{0\} \times \operatorname{mul} A^*)$*. Then* $\tau(\lambda)$ *is* Π*-admissible if and only if*

$$\operatorname{mul}\tau(\lambda) = 0 \quad \textit{and} \quad S_0 \ (= \ker\chi_0) \quad \textit{is an operator}, \tag{6.16}$$

where χ *is the induced boundary relation with the Weyl family* $\tau(\lambda)$*. If* $\tau(\lambda) \in R[\mathcal{H}]$*, then* (6.16) *is equivalent to*

$$\text{s-}\lim_{y\uparrow\infty} \tau(iy)/y = 0. \tag{6.17}$$

Proof. Since $A_0 = A \widehat{+} (\{0\} \times \operatorname{mul} A^*)$, we have $\mathcal{F}_\Gamma = \Gamma(\{0\} \times \operatorname{mul} A^*) = \Gamma(A_0) = \{0\} \times \mathcal{H}$. If (6.16) holds, then $\mathcal{F}_{0,\chi} \cap \mathcal{F}_\Gamma = \operatorname{mul} \chi \cap (\{0\} \times \mathcal{H}) = \{0\}$; see (6.15). Hence, $\tau(\lambda)$ is Π-admissible by Proposition 6.10 (i). Conversely, if $\tau(\lambda)$ is Π-admissible, then $\operatorname{mul} \tau(\lambda) = \{0\}$ by Proposition 6.10 (ii), since $\operatorname{ran} B_M^{1/2} = \operatorname{mul} \mathcal{F}_\Gamma = \mathcal{H}$; cf. [21]. To prove that S_0 is an operator, assume that $\widehat{f}_2 = \{0, f_2'\} \in S_0$. Then $\{\widehat{f}_2, \{0, h'\}\} \in \chi$ for some $h' \in \mathcal{H}$; therefore, $\{0, h'\} \in \mathcal{F}_{0,\chi}$, and thus $\{0, h'\} \in \mathcal{F}_{0,\chi} \cap \mathcal{F}_\Gamma$. Hence, (6.13) gives $h' = 0$, and S_0 is an operator. This completes the proof of the first assertion.

Assume now that $\tau \in R[\mathcal{H}]$. Then $\operatorname{mul} \tau(\lambda) = 0$ by assumption, and Proposition 3.17 shows that $\operatorname{mul} S_0 = \{0\}$ if and only if (6.17) holds.

Remark 6.12. Other approaches to the admissibility problem were suggested in [40, 41, 21]; see also [14]. Namely, a direct proof of part (ii) in Theorem 6.8 applying the Kreĭn–Naĭmark formula was used in [41]. The proof there is more complicated than the one presented here. Further, under the additional assumption that $A_1 = \ker \Gamma_1$ is an operator, another admissibility criterion (with a rather complicated proof) was obtained in [21]. This criterion is equivalent to assertion (iii) in Theorem 6.8, whereas we still have no direct proof of this equivalence.

Another criterion for the admissibility (without additional assumptions) was proved in [40]. A relationship between Theorem 6.8 and the Langer–Textorious result is discussed in Section 6.3.

Yet another admissibility criterion is obtained in the next proposition, where $\widetilde{A}$ is regarded as an extension of the symmetric intermediate extension H_T defined in Proposition 4.4.

Proposition 6.13. *Let A be a symmetric operator satisfying the assumptions of Theorem* 6.8, *let $\tau(\lambda) = \{\phi(\lambda), \psi(\lambda)\}$ be a Nevanlinna pair in $\mathcal{H}$, and let the matrix $\mathcal{M}(\lambda) = (\mathcal{M}_{ij}(\lambda))_{i,j=1}^2$ be defined by* (5.25)*. Assume that $G \in [\mathcal{H}]$ and define $M_G(\lambda)$ by*

$$M_G(\lambda) = G^* \mathcal{M}_{11}(\lambda) G + G^* \mathcal{M}_{12}(\lambda) + \mathcal{M}_{21}(\lambda) G + \mathcal{M}_{22}(\lambda). \tag{6.18}$$

In this case, for the pair $\{\phi(\lambda), \psi(\lambda)\}$ to be Π-admissible, it is necessary and, provided that $\widetilde{A}_{-G^*} = \ker(\Gamma_1 + G^*\Gamma_0)$ is an operator, it is also sufficient that the following condition be valid:

$$\operatorname*{s-lim}_{y \uparrow \infty} \frac{M_G(iy)}{y} = 0. \tag{6.19}$$

Proof. Let χ be the induced boundary relation in (5.7). It follows from Proposition 4.4 and Theorem 5.12 that $M_G(\lambda)$ is the Weyl function of the symmetric relation

$$H_G = \left\{ \widehat{f}_1 \oplus \widehat{f}_2 \in A^* \oplus S_2^* : \begin{array}{c} \{\widehat{f}_2, \widehat{h}\} \in \chi,\ \Gamma_1 \widehat{f}_1 + G^* \Gamma_0 \widehat{f}_1 = 0, \\ \Gamma_1 \widehat{f}_1 + h' = \Gamma_0 \widehat{f}_1 - h = 0 \end{array} \right\} \tag{6.20}$$

corresponding to the boundary relation

$$\Gamma^G = \left\{ \left\{ \widehat{f}_1 \oplus \widehat{f}_2, \begin{pmatrix} -\Gamma_0 \widehat{f}_1 + h \\ -G^* \Gamma_0 \widehat{f}_1 + h' \end{pmatrix} \right\} : \begin{array}{c} \widehat{f}_1 \in A^*,\ \{\widehat{f}_2, \widehat{h}\} \in \chi, \\ \Gamma_1 \widehat{f}_1 + h' = G(h - \Gamma_0 \widehat{f}_1) \end{array} \right\}. \tag{6.21}$$

The necessity of the condition (6.19) follows immediately from (6.10) and (6.11) in Theorem 6.8. To prove the sufficiency part, it is first shown that

$$\operatorname{mul} \widetilde{A}_{-G^*} = \{0\} \qquad \Longrightarrow \qquad \operatorname{mul} H_G = \{0\}. \tag{6.22}$$

Indeed, if $\widehat{f} = \widehat{f}_1 \oplus \widehat{f}_2 \in H_T$ and $\widehat{f}_i = \{0, f_i'\}$, $i = 1, 2$, then (6.20) implies that $\Gamma_1 \widehat{f}_1 + G^* \Gamma_0 \widehat{f}_1 = 0$, and hence $\widehat{f}_1 \in \widetilde{A}_{-G^*}$. Since $\operatorname{mul} \widetilde{A}_{-G^*} = \{0\}$, we have $\widehat{f}_1 = 0$. Now it follows from (6.20) that $\{\widehat{f}_2, 0\} \in \chi$. Thus, $\widehat{f}_2 \in S_2 = \ker \chi$, and hence $\widehat{f}_2 = 0$, since S_2 is an operator. This proves (6.22).

Note now that $\widetilde{A}$ $(= \widetilde{A}_{-\tau(\lambda)})$ in (5.8) coincides with $\ker \Gamma_0^G$. Therefore, $M_G(\lambda)$ is a Weyl function of the pair $(H_G, \widetilde{A})$. Hence, if $\widetilde{A}_{-G^*}$, and thus H_G is an operator, then condition (6.19) implies that $\widetilde{A}$ is an operator by Proposition 3.17.

Note that, if the induced boundary relation χ in Theorem 5.3 is single-valued, then the linear relation H_G in (6.20) becomes

$$H_G = \left\{ \widehat{f}_1 \oplus \widehat{f}_2 \in A^* \oplus T_2 : \Gamma_0 \widehat{f}_1 - \chi_0 \widehat{f}_2 = \Gamma_1 \widehat{f}_1 + \chi_1 \widehat{f}_2 = \Gamma_1 \widehat{f}_1 + G^* \Gamma_0 \widehat{f}_1 = 0 \right\}. \tag{6.23}$$

6.3. *Langer–Textorius Criterion*

We present here a new explicit proof for the Langer–Textorius criterion in [40] by applying the coupling method.

