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Memristive port-Hamiltonian Systems

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The port-Hamiltonian modelling framework is extended to a class of systems containing memristive elements and phenomena. First, the concept of memristance is generalised to the same generic level as the port-Hamiltonian framework. Second, the underlying Dirac structure is augmented with a memristive port. The inclusion of memristive elements in the port-Hamiltonian framework turns out to be almost as straightforward as the inclusion of resistive elements. Although a memristor is a resistive element, it is also a dynamic element since the associated Ohmian laws are rather expressed in terms of differential equations. This means that the state space manifold, as naturally defined by the storage elements, is augmented by the states associated with the memristive elements. Hence the order of complexity is, in general, defined by the number of storage elements plus the number of memristors in the system. Apart from enlarging our repertoire of modelling building blocks, the inclusion of memristive elements in the existing port-Hamiltonian formalism possibly opens up new ideas for controller synthesis and design.

Keywords: memristor; memristive systems; port-Hamiltonian systems; port-based modelling

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1. Introduction and motivation

In the early 1970s, Chua [1] postulated the existence of a new basic electrical circuit element, called the memristor, defined by a nonlinear relationship between charge and flux linkage. The memristor, a contraction of memory and resistance, referring to a resistor with memory, completes the family of the well-known existing fundamental circuit elements: the resistor, inductor and capacitor. Although a variety of physical devices, including thermistors, discharge tubes, Josephson junctions and even ionic systems such as the Hodgkin–Huxley model of a neuron, were shown to exhibit memristive effects [2, 3], a physical passive two-terminal memristive prototype could not be constructed until very recently, when scientists of Hewlett-Packard Laboratories announced its realisation in nature [4]. Strukov *et al*. show that memristance naturally arises in nanoscale systems when electronic and atomic transport are coupled under an external bias voltage. On the other hand, as pointed out in [5], a tapered dashpot is a mechanical resistor whose resistance depends on the displacement of its terminals. Consequently, a description in terms of its associated force and velocity generally yields some complicated, possibly hysteretic, constitutive relationship. These difficulties are

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circumvented by modelling the tapered dashpot as a mechanical memristive element using the relationship between its displacement and momentum (the mechanical analogies of charge and flux linkages) instead.

One of the main reasons why the memristor concept has not yet played a major role in modelling problems can most likely be explained from the fact that so far the majority of practical devices are reasonably well modelled by some (though often artificial) combination of standard modelling building blocks, such as resistive, inductive and capacitive elements, and their nonlinear and multi-port versions. However, as nanoscale electronic devices become more and more important and complex [2], it might be beneficial, and on the longer term even necessary, to enlarge our repertoire of modelling building blocks that establishes a closer connection between the mathematics and the observed physics.

In this article, we study the inclusion of memristive elements and their properties in the port-Hamiltonian modelling framework. The port-Hamiltonian formalism naturally arises from network modelling of physical systems in a variety of domains (e.g., mechanical, electrical, electromechanical, hydrodynamical and thermodynamical); see [6] for a comprehensive summary of the developments of this framework over the past decade. Exposing the relation between the energy storage, dissipation and interconnection structure, this framework underscores the physics of the system. The connection with network (bond-graph) modelling is further formalised with the notion of a so-called Dirac structure on the space of flows and efforts. One of the strong aspects of the port-Hamiltonian formalism is that a power-preserving interconnection between port-Hamiltonian systems results in another port-Hamiltonian system with composite energy, dissipation and interconnection structure. Based on this principle, complex, multi-domain systems can be modelled by interconnecting port-Hamiltonian descriptions of its subsystems. Moreover, several control design methodologies are available that can be directly applied to such port-Hamiltonian descriptions of complex nonlinear systems. It is precisely in this context that a memristive port-Hamiltonian description can be of added value.

The remainder of the article is organised as follows. In Section 2, we briefly recall the basic properties of port-Hamiltonian systems defined with respect to a Dirac structure. Section 3 gives the generalisation of the concept of memristance to the same generic level as the port-Hamiltonian framework. The extension of the input-state-output port-Hamiltonian formulation with a generalised memristive port and some of its basic properties are highlighted in Section 4. Section 5 illustrates some aspects of the theory by using three simple examples. The extension of the port-Hamiltonian framework to include memristive systems, which extends the basic memristor concept to a much broader class of dynamical systems, is discussed in Section 6. The article concludes with some final remarks.

Notation. All vectors, including the gradient of a function, defined in the article are column vectors.

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2. The port-Hamiltonian formalism

The basic ingredient of any port-Hamiltonian system is the power-conserving interconnection structure, mathematically formalised as a Dirac structure, linking the various power ports of the system; see Figure 1. Power ports (henceforth simply called ports) carry two sets of conjugate variables: a vector of flow variables $f \in \mathscr{F}$ and a vector of effort variables $e \in \mathscr{E}$, with product $e^T f$ denoting the power occurring at the port. The Dirac structure captures the basic interconnection laws (like Kirchhoff's laws) together with ideal power-conserving

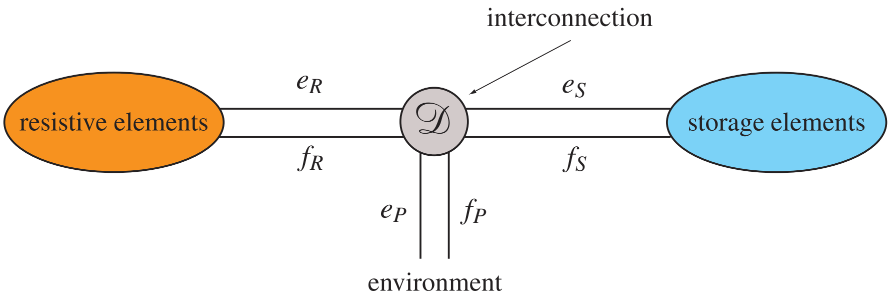

Figure 1. Many physical systems can be characterised by interconnections between energy storage elements, resistive elements and the environment. The key concept in the formulation of port-based network models of physical systems as port-Hamiltonian systems is the geometric notion of a Dirac structure $\mathscr{D}$.

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elements like transformers, gyrators and ideal constraints, and generalises Tellegen's theorem and d'Alembert's principle.

In contrast to common port-based modelling approaches, such as the standard bond-graph formalism [7] or classical energy- and power-based approaches [8], the port-Hamiltonian framework uses only one type of storage. For example, in modelling mechanical systems or electrical networks it is common to consider two types of storage: capacitive or C-type storage, such as a spring or a capacitor, and inertial or I-type storage, such as a mass or an inductor. This approach disables the distinction between flow and effort as rate of change of state and equilibrium-determining variable, or vice versa. Based on the generalised bond-graph (GBG) framework introduced in [9], the port-Hamiltonian formalism considers the flow as rate of change of state exclusively. The usual physical domains are split into two sub-domains, each associated to only one type of storage: the capacitive or C-type storage. Consequently, we do not speak of mechanical or electrical domains, but of kinetic and potential, or electric and magnetic sub-domains, and so on; see Table 1 for a complete overview. The corresponding sub-domains are connected by a so-called symplectic (or unit) gyrator. An additional advantage of considering only one type of storage is that the concept of mechanical force has no unique meaning as it may play the role of a flow in the kinetic domain or an effort in the potential domain, thus leaving the discussion about the force–voltage versus force–current analogy a non-issue.

