

Linked cluster expansions for open quantum systems on a latticeAlberto Biella,¹ Jiasen Jin,² Oscar Viyuela,^{3,4} Cristiano Ciuti,¹ Rosario Fazio,^{5,6} and Davide Rossini⁷¹*Université Paris Diderot, Sorbonne Paris Cité, Laboratoire Matériaux et Phénomènes Quantiques, CNRS-UMR7162, 75013 Paris, France*²*School of Physics, Dalian University of Technology, 116024 Dalian, China*³*Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA*⁴*Department of Physics, Harvard University, Cambridge, Massachusetts 02318, USA*⁵*ICTP, Strada Costiera 11, I-34151 Trieste, Italy*⁶*NEST, Scuola Normale Superiore and Istituto Nanoscienze-CNR, I-56126 Pisa, Italy*⁷*Dipartimento di Fisica, Università di Pisa and INFN, Largo Pontecorvo 3, I-56127 Pisa, Italy*

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We propose a generalization of the linked-cluster expansions to study driven-dissipative quantum lattice models, directly accessing the thermodynamic limit of the system. Our method leads to the evaluation of the desired extensive property onto small connected clusters of a given size and topology. We first test this approach on the isotropic spin-1/2 Hamiltonian in two dimensions, where each spin is coupled to an independent environment that induces incoherent spin flips. Then we apply it to the study of an anisotropic model displaying a dissipative phase transition from a magnetically ordered to a disordered phase. By means of a Padé analysis on the series expansions for the average magnetization, we provide a viable route to locate the phase transition and to extrapolate the critical exponent for the magnetic susceptibility.

DOI: [10.1103/PhysRevB.97.035103](https://doi.org/10.1103/PhysRevB.97.035103)**I. INTRODUCTION**

The recent technological breakthroughs in the manipulation of many-body systems coupled to an external bath are setting the foundation for a careful testing of a new wealth of physical phenomena in the quantum realm [1–3]. Specifically, several promising experimental platforms aimed at investigating the scenario emerging from driven-dissipative quantum many-body systems have recently been proposed and realized in the laboratory. The most remarkable ones are atomic and molecular optical systems through the use of Rydberg atoms, trapped ions or atomic ensembles coupled to a condensate reservoir [4], arrays of coupled QED cavities [5], or coupled optomechanical resonators [6]. These implementations are scalable enough to enable the construction of tunable and interacting artificial lattice structures with hundreds of sites.

The coupling between different unit cells can give rise to a plethora of cooperative phenomena determined by the interplay of on-site interactions, nonlocal (typically nearest-neighbor) processes, and dissipation [7–11]. Recently, a large body of theoretical work has been devoted to the investigation of the collective behavior emerging in dynamical response [12], many-body spectroscopy [13–15], transport [16–20], as well as stationary properties. In the latter context, a careful engineering of the coupling between the system and the environment can stabilize interesting many-body phases in the steady state [21,22]. The phase-diagram of such lattice systems has been predicted to be incredibly rich [23–30] and can display spontaneous ordering associated with the breaking of a discrete [31–33] or continuous symmetry [34,35] possessed by the model. Recently, the critical behavior emerging at the onset of phase transitions started to be investigated by means of different analytical and numerical approaches [36–39].

Theoretically, while at equilibrium we have reached a fairly good understanding of several aspects of the many-body problem under the framework of textbook statistical mechanics, this is no longer the case for quantum systems coupled to some external bath. In such a case, we are indeed facing an inherently out-of-equilibrium situation, where the Hamiltonian of the system \hat{H} is no longer capable of describing it in its whole complexity, and the environmental coupling needs to be accounted for and suitably modeled. Due to the intrinsic difficulty of the problem, a number of approximations are usually considered, which assume a weak system-bath coupling, neglect memory effects in the bath, and discard fast oscillating terms. In most of the experimental situations with photonic lattices, these assumptions are typically met [5,40].

As a result, in many cases of relevance, the coupling to the environment leads to a Markovian dynamics of the system's density matrix ρ , according to a master equation in the Lindblad form [41]:

$$\partial_t \rho = \mathbb{L}[\rho] = -i[\hat{H}, \rho] + \mathbb{D}[\rho], \quad (1)$$

where \mathbb{L} denotes the so called Liouvillian superoperator (we will work in units of $\hbar = 1$). While the commutator on the right-hand side of Eq. (1) accounts for the unitary part of the dynamics, the dissipative processes are ruled by

$$\mathbb{D}[\rho] = \sum_j \left[\hat{L}_j \rho \hat{L}_j^\dagger - \frac{1}{2} \{ \hat{L}_j^\dagger \hat{L}_j, \rho \} \right], \quad (2)$$

where \hat{L}_j are suitable local jump operators that describe the incoherent coupling to the environment. To give a specific example, in the case of spin-1/2 systems, $\hat{L}_j = \sqrt{\Gamma}(\hat{\sigma}_j^x - i\hat{\sigma}_j^y)/2$ and $\sqrt{\Gamma'}\hat{\sigma}_j^z$ [$\hat{\sigma}_j^\beta$ ($\beta = x, y, z$) being the Pauli matrices] describe a local relaxation process along

the z direction (at a rate Γ) and a dephasing noise (at a rate Γ') on the j th spin, respectively [see, e.g., Eq. (9)]. The master equation (1) covers a pivotal role in the treatment of open quantum systems, since it represents the most general completely positive trace-preserving dynamical semigroup [42]. In the following, we will restrict our attention to it, and specifically address the steady-state (long-time limit) solution $\rho_{SS} = \lim_{t \rightarrow \infty} \exp(\mathbb{L}t)\rho(0)$ (and thus $\partial_t \rho_{SS} = 0$) in situations where the steady state is guaranteed to be unique [43].