Following [40], introduce the operator function Q^{τ}_{LT} with values in $[\mathcal{H}]$ by

$$Q^{\tau}_{\mathrm{LT}}(\lambda; z_0) := M(\lambda) - (M(\lambda) - M(z_0)^*)(M(\lambda) + \tau(\lambda))^{-1}(M(\lambda) - M(z_0)). \tag{6.24}$$

It is noted in [40] that there is a symmetric restriction, say, H_{LT}, of $A^{(\tau)}$ such that the function $Q^{\tau}_{\mathrm{LT}}(\lambda; z_0)$ is a Q-function of the pair $(H_{\mathrm{LT}}, A^{(\tau)})$; here $A^{(\tau)}$ is a minimal selfadjoint exit space extension of A corresponding to $\tau(\lambda)$ in (6.5), i.e., one can take $A^{(\tau)} = \widetilde{A}_{-\tau(\lambda)} = \widetilde{A}$ as in (5.8). In the next proposition, the symmetric restriction H_{LT} is expressed in explicit terms. This enables us to derive the Langer–Textorius criterion from Proposition 6.13.

Proposition 6.14. *Let the assumptions be as in Proposition* 6.13, *and let* $z_0 \in \mathbb{C}_+$ *be chosen. Then*

(i) *the linear relation* $H_{\mathrm{LT}} := H_{-M(z_0)}$ *defined by* (6.20) *is a closed symmetric operator on* $\mathfrak{H}_1 \oplus \mathfrak{H}_2$;

(ii) *a linear relation* $\Gamma^{\mathrm{LT}} := \Gamma^{-M(z_0)}$ *defined by* (6.21) *is a boundary relation for* H^*_{LT};

(iii) *the Weyl function corresponding to* Γ^{LT} *is given by*

$$M_{\mathrm{LT}}(\lambda) = Q^{\tau}_{\mathrm{LT}}(\lambda; z_0) - 2\,\mathrm{Re}\,M(z_0); \tag{6.25}$$

(iv) $\tau(\lambda)$ *is admissible if and only if*

$$s - \lim_{y\uparrow\infty} M_{\mathrm{LT}}(iy)/y = 0. \tag{6.26}$$

Proof. The linear relation H_{LT} coincides with H_G in (6.20) for $G = -M(z_0)$. As was shown in Proposition 6.13, H_{LT} is a closed symmetric relation on $\mathfrak{H}_1 \oplus \mathfrak{H}_2$. Moreover, the linear relation $\widetilde{A}_{M(z_0)^*}$ becomes $\widetilde{A}_{M(z_0)^*} = A \widehat{+} \widehat{\mathfrak{N}}_{\bar z_0}$. Since $\mathfrak{N}_{\bar z_0} \cap \operatorname{dom} A = \{0\}$, $\widetilde{A}_{M(z_0)^*}$ is an operator. Now it follows from (6.22) that H_{LT} is an operator, and criterion (iv) is obtained from Proposition 6.13. Equation (6.25) is derived from (5.25), (6.18), and (6.24).

The functions $Q^{\tau}_{\mathrm{LT}}(\lambda; z_0)$ and $M_{\mathrm{LT}}(\lambda)$ are related by (6.25), and thus Proposition 6.14 (iv) yields the following theorem in [40].

Theorem 6.15 [40]. *Let* $z_0 \in \mathbb{C}_+$. *Then the minimal selfadjoint extension* $A^{(\tau)}$ *of* A *in the Kreĭn's formula* (6.5) *is an operator if and only if*

$$\lim_{y\uparrow\infty} (Q^{\tau}_{\mathrm{LT}}(iy; z_0)h, h)/y = 0, \qquad h \in \mathcal{H}. \tag{6.27}$$

Remark 6.16. If $M \in R^u[\mathcal{H}]$, then it follows from Proposition 6.14 (Theorem 6.15) that condition (6.26) (condition (6.27)) follows from conditions (3.28) and (3.29). Again, no direct (analytic) proof of this fact is known to us; cf. Remark 6.9.

6.4. *Langer–Textorius Problem*

As mentioned above, Langer and Textorius [40] showed that a function $Q^{\tau}_{\mathrm{LT}}(\lambda; z_0)$ of the form (6.24) is a Q-function of a pair $(H_{\mathrm{LT}}, A^{(\tau)})$ of some symmetric operator H_{LT}; the operator was not described in [40]. In addition, they posed the question whether or not the operator H_{LT} is simple. The problem is studied in this subsection by using the above approach to the proof of Theorem 6.15. This yields an affirmative answer to their question if we assume in addition that $A^{(\tau)}$ has a pure point spectrum and a negative answer in a more general situation, which is shown by (counter-)examples. We stress that the considerations below substantially depend on the explicit form (6.20) of the symmetric operator H_{LT}, see Proposition 6.14.

Proposition 6.17. *Suppose that the symmetric operators* A *and* S_2 *introduced above are simple and* G *is a bounded dissipative operator on* $\mathcal{H}$ *with* $\ker(G - G^*) = \{0\}$. *Then*

(i) *the operator* H_G *given by* (6.20) *has no eigenvalues*;

(ii) *if* $\widetilde{A}$ *has pure point spectrum,* $\sigma(\widetilde{A}) = \sigma_{pp}(\widetilde{A})$, *then* H_G *is simple*;

(iii) *if the multiplicity* $N(\widetilde{A})$ *of the spectrum of* $\widetilde{A}$ *exceeds* $n_{\pm}(A) = n < \infty$, $N(\widetilde{A}) > n$, *then* H_G *is not simple.*

Proof. (i) Assume for simplicity that χ can be decomposed in the form $\chi = (\chi_0, \chi_1)$. Then it readily follows from (6.20) that

$$(\Gamma_1 + G^*\Gamma_0)\widehat{f}_1 = 0, \qquad (\chi_1 - G^*\chi_0)\widehat{f}_2 = 0, \qquad G^*\Gamma_0\widehat{f}_1 - \chi_1\widehat{f}_2 = 0 \tag{6.28}$$

for any $\widehat{f}_1 \oplus \widehat{f}_2 \in H_G$. Note that, if $\ker G^* = \{0\}$, then this system is equivalent to the system (6.20) determining the operator H_G.

Suppose that $\lambda = \bar{\lambda}$ is an eigenvalue of H_G and that $f_\lambda = f_{1\lambda} \oplus f_{2\lambda}$ $(\neq 0)$ is the corresponding eigenfunction, i.e., $H_G f_\lambda = \lambda f_\lambda$. It follows that equations (6.28) are satisfied with $\widehat{f}_{1\lambda} = \{f_{1\lambda}, \lambda f_{1\lambda}\}$ and $\widehat{f}_{2\lambda} = \{f_{2\lambda}, \lambda f_{2\lambda}\}$ instead of $\widehat{f}_1$ and $\widehat{f}_2$, respectively, i.e.,

$$(\Gamma_1 + G^*\Gamma_0)\widehat{f}_{1\lambda} = 0, \qquad (\chi_1 - G^*\chi_0)\widehat{f}_{2\lambda} = 0. \tag{6.29}$$

Further, $A_{-G^*} = A^* \restriction \ker(\Gamma_1 + G^*\Gamma_0)$ is a maximal dissipative operator, because so is $-G^*$. Therefore, in addition to the equation $A_{-G^*} f_{1\lambda} = \lambda f_{1\lambda}$, one has $A_{-G} f_{1\lambda} = (A_{-G^*})^* f_{1\lambda} = \lambda f_{1\lambda}$. The latter relation is equivalent to $(\Gamma_1 + G\Gamma_0)\widehat{f}_{1\lambda} = 0$. Combining this with the first equation in (6.29) gives $(G - G^*)\Gamma_0\widehat{f}_{1\lambda} = 0$. Since $\ker(G - G^*) = \{0\}$, one has $\Gamma_0\widehat{f}_{1\lambda} = 0$. Combining this relation with (6.29) yields $\Gamma_1\widehat{f}_{1\lambda} = \Gamma_0\widehat{f}_{1\lambda} = 0$, $\widehat{f}_{1\lambda} \in A^*$. Thus, $\widehat{f}_{1\lambda} \in A$ and, since A is simple, it follows that $\widehat{f}_{1\lambda} = 0$.