Table 1. Domains and variables used in the port-Hamiltonian framework.

Physical sub-domain	Flow $f \in \mathscr{F}$	Effort $e \in \mathscr{E}$	State variable $x = \int f \, dt$
electric	current	voltage	charge
magnetic	voltage	current	flux linkage
potential translation	velocity	force	displacement
kinetic translation	force	velocity	momentum
potential rotation	angular velocity	torque	angular displacement
kinetic rotation	torque	angular velocity	angular momentum
potential hydraulic	volume flow	pressure	volume
kinetic hydraulic	pressure	volume flow	flow tube momentum
chemical	molar flow	chemical potential	number of moles
thermal	entropy flow	temperature	entropy

A recent exposition of the GBG framework and its relation to the commonly used port-based modelling approaches is given in the first chapter of [6]. Details about port-Hamiltonian systems, including a wealth of modelling examples, can be found in three successive chapters of the same reference. In the following subsections we briefly recall the basic definitions that are necessary for the developments thereafter.

2.1. Ports, dirac structures and passivity

In order to define a Dirac structure, the spaces of flows and efforts are naturally partitioned as $\mathscr{F} := \mathscr{F}_S \times \mathscr{F}_R \times \mathscr{F}_P$ and $\mathscr{E} := \mathscr{E}_S \times \mathscr{E}_R \times \mathscr{E}_P$, each corresponding to the following set of ports:

- The energy storage port, with port variables $(f_S, e_S) \in \mathscr{F}_S \times \mathscr{E}_S$, is interconnected with the energy storage of the system, which in turn is characterised by an n_S-dimensional space $\mathscr{X}$ of state variables, locally represented by $x \in \mathscr{X}$, together with a Hamiltonian function $H : \mathscr{X} \to \mathbb{R}$ denoting the total stored energy. The corresponding flow variables are given by the rate of change of the state variables. This is accomplished by setting
$$\begin{aligned} f_S &= \dot{x}, \\ e_S &= \frac{\partial H}{\partial x}(x). \end{aligned} \tag{1}$$
Hence, the power at the energy storage port can be written as
$$\dot{H}(x) = \left(\frac{\partial H}{\partial x}(x)\right)^T \dot{x} = e_S^T f_S. \tag{2}$$
- The resistive port corresponds to internal energy dissipation (e.g., friction, electrical resistance, and so on), and its port variables $(f_R, e_R) \in \mathscr{F}_R \times \mathscr{E}_R$ are terminated by a static resistive relation of the form
$$f_R = \hat{f}_R(e_R), \tag{3}$$
with $\hat{f}_R : \mathscr{E}_R \to \mathscr{F}_R$. In many cases, f_R can be derived from a so-called 'content' function $D : \mathscr{E}_R \to \mathbb{R}$ in the sense that $f_R = \partial D(e_R)/\partial e_R$.[1] Note that for passive resistors $e_R^T f_R \geq 0$.
- Finally, the remaining port, with port variables $(f_P, e_P) \in \mathscr{F}_P \times \mathscr{E}_P$, denotes the interaction port of the system, modelling its interaction with other system components or the environment. The power delivered or extracted from the interaction port equals $e_P^T f_P$, which in the sequel is referred to as the supply rate.

The Dirac structure $\mathscr{D}$ is a linear relation between all the port variables that satisfy the power conservation property
$$e_S^T f_S + e_R^T f_R = e_P^T f_P, \tag{4}$$
and has maximal dimension with respect to this property.[2] More specifically, locally around a point $x \in \mathscr{X}$, we can represent $\mathscr{D}$ as
$$\begin{aligned} \mathscr{D} = \{ & (f_S, e_S, f_R, e_R, f_P, e_P) \in \mathscr{F} \times \mathscr{E} \,|\, F_S f_S + E_S e_S \\ & + F_R f_R + E_R e_R = F_P f_P + E_P e_P \}, \end{aligned} \tag{5}$$

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for some matrices F_S, E_S, F_R, E_R, F_P and E_P satisfying

$$F_S E_S^T + E_S F_S^T + F_R E_R^T + E_R F_R^T = F_P E_P^T + E_P F_P^T,$$

and $\mathrm{rank}(F_S|E_S|F_R|E_R|F_P|E_P) = dim(\mathscr{F})$.

As a direct consequence of (4), any port-Hamiltonian system with passive resistive elements satisfies the power-balance inequality

$$\dot{H}(x) = e_S^T f_S = e_P^T f_P - e_R^T f_R \leq e_P^T f_P, \tag{6}$$

since $e_R^T f_R \geq 0$. Integrating the latter from initial time t_0 to t yields the energy balance inequality

$$\begin{aligned} H[x(t)] - H[x(t_0)] &= \int_{t_0}^{t} e_P^T(\tau) f_P(\tau)\mathrm{d}\tau - \int_{t_0}^{t} e_R^T(\tau) f_R(\tau)\mathrm{d}\tau \\ &\leq \int_{t_0}^{t} e_P^T(\tau) f_P(\tau)\mathrm{d}\tau. \end{aligned} \tag{7}$$

If the Hamiltonian function $H(x)$ is bounded from below, then port-Hamiltonian systems are passive with respect to the supply rate $e_P^T f_P$ and the Hamilton as storage function. Note that, recalling Lyapunov stability theory, together with the sufficient conditions for the stability of an equilibrium point, it can be shown that the Hamiltonian is often a bonafide candidate Lyapunov function [10].

2.2. *Input-state-output representation*

An important special case of port-Hamiltonian systems is the class of input-state-output port-Hamiltonian systems, where there are no algebraic constraints on the state variables, and the flow and effort variables at all the other ports have been split into power-conjugated input–output pairs. The corresponding Dirac structure is defined by

$$\begin{aligned} \mathscr{D} = \{ &(f_S, e_S, f_R, e_R, f_P, e_P) \in \mathscr{F} \times \mathscr{E} \,|\, f_S - Je_S + G_R f_R - G_P f_P = 0, \\ &-G_R^T e_S + e_R = 0,\, G_P^T e_S - e_P = 0 \}, \end{aligned} \tag{8}$$

where $J = -J^T$, G_R and G_P are matrices of appropriate dimensions depending on the interconnection, resistive and input–output structure of the system, respectively. Furthermore, assuming that the resistive elements are linear, the constitutive relationship (3) simplifies to

$$f_R = R_e e_R, \tag{9}$$

with $R_e = R_e^T$ being some constant resistance matrix. Then, around $x \in \mathscr{X}$, by utilising Equations (1) and (3), the dynamics on $\mathscr{D}$ take the form

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$$\dot{x} - J\frac{\partial H}{\partial x}(x) + G_R R_e e_R - G_P f_P = 0,$$

$$-G_R^T \frac{\partial H}{\partial x}(x) + e_R = 0,$$

$$G_P^T \frac{\partial H}{\partial x}(x) - e_P = 0,$$

which, after substitution of the second equation into the first and a slight rearrangement, yields the well-known input-state-output port-Hamiltonian representation

$$\Sigma_P : \begin{cases} \dot{x} = (J - R)\dfrac{\partial H}{\partial x}(x) + G_P f_P \\ e_P = G_P^T \dfrac{\partial H}{\partial x}(x), \end{cases} \tag{10}$$

with resistive structure matrix $R := G_R R_e G_R^T$. Consequently, the power-balance inequality (6) can be written as

$$\dot{H}(x) = e_P^T f_P - \left(\frac{\partial H}{\partial x}(x)\right)^T R \frac{\partial H}{\partial x}(x) \leq e_P^T f_P, \tag{11}$$

under the condition that $R \succeq 0$. Note that in this framework, the flow and effort related to the environment are naturally defined as the input and output of the system, respectively.

