Observer-based offset-free internal model control

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Abstract: A linear feedback control structure is proposed that allows internal model control design principles to be applied to unstable and marginally stable plants. The control structure comprises an observer using an augmented plant model, state estimate feedback and disturbance estimate feedback. Conditions are given for both nominal internal stability and offset-free action even in the case of plant-model mismatch. The Youla parameterization is recovered as a limiting case with reduced order observers. The simple design methodology is illustrated for a marginally stable plant with delay.

Keywords: Control system design, disturbance rejection, stabilizing controllers, state estimation, coprime factorization.

1. INTRODUCTION

Integral action is an essential component of many feedback control systems. It is used to ensure zero steady-state error (so-called "offset-free" action) in the face of setpoint demands and plant disturbances. There are many ways to ensure integral action in control design. For linear control design (for example $\mathcal{H}_\infty$ design) integral action is usually achieved via interpolation zeros (Green and Limebeer, 1995). Meanwhile there has been considerable recent interest in the use of disturbance estimates to achieve offset-free control in model predictive control (Pannocchia, 2015). In this work we consider how such disturbance estimates may be associated with internal model control (IMC) and the Youla parameterization.

The Youla parameter in state space is expressed as Q acting on the "innovations" (i.e. on $y - C\hat{x}$ where y is the plant output, $\hat{x}$ a state estimate and C is from the state-space model) which can be considered as an estimate of a plant output disturbance. This gives inherent offset-free action for output step disturbances. With open loop stable plants this also translates to input step disturbances and hence the simple tuning rules associated with IMC. However these simple rules may fail for integrating and open-loop unstable plants. We propose a more general control structure where disturbance estimates derived from an augmented plant model are used for feedback control. These in turn translate into generalizations of IMC and simple tuning rules for a wider class of plants.

The paper is organized as follows. In Section 2 we discuss the relevant literature on IMC, the Youla parameterization and offset-free design for model predictive control. In Section 3 we propose our control structure and show both that it is nominally stabilizing and that it ensures offset-free control even in the face of plant-model mismatch. In Section 4 we discuss its close relation with the Youla parameterization, which can be recovered as a special case with a reduced order observer. We illustrate the method for an integrating plant with delay in Section 5. Finally in the Conclusions (Section 6) we discuss possible extensions and the further work necessary for this to become a viable control design method for large scale plants.

2. RELATED WORK

2.1 IMC fundamentals and design

Internal model control (IMC) was introduced by Garcia and Morari (1982), "originally as a way of understanding predictive control" (Maciejowski, 2001). In fact it has more in common with linear control design: as a design method in its own right, with its close relation to direct synthesis, to the Smith predictor and Dahlin controller for systems with delays, and as a special case of the Youla parameterization. There is a comprehensive treatment by Morari and Zafiriou (1989). It has found widespread industrial application; for example, Gayadeen and Duncan (2016) report a recent large scale implementation.

The control stucture is shown in Fig. 1. If the plant model is $P(s)$, the control input is calculated as

$$u = Qr - Q(y - Pu), \tag{1}$$

where r is the setpoint and y the plant output. A natural interpretation is that the feedback path includes an estimate of additive disturbance at the plant output. Specifically:

$$\hat{d} = y - Pu. \tag{2}$$

If the plant model $P(s)$ is open-loop stable then it admits a simple design procedure. Specifically, for this case, the control structure in Fig. 1 is nominally stable if and only if $Q(s)$ is stable. Furthermore, both the nominal loop complementary sensitivity and the nominal setpoint-to-output response are given by $P(s)Q(s)$. Thus if $P(s)$ is factorized as $P(s) = P_1(s)P_2(s)$ where $P_2(s)$ is minimum phase (for example via an inner-outer factorization) then $Q(s)$ can be set to $Q(s) = P_2(s)^{-1}F(s)$ for some appropriate stable $F(s)$ which can be shaped to give a suitable nominal response. In particular, offset-free control is guaranteed if

$$P(0)Q(0) = I. \tag{3}$$

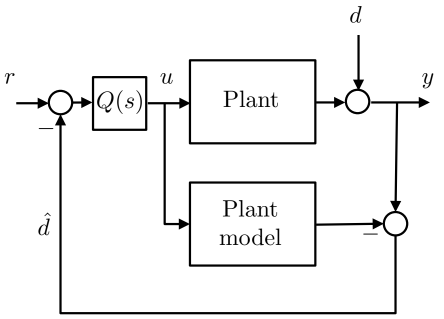

Fig. 1. Internal model control structure.