Similarly, starting with the second equation in (6.29) and using the simplicity of S_2 gives $\widehat{f}_{2\lambda} = 0$. Hence, $f_\lambda = f_{1\lambda} \oplus f_{2\lambda} = 0$, which contradicts the assumption.

(ii) If the operator H_G is not simple, then its point spectrum is nonempty since $\sigma(\widetilde{A})$ is a purely point spectrum. This contradicts (i), i.e., H_G is simple.

(iii) By (6.20), H_G is the restriction of the "coupling" $\widetilde{A} = \widetilde{A}^*$ given by $H_G = \widetilde{A} \restriction \ker(\Gamma_1 + G^*\Gamma_0)$. Therefore, $n_\pm(H_G) = n_\pm(A) = n$. If H_G is simple, then the multiplicity of the spectrum of every selfadjoint extension of H_G does not exceed n (see [1]). Therefore, H_G is not simple.

Corollary 6.18. *Let the conditions of Proposition* 6.14 *be satisfied, and let* $H_{\mathrm{LT}} := H_{-M(z_0)}$ *be the symmetric operator in* (6.20) *with* $G = -M(z_0)$. *If* $\widetilde{A} = A^{(\tau)}$ *has a pure point spectrum, then the operator* H_{LT} *is simple.*

Moreover, if $n_\pm(A) = n < \infty$ *and* $N(\widetilde{A}) > n$, *then* H_{LT} *is not simple.*

Proof. Note that $M(z_0)$ is a bounded dissipative operator and $\ker\big(\operatorname{Im} M(z_0)\big) = \{0\}$; in fact, $0 \in \rho\big(\operatorname{Im} M(z_0)\big)$. Thus, it suffices to apply Proposition 6.17 with $G = -M(z_0)$. The last statement follows from Proposition 6.17 (iii).

Example 6.19. Let $A = S_1$ and S_2 be as in Example 5.9, and let H_{LT} be a symmetric operator defined by (6.23). Then the operator H_{LT} is simple for every $z_0 \in \mathbb{C}_+$. This is immediate from Proposition 6.17 since both A and S_2 are simple. However, this can readily be derived from the definition of H_{LT} by using the form of the Weyl function (see [20])) corresponding to the boundary triplet $\Pi_A = \{\mathbb{C}^2, \Gamma_0, \Gamma_1\}$ for A^* defined in (5.18).

The next example shows that the operator H_{LT} is not simple in general; furthermore, the multiplicity $N(\widetilde{A})$ of the spectrum of $\widetilde{A}$ exceeds $n_\pm(A) = n < \infty$; cf. Proposition 6.17.

Example 6.20. Let the symmetric operator S_j on $\mathfrak{H}_j = L_2(\mathbb{R})$ be defined by

$$S_j = \frac{1}{i}\frac{d}{dx}, \qquad \operatorname{dom} S_2 = \{\, f_j \in W_2^1(\mathbb{R}) : \ f_j(0) = 0 \,\}, \qquad j = 1, 2, \tag{6.30}$$

and let $A = S_1$. Then S_j is densely defined, it has the defect numbers $(1, 1)$, and the adjoint S_j^* is given by the same differential expression on the domain $\operatorname{dom} S_j^* = W_2^1(\mathbb{R}_+) \oplus W_2^1(\mathbb{R}_-)$; $j = 1, 2$. Moreover, by setting

$$\Gamma_0 f_1 = \frac{1}{\sqrt{2}}[f_1(0+) - f_1(0-)], \qquad \Gamma_1 f_1 = \frac{i}{\sqrt{2}}[f_1(0+) + f_1(0-)], \qquad f_1 \in \operatorname{dom} S_1^*,$$

one obtains a boundary triplet $\Pi_A = \{\mathbb{C}, \Gamma_0, \Gamma_1\}$ for $A^* = S_1^*$, and the corresponding Weyl function is given by $M(\lambda) = \pm i,\ \lambda \in \mathbb{C}_\pm$; cf. [10]. Similarly, associate a boundary triplet $\Pi_{S_2} = \{\mathbb{C}, \chi_0, \chi_1\}$ for S_2^* by setting

$$\chi_0 f_2 = \frac{1}{\sqrt{2}}[f_2(0+) + f_2(0-)], \qquad \chi_1 f_2 = \frac{i}{\sqrt{2}}[f_2(0+) - f_2(0-)], \qquad f_2 \in \operatorname{dom} S_2^*.$$

Now the corresponding Weyl function is given by $\tau(\lambda) = -M(\lambda)^{-1} = M(\lambda) = \pm i$, $\lambda \in \mathbb{C}_\pm$. It can readily be seen that the symmetric operator $H_{\mathrm{LT}} = H_{-M(z_0)}$ in (6.23) with $G = -M(z_0) = -i$ and $z_0 \in \mathbb{C}_+$ is determined by the boundary conditions

$$f_1(0+) = f_2(0-) = 0, \qquad f_1(0-) = -f_2(0+), \tag{6.31}$$

whereas $\widetilde{A}_{-\tau(\lambda)} = \ker \Gamma_0^G$ (see (6.21)) is determined by the boundary conditions

$$f_1(0+) = f_2(0-), \qquad f_1(0-) = -f_2(0+). \tag{6.32}$$

The boundary conditions in (6.31) mean that $H_{\mathrm{LT}} = \widetilde{S}_1 \oplus \widetilde{A}_2$, where the symmetric operator $\widetilde{S}_1$ and the selfadjoint operator $\widetilde{A}_2$ are determined by the same differential expression $-iD$ on the domains

$$\operatorname{dom} \widetilde{S}_1 = \left\{ f_{1,+} \oplus f_{2,-} \in W_2^1(\mathbb{R}) :\ f_1(0+) = f_2(0-) = 0 \right\},$$
$$\operatorname{dom} \widetilde{A}_2 = \left\{ f_{1,-} \oplus f_{2,+} \in W_2^1(\mathbb{R}) :\ f_1(0-) = -f_2(0+) \right\},$$

respectively; cf. [15, Ex. 6.7]. Similarly, (6.32) shows that the coupling $\widetilde{A} = \widetilde{A}_{-\tau(\lambda)}$ in Corollary 6.18 is of the form

$$\widetilde{A} = \widetilde{A}_1 \oplus \widetilde{A}_2, \quad \text{where} \quad \operatorname{dom} \widetilde{A}_1 = \left\{ f_{1,+} \oplus f_{2,-} \in W_2^1(\mathbb{R}) :\ f_1(0+) = f_2(0-) \right\}.$$

Therefore, H_{LT} has the defect numbers $(1,1)$ and is not simple. The multiplicity $N(\widetilde{A})$ of the spectrum of $\widetilde{A}$ is now clearly equal to 2. Note that the selfadjoint operators $\widetilde{A}_1$ and $\widetilde{A}_2$ can be identified with each other (they are also unitarily equivalent to the selfadjoint extensions $A_0 = \ker \Gamma_0$ and $A_1 = \ker \Gamma_1$) and, moreover, they are absolutely continuous with $\sigma_{\mathrm{ac}}(\widetilde{A}_j) = \mathbb{R}$, $j = 1, 2$. Finally, observe that here $M_{\mathrm{LT}}(\lambda) = Q_{\mathrm{LT}}^\tau(\lambda; z_0) = M(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$; see (6.25).

7. CHARACTERIZATION OF NAĬMARK EXTENSIONS USING THE SPECTRAL PARAMETER IN THE KREĬN–NAĬMARK FORMULA

In this section, selfadjoint exit space extensions of a densely defined symmetric operator A on $\mathfrak{H}$ are studied. The main result gives a description of the set of all minimal exit space extensions $\widetilde{A}$ of A such that $\operatorname{dom} \widetilde{A} \cap \mathfrak{H} = \operatorname{dom} \widetilde{A}$ in terms of the spectral parameter $\tau(\cdot)$ corresponding to $\widetilde{A}$ in the Kreĭn–Naĭmark formula (6.5).