Remark 2.1: For many systems, especially those with three-dimensional (3D) mechanical components, the Dirac structure will in general be modulated by the state variables x. In such a case, the structure matrices J, G_R and G_P are replaced by their modulated versions $J(x)$, $G_R(x)$ and $G_P(x)$, respectively. We come back to modulated Dirac structures in Section 6. More details on the geometric properties of Dirac structures and port-Hamiltonian systems can be found in [6, 10–12]. Note that, in comparison to these works, we have adopted a different sign convention for the direction of power flow at the resistive and storage ports.

3. Properties of the memristor

Before generalising the concept of memristance to fit the definitions of the port-Hamiltonian framework discussed in the previous section, we will first briefly recall the basic properties of the electrical memristor.

3.1. Chua's memristor

Since electronics was developed, engineers have designed circuits using combinations of three basic two-terminal elements: resistors, inductors and capacitors. From a mathematical perspective, the behaviour of each of these elements, whether linear or nonlinear, is described by relationships between two of the four basic electrical variables: voltage, current, charge and flux linkage. A resistor is described by the relationship of current and voltage, a capacitor by that of voltage and charge and an inductor by that of current and flux linkage. But what about the relationship between charge and flux linkage? As argued by Chua in the early 1970s [1], a fourth element should be added to complete the symmetry. He coined this 'missing element' the 'memristor', referring to a resistor with memory.

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The memory aspect stems from the fact that a memristor 'remembers' the amount of current that has passed through it together with the total applied voltage. More specifically, if q denotes the charge and ϕ denotes the flux linkage, then a two-terminal or one-port *charge-controlled* memristor is defined by the constitutive relationship

$$\phi = \hat{\phi}(q).$$

Since flux linkage is the time integral of voltage u (like in Faraday's law), and charge is the time integral of current i, or equivalently, $u = \dot{\phi}$ and $i = \dot{q}$, we obtain

$$u = M_i(q)i, \tag{12}$$

where $M_i(q) := \mathrm{d}\hat{\phi}(q)/\mathrm{d}q$ is called the incremental memristance.

Similarly, a two-terminal or one-port *flux-controlled* memristor (memductor) is defined by

$$q = \hat{q}(\phi),$$

Differentiation yields the dual of Equation (12),

$$i = M_u(\phi)u, \tag{13}$$

where $M_u(\phi) := \mathrm{d}\hat{q}(\phi)/\mathrm{d}\phi$ is called the incremental 'memductance'.

Observe that Equations (12) and (13) are just the charge- and flux-modulated versions of Ohm's law, respectively. It is important to realise that for the special cases that the constitutive relations are linear, that is, when the incremental memristance M_i or the incremental memductance M_u is constant, a memristor or memductor becomes an ordinary resistor or conductor. Hence, memristors and memductors are only relevant in nonlinear circuits, which may account in part for their neglect in linear network and systems theory. Furthermore, it is directly noticed from Equation (12) (resp. (13)) that $u \equiv 0$ (resp. $i \equiv 0$) whenever $i \equiv 0$ (resp. $u \equiv 0$), regardless of q (resp. ϕ) which incorporates the memory effect. This characteristic feature is the so-called 'no energy discharge property' [2, 13], which is related to the fact that, unlike an inductor or a capacitor, a memristor does not store energy.

Before the effect of memristive elements can be studied in the port-Hamiltonian framework, we first need to bring the concept to the same generic level. This is accomplished by generalising the constitutive relationships (12) and (13) to their multi-terminal or multi-port versions on the level of flows and efforts.

3.2. The generalised memristor

In view of the classifications and analogies of Table 1, the multi-port generalisation of the charge-controlled memristor (12) or the flux-controlled memductor (13) is easily deduced as follows. Let $x_f \in \mathcal{X}_f$ denote the vector of integrated flows, and let $x_e \in \mathcal{X}_e$ denote the vector of integrated efforts, or equivalently, $\dot{x}_f = f$, and $\dot{x}_e = e$, respectively, then the relationship

$$x_e = \hat{x}_e(x_f)$$

constitutes a multi-port x_f-controlled memristor, i.e.,

$$e = M_f(x_f)f, \tag{14}$$

with generalised memristance matrix $M_f(x_f) := \partial\hat{x}_e(x_f)/\partial x_f$.

Note that Equation (14) contains both the original memristive relationships (12) and (13). Moreover, adopting the storage element-based state variable definition of Table 1, the charge-controlled memristor (12), with $x_f = q, f = i$ and $e = u$, belongs to the electric

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sub-domain, while the flux-controlled memductor (13), with $x_f = \phi, f = u$ and $e = i$, belongs to the magnetic sub-domain.

On the other hand, by interchanging the roles of the (integrated) flow and effort, we might as well consider

$$x_f = \hat{x}_f(x_e)$$

yielding a multi-port x_e-controlled memristor, i.e.,

$$f = M_e(x_e)e, \tag{15}$$

with generalised memristance matrix $M_e(x_e) := \partial \hat{x}_f(x_e)/\partial x_e$.

In a similar fashion as the storage and resistive elements, the constitutive relationship of a memristive element will in many cases be derivable from a so-called memristive 'action' function $A_f : \mathscr{X}_f \to \mathbb{R}$ (resp., $A_e : \mathscr{X}_e \to \mathbb{R}$) in the sense that

$$x_e = \frac{\partial A_f}{\partial x_f}(x_f) \quad \left(\text{resp., } x_f = \frac{\partial A_e}{\partial x_e}(x_e)\right). \tag{16}$$

Obviously, A_f and A_e are related by the Legendre transform

$$A_f(x_f) + A_e(x_e) = x_f x_e. \tag{17}$$

More details on the memristive action and some of its applications in a circuit-theoretic context can be found in [1, 14].

