

University of Groningen

POINTS AND TOPOLOGIES IN RIGID GEOMETRY

Put, M. van der

Published in:
Mathematische annalen

IMPORTANT NOTE: You are advised to consult the publisher's version (publisher's PDF) if you wish to cite from it. Please check the document version below.

Document Version
Publisher's PDF, also known as Version of record

Publication date:
1995

[Link to publication in University of Groningen/UMCG research database](#)

Citation for published version (APA):
Put, M. V. D. (1995). POINTS AND TOPOLOGIES IN RIGID GEOMETRY. *Mathematische annalen*, 302(1), 81-103.

Copyright

Other than for strictly personal use, it is not permitted to download or to forward/distribute the text or part of it without the consent of the author(s) and/or copyright holder(s), unless the work is under an open content license (like Creative Commons).

Take-down policy

If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

Downloaded from the University of Groningen/UMCG research database (Pure): <http://www.rug.nl/research/portal>. For technical reasons the number of authors shown on this cover page is limited to 10 maximum.

Points and topologies in rigid geometry

M. van der Put¹, P. Schneider²

¹Rijksuniversiteit Groningen, Mathematisch Instituut, Postbus 800, NL-9700 AV Groningen, The Netherlands

²Fachbereich Mathematik, Universität Münster, Einsteinstrasse 62, D-48149 Münster, Germany

Received: 10 March 1994

1 Introduction

This paper has its origin in notes of the second author and remarks in [P]. Its purpose is to clarify the various concepts of points of an affinoid space and topologies on sets of points which can be found in [B, D, H, P, S].

The set of ordinary points of an affinoid space X over a complete non-archimedean valued field k is too small for the collection of abelian sheaves on X . The family of stalks in the ordinary points does not detect the vanishing of a sheaf. One introduces the notion of a prime filter p on X to remedy this. The collection of all prime filters is called $\mathcal{P}(X)$. For every $p \in \mathcal{P}(X)$ and every abelian sheaf F on X one can form a stalk F_p which is an abelian group. The functor $F \mapsto F_p$ is exact and $F = 0$ if all F_p are zero.

The set $\mathcal{P}(X)$ has a natural topology on it. For this topology $\mathcal{P}(X)$ is quasi-compact but not Hausdorff. The topological space $\mathcal{P}(X)$ can be identified with the projective limit of a certain family of schemes of finite type over the residue field of k . For a field k with a discrete valuation this family consists of the special fibers of all formal schemes \mathcal{X} of finite type and flat over the valuation ring k^0 of k such that the “generic fiber” $\mathcal{X} \otimes k$ is isomorphic to X .

The set $\mathcal{P}(X)$ is shown to be isomorphic to a collection of valuations $\text{Val}(X)$ on the affinoid algebra $\mathcal{O}(X)$ of X . The latter space is the building block in Huber’s approach to analytic spaces. The valuations of (real) rank 1 correspond to a certain subset $\mathcal{M}(X)$ of $\mathcal{P}(X)$. This subset consists of the maximal filters on X . There is a natural retraction map $r: \mathcal{P}(X) \rightarrow \mathcal{M}(X)$. Let $\mathcal{M}(X)_{qt}$ denote the set $\mathcal{M}(X)$ provided with the quotient topology. It turns out to be a compact Hausdorff space. The set $\mathcal{M}(X)$ plays a central role in Berkovich theory of analytic spaces. He provides $\mathcal{M}(X)$ with a certain topology which we will show to coincide with the quotient topology. This answers a question in [D].

The category of abelian sheaves on X turns out to be equivalent to the category of abelian sheaves on the topological space $\mathcal{P}(X)$. Using the retraction map $r : \mathcal{P}(X) \rightarrow \mathcal{M}(X)$ and the nice topological structure of $\mathcal{M}(X)_{qt}$ one finds that the category of overconvergent sheaves on X coincides with the category of abelian sheaves on $\mathcal{M}(X)_{qt}$. This fact was already proven in [S] and led to an easy proof of a base change theorem for rigid analytic spaces.

For part of our results about the spaces $\mathcal{P}(X)$ and $\text{Val}(X)$ one can find brief indications already in [H].

In the last section we extend our results to general rigid spaces.

2 Points and valuations

Let X be an affinoid space over a complete non-archimedean valued field k . We will use the notations $k^0 := \{\lambda \in k : |\lambda| \leq 1\}$ and \bar{k} for the residue field of k . Further π denotes some element in k with $0 < |\pi| < 1$. The algebra of functions on X will be denoted by $\mathcal{O}(X)$. This algebra has a presentation $\alpha : k\langle T_1, \dots, T_m \rangle \rightarrow \mathcal{O}(X)$ which induces a norm $|\cdot|_\alpha$ on $\mathcal{O}(X)$ (compare [BGR] 6.1.1). The subring of elements with norm ≤ 1 is denoted by $\mathcal{O}(X)_\alpha^0$. The subring of the elements with spectral (semi-)norm ≤ 1 ([BGR] 6.2.1.2) is denoted by $\mathcal{O}(X)^0$. One defines $\mathcal{O}(X)^{00} \subseteq \mathcal{O}(X)^0$ to consist of the elements f such that $\lim_{n \rightarrow \infty} |f^n|_\alpha = 0$. An $f \in \mathcal{O}(X)$ belongs to $\mathcal{O}(X)^{00}$ if and only if its spectral (semi-)norm is < 1 ([BGR] 6.2.3.2). The ring $\mathcal{O}(X)^0 / \mathcal{O}(X)^{00}$ is a finitely generated reduced \bar{k} -algebra and is called the *canonical reduction* of $\mathcal{O}(X)$. The corresponding affine scheme over \bar{k} is denoted by \bar{X} and its set of closed points by \bar{X}_{cl} . The dimension of \bar{X} is equal to the dimension of X .

The space X as a set consists of the maximal ideals in $\mathcal{O}(X)$. The finite unions of affinoid subdomains in X are called special subsets. We always give X the following Grothendieck topology:

(1) The admissible subsets of X are the special subsets.

(2) For a special subset U the family $\text{Cov}(U)$ consists of the coverings by special subsets which have a finite subcovering.

A rational set $R(f_0, \dots, f_n)$ in X is given by elements $f_0, \dots, f_n \in \mathcal{O}(X)$ generating the unit ideal and is defined to be

$$\begin{aligned} R(f_0, \dots, f_n) &:= R_X(f_0, \dots, f_n) \\ &:= \{x \in X : |f_0(x)| \geq |f_i(x)| \text{ for all } 1 \leq i \leq n\}. \end{aligned}$$

For any rational set $U \subseteq X$ the coverings of the form

$$U = \bigcup_{0 \leq i \leq n} R_U(f_i, f_0, \dots, f_{i-1}, f_{i+1}, \dots, f_n)$$

are called *rational coverings*. The rational sets together with their rational coverings generate the Grothendieck topology of X ([BGR] 8.2.2.2). This has two simple consequences for abelian sheaves on X . Firstly a sheaf is determined by its sections in rational sets. Secondly let P be a presheaf on X and let F

be its sheafification; if P satisfies the sheaf axiom for rational coverings then we have $P(U) = F(U)$ for any rational set U .

A *filter* f on X is a collection of special subsets of X satisfying:

(p1) $X \in f$ and $\emptyset \notin f$;

(p2) if $U_1, U_2 \in f$ then $U_1 \cap U_2 \in f$;

(p3) if $U \in f$ and the special subset V contains U then $V \in f$.

A *prime filter* p is a filter which in addition fulfills:

(p4) If $U \in p$ and $U = U_1 \cup U_2$ with special subsets U_i then U_1 or $U_2 \in p$.

This is, of course, equivalent to the condition:

(p4)' If $U = \bigcup_{i \in I} U_i$ is an admissible covering of $U \in p$ then $U_{i_0} \in p$ for some $i_0 \in I$.

Let $\mathcal{P}(X)$ denote the set of all prime filters on X . The filters on X are ordered with respect to inclusion. Let $\mathcal{M}(X)$ be the set of maximal filters. We have $\mathcal{M}(X) \subseteq \mathcal{P}(X)$. This is a special case of the following basic argument which later on will be used several times.

Remark 1. Let s be any family of special subsets of X which is closed with respect to finite unions. Assume that there is a filter f on X such that $f \cap s = \emptyset$. Then there is a filter p on X containing f which is maximal with respect to $p \cap s = \emptyset$ and any such filter is a prime filter.

Proof. The existence of p follows by Zorn's lemma. It remains to verify (p4) for p . Let U, U_1, U_2 be special subsets in X such that $U \in p$ and $U = U_1 \cup U_2$. If $U_1 \notin p$ then the collection p_1 of special subsets of X defined by $V \in p_1$ if V contains some $W \cap U_1$ with $W \in p$ still satisfies $X \in p_1$, (p2), and (p3). Since p_1 is larger than p we must have, by the maximality of p , that p_1 contains a member of $s \cup \{\emptyset\}$. Assume now that neither U_1 nor U_2 belong to p . Then there are $V_i \in s \cup \{\emptyset\}$ and $W_i \in p$ such that $W_i \cap U_i \subseteq V_i$. Hence $W := W_1 \cap W_2 \in p$ satisfies $W \cap U \subseteq V_1 \cup V_2 \in s \cup \{\emptyset\}$. This is a contradiction.

A *valuation* (\mathfrak{p}, A) on X is a pair consisting of a prime ideal $\mathfrak{p} \subseteq \mathcal{O}(X)$ and a valuation ring A in the field of fractions of $\mathcal{O}(X)/\mathfrak{p}$ such that:

(v1) $\mathcal{O}(X)^\circ + \mathfrak{p}/\mathfrak{p} \subseteq A$;

(v2) the intersection of all $\pi^n A$ is 0.

Usually we will denote by ϕ the residue class map from $\mathcal{O}(X)$ into the field of fractions K of $\mathcal{O}(X)/\mathfrak{p}$. Let $\text{Val}(X)$ be the set of all valuations on X .

