

# SPHERICAL NILPOTENT ORBITS AND ABELIAN SUBALGEBRAS IN ISOTROPY REPRESENTATIONS

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**ABSTRACT.** Let  $G$  be a simply connected semisimple algebraic group with Lie algebra  $\mathfrak{g}$ , let  $G_0 \subset G$  be the symmetric subgroup defined by an algebraic involution  $\sigma$  and let  $\mathfrak{g}_1 \subset \mathfrak{g}$  be the isotropy representation of  $G_0$ . Given an abelian subalgebra  $\mathfrak{a}$  of  $\mathfrak{g}$  contained in  $\mathfrak{g}_1$  and stable under the action of some Borel subgroup  $B_0 \subset G_0$ , we classify the  $B_0$ -orbits in  $\mathfrak{a}$  and we characterize the sphericity of  $G_0\mathfrak{a}$ . Our main tool is the combinatorics of  $\sigma$ -minuscule elements in the affine Weyl group of  $\mathfrak{g}$  and that of strongly orthogonal roots in Hermitian symmetric spaces.

## 1. INTRODUCTION

Let  $G$  be a connected simply connected semisimple complex algebraic group with Lie algebra  $\mathfrak{g}$ . Let  $B$  be a Borel subgroup, and set  $\mathfrak{b} = \text{Lie}B$ . Recall that a  $G$ -variety  $X$  is called  *$G$ -spherical* if it possesses an open  $B$ -orbit. The relationships between spherical nilpotent orbits and abelian ideals of  $\mathfrak{b}$  have been first investigated in [22]. There it is shown that if  $\mathfrak{a}$  is an abelian ideal of  $\mathfrak{b}$ , then any nilpotent orbit meeting  $\mathfrak{a}$  is a  $G$ -spherical variety and  $G\mathfrak{a}$  is the closure a spherical nilpotent orbit. In particular,  $B$  acts on  $\mathfrak{a}$  with finitely many orbits.

Subsequently, Panyushev [20] dealt with similar questions in the  $\mathbb{Z}_2$ -graded case. Let  $\sigma$  be an involution of  $G$  and  $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$  be the corresponding eigenspace decomposition at the Lie algebra level. Let  $G_0$  be the connected subgroup of  $G$  corresponding to  $\mathfrak{g}_0$  and  $B_0 \subset G_0$  a Borel subgroup of  $G_0$ . The “graded” analog of the set of abelian ideals of  $\mathfrak{b}$  is the set  $\mathcal{I}_{ab}^\sigma$  of (abelian)  $B_0$ -stable subalgebras of  $\mathfrak{g}_1$ .

**Definition 1.1.** We say that  $\mathfrak{a} \in \mathcal{I}_{ab}^\sigma$  is  *$G$ -spherical* (resp.  *$G_0$ -spherical*) if all orbits  $Gx$ ,  $x \in \mathfrak{a}$  are  $G$ -spherical (resp. if all orbits  $G_0x$ ,  $x \in \mathfrak{a}$  are  $G_0$ -spherical).

Panyushev [18] started the classification of the spherical nilpotent  $G_0$ -orbits in  $\mathfrak{g}_1$ . The classification of the spherical nilpotent  $G_0$ -orbits in  $\mathfrak{g}_1$  was then completed by King [12] (see also [2], where the classification is reviewed and a missing case is pointed out). Shortly afterwards, Panyushev [20] noticed the emergence of non-spherical subalgebras  $\mathfrak{a} \in \mathcal{I}_{ab}^\sigma$ , and classified the involutions  $\sigma$  for which these subalgebras exist. After explicit verifications, he also noticed that an element  $\mathfrak{a} \in \mathcal{I}_{ab}^\sigma$  is  $G$ -spherical if and only if it is  $G_0$ -spherical, but no verification was given, and no conceptual proof was known.

The purpose of the present paper is to deepen and expand the results quoted above in the following directions. Let  $\mathfrak{a} \in \mathcal{I}_{ab}^\sigma$ .

- i) We clarify the connections between  $G_0$ -orbits of nilpotent elements in  $\mathfrak{g}_1$ , spherical  $G$ -orbits of nilpotent elements in  $\mathfrak{g}_1$  and  $G_0$ -orbits of abelian subalgebras in  $\mathfrak{g}_1$  which are stable under some Borel subalgebra of  $\mathfrak{g}_0$ .

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- ii) We prove that  $B_0$  acts on  $\mathfrak{a}$  with finitely many orbits, independently of its sphericity. Moreover, we parametrize orbits via orthogonal set of weights of  $\mathfrak{a}$ .
- iii) Assume that there exist non-spherical subalgebras. We give a construction of a canonical non-spherical subalgebra  $\mathfrak{a}_p$ .
- iv) We give a simple criterion to decide whether  $\mathfrak{a}$  is spherical or not: in Theorem 6.7 we show that there exists  $\bar{\mathfrak{a}} \in \mathcal{I}_{ab}^\sigma$  such that  $\mathfrak{a}$  is non-spherical if and only if  $\mathfrak{a} \supset \bar{\mathfrak{a}}$ .

One important feature of our approach lies in the methods used. The theory of abelian ideals and its graded version rely on a strict relationship with the geometry of alcoves of the affine Weyl group  $\widehat{W}$  of  $\mathfrak{g}$  [13], [4], and, for the graded case, with Kac's classification of finite order automorphisms of semisimple Lie algebras [11], [6], [7].

The main link is that a  $B_0$ -stable subalgebra  $\mathfrak{a}$  can be encoded by an element  $w_{\mathfrak{a}} \in \widehat{W}$  defined through its set of inversions  $N(w_{\mathfrak{a}})$  (cf. (2.1)). The elements so obtained, called  $\sigma$ -minuscule (Definition 2.4), pave a convex polytope in the dual space of a Cartan subalgebra of  $\mathfrak{g}$  and have remarkable properties: see Section 2.3 for a recollection of these facts. It has been explicitly asked (e.g., in [20]) to use the above connections as a tool for dealing with problems about sphericity. This is what we do here.

We start discussing items i)-iv) by making the content of i) more precise. Define the *height* of a nilpotent element  $x \in \mathfrak{g}$  as

$$\text{ht}(x) = \max\{n \in \mathbb{N} \mid \text{ad}(x)^n \neq 0\}.$$

In the adjoint case, Panyushev [18] completely characterized the spherical nilpotent  $G$ -orbits in  $\mathfrak{g}$  by showing that, for  $x \in \mathfrak{g}$ , the orbit  $Gx$  is spherical if and only if  $\text{ht}(x) \leq 3$ . Subsequently, Panyushev and Röhrle [22] proved that, if  $\mathfrak{a} \subset \mathfrak{b}$  is an abelian ideal, then the saturation  $G\mathfrak{a}$  is spherical. On the other hand, if  $Gx$  is spherical, by choosing  $\mathfrak{b}$  properly it is always possible to construct an abelian ideal  $\mathfrak{a} \subset \mathfrak{b}$  such that  $G\mathfrak{a} = \overline{Gx}$ . Therefore we may regard both these properties as consequences of the small height of the nilpotent element  $x$ .

For  $i = 0, 1$  define the  $i$ -height of a nilpotent element  $x \in \mathfrak{g}_1$  as

$$\text{ht}_i(x) = \max\{n \in \mathbb{N} \mid \text{ad}(x)_{[\mathfrak{g}_i]}^n \neq 0\}.$$

In [18] Panyushev showed that, for  $x \in \mathfrak{g}_1$ , the following implications hold

$$\text{ht}(x) \leq 3 \implies G_0x \text{ spherical} \implies \text{ht}_0(x) \leq 4, \text{ht}_1(x) \leq 3.$$

In Corollary 6.3, we show the following result.

**Theorem.** *If  $\mathfrak{a} \in \mathcal{I}_{ab}^\sigma$  and  $x \in \mathfrak{a}$ , then  $\text{ht}_0(x) \leq 3$  and  $\text{ht}_1(x) \leq 4$ .*

These properties completely characterize the elements of abelian subalgebras of  $\mathfrak{g}_1$  which are stable under some Borel subgroup of  $G_0$  (see Section 5.1). As a corollary, using Panyushev's criterion for the  $G_0$ -sphericity of a nilpotent element in  $\mathfrak{g}_1$ , it follows that  $\mathfrak{a} \in \mathcal{I}_{ab}^\sigma$  is  $G_0$ -spherical if and only if it is  $G$ -spherical.

Regarding ii), a well known result independently due to Brion [3] and Vinberg [26], states that every spherical  $G$ -variety contains finitely many  $B$ -orbits. In particular, every abelian ideal  $\mathfrak{a}$  of  $\mathfrak{b}$  contains finitely many  $B$ -orbits. In [21], the same result has been proved avoiding the use of the sphericity of  $G\mathfrak{a}$ . In Section 3, we prove, along the same lines, the finiteness theorem quoted in ii), in the more general context of finite order automorphisms of  $G$  (see Theorem 3.1).

To streamline our approach to iii), recall that involutions of  $\mathfrak{g}$  are encoded by the datum of one or two simple roots (with suitable features) in the extended Dynkin diagram  $\widehat{\Pi}$  of  $\mathfrak{g}$ . The main result of [20] has been rephrased by Panyushev

as follows: there exists a non spherical  $\mathfrak{a} \in \mathcal{I}_{ab}^\sigma$  if and only if  $\sigma$  is defined by a single simple root  $\alpha_p$ , which is long and non-complex (see Definition 2.3). (As usual, if  $\widehat{\Pi}$  is simply laced, every root is regarded as long). However this claim was obtained as a by-product of direct considerations on various classes of involutions and by constructing case-by-case a non-spherical element  $\mathfrak{a} \in \mathcal{I}_{ab}^\sigma$  for all involution satisfying the previous condition.

In this paper we observe that, precisely when  $\alpha_p$  is long and non-complex, there exists a *special* element  $\mathfrak{a}_p \in \mathcal{I}_{ab}^\sigma$ , which plays a role in the classification of maximal elements in  $\mathcal{I}_{ab}^\sigma$  performed in [7]. In Section 5 we study the properties of  $\mathfrak{a}_p$ . In particular, using the combinatorics of  $N(w_{\mathfrak{a}_p})$ , we prove that  $\mathfrak{a}_p$  is not  $G_0$ -spherical. The method is combinatorial: we associate to any orthogonal set of maximal cardinality in  $N(w_{\mathfrak{a}_p}) \setminus \{\alpha_p\}$  a generalized Cartan matrix of affine type, whose type is severely restricted (see Proposition 5.3). The information we obtain from this Cartan matrix allows us to build up a generic element  $x \in \mathfrak{a}_p$  with  $\text{ht}_1(x) = 4$ , proving that  $\mathfrak{a}_p$  is not  $G_0$ -spherical.

The same strategy is applied in a wider context in Section 6, and it enables us to classify the spherical elements of  $\mathcal{I}_{ab}^\sigma$ , as outlined in iv). The construction of the minimal non-spherical subalgebra  $\bar{\mathfrak{a}}$  is based on the combinatorics of strongly orthogonal roots in Hermitian symmetric spaces. Many related technical results might be of independent interest, and they are displayed in Section 4. To construct  $\bar{\mathfrak{a}}$ , decompose  $\widehat{\Pi} \setminus \{\alpha_p\}$  into a disjoint union of connected components  $\Sigma$ . Then in each  $\Sigma$  there exists a unique simple root  $\alpha_\Sigma$  non-orthogonal to  $\alpha_p$ , and it turns out that  $\alpha_\Sigma$  determines an Hermitian involution of tube type of the Lie algebra  $\mathfrak{g}_\Sigma$  having  $\Sigma$  as set of simple roots (see Subsection 4.4 and Proposition 5.7). If  $\Phi(\Sigma)_1^+$  denotes the set of positive roots of  $\mathfrak{g}_\Sigma$  having  $\alpha_\Sigma$  in their support, in Lemma 4.12 we prove that there exists a unique subset  $\mathcal{A}_\Sigma$  which is an antichain in  $\Phi(\Sigma)_1^+$  w.r.t. the dominance order  $\leq_\Sigma$  defined by  $\Sigma$  and which is a maximal orthogonal subset of  $\Phi(\Sigma)_1^+$ . Next, we prove that  $\bigcup_\Sigma \bigcup_{\eta \in \mathcal{A}_\Sigma} \{\xi + \alpha_p \mid \xi \leq_\Sigma \eta\}$  is the set of inversions of a  $\sigma$ -minuscule element, hence it determines an element  $\bar{\mathfrak{a}} \in \mathcal{I}_{ab}^\sigma$ , which turns out to have the property described in iv).

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## 2. SETUP

Let  $G$  be a semisimple, connected and simply connected complex algebraic group with Lie algebra  $\mathfrak{g}$ , and let  $B \subset G$  be a Borel subgroup with Lie algebra  $\mathfrak{b}$ . Throughout the paper,  $\sigma : G \rightarrow G$  will be an indecomposable automorphism of finite order  $m$ . Then  $\sigma$  induces an automorphism of  $\mathfrak{g}$  as well, still denoted by  $\sigma$ . Fix a primitive  $m^{\text{th}}$ -root of unity  $\zeta$  and consider the corresponding  $\mathbb{Z}_m$ -grading

$$\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}_m} \mathfrak{g}_i,$$

where  $\mathfrak{g}_i$  denotes the eigenspace of  $\sigma$  of weight  $\zeta^i$ . Then  $\mathfrak{g}_0$  is a reductive subalgebra of  $\mathfrak{g}$  (see [11, Lemma 8.1]), and the connected reductive subgroup  $G_0 \subset G$  defined by  $\mathfrak{g}_0$  coincides with the set of fixed points of  $\sigma$ . Fix a Cartan subalgebra  $\mathfrak{h}_0 \subset \mathfrak{g}_0$ , which is abelian since  $\mathfrak{g}_0$  is reductive. If  $\mathfrak{a} \subset \mathfrak{g}$  is a  $\mathfrak{h}_0$ -stable subspace, we let  $\Psi(\mathfrak{a})$  denote its set of  $\mathfrak{h}_0$ -weights and, for  $\lambda \in \Psi(\mathfrak{a})$ , we let  $\mathfrak{a}^\lambda$  be the corresponding weight space.

Every eigenspace  $\mathfrak{g}_i$  is a  $G_0$ -module under the restriction of the adjoint action. If  $i \in \mathbb{Z}_m$ , we denote by  $\Phi_i$  the set of the non-zero  $\mathfrak{h}_0$ -weights in  $\mathfrak{g}_i$ . Denote finally  $\Phi = \cup_i \Phi_i$  the set of non-zero weights. We say  $\mu, \nu \in \Phi$  are *strongly orthogonal* if  $(\mu, \nu) = 0$  and  $\mu \pm \nu \notin \Phi$ . (This definition agrees with the usual notion of strongly orthogonal roots in a semisimple Lie algebra).

Observe that  $\Phi_0$  is the set of  $\mathfrak{h}_0$ -roots for  $\mathfrak{g}_0$ . As shown in [11, Chapter 8],  $\mathfrak{h}_0$  contains a regular element  $h_{reg}$  of  $\mathfrak{g}$ . In particular the centralizer  $\text{Cent}(\mathfrak{h}_0)$  of  $\mathfrak{h}_0$  in  $\mathfrak{g}$  is a Cartan subalgebra of  $\mathfrak{g}$  and  $h_{reg}$  defines a set of positive roots in the set of roots of  $(\mathfrak{g}, \text{Cent}(\mathfrak{h}_0))$  and a set  $\Phi_0^+$  of positive roots in  $\Phi_0$ . We let  $\Pi_0$  be the corresponding set of simple roots,  $\mathfrak{b}_0$  the corresponding Borel subalgebra, and  $B_0 \subset G_0$  the corresponding Borel subgroup.

**2.1. The grading associated to a nilpotent element  $x \in \mathfrak{g}_1$ .** We fix in this subsection notation concerning the grading of  $\mathfrak{g}$  associated to nilpotent elements in  $\mathfrak{g}_1$  and the corresponding notion of height. The main references for this subsection are [25] and [18]. By [25], an element  $x \in \mathfrak{g}_1$  is semisimple if and only if  $G_0x$  is closed, whereas it is nilpotent if and only if  $0 \in \overline{G_0x}$ .

Let  $x \in \mathfrak{g}_1$  be a nilpotent element. By a modification of the Jacobson-Morozov theorem, there exists a  $\mathfrak{sl}(2)$ -triple  $(x, h, y)$  with  $y \in \mathfrak{g}_1$  and  $h \in \mathfrak{g}_0$ . Such triples are usually called *normal triples*, or *adapted triples*. Let

$$\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}(i)$$

be the  $\mathbb{Z}$ -grading defined by  $h$ ; then we get a bigrading of  $\mathfrak{g}$  by setting

$$\mathfrak{g}_j(i) = \mathfrak{g}_j \cap \mathfrak{g}(i).$$

Since all normal triples containing  $x$  are conjugated by the stabilizer of  $x$  in  $G_0$  (see [25, Theorem 1]), it follows that the structure of this bigrading does not depend on the choice of the normal triple.

Following Panyushev [18], define the *height* of  $x$  as

$$\text{ht}(x) = \max\{n \in \mathbb{N} \mid \mathfrak{g}(n) \neq 0\}.$$

Since  $[x, \mathfrak{g}(i)] = \mathfrak{g}(i+2)$ , this notion agrees with the height defined in the Introduction, namely the maximum  $n$  such that  $\text{ad}(x)^n \neq 0$ .

**2.2. Twisted loop algebra and finite order automorphisms.** Since  $\sigma$  fixes  $h_{reg}$ , we see that the action of  $\sigma$  on the positive roots defines, once Chevalley generators are fixed, a diagram automorphism  $\eta$  of  $\mathfrak{g}$  that, clearly, fixes  $\mathfrak{h}_0$ . Set, using the notation of [11],  $\widehat{\mathfrak{h}} = \mathfrak{h}_0 \oplus \mathbb{C}K \oplus \mathbb{C}d$ . Recall that  $d$  is the element of

$$\widehat{L}(\mathfrak{g}, \sigma) = (\mathbb{C}[t, t^{-1}] \otimes \mathfrak{g}) \oplus \mathbb{C}K \oplus \mathbb{C}d$$

acting on  $\mathbb{C}[t, t^{-1}] \otimes \mathfrak{g}$  as  $t \frac{d}{dt}$ , while  $K$  is a central element. Define  $\delta' \in \widehat{\mathfrak{h}}^*$  by setting  $\delta'(d) = 1$  and  $\delta'(\mathfrak{h}_0) = \delta'(K) = 0$  and let  $\lambda \mapsto \bar{\lambda}$  be the restriction map  $\widehat{\mathfrak{h}} \rightarrow \mathfrak{h}_0$ . There is a unique extension, still denoted by  $(\cdot, \cdot)$ , of the Killing form of  $\mathfrak{g}$  to a nondegenerate symmetric bilinear invariant form on  $\widehat{L}(\mathfrak{g}, \sigma)$ . Let  $\nu : \widehat{\mathfrak{h}} \rightarrow \widehat{\mathfrak{h}}^*$  be the isomorphism induced by the form  $(\cdot, \cdot)$ , and denote again by  $(\cdot, \cdot)$  the form induced on  $\widehat{\mathfrak{h}}^*$ . One has  $(\delta', \delta') = (\delta', \mathfrak{h}_0^*) = 0$ .

We let  $\widehat{\Phi}$  be the set of  $\widehat{\mathfrak{h}}$ -roots of  $\widehat{L}(\mathfrak{g}, \sigma)$ . We can choose as set of positive roots  $\widehat{\Phi}^+ = \Phi_0^+ \cup \{\alpha \in \widehat{\Phi} \mid \alpha(d) > 0\}$ . We let  $\widehat{\Pi} = \{\alpha_0, \dots, \alpha_n\}$  be the corresponding set of simple roots. It is known that  $n$  is the rank of  $\mathfrak{g}_0$ . Recall that any  $\widehat{L}(\mathfrak{g}, \sigma)$  is a Kac-Moody Lie algebra  $\mathfrak{g}(A)$  defined by generators and relations starting from a generalized Cartan matrix  $A$  of affine type. These matrices are classified by means of Dynkin diagrams listed in [11].

Let  $\widehat{W}$  be the Weyl group of  $\widehat{L}(\mathfrak{g}, \sigma)$  and let  $\widehat{\Phi}_{re} = \widehat{W}\widehat{\Pi}$  be the set of real roots of  $\widehat{L}(\mathfrak{g}, \sigma)$ . Recall that if  $\beta = w(\alpha)$ ,  $\alpha \in \widehat{\Pi}$ , one defines  $\beta^\vee = w(\alpha^\vee)$ .

If  $\gamma \in \mathfrak{h}_0^*$ , we set

$$h_\gamma = \nu^{-1}(\gamma), \quad \gamma^\vee = \frac{2h_\gamma}{(\gamma, \gamma)} \quad (\gamma \neq 0).$$

By [11, 5.1] if  $\lambda \in \mathbb{C}\delta' + \mathfrak{h}_0^*$  and  $\beta \in \widehat{\Phi}_{re}$ , then

$$\lambda(\beta^\vee) = 2 \frac{(\lambda, \beta)}{(\beta, \beta)} = 2 \frac{(\lambda, \bar{\beta})}{(\bar{\beta}, \bar{\beta})} = \lambda(\bar{\beta}^\vee) = 2 \frac{(\bar{\lambda}, \bar{\beta})}{(\bar{\beta}, \bar{\beta})} = \bar{\lambda}(\bar{\beta}^\vee).$$

If  $\lambda, \mu \in \mathbb{C}\delta' + \mathfrak{h}_0^*$  and  $(\mu, \mu) \neq 0$ , we set

$$\langle \lambda, \mu^\vee \rangle = 2 \frac{(\lambda, \mu)}{(\mu, \mu)}.$$

In particular, if  $\alpha \in \widehat{\Phi}$  and  $\beta \in \widehat{\Phi}_{re}$ ,

$$\langle \alpha, \beta^\vee \rangle = \alpha(\beta^\vee) = \langle \bar{\alpha}, \bar{\beta}^\vee \rangle = \bar{\alpha}(\bar{\beta}^\vee).$$

We will use these equalities many times without comment.

Following [11, Chapter 8], we can assume that  $\sigma$  is the automorphism of type  $(\eta; s_0, \dots, s_n)$ , where  $\eta$  is the diagram automorphism defined above. Recall that, if  $a_0, \dots, a_n$  are the labels of the Dynkin diagram of  $\widehat{L}(\mathfrak{g}, \sigma)$  and  $k$  is the order of  $\eta$ , then  $k(\sum_{i=0}^n s_i a_i) = m$ . Recall also that  $s_0, \dots, s_n$  are relatively prime so, in the case of involutions ( $m = 2$ ), we must have that  $s_i \in \{0, 1\}$  and  $s_i = 0$  for all but at most two indices.