Remark 3.1: Since (14) already contains both the original memristive relationships (12) and (13), the form (15) should just be considered as the corresponding dual form – in the same sense that $e_R = R_f f_R$ is the dual form of $f_R = R_e e_R$, with $R_f = R_e^{-1}$. To this end, definitions (14) and (15) are in some sense exchangeable. For energy storage elements the distinction between flow and effort as the equilibrium-establishing (rate of change of state) and the equilibrium-determining variable, respectively, is clear since a storage element is defined by a constitutive relationship between effort and integrated flow (state), or in thermodynamic parlance, between an intensive state and extensive state, i.e., $e = \hat{e}(x)$ or $x = \hat{x}(e)$, with $\dot{x} = f$. In terms of input–output causality, the constitutive relationship $e = \hat{e}(x)$ yields a so-called integral causal form in which the flow can be considered as input and the effort as output; see Figure 2(a). This is the form generally considered in the

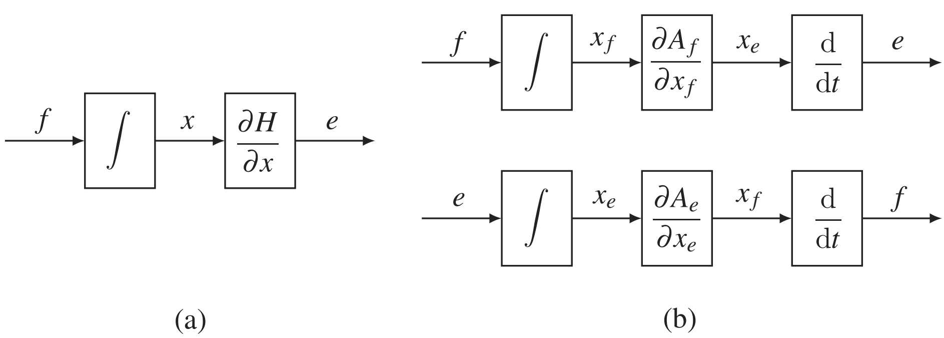

Figure 2. (a) Preferred causality of a storage element; (b) causally natural character of a memristive element.

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port-Hamiltonian framework. The dual or co-energy form, $x = \hat{x}(e)$, yields a differential causal form, considering effort as input and flow as output. Clearly, since both an integration and a differentiation is involved in 'lifting' the memristor to the space of flows and efforts, the memristor, like the resistor, is causally neutral, i.e., there is no fixed or preferred causality, so that it can accept either a flow or an effort as input variable; see Figure 2(b). Furthermore, a generalised memristor does not store integrated flow or integrated effort; it just bookkeeps the amount of integrated flow or integrated effort that passed its port. Hence it does not distinguish between the various sub-domains outlined in Table 1. This means that, starting from Equation (15), the charge-controlled memristor (12) might as well be associated with the magnetic sub-domain, whereas the flux-controlled memductor then belongs to the electric sub-domain.

Remark 3.2: The definitions above can be further generalised as follows. Let the memristive structure be represented by an m-dimensional submanifold $\mathscr{M}$ of $\mathscr{X}_f \times \mathscr{X}_e$, where $m = \dim(\mathscr{X}_f) = \dim(\mathscr{X}_e)$. The tangent space to this submanifold then defines the linear relationship between f and e, like in Equation (14) or (15). Furthermore, if $\mathscr{M}$ is a Lagrangian sub-manifold of $\mathscr{X}_f \times \mathscr{X}_e$, where $\mathscr{X}_f$ is the dual of $\mathscr{X}_e$, then the associated memristive action function corresponds to the generating function of $\mathscr{M}$. The interested reader is referred to [15] for a similar exposition in the context of nonlinear RLC networks.

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4. Port-Hamiltonian systems with memristive dissipation

We are now ready to extend the port-Hamiltonian formalism, as introduced in Section 2, by adding a memristive port, with port variables $(f_M, e_M) \in \mathscr{F}_M \times \mathscr{E}_M$, to the Dirac structure.

4.1. Memristors as port-Hamiltonian systems: the null-Hamiltonian

Assuming that the memristive port can be described by an x_f-controlled constitutive relationship of the form (14), we define the memristive structure $\mathscr{M}$ as

$$\mathscr{M} = \{(f_M, e_M) \in \mathscr{F}_M \times \mathscr{E}_M \,|\, \dot{x}_f - f_M = 0,\; e_M - M_f(x_f) f_M = 0\}, \tag{18}$$

where the generalised memristance $M_f(x_f) = M_f^T(x_f)$ is a matrix of appropriate dimensions. Now, a key observation is that locally around $x_f \in \mathscr{X}_f$ the memristive structure (18) defines a non-energetic port-Hamiltonian system with a direct feed-through term. Indeed, let $H_M : \mathscr{X}_f \to 0$; then the dynamics on $\mathscr{M}$ take the form

$$\Sigma_M : \begin{cases} \dot{x}_f = f_M, \\ e_M = \dfrac{\partial H_M}{\partial x_f}(x_f) + M_f(x_f) f_M, \end{cases} \tag{19}$$

where the memristive port variables f_M and e_M can be considered as the inputs and outputs, respectively. Non-energicness follows from the fact that $H_M(x_f) \equiv 0$, for all $x_f \in \mathscr{X}_f$, which, together with the fact that $e_M \equiv 0$ whenever $f_M \equiv 0$ and regardless of the internal state x_f, clearly underscores the 'no energy discharge property' as discussed in Section 3.1. For this reason we refer to H_M as the 'null-Hamiltonian'.

Remark 4.1: In the light of Remark 3.2, a memristive port can be generally represented by an implicit port-Hamiltonian system (with null-Hamiltonian) of the form

$$\Sigma_M : \begin{cases} \dot{x}_f = f_M, \\ \dot{x}_e = e_M, \quad (x_f, x_e) \in \mathscr{M}. \end{cases}$$

4.2. Input-state-output representation

In order to interconnect the memristive port (19) with the port-Hamiltonian system (10), we need to consider the composition of the Dirac structure (8) and the memristive structure (18). This is tantamount to interconnect (some of) the interconnection ports of (8) with (19) via the (negative) feedback interconnection

$$f_P = -G_M e_M + \widetilde{G}_P \widetilde{f}_P, \quad f_M = G_M^T e_P, \tag{20}$$

where G_M and $\widetilde{G}_P$ are matrices of appropriate dimensions, and $\widetilde{f}_P \in \mathscr{F}_P$ denotes a new input; see Figure 3. If, for simplicity, it is assumed that the resistive port of (10) is vacuous ($R = 0$), and $G_P = I$ we obtain the 'closed-loop' port-Hamiltonian system

$$\begin{pmatrix} \dot{x} \\ \dot{x}_f \end{pmatrix} = \begin{pmatrix} J - M(x_f) & -G_M \\ G_M^T & 0 \end{pmatrix} \begin{pmatrix} \dfrac{\partial H}{\partial x}(x) \\ \dfrac{\partial H_M}{\partial x_f}(x_f) \end{pmatrix} + \begin{pmatrix} \widetilde{G}_P \\ 0 \end{pmatrix} \widetilde{f}_P, \tag{21}$$

with state space $\mathscr{X} \times \mathscr{X}_f$, memristive structure matrix $M(x_f) := G_M M_f(x_f) G_M^T$ and Hamiltonian $H(x) + H_M(x_f) (= H(x) + 0)$. The new output for the system is naturally defined by

$$\tilde{e}_P = \widetilde{G}_P^T \frac{\partial H}{\partial x}(x). \tag{22}$$

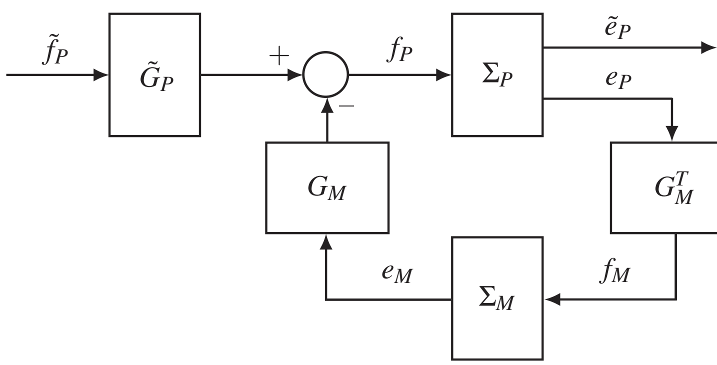

Figure 3. Feedback interpretation of the composition of a port-Hamiltonian system with a memristive port.