7.1. *Definition of Naĭmark Extensions*

Let A be a closed densely defined symmetric operator on a Hilbert space $\mathfrak{H}$ with arbitrary defect numbers, and let $\widetilde{A}$ be a selfadjoint extension of A acting on the Hilbert space $\widetilde{\mathfrak{H}} \supset \mathfrak{H}$. Assign to $\widetilde{A}$ the relations S_j and T_j, $j = 1, 2$, by $S_j = \widetilde{A} \cap \mathfrak{H}_j^2$ and by (5.1) with $\mathfrak{H}_1 = \mathfrak{H} \times \{0\}$ and $\mathfrak{H}_2 = \widetilde{\mathfrak{H}} \ominus \mathfrak{H}$. Then $A \subset S_1 = T_1^*$ and $T_1 \subset S_1^* \subset A^*$. Since A is densely defined, the relations T_1, S_1^*, and A^* are operators on $\mathfrak{H}$, and $\operatorname{mul} \widetilde{A} = \{0\} \times \operatorname{mul} S_2$. If the selfadjoint extension $\widetilde{A}$ is minimal, then $\operatorname{mul} \widetilde{A} = \{0\}$. Recall the Naĭmark classification of the minimal exit space extensions $\widetilde{A}$ of A.

Definition 7.1 [1, 43]. Let $A\,(\neq A^*)$ be a closed densely defined symmetric operator on a Hilbert space $\mathfrak{H}$ with arbitrary defect numbers, and let $\widetilde{A}$ be a minimal selfadjoint extension of A on a Hilbert space $\widetilde{\mathfrak{H}}$ $(\supset \mathfrak{H})$. Then $\widetilde{A}$ is said to be an extension

(i) of the first kind if $\operatorname{dom} \widetilde{A} \cap \mathfrak{H} = \operatorname{dom} \widetilde{A}$;

(ii) of the second kind if $\operatorname{dom} \widetilde{A} \cap \mathfrak{H} = \operatorname{dom} A$;

(iii) of the third kind if $\operatorname{dom} A \subsetneqq \operatorname{dom} \widetilde{A} \cap \mathfrak{H} \subsetneqq \operatorname{dom} \widetilde{A}$.

The set of extensions of A of the first, second, and third kind is denoted by $\mathrm{Nai}_1(A)$, $\mathrm{Nai}_2(A)$, and $\mathrm{Nai}_3(A)$, respectively.

Note that a selfadjoint extension $\widetilde{A}$ of the first kind is a canonical extension, i.e., $\widetilde{A} \subset \mathfrak{H} \times \mathfrak{H}$, due to the minimality of the extension $\widetilde{A}$. Let $\widetilde{A}$ be a minimal selfadjoint extension of A. We have $\operatorname{dom} \widetilde{A} \cap \mathfrak{H} \subset \operatorname{dom} A^*$. Define the restriction N_1 by

$$N_1 := A^* \upharpoonright (\operatorname{dom} \widetilde{A} \cap \mathfrak{H}). \tag{7.1}$$

Proposition 7.2. *Let A be a closed densely defined symmetric operator on $\mathfrak{H}$ with arbitrary defect numbers, let $\widetilde{A}$ be a minimal selfadjoint extension of A acting on $\widetilde{\mathfrak{H}}$, and let N_1 be defined by* (7.1)*. Then N_1 is a densely defined symmetric operator on $\mathfrak{H}$ given by $N_1 = P_{\mathfrak{H}} \widetilde{A} \upharpoonright \mathfrak{H}$ such that*

$$A \subset S_1 \subset N_1 \subset T_1. \tag{7.2}$$

Further, if Δ is the main transform of $\widetilde{A}$ defined by (5.5)*, then* $\ker \Delta_0 = N_1$*. Moreover,*

(i) $\widetilde{A} \in \mathrm{Nai}_1(A) \neq \varnothing$ *if and only if* $N_1 = \widetilde{A}$;
(ii) $\widetilde{A} \in \mathrm{Nai}_2(A) \neq \varnothing$ *if and only if* $N_1 = A$;
(iii) $\widetilde{A} \in \mathrm{Nai}_3(A) \neq \varnothing$ *if and only if* $N_1 \neq \widetilde{A}$ *and* $N_1 \neq A$.

In particular, if A has the defect numbers $(1,1)$*, then $\widetilde{A} \in \mathrm{Nai}_1(A)$ ($\widetilde{A} \in \mathrm{Nai}_2(A)$, $\widetilde{A} \in \mathrm{Nai}_3(A)$) if and only if $A \neq S_1$ ($A = S_1 = N_1$, $A = S_1 \neq N_1$, respectively).*

Proof. As a minimal selfadjoint extension of A, $\widetilde{A}$ (and thus $P_{\mathfrak{H}} \widetilde{A} \upharpoonright \mathfrak{H}$ as well) is an operator. Clearly, $\operatorname{dom} P_{\mathfrak{H}} \widetilde{A} \upharpoonright \mathfrak{H} = \operatorname{dom} \widetilde{A} \cap \mathfrak{H}$ and $A \subset P_{\mathfrak{H}} \widetilde{A} \upharpoonright \mathfrak{H}$, and hence $P_{\mathfrak{H}} \widetilde{A} \upharpoonright \mathfrak{H}$ is a densely defined operator on $\mathfrak{H}$. Since $\widetilde{A}$ is selfadjoint, $\Delta = \mathcal{J}^{-1}(\widetilde{A})\colon \mathfrak{H}_1^2 \to \mathfrak{H}_2^2$ is a unitary relation with

$$S_1 = \ker \Delta = \ker \Delta_0 \cap \ker \Delta_1,$$

and we clearly have $P_{\mathfrak{H}} \widetilde{A} \upharpoonright \mathfrak{H} = \ker \Delta_0$, where Δ_0 and Δ_1 are defined in (3.9). Therefore, $P_{\mathfrak{H}} \widetilde{A} \upharpoonright \mathfrak{H}$ is symmetric and $A \subset P_{\mathfrak{H}} \widetilde{A} \upharpoonright \mathfrak{H} \subset (P_{\mathfrak{H}} \widetilde{A} \upharpoonright \mathfrak{H})^* \subset A^*$. Thus, $P_{\mathfrak{H}} \widetilde{A} \upharpoonright \mathfrak{H} = A^* \upharpoonright (\operatorname{dom} \widetilde{A} \cap \mathfrak{H}) = N_1$. Moreover, since $T_1 = \operatorname{dom} \Delta$, we obtain (7.2).

Assertions (i)–(iii) are clear from (7.1) and Definition 7.1. In particular, let the defect numbers of A be $(1,1)$. If $A \neq S_1$, then S_1 (and therefore, S_2 as well) is selfadjoint, which gives $\widetilde{\mathfrak{H}} = \mathfrak{H}$ (by the simplicity of S_2), i.e., $\widetilde{A} \in \mathrm{Nai}_1(A)$. Conversely, if $\widetilde{A} \in \mathrm{Nai}_1(A)$, then we clearly have $S_1 = N_1 = \widetilde{A}$ and $A \neq S_1$ (since $A \neq A^*$ by definition). The assertions for $\widetilde{A} \in \mathrm{Nai}_j(A)$, $j = 2, 3$, are now obvious.

Proposition 7.3. *Let A be a densely defined symmetric operator on a Hilbert space $\mathfrak{H}$ with arbitrary defect numbers. Then*

(i) $\mathrm{Nai}_1(A) \neq \varnothing$ *if and only if A has equal defect numbers*;
(ii) $\mathrm{Nai}_2(A) \neq \varnothing$;
(iii) $\mathrm{Nai}_3(A) \neq \varnothing$ *if and only if A has positive defect numbers.*

In particular, if A has positive defect numbers, then A has extensions of the second and third kind, whereas, if A has defect numbers $(n,0)$ or $(0,n)$, $n > 0$, then all minimal selfadjoint extensions of A are of the second kind.

Proof. (i) A first-kind extension $\widetilde{A} \in \mathrm{Nai}_1(A)$ is simply a canonical extension of A acting on the original space, and thus $\widetilde{\mathfrak{H}} = \mathfrak{H}$ by the minimality of $\widetilde{A}$.

