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Modeling for Control of a Wobble–Yoke Stirling Engine

Eloisa García–Canseco§, Jacquelien M.A. Scherpen† and Marnix Kuindersma†

§Eindhoven University of Technology, Faculty of Mechanical Engineering, CST, PO Box 513, 5600 MB Eindhoven, The Netherlands.
†University of Groningen, Faculty of Mathematics and Natural Sciences, ITM, Nijenborgh 4, 9747 AG Groningen, The Netherlands.

Email: e.garcia.canseco@tue.nl, j.m.a.scherpen@rug.nl

Abstract—In this paper we derive the dynamic model of a four–cylinder double–acting wobble–yoke Stirling engine introduced originally by [1, 2]. In contrast with the classical thermodynamics methods that dominate the literature of Stirling mechanisms, we present a control system perspective to obtain a useful model for the analysis and synthesis of feedback control laws. The main motivation is the application of this gas engine in a micro–combined heat and power (CHP) generation system.

Keywords: control, micro–CHP, modeling, Stirling engine, cogeneration system.

Nomenclature

- $a_i$: connecting rod bearing center,
- $A_p$: piston area [$m^2$],
- $A_c$: piston rod area [$m^2$],
- $b_i$: wobble yoke–beam bearing center,
- $b_y$: damping coefficient,
- $c_{j}$: nutating bearing center,
- $d$: crankshaft bearing center,
- $e_{j}$: axes of the fixed reference frame,
- $F_{j}$: force [N],
- $g$: acceleration due to gravity [$m/s^2$],
- $I$: mass moment of inertia about the pivot $O$ [$kgm^2$],
- $k_{ps}$: piston spring constant [N/m],
- $l_{j}$: distance [m],
- $m$: piston assembly mass including the connecting rod [kg],
- $m_T$: mass moment of inertia of the working gas [kg],
- $M_i$: angular momentum with respect to the axis $c_i$ [Nm],
- $O$: center of the fixed reference frame, main pivot center,
- $p_{cc}$: pressure in compression space [N/m$^2$],
- $p_{ce}$: crankcase pressure [N/m$^2$],
- $p_{ei}$: pressure in expansion space [N/m$^2$],
- $R$: gas constant [J/(K·mol)],
- $T_h$: hot end temperature [K],
- $T_c$: cold end temperature [K],
- $T_r$: regenerator effective temperature [K],
- $V_{dc}$: compression space volume [$m^3$],
- $V_{de}$: expansion space volume [$m^3$],
- $V_{dc}$: dead volume in compression space [$m^3$],
- $V_{de}$: dead volume in expansion space [$m^3$],
- $V_b$: heater volume [$m^3$],
- $V_c$: cooler volume [$m^3$],
- $V_{sw}$: swept volume in compression space [$m^3$],
- $V_{swe}$: swept volume in expansion space [$m^3$],
- $\theta$: beam angle [rad],
- $\dot{\theta}$: beam angular velocity [rad/s],
- $\ddot{\theta}$: beam angular acceleration [rad/s$^2$],
- $\theta_{max}$: maximum beam angle [rad],
- $\phi$: crankshaft angle [rad],
- $\tau$: shaft torque [Nm].

1. Introduction

In the recent decades, there has been an enormous interest in the application of heat engines for conversion of different forms of heat source into electrical energy [3]. One of the most promising applications is the micro–combined heat and power (CHP) generation, or in other words, the simultaneous production of heat and power at a small–scale [4]. A micro–CHP consist of a gas engine which drives an electrical generator. The main purpose of a micro–CHP system is to replace the conventional boiler in a central heating system. Among the technologies that have been proposed for micro–CHP applications we can mention fuel cells, internal combustion engines and Stirling engines [4, 5].

Since the invention of the first Stirling engine by Robert Stirling in 1816, Stirling engines have been heavily studied, with an increasing interest during the last decades. Nevertheless, most of the studies rely on thermodynamics methods and intuitive design techniques. There exist few literature on the application of dynamics and control methods to investigate their stability and dynamic properties, see for instance [6–9] and the recent work [10]. Moreover, most of the works analyze free–piston Stirling engines.

In this work, we focus on the Whispergen micro–CHP with Stirling engine technology. This micro–CHP unit, developed by WhisperTech Limited [11], was originally designed as a battery charger for marine applications [1]. In contrast with most Stirling engines based on free–piston mechanisms, the Whispergen micro–CHP unit comprises a wobble–yoke mechanism with a four–cylinder double–acting engine configuration.

To the best of the authors knowledge, none of the previous works, namely [1, 2], have approached the study of the wobble–yoke Stirling engine from a control system perspective. Our main contribution is the development of a dynamic model of the wobble–yoke Stirling engine, that can be useful for the analysis and synthesis of feedback control laws for micro–CHP systems.