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4.3. Passivity and the power balance inequality

A memristive port described by Equation (19) is passive if and only if its generalised memristance $M_f(x_f)$ is non-negative. Indeed, differentiating the null-Hamiltonian $H_M(x_f)$ with respect to time and using Equation (19), we have

$$\dot{H}_M(x_f) = \dot{x}_f^T \frac{\partial H_M}{\partial x_f}(x_f) = f_M^T(e_M - M_f(x_f))f_M \equiv 0.$$

Hence, if $M_f(x_f) \succeq 0$, for all $x_f \in \mathscr{X}_f$, the instantaneous power dissipated by the memristive port is given by

$$P_M = f_M^T e_M = f_M^T M_f(x_f) f_M \geq 0,$$

where we recall that the sign convention adopted here is that power supplied to the system carries a negative sign, whereas power extracted from the system carries a positive sign.

The power-balance inequality associated to Equations (21) and (22) takes the form

$$\dot{H}(x) + \dot{H}_M(x_f) = \tilde{e}_P^T \widetilde{f}_P - \left(\frac{\partial H}{\partial x}(x)\right)^T M(x_f) \frac{\partial H}{\partial x}(x) \leq \tilde{e}_P^T \widetilde{f}_P, \tag{23}$$

where $M(x_f) := G_M M_f(x_f) G_M^T \succeq 0$ since $M_f(x_f) \succeq 0$, for all $x_f \in \mathscr{X}_f$. Hence if the Hamiltonian function $H(x)$ is bounded from below, then the system is passive with respect to the supply rate $\tilde{e}_P^T \widetilde{f}_P$ and storage function $H(x)$.

Note that the memristive port (19) appears as an integrated-flow-modulated resistive port. Under the condition that the generalised memristance $M_f(x_f)$ is non-negative the memristive port is dissipative, which is also evident from the first term at the right-hand side of Equation (23). For that reason, we refer to Equations (21) and (22) as a port-Hamiltonian system with memristive dissipation.

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4.4. Degenerate case: linear memristance

In the linear case, i.e., when M_f in Equation (19) is constant, the memristive port reduces to a purely resistive port. This property is consistent with the original definitions of the memristor outlined in Section 3.1.

4.5. Order of complexity

The addition of the memristive port yields that the total state space is in general extended to $\mathscr{X} \times \mathscr{X}_f$. Consequently, in addition to the initial values of the state variables associated with the storage elements, the initial values of the memristors should also be specified in order to find a complete solution of the port-Hamiltonian systems presented above. This means, in general, that the order of complexity [16] of a port-Hamiltonian system with memristive dissipation is determined by

$$n = n_S + n_M,$$

where n_S denotes the number of energy storage elements and n_M the number of memristive elements.

5. Examples

5.1. Josephson junction circuit model

The classical circuit model for a Josephson junction consists of a parallel connection of a linear resistor r, a linear capacitor C and a flux-controlled nonlinear inductor described by the constitutive relationship $i_L = I_o \sin(k\phi_L)$, where I_o is a device parameter and $k = 4\pi\varepsilon/\bar{h}$, with ϵ and $\bar{h}$ denoting the electron charge and Planck's constant, respectively. As discussed in [2], a more rigorous quantum mechanical analysis of the junction dynamics reveals the presence of an additional small current component that can be approximated by $i = g\cos(k_o\phi)u$, for some constants g and k_o. Obviously, the latter can be associated with a flux-controlled memristor (memductor) of the form

$$q_M = \frac{g}{k_o}\sin(k_o\phi_M),$$

with $\dot{q}_M = i, \phi_M = \phi$ and $\dot{\phi}_M = u$. Figure 4 shows the more realistic circuit model for a Josephson junction consisting of a parallel connection of each of the four basic circuit elements.

From a port-Hamiltonian perspective the circuit consists of four ports: an energy storage port defined by the total energy stored in the capacitor and the inductor, a memristive port, a resistive port and an external port. Let the charge q_C and the flux linkage ϕ_L define the state variables (integrated flows) associated with the capacitor and the inductor, respectively, then the Hamiltonian (total stored energy) is given by

$$H(q_C, \phi_L) = \frac{q_C^2}{2C} - \frac{I_o}{k}\cos(k\phi_L),$$

which, according to Equation (1) and Table 1, defines an energy storage port of the form

$$\begin{aligned} f_{S_1} &= \dot{\phi}_L, \\ e_{S_1} &= \frac{\partial H}{\partial \phi_L}, \\ f_{S_2} &= \dot{q}_C, \\ e_{S_2} &= \frac{\partial H}{\partial q_C}. \end{aligned}$$

According to Equation (19), the memristive port is defined by

$$\begin{aligned} \dot{\phi}_M &= f_M, \\ e_M &= g\cos(k_o\phi_M)f_M, \end{aligned}$$

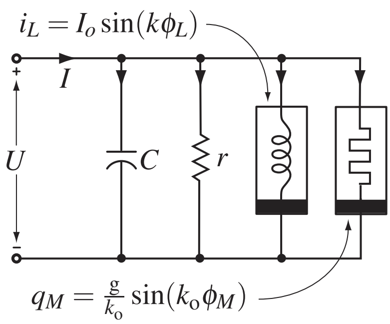

Figure 4. More realistic model of a Josephson junction [2].

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with $x_f = \phi_M$, and according to Kirchhoff's laws we obtain the following set of structure matrices:

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad G_M = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \widetilde{G}_P = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Although the presence of both resistive and memristive elements is not discussed explicitly, the system is easily extended by introducing a resistive port of the form $f_R = e_R/r$ and setting $G_R = (0 \ \ 1)^T$. On the other hand, since the resistor is linear it can also be considered as a degenerate memristor (see Section 4.4). However, in both cases, the following input-state-output port-Hamiltonian system is obtained:

$$\begin{pmatrix} \dot{\phi}_L \\ \dot{q}_C \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -g\cos(k_o\phi_M) - \dfrac{1}{r} \end{pmatrix} \begin{pmatrix} \dfrac{\partial H}{\partial \phi_L} \\ \dfrac{\partial H}{\partial q_C} \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} I,$$

together with

$$U = \frac{\partial H}{\partial q_C}$$

and

$$\dot{\phi}_M = \frac{\partial H}{\partial q_C}.$$

Note that the order of complexity is 3 ($n_S = 2$ and $n_M = 1$) since three initial conditions are needed to solve the system. Interestingly, the system is passive (note that H is bounded from below) under the condition that $rg\cos(k_o\phi_M) \geq -1$, for all admissible ϕ_M.