(ii) The relation $\mathrm{Nai}_2(A) \neq \varnothing$ was proved for every densely defined symmetric operator A by Naĭmark in [43].

(iii) Let $\widetilde{A}$ be a minimal selfadjoint exit space extension of A. If the defect numbers of A are $(n,0)$ or $(0,n)$, $n > 0$, then A is a maximal symmetric operator, and hence $N_1 = A$ (because $N_1 \supset A$ is symmetric by Proposition 7.2). Therefore, in this case, $\widetilde{A} \in \mathrm{Nai}_2(A)$.

Conversely, if both the defect numbers of A are positive, then A admits a proper (even maximal) symmetric extension H acting on the original space $\mathfrak{H}$. If H is not selfadjoint, then every minimal exit space extension of H belongs to $\mathrm{Nai}_3(A)$. It remains to consider the case in which $n_\pm(A) = 1$, and thus every proper symmetric extension H of A on $\mathfrak{H}$ is selfadjoint. Let $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ be an ordinary boundary triplet for A^*. Then the main transform $\widetilde{A} = \mathcal{J}(\Gamma)$ of Γ (see (2.12)) is the selfadjoint relation $\widetilde{A}$ given by

$$\widetilde{A} = \mathcal{J}(\Gamma) = \left\{ \left\{ \begin{pmatrix} f \\ \Gamma_0 \widehat{f} \end{pmatrix}, \begin{pmatrix} f' \\ -\Gamma_1 \widehat{f} \end{pmatrix} \right\} : \widehat{f} \in A^* \right\}. \tag{7.3}$$

Clearly, $\widetilde{A}$ is a minimal exit space extension of A acting on $\mathfrak{H} \oplus \mathbb{C}$. Moreover, $S_1 = \widetilde{A} \cap \mathfrak{H}^2 = A$, and $N_1 = \ker \Gamma_0 \supset A$ is a selfadjoint extension of A acting on $\mathfrak{H}$. By Proposition 7.2, this means that $\widetilde{A} \in \mathrm{Nai}_3(A)$.

7.2. *Geometric Characterizations of Naĭmark Extensions*

The exit space extensions of the second kind, and thus also of the third kind, can be characterized in terms of the operators S_1 and T_2 defined in (5.1).

Theorem 7.4. *Let A be a densely defined symmetric operator on $\mathfrak{H}$ with arbitrary defect numbers, and let $\widetilde{A}$ be a minimal selfadjoint extension of A. Then*

$$\widetilde{A} \in \mathrm{Nai}_2(A) \qquad \Longleftrightarrow \qquad S_1 = A \quad \text{and} \quad \mathrm{mul}\, T_2 = \{0\}. \tag{7.4}$$

In particular, if $\widetilde{A} \in \mathrm{Nai}_2(A)$, then $n_\pm(S_2) = n_\mp(A)$.

Proof. Let S_1 and T_2 be defined by (5.1). Then

$$\mathrm{mul}\, T_2 = \left\{ h' \in \mathfrak{H}_2 : \left\{ \binom{f}{0}, \binom{f'}{h'} \right\} \in \widetilde{A} \right\}. \tag{7.5}$$

Comparing this with the description of $N_1 = P_{\mathfrak{H}} \widetilde{A} \upharpoonright \mathfrak{H}$ in Proposition 7.2, one concludes that

$$S_1 = N_1 \qquad \Longleftrightarrow \qquad \mathrm{mul}\, T_2 = \{0\}. \tag{7.6}$$

The equivalence (7.4) follows now from part (ii) of Proposition 7.2.

If $\widetilde{A} \in \mathrm{Nai}_2(A)$, then $S_1 = A$ by (7.4), and $n_\pm(S_2) = n_\mp(A)$ by Lemma 5.1.

Let A be a symmetric operator on $\mathfrak{H}$. An exit space extension $\widetilde{A} = \widetilde{A}^*$ of A acting on $\widetilde{\mathfrak{H}}$ $(\supset \mathfrak{H})$ is said to be *finite-dimensional* if $\dim(\widetilde{\mathfrak{H}} \ominus \mathfrak{H}) < \infty$. It is clear that A has finite-dimensional selfadjoint extensions if and only if $n_+(A) = n_-(A)$. The next proposition shows that all minimal finite-dimensional extensions of A either are canonical (i.e., $\widetilde{A} \in \mathrm{Nai}_1(A)$) or belong to $\mathrm{Nai}_3(A)$.

Proposition 7.5. *Let A be a densely defined symmetric operator on $\mathfrak{H}$ with arbitrary defect numbers, and let $\widetilde{A}$ be a minimal selfadjoint extension of A. Assume that T_1 is closed. Then*

$$\widetilde{A} \in \mathrm{Nai}_2(A) \qquad \Longleftrightarrow \qquad S_1 = A \quad \text{and} \quad \overline{\mathrm{dom}}\, S_2 = \mathfrak{H}_2. \tag{7.7}$$

Further, if $\widetilde{A} \in \mathrm{Nai}_2(A)$, then S_2 is an unbounded densely defined operator on $\mathfrak{H}_2$. In particular, the set $\mathrm{Nai}_2(A)$ contains no finite-dimensional extensions of A.

Proof. By Lemma 5.1 (ii), T_1 and T_2 are closed or not closed simultaneously. Hence T_2 is closed, and $T_2 = S_2^*$. In this case, $\mathrm{mul}\, T_2 = \{0\}$ if and only if $\overline{\mathrm{dom}}\, S_2 = \mathfrak{H}_2$. Therefore, the equivalence in (7.7) follows from Theorem 7.4.

If $\widetilde{A} \in \mathrm{Nai}_2(A)$, then S_2 is a densely defined operator on $\mathfrak{H}_2$. Further, S_2 is unbounded. Otherwise $\mathrm{dom}\, S_2 = \mathfrak{H}_2$, since S_2 is closed, and this implies that S_2 is selfadjoint. This leads to a contradiction because $n_\pm(S_2) = n_\mp(A)$ by Theorem 7.4.

If $\widetilde{A}$ is a minimal exit space extension of A and $\dim \mathfrak{H}_2 < \infty$, then S_2 is a bounded nondensely defined operator. Hence, in this case, $\widetilde{A} \in \mathrm{Nai}_3(A)$.

If A has finite defect numbers, then the Naĭmark extensions of the second kind can be characterized by means of the operator S_2 by itself.

Corollary 7.6. *Let A be a densely defined symmetric operator on $\mathfrak{H}$ with defect numbers $n_\pm(A) < \infty$ and let $\widetilde{A}$ be a minimal selfadjoint extension of A. Then*

$$\widetilde{A} \in \mathrm{Nai}_2(A) \qquad \Longleftrightarrow \qquad \overline{\mathrm{dom}}\, S_2 = \mathfrak{H}_2 \quad \text{and} \quad n_\pm(S_2) = n_\mp(A). \tag{7.8}$$

Proof. Since $n_\pm(A) < \infty$, T_1 is closed as a finite-dimensional extension of A. Hence, the implication $\Rightarrow$ follows from Theorem 7.4 and Proposition 7.5.

Conversely, assume that $\overline{\operatorname{dom} S_2} = \mathfrak{H}_2$ and $n_\pm(S_2) = n_\mp(A) < \infty$. Then T_2 is closed, and $T_2 = S_2^*$ is an operator, since $\overline{\operatorname{dom} S_2} = \mathfrak{H}_2$. Further, $n_\pm(S_1) = n_\pm(A) < \infty$ by Lemma 5.1 and, since $S_1 \supset A$, this yields $A = S_1$. Therefore, $\widetilde{A} \in \operatorname{Nai}_2(A)$ by Theorem 7.4.

Remark 7.7. (a) If $n_\pm(A) = \infty$, then the relations $\overline{\operatorname{dom} S_2} = \mathfrak{H}_2$ and $n_\pm(S_2) = \infty$ do not imply the inclusion $\widetilde{A} \in \operatorname{Nai}_2(A)$, even if T_1 is closed.