Since  $\sigma$  is the automorphism of type  $(\eta; s_0, \dots, s_n)$ , we can write  $\alpha_i = s_i \delta' + \bar{\alpha}_i$  and it turns out that  $\Pi_0 = \{\alpha_i \mid s_i = 0\}$ . Set also  $\Pi_1 = \widehat{\Pi} \setminus \Pi_0$ . Introduce  $\delta = \sum_{i=0}^n a_i \alpha_i$  and note that  $\delta = (\sum_{i=0}^n a_i s_i) \delta' = \frac{m}{k} \delta'$ .

Given  $\lambda \in \widehat{\Phi}$  we denote by  $\widehat{L}(\mathfrak{g}, \sigma)_\lambda$  the corresponding root space in  $\widehat{L}(\mathfrak{g}, \sigma)$ . Recall the following properties (see [11, Exercise 8.2]).

**Proposition 2.1.** *Let  $\lambda \in \widehat{\Phi}_{re}$ , then the following holds:*

- i)  $\dim \widehat{L}(\mathfrak{g}, \sigma)_\lambda = 1$ .
- ii) *If  $\mu \in \widehat{\Phi}$ , then the set of  $\mu + i\lambda \in \widehat{\Phi} \cup \{0\}$  is a string  $\mu - p\lambda, \dots, \mu + q\lambda$ , where  $p, q$  are non-negative integers such that  $p - q = \langle \mu, \lambda^\vee \rangle$ .*
- iii) *If  $\mu \in \widehat{\Phi}$  and  $\mu + \lambda \in \widehat{\Phi}$ , then  $[\widehat{L}(\mathfrak{g}, \sigma)_\lambda, \widehat{L}(\mathfrak{g}, \sigma)_\mu] \neq 0$ .*

Notice that, if  $\lambda = i\delta' + \alpha \in \widehat{\Phi}_{re}$ , then

$$\widehat{L}(\mathfrak{g}, \sigma)_\lambda = t^i \otimes \mathfrak{g}_i^\alpha.$$

This implies that we can rephrase the previous proposition in terms of  $\mathfrak{h}_0$ -weights in  $\mathfrak{g}$  as follows.

**Corollary 2.2.** *Let  $\alpha \in \Phi_i$  and  $\beta \in \Phi_j$ . If  $i \equiv j \pmod{m}$ , assume also that  $\alpha \neq \beta$ .*

- i)  $\dim \mathfrak{g}_i^\alpha = 1$ , and if  $-\alpha \in \Phi_i$  then  $[\mathfrak{g}_i^\alpha, \mathfrak{g}_i^{-\alpha}] \neq 0$ .
- ii) *If  $(\alpha, \beta) < 0$  then  $\alpha + \beta \in \Phi_{i+j}$ , and if  $(\alpha, \beta) > 0$  then  $\alpha - \beta \in \Phi_{i-j}$ .*
- iii) *If  $\alpha + \beta \in \Phi_{i+j}$ , then  $[\mathfrak{g}_i^\alpha, \mathfrak{g}_j^\beta] \neq 0$ .*

In general,  $\Pi_0$  is disconnected and we write  $\Sigma|\Pi_0$  to mean that  $\Sigma$  is a connected component of  $\Pi_0$ . Clearly, the Weyl group  $W_0$  of  $\mathfrak{g}_0$  is the direct product of the  $W(\Sigma)$ ,  $\Sigma|\Pi_0$ . If  $\theta_\Sigma$  is the highest root of  $\Phi(\Sigma)$ , set

$$\begin{aligned} \widehat{\Phi}_0 &= \{\alpha + \mathbb{Z}k\delta \mid \alpha \in \Phi_0\} \cup \pm \mathbb{N}k\delta, \\ \widehat{\Pi}_0 &= \Pi_0 \cup \{k\delta - \theta_\Sigma \mid \Sigma|\Pi_0\}, \\ \widehat{\Phi}_0^+ &= \Phi_0^+ \cup \{\alpha \in \widehat{\Phi}_0 \mid \alpha(d) > 0\}. \end{aligned}$$

Denote by  $\widehat{W}_0$  the Weyl group of  $\widehat{\Phi}_0$ . If  $\alpha \in \widehat{\Phi}$ , let  $[\alpha : \alpha_i]$  be the coefficient of  $\alpha_i$  in the expansion of  $\alpha$  in terms of  $\widehat{\Pi}$ . Set

$$\text{ht}_\sigma(\alpha) = \sum_{i=0}^n s_i [\alpha : \alpha_i]$$

and, for  $i \in \mathbb{Z}$ ,

$$\widehat{\Phi}_i = \{\alpha \in \widehat{\Phi} \mid \text{ht}_\sigma(\alpha) = i\}.$$

Note that if  $\alpha \in \widehat{\Phi}_i$ , then  $\bar{\alpha}$  is a weight of  $\mathfrak{g}_i$ .

**2.3.  $B_0$ -stable subalgebras in  $\mathfrak{g}_1$  and  $\sigma$ -minuscule elements.** In this subsection we assume that  $\sigma$  is an (indecomposable) involution. With this assumption,  $\Pi_1$  has at most two elements. If  $\widehat{\Pi}$  is simply laced, the real roots of  $\widehat{\Phi}$  are regarded as long.

**Definition 2.3.** We say that  $\eta \in \widehat{\Phi}$  is complex if  $\bar{\eta} \in \Phi_0 \cap \Phi_1$ .

It is clear that complex roots can occur only if  $\text{rk } \mathfrak{g}_0 < \text{rk } \mathfrak{g}$ . Moreover, if  $\mathfrak{g}$  is simple and  $\text{rk } \mathfrak{g}_0 < \text{rk } \mathfrak{g}$ , then  $\eta \in \widehat{\Phi}$  is complex if and only if it belongs to  $\widehat{\Phi}_{r_e}$  and it is not long (see [7]). The case of  $\mathfrak{g}$  semisimple and not simple corresponds to  $\mathfrak{g}$  equal to the sum  $\mathfrak{k} \oplus \mathfrak{k}$  of two isomorphic simple ideals,  $\sigma$  the flip involution,  $\mathfrak{g}_0 = \mathfrak{k}$ , and  $\mathfrak{g}_1 \simeq \mathfrak{k}$  with  $\mathfrak{k}$  acting on itself via the adjoint representation.

For  $w \in \widehat{W}$ , define its set of inversions

$$(2.1) \quad N(w) = \{\alpha \in \widehat{\Phi}^+ \mid w^{-1}(\alpha) \in -\widehat{\Phi}^+\}.$$

Recall that a finite subset  $A$  of positive roots of an affine root system is of the form  $N(w)$  for some  $w \in \widehat{W}$  if and only if both  $A$  and  $\widehat{\Phi}^+ \setminus A$  are closed under root addition (see e.g. [5]). We will refer to this property as *biconvexity*.

If  $\alpha$  is a real root in  $\widehat{\Phi}^+$ , we let  $s_\alpha$  denote the reflection in  $\alpha$ . If  $\alpha_i$  is a simple root we set  $s_i = s_{\alpha_i}$ .

Recall from [6] the following

**Definition 2.4.** An element  $w \in \widehat{W}$  is called  $\sigma$ -minuscule if  $N(w) \subset \widehat{\Phi}_1$ .

We denote by  $\mathcal{W}_\sigma^{ab}$  the set of  $\sigma$ -minuscule elements of  $\widehat{W}$ , and we regard it as a poset under the weak Bruhat order.

We let  $\mathcal{I}_{ab}^\sigma$  be the set of abelian subalgebras of  $\mathfrak{g}$  contained in  $\mathfrak{g}_1$  that are stable under the action of the Borel subalgebra  $\mathfrak{b}_0$  of  $\mathfrak{g}_0$  corresponding to  $\Phi_0^+$ , or equivalently under the action of the Borel subgroup  $B_0 \subset G_0$  with Lie algebra  $\mathfrak{b}_0$ . Inclusion turns  $\mathcal{I}_{ab}^\sigma$  into a poset.

**Proposition 2.5.** [6, Theorem 3.2] *Let  $w \in \mathcal{W}_\sigma^{ab}$ . Suppose  $N(w) = \{\beta_1, \dots, \beta_k\}$ . The map  $\Theta : \mathcal{W}_\sigma^{ab} \rightarrow \mathcal{I}_{ab}^\sigma$  defined by*

$$w \mapsto \bigoplus_{i=1}^k \mathfrak{g}_1^{-\bar{\beta}_i}$$

*is a poset isomorphism.*

Assume that  $\mathfrak{g}_0$  is semisimple; then there is an index  $p$  such that  $\Pi_0 = \widehat{\Pi} \setminus \{\alpha_p\}$ . Assume furthermore that  $\alpha_p$  is non-complex (in particular,  $\mathfrak{g}$  is simple). Set  $\Pi_{0,\alpha_p} = \Pi_0 \cap \alpha_p^\perp$ ,  $W_{0,\alpha_p} = W(\Pi_{0,\alpha_p})$ , and denote by  $w_{0,\alpha_p}$  the longest element of  $W_{0,\alpha_p}$ . Let  $w_0$  be the longest element of  $W_0$ . Set

$$w_p = s_p w_{0,\alpha_p} w_0$$

In [7] it is proved that  $w_p \in \mathcal{W}_\sigma^{ab}$  if and only if  $\alpha_p$  is long; in such a case, the abelian  $B_0$ -stable subalgebra  $\mathfrak{a}_p$  corresponding to  $w_p$  is maximal.

3.  $B_0$ -ORBITS IN  $B_0$ -STABLE SUBALGEBRAS CONTAINED IN  $\mathfrak{g}_1$ 

Throughout this section, we will assume that  $\sigma : \mathfrak{g} \rightarrow \mathfrak{g}$  is an (indecomposable) automorphism of order  $m$ , and that  $\mathfrak{a}$  is a  $B_0$ -stable subalgebra of  $\mathfrak{g}$  contained in  $\mathfrak{g}_1$ . By [19, Proposition 4.9],  $\mathfrak{a}$  contains no semisimple element. In particular,  $\mathfrak{a} \cap \mathfrak{g}_1^0 = 0$  and  $\mathfrak{a}$  is completely determined by its set of weights  $\Psi(\mathfrak{a}) \subset \Phi_1$ :

$$\mathfrak{a} = \bigoplus_{\alpha \in \Psi(\mathfrak{a})} \mathfrak{g}_1^\alpha.$$

Since  $\mathfrak{a}$  is  $B_0$ -stable it follows that  $G_0\mathfrak{a}$  is closed; since it contains no semisimple element it follows that every element in  $\mathfrak{a}$  is nilpotent. Since there are only finitely many nilpotent  $G_0$ -orbits in  $\mathfrak{g}_1$ , it follows that  $G_0\mathfrak{a}$  is the closure of such an orbit.

For all  $\alpha \in \Phi_i$ , fix a non-zero element  $x_i^\alpha \in \mathfrak{g}_i^\alpha$ . If  $v \in \mathfrak{a}$  and  $v = \sum_{\alpha} c_{\alpha} x_1^{\alpha}$ , then we set  $\text{supp}(v) = \{\alpha \in \Psi(\mathfrak{a}) \mid c_{\alpha} \neq 0\}$ . If  $\mathcal{S} \subset \Psi(\mathfrak{a})$  we set

$$x_{\mathcal{S}} = \sum_{\alpha \in \mathcal{S}} x_1^{\alpha}.$$

**Theorem 3.1.** *Let  $\mathfrak{a}$  be a  $B_0$ -stable abelian subalgebra in  $\mathfrak{g}_1$ . For all  $x \in \mathfrak{a}$ , there is a unique orthogonal subset  $\mathcal{S}$  of  $\Psi(\mathfrak{a})$  such that  $B_0x = B_0x_{\mathcal{S}}$ . In particular,  $B_0$  acts on  $\mathfrak{a}$  with finitely many orbits, which are parametrized by the orthogonal subsets of  $\Psi(\mathfrak{a})$ .*

In the special case of the involution  $\sigma : \mathfrak{g} \oplus \mathfrak{g} \rightarrow \mathfrak{g} \oplus \mathfrak{g}$ ,  $(x, y) \mapsto (y, x)$ ,  $\mathcal{I}_{ab}^{\sigma}$  is the set of abelian ideals of  $\mathfrak{b}$  and, for such an ideal  $\mathfrak{a}$ ,  $G\mathfrak{a}$  is always the closure of a spherical nilpotent  $G$ -orbit in  $\mathfrak{g}$  [22]. Moreover, the closure of a spherical nilpotent orbit in  $\mathfrak{g}$  can be realized as  $G\mathfrak{a}$  for some abelian ideal  $\mathfrak{a}$  (up to choosing the Borel subalgebra  $\mathfrak{b}$  in a compatible way). In this case, Panyushev [21] has recently given a new proof of the finiteness of the  $B$ -orbits, by giving an explicit parametrization of the  $B$ -orbits in  $\mathfrak{a}$ . Our proof of Theorem 3.1 will follow closely the proof of [21, Theorem 2.2].

By Corollary 2.2, the following properties for a  $B_0$ -stable abelian subalgebra  $\mathfrak{a} \subset \mathfrak{g}_1$  hold.

- (A1) If  $\alpha \in \Psi(\mathfrak{a})$ , then  $-\alpha \notin \Psi(\mathfrak{a})$ .
- (A2) Let  $\alpha, \beta \in \Psi(\mathfrak{a})$ , then  $\alpha + \beta \notin \Phi_2$ .
- (A3) Let  $\alpha \in \Psi(\mathfrak{a})$  and  $\gamma \in \Phi_0^+$  be such that  $\alpha + \gamma \in \Phi_1$ , then  $\alpha + \gamma \in \Psi(\mathfrak{a})$ .

**Lemma 3.2.** *For  $\alpha, \beta \in \Psi(\mathfrak{a})$ ,  $\alpha \neq \beta$ , the following statements are equivalent:*

- i)  $\alpha, \beta$  are orthogonal;
- ii)  $\alpha - \beta \notin \Phi_0$ ;
- iii)  $\delta' + \alpha$  and  $\delta' + \beta$  are strongly orthogonal in  $\widehat{\Phi}$ .

*Proof.* (1)  $\Rightarrow$  (2) Suppose that  $(\alpha, \beta) = 0$  and  $\alpha - \beta \in \Phi_0$ . Then  $\delta' + \alpha, \delta' + \beta \in \widehat{\Phi}$  are orthogonal as well, and

$$s_{\delta'+\beta}(\alpha - \beta) = s_{\delta'+\beta}(\delta' + \alpha) - s_{\delta'+\beta}(\delta' + \beta) = 2\delta' + \alpha + \beta.$$

It follows that  $2\delta' + \alpha + \beta \in \widehat{\Phi}$ , that is  $\alpha + \beta \in \Phi_2$ , contradicting (A2).

(2)  $\Rightarrow$  (1) Suppose that  $(\alpha, \beta) \neq 0$ . If  $(\alpha, \beta) < 0$ , then Corollary 2.2 implies  $\alpha + \beta \in \Phi_2$ , contradicting (A2). Therefore it must be  $(\alpha, \beta) > 0$ , and again by Corollary 2.2 we get  $\alpha - \beta \in \Phi_0$ .

Statement (3) is clearly equivalent to the others.  $\square$

As in [21], the key step to prove Theorem 3.1 is the following combinatorial lemma, which generalizes [21, Lemma 1.2] to the graded setting.

**Lemma 3.3.** *Let  $\alpha, \beta \in \Psi(\mathfrak{a})$  be orthogonal weights and let  $\gamma \in \Phi_0$ . If  $\alpha + \gamma \in \Psi(\mathfrak{a})$ , then  $\beta + \gamma \notin \Psi(\mathfrak{a})$*

*Proof.* Assume that both  $\alpha + \gamma$  and  $\beta + \gamma$  belong to  $\Psi(\mathfrak{a})$ . Suppose that  $(\alpha, \gamma) < 0$ : then  $(\beta + \gamma, \alpha) < 0$  as well, and Corollary 2.2 implies  $\alpha + \beta + \gamma \in \Phi_2$ , against (A2). Similarly it cannot be  $(\beta, \gamma) < 0$ . Suppose that  $(\alpha, \gamma) = (\beta, \gamma) = 0$ : then  $(\alpha + \gamma, \beta + \gamma) > 0$ , hence Corollary 2.2 ii) implies  $\alpha - \beta \in \Phi_0$ , which contradicts the fact that  $\alpha$  and  $\beta$  are orthogonal by Lemma 3.2. Therefore it must be  $(\alpha, \gamma) \geq 0$  and  $(\beta, \gamma) \geq 0$ .

On the other hand, again by Lemma 3.2, we have that

$$(\alpha, \beta) = 0 \iff \alpha - \beta \notin \Phi_0 \iff (\alpha + \gamma) - (\beta + \gamma) \notin \Phi_0 \iff (\alpha + \gamma, \beta + \gamma) = 0.$$

In turn, by the orthogonality of  $\alpha$  and  $\beta$ , the last equality implies that either  $(\alpha, \gamma) < 0$  or  $(\beta, \gamma) < 0$ , which is a contradiction.  $\square$

We denote by  $\leq_0$  the dominance order on  $\mathfrak{h}_0^*$ :  $\lambda \leq_0 \mu$  if  $\mu - \lambda \in \mathbb{N}\Phi_0^+$ . If  $\mathcal{S}$  is subset of  $\Psi(\mathfrak{a})$  we denote by  $\min(\mathcal{S})$  the set of the minimal elements of  $\mathcal{S}$  w.r.t.  $\leq_0$ , and define two subsets of  $\Psi(\mathfrak{a})$  as follows

$$\begin{aligned} \Psi_{\mathcal{S}} &= \{\beta \in \Phi_1 \mid \text{there is } \alpha \in \mathcal{S} \text{ with } \beta - \alpha \in \Phi_0^+\}, \\ \mathcal{S}^{\geq_0} &= \{\beta \in \Phi_1 \mid \text{there is } \alpha \in \mathcal{S} \text{ with } \alpha \leq_0 \beta\}. \end{aligned}$$

We also denote by  $\mathfrak{a}_{\mathcal{S}}$  the minimal  $B_0$ -stable subalgebra of  $\mathfrak{a}$  containing the weight space  $\mathfrak{g}_1^\alpha$  for all  $\alpha \in \mathcal{S}$ , namely

$$\mathfrak{a}_{\mathcal{S}} = \bigoplus_{\beta \in \mathcal{S}^{\geq_0}} \mathfrak{g}_1^\beta.$$

**Lemma 3.4.** *Let  $\mathfrak{a} \in \mathcal{I}_{ab}^\sigma$ . Let  $\mathcal{S}$  be an orthogonal subset of  $\Psi(\mathfrak{a})$ . Let  $x \in \mathfrak{a}$  be such that  $\mathcal{S} \subset \text{supp}(x)$  and  $\mathcal{S}$  is a lower order ideal in  $\text{supp}(x)$ . Then there is  $y \in B_0x$  with the same property such that  $\text{supp}(y) \cap \Psi_{\mathcal{S}} = \emptyset$ .*

*Proof.* Set  $Z = \{x \in \mathfrak{a} \mid \mathcal{S} \text{ is a lower order ideal in } \text{supp}(x)\}$ . If  $x \in Z$ , we set  $\mathcal{T}(x) = \text{supp}(x) \setminus \mathcal{S}$ . We prove the claim by induction on  $\dim \mathfrak{a}_{\mathcal{T}(x)}$ . If  $\dim \mathfrak{a}_{\mathcal{T}(x)} = 0$ , then  $\text{supp}(x) = \mathcal{S}$  and there is nothing to prove.

Assume  $\dim \mathfrak{a}_{\mathcal{T}(x)} > 0$ . Set  $\mathcal{S}' = \Psi_{\mathcal{S}}$ , and for  $v \in \mathfrak{a}$ , let  $\mathcal{S}'(v) = \mathcal{S}' \cap \text{supp}(v)$ . We can assume that  $\mathcal{S}'(x) \neq \emptyset$ , for, otherwise, we can take  $y = x$ . Then there are  $\alpha \in \mathcal{S}$  and  $\gamma \in \Phi_0^+$  such that  $\alpha + \gamma \in \text{supp}(x)$ . By Lemma 3.3, it follows that  $\varepsilon + \gamma \notin \Psi(\mathfrak{a})$  for all  $\varepsilon \in \mathcal{S} \setminus \{\alpha\}$ . If  $u_\gamma(\xi) \in B_0$  is the element defined by exponentiating  $\xi x_0^\gamma$  ( $\xi \in \mathbb{C}$ ), it follows that

$$(3.1) \quad u_\gamma(\xi)x_1^\beta = x_1^\beta \text{ if } \beta \in \mathcal{S} \setminus \{\alpha\}, \quad u_\gamma(\xi)x_1^\beta = x_1^\beta + \xi[x_0^\gamma, x_1^\beta] + \frac{\xi^2}{2}[x_0^\gamma, [x_0^\gamma, x_1^\beta]] + \dots \text{ otherwise.}$$

Let  $\pi : \mathfrak{a} \rightarrow \bigoplus_{\alpha \in \mathcal{S}} \mathfrak{g}_1^\alpha$  be the projection. We claim that  $\pi(u_\gamma(\xi)x) = \pi(x)$  for all  $\xi \in \mathbb{C}$ . In fact, if  $\varepsilon \in \mathcal{S} \setminus \{\alpha\}$  then, by (3.1),  $\pi(u_\gamma(\xi)x_1^\varepsilon) = x_1^\varepsilon$ . Since  $\mathcal{S}$  is strongly orthogonal,  $\alpha + k\gamma \notin \mathcal{S}$  so  $\pi(u_\gamma(\xi)x_1^\alpha) = x_1^\alpha$ . Finally, if  $\beta \notin \mathcal{S}$  and  $\beta \in \text{supp}(x)$ , then, since  $\mathcal{S}$  is a lower order ideal in  $\text{supp}(x)$ ,  $\beta + k\gamma \notin \mathcal{S}$ , so  $\pi(u_\gamma(\xi)x_1^\beta) = \pi(x_1^\beta) = 0$ . Choose  $\xi_0 \in \mathbb{C}$  such that  $\alpha + \gamma \notin \text{supp}(u_\gamma(\xi_0)x)$  and set  $x' = u_\gamma(\xi_0)x$ . By construction  $\mathcal{S} \subset \text{supp}(x')$ . We have to prove that  $\mathcal{S}$  is a lower order ideal in  $\text{supp}(x')$ . Take  $\eta \in \text{supp}(x')$  such that there exists  $\beta \in \mathcal{S}$  with  $\eta \leq \beta$ . We know that  $\eta = \zeta + r\gamma$ ,  $\zeta \in \text{supp}(x)$ . Since  $\zeta \leq_0 \beta$ , we have that  $\zeta \in \mathcal{S}$ . If  $\zeta \neq \alpha$ , then, by Lemma 3.3,  $\zeta + \gamma$  is not in  $\Phi_1$ , in particular  $\eta \notin \text{supp}(x')$ . If  $\zeta = \alpha$  then, by construction, either  $r > 1$  or  $\eta = \zeta$ . If  $r > 1$  then  $\alpha + \gamma \leq_0 \alpha + r\gamma \leq_0 \beta \in \mathcal{S}$  and  $\alpha + \gamma \in \text{supp}(x)$ , so  $\alpha + \gamma \in \mathcal{S}$ . We already observed that this is not possible, hence  $\eta = \zeta \in \mathcal{S}$ .