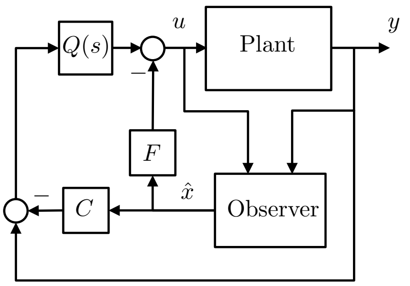

Fig. 2. Structure for Youla parameterization; setpoint omittted.

However, such a simple design procedure cannot be applied if the plant model has poles on the imaginary axis or in the right half plane (Morari and Zafiriou, 1989; Goodwin et al., 2001). In particular, neither internal stability nor offset-free action are guaranteed without further design constraints. As an example, Lee et al. (2000) propose PID tuning rules based on IMC design for integrating and unstable processes; however the design requires careful prescription of zeros and, if the plant is integrating, the model must be perturbed. Similar considerations are required when generalising the Smith predictor to integrating and unstable plants (Majhi and Atherton, 1998).

2.2 *Youla parameterization*

The Youla parameterization, named after Youla et al. (1976), gives a parameterization of all stabilizing controllers. Its structure is shown in Fig. 2; the control is given as

$$u = -F\hat{x} + Q(s)(y - C\hat{x}). \tag{4}$$

In particular, if $u = -Fx$ is a stabilizing state-feedback control and $\hat{x}$ is the state estimate of a stable observer then the closed-loop system is stable if and only if $Q(s)$ is stable. In this form, it is probably best attributed to Desoer et al. (1980). It is a mainstay of linear multivariable control design, and in particular the development of $\mathscr{H}_\infty$ design (Maciejowski, 1989; Green and Limebeer, 1995). Kučera (2011) gives a recent overview and historical perspective.

IMC is well-known to be a special case of the Youla parameterization when the plant is stable. However, when IMC is generalized to the Youla parameterization the nice design features of IMC are usually lost. In particular the literature appears to have no simple generalization from IMC to the Youla parameterization of the design rule (3) to ensure offset-free control action.

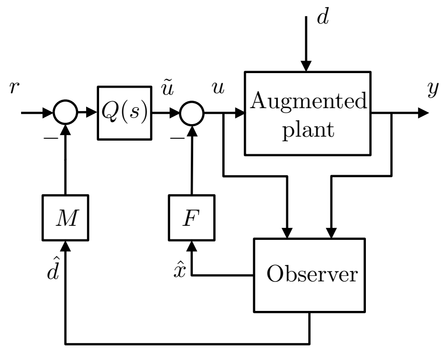

Fig. 3. Proposed control structure.

2.3 *Offset-free MPC principles*

The goal of offset-free MPC is to track (admissible) setpoints despite possible unknown disturbances and/or plant/model mismatch. The general approach is to augment the nominal model with additional states d, referred to as "disturbances", following integral dynamics ($\dot{d} = 0$) which are then estimated along with the original states from the measured outputs. The use of such augmented models can be traced back to Johnson (1971) and Wonham (1979). These disturbances are not controllable but their effect is taken into account to update the nominal model in a way that it is consistent in steady-state with the true unknown plant, i.e. for a given equilibrium input u_s the corresponding outputs of the true process and of the augmented model are equal.

The following result summarizes the offset-free property (Muske and Badgwell, 2002; Pannocchia and Rawlings, 2003; Maeder et al., 2009).

Proposition 1. If the closed-loop system reaches an equilibrium, i.e. $\lim_{t\to\infty} y(t) = y_\infty$ and $\lim_{t\to\infty} u(t) = u_\infty$, then it follows that $y_\infty = r$.

We note that Proposition 1 makes no specific assumption on the actual plant dynamics: as long as the closed-loop system reaches a steady state, the output tracks the desired setpoint r.

For the sake of brevity we omit a detailed description of offset-free linear MPC, but the interested reader is referred to a recent comprehensive review on this topic (Pannocchia, 2015).