5.2. Mechanical system

Consider the mechanical system depicted in Figure 5. The system consists of two carts with masses m_1 and m_2, interconnected by a linear spring with elastance k, and a tapered dashpot d. Since the storage elements are linear, we have $v_1 = p_1/m_1$, $v_2 = p_2/m_2$ and $\mathrm{F}_k = kx_k$, where v_1, v_2, p_1 and p_2 are, respectively, the velocities and momenta of the two masses, and F_k and x_k are, respectively, the force and displacement of the spring. The Hamiltonian (total stored energy) is given by

$$H(p_1, p_2, x_k) = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + \frac{kx_k^2}{2}.$$

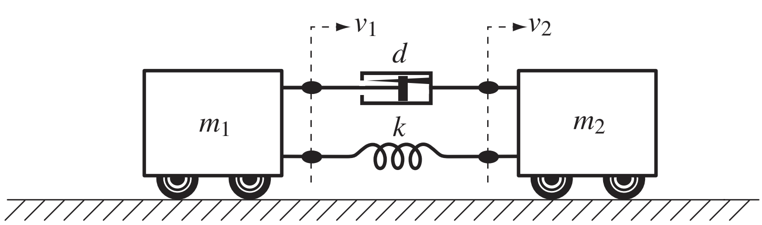

Figure 5. A mechanical mass-spring system with a tapered dashpot. Note that the shape of the pin may be machined to produce any desired memristance curve.

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According to Equation (1) and Table 1, the energy storage port assumes the form

$$\begin{aligned} f_{S_1} &= \dot{p}_1, \\ e_{S_1} &= \frac{\partial H}{\partial p_1}, \\ f_{S_2} &= \dot{p}_2, \\ e_{S_2} &= \frac{\partial H}{\partial p_2}, \\ f_{S_3} &= \dot{x}_k, \\ e_{S_3} &= \frac{\partial H}{\partial x_k}. \end{aligned} \tag{24}$$

As argued in [5], a tapered dashpot can, in principle, not be treated as an ordinary damper since the incremental damping coefficient, i.e., the mechanical resistance, depends on the piston displacement. Hence a description in terms of its associated force F_d and velocity v_d generally yields some complicated (possibly hysteretic) constitutive relationship. These difficulties are circumvented by modelling the tapered dashpot as a memristive element. Indeed, suppose that the constitutive relationship is given by a monotonically increasing function $p_d = \hat{p}_d(x_d)$, where p_d and x_d denote the memristor's momentum and displacement, respectively, then $\mathrm{F}_d = M_v(x_d)v_d$, with mechanical memristance $M_v(x_d) := \mathrm{d}\hat{p}_d(x_d)/\mathrm{d}x_d$, where $\dot{p}_d = \mathrm{F}_d$ and $\dot{x}_d = v_d$. Hence the memristive port is defined by

$$\begin{aligned} \dot{x}_d &= f_M, \\ e_M &= M_v(x_d) f_M. \end{aligned} \tag{25}$$

Since there are no inputs and outputs, the interaction port is vacuous and $G_P = 0$. Furthermore, the interconnective relationships dictate the remaining structure matrices

$$J = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}, \quad G_M = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix},$$

which by Equation (21) yield the following port-Hamiltonian equations:

$$\begin{pmatrix} \dot{p}_1 \\ \dot{p}_2 \\ \dot{x}_k \end{pmatrix} = \underbrace{\begin{pmatrix} -M_v(x_d) & M_v(x_d) & -1 \\ M_v(x_d) & -M_v(x_d) & 1 \\ 1 & -1 & 0 \end{pmatrix}}_{J-M(x_d)} \begin{pmatrix} \frac{\partial H}{\partial p_1} \\ \frac{\partial H}{\partial p_2} \\ \frac{\partial H}{\partial x_k} \end{pmatrix}, \tag{26}$$

together with

$$\dot{x}_d = (1 \;\; -1 \;\; 0) \begin{pmatrix} \frac{\partial H}{\partial p_1} \\ \frac{\partial H}{\partial p_2} \\ \frac{\partial H}{\partial x_k} \end{pmatrix} = v_1 - v_2. \tag{27}$$

Differentiating the Hamiltonian $H(x)$, where $x = (p_1, p_2, x_k)^T$, along the trajectories of the system yields the power balance of the system, i.e.,

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$$\dot{H}(x) = -\left(\frac{\partial H}{\partial x}\right)^T M(x_d)\frac{\partial H}{\partial x} \leq 0,$$

where the inequality stems from the fact that $M(x_d) := G_M M_v(x_d) G_M^T \succeq 0$, for all x_d, by assumption. This implies that the mechanical system is passive – as should be expected.

It should be pointed out, however, that for this particular system it is a coincidence that it is possible to represent the tapered dashpot as a modulated resistor since its displacement coincides with the displacement of the spring, which, in turn, is proportional to the force in the spring. Hence the system contains a conserved quantity $x_d = x_k + \mathrm{c}$, where the constant c depends on the initial condition of the overall system. In general, the states of the memristive elements in a system are independent from the states of the energy storage elements, like in the Josephson junction circuit model.

Another example of a system in which memristance plays a crucial role is the electrolytic tank system discussed in [5]. An example for which the minimal number of state equations is less than the order of complexity is briefly discussed next.

5.3. Electrical network

Consider a flux-controlled memristor (memductor), with a constitutive relationship $q_M = \hat{q}_M(\phi_M)$, connected in parallel with a linear capacitor described by $u_C = q_C/C$. Following the ideas exposed in Section 4, we obtain the following port-Hamiltonian description:

$$\dot{q}_C = -M_u(\phi_M)\frac{\partial H}{\partial q_C},$$
$$\dot{\phi}_M = \frac{\partial H}{\partial q_C},$$

with $H(q_C) = \frac{1}{2C}q_C^2$ and $M_u(\phi_M) := \mathrm{d}\hat{q}_M(\phi_M)/\mathrm{d}\phi_M$. Clearly, the system has two independent initial conditions $\phi_M(t_0)$ and $q_C(t_0)$. However, since $q_C(t) - q_C(t_0) = -[q_M(t) - q_M(t_0)]$, the system can be reduced to a single first-order differential equation,

$$\dot{\phi}_M = -\frac{1}{C}[\hat{q}_M(\phi_M) - \hat{q}_M(\phi_M(t_0)) - q_C(t_0)],$$

but still two initial conditions are needed to solve the latter.

6. Memristive systems

As pointed out in [3, 13], memristors are just a special case of a much broader class of dynamical systems called 'memristive' systems. In contrast to the basic mathematical descriptions of the memristor outlined in Section 3.1, the flux linkage in memristive systems is no longer uniquely defined by the charge, or vice versa. In [3], a current-controlled memristive one-port system is represented by

$$\begin{aligned} \dot{z} &= g_i(z,i), \\ u &= M_i(z,i)i, \end{aligned} \tag{28}$$

and a voltage-controlled memristive one-port system is represented by

$$\begin{aligned} \dot{z} &= g_u(z,u), \\ i &= M_u(z,u)u. \end{aligned} \tag{29}$$

Here i and u denote the port current and voltage, respectively, and z denotes the internal state of the system. The functions g_i and g_u are continuous vector functions of the same dimension

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as z, and M_i and M_u are scalar functions similarly defined as the memristance and memductance in Equations (12) and (13), respectively. The main peculiarity which distinguishes a memristive system from an arbitrary dynamical system is the form of the output equations or read-out maps. Indeed, as with Equation (12) (resp. (13)), it is noticed from Equation (28) [resp. (29)] that the output u (resp. i) is zero whenever the input i (resp. u) is zero, regardless of the state z which incorporates the systems memory effect, i.e., the 'no energy discharge property'. Typical examples of systems that can be modelled by Equation (28) or (29) are thermistors and discharge tubes. The next subsections show how these systems can be captured in the port-Hamiltonian framework.