Solving the long-time dynamics ruled by Eq. (1) for a many-body system is a formidable, yet important, task. Indeed contrary to equilibrium situations, the effect of short-range correlations can be dramatic in a driven-dissipative context, and thus they deserve an accurate treatment through the (in principle) full many-body problem. Exact solutions are restricted to very limited classes of systems, which are typically represented by quadratic forms in the field operators and specific jump terms [44]. A number of viable routes have thus been proposed in the past few years. Under certain hypotheses, analytic approaches such as perturbation theory [45] or renormalization-group techniques based on the Keldysh formalism [9,46] are possible. However, their limited regime of validity calls for more general numerical methods that do not have these limitations.

From a computational point of view, the main difficulty resides in the exponential growth of the many-body Hilbert space with the number N of lattice sites. Moreover, the non-Hermitian Liouvillian superoperator \mathbb{L} acts on the space of density matrices (whose dimension is the square of the corresponding Hilbert space dimension), and its spectral properties are generally much more difficult to address than the low-lying eigenstates of a Hamiltonian system. The difficulty remains even for the fixed point of the dynamics ρ_{SS} , that is, the density matrix associated with the zero eigenvalue of \mathbb{L} .

While in one dimension tensor-network approaches based on a straightforward generalization of matrix product states to operators can be effective [47–49], and alternative strategies have been proposed in order to improve their performances [50–52], going to higher dimensions is much harder. Numerical strategies specifically suited for this purpose have recently been put forward, including cluster mean-field [53], correlated variational *Ansätze* [54,55], truncated correlation hierarchy schemes [56], corner-space renormalization methods [57], and even two-dimensional tensor-network structures [58]. The nonequilibrium extension of the dynamical mean-field theory (which works directly in the thermodynamic limit) has also been proved to be very effective in a wide class of lattice systems [59–61]. Each of such methods presents advantages and limitations, and typically performs better on specific regimes.

In this paper, we will adapt a class of techniques that, in the past, has been shown to be extremely useful and versatile in the study of thermal and quantum phase transitions [62]. The key idea consists in computing extensive properties of lattice systems in the thermodynamic limit, out of certain numerical series expansions. The method, dubbed the linked-cluster expansion (LCE), sums over different contributions associated with clusters of physical sites. In combination with perturbation theories, LCEs have already proved their worth in the context of equilibrium statistical mechanics, both in classical and quantum systems (see Ref. [62] and references

therein). Their predictive power lies beyond the range of validity of the perturbation expansion: using established tools for the analysis of truncated series [63], it has been possible to study equilibrium quantum phase transitions, and to extract critical exponents. Here we focus on numerical linked-cluster expansions (NLCEs), where the k th-order contribution in the LCE is obtained by means of exact diagonalization techniques on finite-size clusters with k sites [64]. The NLCE has been successfully employed in order to evaluate static properties at zero [65] and finite temperature [66], as well as to study the long-time dynamics and thermalization in out-of-equilibrium closed systems [67,68]. Moreover, it has also revealed its flexibility in combination with other numerical methods that can be used to address finite-size clusters, such as density-matrix renormalization-group algorithms [69]. Nonetheless, to the best of our knowledge, it has never been applied in the context of open quantum systems.

Here we see NLCE at work in an interacting two-dimensional spin-1/2 model with incoherent spin relaxation [32], which is believed to exhibit a rich phase diagram, and it represents a testing ground for strongly correlated open quantum systems [39,53,58]. We will test our method both far from critical points and in proximity to a phase transition: in the first case, NLCE allows us to accurately compute the value of the magnetization, while in the second case we are able to estimate the critical point as well as the critical exponent γ for the divergent susceptibility.

The paper is organized as follows. In Sec. II we introduce our NLCE method and discuss how it can be applied to the study of the steady-state of a Markovian Lindblad master equation. The NLCE is then benchmarked in a dissipative two-dimensional spin-1/2 XYZ model (Sec. III). By properly tuning the coupling constants of the Hamiltonian, we are able to study steady-state properties far away from any phase boundary (Sec. III A), and a more interesting scenario exhibiting a quantum phase transition from a paramagnetic to a ferromagnetic phase (Sec. III B). In the latter case, we discuss a simple strategy (based on the Padé analysis of the expansion) in order to locate the critical point and to extrapolate the critical exponent γ . Finally, Sec. IV is devoted to the conclusions.

II. LINKED-CLUSTER METHOD

We start with a presentation of the NLCE formalism [64], unveiling its natural applicability to the study of driven-dissipative quantum systems whose dynamics is governed by a Lindblad master equation. We follow an approach that is routinely employed in series expansions for lattice models, such as high-temperature classical expansions [62]. Since we are interested in the steady-state properties of the system, our target objects will be the expectation values of generic extensive observables \hat{O} onto the asymptotic long-time limit solution ρ_{SS} of the master equation: $O = \text{Tr}[\hat{O}\rho_{SS}]$. In practice, for each cluster appearing in the expansion, the steady-state density matrix ρ_{SS} is reached by time-evolving a random initial state according to the master equation (1) by means of a fourth-order Runge-Kutta method. We stress that there are no restrictions in the limits of applicability of this approach to different scenarios for homogeneous systems, which can be straightforwardly extended to the case of generic

non-Markovian master equations and/or nonequilibrium states $\rho(t)$. Therefore, boundary-driven systems [16,18–20,70] and disordered lattices [71] do not fit within this framework.