We note that one can replace (v1) by the weaker condition

(v1)' $\phi(\mathcal{O}(X)^\circ_\alpha) \subseteq A$.

Indeed, by Noether normalization we have a finite monomorphism $k\langle T_1, \dots, T_d \rangle \hookrightarrow \mathcal{O}(X)$ such that the T_i are mapped into $\mathcal{O}(X)^\circ_\alpha$. An element $f \in \mathcal{O}(X)$ with spectral (semi-)norm ≤ 1 is integral over $k^\circ\langle T_1, \dots, T_d \rangle$ ([BGR] 6.3.4.1). This last ring is mapped by ϕ into A . Hence $\phi(f)$ is integral over A and therefore lies in A .

Also note that (v2) means that the valuation topology on A is the π -adic topology. It is sometimes convenient to replace A by its completion \hat{A} , the projective limit of the $A/\pi^n A$. It follows from (v2) that A is a subring of \hat{A} .

One easily verifies that \hat{A} has no zero divisors and is in fact a valuation ring. The field of fractions of \hat{A} will be denoted by \hat{K} . Let \mathfrak{m}_A be the maximal ideal in A . The map $A \rightarrow A/\mathfrak{m}_A$ extends to a map $\hat{A} \rightarrow A/\mathfrak{m}_A$ and therefore the maximal ideal $\mathfrak{m}_{\hat{A}}$ contains \mathfrak{m}_A . Suppose now that $a \in K \setminus A$. Then $a^{-1} \in \mathfrak{m}_A$. Hence $a^{-1} \in \mathfrak{m}_{\hat{A}}$ and $a \notin \hat{A}$. This shows that $\hat{A} \cap K = A$. More general, for any $a \in K$ one has $a\hat{A} \cap K = aA$.

The aim of this section is to find a natural bijection $\text{Val}(X) \rightarrow \mathcal{P}(X)$. Let a valuation (\mathfrak{p}, A) be given. Define a family $p = p(\mathfrak{p}, A)$ of special subsets of X as follows:

U belongs to p if it contains a rational set $R(f_0, \dots, f_n)$ such that $\phi(f_i) \in \phi(f_0)A$ for all $1 \leq i \leq n$.

Proposition 2. i) If $R(f_0, \dots, f_n) \subseteq R(g_0, \dots, g_m)$ and $\phi(f_i) \in \phi(f_0)A$ for all i then $\phi(g_j) \in \phi(g_0)A$ for all j ;

ii) p is a prime filter on X .

Proof. i. The homomorphism $\phi : \mathcal{O}(X)_\alpha^0 \rightarrow A$ extends to the homomorphism

$$\begin{aligned} \phi_1 : \mathcal{O}(X)_\alpha^0[T_1, \dots, T_n] &\rightarrow A \\ T_i &\mapsto \frac{\phi(f_i)}{\phi(f_0)}. \end{aligned}$$

By taking limits ϕ_1 extends uniquely to a homomorphism $\phi_2 : \mathcal{O}(X)_\alpha^0(T_1, \dots, T_n) \rightarrow \hat{A}$. The corresponding homomorphism $\mathcal{O}(X)\langle T_1, \dots, T_n \rangle \rightarrow \hat{K}$ factors over $\phi_3 : R := \mathcal{O}(X)\langle T_1, \dots, T_n \rangle / (f_1 - f_0 T_0, \dots, f_n - f_0 T_n) \rightarrow \hat{K}$. We have $R = \mathcal{O}(R(f_0, \dots, f_n))$. Let us note here that since the image of ϕ_3 is not $\{0\}$ it follows that $R(f_0, \dots, f_n)$ is not empty. By construction $\phi_3(R_\beta^0) \subseteq \hat{A}$ where $|\cdot|_\beta$ is the norm on R induced by the given presentations of R and $\mathcal{O}(X)$. As noted before, this implies that also $\phi_3(R^0) \subseteq \hat{A}$.

We consider now the restriction map $\mathcal{O}(R(g_0, \dots, g_m)) \rightarrow \mathcal{O}(R(f_0, \dots, f_n))$. The images of the elements $\frac{g_j}{g_0}$ have spectral (semi-)norm ≤ 1 in R . Hence

$$\frac{\phi(g_j)}{\phi(g_0)} = \phi_3 \left(\frac{g_j}{g_0} \right) \in \hat{A} \cap K = A.$$

ii. We have already seen that the empty set does not belong to p . Further $X = R(1, 0)$ obviously belongs to p . This proves (p1). The formula $R(f_0, \dots, f_n) \cap R(g_0, \dots, g_m) = R(\dots, f_i g_j, \dots)$ proves (p2). The condition (p3) holds by construction. For (p4) we first consider the case of a rational covering of X given by $g_0, \dots, g_m \in \mathcal{O}(X)$ generating the unit ideal. The fractional ideal $\phi(g_0)A + \dots + \phi(g_m)A$ in K is generated by some $\phi(g_{i_0})$. Hence $R(g_{i_0}, g_0, \dots, g_m) \in p$. In order to prove (p4) in general it suffices, by the definition of p , to consider a rational set $U = R(f_0, \dots, f_m)$. Because p is already known to be a filter (p4) for U is a consequence of (p4) for X by the following fact: For any admissible covering of U there exists a rational covering of

X whose restriction to U is finer than the given covering. This is well-known but for the convenience of the reader we include the argument.

First we refine the given covering into a rational covering

$$U = \bigcup_{0 \leq j \leq m} U_j \quad \text{where } U_j = R_U(g_j, g_0, \dots, g_{j-1}, g_{j+1}, \dots, g_m)$$

with $g_0, \dots, g_m \in \mathcal{O}(U)$ generating the unit ideal. Since g_j is invertible on U_j we find an $\varepsilon > 0$ such that

$$|g_j(x)| > \varepsilon \quad \text{for any } x \in U_j \text{ and any } 0 \leq j \leq m.$$

Now we choose elements $g'_0, \dots, g'_m \in \mathcal{O}(X)[f_0^{-1}]$ such that

$$|g_j(x) - g'_j(x)| < \varepsilon \quad \text{for any } x \in U \text{ and any } 0 \leq j \leq m.$$

Then the g'_j cannot have a common zero in U and therefore generate the unit ideal in $\mathcal{O}(U)$; also by construction we have

$$U_j = R_U(g'_j, g'_0, \dots, g'_{j-1}, g'_{j+1}, \dots, g'_m) \quad \text{for any } 0 \leq j \leq m.$$

Moreover we may multiply all the g'_j by an appropriate power of f_0 without changing the U_j . This shows that from the start we may assume without changing the given rational covering that the g_0, \dots, g_m extend to functions on X . In this situation we choose a $\lambda \in k^\times$ such that $|\lambda| < \varepsilon$. The rational covering of X given by the functions λ, g_0, \dots, g_m then has the wanted property.

We next construct a map $\mathcal{P}(X) \rightarrow \text{Val}(X)$. Let p be a prime filter on X . We put

$$\begin{aligned} \mathcal{O}_p &:= \varinjlim_{U \in p} \mathcal{O}(U) = \varinjlim \{ \mathcal{O}(U) : U \in p \text{ rational} \} \quad \text{and} \\ \mathcal{O}_p^0 &:= \varinjlim_{U \in p} \mathcal{O}^0(U) = \varinjlim \{ \mathcal{O}^0(U) : U \in p \text{ rational} \}. \end{aligned}$$

Define

$$\|f\|_p := \inf \|f\|_U \quad \text{for } f \in \mathcal{O}_p;$$

here $\| \cdot \|_U$ denotes the spectral (semi-)norm on U and the infimum is taken over all $U \in p$ such that f is defined on U . Let

$$\mathfrak{m}_p := \{ f \in \mathcal{O}_p : \|f\|_p = 0 \}.$$

Clearly \mathfrak{m}_p is an ideal in both rings \mathcal{O}_p and \mathcal{O}_p^0 . Put

$$k_p := \mathcal{O}_p / \mathfrak{m}_p \quad \text{and} \quad k_p^0 := \mathcal{O}_p^0 / \mathfrak{m}_p.$$

Proposition 3. i. \mathfrak{m}_p is the unique maximal ideal of \mathcal{O}_p ;

ii. k_p^0 is a valuation ring with field of fractions k_p ;

iii. let \mathfrak{p} be the kernel of the homomorphism $\mathcal{O}(X) \rightarrow k_p$ and let A be the preimage of k_p^0 in the field of fractions of $\mathcal{O}(X)/\mathfrak{p}$; then $\text{val}(p) := (\mathfrak{p}, A)$ is a valuation on X .

Proof. i. Let $f \in \mathcal{O}_p$ be defined on some $U \in p$. If for some $\varepsilon \in |k|$, $\varepsilon > 0$, the set $\{x \in U : |f(x)| \geq \varepsilon\}$ belongs to p then f is an invertible element of \mathcal{O}_p . If not then $\{x \in U : |f(x)| \leq \varepsilon\} \in p$ for all ε and so $f \in \mathfrak{m}_p$.

ii. We will not distinguish in notation between $f \in \mathcal{O}_p$ and its image in k_p . Suppose that $f \in k_p$ does not lie in k_p^0 . Let f be defined on some $U \in p$. Then $\{x \in U : |f(x)| \leq 1\}$ does not belong to p . Hence the set $\{x \in U : |f(x)| \geq 1\} \in p$ and so $f^{-1} \in k_p^0$.

iii. The only non-trivial item to verify is (v2). But $\pi^n A \subseteq \pi^n k_p^0$. Let $f \in \mathcal{O}_p$ represent an element in the intersection of all $\pi^n k_p^0$. Then clearly $\|f\|_p = 0$ and the image of f in k_p^0 is 0.