We now prove that  $\mathfrak{a}_{\mathcal{T}(x')} \subset \mathfrak{a}_{\mathcal{T}(x)}$ . It suffices to prove that, if  $\beta \in \mathcal{T}(x')$  then  $\beta \in \Psi(\mathfrak{a}_{\mathcal{T}(x)})$ . Write  $\beta = \zeta + r\gamma$  with  $r \geq 0$  and  $\zeta \in \text{supp}(x)$ . If  $\zeta \notin \mathcal{S}$  then we are already done. If  $\zeta \in \mathcal{S}$  then, as shown above, we have  $\zeta = \alpha$ . Note that  $r > 0$ , for,



otherwise  $\beta \in \mathcal{S}$ . Now, if  $r > 0$ , as  $\alpha + \gamma \in \mathcal{T}(x)$  and  $\alpha + \gamma \leq_0 \beta$ , we have that  $\beta \in \Psi(\mathfrak{a}_{\mathcal{T}(x)})$ .

Since  $\alpha + \gamma \notin \text{supp}(x')$ , it follows that  $\mathfrak{a}_{\mathcal{T}(x')} \subsetneq \mathfrak{a}_{\mathcal{T}(x)}$ . By the induction hypothesis, it follows that  $B_0x'$  contains an element  $y$  such that  $\mathcal{S}$  is a lower order ideal in  $\text{supp}(y)$  and  $\text{supp}(y) \cap \Psi_{\mathcal{S}} = \emptyset$ . Since  $x' \in B_0x$ , we have that  $y \in B_0x$ .  $\square$

Thanks to previous lemmas, we can now reproduce the same argument given in [21, Theorem 2.2] to prove Theorem 3.1.

*Proof of Theorem 3.1.* We first show that every  $B_0$ -orbit in  $\mathfrak{a}$  possesses a representative of the form  $x_{\mathcal{S}}$ , for some orthogonal subset  $\mathcal{S} \subset \Psi(\mathfrak{a})$ .

Let  $v \in \mathfrak{a}$ . Set  $v_0 = v$  and  $\mathcal{S}_0 = \min \text{supp}(v)$ . Notice that  $\mathcal{S}_0$  is orthogonal by Lemma 3.2: indeed for all  $\alpha, \beta \in \mathcal{S}_0$  we have by construction that  $\alpha - \beta \notin \Phi_0$ . Therefore  $v_0$  and  $\mathcal{S}_0$  satisfy the assumptions of Lemma 3.4, and there is  $v_1 \in B_0v_0$  such that  $\mathcal{S}_0 \subset \text{supp}(v_1)$  and  $\text{supp}(v_1) \cap \Psi_{\mathcal{S}_0} = \emptyset$ . Define  $\mathcal{S}_1 = \mathcal{S}_0 \cup \min(\text{supp}(v_1) \setminus \mathcal{S}_0)$ : then by construction we still have  $\alpha - \beta \notin \Phi_0$  for all  $\alpha, \beta \in \mathcal{S}_1$ , so that  $\mathcal{S}_1$  is again orthogonal by Lemma 3.2. It is clear that  $\mathcal{S}_1$  is lower order ideal in  $\text{supp}(v_1)$ .

More generally, let  $i \geq 0$  and suppose that  $v_i$  and  $\mathcal{S}_i$  are defined. Then there is  $v_{i+1} \in B_0v_i = B_0v_0$  such that  $\mathcal{S}_i$  is a lower order ideal in  $\text{supp}(v_{i+1})$ ,  $\text{supp}(v_{i+1}) \cap \Psi_{\mathcal{S}_i} = \emptyset$ , and

$$\mathcal{S}_{i+1} = \mathcal{S}_i \cup \min(\text{supp}(v_{i+1}) \setminus \mathcal{S}_i)$$

is an orthogonal subset by Lemma 3.2 that is a lower order ideal in  $\text{supp}(v_{i+1})$ .

Clearly,  $\mathcal{S}_{i+1}$  is strictly bigger than  $\mathcal{S}_i$ , unless  $\text{supp}(v_{i+1}) = \mathcal{S}_i$ . Therefore, proceeding inductively, we find an element  $v_{k+1} \in B_0v$  whose support equals  $\mathcal{S}_k$ , which is an orthogonal subset, hence  $B_0v = B_0v_{k+1} = B_0x_{\mathcal{S}_k}$ .

We now show that every  $B_0$ -orbit contains a unique orthogonal representative  $x_{\mathcal{S}}$ . If  $\mathcal{S} \subset \Psi(\mathfrak{a})$ , notice that the vector space  $\langle B_0x_{\mathcal{S}} \rangle$  generated the orbit of  $x_{\mathcal{S}}$  is  $B_0$ -stable, therefore it coincides with  $\mathfrak{a}_{\mathcal{S}}$ .

Let  $\mathcal{S}, \mathcal{S}'$  be orthogonal subsets of  $\Psi(\mathfrak{a})$  and suppose that  $B_0x_{\mathcal{S}} = B_0x_{\mathcal{S}'}$ . Then  $\mathfrak{a}_{\mathcal{S}} = \mathfrak{a}_{\mathcal{S}'}$ , and we set  $\Gamma = \min \Psi(\mathfrak{a}_{\mathcal{S}})$ . Notice that  $\Gamma \subset \mathcal{S} \cap \mathcal{S}'$ , therefore setting  $\mathcal{R} = \mathcal{S} \setminus \Gamma$  and  $\mathcal{R}' = \mathcal{S}' \setminus \Gamma$  we can decompose  $x_{\mathcal{S}} = x_{\Gamma} + x_{\mathcal{R}}$  and  $x_{\mathcal{S}'} = x_{\Gamma} + x_{\mathcal{R}'}$ . Let  $B_0 = T_0U_0$  be the Levi decomposition of  $B_0$ . Let  $b \in B_0$  be such that  $bx_{\mathcal{S}} = x_{\mathcal{S}'}$  and write  $b = t^{-1}u$  with  $t \in T_0$  and  $u \in U_0$ . Then  $ux_{\Gamma} + ux_{\mathcal{R}} = tx_{\Gamma} + tx_{\mathcal{R}'}$ . It follows that  $tx_{\Gamma} = ux_{\Gamma} = x_{\Gamma}$ , hence  $x_{\mathcal{R}'} \in B_0x_{\mathcal{R}}$ . Since  $\mathfrak{a}_{\mathcal{R}} \subset \mathfrak{a}_{\mathcal{S}} \cap \mathfrak{a}_{\mathcal{S}'}$  is a smaller  $B_0$ -stable abelian subalgebra in  $\mathfrak{g}_1$  and since  $\mathcal{R}$  and  $\mathcal{R}'$  are orthogonal subsets in  $\Psi(\mathfrak{a}_{\mathcal{R}})$ , the claim follows proceeding by downward induction.  $\square$

The following facts are also proved by adapting the same proofs of [21]. Denote by  $\mathcal{S}_{\mathfrak{a}} \subset \Psi(\mathfrak{a})$  the subset constructed as follows: set  $\mathcal{S}_1 = \min \Psi(\mathfrak{a})$ , and for  $i > 1$  define inductively

$$\mathcal{S}_i = \min \left( \Psi(\mathfrak{a}) \setminus \bigcup_{j < i} (\mathcal{S}_j \cup \Psi_{\mathcal{S}_j}) \right).$$

Define  $\mathcal{S}_{\mathfrak{a}} = \bigcup_{i > 0} \mathcal{S}_i$ , which is an orthogonal subset thanks to Lemma 3.2.

**Proposition 3.5.** *Let  $\mathcal{S} \subset \Psi(\mathfrak{a})$  be an orthogonal subset.*

- i)  $B_0x_{\mathcal{S}}$  is open in  $\mathfrak{a}$  if and only if  $\mathcal{S} = \mathcal{S}_{\mathfrak{a}}$ .
- ii) As a  $T_0$ -module, the tangent space  $T_{x_{\mathcal{S}}}(B_0x_{\mathcal{S}})$  decomposes as follows:

$$T_{x_{\mathcal{S}}}(B_0x_{\mathcal{S}}) = \bigoplus_{\alpha \in \text{SU}\Psi_{\mathcal{S}}} \mathfrak{g}_1^{\alpha}.$$

In particular,  $\dim B_0x_{\mathcal{S}} = |\mathcal{S}| + |\Psi_{\mathcal{S}}|$ .

## 4. ANTICHAINS OF ORTHOGONAL ROOTS IN HERMITIAN SYMMETRIC SPACES

Suppose that  $\mathfrak{g}$  is a simple Lie algebra and let  $\Pi$  be a set of simple roots. Let  $\theta$  be the corresponding highest root and let  $\alpha_q \in \Pi$  be a simple root with  $[\theta : \alpha_q] = 1$ . Let  $\mathfrak{p}^+ \subset \mathfrak{g}$  be the maximal parabolic subalgebra associated to the set of simple roots  $\Pi \setminus \{\alpha_q\}$ . Then its nilradical  $\mathfrak{p}_u^+$  is abelian; conversely any standard parabolic subalgebra with abelian nilradical arises in this way.

Let  $\mathfrak{p}^+ = \mathfrak{l} \oplus \mathfrak{p}_u^+$  be the Levi decomposition, and let  $\mathfrak{p}^-$  be the opposite parabolic subalgebra of  $\mathfrak{p}^+$ . The decomposition into  $\mathfrak{l}$ -submodules  $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{p}_u^+ \oplus \mathfrak{p}_u^-$  defines an involution  $\sigma$  of  $\mathfrak{g}$  by setting  $\sigma(x) = x$  if  $x \in \mathfrak{l}$  and  $\sigma(x) = -x$  if  $x \in \mathfrak{p}_u^+ \oplus \mathfrak{p}_u^-$ . It is then clear that

$$\mathfrak{g}_0 = \mathfrak{l}, \quad \mathfrak{g}_1 = \mathfrak{p}_u^+ \oplus \mathfrak{p}_u^-.$$

Recalling the notation introduced in Section 2.2, the sets  $\Phi_i$  attached to  $\sigma$  are  $\Phi_1 = \Phi_1^+ \cup -\Phi_1^+$ , where  $\Phi_1^+$  is the set of roots  $\beta$  such  $[\beta : \alpha_q] = 1$ , while  $\Phi_0$  is the set of roots  $\beta$  such that  $[\beta : \alpha_q] = 0$ . Observe that in this case  $\Phi = \Phi_0 \cup \Phi_1$  is the set of roots of  $\mathfrak{g}$ . Since in this case  $\mathfrak{h}_0$  is a Cartan subalgebra of  $\mathfrak{g}$ , we can choose  $h_{reg} \in \mathfrak{h}_0$  so that  $\alpha(h_{reg}) > 0$  for all  $\alpha \in \Pi$ . With this choice, letting  $\Phi^+$  denote the set of positive roots of  $\mathfrak{g}$  corresponding to the choice of  $\Pi$ , we have that

$$\begin{aligned} \Phi_0^+ &= \{\beta \in \Phi^+ \mid [\beta : \alpha_q] = 0\}, \\ \Phi_1^+ &= \{\beta \in \Phi^+ \mid [\beta : \alpha_q] > 0\}, \\ \Pi_0 &= \Pi \setminus \{\alpha_q\}. \end{aligned}$$

We set  $\Phi_i^- = -\Phi_i^+$  ( $i = 0, 1$ ). Clearly,  $\Phi_1^+$  is the set of weights of  $\mathfrak{h}_0$  in  $\mathfrak{p}_u^\pm$ . We let  $\mathfrak{b}_0^\pm$  be the Borel subalgebra of  $\mathfrak{g}_0$  corresponding to  $\Phi_0^\pm$ . Recall that  $W_0$  denotes the Weyl group of  $\mathfrak{g}_0$ .

Let  $\text{Ort}(\Phi_1^+)$  be the collection of the orthogonal subsets of  $\Phi_1^+$ , and let  $\text{Ort}_{\max}(\Phi_1^+)$  be the collection of the orthogonal subsets of  $\Phi_1^+$  which are maximal with respect to inclusion. Regard  $\Phi_1^+$  as a partial ordered set via  $\leq_0$ . Since  $\mathfrak{p}_u^+$  is an abelian subalgebra of  $\mathfrak{g}$ , by Lemma 3.2 two elements  $\alpha, \beta \in \Phi_1^+$  are orthogonal if and only if they are strongly orthogonal, if and only if  $\alpha - \beta \notin \Phi_0$ . In particular, every antichain  $\mathcal{A} \subset \Phi_1^+$  is an orthogonal subset.

Given  $\mathcal{B} \in \text{Ort}(\Phi_1^+)$ , let  $\mathfrak{a}_{\mathcal{B}} \subset \mathfrak{p}_u^+$  be the  $B_0$ -stable subalgebra generated by  $\mathcal{B}$ . We define a preorder  $\vdash$  on  $\text{Ort}(\Phi_1^+)$  as follows: if  $\mathcal{B}_1, \mathcal{B}_2 \in \text{Ort}(\Phi_1^+)$ , then  $\mathcal{B}_1 \vdash \mathcal{B}_2$  if  $\mathfrak{a}_{\mathcal{B}_1} \subset \mathfrak{a}_{\mathcal{B}_2}$ . Equivalently,  $\mathcal{B}_1 \vdash \mathcal{B}_2$  if and only if  $\mathcal{B}_1 \subset \mathcal{B}_2^{\geq 0}$ , where, if  $\mathcal{B} \subset \Phi_1^+$ , we set

$$\mathcal{B}^{\geq 0} = \{\alpha \in \Phi_1^+ \mid \text{there is } \beta \in \mathcal{B} \text{ such that } \beta \leq_0 \alpha\}.$$

Given  $m \leq r$ , we will denote by  $\text{Ort}_m(\Phi_1^+)$  the set of the orthogonal subsets of cardinality  $m$ . If  $\Phi$  is not simply laced, we will say  $\mathcal{B} \in \text{Ort}(\Phi_1^+)$  is of type  $(h, k)$  if it contains exactly  $h$  short roots and  $k$  long roots, and we denote by  $\text{Ort}_{(h,k)}(\Phi_1^+)$  the set of the orthogonal subsets of type  $(h, k)$ . To unify some notations, in the simply laced case we will regard every root as a long root. Therefore if  $\Phi$  is simply laced we have  $\text{Ort}_m(\Phi_1^+) = \text{Ort}_{(0,m)}(\Phi_1^+)$ .

In this section we will study the antichains of  $\Phi_1^+$ . We will show that for every orthogonal subset  $\mathcal{B} \subset \Phi_1^+$  there is always an antichain  $\mathcal{A} \subset \Phi_1^+$  such that  $\mathcal{A} \vdash \mathcal{B}$ . We first discuss the simply laced case uniformly; the two remaining cases  $(B_n, \alpha_1)$  and  $(C_n, \alpha_n)$  will be treated separately. We summarize our results in the following theorem.

**Theorem 4.1.**

- i) *Suppose that  $\Phi$  is simply laced. For all  $\mathcal{B} \in \text{Ort}(\Phi_1^+)$ , there is an antichain  $\mathcal{A} \subset \Phi_1^+$  such that  $|\mathcal{A}| = |\mathcal{B}|$  and  $\mathcal{A} \vdash \mathcal{B}$ .*

- ii) Suppose that  $\Phi$  is not simply laced. For all  $\mathcal{B} \in \text{Ort}(\Phi_1^+)$  of type  $(h, k)$ , there is an antichain  $\mathcal{A} \subset \Phi_1^+$  of type  $(h + \lfloor k/2 \rfloor, k - 2\lfloor k/2 \rfloor)$  such that  $\mathcal{A} \vdash \mathcal{B}$ .

*Proof.* The claim follows combining Proposition 4.6, Proposition 4.7, Proposition 4.10.  $\square$

Let  $P \subset G$  be the parabolic subgroup corresponding to  $\mathfrak{p}^+$ . It is well known that  $G/P$  is an irreducible simply connected Hermitian symmetric space of compact type, whose corresponding involution of  $G$  is  $\sigma$ , and every such a symmetric space arises in this way (see e.g. [23, Section 5.5]). Therefore we will refer to the pair  $(\Pi, \alpha_q)$  as a *Hermitian pair*, and we will say that  $\sigma$  is an *involution of Hermitian type*, or simply a *Hermitian involution*. Correspondingly, we get also a symmetric variety  $G/G_0$ , where  $G_0 = G^\sigma$  is the Levi factor of  $P$ .

Since  $\mathfrak{p}_u^+ \subset \mathfrak{g}_1$  is a  $B_0$ -stable abelian subalgebra of  $\mathfrak{g}$ , by Theorem 3.1 it possesses finitely many  $B_0$ -orbits, which are classified by  $\text{Ort}(\Phi_1^+)$ . In this situation, the description of the  $B_0$ -orbits already follows by [21]: since  $\mathfrak{p}_u^+$  is abelian, the unipotent radical  $P_u$  acts trivially on its Lie algebra  $\mathfrak{p}_u^+$ , therefore every  $B$ -orbit is actually a  $B_0$ -orbit. The  $G_0$ -orbits in  $\mathfrak{p}_u^+$  were studied by Muller, Rubenthaler, and Schiffmann [16] and by Richardson, Röhrle and Steinberg [23]. In the latter reference it is shown that they are parametrized by the  $W_0$ -orbits in  $\text{Ort}(\Phi_{1,\ell}^+)$ , where  $\Phi_{1,\ell}^+ \subset \Phi_1^+$  denotes the subset of the long roots.

Let  $\mathcal{S}_{\Pi, \alpha_q} \subset \Phi_1^+$  be the orthogonal subset corresponding to the open  $B_0$ -orbit of  $\mathfrak{p}_u^+$ , constructed recursively as in Proposition 3.5. In this case  $\mathcal{S}_{\Pi, \alpha_q}$  is well known, and it coincides with the set of Harish-Chandra strongly orthogonal roots (see [9], [15]). Denote by  $r = |\mathcal{S}_{\Pi, \alpha_q}|$  the rank of the symmetric variety  $G/G_0$ . By [15, Theorem 2] we have  $\mathcal{S}_{\Pi, \alpha_q} \subset \Phi_{1,\ell}^+$ , in particular  $\mathcal{S}_{\Pi, \alpha_q}$  is a maximal orthogonal subset of  $\Phi_1^+$  consisting of long roots, and by [16, Theorem 2.12 and Proposition 2.13] it follows that  $\mathcal{S}_{\Pi, \alpha_q}$  is an orthogonal subset of maximal cardinality in  $\Phi_1^+$ . In particular,  $\mathcal{S}_{\Pi, \alpha_q} \in \text{Ort}_{\max}(\Phi_1^+)$ , and  $|\mathcal{B}| \leq r$  for all  $\mathcal{B} \in \text{Ort}(\Phi_1^+)$ .

We report in Table 1 the classification of the Hermitian pairs, together with the rank of the corresponding symmetric varieties  $G/G_0$  (where  $\Pi = \{\alpha_1, \dots, \alpha_n\}$  is enumerated as in [1]).

| $(\Pi, \alpha_q)$                      | $\text{rk}(G/G_0)$            |
|--|-------------------------------|
| $(A_n, \alpha_q) (1 \leq q \leq n)$    | $\min\{q, n + 1 - q\}$        |
| $(B_n, \alpha_1)$                      | 2                             |
| $(C_n, \alpha_n)$                      | $n$                           |
| $(D_n, \alpha_1)$                      | 2                             |
| $(D_n, \alpha_{n-1}), (D_n, \alpha_n)$ | $\lfloor \frac{n}{2} \rfloor$ |
| $(E_6, \alpha_1), (E_6, \alpha_6)$     | 2                             |
| $(E_7, \alpha_7)$                      | 3                             |

**Table 1** – Hermitian pairs and ranks of the corresponding symmetric varieties.

*Remark 4.2.* By [23, Proposition 2.8 and Remark], the Weyl group  $W_0$  acts transitively on  $\text{Ort}_{(h,k)}(\Phi_1^+)$  for all  $h, k$ . In particular, we see that if  $\Phi$  is simply laced then  $\text{Ort}_{\max}(\Phi_1^+) = \text{Ort}_r(\Phi_1^+)$  coincides with the collection of the orthogonal subsets of maximal cardinality. On the other hand in the non-simply laced cases, corresponding to the Hermitian pairs  $(B_n, \alpha_1)$  and  $(C_n, \alpha_n)$ , we will easily see that if  $\mathcal{B}$  is an orthogonal subset of type  $(h, k)$ , then  $\mathcal{B} \in \text{Ort}_{\max}(\Phi_1^+)$  if and only if  $2h + k = r$ . In particular, it follows that, if  $\mathcal{B} \in \text{Ort}(\Phi_1^+)$  has maximal cardinality, then every

root in  $\mathcal{B}$  is long. Hence the orthogonal subsets of maximal cardinality coincide with the elements of  $\text{Ort}_{(0,r)}(\Phi_1^+)$ , and  $W_0$  acts transitively on these subsets. As well, it follows that, by choosing properly the Borel subgroup  $B_0 \subset G_0$ , every subset of orthogonal roots of maximal cardinality can be made into a set of strongly orthogonal Harish-Chandra roots for  $\Phi_1^+$ .

**4.1. The simply laced case.** We start by recording a well known fact that holds for any root system. See e.g. [24, Lemma 3.2].

**Lemma 4.3.** *Let  $\beta, \beta' \in \Phi^+$  and suppose that  $\beta' - \beta$  is a sum of positive roots. Then there are  $\gamma_1, \dots, \gamma_m \in \Phi^+$  such that  $\beta' - \beta = \gamma_1 + \dots + \gamma_m$  and  $\beta + \gamma_1 + \dots + \gamma_i \in \Phi^+$  for all  $i \leq m$ .*

In the simply laced case, Lemma 4.3 can be improved as follows:

**Proposition 4.4.** *Suppose that  $\Phi$  is simply laced. Let  $\beta, \beta' \in \Phi^+$  and suppose that  $\beta' - \beta$  is a sum of positive roots. Then  $\beta' - \beta$  is a sum of positive pairwise orthogonal roots.*

*Proof.* By Lemma 4.3, there are  $\gamma_1, \dots, \gamma_m \in \Phi^+$  such that  $\beta' - \beta = \gamma_1 + \dots + \gamma_m$  and  $\beta + \gamma_1 + \dots + \gamma_i$  is a positive root for all  $i \leq m$ . Let  $m$  be minimal with the previous property, fix  $\gamma_1, \dots, \gamma_m \in \Phi^+$  as above and, if  $0 \leq i \leq m$ , denote  $\beta_i = \beta + \gamma_1 + \dots + \gamma_i$ . We claim that  $\gamma_1, \dots, \gamma_m$  are pairwise orthogonal.