3. PROPOSED OBSERVER BASED IMC

3.1 *Control structure*

The proposed control structure is illustrated in Fig 3. Let the plant model have dynamics

$$P(s) = C(sI - A)^{-1}B, \tag{5}$$

with $A \in \mathbb{R}^{n_x \times n_x}$, $B \in \mathbb{R}^{n_x \times n_u}$, $C \in \mathbb{R}^{n_y \times n_x}$.

Assumption 2. The pair (A,B) is controllable (stabilizable), the pair (C,A) observable (detectable) and

$$\text{rank}\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = n_x + n_y. \tag{6}$$

Remark 3. Condition (6), as thoroughly discussed in (Davison and Smith, 1974; Smith and Davison, 1972), implies that for any given reference $r \in \mathbb{R}^{n_y}$, there exists an equilibrium input and state pair (x,u) such that the corresponding output is equal to r, i.e. the following system has a solution:

$$
\begin{aligned}
0 &= Ax + Bu, \\
r &= Cx.
\end{aligned} \tag{7}
$$

Clearly, it implies that $n_y \le n_u$. Furthermore, the frequency interpretation of (6) is that the system does not have any transmission zero at the origin.

In order to include integral action in the proposed control, following the same approach used in offset-free MPC design, we consider an augmented plant model:

$$
\begin{aligned}
\dot{x} &= Ax + Bu + B_d d, \\
\dot{d} &= 0, \\
y &= Cx + C_d d,
\end{aligned} \tag{8}
$$

with $B_d \in \mathbb{R}^{n_x \times n_d}$, and $C_d \in \mathbb{R}^{n_y \times n_d}$.

Assumption 4. The matrices (B_d, C_d) are such that:

$$
\operatorname{rank} \begin{bmatrix} A & B_d \\ C & C_d \end{bmatrix} = n_x + n_d. \tag{9}
$$

Proposition 5. The augmented model (8) is observable.

Remark 6. The proof of Proposition 5 follows that given by Pannocchia and Rawlings (2003) for discrete time. We note that as long as Assumption 4 is satisfied, the choice of (B_d, C_d) is arbitrary. A typical choice in industrial MPC algorithms is $B_d = 0$, $C_d = I$. However, if the nominal system has poles at the origin, it must be $B_d \neq 0$.

Condition (9) implies that $n_d \le n_y$ (Pannocchia and Rawlings, 2003). On the other hand, in order to achieve integral action in all outputs we require $n_d \ge n_y$. Coupling the two conditions we make the following assumption.

Assumption 7. $n_d = n_y$.

Let F be a stabilizing state feedback gain (so that $A - BF$ is Hurwitz) and let the state and disturbance estimates $\hat{x}$ and $\hat{d}$ be obtained from the stable observer

$$
\begin{aligned}
\dot{\hat{x}} &= A\hat{x} + B_d\hat{d} + Bu + L_x(y - C\hat{x} - C_d\hat{d}), \\
\dot{\hat{d}} &= L_d(y - C\hat{x} - C_d\hat{d}).
\end{aligned} \tag{10}
$$

The observer is stable if the matrix $A_a - L_a C_a$ is Hurwitz, with

$$
A_a = \begin{bmatrix} A & B_d \\ 0 & 0 \end{bmatrix}, L_a = \begin{bmatrix} L_x \\ L_d \end{bmatrix}, C_a = \begin{bmatrix} C & C_d \end{bmatrix}. \tag{11}
$$

The control is then given as

$$
u = \tilde{u} - F\hat{x} \quad \text{with} \quad \tilde{u} = Q(s)\left(r - M\hat{d}\right), \tag{12}
$$

where $Q(s) \sim \left[\begin{array}{c|c} A_Q & B_Q \\ \hline C_Q & D_Q \end{array}\right]$ is stable and $M \in \mathbb{R}^{n_y \times n_d}$.

3.2 Nominal stability and setpoint response

Suppose the plant itself evolves according to

$$
\begin{aligned}
\dot{x} &= Ax + Bu + \bar{B}_d d, \\
y &= Cx + \bar{C}_d d.
\end{aligned} \tag{13}
$$

That is to say the with the same A, B and C as (8) but with the pairs (B_d, C_d) and $(\bar{B}_d, \bar{C}_d)$ not necessarily equal. Nominal stability can be established.

Theorem 8. The closed-loop system formed by plant (13) and controller given by (12) with observer given in (10) is exponentially stable.