Theorem 4. *The maps $p(\cdot)$ and $\text{val}(\cdot)$ between $\text{Val}(X)$ and $\mathcal{P}(X)$ are each others inverses.*

Proof. Let (\mathfrak{p}, A) be given and let $p := p(\mathfrak{p}, A)$. In the proof of Proposition 2 we have seen that, for any $U = R(f_0, \dots, f_n) \in p$, there is a unique continuous extension $\phi_U : \mathcal{O}(U) \rightarrow \hat{K}$ of ϕ satisfying $\phi_U(\mathcal{O}(U)^0) \subseteq \hat{A}$. This induces a homomorphism $\phi_p : \mathcal{O}_p \rightarrow \hat{K}$ such that $\phi_p(\mathcal{O}_p^0) \subseteq \hat{A}$. For $f \in \mathfrak{m}_p$ one has

$$f \in \bigcap_{n \in \mathbb{N}} \pi^n \mathcal{O}_p^0 \quad \text{and so} \quad \phi_p(f) \in \bigcap_{n \in \mathbb{N}} \pi^n \hat{A} = \{0\}.$$

Hence ϕ_p induces an injection $k_p \subseteq \hat{K}$ such that $k_p^0 \subseteq \hat{A}$. Let $f \in k_p \cap \hat{A}$. One can represent f by some element $g \in \mathcal{O}(U)$ with $U \in p$. The condition $\phi_U(g) \in \hat{A}$ implies that $\{x \in U : |g(x)| \leq 1\} \in p$ and therefore $f \in k_p^0$. Hence $k_p^0 = k_p \cap \hat{A}$. This implies $k_p^0 \cap K = A$ and we have proved that $\text{val}(p) = (\mathfrak{p}, A)$. Let a prime filter p be given and let $(\mathfrak{p}, A) := \text{val}(p)$. The filter induces a homomorphism $\psi : \mathcal{O}(X) \rightarrow k_p$. It is easily seen that a rational subset $U = R(f_0, \dots, f_n)$ is contained in p if and only if $\psi(f_i) \in \psi(f_0)k_p^0$ for all $1 \leq i \leq n$. Since $A = K \cap k_p^0$ it follows that $p(\mathfrak{p}, A) = p$.

Corollary 5. *Let p be a prime filter on X and let (\mathfrak{p}, A) be the corresponding valuation; we then have:*

- i. \mathcal{O}_p is a Henselian local ring;
- ii. k_p^0 is Henselian with respect to its prime ideal $\sqrt{\pi k_p^0}$;
- iii. $\hat{A} = \varinjlim k_p^0 / \pi^n k_p^0$; in general $\hat{A} \neq k_p^0$;
- iv. there is a chain of subfields $\hat{K} \supseteq M \supseteq L$ such that
 - \hat{K} is finite over M ,
 - $L = k(T_1, \dots, T_d)$ is purely transcendental over k with $d \leq \dim(X)$, and
 - L is π -dense in M , i.e., for any $m \in M$ there is a $\ell \in L$ with $\ell - m \in \pi \hat{A}$;
- v. $\text{Krull dim}(A) \leq \dim(X) + 1$.

Proof. i. First of all we observe that by Hensel's lemma \mathcal{O}_p is Henselian if $(\mathcal{O}_p)_{\text{red}}$ is Henselian. Hence we may assume that X is reduced. According to one of the equivalent conditions for being Henselian we have to show that

any finite free \mathcal{O}_p -algebra R is a product of local rings. In other words we have to show that any idempotent \bar{e} in $R \otimes_{\mathcal{O}_p} k_p$ lifts to an idempotent in R . We clearly find an $U \in p$ and a finite free $\mathcal{O}(U)$ -algebra R' such that $R = R' \otimes_{\mathcal{O}(U)} \mathcal{O}_p$. Since X is reduced $\| \cdot \|_U$ is a complete norm on $\mathcal{O}(U)$ ([BGR] 6.2.4.1). According to [BGR] 6.1.1.6 and 6.1.3.3 there is a Banach algebra norm $\| \cdot \|$ on R' such that $(R', \| \cdot \|)$ is a normed $(\mathcal{O}(U), \| \cdot \|_U)$ -algebra. After replacing U by a smaller set in p we may assume that \bar{e} lifts to an element $e_0 \in R'$ such that $\|e_0^2 - \bar{e}\| < 1$. Define a sequence $(e_n)_{n \geq 0}$ of elements in R' by

$$e_{n+1} := 3e_n^2 - 2e_n^3.$$

By construction one has

$$\begin{aligned} e_{n+1} - e_n &= (e_n^2 - e_n)(1 - 2e_n) \quad \text{and} \\ e_{n+1}^2 - e_{n+1} &= 4(e_n^2 - e_n)^3 - 3(e_n^2 - e_n)^2. \end{aligned}$$

It follows that the sequence $(e_n)_{n \geq 0}$ converges to an idempotent $e \in R'$ which lifts \bar{e} .

ii. This is proved in a similar way as the first assertion.

iii. The first statement is easily verified. An example for the inequality is provided by $X := Sp(k\langle T \rangle)$ and the valuation ring $A := \left\{ \frac{f}{g} : f, g \in k\langle T \rangle, g \neq 0, \text{ and } \|f\|_X \leq \|g\|_X \right\}$.

iv. After dividing $\mathcal{O}(X)$ by a prime ideal we may suppose that ϕ is injective. The field of fractions K of $\mathcal{O}(X)$ is π -dense in \hat{K} . The algebra $\mathcal{O}(X)$ is finite over some $R := k\langle T_1, \dots, T_d \rangle$ with d equal to the dimension of X . Let $M \subseteq \hat{K}$ denote the π -completion (in the obvious sense) of the field of fractions of R . Then \hat{K} is finite over M and M has $k\langle T_1, \dots, T_d \rangle$ as π -dense subfield.

v. In the valuation ring A the smallest non-zero prime ideal is $I := \sqrt{\pi A}$. Indeed, let $J \subset I$ be a prime ideal which is smaller than I . Choose $a \in J$. Let $n \in \mathbb{N}$ and suppose that $aA + \pi^n A = aA$. Then $\pi^n \in aA$ and $\pi \in J$ and one finds the contradiction $I \subseteq J$. Hence $aA + \pi^n A = \pi^n A$ for all $n \in \mathbb{N}$, hence $a \in \bigcap_{n \in \mathbb{N}} \pi^n A = \{0\}$, and hence $J = 0$.

We will show that $\text{Krull dim}(A/I) \leq \dim(X)$. The map $\phi : \mathcal{O}(X) \rightarrow K$ has the properties $\phi(\mathcal{O}(X)^0) \subseteq A$ and $\phi(\mathcal{O}(X)^{00}) \subseteq I$. The kernel of the induced homomorphism $\mathcal{O}(\bar{X}) \rightarrow A_1 := A/I$ is a prime ideal corresponding to a closed irreducible (and reduced) subset $Y \subseteq \bar{X}$. It can be seen that the induced injective map $\mathcal{O}(Y) \rightarrow A_1$ gives a bijection between the fields of fractions of the two rings. In other terms: A_1 is a valuation ring of the field of fractions of $\mathcal{O}(Y)$ containing $\mathcal{O}(Y)$. It is well known that the Krull dimension of such a valuation ring is \leq the dimension of Y .

An ordinary point $x \in X$ can be identified with the prime filter $\{U : x \in U\}$. The valuation corresponding to x is given by the homomorphism $\mathcal{O}(X) \rightarrow \mathcal{O}(X)/\mathfrak{m}_x$ where \mathfrak{m}_x denotes the maximal ideal corresponding to x . In this way X can be embedded in $\mathcal{P}(X)$ and $\text{Val}(X)$, respectively.

A valuation (p, A) on X is called an *analytic point* of X if the valuation ring A has rank 1. Recall that a valuation ring has rank 1 if and only if its

Krull dimension is 1 ([M] 10.7). Obviously any ordinary point gives rise to an analytic point. Let p_1 and p_2 be two prime filters on X with corresponding valuations (\mathfrak{p}_i, A_i) . Then $p_1 \subseteq p_2$ if and only if $\mathfrak{p}_1 = \mathfrak{p}_2$ and A_2 is a localization of A_1 with respect to some prime ideal of A_1 . This follows easily from Theorem 4. In particular the prime filter p is maximal if and only if the corresponding valuation ring A has rank 1. In other words: The subset of analytic points corresponds to the subset $\mathcal{M}(X)$ of maximal filters.

The analytic points of X can also be described in the following way. An analytic point a of X is a semi-norm $|\cdot|_a : \mathcal{O}(X) \rightarrow \mathbb{R}_{\geq 0}$ on the affinoid algebra $\mathcal{O}(X)$ of X satisfying:

$$(a1) \quad |f + g|_a \leq \max(|f|_a, |g|_a) \text{ for all } f, g \in \mathcal{O}(X);$$

$$(a2) \quad |fg|_a = |f|_a |g|_a \text{ for all } f, g \in \mathcal{O}(X);$$

$$(a3) \quad |\lambda|_a = |\lambda| \text{ for any } \lambda \in k;$$

(a4) $|\cdot|_a : \mathcal{O}(X) \rightarrow \mathbb{R}_{\geq 0}$ is continuous with respect to the norm topology on $\mathcal{O}(X)$.

Still another characterization of analytic points will be given in Lemma 6.

As an example we give an explicit description of the prime filters on the disk $D = Sp(k\langle T \rangle)$. For convenience we suppose that k is algebraically closed. For any $d \in D$ and any $\rho \in |k^\times|$ with $0 < \rho \leq 1$ we write $D(d, \rho) := \{x \in D : |x - d| \leq \rho\}$. This is a closed disk. We will write $D(d, \rho)^*$ for any subset of D of the form

$$\{x \in D(d, \rho) : |x - a_i| = \rho \text{ for } i = 1, \dots, m\} \quad \text{with } a_1, \dots, a_m \in D(d, \rho).$$

This is a formal open subset of $D(d, \rho)$. We start by describing the analytic points of D (compare also [B] 1.4.4).