2. Description of the System

Figure 1 shows the schematic representation of the four–cylinder double–acting Stirling engine. The four cylinders...
are phased at 90° from each other with respect to φ. The links connecting the cylinders sketch the wobble–yoke mechanism whose function is to translate the vertical motion of the cylinders into the rotational motion through the shaft angle φ. The wobble yoke drive is based on the well-known spherical crank–rocker four–bar linkage [12, 13]. Its working principle can be explained by reference to Figure 2. The mechanism is based on a beam which pivots about its center O in one plane (e2e1 for beam 1, and e3e1 for beam 2). Each beam is attached to the cylinders with connecting rods at each end via bearings a1 and a3. An eccentric bearing e1 is attached to the drive shaft and it is connected to the beam via two bearings b1 and b2. The eccentric bearing e1 induces a rotational motion on bearing c1. Due to the geometrical and physical configuration of the mechanism, bearing c1 describes a circle of radius l. The axis of bearings b1, b3, c1 and d must intersect the center O, so that the kinematic constraints of the spherical crank rocker are satisfied [12, 13]. An analogous discussion applies to the second beam. We refer the reader to [1, 2] for more details about the wobble–yoke Stirling engine.

3. Modeling for control

In this section we derive the equations of motion for the wobble–yoke Stirling engine depicted in Figure 1. We make the following fundamental Assumption throughout the work:

\( a1 \) Small motion: Let \(-15° < \theta_i < 15°\), then \( \cos \theta_j \approx 1 \), \( \sin \theta_i \approx \theta_i \), \( \theta_i^2 \approx 0 \).

During operation of the engine, the beam angle \( \theta_j \) between the beam and the horizontal axis (e2 for beam 1 and e1 for beam 2)—cf. Figure 2—varies between its maximum \( \theta_{\text{max}} \) and its minimum \( -\theta_{\text{min}} \). Due to physical constraints of the engine, the maximum beam angle is approximately 10.21°, thus, Assumption \( a1 \) is physically correct.

3.1. Kinematics

The design of the wobble yoke mechanism is based on the classical spherical four–bar linkage [2]. These kind of linkages, which are well known in robotics, have the property that every link in the system rotates about the same fixed point [12, 13]. Hence, as indicated by its name, the trajectories of the points at the end of each link lie on concentric spheres. In robotics, only the revolute joint is compatible with this rotational movement and its axis must pass through the fixed point. The wobble yoke is indeed a particular class of the spherical linkage known as spherical crank rocker [2].

The revolute joints are replaced by the spherical bearings located at points \( b_1, b_3, c_1 \) and \( d \) (cf. Figure 2). The axis of the aforementioned bearings must intersect the sphere center \( O \). Further details about spherical linkages are given in [12, 13].

Consider the schematic representation of beam 1 shown in Figure 2. We define the reference frame \( e_n, n = 1, \ldots, 3 \), which is fixed at the pivot center of the beam \( O \). As was explained in Section 2, the vertical motion of the pistons (not shown in Fig. 2) inside the cylinders, leads to a rotation of the beam around \( O \). This rotation is represented by the instantaneous value of \( \theta_1 \). The variation on \( \theta_1 \) causes as well a rotational movement around the axis \( e_1 \), which is represented by the crank angle \( \phi \).

Then, the kinematic problem for the wobble yoke Stirling engine consist in finding the equations of the angular displacements \( \theta_j, j = 1, \ldots, 2 \), and the vertical displacements \( z_i, i = 1, \ldots, 4 \) of the connecting rod bearing center \( a_i, i = 1, \ldots, 4 \), in terms of the crank angle \( \phi \).

Following the same procedure as [2], we have that—if some straightforward but cumbersome computations—the kinematic equation relating the angular displacement, velocity and acceleration \( \theta_1, \dot{\theta}_1 \) and \( \ddot{\theta}_1 \) in terms of the crank angle \( \phi \) are given respectively by

\[
\begin{align*}
\theta & = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{bmatrix} = \begin{bmatrix} \tan^{-1}(\kappa \sin \phi) \\ \tan^{-1}(\kappa \cos \phi) \\ \kappa \phi \cos \phi \\ -\kappa \phi \sin \phi \end{bmatrix}, \\
\theta & = \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \\ \dot{\theta}_4 \end{bmatrix} = \begin{bmatrix} \kappa \phi \cos \phi \\ \kappa \phi \sin \phi \\ -\kappa \phi \cos \phi + 2 \tan^{-1}(\kappa \sin \phi) \cos^2 \phi \\ -\kappa \phi \sin \phi - \kappa \phi \cos \phi + 2 \tan^{-1}(\kappa \cos \phi) \sin^2 \phi \end{bmatrix}, \\
\ddot{\theta} & = \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{\theta}_3 \\ \ddot{\theta}_4 \end{bmatrix} = \begin{bmatrix} \kappa \phi^2 \cos \phi \\ \kappa \phi^2 \sin \phi \\ -\kappa \phi \sin \phi - \kappa \phi \cos \phi + 2 \tan^{-1}(\kappa \cos \phi) \sin^2 \phi \end{bmatrix},
\end{align*}
\]

where \( \kappa = \tan \theta_{\text{max}} \). For the vertical displacements \( z_i \), we only have two independent equations, namely, \( z_1 \) and \( z_2 \)
and the output shaft–