If  $m = 1$  there is nothing to prove. Assume  $m > 1$ , and suppose that  $\gamma_1, \dots, \gamma_m$  are not orthogonal. Let  $i_0 \leq m$  be the minimum such that  $(\gamma_i, \gamma_j) = 0$  for all  $i, j < i_0$  with  $i \neq j$ , and let  $j_0 < i_0$  be such that  $(\gamma_{j_0}, \gamma_{i_0}) \neq 0$ . Since  $\langle \beta, \gamma_{i_0}^\vee \rangle \geq -1$  and  $\langle \gamma_i, \gamma_{i_0}^\vee \rangle \geq -1$ , we can assume that  $\langle \gamma_{j_0}, \gamma_{i_0}^\vee \rangle = -1$ , thus  $\gamma_{j_0} + \gamma_{i_0}$  is a root. To reach a contradiction, we show that the  $m - 1$  positive roots

$$\gamma_1, \dots, \gamma_{j_0-1}, \gamma_{j_0+1}, \dots, \gamma_{i_0-1}, \gamma_{j_0} + \gamma_{i_0}, \gamma_{i_0+1}, \dots, \gamma_m$$

also satisfy the assumptions of  $\gamma_1, \dots, \gamma_m$ , contradicting the minimality of  $m$ . That is, we show that  $\beta_i - \gamma_{j_0} \in \Phi^+$  whenever  $j_0 < i < i_0$ .

Indeed, if  $i < i_0$ , then  $\beta_{i-1} + \gamma_i = \beta_i \in \Phi^+$ , therefore  $\langle \beta_{i-1}, \gamma_i^\vee \rangle = -1$ , and being  $(\gamma_i, \gamma_j) = 0$  for all  $j < i$  it follows that  $\langle \beta, \gamma_i^\vee \rangle = -1$ . Therefore, if  $j_0 < i < i_0$ , then it follows  $\langle \beta_i, \gamma_{j_0}^\vee \rangle = \langle \beta, \gamma_{j_0}^\vee \rangle + \langle \gamma_{j_0}, \gamma_{j_0}^\vee \rangle = 1$ , and the claim follows.  $\square$

**Lemma 4.5.** *Suppose that  $\Phi$  is simply laced. Let  $\mathcal{B} \in \text{Ort}(\Phi_1^+)$  and suppose that it is not an antichain, then there exists  $\mathcal{B}' \in \text{Ort}(\Phi_1^+)$  with  $|\mathcal{B}'| = |\mathcal{B}|$  such that  $\mathcal{B}' \vdash \mathcal{B}$ , and  $\dim \mathfrak{a}_{\mathcal{B}'} < \dim \mathfrak{a}_{\mathcal{B}}$ .*

*Proof.* Notice that  $W_0$  acts on  $\text{Ort}(\Phi_1^+)$ , we will find  $\mathcal{B}'$  in the  $W_0$ -orbit of  $\mathcal{B}$ . Let  $\beta \in \mathcal{B}$  be a minimal element and let  $\beta' \in \mathcal{B}$  with  $\beta < \beta'$ . Write  $\beta' - \beta = \gamma_1 + \dots + \gamma_m$  for some pairwise orthogonal roots  $\gamma_1, \dots, \gamma_m \in \Phi^+$  as in Proposition 4.4, and notice that  $\gamma_i \in \Phi_0^+$  for all  $i$ . Set  $\gamma = \gamma_1$ . Then  $\langle \beta' - \beta, \gamma^\vee \rangle = 2$ , and since  $\Phi$  is simply laced it follows that  $s_\gamma(\beta) = \beta + \gamma$  and  $s_\gamma(\beta') = \beta' - \gamma$ . On the other hand by Lemma 3.3  $\gamma$  is orthogonal to every root in  $\mathcal{B} \setminus \{\beta, \beta'\}$ , therefore

$$s_\gamma(\mathcal{B}) = (\mathcal{B} \setminus \{\beta, \beta'\}) \cup \{s_\gamma(\beta), s_\gamma(\beta')\}.$$

Being  $\gamma < \beta' - \beta$ , we have  $\beta < s_\gamma(\beta)$  and  $\beta < s_\gamma(\beta')$ . Hence  $s_\gamma(\mathcal{B}) \vdash \mathcal{B}$ , and since  $\beta$  is minimal in  $\mathcal{B}$  we get  $\beta \notin \Psi(\mathfrak{a}_{\mathcal{B}'})$ .  $\square$

**Proposition 4.6.** *Suppose that  $\Phi$  is simply laced. Let  $\mathcal{B} \in \text{Ort}(\Phi_1^+)$ , then there is an antichain  $\mathcal{A} \in \text{Ort}(\Phi_1^+)$  with  $|\mathcal{A}| = |\mathcal{B}|$  such that  $\mathcal{A} \vdash \mathcal{B}$ .*

*Proof.* Notice that  $W_0$  acts on the orthogonal subsets of cardinality  $m = |\mathcal{B}|$ . Suppose that  $\mathcal{B}$  is not an antichain, then by Lemma 4.5 there is  $\mathcal{B}_1 \in \text{Ort}_{(m,0)}(\Phi_1^+)$  such that  $\mathcal{B}_1 \vdash \mathcal{B}$  and  $\dim \mathfrak{a}_{\mathcal{B}_1} < \dim \mathfrak{a}_{\mathcal{B}}$ . Let  $i \geq 1$  and suppose that  $\mathcal{B}_1, \dots, \mathcal{B}_i \in \text{Ort}_{(m,0)}(\Phi_1^+)$  are such that  $\mathcal{B}_i \vdash \dots \vdash \mathcal{B}_1 \vdash \mathcal{B}$  and  $\dim \mathfrak{a}_{\mathcal{B}_i} < \dim \mathfrak{a}_{\mathcal{B}_{i-1}} < \dots <$

$\dim \mathfrak{a}_{\mathcal{B}}$ . If  $\mathcal{B}_i$  is not an antichain, then we can apply Lemma 4.5 again, and we find  $\mathcal{B}_{i+1} \in \text{Ort}_{(m,0)}(\Phi_1^+)$  such that  $\mathcal{B}_{i+1} \vdash \mathcal{B}_i$  and  $\dim \mathfrak{a}_{\mathcal{B}_{i+1}} < \dim \mathfrak{a}_{\mathcal{B}_i}$ . Since  $\mathcal{B}_{i+1}$  is not empty it must be  $\dim \mathfrak{a}_{\mathcal{B}_{i+1}} > 0$ , therefore the process must stop for some  $k$ , and  $\mathcal{B}_k$  is an antichain.  $\square$

**4.2. The odd orthogonal case.** Consider the Hermitian pair  $(B_n, \alpha_1)$ . We enumerate the set of simple roots  $\Pi = \{\alpha_1, \dots, \alpha_n\}$  as in [1]. Given  $i, j$  such that  $1 \leq i \leq n$  and  $1 \leq j < n$  we set

$$\begin{aligned}\beta_i &= \alpha_1 + \dots + \alpha_i, \\ \beta'_j &= \alpha_1 + \dots + \alpha_j + 2\alpha_{j+1} + \dots + 2\alpha_n.\end{aligned}$$

Then  $\Phi_1^+ = \{\beta_i \mid 1 \leq i \leq n\} \cup \{\beta'_j \mid 1 \leq j < n\}$ . Notice that  $\Phi_1^+$  contains a unique short root, namely  $\beta_n$ .

In this case  $\text{Ort}_{\max}(\Phi_1^+) = \{\mathcal{B}_1, \dots, \mathcal{B}_n\}$ , where we set  $\mathcal{B}_i = \{\beta_i, \beta'_i\}$  for all  $i < n$ , and  $\mathcal{B}_n = \{\beta_n\}$ . In particular, the only possible types for an orthogonal subset are  $(0, 2)$ ,  $(1, 0)$  and  $(0, 1)$ . Moreover  $\mathcal{B}_n \vdash \mathcal{B}_{n-1} \vdash \dots \vdash \mathcal{B}_1$ , and  $\mathcal{B}_n$  is the unique antichain in  $\text{Ort}_{\max}(\Phi_1^+)$ . In particular, the following proposition trivially holds.

**Proposition 4.7.** *Consider the Hermitian pair  $(B_n, \alpha_1)$ , and let  $\Phi^+ = \Phi_0^+ \cup \Phi_1^+$  be the corresponding decomposition.*

- i) *Let  $\mathcal{B} \in \text{Ort}(\Phi_1^+)$  of type  $(h, k)$ , then there is an antichain  $\mathcal{A} \subset \Phi_1^+$  of type  $(h + \lfloor \frac{k}{2} \rfloor, k - 2\lfloor \frac{k}{2} \rfloor)$  such that  $\mathcal{A} \vdash \mathcal{B}$ .*
- ii) *There exists a unique antichain  $\mathcal{A}_* \in \text{Ort}_{\max}(\Phi_1^+)$ , and  $\mathcal{A}_* \vdash \mathcal{B}$  for all  $\mathcal{B} \in \text{Ort}_{\max}(\Phi_1^+)$ .*

**4.3. The symplectic case.** Consider the Hermitian pair  $(C_n, \alpha_n)$ . We enumerate the set of simple roots  $\Pi = \{\alpha_1, \dots, \alpha_n\}$  as in [1], and we embed  $\Phi$  into the euclidean vector space  $\mathbb{R}^n$  with orthonormal basis  $\varepsilon_1, \dots, \varepsilon_n$  by setting  $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$  for all  $i < n$  and  $\alpha_n = 2\varepsilon_n$ . Then

$$\Phi = \{\pm(\varepsilon_i \pm \varepsilon_j) \mid 1 \leq i, j \leq n\} \setminus \{0\},$$

and  $\Phi_1^+ = \{\varepsilon_i + \varepsilon_j \mid 1 \leq i \leq j \leq n\}$ . Notice that, for  $1 \leq i \leq j \leq n$ , we have

$$\varepsilon_i + \varepsilon_j = \alpha_i + \dots + \alpha_{j-1} + 2\alpha_j + \dots + 2\alpha_{n-1} + \alpha_n.$$

In particular  $\varepsilon_i + \varepsilon_j \leq \varepsilon_h + \varepsilon_k$  if and only if  $h \leq i$  and  $k \leq j$ . Notice that  $\mathcal{S}_{C_n, \alpha_1} = \{2\varepsilon_1, \dots, 2\varepsilon_n\}$ , so that  $r = n$ .

Let  $\mathcal{B} \in \text{Ort}(\Phi_1^+)$ , and write  $\mathcal{B} = \{\varepsilon_{i_1} + \varepsilon_{j_1}, \dots, \varepsilon_{i_m} + \varepsilon_{j_m}\}$  for some indices  $i_1 \leq j_1, \dots, i_m \leq j_m$ . Correspondingly, we have a disjoint union  $\bigcup_{k=1}^m \{i_k, j_k\}$ , and  $\mathcal{B} \in \text{Ort}_{\max}(\Phi_1^+)$  if and only if  $\{1, \dots, n\} = \bigcup_{k=1}^m \{i_k, j_k\}$ , and it immediately follows that, if  $\mathcal{B}$  has type  $(h, k)$ , then  $\mathcal{B} \in \text{Ort}_{\max}(\Phi_1^+)$  if and only if  $2h + k = n$ . Notice moreover that  $\mathcal{B}$  is an antichain if and only if, up to some permutation of  $\{1, \dots, m\}$ , we have

$$i_1 < i_2 < \dots < i_{m-1} < i_m \leq j_m < j_{m-1} < \dots < j_2 < j_1.$$

It follows that there is a unique  $\mathcal{B} \in \text{Ort}_{\max}(\Phi_1^+)$  which satisfies the previous inequalities, therefore there is a unique antichain  $\mathcal{A}_* \in \text{Ort}_{\max}(\Phi_1^+)$ .

The following lemma is an easy consequence of previous description of  $\Phi_1^+$ .

**Lemma 4.8.** *Let  $\beta, \beta' \in \Phi_1^+$  be orthogonal roots.*

- i) *Suppose that  $\{\beta, \beta'\}$  is of type  $(1, 1)$  and suppose that  $\beta < \beta'$ . Then  $\beta' - \beta = 2\alpha + \alpha'$  for some short roots  $\alpha, \alpha' \in \Phi_0^+$  with  $\langle \alpha', \alpha^\vee \rangle = -1$ .*
- ii) *Suppose that  $\{\beta, \beta'\}$  is of type  $(2, 0)$ , and suppose that  $\beta < \beta'$ . Then  $\beta' - \beta = \alpha + \alpha'$  for some orthogonal short roots  $\alpha, \alpha' \in \Phi_0^+$ .*
- iii) *Suppose that  $\{\beta, \beta'\}$  is of type  $(0, 2)$ , then  $\beta - \beta' = 2\alpha$  for some short root  $\alpha \in \Phi_0$ .*

*Proof.* Assume  $\beta = \varepsilon_i + \varepsilon_j$  and  $\beta' = \varepsilon_h + \varepsilon_k$ , for some  $i \leq j$  and  $h \leq k$ . The orthogonality implies that  $i \neq h$  and  $j \neq k$ .

i) We have in this case  $h \leq k < i \leq j$ , and since  $\{\beta, \beta'\}$  contains exactly one long root, either  $h = k < i < j$  or  $h < k < i = j$ . Therefore the claim follows by setting  $\alpha = \varepsilon_k - \varepsilon_i$  and  $\alpha' = \varepsilon_h - \varepsilon_k + \varepsilon_i - \varepsilon_j$ .

ii) We have in this case  $i < j$  and  $h < k$ , and the claim follows by setting  $\alpha = \varepsilon_i - \varepsilon_j = \alpha_k + \dots + \alpha_{j-1}$  and  $\alpha' = \varepsilon_h - \varepsilon_k = \alpha_h + \dots + \alpha_{k-1}$ .

iii) We have in this case  $i = j$  and  $h = k$ , and the claim follows by setting  $\alpha = \varepsilon_h - \varepsilon_i$ .  $\square$

**Lemma 4.9.** *Let  $\mathcal{B} \subset \Phi_1^+$  be an orthogonal subset of type  $(h, k)$ , set  $k' = \lfloor \frac{k}{2} \rfloor$  and suppose that  $\mathcal{B}$  is not an antichain. Then there exists an orthogonal subset  $\mathcal{B}'$  of type  $(h + k', k - 2k')$  such that  $\mathcal{B}' \vdash \mathcal{B}$ , and  $\dim \mathfrak{a}_{\mathcal{B}'} < \dim \mathfrak{a}_{\mathcal{B}}$ .*

*Proof.* Let  $\beta \in \mathcal{B}$  be a minimal element, and suppose that  $\beta < \beta'$  for some  $\beta' \in \mathcal{B}$ . We construct an orthogonal subset  $\mathcal{B}'$  such that  $\mathcal{B}' \vdash \mathcal{B}$  and  $\dim \mathfrak{a}_{\mathcal{B}'} < \dim \mathfrak{a}_{\mathcal{B}}$ , whose type is  $(h + 1, k - 2)$  if  $\beta, \beta'$  are both long, and  $(h, k)$  otherwise. Since two positive long roots in a root system of type  $C_n$  are always comparable, the claim will follow repeating the argument until  $\mathcal{B}'$  contains at most a single long root.

If  $\beta, \beta'$  are both long, then by Lemma 4.8  $\beta' - \beta = 2\alpha$  for some short root  $\alpha \in \Phi_0^+$ . Denote  $\mathcal{B}' = (\mathcal{B} \setminus \{\beta, \beta'\}) \cup \{\alpha + \beta\}$ . Since  $\alpha + \beta = \beta' - \alpha \in \Phi_1^+$ , Lemma 3.3 implies that  $(\alpha, \beta) = 0$  for all  $\beta \in \mathcal{B} \setminus \{\beta, \beta'\}$ . Therefore  $\mathcal{B}'$  is orthogonal, and it is of type  $(h + 1, k - 2)$  since  $\alpha + \beta$  is a short root. Moreover  $\mathcal{B}' \vdash \mathcal{B}$ , and since  $\beta$  is minimal in  $\mathcal{B}$  we get  $\dim \mathfrak{a}_{\mathcal{B}'} < \dim \mathfrak{a}_{\mathcal{B}}$  as well.

Suppose that  $\beta, \beta'$  are both short roots. Following Lemma 4.8, write  $\beta' - \beta = \alpha + \alpha'$  with  $\alpha, \alpha' \in \Phi_0^+$  short orthogonal roots. In particular, it must be  $\langle \beta', \alpha^\vee \rangle = -\langle \beta, \alpha^\vee \rangle = 1$ , and by Lemma 3.3 it follows  $(\alpha, \beta'') = 0$  for all  $\beta'' \in \mathcal{B} \setminus \{\beta, \beta'\}$ . Therefore

$$s_\alpha(\mathcal{B}) = (\mathcal{B} \setminus \{\beta, \beta'\}) \cup \{\alpha + \beta, \beta' - \alpha\}.$$

On the other hand, being  $\beta' - \beta = \alpha + \alpha'$ , we get  $\beta < s_\alpha(\beta)$  and  $\beta < s_\alpha(\beta')$ . Therefore  $s_\alpha(\mathcal{B}) \vdash \mathcal{B}$ , and since  $\beta$  is minimal in  $\mathcal{B}$  it follows  $\dim \mathfrak{a}_{s_\alpha(\mathcal{B})} < \dim \mathfrak{a}_{\mathcal{B}}$ .

Suppose finally that  $\|\beta\| \neq \|\beta'\|$ . Following Lemma 4.8, we can write  $\beta' - \beta = 2\alpha + \alpha'$  where  $\alpha, \alpha' \in \Phi_0^+$  are short roots with  $\langle \alpha', \alpha^\vee \rangle = -1$ . In particular we get  $\alpha + \alpha' \in \Phi_0^+$ , hence  $\beta + \alpha, \beta' - \alpha \in \Phi_1^+$ , and by Lemma 3.3 it follows  $(\alpha, \beta'') = 0$  for all  $\beta'' \in \mathcal{B} \setminus \{\beta, \beta'\}$ . Therefore

$$s_\alpha(\mathcal{B}) = (\mathcal{B} \setminus \{\beta, \beta'\}) \cup \{s_\alpha(\beta), s_\alpha(\beta')\}.$$

On the other hand, being  $\beta' - \beta = 2\alpha + \alpha'$ , we get  $\beta < s_\alpha(\beta)$  and  $\beta < s_\alpha(\beta')$ . Therefore  $s_\alpha(\mathcal{B}) \vdash \mathcal{B}$ , and since  $\beta$  is minimal in  $\mathcal{B}$  we get  $\dim \mathfrak{a}_{s_\alpha(\mathcal{B})} < \dim \mathfrak{a}_{\mathcal{B}}$  as well.  $\square$

**Proposition 4.10.** *Consider the Hermitian pair  $(C_n, \alpha_1)$ .*

- i) *Let  $\mathcal{B} \subset \Phi_1^+$  be an orthogonal subset of type  $(h, k)$ . Then there is an antichain  $\mathcal{A} \subset \Phi_1^+$  of type  $(h + \lfloor \frac{k}{2} \rfloor, k - 2\lfloor \frac{k}{2} \rfloor)$  such that  $\mathcal{A} \vdash \mathcal{B}$ .*
- ii) *There exists a unique antichain  $\mathcal{A}_* \in \text{Ort}_{\max}(\Phi_1^+)$ , and  $\mathcal{A}_* \vdash \mathcal{B}$  for all  $\mathcal{B} \in \text{Ort}_{\max}(\Phi_1^+)$ .*

*Proof.* i) Suppose that  $\mathcal{B}$  is not an antichain, by Lemma 4.9 there is an orthogonal subset  $\mathcal{B}_1 \subset \Phi_1^+$  of type  $(h + \lfloor \frac{k}{2} \rfloor, k - 2\lfloor \frac{k}{2} \rfloor)$  such that  $\mathcal{B}_1 \vdash \mathcal{B}$  and  $\dim \mathfrak{a}_{\mathcal{B}_1} < \dim \mathfrak{a}_{\mathcal{B}}$ . Suppose that  $\mathcal{B}_i$  is defined, and suppose that  $\mathcal{B}_i$  is not an antichain. Then we can apply Lemma 4.9 again, and we find an orthogonal subset  $\mathcal{B}_{i+1} \subset \Phi_1^+$  such that  $\mathcal{B}_{i+1} \vdash \mathcal{B}_i$  and  $\dim \mathfrak{a}_{\mathcal{B}_{i+1}} < \dim \mathfrak{a}_{\mathcal{B}_i}$ . Since  $\mathcal{B}_{i+1}$  is not empty,  $\mathfrak{a}_{\mathcal{B}_{i+1}}$  cannot be zero, therefore the process must stop for some  $k$ , and  $\mathcal{B}_k$  is an antichain.

ii) As we already noticed, if  $\mathcal{B} \in \text{Ort}(\Phi_1^+)$  has type  $(h, k)$ , then  $\mathcal{B}$  is maximal if and only if  $2h + k = n$ . Therefore by i) for all  $\mathcal{B} \in \text{Ort}_{\max}(\Phi_1^+)$  there exists an antichain  $\mathcal{A} \in \text{Ort}_{\max}(\Phi_1^+)$  such that  $\mathcal{A} \vdash \mathcal{B}$ . We also already noticed that there is a unique antichain  $\mathcal{A}_* \in \text{Ort}_{\max}(\Phi_1^+)$ , therefore  $\mathcal{A}_* \vdash \mathcal{B}$  for all  $\mathcal{B} \in \text{Ort}_{\max}(\Phi_1^+)$ .  $\square$

**4.4. Hermitian symmetric spaces of tube type.** Let  $(\Pi, \alpha_q)$  be a Hermitian pair and let  $\mathfrak{p}^+$  be the corresponding standard parabolic subalgebra of  $\mathfrak{g}$ . Let  $\mathcal{S}_{\Pi, \alpha_q} = \{\gamma_1, \dots, \gamma_r\}$  be the set of Harish-Chandra strongly orthogonal roots, and set  $\mathfrak{h}^- = \text{span}(\gamma_i^\vee \mid i = 1, \dots, r)$ . By [9], [15], a root  $\alpha \in \Phi$  is in  $\Phi_1^+$  if and only if either  $\alpha|_{\mathfrak{h}^-} = \frac{1}{2}(\gamma_i + \gamma_j)$ , for some  $i \leq j$ , or  $\alpha|_{\mathfrak{h}^-} = \frac{1}{2}\gamma_i$  for some  $i$ . A root  $\alpha \in \Phi$  is in  $\Phi_0^+$  if and only if either  $\alpha|_{\mathfrak{h}^-} = \frac{1}{2}(\gamma_i - \gamma_j)$ , for some  $i \leq j$ , or  $\alpha|_{\mathfrak{h}^-} = \pm \frac{1}{2}\gamma_i$ , for some  $i$ . Recall that the Hermitian symmetric space  $G/P$  is called of *tube type* if it is holomorphically equivalent to the tube over a self dual cone. It is known (cf. [14]) that Hermitian symmetric spaces of tube type correspond to Hermitian involutions such that  $\alpha \in \Phi$  is in  $\Phi_1^+$  if and only if  $\alpha|_{\mathfrak{h}^-} = \frac{1}{2}(\gamma_i + \gamma_j)$  for some  $i \leq j$ , and a root  $\alpha \in \Phi$  is in  $\Phi_0^+$  if and only if  $\alpha|_{\mathfrak{h}^-} = \frac{1}{2}(\gamma_i - \gamma_j)$ , for some  $i \leq j$ . We will call such involutions *Hermitian involutions of tube type*. Observe that a Hermitian involution is of tube type if and only if

$$(4.1) \quad \left( \sum_{i=1}^r \gamma_i, \alpha \right) = (\alpha_q, \alpha_q) i, \text{ for all } \alpha \in \Phi_i, i = 0, 1.$$

Hermitian symmetric spaces of tube type are classified by the Hermitian pairs  $(\Pi, \alpha_q)$  such that  $w_0(\alpha_q) = -\alpha_q$ , in which case we say that  $(\Pi, \alpha_q)$  is a *Hermitian pair of tube type* (see e.g. [10, Ch. X, D.4 pg. 528]). In particular, we have the following possibilities:

- i)  $(A_{2q-1}, \alpha_q)$ ;
- ii)  $(B_n, \alpha_1)$ ;
- iii)  $(C_n, \alpha_n)$ ;
- iv)  $(D_n, \alpha_{n-1})$  with  $n$  even;  $(D_n, \alpha_n)$  with  $n$  even;  $(D_n, \alpha_1)$  for all  $n$ ;
- v)  $(E_7, \alpha_7)$ .