Let $|\cdot|_a$ be an analytic point. For every $d \in D$ one defines the real number $\rho(d) := |T - d|_a$. Then $D(d, \rho)$ belongs to a if and only if $\rho(d) \leq \rho$. This follows from the definition of the filter attached to $|\cdot|_a$.

For convenience we extend the notation, namely $D(d, \rho) := \{x \in D : |x - d| \leq \rho\}$ for any $\rho \in \mathbb{R}$, $0 \leq \rho \leq 1$.

For two points $d_1, d_2 \in D$ one has

$$D(d_1, \rho(d_1)) \subseteq D(d_2, \rho(d_2)) \text{ or } D(d_2, \rho(d_2)) \subseteq D(d_1, \rho(d_1))$$

because $|d_1 - d_2| = |(T - d_1) - (T - d_2)|_a \leq \max(|T - d_1|_a, |T - d_2|_a)$. Now there are several possibilities: If $\bigcap \{D(d, \rho(d)) : d \in D\}$ is

(1) an (ordinary) point $x \in D$ then a is the filter of all special subsets containing x ;

(2) equal to $D(d_0, \rho_0)$ with $\rho_0 \in |k^\times|$ then a consists of the special subsets containing some $D(d_0, \rho_0)^*$; further $|\cdot|_a$ is the spectral norm (or supremum norm) on $D(d_0, \rho_0)$;

(3) equal to $D(d_0, \rho_0)$ with $\rho_0 \notin |k^\times|$ then $|\cdot|_a$ still is the supremum norm on $D(d_0, \rho_0)$; further $D(d, \rho)^*$ belongs to a if and only if $D(d, \rho)^* \supseteq \{x \in D : \rho_1 \leq |x - d_0| \leq \rho_2\}$ with $\rho_1 < \rho_0 < \rho_2$ and $\rho_1, \rho_2 \in |k^\times|$;

(4) empty then $|\cdot|_a$ is the infimum of the supremum norms taken over the sets $D(d, \rho(d))$; a special subset belongs to a if and only if it contains some $D(d, \rho(d))$; this case can only occur if the field k is not maximally complete.

The description of the prime filters p which are not maximal is more complicated. We will use the results and notations of the subsequent paragraph for this. Let $a := r(p)$. For an a of the type (1), (3), or (4) above one sees that $\tilde{a} = a$; hence $p = a$. For a of type (2), we take for notational convenience $D = (d_0, \rho_0)$. For any $\bar{\lambda} \in \bar{k}$ one chooses a $\lambda \in k^0$ with residue $\bar{\lambda}$. Define $p_{\bar{\lambda}}$ to be the family of special subsets of D containing for some $\rho \in |k^\times|$, $\rho < 1$, the ring domain $\{x \in D : \rho < |x - \lambda| < 1\}$. This is in fact a prime filter and $r(p_{\bar{\lambda}}) = a$. Using Lemma 6 one can show that any $p \neq a$ with $r(p) = a$ is equal to $p_{\bar{\lambda}}$ for a unique $\bar{\lambda} \in \bar{k}$. This finishes the description of all prime filters on D .

Coming back to the general situation we finally describe a natural retraction map

$$r : \mathcal{P}(X) \rightarrow \mathcal{M}(X).$$

Let p be any prime filter on X . The map $f \mapsto \|f\|_p$ satisfies (a1)–(a4), as is easily seen. We denote by $r(p)$ the analytic point of X (resp. the maximal filter) which corresponds to $\| \cdot \|_p$. One has that $R(f_1, \dots, f_n)$ belongs to $r(p)$ if and only if $\|f_0\|_p \geq \|f_i\|_p$ for all $1 \leq i \leq n$. Moreover, this is equivalent to $R(\rho f_0, f_1, \dots, f_n) \in p$ for every $\rho \in \sqrt{|k^\times|}$, $\rho > 1$. The notation which we use here and in the sequel is

$$R(\rho f_0, f_1, \dots, f_n) := \{x \in X : \rho |f_0(x)| \geq |f_i(x)| \text{ for all } 1 \leq i \leq n\};$$

this is a rational subset ([BGR] 7.2.3).

Lemma 6. i. A prime filter p is maximal if and only if it has the property that $R(f_0, \dots, f_n) \in p$ if $R(\rho f_0, f_1, \dots, f_n) \in p$ for every $\rho \in \sqrt{|k^\times|}$, $\rho > 1$;

ii. for a prime filter p the filter $r(p)$ is the unique maximal filter containing p ;

iii. if A is the valuation ring corresponding to the prime filter p then the valuation ring corresponding to $r(p)$ is the localization of A at the prime ideal $\sqrt{\pi}A$.

Proof. Obvious.

We will need some further properties of this retraction map.

Lemma 7. Let a be an analytic point of X with corresponding valuation (\mathfrak{p}, A) ; we have:

i. define \tilde{a} to be the family of all special subsets of X which contain $R(\rho f_0, f_1, \dots, f_n)$ for some $R(f_0, \dots, f_n) \in a$ and some $\rho \in \sqrt{|k^\times|}$, $\rho > 1$; then \tilde{a} is equal to the intersection of all prime filters p with $r(p) = a$;

ii. the set $r^{-1}(\{a\})$ can be identified with the set of all valuation rings in the residue field A/\mathfrak{m}_A of A which contain the image of the map $\mathcal{O}(X)^0 \rightarrow \mathcal{O}(\bar{X}) \rightarrow A/\mathfrak{m}_A$.

Proof. i. We know already that the prime filters p containing \tilde{a} are precisely those with $r(p) = a$. In particular \tilde{a} is contained in the intersection of all those

prime filters. In order to see the reverse inclusion observe first that \tilde{a} is a filter. If $U \notin \tilde{a}$ is any special subset then applying Remark 1 to the family $s := \{U\}$ we obtain a prime filter containing \tilde{a} but not U . ii. This follows from Lemma 6.iii and [M] 10.1.

3 Topologies on $\mathcal{P}(X)$ and $\mathcal{M}(X)$

Let Z be a scheme of finite type over a field F . We denote by Z_{cl} the variety over F consisting of the closed points of Z . A prime filter on Z_{cl} is a collection of Zariski open subsets of Z_{cl} having the properties (p1)–(p4) as in Sect. 2. Let $\mathcal{P}(Z_{cl})$ be the set of all those prime filters. Every point $z \in Z$ defines an irreducible closed subset $\overline{\{z\}}$ and a prime filter p_z consisting of the open subsets $U \subseteq Z_{cl}$ such that $U \cap \overline{\{z\}} \neq \emptyset$. We want to prove the following amusing fact.

Lemma 1. *The map $Z \rightarrow \mathcal{P}(Z_{cl})$ given by $z \mapsto p_z$ is a bijection.*

Proof. The injectivity of the map is clear. For the surjectivity it suffices to consider an affine scheme Z . Let p be a prime filter on Z_{cl} and define

$$\mathcal{O}_p := \varinjlim_{U \in p} \mathcal{O}(U) \quad \text{and} \quad \mathfrak{m}_p := \varinjlim_{U \in p} \{f \in \mathcal{O}(U) : \{y \in U : f(y) \neq 0\} \notin p\}.$$

Here \mathcal{O} denotes the structure sheaf of Z as well as its inverse image on Z_{cl} . Clearly \mathfrak{m}_p is an ideal in \mathcal{O}_p . If $f \in \mathcal{O}_p \setminus \mathfrak{m}_p$ is defined on $U \in p$ then the set $\{y \in U : f(y) \neq 0\}$ belongs to p and so f is invertible in \mathcal{O}_p . Hence \mathcal{O}_p is a local ring. The kernel of $\phi : \mathcal{O}(Z) \rightarrow \mathcal{O}_p \rightarrow \mathcal{O}_p/\mathfrak{m}_p$ is some prime ideal I of $\mathcal{O}(Z)$. For $f \in \mathcal{O}(Z)$ put $Z_f := \{y \in Z : f(y) \neq 0\}$ which is Zariski open in Z ; then $Z_f \cap Z_{cl} \in p$ if and only if $f \notin I$. Since the Z_f form a basis of the Zariski topology it follows from (p4) that \mathcal{O}_p is the direct limit of all $\mathcal{O}(Z_f)$ with $f \notin I$. Therefore \mathcal{O}_p is the localization of $\mathcal{O}(Z)$ with respect to the prime ideal I . The prime ideal I is a point z of Z and $p = p_z$.

On $\mathcal{P}(Z_{cl})$ we put the topology induced by its bijection with Z . One can verify that the open sets in $\mathcal{P}(Z_{cl})$ are the sets $\{p \in \mathcal{P}(Z_{cl}) : U \in p\}$ where $U \subseteq Z_{cl}$ is a Zariski open set.

We return now to the affinoid space X over k and its set of prime filters $\mathcal{P}(X)$. On this set we define a topology by taking as a basis for the open sets the sets $\tilde{U} := \{p \in \mathcal{P}(X) : U \in p\}$ where $U \subseteq X$ is a special subset. If U is an affinoid subdomain of X then the obvious bijection $\mathcal{P}(U) \xrightarrow{\sim} \tilde{U}$ is a homeomorphism. The next fact is a central result in [H].

Lemma 2. i. *The space $\mathcal{P}(X)$ is quasi-compact;*
 ii. *for any analytic point $a \in \mathcal{M}(X) \subseteq \mathcal{P}(X)$ the closure of $\{a\}$ is the set $r^{-1}(\{a\})$, where r is the retraction map.*

Proof. i. Let $\{\tilde{U}_j\}_{j \in J}$ be a covering of $\mathcal{P}(X)$. Assume that this covering has no finite subcovering. By (p4) this means that X does not belong to the family

s of all finite unions of sets U_j . Applying Remark 2.1 to the filter $\{X\}$ we obtain a prime filter p such that $p \cap s = \emptyset$. On the other hand we have $p \in \tilde{U}_{j_0}$ for some $j_0 \in J$ and hence $U_{j_0} \in p \cap s$. This is a contradiction. ii. Let a prime filter p be given. By the definition of the topology we have

$$\mathcal{P}(X) \setminus \overline{\{p\}} = \{p' : p' \text{ contains some special subset } U \notin p\}$$

and hence $\overline{\{p\}} = \{p' : p' \subseteq p\}$. The assertion follows now from Lemma 2.6.ii.