Let us first write the Liouvillian operator \mathbb{L} as a sum of local terms \mathbb{L}_k , each of them supposedly acting on few neighboring sites. For the sake of simplicity and without loss of generality, each term \mathbb{L}_k only couples two neighboring sites:

$$\mathbb{L} = \sum_k \alpha_k \mathbb{L}_k = \sum_{(i,j)} \alpha_{ij} \mathbb{L}_{ij}, \quad (3)$$

where α_{ij} denotes the local coupling strength, and the index $k = (i, j)$ is a shorthand notation for the couple of i - j sites. The terms of \mathbb{L} acting exclusively on the i th site can be arbitrary absorbed in the terms of the sum such that $i \in k$. The observable O can always be arranged in a multivariable expansion in powers of α_k :

$$O(\{\alpha_k\}) = \sum_{\{n_k\}} O_{\{n_k\}} \prod_k \alpha_k^{n_k}, \quad (4)$$

where n_k runs over all non-negative integers for each k , such that any possible polynomial in the α_k couplings is included. The expansion (4) can then be reorganized in clusters:

$$O = \sum_c W_{\{O\}}(c), \quad (5)$$

where each c represents a nonempty set of k -spatial indexes, which identify the links belonging to the given cluster. Specifically, the so called cluster weight $W_{\{O\}}(c)$ contains all terms of the expansion (4), which have at least one power of α_k , $\forall k \in c$, and no powers of α_k if $k \notin c$. On the contrary, all terms in Eq. (4) can be included in one of these clusters. Using the inclusion-exclusion principle, one can take $W_{\{O\}}(c)$ out of the sum (5) obtaining the recurrence relation

$$W_{\{O\}}(c) = O(c) - \sum_{s \subset c} W_{\{O\}}(s), \quad (6)$$

where $O(c) = \text{Tr}[\hat{O} \rho_{\text{SS}}(c)]$ is the steady-state expectation value of the observable calculated for the finite cluster c , the sum runs over all the subclusters s contained in c , and $\rho_{\text{SS}}(c)$ is the steady state of the Liouvillian $\mathbb{L}(c)$ over the cluster c . An important property of Eq. (6) is that if c is formed out of two disconnected clusters c_1 and c_2 , its weight $W_{\{O\}}(c)$ is zero. This follows from the fact that O is an extensive property [$O(c) = O(c_1) + O(c_2)$] and $c = c_1 + c_2$.

The symmetries of the Liouvillian \mathbb{L} may drastically simplify the summation (5), since it is typically not needed to compute all the contributions coming from each cluster. This can be immediately seen, e.g., for situations in which the interaction term α_k between different couples of sites is homogeneous throughout the lattice. In such cases, it is possible to identify the topologically distinct clusters, so that a representative c_n for each class can be chosen and counted according to its multiplicity $\ell(c_n)$ per lattice site (the lattice constant of the graph c_n). Here the subscript n denotes the number of k -spatial indexes that are grouped in the cluster, that is, its size. The property O per lattice site can thus be

written directly in the thermodynamic limit $L \rightarrow \infty$ as

$$\frac{O}{L} = \sum_{n=1}^{+\infty} \left[\sum_{\{c_n\}} \ell(c_n) W_{\{O\}}(c_n) \right]. \quad (7)$$

The outer sum runs over all possible cluster sizes, while the inner one accounts for all topologically distinct clusters $\{c_n\}$ of a given size n . Let us emphasize that, if the series expansion (7) is truncated up to order $n = R$, only clusters c at most of size R have to be considered. Indeed, each of them should include at least one power of α_k , $\forall k \in c$. Therefore, a cluster of size $R + 1$ or larger does not contribute to the expansion, up to order α^R . As a matter of fact, dealing with open many-body systems significantly reduces our ability to compute large orders in the expansion, with respect to the *closed*-system scenario. The size of the Liouvillian superoperator governing the dynamics scales as $\dim(\mathbb{L}) = d^{2n}$, where d is the dimension of the local Hilbert space and n is the number of sites of a given cluster. In isolated systems, one would need to evaluate the ground state of the cluster Hamiltonian, of size $\dim(\hat{H}) = d^n$. Therefore, for the case of spin-1/2 systems (which have a local dimension $d = 2$), we are able to compute the steady state for clusters up to $n = 8$, such that $\dim(\mathbb{L}) = 2^{2 \times 8} = 65\,536$. The complexity of the problem is thus comparable to what has been done for spin systems at equilibrium, where the NLCE has been computed up to $n = 15$ (see, for example, Refs. [66,72]).

In graph theory, there are established algorithms to compute all topologically distinct clusters for a given size and lattice geometry. This could drastically increase the efficiency of the NLCE algorithm, since for highly symmetric systems the number of topologically distinct clusters is exponentially smaller than the total number of connected clusters. Explaining how to optimize the cluster generation lies beyond the scope of the present work. The basic cluster generation scheme we used is explained in full detail in Ref. [72]. Notice that once all the topologically distinct n -site clusters and their multiplicities have been generated for a given lattice geometry, one can employ NLCE for any observable and Liouvillian within the same spatial symmetry class of the considered lattice.

A remarkable advantage of NLCE over other numerical methods is that it enables us to directly approach the thermodynamic limit by only evaluating the contributions of a finite number of clusters (i.e., using a limited amount of resources). We should stress that, contrary to standard perturbative expansions, there is no perturbative parameter in the system upon which the NLCE is based and can be controlled. Strictly speaking, the actual control parameter is given by the typical length scale of correlations that are present in the system: the convergence of the series (7) with n would be ensured from an order R^* that is larger than the typical length scale of correlations [64,72].