Notice that being of tube type is equivalent to the fact that  $\mathfrak{p}_u^+$  is a regular pre-homogeneous space under the action of  $G_0$ , namely the boundary of the open  $G_0$ -orbit has codimension 1 (see [16] and the references therein).

If  $(\Pi, \alpha_q)$  is a Hermitian pair of tube type and  $\Pi$  is not simply laced, then the short roots in  $\Phi_1$  admit a nice description:

**Lemma 4.11.** *Suppose that  $\Phi$  is not simply laced and let  $\sigma$  be a Hermitian involution of tube type. Let  $\mathcal{S}$  be an orthogonal subset of  $\Phi_1^+$  of maximal cardinality and let  $\beta \in \Phi_1^+$  be a short root. Then  $\beta = \frac{1}{2}(\gamma + \gamma')$  for some distinct elements  $\gamma, \gamma' \in \mathcal{S}$ .*

*Proof.* By Remark 4.2, every  $\gamma \in \mathcal{S}$  is a long root of  $\Phi$ , and we can choose a set of positive roots in  $\Phi_0^+ \subset \Phi_0$  so that  $\mathcal{S}$  is the corresponding set of Harish-Chandra strongly orthogonal roots. Since  $\sigma$  is of tube type, we have that  $\beta = \frac{1}{2}(\gamma + \gamma') + \lambda$ , for some  $\gamma, \gamma' \in \mathcal{S}$  with  $(\gamma, \gamma') = 0$  and some  $\lambda$  with  $(\lambda, \gamma'') = 0$  for all  $\gamma'' \in \mathcal{S}$ , therefore  $\|\beta\|^2 = \frac{1}{4}(\|\gamma\|^2 + \|\gamma'\|^2) + \|\lambda\|^2 = \frac{1}{2}\|\gamma\|^2 + \|\lambda\|^2$ . On the other hand  $\beta$  is a short root, therefore  $\|\beta\|^2 = \frac{1}{2}\|\gamma\|^2$  and it follows  $\lambda = 0$ .  $\square$

If  $\sigma$  is the Hermitian involution of tube type associated to the Hermitian pair  $(B_n, \alpha_1)$  or  $(C_n, \alpha_n)$ , we proved in Proposition 4.7 and Proposition 4.10 that there exists a unique antichain  $\mathcal{A}_* \in \text{Ort}_{\max}(\Phi_1^+)$ , and that  $\mathcal{A}_* \vdash \mathcal{B}$  for all  $\mathcal{B} \in \text{Ort}_{\max}(\Phi_1^+)$ . We now show that this property holds whenever  $\sigma$  is a Hermitian involution of tube type.

**Proposition 4.12.** *Suppose that  $\sigma$  is a Hermitian involution of tube type.*

*Then there exists a unique antichain  $\mathcal{A}_* \in \text{Ort}_{\max}(\Phi_1^+)$ , and  $\mathcal{A}_* \vdash \mathcal{B}$  for all  $\mathcal{B} \in \text{Ort}_{\max}(\Phi_1^+)$ .*

*Proof.* As we noticed, the claim has already been proved if  $\Phi$  is not simply laced. Therefore we will assume that  $\Phi$  is simply laced, so that  $\text{Ort}_{\max}(\Phi_1^+)$  coincides with the collection of the orthogonal subsets of maximal cardinality  $r$ . By Proposition 4.6, for all  $\mathcal{B} \in \text{Ort}_{\max}(\Phi_1^+)$ , there is an antichain  $\mathcal{A} \in \text{Ort}_{\max}(\Phi_1^+)$  such that  $\mathcal{A} \vdash \mathcal{B}$ . Therefore we only need to show the uniqueness of the antichain in  $\text{Ort}_{\max}(\Phi_1^+)$ .

Suppose  $\mathcal{A}, \mathcal{A}' \in \text{Ort}_{\max}(\Phi_1^+)$  are both antichains. Since they are both of maximal cardinality, by Remark 4.2, they are both sets of Harish-Chandra roots for some choice of positive sets of roots in  $\Phi_0$ . Since the pair  $(\Pi, \alpha_q)$  is of tube type we have, by (4.1),

$$\left( \sum_{\gamma \in \mathcal{A}} \gamma, \alpha_q \right) = (\alpha_q, \alpha_q) = \left( \sum_{\gamma' \in \mathcal{A}'} \gamma', \alpha_q \right)$$

and, if  $\alpha \in \Pi_0$ ,

$$\left( \sum_{\gamma \in \mathcal{A}} \gamma, \alpha \right) = 0 = \left( \sum_{\gamma' \in \mathcal{A}'} \gamma', \alpha \right).$$

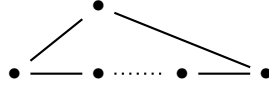
It follows that

$$(4.2) \quad \sum_{\gamma \in \mathcal{A}} \gamma = \sum_{\gamma' \in \mathcal{A}'} \gamma'.$$

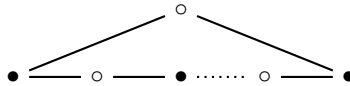
Let  $C$  be the matrix  $((\gamma, \gamma'))_{\gamma \in \mathcal{A}, \gamma' \in \mathcal{A}'}$ . Consider the matrix  $C'$  obtained by replacing the nonzero entries of  $C$  with 1. This is the incidence matrix of a relation. Let  $\mathcal{G}$  be its incidence graph. Write  $\mathcal{G} = \cup_i \mathcal{G}_i$ , where  $\mathcal{G}_i$  are the connected components of  $\mathcal{G}$ . Assume first that  $\mathcal{G}_i$  has more than one node. If  $\gamma \in \mathcal{G}_i \cap \mathcal{A} \cap \mathcal{A}'$  then  $(\gamma, \gamma') = 0$  for all  $\gamma' \in \mathcal{A}' \setminus \{\gamma\}$ , so  $\gamma$  is connected in  $\mathcal{G}$  only to itself. Since  $\mathcal{G}_i$  is connected this is not possible, hence  $\mathcal{G}_i \cap \mathcal{A} \cap \mathcal{A}' = \emptyset$ . If  $\mathcal{G}_i \cap \mathcal{A} = \emptyset$  then any  $\gamma' \in \mathcal{G}_i \cap \mathcal{A}'$  is orthogonal to  $\mathcal{A}$ , contradicting the fact that  $\mathcal{A}$  is in  $\text{Ort}_{\max}(\Phi_1^+)$ . Symmetrically we have that also  $\mathcal{G}_i \cap \mathcal{A}'$  is not empty. Since  $\mathcal{G}_i$  is connected, if  $\gamma_0 \in \mathcal{G}_i \cap \mathcal{A}$ , there must be  $\gamma'_0 \in \mathcal{G}_i \cap \mathcal{A}'$  such that  $\gamma_0 \neq \gamma'_0$  and  $(\gamma_0, \gamma'_0) \neq 0$ . Since

$$\left\langle \sum_{\gamma \in \mathcal{A}} \gamma, \gamma'_0 \right\rangle = 2 = \left\langle \sum_{\gamma' \in \mathcal{A}'} \gamma', \gamma'_0 \right\rangle = 1 + \left\langle \sum_{\gamma' \neq \gamma'_0} \gamma', \gamma'_0 \right\rangle,$$

we see that there are exactly two nodes in  $\mathcal{G}_i$  to which  $\gamma_0$  is connected. Symmetrically, the same property holds for all  $\gamma' \in \mathcal{G}_i \cap \mathcal{A}'$ . Thus every node has degree exactly 2 in  $\mathcal{G}_i$ . It follows that  $\mathcal{G}_i$  is a cycle



Since  $\mathcal{G}_i \cap \mathcal{A} \cap \mathcal{A}' = \emptyset$  and the nodes in  $\mathcal{A}$  connect only to nodes in  $\mathcal{A}'$  we have only this possibility (letting  $\circ$  be the nodes in  $\mathcal{A}$  and  $\bullet$  the nodes in  $\mathcal{A}'$ ):



If  $(\gamma, \gamma') \neq 0$  then  $(\gamma, \gamma') > 0$  so  $\gamma - \gamma'$  is a root. It follows that either  $\gamma > \gamma'$  or  $\gamma' > \gamma$ . We give an orientation to the graph  $\mathcal{G}$  by orienting the edges so that they point from the larger to the smaller root. Since both  $\mathcal{A}$  and  $\mathcal{A}'$  are antichains,

we cannot have consecutive arrows  $\circ \rightarrow \bullet \rightarrow \circ$ ,  $\bullet \rightarrow \circ \rightarrow \bullet$ . It follows that



the nodes  $\circ$  are either all sources or all sinks. By eventually exchanging  $\mathcal{A}$  and  $\mathcal{A}'$ , we can assume that all  $\circ$  are sources and all  $\bullet$  are sinks. This means that we can enumerate  $\mathcal{A} \cap \mathcal{G}_i = \{\gamma_1, \dots, \gamma_s\}$  and  $\mathcal{A}' \cap \mathcal{G}_i = \{\gamma'_1, \dots, \gamma'_s\}$  so that  $\gamma'_i <_0 \gamma_i$ .

It follows that  $\lambda = \sum \gamma_i - \sum \gamma'_i >_0 0$ . On the other hand

$$\begin{aligned} \|\lambda\|^2 &= \left\| \sum \gamma_i \right\|^2 + \left\| \sum \gamma'_i \right\|^2 - \left( \sum \gamma_i, \sum \gamma'_i \right) - \left( \sum \gamma'_i, \sum \gamma_i \right) \\ &= \left\| \sum \gamma_i \right\|^2 + \left\| \sum \gamma'_i \right\|^2 - \left( \sum \gamma_i, \sum_{\gamma' \in \mathcal{A}'} \gamma' \right) - \left( \sum \gamma'_i, \sum_{\gamma \in \mathcal{A}} \gamma \right). \end{aligned}$$

By (4.2),

$$\begin{aligned} \|\lambda\|^2 &= \left\| \sum \gamma_i \right\|^2 + \left\| \sum \gamma'_i \right\|^2 - \left( \sum \gamma_i, \sum_{\gamma \in \mathcal{A}} \gamma \right) - \left( \sum \gamma'_i, \sum_{\gamma' \in \mathcal{A}'} \gamma' \right) \\ &= \left\| \sum \gamma_i \right\|^2 + \left\| \sum \gamma'_i \right\|^2 - \left( \sum \gamma_i, \sum \gamma_i \right) - \left( \sum \gamma'_i, \sum \gamma'_i \right) = 0. \end{aligned}$$

It follows that  $\mathcal{G}_i$  has only one node for all  $i$ . As observed earlier  $\mathcal{G}_i \cap \mathcal{A} \neq \emptyset$  and  $\mathcal{G}_i \cap \mathcal{A}' \neq \emptyset$ , thus, if  $\mathcal{G}_i = \{\gamma\}$ , then  $\gamma \in \mathcal{A} \cap \mathcal{A}'$ . Therefore  $\mathcal{A} \cup \mathcal{A}' = \cup_i \mathcal{G}_i \subset \mathcal{A} \cap \mathcal{A}'$ . Thus  $\mathcal{A} = \mathcal{A}'$ .  $\square$

*Remark 4.13.* The proof of uniqueness of the antichain we have given when  $\Phi$  is simply laced can be extended (with some complications) to a uniform proof for any  $\Phi$ . Since the two non simply laced cases are easily dealt individually, we preferred to omit this more complicated approach.

Given  $\mathcal{B} \subset \Phi_1^+$ , set

$$\mathcal{B}^{\leq_0} = \{\alpha \in \Phi_1^+ \mid \text{there is } \beta \in \mathcal{B} \text{ such that } \alpha \leq_0 \beta\}.$$

Notice that  $\mathcal{B}^{\leq_0} = \Psi(\mathfrak{a}_{\mathcal{B}}^-)$ , where  $\mathfrak{a}_{\mathcal{B}}^- \subset \mathfrak{p}_{\mathfrak{u}}^+$  is the  $B_0^-$ -stable subalgebra generated by  $\mathcal{B}$  and  $B_0^- \subset G_0$  is the opposite Borel subgroup of  $B$ .

If  $\mathcal{A}_* \in \text{Ort}_{\max}(\Phi_1^+)$  is the unique antichain, it follows by Proposition 4.12, that  $\mathcal{A}_* \subset \mathcal{B}^{\geq_0}$  for all  $\mathcal{B} \in \text{Ort}_{\max}(\Phi_1^+)$ . The following corollary shows that  $\mathcal{A}_* \subset \mathcal{B}^{\leq_0}$  as well.

**Corollary 4.14.** *Let  $\mathcal{A}_* \in \text{Ort}_{\max}(\Phi_1^+)$  be the unique antichain. Then  $\mathcal{A}_* \subset \mathcal{B}^{\leq_0}$  for all  $\mathcal{B} \in \text{Ort}_{\max}(\Phi_1^+)$ .*

*Proof.* Let  $\leq'_0$  be the partial order on  $\Phi_1^+$  defined by  $\Phi_0^-$ . Then  $\leq'_0$  is the reverse partial order of  $\leq_0$ , therefore a subset  $\mathcal{A} \subset \Phi_1^+$  is an antichain w.r.t.  $\leq_0$  if and only if it is an antichain w.r.t.  $\leq'_0$ . Therefore, if  $\vdash'$  is the preorder on  $\text{Ort}(\Phi_1^+)$  defined by  $\leq'_0$ , it follows by Proposition 4.12 that  $\mathcal{A}_* \vdash' \mathcal{B}$ , namely  $\mathcal{A}_* \subset \mathcal{B}^{\leq_0}$ .  $\square$

## 5. THE SPECIAL $B_0$ -STABLE ABELIAN SUBALGEBRA

For the rest of the paper we will assume that  $\sigma : G \rightarrow G$  is an (indecomposable) involution. Moreover, throughout this section, we assume that  $\mathfrak{g}_0$  is semisimple and that the simple root  $\alpha_p \in \widehat{\Pi}$  corresponding to  $\sigma$  is long and non-complex.

Recall from Section 2.3 the element  $w_p$  of  $\mathcal{W}_{\sigma}^{ab}$  and the corresponding subalgebra  $\mathfrak{a}_p \in \mathcal{I}_{ab}^{\sigma}$ . We call  $\mathfrak{a}_p$  the *special  $B_0$ -stable abelian subalgebra*.

For each component  $\Sigma$  of  $\Pi_0$ , we let  $\Phi(\Sigma)$  be the root subsystem of  $\Phi_0$  generated by  $\Sigma$ . As shown in [6, Lemma 5.7], there is a unique simple root  $\alpha_{\Sigma} \in \Sigma$  which is connected to  $\alpha_p$ . If moreover  $\theta_{\Sigma}$  is the highest root in  $\Phi(\Sigma)$ , then  $[\theta_{\Sigma} : \alpha_{\Sigma}] = 1$ , therefore  $(\Sigma, \alpha_{\Sigma})$  is a Hermitian pair. It is then clear that  $\gamma \in \Phi(\Sigma)$  is orthogonal to  $\alpha_p$  if and only if  $[\gamma : \alpha_{\Sigma}] = 0$ . Following Section 4, we denote by  $\Phi(\Sigma)_1^+$  the set of roots in  $\Phi(\Sigma)^+$  that have  $\alpha_{\Sigma}$  in their support.

Set  $\mathcal{C}_{\sigma} = \{\alpha \in \widehat{\Phi}^+ \mid \alpha_p + k\delta - \alpha \in \widehat{\Phi}^+\}$  and define

$$\mathcal{C}_{\sigma}^1 = \mathcal{C}_{\sigma} \cap \widehat{\Phi}_1 = \{\alpha \in \mathcal{C}_{\sigma} \mid [\alpha : \alpha_p] = 1\}.$$

Notice that  $\mathcal{C}_\sigma^1 \subset \widehat{\Phi}_{re}^+$ .

**Lemma 5.1.**

- i)  $N(w_p) = \bigcup_{\Sigma} \{\gamma + \alpha_p \mid \gamma \in \Phi(\Sigma)_1^+\} \cup \{\alpha_p\}$ , where  $\Sigma$  ranges among the components of  $\Pi_0$ ;
- ii)  $N(w_p) = \mathcal{C}_\sigma^1$ ;
- iii) The map  $\Upsilon : \mathcal{C}_\sigma^1 \rightarrow \bigcup_{\Sigma} \Phi(\Sigma)_1^+ \cup \{0\}$  mapping  $\eta$  to  $\eta - \alpha_p$  is an order preserving bijection, where  $\Sigma$  varies among the components of  $\Pi_0$ ;
- iv) If  $\gamma, \eta \in \mathcal{C}_\sigma^1 \setminus \{\alpha_p\}$ , then  $(\Upsilon(\gamma), \Upsilon(\eta)) = (\gamma, \eta)$ .
- v) If  $\eta \in \mathcal{C}_\sigma^1 \setminus \{\alpha_p\}$ , then  $(\alpha_p, \Upsilon(\eta)) = -(\alpha_p, \eta)$ .

*Proof.* i) It is well known that  $N(w_{0,\alpha_p}w_0) = \bigcup_{\Sigma} \Phi(\Sigma)_1^+$ . If  $\gamma \in \Phi(\Sigma)_1^+$ , then  $\langle \gamma, \alpha_p^\vee \rangle = \langle \alpha_\Sigma, \alpha_p^\vee \rangle$ , and since  $\alpha_p$  is long, we have  $\langle \alpha_\Sigma, \alpha_p^\vee \rangle = -1$ . It follows that  $s_p(\gamma) = \gamma + \alpha_p \in \widehat{\Phi}^+$ , hence

$$N(w_p) = N(s_p w_{0,\alpha_p} w_0) = \{\alpha_p\} \cup s_p(N(w_{0,\alpha_p} w_0)),$$

and the claim follows.

ii) Clearly  $\alpha_p \in \mathcal{C}_\sigma^1$ . Moreover, if  $\gamma \in \Phi(\Sigma)_1^+$ , then  $k\delta + \alpha_p - (\gamma + \alpha_p) = k\delta - \gamma \in \widehat{\Phi}^+$ . It follows that  $N(w_p) \subseteq \mathcal{C}_\sigma^1$ . Since  $w_p(\alpha_p) = k\delta + \alpha_p$ , it follows that  $\ell(w_p s_p) = \ell(w_p) + 1$ . This implies that  $N(w_p s_p) = N(w_p) \cup \{k\delta + \alpha_p\}$ . Let  $\eta \neq \alpha_p$  be in  $\mathcal{C}_\sigma^1$  so that there is  $\beta$  such that  $\eta + \beta = k\delta + \alpha_p$ . By biconvexity of  $N(w_p s_p)$ , exactly one between  $\eta$  and  $\beta$  is in  $N(w_p)$ . Since  $[\eta : \alpha_p] = 1$  and  $[\beta : \alpha_p] = 2$ , it follows from the fact that  $w_p \in \mathcal{W}_\sigma^{ab}$  that  $\eta \in N(w_p)$ . Thus  $\mathcal{C}_\sigma^1 \subseteq N(w_p)$ , and the claim follows.

iii) Follows by i) and ii).

iv) Since  $\alpha_p$  is long, if  $\alpha \in \Phi(\Sigma)_1^+$  then  $(\alpha, \alpha_p) = -\frac{(\alpha_p, \alpha_p)}{2}$ . Therefore, if  $\gamma, \eta \in \mathcal{C}_\sigma^1 \setminus \{\alpha_p\}$ , we have

$$(\gamma, \eta) = (\gamma - \alpha_p, \eta - \alpha_p) + (\gamma - \alpha_p, \alpha_p) + (\eta - \alpha_p, \alpha_p) + (\alpha_p, \alpha_p) = (\gamma - \alpha_p, \eta - \alpha_p).$$

The second claim follows as well, since  $(\Upsilon(\eta), \Upsilon(\eta)) = (\eta, \eta)$ .

v) Since  $\alpha_p$  is long and  $\Upsilon(\eta) \in \Phi(\Sigma)_1^+$ , it must be  $\langle \Upsilon(\eta), \alpha_p^\vee \rangle = -1$ . Therefore  $\langle \eta, \alpha_p^\vee \rangle = 1$ , and it follows  $(\alpha_p, \Upsilon(\eta)) = -(\alpha_p, \eta)$ .  $\square$

**Lemma 5.2.** *Let  $\{\eta_1, \dots, \eta_t\}$  be an orthogonal set of real roots, let  $\eta_{t+1}$  be a real root such that  $(\eta_i, \eta_{t+1}) < 0$  for all  $i \leq t$ , and set  $A = (\langle \eta_j, \eta_i^\vee \rangle)_{i,j=1,\dots,t+1}$ . Then  $A$  is a generalized Cartan matrix of finite or affine type.*

*Proof.* The fact that  $A$  is a generalized Cartan matrix [11, Section 1.1] is clear. It is also symmetrizable: setting  $D = \text{diag}(\|\eta_1\|^2, \dots, \|\eta_{t+1}\|^2)$  we have that  $DA = 2((\eta_i, \eta_j))$ , which is symmetric. Since  $(\eta_i, \eta_{t+1}) \neq 0$  for all  $i \leq t$ ,  $DA$  is an indecomposable matrix. Therefore it is enough to check that  $DA$  is of finite or affine type. By [11, Lemma 4.5], we need to check that  $DA$  is positive semi-definite of corank less than or equal to 1. Since  $\eta_1, \dots, \eta_t$  are orthogonal  $DA$  has rank at least  $t$ . It is clear that, given a positive semi-definite symmetric bilinear form  $(\cdot, \cdot)$  on a vector space  $V$  and a set of vectors  $R = \{v_1, \dots, v_k\}$  in  $V$ , the matrix  $((v_i, v_j))$  is positive semi-definite. On the other hand, since  $\widehat{\Phi}$  is an affine system, the invariant form  $(\cdot, \cdot)$  is positive semi-definite, and being  $DA = 2((\eta_i, \eta_j))$  it follows that  $DA$  is positive semi-definite.  $\square$

Let  $\mathcal{S} \subset \mathcal{C}_\sigma^1 \setminus \{\alpha_p\}$  be an orthogonal subset of maximal cardinality. Consider the set of roots

$$\Pi_{\mathcal{S}} = \mathcal{S} \cup \{-\alpha_p\}$$

and the matrix  $A_{\mathcal{S}} = (\langle \eta', \eta^\vee \rangle)_{\eta, \eta' \in \Pi_{\mathcal{S}}}$ . We will show that  $G_0 \mathfrak{a}_p$  is not spherical. The following is the key result in this direction.