The maximal ideal corresponding to a point in X is the kernel of a surjective k -algebra homomorphism $\psi : \mathcal{O}(X) \rightarrow \ell$ where ℓ is a finite extension of k . There is an induced \bar{k} -algebra homomorphism $\bar{\psi} : \mathcal{O}(\bar{X}) \rightarrow \bar{\ell}$. The kernel of $\bar{\psi}$ is a maximal ideal of $\mathcal{O}(\bar{X})$. This defines a map

$$red : X \rightarrow \bar{X}_{cl}$$

which is called the *canonical reduction map*. It is surjective and for every Zariski open $V \subseteq \bar{X}_{cl}$ the preimage $red^{-1}V$ is a special subset in X ([BGR] 7.1.5.2 and 4).

Lemma 3. *There is a natural surjective continuous map $red : \mathcal{P}(X) \rightarrow \mathcal{P}(\bar{X}_{cl}) \cong \bar{\mathcal{X}}$.*

Proof. For $p \in \mathcal{P}(X)$ define

$$red(p) := \{V \subseteq \bar{X}_{cl} \text{ open} : red^{-1}V \in p\}.$$

It is clear that $red(p)$ is a prime filter on \bar{X}_{cl} . The continuity of red follows from the definition of the topologies. Let a prime filter q on \bar{X}_{cl} be given. The family

$$f := \text{all special subsets } U \subseteq X \text{ such that } U \supseteq red^{-1}V \text{ for some } V \in q$$

is a filter on X . The family

$$s := \text{all special subsets } U \subseteq X \text{ such that } U \subseteq red^{-1}W \\ \text{for some open } W \subseteq \bar{X}_{cl} \text{ with } W \notin q$$

is closed with respect to finite unions and fulfills $f \cap s = \emptyset$. Applying the Remark 2.1 we obtain a prime filter p on X such that $p \supseteq f$ and $p \cap s = \emptyset$. The former property implies that $q \subseteq red(p)$ and the latter that $red(p) \subseteq q$.

These considerations can be generalized in the following way. Consider the rational covering $\{U_0, \dots, U_n\}$ of X given by elements $f_0, \dots, f_n \in \mathcal{O}(X)$ generating the unit ideal. For every i, j one has that $\bar{U}_i \cap \bar{U}_j$ is an open affine subscheme of \bar{U}_i ([BGR] 7.2.6.3). The affine schemes \bar{U}_i are glued together over these open subschemes. The result is a reduced scheme of finite type over \bar{k} which we denote by $(\bar{X}, f.)$. The canonical reduction maps $U_i \rightarrow (\bar{U}_i)_{cl}$ glue to a map

$$red(f.) : X \rightarrow \overline{(\bar{X}, f.)}_{cl}$$

which is called the reduction map of the rational covering given by $f. = \{f_0, \dots, f_n\}$. Again it is surjective and for every open $V \subseteq \overline{(\bar{X}, f.)}_{cl}$ the

preimage $red(f_\cdot)^{-1}V$ is a special subset in X . As in Lemma 3 one can prove that this map extends naturally to a surjective continuous map

$$red(f_\cdot) : \mathcal{P}(X) \rightarrow \mathcal{P}(\overline{(X, f_\cdot)_{cl}}) \cong \overline{(X, f_\cdot)}.$$

This leads to a continuous map

$$\mathcal{P}(X) \rightarrow \varprojlim \overline{(X, f_\cdot)}$$

in which the last space is provided with the topology of the projective limit.

Theorem 4. $\mathcal{P}(X) \xrightarrow{\sim} \varprojlim \overline{(X, f_\cdot)}$ is a (bijective) homeomorphism.

Proof. First note that for any rational set $U = R(g_0, \dots, g_m)$ in X the subset \bar{U}_{cl} in $\overline{(X, g_\cdot)_{cl}}$ is open and $U = red(g_\cdot)^{-1}\bar{U}_{cl}$. This immediately implies that the map in question is injective. But it also shows that for any $\{p(f_\cdot)\}_f$ in the projective limit on the right hand side the filter

$$p := \text{all special subsets } U \subseteq X \text{ such that } U \supseteq red(f_\cdot)^{-1}V$$

for some f_\cdot and some $V \in p(f_\cdot)$

on X is a prime filter. One easily checks that p is a preimage of $\{p(f_\cdot)\}_f$. Hence our map is bijective. In order to see that the map is open let $U = R(g_0, \dots, g_m)$ be again a rational set in X . The image of $\bar{U} \subseteq \mathcal{P}(X)$ consists of the elements $\{p(f_\cdot)\}_f$ in the projective limit such that $p(g_\cdot)$ contains the open set $\bar{U}_{cl} \subseteq \overline{(X, g_\cdot)_{cl}}$.

Assume the valuation of k to be discrete and let π generate the maximal ideal of k^0 . The Raynaud functor associates to every formal scheme \mathcal{X} of finite type and flat over k^0 a rigid analytic space $X := \mathcal{X} \otimes k$ called the “generic fiber” of \mathcal{X} . If the formal scheme is affine, i.e., \mathcal{X} is the formal spectrum of a flat k^0 -algebra of the type $R = k^0\langle T_1, \dots, T_n \rangle / I$ then X is the affinoid space with algebra $\mathcal{O}(X) = R \otimes_{k^0} k$. One finds a reduction map $red : X \rightarrow \bar{\mathcal{X}}_{cl}$ where $\bar{\mathcal{X}} := \text{Spec}(R \otimes_{k^0} \bar{k})$. In the general case one has a similar reduction map where $\bar{\mathcal{X}} := \mathcal{X} \otimes_{k^0} \bar{k}$ is the special fiber of \mathcal{X} . Blowing ups of \mathcal{X} with respect to an ideal with support in the special fiber do not change X but change the reduction map.

Let X be a reduced affinoid space over k . The ring $\mathcal{O}(X)^0$ is noetherian, π -adically complete and $\mathcal{O}(X)^0 / (\pi)$ is of finite type over \bar{k} ([BGR] 6.4.1.6). This makes $\mathcal{X} := \text{Spf}(\mathcal{O}(X)^0)$ into a formal scheme of finite type and flat over k^0 with generic fiber X . The reduction map corresponding to \mathcal{X} is what we have called the canonical reduction map. For any rational covering $\{U_1, \dots, U_n\}$ of X given by elements f_0, \dots, f_n the glueing of the $\text{Spf}(\mathcal{O}(U_i)^0)$ gives a formal scheme $\mathcal{X}(f_\cdot)$ of finite type and flat over k^0 with generic fiber X and with reduction as explained before Theorem 4. This new formal scheme can be obtained from \mathcal{X} by blowing up some ideal supported in the special fiber. One can show that the family $\mathcal{X}(f_\cdot)$ is cofinal in the collection of all formal

schemes \mathcal{Y} of the type above with X as generic fiber. This is the interpretation of Theorem 4 in terms of formal schemes.

For more general fields this interpretation remains valid but the formulation is more complicated due to the fact that the rings $k^0\langle T_1, \dots, T_n \rangle$ are not noetherian.

Definition. *The Berkovich topology on $\mathcal{M}(X)$ is the coarsest topology such that, for every $f \in \mathcal{O}(X)$, the function $a \mapsto |f|_a$ on $\mathcal{M}(X)$ is continuous.*

The description of analytic points in terms of semi-norms on $\mathcal{O}(X)$ shows that the functions in the above Definition separate points in $\mathcal{M}(X)$. Hence the Berkovich topology is Hausdorff. In the following $\mathcal{M}(X)$ always is equipped with the Berkovich topology.

Theorem 5. *The Berkovich topology on $\mathcal{M}(X)$ coincides with the quotient topology derived from the topology on $\mathcal{P}(X)$ and the retraction map $r : \mathcal{P}(X) \rightarrow \mathcal{M}(X)$; $\mathcal{M}(X)$ is compact.*

Proof. We temporarily write $\mathcal{M}(X)_B$ for $\mathcal{M}(X)$ equipped with the Berkovich topology. It suffices to show that the map $r : \mathcal{P}(X) \rightarrow \mathcal{M}(X)_B$ is continuous. Since the first space is quasi-compact and hence the second one is compact the map then must be a quotient map. It is easy to see that the open subsets

$$M(f_0, \dots, f_n) := \{a \in \mathcal{M}(X) : |f_0|_a > |f_i|_a \text{ for all } 1 \leq i \leq n\}$$

for any $f_0, \dots, f_n \in \mathcal{O}(X)$ generating the unit ideal form a basis of the topology of $\mathcal{M}(X)_B$. Let p be prime filter on X . We have $r(p) \in M(f_0, \dots, f_n)$ if and only if $R(\rho f_0, f_1, \dots, f_n) \in r(p)$ for some $\rho \in \sqrt{|k^\times|}$, $\rho < 1$, which by Lemma 2.7.i is equivalent to $R(\rho' f_0, f_1, \dots, f_n) \in p$ for all $\rho' > \rho$. It follows that

$$(*) \quad r^{-1}M(f_0, \dots, f_n) = \bigcup \{R(\rho f_0, f_1, \dots, f_n)^\sim : \rho \in \sqrt{|k^\times|} \text{ and } \rho < 1\}.$$

The right hand side is open in $\mathcal{P}(X)$.

The second statement in the above Theorem is due to Berkovich ([B]). But our proof is completely different.

4 Sheaves on X , $\mathcal{M}(X)$, and $\mathcal{P}(X)$

The inclusion $X \subseteq \mathcal{P}(X)$ is not very useful for comparing sheaves on the two spaces. But viewing X with its Grothendieck topology as well as the topological space $\mathcal{P}(X)$ as sites the functor $U \mapsto \tilde{U}$ defines a morphism of sites

$$\sigma : \mathcal{P}(X) \rightarrow X.$$

The following result was proved in [H].