In the next sections, we give two illustrative examples of how NLCE performs for two-dimensional (2D) dissipative quantum lattice models of interacting spin-1/2 particles.

III. MODEL

Our model of interest is a spin-1/2 lattice system in two dimensions, whose coherent internal dynamics is governed by

the anisotropic XYZ-Heisenberg Hamiltonian:

$$\hat{H} = \sum_{\langle i,j \rangle} (J_x \hat{\sigma}_i^x \hat{\sigma}_j^x + J_y \hat{\sigma}_i^y \hat{\sigma}_j^y + J_z \hat{\sigma}_i^z \hat{\sigma}_j^z), \quad (8)$$

where $\hat{\sigma}_j^\beta$ ($\beta = x, y, z$) denote the Pauli matrices for the j th spin of the system, and $\langle i, j \rangle$ restricts the summation over all couples of nearest-neighboring spins. Each spin is subject to an incoherent dissipative process that tends to flip it down along the z direction in an independent way with respect to all the other spins. In the Markovian approximation, such a mechanism is faithfully described by the Lindblad jump operator $\hat{L}_j = \sqrt{\Gamma} \hat{\sigma}_j^-$ acting on each spin:

$$\mathbb{D}[\rho] = \Gamma \sum_j \left[\hat{\sigma}_j^- \rho \hat{\sigma}_j^+ - \frac{1}{2} \{ \hat{\sigma}_j^+ \hat{\sigma}_j^-, \rho \} \right], \quad (9)$$

where $\hat{\sigma}_j^\pm = \frac{1}{2}(\hat{\sigma}_j^x \pm i \hat{\sigma}_j^y)$ stands for the corresponding raising and lowering operator along the z axis, while Γ is the rate of the dissipative processes. In the following, we will always work in units of Γ .

The outlined model is particularly relevant as it is considered a prototypical dissipative quantum many-body system: its phase diagram is very rich and has been subject to a number of studies at the mean-field level [32] and even beyond such a regime by means of the cluster mean-field [53], the corner-space renormalization group [39], and the dissipative PEPS [58]. Remarkably, the Lindblad master equation with the Hamiltonian in Eq. (8) and the dissipator in Eq. (9) presents a \mathbb{Z}_2 symmetry that is associated with a π rotation along the z axis: $\hat{\sigma}^x \rightarrow -\hat{\sigma}^x$, $\hat{\sigma}^y \rightarrow -\hat{\sigma}^y$. For certain values of the couplings J_α , it is possible to break up this symmetry, thus leading to a dissipative phase transition from a paramagnetic (PM) to a ferromagnetic (FM) phase, the order parameter being the in-plane xy magnetization. We stress that an XY anisotropy ($J_x \neq J_y$) is necessary to counteract the incoherent spin flips, otherwise the steady-state solution of Eq. (8) would be perfectly polarized, with all the spins pointing down along the z direction.

The existing literature allows us to benchmark our approach, both far from criticality (Sec. III A) where correlations grow in a controllable way, and in proximity to a \mathbb{Z}_2 -symmetry-breaking phase transition (Sec. III B), where correlations diverge in the thermodynamic limit. In the latter, we show how it is possible to exploit the NLCE method in combination with a Padé approximants analysis in order to calculate the location of the critical point as well as the critical exponent γ of the transition, which is associated with a power-law divergence of the magnetic susceptibility to an external field. Contrary to all the other known methods, either being mean-field or dealing with finite-length systems, the NLCE directly addresses the thermodynamic limit, and thus, to the best of our knowledge, at present it represents the only unbiased numerical method to calculate such an exponent.

A. Isotropic case

Let us start our analysis by considering a cut in the parameter space that does not cross any critical line. Specifically, we set

$$\alpha = J_x = -J_y = J_z. \quad (10)$$

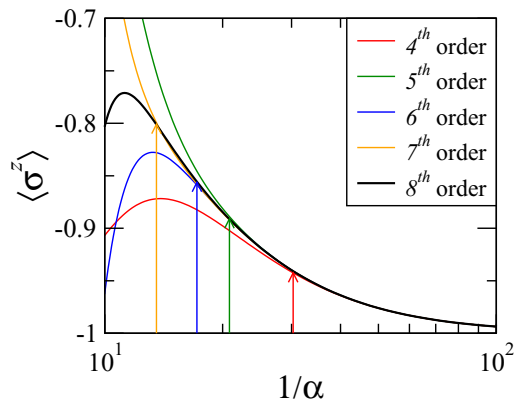


FIG. 1. Steady-state average magnetization along the z direction for the isotropic Heisenberg model, evaluated by means of the NLCE (bare sum) at different orders R in the cluster size, as a function of $1/\alpha$. The arrows indicate the values of α^* at which each curve at the R th order ($R < 8$) starts deviating significantly from the highest accuracy curve ($R = 8$; thick black line) that we have.

For $\alpha = 0$, the coherent dynamics is switched off, the coupling in the x - y plane is thus isotropic, and the dissipative processes cannot be counteracted regardless of the value of the local relaxation rates [32]. As a consequence, regardless of the initial conditions, the steady state is the pure state having all spins pointing down along the z -axis:

$$\rho_{\text{ss}}|_{\alpha=0} = \bigotimes_i |\downarrow\rangle\langle\downarrow|. \quad (11)$$

Thus we expect the NLCE would give the exact thermodynamic limit already at first order in the cluster size. As the parameter α is increased, correlations progressively build up on top of the fully factorizable density matrix (11), therefore higher orders in the expansion of Eq. (7) are needed.