To see this, consider two symmetric operators A' and A'' on $\mathfrak{H}_1'$ and $\mathfrak{H}_1''$, respectively, with $n_\pm(A') = \infty$ and $n_\pm(A'') = n < \infty$. Let $\widetilde{A}' \in \operatorname{Nai}_2(A')$ and $\widetilde{A}'' \in \operatorname{Nai}_3(A'')$ be their selfadjoint extensions acting on $\mathfrak{H}' = \mathfrak{H}_1' \oplus \mathfrak{H}_2'$ and $\mathfrak{H}'' = \mathfrak{H}_1'' \oplus \mathfrak{H}_2''$, respectively. In addition, assume that T_1' is closed. Let $\overline{\operatorname{dom} S_2''} = \mathfrak{H}_2''$. Then, by Proposition 7.5, $\overline{\operatorname{dom} S_2'} = \mathfrak{H}_2$, and $S_2 = S_2' \oplus S_2''$ acts on $\mathfrak{H}_2 = \mathfrak{H}_2' \oplus \mathfrak{H}_2''$ and is densely defined. Moreover, $n_\pm(S_2) = n_\pm(S_2') + n_\pm(S_2'') = \infty$. On the other hand, $\widetilde{A} = \widetilde{A}' \oplus \widetilde{A}'' \in \operatorname{Nai}_3(A)$, where $A = A' \oplus A''$.

(b) Moreover, if $n_\pm(A) = \infty$ and T_1 is not closed, then $\operatorname{dom} S_2$ need not be dense in $\mathfrak{H}_2$, although $\widetilde{A} \in \operatorname{Nai}_2(A)$.

7.3. *Examples of Naĭmark Extensions*

This subsection contains some illustrative examples concerning the Naĭmark extensions of several differential operators.

Example 7.8. Let A be the operator on $L_2(0,1)$ generated by the differential expression $-D^2$ as defined in Example 5.8. The boundary triplet defined by $\Gamma_0 f = f(0)$ and $\Gamma_1 f = f'(0)$, $f \in \operatorname{dom} A^*$, induces a one-dimensional exit space extension $\widetilde{A}$ of A as in (7.3), namely,

$$\widetilde{A} = \left\{ \left\{ \begin{pmatrix} f \\ f(0) \end{pmatrix}, \begin{pmatrix} -f'' \\ -f'(0) \end{pmatrix} \right\} : f \in W_2^2(0,1),\ f(1) = 0 \right\}. \tag{7.9}$$

Here $\operatorname{dom} \widetilde{A} \cap \mathfrak{H} = \left\{ f \in W_2^2(0,1) : f(0) = f(1) = 0 \right\} \supsetneqq \operatorname{dom} A$ and $\widetilde{A} \in \operatorname{Nai}_3(A)$.

Exit space extensions of the second kind naturally arise under restrictions of domains of differential operators or extensions involving *interface conditions*, as is shown by the following examples.

Example 7.9. Let S_2 be the symmetric operator in $L_2(-1,0)$, and let $\widetilde{A}$ be the selfadjoint operator (coupling) on $L_2(-1,1)$ generated by the differential expression $-D^2$ as defined in Example 5.8; cf. (5.16), (5.17). If $f \in \operatorname{dom} \widetilde{A} \cap L_2(0,1)$, then we clearly have $f \in AC[-1,1]$ and $f(x) = 0$ for all $-1 \leqslant x \leqslant 0$. Hence, in this case, $\operatorname{dom} \widetilde{A} \cap L_2(0,1) = \operatorname{dom} A$, and therefore $\widetilde{A} \in \operatorname{Nai}_2(A)$.

Example 7.10. Consider the differential expression

$$l(x,D) = \sum_{j=0}^{n} (-1)^j (d^j/dx^j) p_{n-j}(x) (d^j/dx^j) \tag{7.10}$$

in $\mathfrak{H} = L_2(\mathbb{R})$ with real coefficients satisfying the following integrability conditions:

$$p_0 = 1, \qquad p_n \in L_1(\mathbb{R}) \cap L_\infty(\mathbb{R}), \qquad p_{n-j} \in W_1^j(\mathbb{R}), \qquad j = 1, \ldots, n-1. \tag{7.11}$$

The operator $\widetilde{A}$ generated on $L_2(\mathbb{R})$ by the differential expression (7.10) with the domain $\operatorname{dom} \widetilde{A} = W_2^{2n}(\mathbb{R})$ is selfadjoint (see [44, Sec. 23, Th. 3]). The orthogonal decomposition $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$ with $\mathfrak{H}_1 := L_2(0,1)$ and $\mathfrak{H}_2 := L_2(\mathbb{R} \setminus [0,1])$ induces the operators S_j and T_j, $j = 1,2$, according to (5.1). Let $A = A_{\min}$ be the minimal (symmetric) operator generated on $\mathfrak{H}_1 = L_2(0,1)$ by the differential expression (7.10) on the domain $\operatorname{dom} A = \overset{\circ}{W}{}_2^{2n}(0,1)$. Since the Sobolev space $W_2^{2n}(0,1)$ can be described by $W_2^{2n}(0,1) = \left\{ f : f^{(j)} \in AC[0,1],\ j = 0,1,\ldots,2n-1,\ f^{(2n)} \in L_2(0,1) \right\}$, we readily conclude that

$$\operatorname{dom} N_1 = \operatorname{dom} \widetilde{A} \cap \mathfrak{H}_1 = \operatorname{dom} \widetilde{A} \cap L_2(0,1) = \overset{\circ}{W}{}_2^{2n}(0,1) = \operatorname{dom} A. \tag{7.12}$$

Thus, $A = S_1 = N_1$ and $\widetilde{A} \in \operatorname{Nai}_2(A)$ (by Proposition 7.2). Since

$$P_1(\operatorname{dom} \widetilde{A}) = P_1 W_2^{2n}(\mathbb{R}) = W_2^{2n}(0,1),$$

we obtain $T_1 = A^* = A_{\max}$, and thus T_1 is closed. Similarly, we can see that

$$\operatorname{dom} S_2 = \operatorname{dom} \widetilde{A} \cap \mathfrak{H}_2 = \overset{\circ}{W}{}_2^{2n}(-\infty, 0) \oplus \overset{\circ}{W}{}_2^{2n}(1, \infty), \qquad S_2 = S_2' \oplus S_2'',$$

where S_2' and S_2'' are the minimal operators generated by $l(x, D)$ on $L_2(-\infty, 0)$ and $L_2(1, \infty)$, respectively. In particular, S_2 is densely defined (cf. Proposition 7.5) and, moreover, $T_2 = S_2^* = {S_2'}^* \oplus {S_2''}^*$ with

$$\operatorname{dom} T_2 = P_2(\operatorname{dom} \widetilde{A}) = P_2\, W_2^{2n}(\mathbb{R}) = W_2^{2n}(-\infty, 0) \oplus W_2^{2n}(1, \infty).$$

Example 7.11. Consider the differential expression (7.10) on $\mathfrak{H} := L_2(0, \infty)$ with the same integrability conditions as in (7.11). Let $\widetilde{A}$ be the operator generated on $\mathfrak{H}$ by the differential expression (7.10) with the following Dirichlet boundary conditions at zero:

$$\operatorname{dom} \widetilde{A} = \bigl\{ f \in W_2^{2n}(\mathbb{R}_+) : f(0) = \cdots = f^{(n-1)}(0) = 0 \bigr\}.$$

The above assumptions on the coefficients p_j imply that the operator $\widetilde{A}$ is selfadjoint. The orthogonal decomposition $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$ with $\mathfrak{H}_1 := L_2(0, 1)$ and $\mathfrak{H}_2 := L_2(1, \infty)$ induces the operators S_j and T_j, $j = 1, 2$, according to (5.1). We have $S_1 = N_1$ again, see (7.6), and $\operatorname{dom} S_1 = \operatorname{dom} N_1 = \operatorname{dom} \widetilde{A} \cap L_2(0, 1)$ is given by

$$\bigl\{ f \in W_2^{2n}(0, 1) : f(0) = \cdots = f^{(n-1)}(0) = f(1) = \cdots = f^{(2n-1)}(1) = 0 \bigr\}.$$

Here $n_\pm(S_1) = n$ and $S_1 \supset A$ and $\dim S_1/A = n$ in view of (7.12), and $\widetilde{A} \in \mathrm{Nai}_3(A)$.