Theorem 1. *The functors σ_* and σ^* are quasi-inverse equivalences between the categories of abelian sheaves on $\mathcal{P}(X)$ and on X .*

Proof. Let F be an abelian sheaf on X . Then σ^*F is the sheaf associated with the presheaf P given by

$$P(N) = \varinjlim_{N \subseteq \tilde{U}} F(U) \text{ for any open } N \subseteq \mathcal{P}(X).$$

Consider the case where N is of the form $N = \tilde{V}$ for some special subset $V \subseteq X$. A special subset U such that $\tilde{V} \subseteq \tilde{U}$ satisfies $V \subseteq U$. Indeed, suppose there is a point $x \in V \setminus U$. Its neighbourhood filter lies in \tilde{V} but does not contain U which is a contradiction. It follows that $P(\tilde{V}) = F(V)$. In particular P already satisfies the sheaf axiom for coverings consisting of open sets of the form \tilde{V} . Since the latter form a basis of the topology of $\mathcal{P}(X)$ sheafification leaves the sections in such a set \tilde{V} unchanged: We have

$$(\sigma_*\sigma^*F)(V) = (\sigma^*F)(\tilde{V}) = P(\tilde{V}) = F(V) \text{ and hence } \sigma_*\sigma^*F = F.$$

Also for an abelian sheaf S on $\mathcal{P}(X)$ we obtain

$$(\sigma^*\sigma_*S)(\tilde{V}) = (\sigma_*S)(V) = S(\tilde{V}) \text{ and hence } \sigma^*\sigma_*S = S.$$

This proof in particular shows that for any abelian sheaf F on X and any special subset $U \subseteq X$ we have

$$(\sigma^*F)(\tilde{U}) = F(U).$$

It follows that

$$F_p := \varinjlim_{U \in \mathcal{P}} F(U) = (\sigma^*F)_p \text{ for any } p \in \mathcal{P}(X)$$

where the right hand side is the stalk in the usual sense of the sheaf σ^*F in the point p . As a consequence we obtain that the functors $F \mapsto F_p$ are exact and that $F = 0$ if all $F_p = 0$.

An abelian (pre)sheaf F on X is called *overconvergent* if, for all $f_0, \dots, f_n \in \mathcal{O}(X)$ generating the unit ideal, we have

$$F(R(f_0, \dots, f_n)) = \varinjlim F(R(\rho f_0, f_1, \dots, f_n))$$

where the limit is taken over all $\rho \in \sqrt{|k^\times|}$, $\rho > 1$. One can verify that this is the same notion as that of a conservative sheaf in [S] as well as that of a constructible sheaf in [P]. We note that the sheaf associated to an overconvergent presheaf is also overconvergent. A sheaf on $\mathcal{P}(X)$ will be called overconvergent if the corresponding sheaf on X is overconvergent. The following theorem is one of the main results in [S]. The proof given here is however more direct.

Theorem 2. *The retraction map $r : \mathcal{P}(X) \rightarrow \mathcal{M}(X)$ gives rise to quasi-inverse equivalences r_* and r^* between the category of overconvergent sheaves on $\mathcal{P}(X)$ (or on X) and the category of all abelian sheaves on $\mathcal{M}(X)$.*

Proof. (1) First we show that for any abelian sheaf T on $\mathcal{M}(X)$ the sheaf r^*T is overconvergent. Since sheafification preserves the property of being overconvergent it suffices to check that the presheaf inverse image of T is overconvergent. This is the presheaf P on $\mathcal{P}(X)$ given by

$$P(N) = \varinjlim_{N \subseteq r^{-1}M} T(M).$$

Let $U = R(f_0, \dots, f_n)$ be a rational subset of X . Then

$$r(\tilde{U}) = \{a \in \mathcal{M}(X) : U \in a\} = \{a \in \mathcal{M}(X) : |f_0|_a \geq |f_1|_a, \dots, |f_n|_a\}$$

is a closed subset in $\mathcal{M}(X)$. A fundamental system of open neighbourhoods of $r(\tilde{U})$ is given by $\{M(\varepsilon) : \varepsilon > 0\}$ with

$$M(\varepsilon) := \{a \in \mathcal{M}(X) : (1 + \varepsilon)|f_0|_a > |f_1|_a, \dots, |f_n|_a\}.$$

We have $P(\tilde{U}) = \varinjlim T(M(\varepsilon))$ and this implies that

$$P(\tilde{U}) = \varinjlim P(R(\rho f_0, f_1, \dots, f_n)^\sim)$$

where the direct limit is taken over all $\rho \in \sqrt{|k^\times|}$, $\rho > 1$. The presheaf P therefore is overconvergent.

(2) Next we show that for any overconvergent sheaf S on $\mathcal{P}(X)$ and any analytic point a of X the natural map

$$(r_*S)_a \xrightarrow{\cong} S_a$$

is bijective. By the construction of the topology S_a is the direct limit of all $S(\tilde{U})$ with $U = R(f_0, \dots, f_n) \in a$. Since S is overconvergent S_a is also the direct limit of all $S(\tilde{U})$ where $U = R(\rho f_0, f_1, \dots, f_n)$ with $R(f_0, \dots, f_n) \in a$ and $\rho \in \sqrt{|k^\times|}$, $\rho > 1$. Our claim follows since the open subset

$$M := \{b \in \mathcal{M}(X) : \rho|f_0|_b > |f_1|_b, \dots, |f_n|_b\}$$

in $\mathcal{M}(X)$ satisfies $r^{-1}M \subseteq R(\rho f_0, f_1, \dots, f_n)^\sim$ by the formula (*) in the proof of Theorem 3.5.

(3) A homomorphism of overconvergent sheaves $S \rightarrow S'$ is an isomorphism if for any analytic point $a \in \mathcal{P}(X)$ the homomorphism $S_a \rightarrow S'_a$ is bijective. Indeed, let p be any prime filter contained in the maximal filter a . Then the natural map $S_p \rightarrow S'_p$ is an isomorphism for any overconvergent sheaf S .

(4) For any abelian sheaf T on $\mathcal{M}(X)$ we have $(r_*r^*T)_a = (r^*T)_a$ by (1) and (2) and $(r^*T)_a = T_a$ since $r(a) = a$. Hence $r_*r^*T = T$.

For any overconvergent sheaf S on $\mathcal{P}(X)$ and any analytic point $a \in \mathcal{P}(X)$ we have $(r^*r_*S)_a = (r_*S)_a = S_a$ according to (2). Because of (1) and (3) this proves that $r^*r_*S = S$.

Corollary 3. *Let S be a sheaf on $\mathcal{P}(X)$ and a an analytic point of X ; we have*

$$(r^*r_*S)_a = (r_*S)_a = S_{\tilde{a}} := \varinjlim_{U \in \tilde{a}} S(\tilde{U}).$$

Proof. (\tilde{a} was defined in Lemma 2.7.) Consider the two families

$$\{\tilde{U} : U \in \tilde{a}\}$$

and

$$\{r^{-1}M(f_0, \dots, f_n) : f_0, \dots, f_n \in \mathcal{O}(X) \text{ generating the unit ideal} \\ \text{such that } a \in M(f_0, \dots, f_n)\}$$

of open subsets in $\mathcal{P}(X)$. It is an immediate consequence of the formula (*) in the proof of Theorem 3.5 that every member of the second family contains a member of the first family. The same formula (*) also shows, as noted already in the previous proof, the following: If $R(f_0, \dots, f_n) \in a$ so that $R(\rho f_0, f_1, \dots, f_n) \in \tilde{a}$ for any $\rho \in \sqrt{|k^x|}$, $\rho > 1$, then

$$r^{-1}\{b \in \mathcal{M}(X) : \rho|f_0|_b > |f_i|_b \text{ for all } 1 \leq i \leq n\} \subseteq R(\rho f_0, f_1, \dots, f_n)^\sim.$$

Hence any set in the first family contains a set in the second family.

5 General rigid spaces

In this section X is an arbitrary rigid space over k . The definition of the sets $\mathcal{M}(X) \subseteq \mathcal{P}(X)$ and $\text{Val}(X)$ together with the natural bijection between $\text{Val}(X)$ and $\mathcal{P}(X)$ generalizes in a straightforward way. Of course filters on X now have to be formed among all admissible open subsets of X and prime filters have to be defined by the condition (p4)'. For any affinoid open subset $U \subseteq X$ there are obvious bijections

$$\mathcal{P}(U) \xrightarrow{\sim} \tilde{U} := \{p \in \mathcal{P}(X) : U \in p\}$$

and

$$\mathcal{M}(U) \xrightarrow{\sim} \underline{U} := \{a \in \mathcal{M}(X) : U \in a\}.$$

The retraction map

$$r_X : \mathcal{P}(X) \rightarrow \mathcal{M}(X) \\ p \mapsto \begin{array}{l} \text{unique maximal filter} \\ \text{containing } p \end{array}$$

still is defined. We equip $\mathcal{P}(X)$ with the topology for which the subsets \tilde{U} for $U \subseteq X$ affinoid open form a base. Then $\mathcal{P}(U) \xrightarrow{\sim} \tilde{U}$ is a homeomorphism. We have

$$r_X^{-1}(\{a\}) = \text{closure of } \{a\} \text{ in } \mathcal{P}(X) \quad \text{for any } a \in \mathcal{M}(X).$$

It remains true in general that the categories of abelian sheaves on $\mathcal{P}(X)$ and on X are naturally equivalent. We always give $\mathcal{M}(X)$ the quotient topology with respect to the map r_X .