**Proposition 5.3.** *Let  $\mathcal{S} \subset \mathcal{C}_\sigma^1 \setminus \{\alpha_p\}$  be an orthogonal subset of maximal cardinality. Then  $|\mathcal{S}| \leq 4$  and  $A_{\mathcal{S}}$  is a Cartan matrix of affine type.*

*Proof.* Set  $\mathcal{S} = \{\eta_1, \dots, \eta_t\}$ . By Lemma 5.2,  $A_{\mathcal{S}}$  has to be either of finite or of affine type. By a slight abuse of notation we denote by  $\Pi_{\mathcal{S}}$  the corresponding Dynkin diagram. This diagram has  $t+1$  nodes with  $t$  nodes connected only to the remaining node corresponding to  $\alpha_p$ . This immediately implies that  $t \leq 4$ , and since  $\alpha_p$  is long, the node connected to all other nodes corresponds to a long simple root.

If  $t = 4$  then the diagram is of type  $D_4^{(1)}$ , hence it is affine. If  $t = 3$  the only possibilities are  $D_4$  or  $B_3^{(1)}$ . If  $\Pi_{\mathcal{S}}$  is of type  $D_4$ , then  $\eta = k\delta - (\eta_1 + \eta_2 + \eta_3 - 2\alpha_p)$  is a root in  $\widehat{\Phi}$  with  $[\eta : \alpha_p] = 1$ . Set  $\beta = \eta_1 + \eta_2 + \eta_3 - \alpha_p \in \widehat{\Phi}$ . Then  $\beta \in \widehat{\Phi}$ ,  $[\beta : \alpha_p] = 2$ , and  $\eta + \beta = k\delta + \alpha_p$ . It follows that  $\eta \in \mathcal{C}_\sigma^1$ . Being  $(\eta, \eta_i) = 0$  for  $i = 1, 2, 3$ , we see that  $\{\eta_1, \eta_2, \eta_3\}$  is not a set of maximal cardinality in  $\mathcal{C}_\sigma^1 \setminus \{\alpha_p\}$ . This excludes the possibility that  $\Pi_{\mathcal{S}}$  is of type  $D_4$ , so  $\Pi_{\mathcal{S}}$  is of affine type.

If  $t = 2$  then  $\Pi_{\mathcal{S}}$  can be only of type  $A_3, B_3, G_2^{(1)}, D_3^{(2)}$ . By Lemma 5.1 together with Remark 4.2,  $\{\Upsilon(\eta_1), \Upsilon(\eta_2)\}$  is an orthogonal subset of maximal cardinality in  $\bigcup_{\Sigma} \Phi(\Sigma)_1^+$ , and both  $\Upsilon(\eta_1)$  and  $\Upsilon(\eta_2)$  are long roots in the respective components of  $\Pi_0$ . In particular,  $\Pi_0$  contains at most two components.

Suppose that  $t = 2$  and that  $\Pi_0 = \Sigma_1 \cup \Sigma_2$  is the union of two components, then by Remark 4.2 the Hermitian symmetric spaces corresponding to  $(\Sigma_1, \alpha_{\Sigma_1})$  and  $(\Sigma_2, \alpha_{\Sigma_2})$  have both rank 1, hence  $\Sigma_1, \Sigma_2$  are both of type  $A$ , and  $\alpha_{\Sigma_i}$  is an extremal root in  $\Sigma_i$  (see Table 1). Since  $\alpha_{\Sigma_i}$  and  $\Upsilon(\eta_i)$  are both long in  $\Sigma_i$ , by Lemma 5.1 v) it follows that

$$(5.1) \quad \langle -\alpha_p, \eta_i^\vee \rangle = \langle \alpha_p, \Upsilon(\eta_i)^\vee \rangle = \langle \alpha_p, \alpha_{\Sigma_i}^\vee \rangle.$$

This means that  $\alpha_p$  is connected to  $\alpha_{\Sigma_i}$  in  $\widehat{\Pi}$  with the same number of edges that connect  $-\alpha_p$  to  $\eta_i$  in  $\Pi_{\mathcal{S}}$ . If  $\Pi_{\mathcal{S}}$  is of type  $A_3$ , it follows that  $\widehat{\Pi}$  is of type  $A$ , of the shape

$$\circ \text{ --- } \circ \text{ ..... } \circ \text{ --- } \bullet \text{ --- } \circ \text{ --- } \bullet \text{ --- } \circ \text{ ..... } \circ \text{ --- } \circ$$

(where the roots  $\alpha_{\Sigma_i}$  are denoted by black nodes), which is absurd since  $\widehat{\Pi}$  is not of finite type. If  $\Pi_{\mathcal{S}}$  is of type  $B_3$ , then  $\widehat{\Pi}$  is of the shape

$$\circ \text{ --- } \circ \text{ ..... } \circ \text{ --- } \bullet \text{ --- } \circ \text{ --- } \bullet \text{ --- } \circ \text{ ..... } \circ \text{ --- } \circ$$

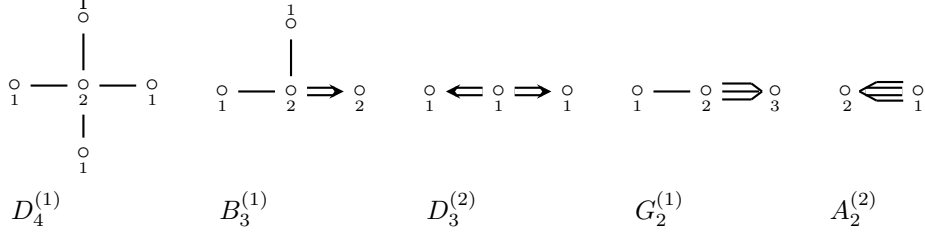
The only affine diagrams of this shape are  $F_4^{(1)}$  and  $E_6^{(2)}$ . If  $\widehat{\Pi}$  is of type  $F_4^{(1)}$  then  $\alpha_p$  has label 3, while if  $\widehat{\Pi}$  is of type  $E_6^{(2)}$ , then  $\alpha_p$  has label 2. Since in both cases the corresponding automorphism  $\sigma$  is not an involution, it follows that  $\Pi_0$  must contain a unique component.

Suppose that  $t = 2$  and that  $\Pi_0$  is connected. By Lemma 5.1 together with Remark 4.2, it follows that  $\{\Upsilon(\eta_1), \Upsilon(\eta_2)\}$  is an orthogonal subset of maximal cardinality in  $\Phi(\Pi_0)_1^+$ , and that  $\|\eta_1\| = \|\eta_2\|$ . Therefore, if  $\Pi_{\mathcal{S}}$  is not affine, it must be of type  $A_3$ . Suppose that this is the case; then by (5.1)  $\alpha_p$  is connected to  $\alpha_{\Pi_0}$  by a single edge. On the other hand, by Remark 4.2 again, the symmetric space corresponding to the Hermitian pair  $(\Pi_0, \alpha_{\Pi_0})$  has rank 2. Up to an automorphism of  $\Pi_0$ , by Table 1 the possibilities for the pair  $(\Pi_0, \alpha_{\Pi_0})$  are the following:  $(A_n, \alpha_2)$ ,  $(B_n, \alpha_1)$ ,  $(D_n, \alpha_1)$ ,  $(D_5, \alpha_5)$ ,  $(E_6, \alpha_1)$ . But then by obvious considerations it follows that  $\widehat{\Pi}$  also of finite type, a contradiction.

It remains to check the case when  $t = 1$ . If  $\Pi_{\mathcal{S}}$  is of finite type, then it is of type  $A_2, C_2$ , or  $G_2$ . Since  $\{\Upsilon(\eta_1)\}$  is a set of orthogonal roots of maximal cardinality in  $\bigcup_{\Sigma} \Phi(\Sigma)_1^+$ , it follows that  $\Pi_0$  is connected, and the Hermitian symmetric variety corresponding to the pair  $(\Pi_0, \alpha_{\Pi_0})$  has rank one, therefore  $\Pi_0$  is of type  $A$  and  $\alpha_{\Pi_0}$

is an extremal root in  $\Pi_0$ . By (5.1),  $\alpha_p$  is connected to  $\alpha_{\Pi_0}$  by the same number of edges that connect  $-\alpha_p$  to  $\eta_1$ . Therefore, if  $\Pi_{\mathcal{S}}$  is of finite type, then  $\widehat{\Pi}$  would be of finite type as well, which is absurd.  $\square$

For the reader's convenience we list here all possible diagrams for the affine root system  $\Pi_{\mathcal{S}}$ , with the corresponding labels.



**Table 2** – Affine root systems corresponding to non-spherical orbits in  $G_0 \mathfrak{a}_p$

*Remark 5.4.* Let  $\Xi$  be one of the affine diagrams of Table 2, let  $k_{\Xi}$  be the integer such that  $\Xi$  is of type  $X_r^{(k_{\Xi})}$  ( $k_{\Xi} \in \{1, 2\}$ ) and let  $\alpha \in \Xi$  be the unique long simple root which is connected to all other simple roots. Let  $\xi \in \Xi$  and let  $a_{\Xi, \xi}$  be the corresponding label in  $\Xi$ , then for all  $\xi \in \Xi$ , we have  $k_{\Xi} a_{\Xi, \xi} = |\langle \alpha, \xi^{\vee} \rangle|$ . In particular, the equality

$$k_{\Xi} \left( \sum_{\xi \neq \alpha} a_{\Xi, \xi} \right) = 4$$

holds. If  $\mathcal{S}$  is an orthogonal subset of  $\mathcal{C}_{\sigma}^1 \setminus \{\alpha_p\}$  of maximal cardinality and  $\xi \in \Pi_{\mathcal{S}}$ , then we will denote  $k_{\Pi_{\mathcal{S}}}$  and  $a_{\Pi_{\mathcal{S}}, \xi}$  simply by  $k_{\mathcal{S}}$  and  $a_{\mathcal{S}, \xi}$ .

As in Section 3, we fix a weight vector  $x_1^{\mu} \in \mathfrak{g}_1^{\mu}$  for all  $\mu \in \Phi_1^+$ . Recall that  $\bar{\alpha}_p$  is the lowest weight in  $\Phi_1$  and that  $\Psi(\mathfrak{a}_p) = \{-\bar{\eta} \mid \eta \in \mathcal{C}_{\sigma}^1\}$ . If  $\mathcal{S} \subset \mathcal{C}_{\sigma}^1 \setminus \{\alpha_p\}$  is an orthogonal subset, we set

$$x_{\mathcal{S}} = \sum_{\eta \in \mathcal{S}} x_1^{-\bar{\eta}} \in \mathfrak{a}_p.$$

For all  $\eta \in \mathcal{S}$ , we choose  $y_1^{\bar{\eta}} \in \mathfrak{g}_1^{\bar{\eta}}$  so that  $[x_1^{-\bar{\eta}}, y_1^{\bar{\eta}}] = -\bar{\eta}^{\vee}$ . Since the weights  $\bar{\eta}$  are strongly orthogonal, setting  $y_{\mathcal{S}} = \sum_{\eta \in \mathcal{S}} y_1^{\bar{\eta}}$  and  $h_{\mathcal{S}} = -\sum_{\eta \in \mathcal{S}} \bar{\eta}^{\vee}$ , we have that  $\{x_{\mathcal{S}}, h_{\mathcal{S}}, y_{\mathcal{S}}\}$  is a normal  $\mathfrak{sl}(2)$ -triple which contains  $x_{\mathcal{S}}$  as a nilpositive element.

**Theorem 5.5.** *Suppose that  $\mathfrak{g}_0$  is semisimple and  $\alpha_p$  is long and non-complex, and let  $\mathcal{S}$  be an orthogonal subset of  $\mathcal{C}_{\sigma}^1 \setminus \{\alpha_p\}$  of maximal cardinality. Then the orbit  $G_0 x_{\mathcal{S}} \subset \mathfrak{g}_1$  is not spherical. In particular,  $G_0 \mathfrak{a}_p$  is not spherical.*

*Proof.* Let  $\mathcal{S} = \{\eta_1, \dots, \eta_t\}$  be an orthogonal subset of  $\mathcal{C}_{\sigma}^1 \setminus \{\alpha_p\}$  of maximal cardinality. Since  $\sum_{\alpha \in \Pi_{\mathcal{S}}} a_{\mathcal{S}, \alpha} \alpha$  is an isotropic vector, it follows that  $k_{\mathcal{S}} (\sum_{\alpha \in \Pi_{\mathcal{S}}} a_{\mathcal{S}, \alpha} \alpha)$  is a multiple of  $\delta$ . Since

$$\left[ k_{\mathcal{S}} \left( \sum_{\alpha \in \Pi_{\mathcal{S}}} a_{\mathcal{S}, \alpha} \alpha \right) : \alpha_p \right] = \left[ k_{\mathcal{S}} \left( \sum_{i=1}^t a_{\mathcal{S}, \eta_i} \eta_i \right) : \alpha_p \right] - k_{\mathcal{S}} a_{-\alpha_p, \mathcal{S}} = 4 - 2 = 2,$$

it follows that  $k_{\mathcal{S}} (\sum_{\alpha \in \Pi_{\mathcal{S}}} a_{\mathcal{S}, \alpha} \alpha) = k\delta$ , hence

$$(5.2) \quad \sum_{i=1}^t k_{\mathcal{S}} a_{\mathcal{S}, \eta_i} \eta_i - \alpha_p = k\delta + \alpha_p.$$

It follows moreover that  $k_{\mathcal{S}} a_{\mathcal{S}, \eta_i}(\eta_i, \eta_i) = 2(\alpha_p, \eta_i)$ , namely

$$(5.3) \quad k_{\mathcal{S}} a_{\mathcal{S}, \eta_i} = \langle \alpha_p, \eta_i^\vee \rangle.$$

The equalities (5.2) and (5.3) show that

$$\sum_{i=1}^t \langle \alpha_p, \eta_i^\vee \rangle \bar{\eta}_i = 2\bar{\alpha}_p,$$

hence  $\sum_{i=1}^t \langle \alpha_p, \eta_i^\vee \rangle \langle \bar{\eta}_i, \alpha_p^\vee \rangle = 4$ . Since  $\alpha_p$  is long, we have that  $1 = \langle \eta_i, \alpha_p^\vee \rangle = \langle \bar{\eta}_i, \alpha_p^\vee \rangle$ , therefore considering the normal  $\mathfrak{sl}(2)$ -triple  $\{x_{\mathcal{S}}, h_{\mathcal{S}}, y_{\mathcal{S}}\}$  we get

$$-\bar{\alpha}_p(h_{\mathcal{S}}) = \sum_{i=1}^t \langle \alpha_p, \bar{\eta}_i^\vee \rangle = \sum_{i=1}^t \langle \alpha_p, \eta_i^\vee \rangle = 4.$$

Considering the grading  $\bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_1(i)$  induced by  $h_{\mathcal{S}}$ , we see that  $\mathfrak{g}_1^{-\bar{\alpha}_p} \subset \mathfrak{g}_1(4)$ , therefore  $G_0 x_{\mathcal{S}}$  is not spherical by [18, Theorem 5.6].  $\square$

Let  $\Sigma \subset \Pi_0$  be a component, and let  $e_{\Sigma} = -\langle \alpha_p, \alpha_{\Sigma}^\vee \rangle$  be the number of edges connecting  $\alpha_{\Sigma}$  with  $\alpha_p$ . As a corollary of the previous proof we get the following.

**Corollary 5.6.** *Let  $\mathcal{S} = \{\eta_1, \dots, \eta_t\}$  be an orthogonal subset of maximal cardinality of  $\mathcal{C}_{\sigma}^1 \setminus \{\alpha_p\}$  and let  $\Sigma \subset \Pi_0$  be a component.*

- i) *If  $\Upsilon(\eta_i) \in \Phi(\Sigma)$ , then  $k_{\mathcal{S}} a_{\mathcal{S}, \eta_i} = \langle \alpha_p, \eta_i^\vee \rangle = e_{\Sigma}$ .*
- ii) *Let  $(k\delta - 2\alpha_p)_{\Sigma}$  be the orthogonal projection of  $k\delta - 2\alpha_p$  onto  $\mathfrak{h}_{\Sigma} = \text{span}(h_{\alpha} \mid \alpha \in \Sigma)$  and set  $I_{\Sigma} = \{i \mid \Upsilon(\eta_i) \in \Phi(\Sigma)\}$ . Then*

$$\sum_{i \in I_{\Sigma}} \Upsilon(\eta_i) = \frac{1}{e_{\Sigma}} (k\delta - 2\alpha_p)_{\Sigma}.$$

*Proof.* i) This follows immediately by formulas (5.1) and (5.3).

- ii) Since  $k_{\mathcal{S}}(\sum_{i=1}^t a_{\mathcal{S}, \eta_i}) = 4$ , formula (5.2) implies that

$$\sum_{i=1}^t k_{\mathcal{S}} a_{\mathcal{S}, \eta_i} \Upsilon(\eta_i) = k\delta - 2\alpha_p.$$

On the other hand  $(k\delta - 2\alpha_p)_{\Sigma} = \sum_{i \in I_{\Sigma}} k_{\mathcal{S}} a_{\mathcal{S}, \eta_i} \Upsilon(\eta_i)$ , therefore the claim follows by i).  $\square$

We already noticed at the beginning of the section that  $(\Sigma, \alpha_{\Sigma})$  is a Hermitian pair. More precisely, we have the following.

**Proposition 5.7.** *Let  $\Sigma \subset \Pi_0$  be a component, then  $(\Sigma, \alpha_{\Sigma})$  is a Hermitian pair of tube type.*

*Proof.* For all  $\alpha \in \Sigma \setminus \{\alpha_{\Sigma}\}$ , it clearly holds  $(\alpha, k\delta - 2\alpha_p) = (\alpha, (k\delta - 2\alpha_p)_{\Sigma}) = 0$ . By Corollary 5.6 we get then the equality

$$(\alpha, \sum_{i \in I_{\Sigma}} \Upsilon(\eta_i)) = i(\alpha_{\Sigma}, \alpha_{\Sigma}) \quad \text{for all } \alpha \in \Phi(\Sigma)_i^+, \quad i = 0, 1.$$

By Lemma 5.1,  $\{\Upsilon(\eta_i) \mid i \in I_{\Sigma}\}$  is an orthogonal set of maximal cardinality in  $\Phi(\Sigma)_1^+$ . By Remark 4.2 we can then choose a set of positive roots for  $\Phi(\Sigma)_0$  in such a way that  $\{\Upsilon(\eta_i) \mid i \in I_{\Sigma}\}$  is the corresponding set of strongly orthogonal roots in  $\Phi(\Sigma)_1^+$ . Therefore the claim follows immediately from (4.1).  $\square$

In Theorem 5.5 we constructed a non-spherical orbit  $G_0 x_{\mathcal{S}} \subset G_0 \mathfrak{a}_p$  starting from an orthogonal subset  $\mathcal{S} \subset \mathcal{C}_{\sigma}^1$  of maximal cardinality. We now show that this construction extends to any maximal orthogonal subset  $\mathcal{S} \subset \mathcal{C}_{\sigma}^1$ , and it always gives rise to the same  $G_0$ -orbit.

**Theorem 5.8.** *Let  $\mathcal{S} \subset \mathcal{C}_\sigma^1 \setminus \{\alpha_p\}$  be a maximal orthogonal subset.*

- i) *For all  $\alpha \in \Phi$  it holds  $\alpha(h_{\mathcal{S}}) = -2\langle \alpha, \alpha_p^\vee \rangle$ .*
- ii) *The orbit  $G_0x_{\mathcal{S}} \subset \mathfrak{g}_1$  is not spherical, and  $G_0x_{\mathcal{S}} = G_0x_{\mathcal{T}}$  for all maximal orthogonal subset  $\mathcal{T} \subset \mathcal{C}_\sigma^1 \setminus \{\alpha_p\}$ .*

*Proof.* If  $\Sigma \subset \Pi_0$  is a component, set  $\mathcal{S}_\Sigma = \Upsilon(\mathcal{S}) \cap \Phi(\Sigma)_1^+$ , a maximal orthogonal subset in  $\Phi(\Sigma)_1^+$ . Write explicitly  $\mathcal{S}_\Sigma = \{\gamma_1, \dots, \gamma_s, \beta_1, \dots, \beta_t\}$  with  $\gamma_1, \dots, \gamma_s$  long roots and  $\beta_1, \dots, \beta_t$  short roots. Let  $r_\Sigma$  be the rank of the Hermitian symmetric space corresponding to the pair  $(\Sigma, \alpha_\Sigma)$  and let  $\mathcal{S}'_\Sigma = \{\gamma_1, \dots, \gamma_{r_\Sigma}\}$  be an orthogonal subset of maximal cardinality of  $\Phi(\Sigma)_1^+$  containing  $\{\gamma_1, \dots, \gamma_s\}$ . Then  $\mathcal{S}' = \bigcup_\Sigma \Upsilon^{-1}(\mathcal{S}'_\Sigma)$  is an orthogonal subset of  $\mathcal{C}_\sigma^1 \setminus \{\alpha_p\}$  of maximal cardinality.