This is exactly what we observe in Fig. 1, where we show the steady-state value of the average magnetization along the z direction, $O/L = \langle \hat{\sigma}_j^z \rangle$, evaluated by means of the NLCE in Eq. (7) up to a given order R as function of α . Note that, as long as R is increased, the convergence of the NLCE to the most accurate data (highest order that we have) progressively improves. This shows that, in the region where different curves overlap, correlations among the different sites are well captured by the clusters that we are considering in the expansion, up to a given order. When α is increased, the range of correlations grows as well, and one needs to perform the expansion to larger orders. For $\alpha \gtrsim 0.075$, orders higher than $R = 8$ are needed to obtain a good convergence in the bare data.

It is possible, however, to improve the convergence of the expansion without increasing the size of the considered clusters by simply exploiting two resummation algorithms that have already been shown to be very useful in the context of NLCEs of given thermodynamic properties [64,72]. Specifically we employ Wynn's algorithm [73] and the Euler transformation [74,75]. A detailed explanation of how such resummation schemes can be exploited in the context of NLCE can be found in Ref. [72].

The results for $\langle \hat{\sigma}^z \rangle$ as a function of α are shown in Fig. 2 for various orders in the two resummation schemes

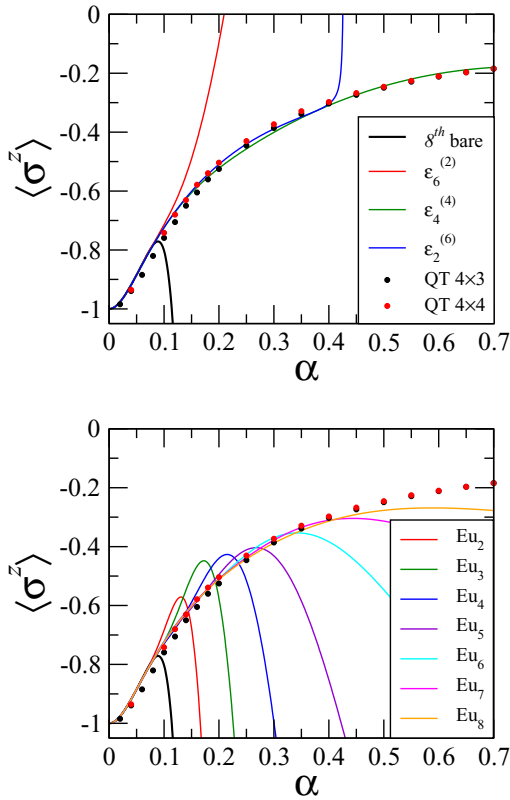


FIG. 2. Steady-state average z -magnetization as a function of α , after implementing two resummation techniques on the bare data at order $R = 8$ (black curve, same as in Fig. 1): Wynn's algorithm (colored curves in the upper panel) and Euler transformation (colored curves in the lower panel). The symbols denote the results of QT simulations for a finite system using periodic boundary conditions, with a 4×4 square plaquette (red circles) and a 4×3 plaquette (black circles) constructed from the previous one after removing the four sites at the corners. Note that quite counterintuitively, in the case of Wynn's algorithm, $\epsilon_4^{(4)}$ seems to converge for larger values of α , with respect to $\epsilon_2^{(6)}$.

(see the legends for details). It can be seen immediately that the convergence of the expansion is drastically improved of about one order of magnitude. A comparison of NLCEs data with the outcome of simulations obtained by means of quantum trajectories (QTs) [76] for finite-size plaquettes shows that the resummed data give qualitatively analogous results up to $\alpha \approx 0.7$ for Wynn's algorithm and $\alpha \approx 0.4$ for the Euler transformation, despite a slight discrepancy between them. Such a difference is due to the fact that, even if for small α correlations are very small, finite-system effects are non-negligible: while NLCE data are directly obtained in the thermodynamic limit, QTs are inevitably affected by such effects. As long as α is decreased, the discrepancy between the two approaches decreases, both leading to $\langle \hat{\sigma}^z \rangle \rightarrow -1$ in the limit $\alpha \rightarrow 0$ of Eq. (11).

B. Anisotropic case and the paramagnetic-to-ferromagnetic phase transition

We now discuss the more interesting scenario of an anisotropic Heisenberg model ($J_x \neq J_y \neq J_z$), where the sys-

tem can cross a critical line and exhibit a dissipative phase transition [32]. Toward that end, we set

$$J_x = 0.9, \quad J_y = 0.9 + \alpha, \quad J_z = 1, \quad (12)$$

with $\alpha \in [0, 0.25]$. For $\alpha = 0$ (i.e., $J_x = J_y$), we come back to the trivial situation in which the Hamiltonian conserves the magnetization along the z direction, and the steady state is the pure state in Eq. (11), with all the spins pointing down in the z direction. Away from this singular point, for a certain $\alpha_c > 0$ the system undergoes a second-order phase transition associated with the spontaneous breaking of the \mathbb{Z}_2 symmetry possessed by the master equation (1), from a paramagnetic (PM) for $\alpha < \alpha_c$ to a ferromagnetic (FM) phase for $\alpha > \alpha_c$. In the FM phase, a finite magnetization in the x - y plane develops: $\langle \hat{\sigma}^x \rangle, \langle \hat{\sigma}^y \rangle \neq 0$, which also defines the order parameter of the transition.

The phenomenology of this phase transition has recently received a lot of attention, and it has been investigated at a Gutzwiller mean-field level [32] and by means of more sophisticated methods, including the cluster mean-field approach [53], the corner-space renormalization technique [39], and the projected entangled pair operators [58]. The phase-transition point for the same choice of parameters of Eq. (12) has been estimated to be $\alpha_c = 0.1$ [32], 0.14 ± 0.01 [53], and 0.17 ± 0.02 [39].