6.1. The thermistor

The first example in [3] is a negative-temperature-coefficient thermistor characterised by

$$\begin{aligned} \dot{T} &= -\delta\frac{T}{C} + \frac{R_0(T_0)}{C}\exp\left[\beta\left(\frac{1}{T}-\frac{1}{T_0}\right)\right]i^2 =: g(T,i), \\ u &= \mathrm{R}_0(T_0)\exp\left[\beta\left(\frac{1}{T}-\frac{1}{T_0}\right)\right]i =: M_i(T)i, \end{aligned} \tag{30}$$

where T is the absolute body temperature of the thermistor, T_0 is the ambient temperature, C is the heat capacity, δ a dissipation constant, β is some material constant and the constant $\mathrm{R}_0(T_0)$ denotes the cold temperature resistance at $T = T_0$.

Before Equation (30) can be expressed in a port-Hamiltonian fashion, we first need to perform a change of variables. According to Table 1, the natural state variable for the thermal domain is the entropy, say S. The associated flow is the entropy flow $\dot{S}$, whereas the effort is represented by the temperature T. Since the heat capacity C is assumed constant, the relationship between S and T is given by the linear expression $S = CT$. Defining the Hamiltonian $H(S) = \frac{1}{2C}S^2$, we can rewrite 30 as

$$\begin{aligned} \dot{S} &= -\delta\frac{\mathrm{d}H}{\mathrm{d}S}(S) + \widetilde{M}_i(S)i^2, \\ u &= \widetilde{M}_i(S)i, \end{aligned} \tag{31}$$

with

$$\widetilde{M}_i(S) := \mathrm{R}_0(T_0)\exp\left[\beta C\left(\frac{1}{S}-\frac{1}{S_0}\right)\right]. \tag{32}$$

Clearly, these expressions do not yet define a proper input-state-output port-Hamiltonian system since the term i^2 renders (31) non-affine in the input. To circumvent this problem, we must extend our definition of an input-state-output port-Hamiltonian system to a description that allows for both state and input modulation in the structure matrices. In fact, system (31) is a special case of a port-Hamiltonian system with direct feed-through terms [6] of the form

$$\begin{aligned} \dot{z} &= [J(z,f) - R(z,f)]\frac{\partial H}{\partial z}(z) + [G(z,f) - P(z,f)]f, \\ e &= [G(z,f) + P(z,f)]^T\frac{\partial H}{\partial z}(z) + [K(z,f) + N(z,f)]f, \end{aligned} \tag{33}$$

where the matrices $J(z,f)$, $R(z,f)$ and $G(z,f)$ are similarly defined as J, R and G_P in Equation (10), $K(z,f)$ is a skew-symmetric matrix and $N(z,f)$ is a symmetric matrix that plays a role similar to M_f in Equation (19). Furthermore, it follows that

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$$\dot{H}(z) = e^T f - \begin{pmatrix} \frac{\partial H}{\partial z}(z) \\ f \end{pmatrix}^T \begin{pmatrix} R(z,f) & P(z,f) \\ P^T(z,f) & N(z,f) \end{pmatrix} \begin{pmatrix} \frac{\partial H}{\partial z}(z) \\ f \end{pmatrix}, \tag{34}$$

which, under the condition that

$$\begin{pmatrix} R(z,f) & P(z,f) \\ P^T(z,f) & N(z,f) \end{pmatrix} \succeq 0, \tag{35}$$

for all z, f, implies that system (33) is passive with respect to the supply rate $e^T f$ and storage function $H(z)$.

To show that (31) is indeed a special case of (33), we readily observe that (31) can be cast in the form of (33) by letting $z = S$, $f = i$, $e = u$, $J(S,i) = 0$, $R(S,i) = \delta$, $G(S,i) = \frac{1}{2}\widetilde{M}_i(S)i$, $P(S,i) = -G(S,i)$, $K(S,i) = 0$ and $N(S,i) = \widetilde{M}_i(S)$. Substituting the latter into (33) yields

$$\begin{aligned} \dot{S} &= -\delta \frac{\mathrm{d}H}{\mathrm{d}S}(S) + \left[\frac{1}{2}\widetilde{M}_i(S)i - \left(-\frac{1}{2}\widetilde{M}_i(S)i\right)\right] i, \\ u &= \underbrace{\left[\frac{1}{2}\widetilde{M}_i(S)i + \left(-\frac{1}{2}\widetilde{M}_i(S)i\right)\right]}_{=0} \frac{\mathrm{d}H}{\mathrm{d}S}(S) + \widetilde{M}_i(S)i. \end{aligned} \tag{36}$$

The power occurring at the port equals $ui = \widetilde{M}_i(S)i^2$. Hence, from an input–output perspective, the system is passive if and only if $\widetilde{M}_i(S) \geq 0$, for all S, as is already concluded in [3]. However, for the overall system to be passive with respect to the supply rate ui and storage function $H(S)$, we naturally need to pose the extra condition that $\delta \geq 0$ in order to satisfy the matrix inequality (35).

6.2. Discharge tubes

The dynamics of a discharge tube can be described by [17] as

$$\begin{aligned} \dot{n} &= -\beta n + \alpha \frac{n}{F} u^2 =: g(n,u), \\ i &= \frac{n}{F} u =: M_u(n)u, \end{aligned} \tag{37}$$

where n denotes the electron density of the tube, and α, β and F are constants depending on the dimension of the tube and the gas fillings.[3] Based on the previous developments, the port-Hamiltonian structure of a discharge tube is deduced as follows. Although there is no direct classification in terms of the domains listed in Table 1, selecting the electron density as the state, defining the Hamiltonian $H(n) = \frac{1}{2}n^2$, letting $J(n,u) = 0$, $R(n,u) = \beta$,

$G(n,u) = \frac{1}{2}\alpha M_u(n)u$, $P(n,u) = -G(n,u)$, $K(n,u) = 0$ and $N(n,u) = M_u(n)$ yields

$$\begin{aligned} \dot{n} &= -\beta \frac{\mathrm{d}H}{\mathrm{d}n}(n) + \left[\frac{1}{2}\alpha M_u(n)u - \left(-\frac{1}{2}\alpha M_u(n)u\right)\right] u, \\ i &= \underbrace{\left[\frac{1}{2}\alpha M_u(n)u + \left(-\frac{1}{2}\alpha M_u(n)u\right)\right]}_{=0} \frac{\mathrm{d}H}{\mathrm{d}n}(n) + M_u(n)u. \end{aligned} \tag{38}$$

It follows that the overall system is passive with respect to the supply rate ui and storage function $H(S)$ if and only if $\beta, M_u(n) \geq 0$, for all n.