Lemma 1. *Assume X to be affinoid; then the subsets $\mathcal{M}(X) \setminus \underline{U}$ with U running through the affinoid subdomains of X are open and generate the topology of $\mathcal{M}(X)$.*

Proof. Clearly the natural map $\mathcal{M}(U) \rightarrow \mathcal{M}(X)$ is continuous. Since both sides are compact its image \underline{U} is closed. Using the Berkovich topology we know that the sets $\{a \in \mathcal{M}(X) : |f|_a < \rho\}$ and $\{a \in \mathcal{M}(X) : |f|_a > \rho\}$ with $f \in \mathcal{O}(X)$ and $\rho \in \sqrt{|k^\times|}$ generate the topology of $\mathcal{M}(X)$. Those sets obviously are of the form $\mathcal{M}(X) \setminus \underline{U}$.

Lemma 2. *Assume X to be quasi-separated and let $U \subseteq X$ be affinoid open; then the natural map $\mathcal{M}(U) \rightarrow \mathcal{M}(X)$ is a homeomorphism onto its image \underline{U} which is closed in $\mathcal{M}(X)$.*

Proof. Let $X = \bigcup_{i \in I} U_i$ be an admissible affinoid open covering. Making obvious identifications we have

$$\begin{aligned} \mathcal{P}(X) \setminus r_X^{-1} \underline{U} &= \bigcup_{i \in I} \mathcal{P}(U_i) \setminus r_X^{-1} \underline{U} = \bigcup_{i \in I} \mathcal{P}(U_i) \setminus r_{U_i}^{-1} (\underline{U} \cap \underline{U}_i) \\ &= \bigcup_{i \in I} r_{U_i}^{-1} (\mathcal{M}(U_i) \setminus (\underline{U} \cap \underline{U}_i)). \end{aligned}$$

Since X is quasi-separated it follows from Lemma 1 that $\mathcal{M}(U_i) \setminus (\underline{U} \cap \underline{U}_i)$ is open in $\mathcal{M}(U_i)$. This implies that \underline{U} is closed in $\mathcal{M}(X)$. In this way we see that the natural continuous map $\mathcal{M}(U) \rightarrow \mathcal{M}(X)$ has the property that the image \underline{V} of $\mathcal{M}(V)$ for any affinoid subdomain $V \subseteq U$ is closed in $\mathcal{M}(X)$. Again by Lemma 1 this map therefore is closed.

Lemma 3. *Let $X = \bigcup_{i \in I} U_i$ be an admissible affinoid open covering; we then have: A subset M of $\mathcal{M}(X)$ is open (closed) if and only if the preimage of M in $\mathcal{M}(U_i)$ is open (closed) for any $i \in I$.*

Proof. The direct implication is trivial. Also the assertion about closedness follows from the one about openness. Therefore assume that the preimage of M in $\mathcal{M}(U_i)$ is open for any $i \in I$. We have to show that $r_X^{-1} M$ is open. But (again with the obvious identifications)

$$r_X^{-1} M = \bigcup_{i \in I} (r_X^{-1} M \cap \tilde{U}_i) = \bigcup_{i \in I} r_{U_i}^{-1} (M \cap \mathcal{M}(U_i)).$$

The next results should be compared with [Be] Sect. 1.6. They say that as far as quasi-separated spaces X with the subsequent condition (*) are concerned the theory of Berkovich is the theory of the space $\mathcal{M}(X)$. In the following we extend the notations \tilde{U} and \underline{U} in the obvious way to arbitrary admissible open subsets $U \subseteq X$.

Proposition 4. *Suppose that X is quasi-separated and that it has an admissible affinoid open covering $X = \bigcup_{i \in I} U_i$ such that*

(*) for any $i \in I$ there are only finitely many $j \in I$ with $U_i \cap U_j \neq \emptyset$;

then $\mathcal{M}(X)$ is a locally compact and paracompact Hausdorff space. Moreover any point in $\mathcal{M}(X)$ has a fundamental system of compact neighbourhoods of the form \underline{V} for some quasi-compact admissible open subset $V \subseteq X$.

Proof. Let a, b denote distinct points in $\mathcal{M}(X)$. Since the $\mathcal{M}(U_i)$ are Hausdorff it follows from Lemma 2 that we can choose, for any $i \in I$, closed subsets $M_i, N_i \subseteq U_i$ with union \underline{U}_i and such that $a \notin M_i$, $b \notin N_i$. Define $M := \bigcup_{i \in I} M_i$ and $N := \bigcup_{i \in I} N_i$. The condition (*) implies that the intersections of M and N with \underline{U}_i are closed. Therefore M and N are closed in $\mathcal{M}(X)$ by Lemma 3; their union is $\mathcal{M}(X)$ and $a \notin M$, $b \notin N$. This shows that $\mathcal{M}(X)$ is Hausdorff. By a similar reasoning the union of \underline{U}_i with i running through any subset of I is closed in $\mathcal{M}(X)$. Applying this to those \underline{U}_i which do not contain a given point $a \in \mathcal{M}(X)$ we see that the union of the finitely many other \underline{U}_i which do contain a is a compact neighbourhood of a . Hence $\mathcal{M}(X)$ is locally compact. It is also clear now that the \underline{U}_i for $i \in I$ form a locally finite covering of $\mathcal{M}(X)$ by compact subsets. This implies that $\mathcal{M}(X)$ is paracompact ([E] 5.1.34). The above argument showed that a point $a \in \mathcal{M}(X)$ has at least one neighbourhood of the form \underline{V} for some quasi-compact admissible open subset $V \subseteq X$. Using a finite affinoid open covering of V a simple topological argument therefore implies that our second assertion only has to be checked in the case of an affinoid space X . This is done in [B] 2.2.3 (iii).

Of course, in the situation of Proposition 4 the space $\mathcal{M}(U)$, for any admissible open subset $U \subseteq X$, is Hausdorff.

An abelian sheaf F on X (or $\mathcal{P}(X)$) is called *overconvergent* if its restriction to any affinoid open subset $U \subseteq X$ (or $\mathcal{P}(U)$) is overconvergent. It is shown in [S] Sect. 2 that it suffices to test this condition for an admissible affinoid open covering of X . It is immediate from the affinoid case that the functor r_X^* maps any abelian sheaf on $\mathcal{M}(X)$ to an overconvergent sheaf on $\mathcal{P}(X)$ (or X).

Lemma 5. *With the same assumptions as in Proposition 4 let $U \subseteq X$ be a quasi-compact admissible open subset and consider the commutative diagram*

$$\begin{array}{ccc} \mathcal{P}(U) & \xrightarrow{\varphi} & \mathcal{P}(X) \\ r_U \downarrow & & \downarrow r_X \\ \mathcal{M}(U) & \xrightarrow{\psi} & \mathcal{M}(X) \end{array}$$

where φ and ψ are the natural maps; for any overconvergent sheaf S on $\mathcal{P}(X)$ the base change map

$$\psi^* r_{X*} S \xrightarrow{\cong} r_{U*} \varphi^* S$$

is an isomorphism.

Proof. (The above diagram in general is not cartesian. The map φ is an open immersion whereas ψ , by Lemma 2, is a closed immersion.) It is convenient to introduce the following notation. For any two admissible open subsets $V \subseteq W \subseteq X$ let $S(V, W)$ denote the sheaf on \tilde{W} which is the direct image of $S|_{\tilde{V}}$. The proof proceeds in several steps by imposing additional assumptions which gradually will be weakened.

Step 1. U and X both are affinoid. Then the assertion is an immediate consequence of Theorem 4.2.

Step 2. X is separated, U is affinoid, and the sheaf is of the form $S(V, X)$ for some affinoid open subset $V \subseteq X$. Since by [S] 2.4 with S also $S(V, X)$ is overconvergent we may apply Step 1 to $V \cap U \subseteq V$ and obtain

$$\begin{aligned} r_{U*}\varphi^*S(V, X) &= r_{U*}S(V \cap U, U) \\ &= \text{direct image on } \underline{U} \text{ of } r_{V \cap U*}(S|_{\tilde{V} \cap \tilde{U}}) \\ &= \text{direct image on } \underline{U} \text{ of } (r_{V*}(S|_{\tilde{V}}))|_{\underline{V} \cap \underline{U}} \\ &= \psi^*(\text{direct image on } \mathcal{M}(X) \text{ of } r_{V*}(S|_{\tilde{V}})) \\ &= \psi^*r_{X*}S(V, X). \end{aligned}$$

Step 3. X is separated and quasi-compact and U is affinoid. Then we have the exact sequence of sheaves

$$0 \rightarrow S \rightarrow \bigoplus_{i \in I} S(U_i, X) \rightarrow \bigoplus_{i, j \in I} S(U_i \cap U_j, X)$$

where I can be taken to be finite. We apply now Step 2 to the middle and the right hand terms and we use that the functors involved in the base change map are left exact and commute with finite direct sums.

Step 4. X is separated and quasi-compact. Let $U = V_1 \cup \dots \cup V_r$ be a covering by affinoid open subsets $V_j \subseteq U$ and let $\varphi_j : \mathcal{P}(V_j) \rightarrow \mathcal{P}(U)$ and $\psi_j : \mathcal{M}(V_j) \rightarrow \mathcal{M}(U)$ be the natural maps. The assertion may be checked after restricting to $\mathcal{M}(V_j)$ for all $1 \leq j \leq r$. But then using Step 3 twice for $V_j \subseteq X$ and $V_j \subseteq U$ we obtain

$$\psi_j^*\psi^*r_{X*}S = (\psi\psi_j)^*r_{X*}S = r_{V_j*}(\varphi\varphi_j)^*S = r_{V_j*}\varphi_j^*\varphi^*S = \psi_j^*r_{U*}\varphi^*S.$$

Step 5. The sheaf is of the form $\mathcal{S}(V, X)$ for some separated and quasi-compact admissible open subset $V \subseteq X$. Redo Step 2 but now using Step 4 instead of Step 1.

Step 6. In the general situation we consider the exact sequence of sheaves

$$0 \rightarrow S \rightarrow \prod_{i \in I} S(U_i, X) \rightarrow \prod_{i, j \in I} S(U_i \cap U_j, X).$$

The condition (*) implies that the infinite products appearing in this sequence coincide with the corresponding direct sums. Therefore we can redo Step 3 now based on Step 5.