Example 7.12. Consider the differential operator in Example 1.4. We have $S_1 = N_1$ again, and $f \in \operatorname{dom} S_1 = \operatorname{dom} \widetilde{A} \cap L_2(\Omega)$ if and only if $f \in W_2^2(\mathbb{R}^n)$ and $f(x) = 0$ for $x \in \Omega_2$. On the other hand, since $\partial\Omega$ is smooth, the trace theorem (see [53]) implies that, for each $f \in W_2^2(\mathbb{R}^n)$, there exist traces

$$\bigl\{ \gamma_0 f := f \restriction \partial\Omega,\ \gamma_1 f := (\partial f/\partial n) \restriction \partial\Omega \bigr\} \in W_2^{3/2}(\partial\Omega) \times W^{1/2}(\partial\Omega).$$

If $f \in \operatorname{dom} S_1$, then $f \in W_2^2(\Omega_1)$ and $\gamma_0 f = \gamma_1 f = 0$ because $f(x) = 0$ for $x \in \Omega_2$. Thus, $f \in \overset{\circ}{W}{}_2^2(\Omega_1)$ (again by the trace theorem), and consequently

$$\operatorname{dom} S_1 = \operatorname{dom} \widetilde{A} \cap L_2(\Omega) = W_2^2(\mathbb{R}^n) \cap L_2(\Omega) = \overset{\circ}{W}{}_2^2(\Omega). \tag{7.13}$$

Combining this relation with $\operatorname{dom} A = \overset{\circ}{W}{}_2^2(\Omega)$, we see that $N_1 = S_1 = A$, and hence $\widetilde{A} \in \mathrm{Nai}_2(A)$. In contrast to the previous example, the operators T_j, $j = 1, 2$, are not closed here. As a conclusion, $\operatorname{dom} T_1 = W_2^2(\Omega)$ is a proper subset of $\operatorname{dom} A_{\max} = \operatorname{dom} S_1^*$, and the maximal operator $A_{\max}$ generated on $L_2(\Omega)$ by the differential expression (1.12) depends on the coefficients a_{ij}.

7.4. *Analytic Characterization of Naĭmark Extensions of the Second Kind*

In this subsection, exit space extensions of the second kind are characterized in terms of the spectral parameter $\tau(\cdot)$ arising in the Kreĭn–Naĭmark formula (6.5) for the generalized resolvents. We stress that the treatment of $\tau(\cdot)$ as the Weyl family (Weyl function) of the induced boundary relation χ constructed for S_2^* in Theorem 5.3 (see Proposition 5.6) plays a crucial role when establishing the relationship between geometric and analytic considerations. The treatment below is divided into two cases.

7.4.1. Case of bounded $\tau(\cdot) \in R[\mathcal{H}]$. Selfadjoint exit space extensions of the second kind generated by bounded spectral parameters $\tau(\cdot) \in R[\mathcal{H}]$ are characterized by the following theorem.

Theorem 7.13. *Let A be a closed densely defined symmetric operator on $\mathfrak{H}$ with equal defect numbers $n_\pm(A) \leqslant \infty$, and let $\widetilde{A} = \widetilde{A}_{-\tau(\lambda)}$ be a minimal exit space extension of A corresponding to $\tau(\lambda)$ in* (6.5). *Moreover, assume that $\tau(\cdot) \in R[\mathcal{H}]$. Then $\widetilde{A} \in \mathrm{Nai}_2(A)$ if and only if the following two conditions are satisfied*:

$$\lim_{y \uparrow \infty} y^{-1} \tau(iy) = 0 \quad \textit{and} \quad \lim_{y \uparrow \infty} y \,\mathrm{Im}\,(\tau(iy)h, h) = \infty, \qquad h \in \mathcal{H} \setminus \{0\}. \tag{7.14}$$

Proof. Let $\tau(\cdot) \in R[\mathcal{H}]$. Then, by Proposition 5.6, $\tau(\cdot)$ is the Weyl function of the induced boundary relation χ constructed for S_2^* in Theorem 5.3. Since $\tau(\cdot) \in R[\mathcal{H}]$, the boundary relation χ satisfies condition (3.26), and hence it has also properties (B1)–(B3); see Proposition 3.16.

Now assume that $\widetilde{A} = \widetilde{A}_{-\tau(\lambda)} \in \mathrm{Nai}_2(A)$. Then $S_1 = A$ and $\mathrm{mul}\, T_2 = \{0\}$ by Theorem 7.4. Proposition 5.11 shows that χ is single-valued. Further, applying part (ii) of Proposition 3.17 to χ, we see from $\mathrm{mul}\, T_2 = \{0\}$ that $\tau(\lambda)$ satisfies conditions (3.28) and (3.29) with $\mathcal{H}_0 = \mathrm{mul}\, \chi_0 = \{0\}$, i.e., (7.14) holds.

Conversely, assume that $\tau(\cdot) \in R[\mathcal{H}]$ satisfies the conditions in (7.14). The second condition in (7.14) implies that $\ker \mathrm{Im}\, \tau(iy) = \{0\}$. Hence, $\tau(\lambda) \in R[\mathcal{H}]$ is strict, and thus χ is single-valued. It follows from Proposition 5.11 that $S_1 = A$. According to Proposition 3.17, conditions (7.14) mean that $\mathrm{mul}\, T_2 = \{0\}$. Therefore, $\widetilde{A} = \widetilde{A}_{-\tau(\lambda)} \in \mathrm{Nai}_2(A)$ by Theorem 7.4.

The following analytic characterizations are related to Proposition 7.5 and Corollary 7.6.

Proposition 7.14. *Let A be a closed densely defined symmetric operator in $\mathfrak{H}$ with equal defect numbers $n_\pm(A) \leqslant \infty$, and let $\widetilde{A} = \widetilde{A}_{-\tau(\lambda)}$ be a minimal exit space extension of A corresponding to $\tau(\lambda)$ in* (6.5). *Then the following statements are equivalent*:

(i) $\widetilde{A} \in \mathrm{Nai}_2(A)$ *and T_1 is closed*;
(ii) $\tau(\cdot) \in R^u[\mathcal{H}]$ *and the conditions in* (7.14) *are satisfied.*

Moreover, if T_1 is closed (*in particular, if $n_\pm(A) = n < \infty$*), *then the following assertions are equivalent*:

(iii) $\widetilde{A} \in \mathrm{Nai}_2(A)$;
(iv) $\tau(\cdot) \in R^u[\mathcal{H}]$ *and the conditions in* (7.14) *are satisfied*;
(v) $\tau(\cdot) \in R[\mathcal{H}]$ *and the conditions in* (7.14) *are satisfied.*

Proof. (i) $\Rightarrow$ (ii) Let $\widetilde{A} \in \mathrm{Nai}_2(A)$, and let T_1 be closed. Then $S_1 = A$ and $\overline{\mathrm{dom} S_2} = \mathfrak{H}_2$ by Proposition 7.5. The induced boundary relation $\chi\colon \mathfrak{H}_2^2 \to \mathcal{H}^2$ is single-valued by Proposition 5.11. Moreover, $\mathrm{dom}\, \chi = T_2$ by construction (see Theorem 5.3) and, since T_1 is closed, we have $T_2 = S_2^*$ by parts (i), (ii) of Lemma 5.1. Therefore, $\mathrm{ran}\, \chi = \mathcal{H}^2$, which means that $\tau(\cdot) \in R^u[\mathcal{H}]$; see [15, Prop. 5.3]. The conditions in (7.14) now follow from Theorem 7.13.

(ii) $\Rightarrow$ (i) If $\tau(\cdot) \in R^u[\mathcal{H}]$, then $\mathrm{ran}\, \chi = \mathcal{H}^2$ is closed. Therefore, $T_2 = \mathrm{dom}\, \chi$ and T_1 are also closed by Proposition 2.4 and Lemma 5.1. The assertion $\widetilde{A} \in \mathrm{Nai}_2(A)$ follows from Theorem 7.13.