By Proposition 5.7, the Hermitian pair  $(\Sigma, \alpha_\Sigma)$  is of tube type, therefore by Lemma 4.11, upon relabelling  $\gamma_{s+1}, \dots, \gamma_{r_\Sigma}$ , we may assume that  $\beta_i = \frac{1}{2}(\gamma_{s+2i-1} + \gamma_{s+2i})$  for all  $i = 1, \dots, t$ . By Corollary 5.6 it follows that

$$\sum_{i=1}^s \gamma_i + \sum_{j=1}^t 2\beta_j = \sum_{i=1}^{r_\Sigma} \gamma_i = \frac{1}{e_\Sigma} (k\delta - 2\alpha_p)_\Sigma.$$

Thus

$$e_\Sigma \left( \sum_{i=1}^s \Upsilon^{-1}(\gamma_i) + \sum_{j=1}^t 2\Upsilon^{-1}(\beta_j) \right) = (k\delta - 2\alpha_p)_\Sigma + e_\Sigma r_\Sigma \alpha_p.$$

Summing over all components  $\Sigma \subset \Pi_0$ , we get

$$\sum_{\eta \in \mathcal{S}} e_\eta \eta = k\delta - 2\alpha_p + \sum_{\Sigma} e_\Sigma r_\Sigma \alpha_p,$$

where for  $\eta \in \mathcal{C}_\sigma^1 \setminus \{\alpha_p\}$  we define

$$e_\eta = -\langle \alpha_p, \Upsilon(\eta)^\vee \rangle = \begin{cases} e_\Sigma & \text{if } \Upsilon(\eta) \in \Phi(\Sigma)_1^+ \text{ is a long root,} \\ 2e_\Sigma & \text{if } \Upsilon(\eta) \in \Phi(\Sigma)_1^+ \text{ is a short root.} \end{cases}$$

Notice that by Lemma 5.1 and Corollary 5.6 i) we have  $e_\eta = \langle \alpha_p, \eta^\vee \rangle = k_{\mathcal{S}} a_{\eta, \mathcal{S}}$  for all  $\eta \in \mathcal{S}'$ . Therefore by Corollary 5.6 ii) and Remark 5.4 we obtain

$$\sum_{\Sigma} e_\Sigma r_\Sigma = \sum_{\eta \in \mathcal{S}'} e_\eta = \sum_{\eta \in \mathcal{S}'} k_{\mathcal{S}'} a_{\eta, \mathcal{S}'} = 4.$$

On the other hand, by Lemma 5.1, we have  $e_\eta = \langle \alpha_p, \eta^\vee \rangle$  for all  $\eta \in \mathcal{S}$ , therefore we get

$$(5.4) \quad \sum_{\eta \in \mathcal{S}} \langle \alpha_p, \eta^\vee \rangle \eta = \sum_{\eta \in \mathcal{S}} e_\eta \eta = k\delta + 2\alpha_p.$$

We now conclude the proof by showing that the orbit  $G_0x_{\mathcal{S}}$  does not depend on the maximal orthogonal subset  $\mathcal{S} \subset \mathcal{C}_\sigma^1$ , hence it is non-spherical by Theorem 5.5. We identify the orbit  $G_0x_{\mathcal{S}} \subset \mathfrak{g}_1$  by computing its weighted Dynkin diagram (see [8, Section 9.5]).

By Lemma 5.1 we have  $\langle \eta, \alpha_p^\vee \rangle = 1$  for all  $\eta \in \mathcal{C}_\sigma^1 \setminus \{\alpha_p\}$ . If  $\alpha \in \Phi$ , considering the normal  $\mathfrak{sl}(2)$ -triple  $\{x_{\mathcal{S}}, h_{\mathcal{S}}, y_{\mathcal{S}}\}$ , it follows that

$$\alpha(h_{\mathcal{S}}) = - \sum_{\eta \in \mathcal{S}} \langle \alpha, \eta^\vee \rangle \langle \eta, \alpha_p^\vee \rangle = - \frac{2}{\|\alpha_p\|^2} (\alpha, \sum_{\eta \in \mathcal{S}} \langle \alpha_p, \eta^\vee \rangle \eta).$$

By (5.4) we get then

$$\alpha(h_{\mathcal{S}}) = - \frac{2}{\|\alpha_p\|^2} (\alpha, k\delta + 2\alpha_p) = -2\langle \alpha, \alpha_p^\vee \rangle.$$

Since the right hand side of previous equality does not depend on  $\mathcal{S}$ , it follows that  $G_0x_{\mathcal{S}}$  does not depend on  $\mathcal{S}$  either.  $\square$

We conclude this Section by showing that  $G_0\mathfrak{a}_p$  has complexity one. Recall that the complexity of an irreducible  $G_0$ -variety is defined as

$$c_{G_0}(X) = \min_{x \in X} \text{codim} B_0 x.$$

In particular, the spherical  $G_0$ -varieties coincide with the  $G_0$ -varieties of complexity zero, and the complexity of a  $G_0$ -variety can be regarded as a measure of its non-sphericity. By Theorem 3.1 together with Theorem 5.8, if  $\mathcal{S} \subset \mathcal{C}_\sigma^1$  is a maximal orthogonal subset, then  $x_{\mathcal{S}}$  is in the open  $G_0$ -orbit of  $G_0\mathfrak{a}_p$ . Since the complexity of a  $G_0$ -variety coincides with that of its  $G_0$ -stable open subsets, it follows in particular that  $c_{G_0}(G_0 x_{\mathcal{S}}) = 1$  for all maximal orthogonal subset  $\mathcal{S} \subset \mathcal{C}_\sigma^1 \setminus \{\alpha_p\}$ .

We start by recalling and proving some general results about the gradings associated to nilpotent elements of small height in  $\mathbb{Z}_2$ -graded Lie algebras.

**5.1. Remarks on the gradings associated to nilpotent elements of small height in  $\mathfrak{g}_1$ .** Let  $x_1 \in \mathfrak{g}_1$  be a nilpotent element and let  $\{x_1, h_0, y_1\}$  be a normal  $\mathfrak{sl}(2)$ -triple containing  $x_1$ . Denote  $\mathfrak{p}_0 = \mathfrak{g}_0(\geq 0)$ ,  $\mathfrak{u}_0 = \mathfrak{g}_0(\geq 1)$  and  $\mathfrak{l}_0 = \mathfrak{g}_0(0)$ , then  $\mathfrak{p}_0$  is a parabolic subalgebra of  $\mathfrak{g}_0$  with Levi factor  $\mathfrak{l}_0$  and with nilradical  $\mathfrak{u}_0$ . Fix a Borel subalgebra  $\mathfrak{b}_0 \subset \mathfrak{g}_0$  contained in  $\mathfrak{p}_0$  and set  $\mathfrak{b}_{00} = \mathfrak{b}_0 \cap \mathfrak{g}_0(0)$ , a Borel subalgebra of  $\mathfrak{l}_0$ , so that  $\mathfrak{b}_0 = \mathfrak{b}_{00} \oplus \mathfrak{u}_0$ . Notice that  $\mathfrak{g}_1(i)$  is  $\mathfrak{l}_0$ -stable for all  $i \in \mathbb{Z}$ .

Let  $P_0$  be the parabolic subgroup of  $G_0$  corresponding to  $\mathfrak{p}_0$ ,  $L_0 \subset P_0$  the Levi factor corresponding to  $\mathfrak{l}_0$  and  $U_0$  the unipotent radical of  $P_0$ ,  $B_0 \subset G_0$  the Borel subgroup corresponding to  $\mathfrak{b}_0$ , and  $B_{00} \subset L_0$  be the Borel subgroup of  $L_0$  corresponding to  $\mathfrak{b}_{00}$ . Recall that in this section the fixed points set  $\mathfrak{g}_0$  of the involution is assumed to be semisimple and we denote by  $\alpha_p$  the simple root in  $\hat{\Pi}$  corresponding to  $\sigma$ .

**Proposition 5.9.** *Suppose that  $\mathfrak{g}_0$  is semisimple with corresponding simple root  $\alpha_p \in \hat{\Pi}$ , and let  $x_1 \in \mathfrak{g}_1$  be a nilpotent element with  $\text{ht}(x_1) = m$ . Then  $\mathfrak{g}_1(m)$  is an irreducible  $L_0$ -module. If moreover  $\alpha(h_0) \geq 0$  for all  $\alpha \in \Phi_0^+$ , the highest weight vector of  $\mathfrak{g}_1(m)$  is  $x_1^{-\bar{\alpha}_p}$ .*

*Proof.* Up to conjugating  $x_1$  we may assume that  $\alpha(h_0) \geq 0$  for all  $\alpha \in \Phi_0^+$ . Notice that  $\mathfrak{u}_0$  acts trivially on  $\mathfrak{g}_1(m)$ , therefore every highest weight of  $\mathfrak{g}_1(m)$  as a  $\mathfrak{l}_0$ -module is actually a highest weight for  $\mathfrak{g}_1$  as a  $\mathfrak{g}_0$ -module. On the other hand, since  $\mathfrak{g}_0$  is semisimple,  $\mathfrak{g}_1$  is an irreducible  $\mathfrak{g}_0$ -module with highest weight vector  $x_1^{-\bar{\alpha}_p}$ , therefore  $\mathfrak{g}_1(m)$  is an irreducible  $\mathfrak{l}_0$ -module as well, with highest weight vector  $x_1^{-\bar{\alpha}_p}$ .  $\square$

Assume furthermore that  $\text{ht}(x_1) \leq 4$  and  $\mathfrak{g}_0(4) = 0$ . For  $i \geq 2$ , set  $\mathfrak{a}_i = \mathfrak{g}_1(\geq i)$ . Notice that  $\mathfrak{a}_i$  is a  $B_0$ -stable abelian subalgebra of  $\mathfrak{g}$ : indeed  $\mathfrak{a}_i$  is  $\mathfrak{p}_0$ -stable, hence  $P_0$ -stable, and being  $\text{ht}(x_1) \leq 4$  and  $\mathfrak{g}_0(4) = 0$ , it follows that  $[\mathfrak{a}_i, \mathfrak{a}_i] \subset \mathfrak{g}_0(4) = 0$ . It follows then by Theorem 3.1 that  $\mathfrak{a}_i$  possesses finitely many  $B_0$ -orbits, parametrized by the orthogonal subsets of  $\Psi(\mathfrak{a}_i)$ .

**Proposition 5.10.** *Suppose that  $\text{ht}(x_1) \leq 4$  and that  $\mathfrak{g}_0(4) = 0$ . Then  $\mathfrak{g}_1(i)$  is a spherical  $L_0$ -module for all  $i \geq 2$ .*

*Proof.* Let  $i \geq 2$  and consider the  $B_0$ -stable subalgebra  $\mathfrak{a}_i$ . By Theorem 3.1, there is  $v_i \in \mathfrak{a}_i$  such that  $\overline{B_0 v_i} = \mathfrak{a}_i$ . Since  $\mathfrak{a}_i = \mathfrak{g}_1(i) \oplus \mathfrak{a}_{i+1}$  and since  $\mathfrak{a}_{i+1}$  is also  $B_0$ -stable, we may write  $v_i = u_i + u'_i$ , for some  $u_i \in \mathfrak{g}_1(2)$  and  $u'_i \in \mathfrak{a}_{i+1}$  with  $u_i \neq 0$ . Therefore

$$\mathfrak{g}_1(i) \oplus \mathfrak{a}_{i+1} = \mathfrak{a}_i = [\mathfrak{b}_0, v_i] \subset [\mathfrak{b}_{00}, u_i] \oplus ([\mathfrak{u}_0, u_i] + [\mathfrak{b}_0, u'_i]).$$

Since  $[\mathfrak{b}_{00}, u_i] \subset \mathfrak{g}_1(i)$  and  $[\mathfrak{u}_0, u_i] + [\mathfrak{b}_0, u'_i] \subset \mathfrak{a}_{i+1}$ , the equality  $[\mathfrak{b}_{00}, u_i] = \mathfrak{g}_1(2)$  follows. Therefore  $\mathfrak{g}_1(2) = \overline{B_{00} u_i}$  is a spherical  $L_0$ -module.  $\square$

**5.2. The complexity of  $G_0\mathfrak{a}_p$ .** We now apply the results of previous subsection to compute the complexity of  $G_0\mathfrak{a}_p$ . As we already noticed, it is enough to show that  $c_{G_0}(G_0x_S) = 1$  when  $\mathcal{S} \subset \mathcal{C}_\sigma^1 \setminus \{\alpha_p\}$  is an orthogonal subset of maximal cardinality. In particular, under the bijection  $\Upsilon : \mathcal{C}_\sigma^1 \setminus \{\alpha_p\} \rightarrow \bigcup_\Sigma \Phi(\Sigma)_1^+$  of Lemma 5.1, we may assume that  $\alpha_\Sigma \in \Upsilon(\mathcal{S})$  for all components  $\Sigma \subset \Pi_0$ .

Let  $\mathcal{S} \subset \mathcal{C}_\sigma^1 \setminus \{\alpha_p\}$  be an orthogonal subset of maximal cardinality, let  $\{x_S, h_S, y_S\}$  be the corresponding normal  $\mathfrak{sl}(2)$ -triple, let  $\mathfrak{g} = \bigoplus \mathfrak{g}(i)$ , and for  $j = 0, 1$  set  $\mathfrak{g}_j(i) = \mathfrak{g}_j \cap \mathfrak{g}(i)$ . We keep the notation of previous subsection.

Let  $K_0 \subset L_0$  be the identity component of the stabilizer of  $x_S \in \mathfrak{g}_1(2)$ . Then  $K_0$  is reductive and  $\mathfrak{g}_1(2)$  is a  $K_0$ -orthogonal module, therefore by a theorem of Luna there exists a reductive subgroup  $M \subset K_0$  and a  $K_0$ -stable open subset  $Z \subset \mathfrak{g}_1(2)$  such that every  $K_0$ -orbit in  $Z$  is isomorphic to  $K_0/M$  (see [18, Section 5]). Then by [18, Theorem 5.4] the following formula holds:

$$(5.5) \quad c_{G_0}(G_0x_S) = c_{L_0}(\mathfrak{g}_1(2)) + c_M(\mathfrak{g}_1(\geq 3)).$$

**Proposition 5.11.** *We have  $\mathfrak{g}_1(3) = \mathfrak{g}_0(4) = \{0\}$  and  $\mathfrak{g}_1(4) = \mathfrak{g}_1^{-\bar{\alpha}_p}$  is the trivial one-dimensional representation of  $(L_0, L_0)$ .*

*Proof.* By Theorem 5.5, we have  $\alpha(h_S) = -2\langle \alpha, \alpha_p^\vee \rangle$  for all  $\alpha \in \Phi$ . In particular  $\mathfrak{g}(3) = \{0\}$ . If  $\mathfrak{g}(4)^\alpha \neq \{0\}$ , then  $\langle \alpha, \alpha_p^\vee \rangle = -2$ . Since  $\alpha_p$  is long, it follows that  $\alpha = -\bar{\alpha}_p$ . Since  $\alpha_p$  is non-complex,  $\mathfrak{g}_0(4) = \{0\}$  and  $\mathfrak{g}_1(4) = \mathfrak{g}_1^{-\bar{\alpha}_p}$ .

By definition,  $L_0$  is the Levi subgroup of  $G_0$  whose set of simple roots is

$$\Pi_{00} = \bigcup_\Sigma \{\alpha \in \Sigma \mid \alpha(h_S) = 0\}.$$

Let  $\Sigma \subset \Pi_0$  be a component. Recall that  $\alpha_\Sigma \in \Sigma$  is the unique simple root non-orthogonal to  $\alpha_p$ . On the other hand by Theorem 5.8 we have  $\alpha(h_S) = -2\langle \alpha, \alpha_p^\vee \rangle$ , therefore  $\Pi_{00} = \bigcup_\Sigma (\Sigma \setminus \{\alpha_\Sigma\})$  and it follows that every simple root of  $L_0$  is orthogonal to  $-\bar{\alpha}_p$ . This show that  $\mathfrak{g}_1(4) = \mathfrak{g}_1^{-\bar{\alpha}_p}$  is the trivial one-dimensional representation of  $(L_0, L_0)$ .  $\square$

**Corollary 5.12.** *Let  $\mathcal{S} \subset \mathcal{C}_\sigma^1 \setminus \{\alpha_p\}$  be an orthogonal subset of maximal cardinality, then  $c_{G_0}(G_0x_S) = 1$ . In particular,  $c_{G_0}(G_0\mathfrak{a}_p) = 1$ .*

*Proof.* Proposition 5.10 implies that  $c_{L_0}(\mathfrak{g}_1(2)) = 0$ , whereas Proposition 5.11 shows that  $\mathfrak{g}_1(\geq 3) = \mathfrak{g}_1(4)$  is one-dimensional, and by Theorem 5.8 together with (5.5) we get  $1 \leq c_{G_0}(G_0x_S) = c_M(\mathfrak{g}_1(4)) \leq 1$ .  $\square$

## 6. CLASSIFICATION OF $B_0$ -STABLE SUBALGEBRAS OF $\mathfrak{g}_1$

In this section  $\mathfrak{g}$  is a semisimple Lie algebra and  $\sigma$  is an (indecomposable) involution of  $\mathfrak{g}$ . Theorem 3.1 prompts us to study the orbits  $G_0x \subset \mathfrak{g}_0 \oplus \mathfrak{g}_1$  with  $x$  of the form

$$(6.1) \quad x_S = \sum_{\gamma \in \mathcal{S}_0} x_0^\gamma + \sum_{\gamma \in \mathcal{S}_1} x_1^\gamma,$$

where  $\mathcal{S}_0 \subset \Phi_0$ ,  $\mathcal{S}_1 \subset \Phi_1$ ,  $\mathcal{S}_0 \cap \mathcal{S}_1 = \emptyset$ , and  $\mathcal{S}_0 \cup \mathcal{S}_1$  is a set of strongly orthogonal weights in  $\Phi = \Phi_0 \cup \Phi_1$ . We denote by  $\mathcal{S}$  the disjoint union of  $\mathcal{S}_0$  and  $\mathcal{S}_1$ . Notice that all  $x_S$  are nilpotent: indeed setting  $y_S = \sum_{\gamma \in \mathcal{S}_0} y_0^{-\gamma} + \sum_{\gamma \in \mathcal{S}_1} y_1^{-\gamma}$  and  $h_S = \sum_{\gamma \in \mathcal{S}_1} \gamma^\vee + \sum_{\gamma \in \mathcal{S}_2} \gamma^\vee$  we get a  $\mathfrak{sl}(2)$ -triple  $\{x_S, h_S, y_S\}$ .

Let  $m$  be the height of  $x_S$ . Since  $h_S \in \mathfrak{h}_0$ , we can choose a weight  $\alpha \in \Phi$  such that  $\mathfrak{g}^\alpha \subset \mathfrak{g}(m)$ , namely such that

$$\alpha(h_S) = \sum_{\gamma \in \mathcal{S}} \langle \alpha, \gamma^\vee \rangle = m.$$



Set  $\mathcal{S}^+(\alpha) = \{\gamma \in \mathcal{S} \mid \langle \alpha, \gamma^\vee \rangle > 0\}$ , so that

$$(6.2) \quad \sum_{\gamma \in \mathcal{S}^+(\alpha)} \langle \alpha, \gamma^\vee \rangle \geq m.$$

Define  $\hat{\alpha} = \alpha$  if  $\alpha \in \Phi_0$ , and  $\hat{\alpha} = \delta' + \alpha$  if  $\alpha \in \Phi_1 \setminus \Phi_0$ . Choose for each  $\gamma \in \mathcal{S}^+(\alpha)$  a root  $\hat{\gamma} \in \widehat{\Phi}$  such that  $\bar{\hat{\gamma}} = \gamma$ , and define  $\widehat{\mathcal{S}^+(\alpha)} = \{\hat{\gamma} \mid \gamma \in \mathcal{S}^+(\alpha)\}$  and  $\Pi_{\mathcal{S}, \alpha} = \widehat{\mathcal{S}^+(\alpha)} \cup \{-\hat{\alpha}\}$ . As  $\mathcal{S}^+(\alpha) \cup \{-\alpha\} \subset \Phi$ , we have that  $\Pi_{\mathcal{S}, \alpha} \subset \widehat{\Phi}^{re}$ , so the matrix  $A(\mathcal{S}, \alpha) = (\langle \beta, \xi^\vee \rangle)_{\beta, \xi \in \Pi_{\mathcal{S}, \alpha}}$  is a generalized Cartan matrix, which is of finite or affine type by Lemma 5.2.

**Lemma 6.1.** *Let  $\mathcal{S} \subset \Phi$  be a strongly orthogonal subset, and let  $\alpha \in \Phi$  be such that  $\alpha(h_{\mathcal{S}}) = \text{ht}(x_{\mathcal{S}})$ . Then the following statements hold.*

- i)  $\text{ht}(x_{\mathcal{S}})$  is less than or equal to the degree of  $-\hat{\alpha}$  in  $\Pi_{\mathcal{S}, \alpha}$ . In particular  $\text{ht}(x_{\mathcal{S}}) \leq 4$ .
- ii) If  $Gx_{\mathcal{S}}$  is not spherical, then  $\text{ht}(x_{\mathcal{S}}) = 4$  and  $\Pi_{\mathcal{S}, \alpha}$  is of affine type, in which case its diagram is one of those listed in Table 2.

*Proof.* i) If  $\gamma \in \mathcal{S}^+(\alpha)$ , notice that  $\langle \alpha, \gamma^\vee \rangle = \langle \hat{\alpha}, \hat{\gamma}^\vee \rangle$ , and that this number is less or equal to the number of edges connecting  $-\hat{\alpha}$  with  $\hat{\gamma}$ . Therefore the claim follows by formula (6.2), by observing that the degree of any node in a finite or affine diagram is at most 4.

ii) By [17, Theorem 3.1], if  $Gx_{\mathcal{S}}$  is not spherical then  $\text{ht}(x_{\mathcal{S}}) \geq 4$ , hence  $\text{ht}(x_{\mathcal{S}}) = 4$  by i). Since in a Dynkin diagram of finite type any node has degree at most 3, it follows that  $\Pi_{\mathcal{S}, \alpha}$  is affine. Moreover, if  $\langle \hat{\alpha}, \hat{\gamma}^\vee \rangle$  is less than the number of edges connecting  $\hat{\alpha}$  and  $\hat{\gamma}$  for some  $\gamma \in \mathcal{S}^+(\alpha)$ , then  $\sum_{\gamma \in \mathcal{S}^+(\alpha)} \langle \hat{\alpha}, \hat{\gamma}^\vee \rangle < 4$ . Thus, for all  $\gamma \in \mathcal{S}^+(\alpha)$ ,  $\langle \hat{\alpha}, \hat{\gamma}^\vee \rangle$  equals the number of edges connecting  $\hat{\alpha}$  and  $\hat{\gamma}$ , and  $\hat{\alpha}$  is long in  $\Pi_{\mathcal{S}, \alpha}$ . It follows that the diagram  $\Pi_{\mathcal{S}, \alpha}$  is one of the affine diagrams listed in Table 2.  $\square$

In the next result we use the main idea of Proposition 2.2 of [22].

**Lemma 6.2.** *Let  $\mathcal{S} \subset \Phi$  be a strongly orthogonal subset and suppose that  $\mathcal{S} \subset \Psi(\mathfrak{a})$  for some  $\mathfrak{a} \in \mathcal{I}_{ab}^\sigma$ . Let  $\alpha \in \Phi$  be such that  $\alpha(h_{\mathcal{S}}) = \text{ht}(x_{\mathcal{S}})$ , then  $\alpha \in \Phi_1 \setminus \Phi_0$  and*

$$\sum_{\gamma \in \mathcal{S}^+(\alpha)} \langle \alpha, \gamma^\vee \rangle \hat{\gamma} = k\delta + 2\hat{\alpha}.$$

*In particular,  $\text{ht}_0(x_{\mathcal{S}}) \leq 3$  and  $\sum_{\gamma \in \mathcal{S}^+(\alpha)} \langle \beta, \gamma^\vee \rangle = 2\langle \beta, \alpha^\vee \rangle$  for all  $\beta \in \Phi \cup \{0\}$ .*

*Proof.* Since  $\mathcal{S}^+(\alpha) \subset \Phi_1$  we can assume that  $\hat{\gamma} = \delta' + \gamma$  for all  $\gamma \in \mathcal{S}^+$ . As in the proof of Theorem 5.5, we find that, if  $\Pi_{\mathcal{S}, \alpha} = \Xi$ , then  $k_{\Xi}(\sum_{\xi \in \Xi} a_{\Xi, \xi} \xi)$  is an isotropic vector, hence it is a multiple of  $\delta$ , say

$$(6.3) \quad k_{\Xi}(\sum_{\xi \in \Xi} a_{\Xi, \xi} \xi) = s\delta.$$

The coefficient  $s$  can be computed by counting the occurrences of roots in  $\Phi_1^+$  in the left hand side of (6.3). It follows that  $s = 2k$  if  $\alpha \in \Phi_0$ , and  $s = k$  if  $\alpha \in \Phi_1 \setminus \Phi_0$ .