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7. Final remarks

In this article, we have extended the existing port-Hamiltonian formalism with the inclusion of generalised memristive elements. Besides being a resistive element, a memristor also exhibits dynamics because the associated Ohmian laws are rather expressed in terms of differential equations. As a result, the state space manifold, as naturally defined by the storage elements, is augmented by the states associated with the memristive elements, and thus the order of complexity is, in general, defined by the total number of storage elements and memristors in the system. However, depending on the physical structure, there can exist constraints among some of the variables leading to conserved quantities. An example is provided by the mechanical system discussed in Section 5.2. Although memristors, like storage elements, exhibit dynamics and thus possess memory, they do not store energy. This fact is underscored by associating with the memristive port a so-called null-Hamiltonian.

In conclusion, the following remarks are in order:

- In the port-Hamiltonian formalism we can combine both the resistive and memristive ports into a single 'dissipative' port; see Figure 6. Such port can be described by

$$\begin{aligned}\mathscr{M}_{\mathscr{R}} = \; & \{(f_D, e_D) \in \mathscr{F}_D \times \mathscr{E}_D \,|\, \dot{x}_f - f_D = 0, \\ & \dot{x}_e - e_D = 0,\; D_E(\cdot)e_D - D_F(\cdot)f_D = 0\},\end{aligned}$$

  where the matrices $D_E(\cdot)$ and $D_F(\cdot)$, in general, depend on x_f and/or x_e, and satisfy

$$D_E(\cdot)D_F^T(\cdot) = D_F(\cdot)D_E^T(\cdot) \succeq 0.$$

  Note that if the dissipative port only contains purely resistive elements, we identify $D_E = R_e$ and $D_F = I$. Similarly, letting $D_E = I$ and $D_F(x_f) = M_f(x_f)$, we obtain the memristive structure (18).

- The broad generalisation of memristors, called memristive systems, are shown to be representable in an input-state-output port-Hamiltonian description with direct feed-through terms and structure matrices that are modulated by both state variables and input flows. However, the dependence of the underlying interconnection structure $\mathscr{D}$ on the input flows does not fit the definition of a Dirac structure. A further generalisation of the notion of a Dirac structure is necessary to formalise systems of the form (33). It should be noted that this problem does not occur in the context of

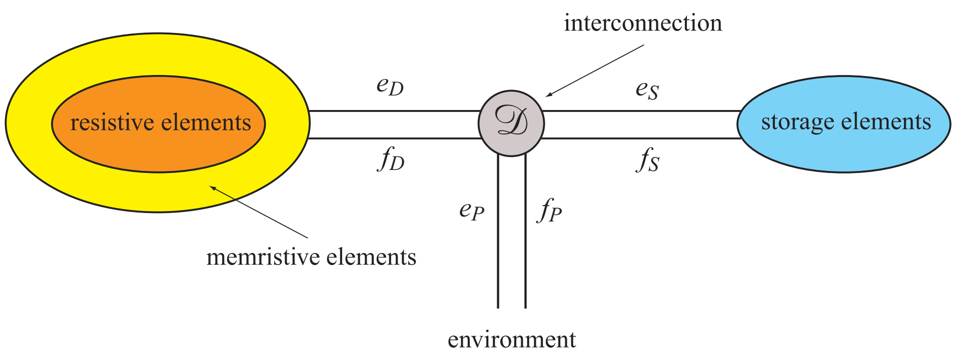

Figure 6. Port-Hamiltonian system with a single dissipative port containing memristors and linear resistors.

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switching networks because then the Dirac structure depends on non-energetic variables, such as externally controlled switches, instead of external flows or efforts [6].

- Recently, memristive systems are accompanied by two new types of systems, called 'meminductive' and 'memcapacitive' systems [13]. The resulting memory devices share many of the characteristics of memristive systems, but with a fundamental difference: they do store energy. The next step is to study under what conditions these systems can also be captured in the port-Hamiltonian formalism.

Notes

1. For linear mechanical dissipation the content function coincides with the usual Rayleigh dissipation function; see e.g. [8].
2. Note that Equation (4) is a generalisation of Tellegen's theorem; see [6] for more details.
3. Note that the choice of the input and output differs from the choice in [3].

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References

[1] L.O. Chua, Memristor – the missing circuit element, *IEEE Trans. on Circ. Theory*, CT-18(2) (September 1971), pp. 507–519.
[2] L.O. Chua, Nonlinear circuit foundations for nanodevices, part I: the four-element torus, *Proc. IEEE* 91 (November 2003), pp. 1830–1859.
[3] L.O. Chua and S.M. Kang, Memristive devices and systems, *Proc. IEEE*, 64(1976), pp. 209–223.
[4] D.B. Strukov, G.S. Snider, D.R. Stewart, and R.S. Williams, The missing memristor found, *Nature* 453 (May 2008), pp. 80–83.
[5] G.F. Oster and D.M. Auslander, The memristor: a new bond graph element, *Trans. ASME Dyn. Sys. Contr.* 94(3) (1973), pp. 249–252.
[6] V. Duindam, A. Macchelli, S. Stramigioli, and H. Bruyninckx (ed.), *Modeling and Control of Complex Physical Systems: The Port-Hamiltonian Approach*, Springer, Berlin Heidelberg 2009.
[7] P.J. Gawthrop and G.P. Bevan, Bond-graph modeling, *IEEE Control Syst. Mag.* 27(2) (2007), pp. 24–25.
[8] D. Jeltsema and J.M.A. Scherpen, Multidomain modeling of nonlinear networks and systems: energy- and power-based perspectives, *IEEE Control Syst. Mag.* 29(4) (2009), pp. 28–59.
[9] P.C. Breedveld, *Thermodynamic bond graphs and the problem of thermal inertance*, J. Franklin Inst. 314(1) (July 1982), pp. 15–40.
[10] A.J.Van der Schaft, $\mathscr{L}_2$-Gain and Passivity Techniques in Nonlinear Control, Springer-Verlag, London, 2000.
[11] J. Cervera, A.J. van der Schaft, and A. Banos, Interconnections of port-Hamiltonian systems and composition of Dirac structures, *Automatica* 43(2007), pp. 212–225.
[12] M. Dalsmo and A.J. van der Schaft, On representations and integrability of mathematical structures in energy-conserving physical systems, *SIAM J. Contr. Opt.* 37 (1999), 54–91.
[13] M. Di Ventra, Y.V. Pershin, and L.O. Chua, Circuit elements with memory: memristors, memcapacitors and meminductors, *arXiv:0901.3682*, 2009.
[14] M.M. Milić and L.A. Novak, The anti-Lagrangian equations: a missing network description, *J. Franklin Inst.* 307 (1979), pp. 183–191.
[15] R.K. Brayton. *Nonlinear reciprocal networks, Mathem. Aspects of Electrical Network Analysis. Providence, RI: American Math. Soc. (AMS)*, p. 1–15, 1971.
[16] Y.F. Lam, Formulation of normal form equations of nonlinear networks containing memristors and coupled elements, *IEEE Trans. Circ. Theory*, CT-19(6) (November 1972), pp. 585–594.
[17] K. Hong Loo, D.A. Stone, R.C. Tozer, and R. Devonshire, A dynamic conductance model of fluorescent lamp for electronic ballast design simulation, *IEEE Trans. Power Elec.* 20(5) (September 2005), pp. 1178–1185.