If T_1 is closed, then (iii), (iv), and (v) are equivalent. This follows from the equivalence of (i) and (ii) together with Theorem 7.13. Finally, note that, if $n_\pm(A) = n < \infty$, then T_1 is closed; cf. Corollary 7.6.

7.4.2. General case $\tau(\cdot) \in \widetilde{R}(\mathcal{H})$. The general case, where $\tau(\cdot) \in \widetilde{R}(\mathcal{H})$ is an arbitrary Nevanlinna family, is reduced to the case of $\tau(\cdot) \in R[\mathcal{H}]$. To this end, we use Proposition 5.15, where a suitable "coupling" of the boundary relation $\chi\colon \mathfrak{H}_2^2 \to \mathcal{H}^2$ whose Weyl family is $\tau(\cdot)$ with an appropriate boundary triplet $\Gamma'\colon \mathfrak{H}_1^2 \to \mathcal{H}^2$ is considered.

Theorem 7.15. *Let A be a closed densely defined symmetric operator on $\mathfrak{H}$ with equal defect numbers $n_+(A) = n_-(A)$, and let $\widetilde{A} = \widetilde{A}_{-\tau(\lambda)}$ be a minimal exit space extension of A corresponding to $\tau(\lambda)$ in* (6.5). *In this case, $\widetilde{A} \in \mathrm{Nai}_2(A)$ if and only if $\tau(\cdot) \in R^s(\mathcal{H})$ and the operator function*

$$\tau^{(1)}(\lambda) = -(\tau(\lambda) - 1/\lambda)^{-1} \tag{7.15}$$

satisfies the limit conditions in (7.14), *i.e.,*

$$\lim_{y\uparrow\infty} y^{-1}\tau^{(1)}(iy) = 0 \quad \text{and} \quad \lim_{y\uparrow\infty} y\, \mathrm{Im}\left(\tau^{(1)}(iy)h, h\right) = \infty, \qquad h \in \mathcal{H} \setminus \{0\}. \tag{7.16}$$

Proof. By Proposition 5.11, $S_1 = A$ holds if and only if $\tau(\cdot) \in R^s(\mathcal{H})$ or, equivalently, if and only if the induced boundary relation χ is single-valued. By Theorem 7.4, it remains to prove that the limit conditions in (7.16) are equivalent to $\mathrm{mul}\, T_2 = \{0\}$.

Consider a trivial symmetric operator $A' = \{0, 0\}$ in $\mathcal{H}$, which implies that $A'^* = \mathcal{H}^2$, and define a boundary triplet $\Pi' = \{\mathcal{H}, \Gamma_0', \Gamma_1'\}$ for A'^* by the rule

$$\Gamma_0'\widehat{f} = f', \qquad \Gamma_1'\widehat{f} = -f, \qquad \widehat{f} = \{f, f'\} \in \mathcal{H}^2. \tag{7.17}$$

In this case, $\ker \Gamma_1' = \{0\} \times \mathcal{H}$ is selfadjoint extension of A' of the form (5.40), and the corresponding Weyl function is $-(1/\lambda)I_{\mathcal{H}}$. Hence, one can apply Proposition 5.15 to the boundary relation χ ($\ker \chi = S_2$, $\operatorname{dom} \chi = T_2$) and the boundary triplet Π' (instead of Π). Then the linear relation $H^{(1)}$ of the form (5.44) determines a single-valued symmetric operator on $\mathcal{H} \oplus \mathfrak{H}_2$ by Proposition 5.15 (ii) because $\operatorname{mul} S_2 = \{0\}$. According to Theorem 5.14, the operator function $\tau^{(1)}(\lambda)$ in (7.15) is the Weyl function corresponding to the boundary relation $\Gamma^{(1)}$ whose domain $T^{(1)}$ is given by (5.41). Further, by Theorem 5.14, the boundary relation $\Gamma^{(1)}$ satisfies conditions (B1)–(B3), and (5.35) shows that $\Gamma^{(1)}$ is single-valued, since χ is single-valued. Proposition 5.15 (i) shows that

$$\operatorname{mul} T_2 = \{0\} \qquad \Longleftrightarrow \qquad \operatorname{mul} T^{(1)} = \{0\}. \tag{7.18}$$

Finally, it follows from Proposition 3.17 that $\operatorname{mul} T^{(1)} = \{0\}$ if and only if the corresponding Weyl family $\tau^{(1)}(\cdot)$ satisfies the limit conditions (7.16), since $\mathcal{H}_0 = \operatorname{mul} \Gamma_0^{(1)} = \{0\}$ here. In view of (7.18), this completes the proof.

Combining Theorem 7.13 with Theorems 3.6 and 7.15 leads to the following result (we again know no direct analytic proof, even in the matrix case).

Proposition 7.16. *Let* $\tau(\cdot) \in R[\mathcal{H}]$ *satisfy conditions* (7.14)*. Then the function*

$$\tau^{(1)}(\lambda) := -\big(\tau(\lambda) - 1/\lambda\big)^{-1}$$

satisfies conditions (7.16)*. Conversely, if* $\tau^{(1)}(\cdot)$ *satisfies conditions* (7.16) *and, in addition,* $\tau(\cdot) \in R^s[\mathcal{H}]$*, then* $\tau(\cdot)$ *satisfies conditions* (7.14)*.*

Proof. By Theorem 3.6, there exists a boundary relation such that the corresponding Weyl function is $\tau(\cdot)$. If $\tau(\cdot) \in R[\mathcal{H}]$ satisfies conditions (7.14), then $\tau(\cdot) \in R^s[\mathcal{H}]$ and $\widetilde{A} = \widetilde{A}_{-\tau} \in \operatorname{Nai}_2(A)$ by Theorem 7.13. Theorem 7.15 now shows that $\tau^{(1)}(\cdot)$ satisfies conditions (7.16).

Conversely, if $\tau^{(1)}(\cdot)$ satisfies the conditions in (7.16) and if $\tau(\cdot) \in R^s[\mathcal{H}]$, then we have $\widetilde{A} = \widetilde{A}_{-\tau} \in \operatorname{Nai}_2(A)$ by Theorem 7.15, and (7.14) holds by Theorem 7.13.

If $\tau(\cdot) \in R^s(\mathcal{H}) \setminus R^s[\mathcal{H}]$, it seems natural to characterize the inclusion $\widetilde{A} = \widetilde{A}_{-\tau(\lambda)} \in \operatorname{Nai}_2(A)$ in terms of strong resolvent limits of $\tau(\cdot)$, i.e., to express them in terms of the function

$$\tau^{(2)}(\cdot) := -(\tau(\cdot) + i \cdot I)^{-1}$$

instead of the limits of $\tau^{(1)}(\cdot)$ in (7.16). Using the arguments of Theorems 7.15 and 5.14 and replacing the boundary triplet Π' with the boundary triplet Π'' corresponding to the Weyl function $W''(\lambda) = iI_{\mathcal{H}}$, $\lambda \in \mathbb{C}_+$ (see Example 6.20), we obtain the following necessary condition only.

Proposition 7.17. *Let the conditions of Theorem* 7.15 *be satisfied and assume that*

$$\widetilde{A} = \widetilde{A}_{-\tau(\lambda)} \in \operatorname{Nai}_2(A).$$

Then the operator function

$$\tau^{(2)}(\cdot) = -(\tau(\cdot) + i \cdot I)^{-1}$$

satisfies the analogs of the limit conditions in (7.16)*.*

Remark 7.18. The converse to Proposition 7.17 fails to hold even in the scalar case, because the operator $H^{(1)}$ can be densely defined despite the fact that the operator $A \oplus S_2$ is not. For instance, if $\tau(\lambda) = -1/\lambda$, then $\tau^{(2)}(\lambda) = \lambda(1 - i\lambda)^{-1}$ satisfies the limit conditions similar to (7.16), whereas $\tau(\cdot)$ does not satisfy these properties. This partially explains the choice of $\tau^{(1)}(\cdot)$ instead of $\tau^{(2)}(\cdot)$ in Theorem 7.15.

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