Here we follow the approach of Rota *et al.* [39] and discuss the magnetic linear response to an applied magnetic field in the x - y plane, which modifies the Hamiltonian in Eq. (8) according to

$$\hat{H} \rightarrow \hat{H} + \sum_j h(\hat{\sigma}_j^x \cos \theta + \hat{\sigma}_j^y \sin \theta), \quad (13)$$

where θ denotes the field direction, $[\vec{h}(\theta)] = (h_x, h_y)^T$, and $h_x = h \cos \theta$, $h_y = h \sin \theta$. Such a response is well captured by the susceptibility tensor χ , with matrix elements $\chi_{\alpha\beta} = \lim_{h_\beta \rightarrow 0} \langle \hat{\sigma}^\alpha \rangle / h_\beta$. In particular, we concentrate on the angularly averaged magnetic susceptibility

$$\chi_{av} = \lim_{h \rightarrow 0} \frac{1}{2\pi} \int_0^{2\pi} d\theta \frac{|\vec{M}(\theta)|}{h}, \quad (14)$$

where $\vec{M}(\theta) = \chi \cdot \vec{h}(\theta)$ is the induced magnetization along an arbitrary direction of the field.

We start by computing the NLCE for the magnetic susceptibility χ_{av} in the parameter range $0 \leq \alpha \leq 0.25$, and improving the convergence of the series up to a given order, by exploiting the Euler algorithm. Along this specific cut in the parameter space, the latter has been proven to perform better (contrary to what we observed far from criticality; see Fig. 2). The relevant numerical data are shown in Fig. 3, and they are put in direct comparison with those obtained with an alternative method (the corner-space renormalization group) in Ref. [39]. We observe fairly good agreement with the two approaches, in the small- α parameter range ($0 \leq \alpha \lesssim 0.02$), and we point out that in both cases a sudden increase of χ_{av} for $\alpha \gtrsim 0.1$ supports the presence of a phase transition in that region. It is important to remark that the result of the expansion at different orders beyond the shaded region $\alpha \gtrsim 0.02$ has no physical meaning. However, as we will show in the next section, by analyzing how the expansion behaves when approaching criticality, it is

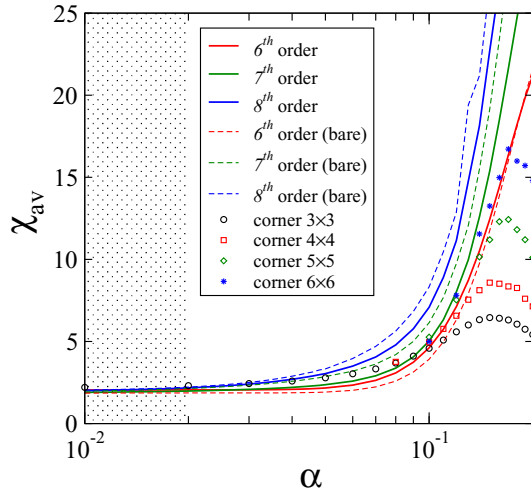


FIG. 3. Angularly averaged magnetic susceptibility to an external field in the x - y plane, as a function of $\alpha = J_y - 0.9$. The continuous curves denote Euler resummed data, to the best achievable expansion, of the bare NLCE results up to the order $R = 8$. The dashed lines are the results of the bare expansions. Symbols are the results from the corner-space renormalization method, taken from Ref. [39]. The dotted area highlights the region of α $0 \leq \alpha \leq 0.02$, for which the NLCE converges.

possible to provide an estimate of the critical point α_c , as well as of the critical exponent γ . We also note that, contrary to the isotropic case, here we do not observe an *exact* data collapse of the NLCEs for χ_{av} , even for $\alpha = 0$. The reason resides in the fact that the presence of an external field (13) makes the structure of the steady state nontrivial as soon as $h \neq 0$, thus allowing correlations to set in.

1. Critical behavior

We now show how to exploit the above NLCE data (in combination with a Padé analysis [62]) in order to locate the critical point α_c for the PM-FM transition, and extract the critical exponent γ of the magnetic susceptibility [77] $\chi_{av} \sim |\alpha - \alpha_c|^{-\gamma}$. The possibility to extrapolate the critical exponents for a dissipative quantum phase transition is very intriguing, since, to the best of our knowledge, the only numerical work in this context that is present in the literature is Ref. [39]. However, since finite-size systems are considered there, it was only possible to estimate the finite-size ratio γ/ν , where ν denotes the critical exponent associated with the divergent behavior of the correlation length. The present work offers a complementary point of view since here we are able to provide an independent estimate of the critical exponent γ by directly accessing the thermodynamic limit.

To achieve this goal, we study the logarithmic derivative of the averaged magnetic susceptibility, which converts an algebraic singularity into a simple pole [62]:

$$\text{Dlog } \chi_{av}(\alpha) \equiv \frac{\chi'_{av}(\alpha)}{\chi_{av}(\alpha)}. \quad (15)$$

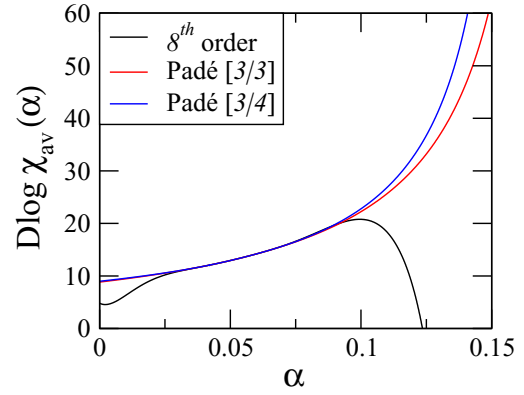


FIG. 4. Logarithmic derivative of χ_{av} as a function of α . The black line is obtained from Euler resummed data to the order $R = 8$ (blue line of Fig. 3). The red and blue lines are the results of the Padé analysis with different [3/3] and [3/4] approximants, respectively.