Define a multiset  $\{\hat{\gamma}_i \mid i = 1, \dots, 4\}$  by listing every  $\hat{\gamma} \in \widehat{\mathcal{S}^+(\alpha)}$  with multiplicity  $\langle \hat{\alpha}, \hat{\gamma}^\vee \rangle = k_{\Xi} a_{\Xi, \hat{\gamma}}$  (see Remark 5.4). Set  $\beta = k\delta + \hat{\alpha}$ . Since  $\hat{\alpha} = \alpha \in \Phi_0$ , it follows that  $\beta$  is a root. Notice that  $\beta - \hat{\gamma}_i - \hat{\gamma}_j$  is a root for all  $i \neq j$ . If indeed  $\hat{\gamma}_i = \hat{\gamma}_j$  for some  $i \neq j$ , then  $\langle \hat{\alpha}, \hat{\gamma}_i^\vee \rangle \geq 2$ , hence  $\langle \beta, \hat{\gamma}_i^\vee \rangle \geq 2$  and  $\beta - 2\hat{\gamma}_i$  is a root. If instead  $\hat{\gamma}_i \neq \hat{\gamma}_j$ , then  $\langle \hat{\alpha}, \hat{\gamma}_i^\vee \rangle > 0$  and  $\langle \hat{\alpha} - \hat{\gamma}_i, \hat{\gamma}_j^\vee \rangle > 0$ , so  $\beta - \hat{\gamma}_i - \hat{\gamma}_j$  is either a root or zero. On the other hand  $\hat{\gamma}_i + \hat{\gamma}_j$  cannot be a root because  $\mathfrak{a}$  is abelian, therefore  $\beta - \hat{\gamma}_i - \hat{\gamma}_j$  is a root also in this case.

Since  $\sum_{i=1}^4 \hat{\gamma}_i = \sum_{\gamma \in \mathcal{S}^+(\alpha)} k_{\Xi} a_{\Xi, \hat{\gamma}} = 2k\delta + 2\hat{\alpha} = 2\beta$ , we have

$$\sum_{i < j} (\beta - \hat{\gamma}_i - \hat{\gamma}_j) = 6\beta - 3 \sum_{i=1}^4 \hat{\gamma}_i = 0,$$

thus  $\beta - \hat{\gamma}_i - \hat{\gamma}_j$  is a positive root for some  $i < j$ . Since  $\hat{\gamma}_i \in \widehat{\Phi}_1^+$  for each  $i$ ,  $\beta - \hat{\gamma}_i - \hat{\gamma}_j$  is a root in  $\widehat{\Phi}_0^+$ . Since  $\langle \beta, \hat{\gamma}_i^\vee \rangle > 0$  and  $\text{ht}_\sigma(\beta - \hat{\gamma}_i) = 1$ ,  $\beta - \hat{\gamma}_i \in \widehat{\Phi}_1$ . Since  $\beta - \hat{\gamma}_i = \hat{\gamma}_j + (\beta - \hat{\gamma}_i - \hat{\gamma}_j)$ , we see that  $\beta - \hat{\gamma}_i \in \Psi(\mathfrak{a})$ . Since  $\beta - \hat{\gamma}_i$  and  $\gamma_i$  are both in  $\Psi(\mathfrak{a})$  and  $\beta - \hat{\gamma}_i + \hat{\gamma}_i = \beta$ , we reach a contradiction since  $\mathfrak{a}$  is abelian.

To prove the last claim, notice that  $\alpha$  is long in  $\Pi_{\mathcal{S}, \alpha}$ , hence  $\langle \gamma, \alpha^\vee \rangle = 1$ . If  $\beta \in \Phi_0 \cup \Phi_1 \cup \{0\}$ , we get then the equality

$$\sum_{\gamma \in \mathcal{S}^+(\alpha)} \langle \beta, \gamma^\vee \rangle = \frac{2}{\|\alpha\|^2} \sum_{\gamma \in \mathcal{S}^+(\alpha)} \langle \alpha, \gamma^\vee \rangle (\beta, \hat{\gamma}) = 2 \frac{(\beta, k\delta + 2\hat{\alpha})}{\|\alpha\|^2} = 2\langle \beta, \alpha^\vee \rangle.$$

□

As a consequence of Lemma 6.2, we get the following result.

**Corollary 6.3.** *Let  $\mathfrak{a} \in \mathcal{I}_{ab}^\sigma$  and let  $x \in \mathfrak{a}$ , then  $\text{ht}(x) \leq 4$  and  $\text{ht}_0(x) \leq 3$ . In particular,  $Gx$  is spherical if and only if  $\text{ht}_1(x) \leq 3$ , if and only if  $G_0x$  is spherical.*

*Proof.* By Theorem 3.1, acting with  $B_0$  we may assume that  $x = x_{\mathcal{S}}$  for some orthogonal subset  $\mathcal{S} \subset \Psi(\mathfrak{a})$ . Then by Lemma 6.1 we get  $\text{ht}(x) \leq 4$ , and by Lemma 6.2 we get  $\text{ht}_0(x) \leq 3$ . The last claim follows by [18, Theorem 5.6]. □

If in previous corollary we take  $x$  in the open  $B_0$ -orbit of  $\mathfrak{a}$ , then we get the following.

**Corollary 6.4.** *Let  $\mathfrak{a} \in \mathcal{I}_{ab}^\sigma$ , then  $\mathfrak{a}$  is  $G$ -spherical if and only if it is  $G_0$ -spherical.*

Recall that  $\Pi_1$  contains at most two elements, and that if  $\widehat{\Pi}$  is simply laced, then the real roots of  $\widehat{\Phi}$  are regarded as long. The next result has been proved in [20] as a consequence of a case-by-case inspection. We provide here a conceptual proof that follows from Lemma 6.2 and the results of Section 5.1. Note also that Theorem 6.5 includes Theorem 2.3 of [22].

**Theorem 6.5.** *There exists  $\mathfrak{a} \in \mathcal{I}_{ab}^\sigma$  such that  $G_0\mathfrak{a}$  is not spherical if and only if  $\Pi_1 = \{\alpha_p\}$  and  $\alpha_p$  is long and non-complex.*

*Proof.* If  $\Pi_1 = \{\alpha_p\}$  with  $\alpha_p$  long and non-complex then by Theorem 5.5 the special  $B_0$ -stable subalgebra  $\alpha_p$  gives rise to a non-spherical variety  $G_0\mathfrak{a}_p$ .

Let now  $\mathfrak{a} \in \mathcal{I}_{ab}^\sigma$  and suppose that  $G_0\mathfrak{a}$  is not spherical. By Theorem 3.1 there is an orthogonal subset  $\mathcal{S} \subset \Psi(\mathfrak{a})$  such that  $G_0x_{\mathcal{S}}$  is not spherical, and by Lemma 6.1 we get  $\text{ht}(x_{\mathcal{S}}) = 4$ . Fix  $\alpha \in \Phi$  such that  $\alpha(h_{\mathcal{S}}) = 4$ , then  $\alpha \in \Phi_1 \setminus \Phi_0$  by Lemma 6.2. Set  $\hat{\alpha} = \delta' + \alpha \in \widehat{\Phi}_1$ .

If  $\Pi_1 = \{\alpha_i, \alpha_j\}$  consists of two distinct elements, then  $k = 1$ . Moreover, since  $\hat{\alpha} \in \widehat{\Phi}_1^+$ , we can assume  $[\hat{\alpha} : \alpha_i] = 1$  and  $[\hat{\alpha} : \alpha_j] = 0$ . By Lemma 6.2 we have

$$\sum_{\gamma \in \mathcal{S}^+(\alpha)} \langle \alpha, \gamma^\vee \rangle \hat{\gamma} - 2\hat{\alpha} = \delta.$$

Being  $(\hat{\gamma}, \hat{\alpha}) > 0$ , for all  $\gamma \in \mathcal{S}^+(\alpha)$  relation  $\hat{\gamma} - \hat{\alpha} \in \widehat{\Phi} \cup \{0\}$  holds. Therefore  $[\hat{\gamma} : \alpha_i] = 1$  for all  $\gamma \in \mathcal{S}^+(\alpha)$ , and we get  $[\delta : \alpha_i] = 2$  which is absurd.

Thus  $\Pi_1 = \{\alpha_p\}$  consists of a single element, and  $\mathfrak{g}_0$  is semisimple. By Proposition 5.9, we can choose  $\alpha = w(-\bar{\alpha}_p)$  with  $w \in W_0$ . Since  $\alpha \in \Phi_1 \setminus \Phi_0$ , we see that  $\alpha_p$  cannot be complex. Suppose that  $\alpha_p$  is short. Then also  $\alpha$  is short, and there must be a component  $\Sigma \subset \Pi_0$  such that  $\theta_\Sigma$  is long. Since  $(\theta_\Sigma, \alpha_p) = (\alpha_\Sigma, \alpha_p) < 0$ ,

it follows that  $\langle \theta_\Sigma, \alpha_p^\vee \rangle \leq -2$ , and setting  $\beta = w(\theta_\Sigma)$  we get  $\langle \beta, \alpha^\vee \rangle \geq 2$ . Since  $\alpha(\sum_{\gamma \in \mathcal{S}^+(\alpha)} \gamma^\vee) = \text{ht}(x_\mathcal{S}) = 4$ , it follows that the element  $x_{\mathcal{S}^+(\alpha)} = \sum_{\gamma \in \mathcal{S}^+(\alpha)} x_\gamma$  is still in  $\mathfrak{a}$ , and being  $\text{ht}(x_{\mathcal{S}^+(\alpha)}) \geq 4$  its  $G_0$ -orbit is still non-spherical by Corollary 6.3. Therefore we can assume that  $\mathcal{S} = \mathcal{S}^+(\alpha)$ . By Lemma 6.2 we get then

$$\beta(\sum_{\gamma \in \mathcal{S}} \gamma^\vee) = \beta(\sum_{\gamma \in \mathcal{S}^+(\alpha)} \gamma^\vee) = 2\langle \beta, \alpha^\vee \rangle \geq 4.$$

As  $\beta \in \Phi_0$  and  $\mathfrak{g}_0(i) = 0$  for  $i > 3$ , this is absurd. Therefore  $\alpha_p$  must be long.  $\square$

Assume now that  $\Pi_1 = \{\alpha_p\}$  with  $\alpha_p$  long and non-complex. We give a classification of the subalgebras  $\mathfrak{a} \in \mathcal{I}_{ab}^\sigma$  such that  $G_0\mathfrak{a}$  is non-spherical. By Proposition 4.12, for any component  $\Sigma \subset \Pi_0$  there is a unique maximally orthogonal antichain  $\mathcal{A}_\Sigma$  in  $\Phi(\Sigma)_1^+$ . Regarding  $\mathcal{C}_\sigma^1$  as a poset w.r.t. the dominance order, it is clear from Lemma 5.1 that  $\mathcal{C}_\sigma^1 \setminus \{\alpha_p\}$  contains a unique maximally orthogonal antichain  $\mathcal{A}$ , namely

$$\mathcal{A} = \bigcup_{\Sigma} \Upsilon^{-1}(\mathcal{A}_\Sigma).$$

For  $\Gamma \subset \mathcal{C}_\sigma^1$ , we set  $\Gamma^{\leq 0} = \{\xi \in \mathcal{C}_\sigma^1 \mid \xi \leq_0 \eta \text{ for some } \eta \in \Gamma\}$ .

**Lemma 6.6.** *There is  $\bar{w} \in \mathcal{W}_\sigma^{ab}$  such that  $N(\bar{w}) = \mathcal{A}^{\leq 0}$ .*

*Proof.* We show that, if  $\zeta, \xi \in \widehat{\Phi}^+$  are such that  $\zeta + \xi \in \mathcal{A}^{\leq 0}$ , then exactly one among  $\zeta$  and  $\xi$  is in  $\mathcal{A}^{\leq 0}$ . Since  $\mathcal{A}^{\leq 0} \subset \mathcal{C}_\sigma^1 = N(w_p)$ , then exactly one among  $\zeta$  and  $\xi$  (say  $\zeta$ ) is in  $\mathcal{C}_\sigma^1$ . Since  $\zeta + \xi \in \mathcal{A}^{\leq 0}$ , then  $\zeta \in \mathcal{A}^{\leq 0}$ . This implies that both  $\mathcal{A}^{\leq 0}$  and its complement are closed under root addition. It follows that there is  $\bar{w} \in \widehat{W}$  such that  $N(\bar{w}) = \mathcal{A}^{\leq 0}$ . Since  $N(\bar{w}) \subset N(w_p)$ , it is clear that  $\bar{w} \in \mathcal{W}_\sigma^{ab}$ .  $\square$

Let  $\bar{\mathfrak{a}} = \Theta(\bar{w})$  (see Proposition 2.5).

**Theorem 6.7.** *Suppose that  $\Pi_1 = \{\alpha_p\}$  with  $\alpha_p$  long and non-complex, and let  $\mathfrak{a} \in \mathcal{I}_{ab}^\sigma$ . Then  $G_0\mathfrak{a}$  is not spherical if and only if  $\{-\bar{\eta} \mid \eta \in \mathcal{A}\} \subset \Psi(\mathfrak{a})$ , if and only if  $\bar{\mathfrak{a}} \subset \mathfrak{a}$ .*

*Proof.* Suppose that  $G_0\mathfrak{a}$  is non-spherical, and let  $\mathcal{S} \subset \Psi(\mathfrak{a})$  be an orthogonal subset such that  $G_0x_\mathcal{S}$  is not spherical. Then by Lemma 6.1 we have  $\text{ht}(x_\mathcal{S}) = 4$ , and by Corollary 6.3 there is  $\alpha \in \Phi_1$  such that  $\mathfrak{g}_1(4)^\alpha \neq 0$ , namely  $\alpha(h_\mathcal{S}) = \sum_{\gamma \in \mathcal{S}} \langle \alpha, \gamma^\vee \rangle = 4$ . We may assume that  $\alpha$  is maximal w.r.t.  $\leq_0$  among the weights of  $\mathfrak{g}_1(4)$ . As in the proof of Theorem 6.5, we can assume that  $\mathcal{S} = \mathcal{S}^+(\alpha)$ .

Suppose that  $\alpha = -\bar{\alpha}_p$ . We have that  $\hat{\alpha} = \delta' - \bar{\alpha}_p = -\alpha_p + 2\delta' = k\delta - \alpha_p$ . If  $\gamma \in \mathcal{S}$ , since  $\alpha_p$  is long we get

$$\langle \delta' - \gamma, (k\delta + \alpha_p)^\vee \rangle = \langle \gamma, \alpha^\vee \rangle = 1,$$

hence  $\delta' - \gamma \in \mathcal{C}_\sigma^1$ . Thus  $\mathcal{O} = \{\delta' - \gamma \mid \gamma \in \mathcal{S}\}$  is an orthogonal subset of  $\mathcal{C}_\sigma^1$ . Notice moreover that  $\mathcal{O}$  is a maximal orthogonal subset in  $\mathcal{C}_\sigma^1$ : if indeed  $\eta \in \mathcal{O}^\perp \cap \mathcal{C}_\sigma^1$ , then  $\Pi_{\mathcal{S}, \alpha} \cup \{\eta\}$  gives rise to a generalized Cartan matrix which is neither finite nor affine, contradicting Lemma 5.2. Therefore  $\Upsilon(\mathcal{O})$  is a maximal orthogonal set in  $\bigcup_{\Sigma} \Phi(\Sigma)_1^+$ , and by Corollary 4.14 it follows that  $\bigcup_{\Sigma} \mathcal{A}_\Sigma \subset \Upsilon(\mathcal{O})^{\leq 0}$ . By Lemma 5.1, it follows that  $\mathcal{A} \subset \mathcal{O}^{\leq 0} \subset N(\Theta^{-1}(\mathfrak{a}))$  which in turns means that  $N(\bar{w}) \subset N(\Theta^{-1}(\mathfrak{a}))$ , or, equivalently, that  $\bar{\mathfrak{a}} \subset \mathfrak{a}$ .

Suppose that  $\alpha \neq -\bar{\alpha}_p$ . Then there exists  $\beta \in \Phi_0^+$  such that  $\alpha + \beta \in \Phi_0 \cup \Phi_1 \cup \{0\}$ , and the maximality of  $\alpha$  among the weights of  $\mathfrak{g}_1(4)$  implies that  $\sum_{\gamma \in \mathcal{S}} \langle \alpha + \beta, \gamma^\vee \rangle < 4$ . Therefore by Lemma 6.2 we get

$$(6.4) \quad 2\langle \beta, \alpha^\vee \rangle = \sum_{\gamma \in \mathcal{S}} \langle \beta, \gamma^\vee \rangle < 0.$$

In particular there is  $\gamma$  such that  $\langle \beta, \gamma \rangle < 0$  and, by Lemma 3.3 it follows  $\langle \beta, \gamma' \rangle \geq 0$  for all  $\gamma' \in \mathcal{S} \setminus \{\gamma\}$ .

Suppose that  $\langle \beta, \gamma' \rangle = 0$  for all  $\gamma' \in \mathcal{S}$ . Then  $s_\beta(\mathcal{S})$  is an orthogonal subset of  $\Psi(\mathfrak{a})$  and  $G_0 x_{s_\beta(\mathcal{S})} = G_0 x_{\mathcal{S}}$  is still not spherical, and  $s_\beta(\alpha)$  is a maximal weight in  $\Phi_1$  w.r.t.  $\leq_0$  such that  $s_\beta(\alpha)(h_{s_\beta(\mathcal{S})}) = \text{ht}(x_{s_\beta(\mathcal{S})}) = 4$ . On the other hand by (6.4) we have  $\alpha \leq_0 s_\beta(\alpha)$ , therefore we may proceed inductively by replacing  $x_{\mathcal{S}}$  with  $x_{s_\beta(\mathcal{S})}$  until either  $\alpha = -\bar{\alpha}_p$  or we find a root  $\gamma' \in \mathcal{S}$  such that  $\langle \gamma', \beta^\vee \rangle > 0$ .

If  $\alpha = -\bar{\alpha}_p$  then we are done, therefore we may assume that there are  $\gamma, \gamma' \in \mathcal{S}$  such that  $\langle \gamma, \beta \rangle < 0$  and  $\langle \gamma', \beta \rangle > 0$ . Consider the set  $\Pi_\beta = \{\gamma, \beta, -\gamma'\}$ , then  $A_\beta = (\langle \nu, \xi^\vee \rangle)_{\nu, \xi \in \Pi_\beta}$  is a generalized Cartan matrix, and by Lemma 5.2 it is either of affine or of finite type. Identify  $\Pi_\beta$  with the corresponding Dynkin diagram; since  $\langle \beta, \gamma' \rangle \geq 0$  for all  $\gamma' \in \mathcal{S} \setminus \{\gamma\}$ , by (6.4) we have that  $\langle \beta, \gamma^\vee \rangle \leq -2$ , so  $\Pi_\beta$  is not simply laced and  $\gamma$  is a short node. If moreover  $\langle \beta, \gamma^\vee \rangle = -2$ , then again by (6.4) it follows that  $\langle \beta, \gamma'^\vee \rangle = 1$ . With these conditions at hand, we see that the only possibilities for the diagram of  $\Pi_\beta$  are the following:

$$\begin{array}{ccc} \circ & \Rightarrow & \circ & \Rightarrow & \circ & & \circ & - & \circ & \Rightarrow & \circ & & \circ & - & \circ & \Rightarrow & \circ \\ -\gamma' & & \beta & & \gamma & & -\gamma' & & \beta & & \gamma & & -\gamma' & & \beta & & \gamma \end{array}$$

The first case has to be discarded because, otherwise, the diagram of  $\Pi_{\mathcal{S}, \alpha}$  would have rank three with two nodes  $\gamma, \gamma'$  satisfying  $\frac{\|\gamma'\|^2}{\|\gamma\|^2} = 4$ , and this never occurs for the diagrams of Table 2. The second case also has to be discarded, otherwise by Lemma 6.2 it would follow

$$2\langle \beta, \alpha^\vee \rangle = \sum_{\gamma \in \mathcal{S}} \langle \beta, \gamma^\vee \rangle = -1.$$

Therefore the diagram of  $\Pi_\beta$  is of type  $G_2^{(1)}$ . This is possible only if  $\widehat{L}(\mathfrak{g}, \sigma)$  is of type  $G_2^{(1)}$ . Let  $\widehat{\Pi} = \{\alpha_0, \alpha_1, \alpha_2\}$  be as in [11, Table Aff 1] (in particular,  $\alpha_2$  is short). Since  $\beta$  is long and belongs to  $\Phi_0^+$ , we have that  $\beta = \alpha_0$ . Moreover,  $-\gamma' + 2\beta + 3\gamma$  is isotropic so  $-\gamma' + 2\beta + 3\gamma = \delta$ , hence  $-\gamma' + 3\gamma = 3\alpha_0 + 2\alpha_1 + 3\alpha_2$ . Since  $\hat{\gamma}', \hat{\gamma} \in \widehat{\Phi}_1^+$ , we have that  $[\hat{\gamma}' : \alpha_0] \leq 1$  and  $[\hat{\gamma} : \alpha_0] \leq 1$ . It follows that  $\hat{\gamma}' = \alpha_1 + x\alpha_2$  and  $\hat{\gamma} = \alpha_0 + \alpha_1 + y\alpha_2$ . From  $[-\hat{\gamma}' + 3\hat{\gamma} : \alpha_2] = 3$  and  $(\hat{\gamma}', \hat{\gamma}) = 0$  we obtain that either  $\hat{\gamma}' = \alpha_1, \hat{\gamma} = \alpha_0 + \alpha_1 + \alpha_2$  or  $\hat{\gamma}' = \alpha_1 + 3\alpha_2, \hat{\gamma} = \alpha_0 + \alpha_1 + 2\alpha_2$ . In both cases one easily verifies that  $\hat{\gamma}$  cannot belong to  $\Psi(\mathfrak{a})$  with  $\mathfrak{a} \in \mathcal{I}_{ab}^\sigma$ . Hence we have obtained the desired contradiction. We conclude that  $\alpha = -\bar{\alpha}_p$ , and the proof is complete.  $\square$

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