Theorem 6. *With the same assumptions as in Proposition 4 we have that r_{X*} and r_X^* are quasi-inverse equivalences between the category of overconvergent sheaves on $\mathcal{P}(X)$ (or X) and the category of all abelian sheaves on $\mathcal{M}(X)$.*

Proof. We have to show that the two adjunction maps are isomorphisms. This can be checked after restriction to $\mathcal{P}(U_i)$ and $\mathcal{M}(U_i)$, respectively. But applying Lemma 5 and Theorem 4.2 we obtain

$$(r_X^* r_{X*} S)|_{\mathcal{P}(U_i)} = r_{U_i}^*((r_{X*} S)|_{\mathcal{M}(U_i)}) = r_{U_i}^* r_{U_i*}(S|_{\mathcal{P}(U_i)}) = S|_{\mathcal{P}(U_i)}$$

for any overconvergent sheaf S on $\mathcal{P}(X)$ and similarly

$$(r_{X*} r_X^* T)|_{\mathcal{M}(U_i)} = r_{U_i*}((r_X^* T)|_{\mathcal{P}(U_i)}) = r_{U_i*} r_{U_i}^*(T|_{\mathcal{M}(U_i)}) = T|_{\mathcal{M}(U_i)}$$

for any sheaf T on $\mathcal{M}(X)$.

The assumptions of Proposition 4 are satisfied by any reasonable rigid space. For example, the generic fiber of any formal scheme of finite type and flat over k^0 is quasi-separated and quasi-compact. To discuss the property (*) a little further we first note that obviously any morphism $\beta : X \rightarrow Y$ of rigid spaces over k induces a continuous map

$$\begin{aligned} \mathcal{P}(\beta) : \mathcal{P}(X) &\rightarrow \mathcal{P}(Y) \\ p &\mapsto \{V \subseteq Y \text{ admissible open} : \beta^{-1}V \in p\}. \end{aligned}$$

This map respects maximal filters: To see this we may assume X and Y to be affinoid; then it is a consequence of Lemma 2.6. Hence β also induces a continuous map

$$\mathcal{M}(\beta) : \mathcal{M}(X) \rightarrow \mathcal{M}(Y).$$

If $\iota : U \xrightarrow{\subseteq} X$ is the inclusion of an admissible open subset then $\mathcal{M}(\iota)$ is injective but in general not open. We therefore introduce the following notion.

Definition. *An open immersion $\iota : U \rightarrow X$ is called wide open if $\mathcal{M}(\iota)$ is an open immersion, too.*

In case ι is a wide open inclusion map we call U simply a wide open subset of X . By Lemma 3 this notion is local in X .

Lemma 7. *For any admissible open subset $U \subseteq X$ the following conditions are equivalent:*

- i. U is wide open in X ;
- ii. $r_X^{-1}(U) = \tilde{U}$;
- iii. \tilde{U} is open in $\mathcal{M}(X)$.

Proof. It is trivial that i. implies iii. Assume now iii. to hold. It is clear that $\tilde{U} \subseteq r_X^{-1}(U)$. Let $p \in r_X^{-1}(U)$ be any prime filter. Since $r_X^{-1}(U)$ is open in $\mathcal{P}(X)$ we find an affinoid open subset $V \subseteq X$ such that $V \in p$ and $\tilde{V} \subseteq \tilde{U}$. The latter implies that $V \subseteq U$ and hence $U \in p$ which means that $p \in \tilde{U}$.

The assertion ii. can be expressed by saying that the obvious diagram of topological spaces

$$\begin{array}{ccc} \mathcal{P}(U) & \xrightarrow{\mathcal{P}(t)} & \mathcal{P}(X) \\ r_U \downarrow & & \downarrow r_X \\ \mathcal{M}(U) & \xrightarrow{\mathcal{M}(t)} & \mathcal{M}(X) \end{array}$$

is cartesian. Moreover $\mathcal{P}(t)$ is an open immersion and r_U and r_X are quotient maps. It is straightforward that then $\mathcal{M}(t)$ has to be an open immersion, too.

Proposition 8. *With the same assumptions as in Proposition 4 let $U \subseteq X$ be a wide open subset which possesses a countable admissible affinoid open covering; then U is quasi-separated and satisfies the condition (*) in Proposition 4.*

Proof. It follows from Proposition 4 that $\mathcal{M}(X)$ is locally compact. As an open subset $\mathcal{M}(U)$ is locally compact, too. Of course U is quasi-separated. The assumption about the countable covering then implies by Lemma 2 that $\mathcal{M}(U)$ is a countable union of compact subsets or in other words is countable at infinity and in particular is paracompact.

According to [Bou] I.9.10 Corollary there is a locally finite open covering

$$\mathcal{M}(U) = \bigcup_{i \in J} M_i$$

such that all the \bar{M}_i are compact. Moreover, by [Bou] IX.4.3 Theorem 3 there are open coverings

$$\mathcal{M}(U) = \bigcup_{i \in J} N_i = \bigcup_{i \in J} L_i$$

such that

$$\bar{L}_i \subseteq N_i \subseteq \bar{N}_i \subseteq M_i \quad \text{for any } i \in J.$$

The \bar{N}_i and \bar{L}_i are compact as well. We claim that for any $i \in J$ there are only finitely many $j \in J$ with $\bar{N}_i \cap \bar{N}_j \neq \emptyset$. Fix an $i \in J$. The family $\{\bar{N}_j : j \in J\}$ being locally finite we find, for any $a \in \bar{N}_i$, an open neighbourhood M_a of a in M_i which intersects only finitely many sets \bar{N}_j . Since \bar{N}_i is compact we have

$$\bar{N}_i \subseteq M_{a_1} \cup \dots \cup M_{a_r} \quad \text{for some points } a_1, \dots, a_r.$$

If now $\bar{N}_i \cap \bar{N}_j \neq \emptyset$ then also $M_{a_\rho} \cap \bar{N}_j \neq \emptyset$ for some $1 \leq \rho \leq r$. Hence this can happen only for finitely many $j \in J$.

According to Proposition 4 any point in some \bar{L}_i has a neighbourhood in N_i of the form \underline{V} for some quasi-compact admissible open subset $V \subseteq U$. Since the \bar{L}_i are compact and cover $\mathcal{M}(U)$ it follows that there exists a family $\{V_i\}_{i \in I}$ of affinoid open subsets in U such that

- $U = \bigcup_{i \in I} V_i$,
- for any $i \in I$ there are only finitely many $j \in I$ with $V_i \cap V_j \neq \emptyset$, and
- for any point $a \in \mathcal{M}(U)$ there is an $i \in I$ such that \underline{V}_i is a neighbourhood of a in $\mathcal{M}(U)$.

The latter property implies that the V_i form an admissible covering of U : Let $\beta: Y \rightarrow U$ be any morphism from an affinoid space Y into U . Using the compactness of $\mathcal{M}(Y)$ it easily follows that the covering $\{\beta^{-1}V_i\}_{i \in I}$ of Y has a finite subcovering and hence is admissible.

Now let $X = Z^{an}$ be the rigid space associated to a separated scheme Z of finite type over k ([BGR] 9.3.6). Of course X is separated.

Lemma 9. *If $Z_0 \subseteq Z$ is a Zariski open subscheme then $U := Z_0^{an}$ is wide open in $X = Z^{an}$.*

Proof. Using Lemma 3 and an admissible affinoid open covering of Z^{an} this follows from the fact that any Zariski open subset in an affinoid space is wide open ([S] Sect. 3 Proposition 3 (iii)).

Proposition 10. *$X = Z^{an}$ satisfies the condition (*) in Proposition 4.*

Proof. We want to apply Proposition 8. By Nagata Z is Zariski open in a proper scheme \bar{Z} over k . Since \bar{Z}^{an} is proper and hence quasi-compact it satisfies the assumptions of Proposition 4. According to the previous Lemma Z^{an} is wide open in \bar{Z}^{an} . By Proposition 8 it remains to check that Z^{an} has a countable admissible affinoid open covering. Writing Z as a finite union of affine open subschemes we are reduced to consider the case where Z is affine. But for the affine space and then also for any Zariski closed subscheme in the affine space our claim is obvious.

Using the properties of the Raynaud functor ([Meh] or [BL] 4.1) one can easily establish a version of Theorem 3.4 for any quasi-separated and quasi-compact space X .

References

- [B] V.G. Berkovich, Spectral theory and analytic geometry over non-Archimedean fields, Providence 1990
- [Be] V.G. Berkovich, Etale cohomology for non-archimedean analytic spaces, Publ. Math. IHES 78 (1993), 5–161
- [BGR] S. Bosch, U. Guntzer, R. Remmert, Non-Archimedean Analysis, Berlin-Heidelberg-New York 1984
- [BL] S. Bosch, W. Lütkebohmert, Formal and rigid geometry I. Rigid spaces, Math. Ann. 295 (1993), 291–317
- [Bou] N. Bourbaki, General Topology, Berlin-Heidelberg-New York 1989
- [D] P. Deligne, Letter to Raynaud, 1992
- [E] R. Engelking, General topology, Warschau 1977
- [FP] J. Fresnel, M. van der Put, Géométrie Analytique Rigide et Applications, Progress in Math. vol. 18, 1981
- [GP] L. Gerritzen, M. van der Put, Schottky groups and Mumford curves, Springer Lect. Notes in Math. 817, 1980
- [H] R. Huber, Continuous valuations, Math. Z. 212 (1993), 455–477
- [M] H. Matsumura, Commutative ring theory, Cambridge 1986

- [Meh] F. Mehlmann, Ein Beweis für einen Satz von Raynaud über flache Homomorphismen affinoider Algebren, *Schriften. Math. Univ. Münster, 2. Ser.* 19 (1981)
- [P] M. van der Put, Cohomology on affinoid spaces, *Compositio Math.* 45 (1982), 165–198
- [S] P. Schneider, Points of rigid analytic varieties, *J. reine angew. Math.* 434 (1993), 127–157