If $\chi_{av} \sim |\alpha - \alpha_c|^{-\gamma}$ for $|\alpha - \alpha_c| \ll 1$, the logarithmic derivative behaves as

$$\text{Dlog } \chi_{av}(\alpha) \sim \frac{\gamma}{|\alpha - \alpha_c|}. \quad (16)$$

Studying the divergent behavior of Eq. (16) simplifies the problem, since the function $\text{Dlog } \chi_{av}(\alpha)$ has a simple pole at the critical point $\alpha = \alpha_c$ with a residue corresponding to the critical exponent γ .

In Fig. 4 we show the behavior of the logarithmic derivative calculated from the Euler resummed data to the order $R = 8$ (blue line in Fig. 3), which represents our best approximation for χ_{av} at small α . The behavior at large α of the function $\text{Dlog } \chi_{av}(\alpha)$ is extrapolated exploiting the Padé approximants. A Padé approximant is a representation of a finite power series as a ratio of two polynomials,

$$\text{Dlog } \chi_{av}(\alpha) = \sum_{n=0}^R a_n \alpha^n = \frac{P_L(\alpha)}{Q_M(\alpha)}, \quad (17)$$

where $P_L(\alpha)$ and $Q_M(\alpha)$ are polynomials of degree L and M (with $L + M \leq R$), respectively. This is denoted as the $[L/M]$ approximant, and it can represent functions with simple poles *exactly*. Next, we fit $\text{Dlog } \chi_{av}(\alpha)$ (black line in Fig. 4) with an eighth-degree polynomial between $\alpha_{in} = 0.05$ and $0.06 \leq \alpha_{fin} \leq 0.1$ in order to obtain the coefficients $\{a_n\}_{n=1, \dots, R}$ (with $R = 8$). Once the coefficients $\{a_n\}$ are known, it is straightforward to evaluate the coefficients of the polynomials P_L and Q_M through Eq. (17). Further details about this procedure can be found in Appendix. As is clear from Eq. (17), the position of the critical point α_c can be deduced by studying the zeros of $Q_M(\alpha)$. Typically, only one of the M zeros is real and located in the region of interest. Finally, the critical exponent is evaluated by computing the residue of $Q_M(\alpha)$ at $\alpha = \alpha_c$:

$$\gamma = - \lim_{\alpha \rightarrow \alpha_c} Q_M(\alpha)(\alpha - \alpha_c). \quad (18)$$

Of course, the values of α_c and γ will depend on the specific choice of the approximants $[L, M]$ and on the region over which the fit is performed. The dependence of the results on α_{fin} is shown in Appendix. We found that the Padé analysis gives

stable results for $0.06 \lesssim \alpha_{\text{fin}} \lesssim 0.08$ and $0.06 \lesssim \alpha_{\text{fin}} \lesssim 0.095$ for the approximants [3|3] and [3|4], respectively.

The results of the Padé analysis hint at a divergence at $\alpha_c = 0.179 \pm 0.001$ with $\gamma = 1.85 \pm 0.05$ for [3|3], and $\alpha_c = 0.1665 \pm 0.0005$ with $\gamma = 1.5 \pm 0.05$ for [3|4]. The other approximants $[L|M]$ such that $L + M \leq R = 8$ do not give physical results in this range of parameters. The error bar is underestimated, since it accounts only for the error introduced in the fitting procedure and neglects the propagation of the numerical error made on the steady-state evaluation. Furthermore, the Padé analysis has been performed over a range of α for which the resummed NLCE is not exactly converged (see Fig. 3). Indeed, by performing the same kind of analysis on the Euler resummed data for $R = 7$ in the expansion, we estimate that the error due to the lack of convergence of the series affects the second significant digit of α_c . In the region considered for the fit, we do not get stable results for the exponent γ , making it difficult to estimate the impact of this source of error on this quantity. To overcome this issue, one should be able to compute higher orders in the expansion and to perform a more accurate analysis of the criticality. However, the value of the critical point we found is in agreement with the results reported in Refs. [53] and [39] as of now.

IV. CONCLUSIONS

In this work, we have proposed a numerical algorithm based on the generalization of the linked-cluster expansion to open quantum systems on a lattice, allowing us to access directly the thermodynamic limit and to evaluate the extensive properties of the system. Specifically, we extended the formalism to the Liouvillian case, and we showed how the basic properties of the expansion are translated to the open-system realm. Given its generality, this method can be applied to open fermionic, bosonic, and spin systems in an arbitrary lattice geometry.

We tested our approach with a study of the steady-state properties of the paradigmatic dissipative spin-1/2 XYZ model on a two-dimensional square lattice. Far away from the critical boundaries of the model, we accurately computed the spin magnetization. Upon increasing the order of the expansion, we were able to progressively access regions of the phase diagram that are characterized by a larger amount of correlations among distant sites. The convergence properties of the expansion can be dramatically improved by employing more sophisticated resummation schemes. We then used the numerical linked-cluster expansion across a phase transition in order to study its critical properties. By means of a Padé analysis of the series, we located the critical point and provided an estimate of the critical exponent γ , which determines the divergent behavior of the (average) magnetic susceptibility close to the phase transition.