\( \tau_p \) (cf. Fig. 6), it can be shown [14] that

\[
V_{r_i} = \frac{A_p - A_r}{2} z_i + \frac{V_{swc}}{2} + V_{dc},
\]

(10)

\[
V_{r_i} = -\frac{A_p}{2} z_i + \frac{V_{swc}}{2} + V_{dc}.
\]

(11)

Substituting (10) and (11) into the pressure equation (9) and after grouping the constant terms in \( \beta_1 \), we have that the instantaneous pressure in the compression spaces is given by

\[
p_{r_i} = p_{swc} \left(1 + \frac{A_p - A_r}{2 \rho_c T_k \beta_1} - \frac{A_p}{2 \rho_c T_k \beta_1} z_{i+1}\right)^{-1}
\]

(12)

where \( p_{swc} = MR/\beta_1 \) is the mean pressure in the working spaces, and

\[
\beta_1 = \frac{1}{T_c} \left(V_{dc} + \frac{V_{swc}}{2}\right) + \frac{1}{T_f} \left(V_{dc} + \frac{V_{swc}}{2}\right) + \frac{1}{T_r} \left(V_{dc} + \frac{V_{swc}}{2}\right).
\]

3.4. Rotational movement

In this section we apply a slightly modification of the approach followed by [2] to obtain the equation for the rotational movement. The main differences with [2] are the damping and spring terms in the force \( F_i \), the computation of the equivalent forces \( F_{i+1} \) and \( F_{i-1} \), and the output shaft–torque equation \( \tau \).

We define the net forces acting at the connecting rod bearing \( a_i \) as (cf. Fig. 5)

\[
F_i = m_{ci} \left(A_p - A_r\right) (p_{ci} - p_{ci-1}) + A_p (p_{ci} - p_{ci-1}) - k_c (z_i - z_{i-1}) - b_p \dot{z}_i.
\]

(13)

The difference between the forces \( F_1 \) and \( F_3 \) acting at the beam ends \( a_1 \) and \( a_3 \) (cf. Fig. 5) produces an angular momentum \(^2\) in the pivot center \( O \) pointing in the \( e_1 \) axis direction \( M_1 = I_{o1a}(F_1 - F_3) - I_1 \dot{\theta}_1 \). Similarly, for the second beam, we have an angular momentum in the pivot center \( O \) pointing in the \( e_2 \) axis direction \( M_2 = I_{o2a}(F_2 - F_4) - I_2 \dot{\theta}_2 \).

To get the equation of the output–shaft torque \( \tau \) in the \( e_3 \) direction, we observe that the angular momenta \( M_1 \) and \( M_2 \) can be translated into equivalent forces \( F_{1a} \) and \( F_{2a} \) acting at the points \( c_1 \) and \( c_2 \), for beams 1 and 2 respectively. If we constrain the forces \( F_{1a} \) and \( F_{2a} \) to lie in the plane \( e_1 e_2 \) (cf. Fig. 6), it can be shown [14] that \( F_{1a} = [0, f_{12}, 0]^T \) and \( F_{2a} = [f_{21}, 0, 0]^T \), with \( f_{12} = \frac{M_1}{I_{o1a}} \) and \( f_{21} = \frac{M_2}{I_{o2a}} \). The output–shaft torque \( \tau \) is then computed as (cf. Fig. 6)

\[
\tau = \kappa (M_1 + M_2) \cos \phi = (M_1 + M_2) \tan \theta_2.
\]

(14)

Throughout the document, we follow the right–hand rule for rotational movement, i.e., a counterclockwise rotation produces a positive momentum about the rotational axis.
where we have used $I_{cm} = I_O a_1k$ and (1). By Assumption a1, $\tan\theta_2 \approx \theta_2 \approx \frac{z_2}{a_2}$. Thus, (14) finally becomes

$$\tau = \frac{M_1 + M_2}{I_O a_1} z_2.$$  

(15)

4. Conclusions and Future Work

We have developed a dynamic model of a wobble–yoke Stirling engine mechanism with a four-cylinder double-acting configuration. Because of the pressure dynamics (12), the piston motion equation (7) are nonlinear. However, the system can be studied via linear analysis methods due to the linear spring behavior of the working gas [16]. Current research is underway to validate and investigate the stability properties of the model. See [14] for further details.

Among the issues that are currently being explored are the application of the wobble–yoke Stirling engine in micro–CHP systems, particularly, those composed of an electricity generator driven by the wobble–yoke Stirling engine.

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