This method, together with that of Ref. [58], is a (non-mean-field) numerical approach that allows us to compute the steady-state properties of an open lattice model in two spatial dimensions in the thermodynamic limit. Here the intrinsic limitation is that, in order to compute high-order terms in the expansion (and thus to access strongly correlated regions of phase space), an evaluation of the steady state on a large number of connected sites is required. Furthermore, in the case of bosonic systems, a further complication arises from the local Hilbert space dimension. We believe that a very

interesting perspective left for the future is the combination of the linked-cluster expansion with the corner-space renormalization method [39], and also possibly with Monte Carlo approaches [78]. Additionally, a careful identification of the internal symmetries of the model may help in decreasing the effective dimension of the Liouvillian space.

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APPENDIX: PADÉ APPROXIMANTS

Here we discuss the details related to the Padé analysis of the divergent behavior of the magnetic susceptibility, which

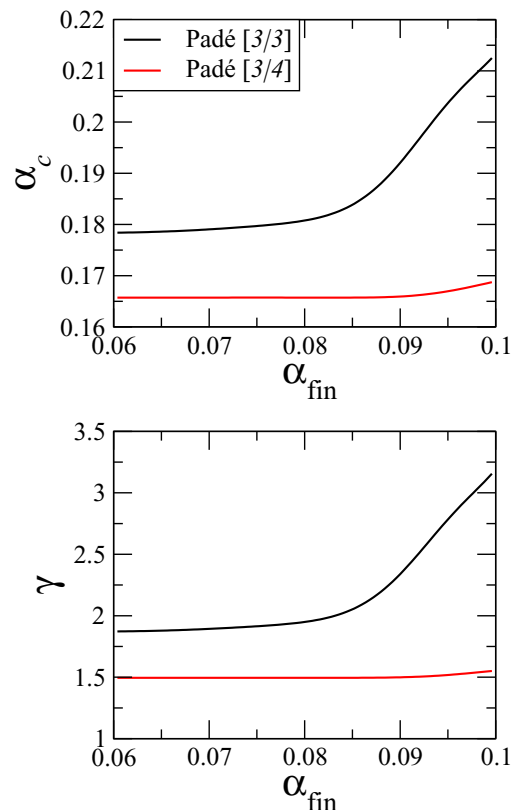


FIG. 5. Position of the critical point α_c (top panel) and value of the critical exponent γ (bottom panel) as a function of the upper boundary of the fitting region α_{fin} .

has been performed in Sec. III B 1. As already introduced in the main text, the Padé approximant is a representation of the first R terms of a power series as a ratio of two polynomials.

Let us consider Eq. (17), where $\text{Dlog}\chi_{\text{av}}(\alpha)$ is the function for which we know the Taylor expansion up to the order R ,

$$\text{Dlog}\chi_{\text{av}}(\alpha) = \sum_{n=0}^R a_n \alpha^n = \frac{P_L(\alpha)}{Q_M(\alpha)}, \quad (\text{A1})$$

and the Padé polynomials are parametrized as follows:

$$P_L(\alpha) = \sum_{n=0}^L p_n \alpha^n, \quad Q_M(\alpha) = 1 + \sum_{n=1}^M q_n \alpha^n, \quad (\text{A2})$$

with $L + M \leq R$. This is denoted as the $[L|M]$ approximant.

Let us start by showing that if the function $\chi_{\text{av}}(\alpha)$ has an algebraic singularity at $\alpha = \alpha_c$, then its logarithmic derivative [see Eq. (15)] has a simple pole at the same value of α . To show this, let us note that for $|\alpha - \alpha_c| \ll 1$,

$$\chi_{\text{av}}(\alpha) = \frac{g(\alpha)}{|\alpha - \alpha_c|^\gamma}, \quad (\text{A3})$$

where $g(\alpha)$ is an analytic function in the range of α in which we are interested. Therefore, Eq. (A3) becomes

$$\text{Dlog}\chi_{\text{av}}(\alpha) = \frac{g'(\alpha)}{g(\alpha)} - \frac{\gamma}{|\alpha - \alpha_c|}. \quad (\text{A4})$$

Given the coefficients $\{a_n\}$ (calculated by fitting the function $\text{Dlog}\chi_{\text{av}}(\alpha)$ with an eighth-degree polynomial from $\alpha_{\text{in}} \leq \alpha \leq \alpha_{\text{fin}}$), it is easy to obtain the coefficients $\{p_n\}$ and $\{q_n\}$ in Eq. (A2) exploiting Eq. (A1). This gives the following set of $L + M + 1$ linear equations:

$$\begin{aligned} a_0 &= p_0, \\ a_1 + a_0 q_1 &= p_1, \\ a_2 + a_1 q_1 + a_0 q_2 &= p_2, \\ &\vdots \\ a_L + a_{L-1} q_1 + \cdots + a_0 q_L &= p_L, \\ a_{L+1} + a_L q_1 + \cdots + a_{L-M+1} q_M &= 0, \\ &\vdots \\ a_{L+M} + a_{L+M-1} q_1 + \cdots + a_L q_M &= 0. \end{aligned} \quad (\text{A5})$$

Once the coefficients $\{p_n\}$ and $\{q_n\}$ have been determined, one can calculate α_c by studying the zeros of $Q_M(\alpha)$ and compute the critical exponent γ by evaluating the residue at $\alpha = \alpha_c$ (see Sec. III B 1). In Fig. 5, we show the position of the critical point α_c (top panel) and the value of the critical exponent γ (bottom panel) as a function of the upper fit boundary α_{fin} for $\alpha_{\text{in}} = 0.05$.

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