The response from setpoint r to output y is given as

$$
\begin{aligned}
y &= \begin{bmatrix} C & 0 & 0 & 0 \end{bmatrix} (sI - A_{CL})^{-1} \begin{bmatrix} BD_Q \\ B_Q \\ 0 \\ 0 \end{bmatrix} r, \\
&= C(sI - A + BF)^{-1}B\left(D_Q + C_Q(sI - A_Q)^{-1}B_Q\right) r \\
&= \tilde{P}(s)Q(s).
\end{aligned} \tag{14}
$$

where A_{CL} is a suitably defined block matrix, and $\tilde{P}(s)$ represents the stabilized plant model:

$$
\tilde{P}(s) = C(sI - A + BF)^{-1}B. \tag{15}
$$

It follows that we can shape the setpoint response by following standard IMC tuning rules but with the stabilized plant model $\tilde{P}(s)$ in place of the open-loop plant model $P(s)$.

As a consequence of the previous discussion, the following design assumption is made (recall that $n_y \le n_u$).

Assumption 9. The transfer function matrix $Q(s)$ can be, and is, designed such that $\tilde{P}(0)Q(0) = I$.

Then, the following result is easily established.

Proposition 10. In absence of disturbance, $d(t) = 0, \forall t$, then for any $r \in \mathbb{R}^{n_y}$ it follows that:

$$
\lim_{t \to \infty} y(t) = r. \tag{16}
$$

3.3 Offset-free properties

In the previous section we showed that in the nominal case and with a suitable choice of $Q(s)$ there is unity steady-state gain from setpoint to output. Here we give further conditions on M that ensure offset-free tracking even when there is plant/model mismatch or a persistent nonzero disturbance, under the assumption of closed-loop stability.

We first establish the following useful result.

Lemma 11. If the observer (10) is stable, then the matrix $L_d \in \mathbb{R}^{n_y \times n_y}$ is invertible.

For the next result, we also need to define the following (stable) transfer function matrix:

$$
\tilde{P}_d(s) = C(sI - A + BF)^{-1}B_d + C_d. \tag{17}
$$

Theorem 12. Let $M = \tilde{P}_d(0)$. Assume that the closed-loop system has reached an equilibrium, with input u and output y. It follows that $y = r$.

Remark 13. This shows that, independent of the plant dynamics, if the closed-loop system reaches an equilibrium the output has reached the setpoint. See Remark 18 below.

3.4 Equivalent compensator and nominal sensitivities

To obtain both the equivalent compensator and the nominal closed-loop sensitivities it is useful to consider factorizations of the plant and the controller when $Q(s) = 0$. The development is similar to standard treatment of the Youla parameterization (e.g. Green and Limebeer, 1995), albeit with an augmented observer.

The plant model P given by (5) has the right coprime factorization

$$
P = ND^{-1}, \tag{18}
$$

with

$$
\begin{bmatrix} N \\ D \end{bmatrix} \sim \left[\begin{array}{c|c} A - BF & B \\ \hline C & 0 \\ -F & I \end{array}\right]. \tag{19}
$$

Similarly P has the left coprime factorization

$$P = \tilde{D}^{-1}\tilde{N}, \tag{20}$$

with

$$[\, \tilde{N} \;\; \tilde{D} \,] \sim \left[\begin{array}{c|cc} A_a - L_a C_a & B_a & -L_a \\ \hline C_a & 0 & I \end{array}\right], \tag{21}$$

where A_a, L_a and C_a are given by (11) and

$$B_a = \begin{bmatrix} B \\ 0 \end{bmatrix}. \tag{22}$$

Define

$$F_a = [\, F \;\; 0 \,]. \tag{23}$$

Then the equivalent compensator when $Q(s) = 0$ can be written

$$K = Y^{-1}X \text{ and } K = \tilde{X}\tilde{Y}^{-1}, \tag{24}$$

with

$$[\, X \;\; Y \,] = \left[\begin{array}{c|cc} A_a - L_a C_a & L_a & B_a \\ \hline F_a & 0 & I \end{array}\right], \tag{25}$$

and

$$\begin{bmatrix} \tilde{X} \\ \tilde{Y} \end{bmatrix} \sim \left[\begin{array}{c|c} A_a - B_a F_a & L_a \\ \hline F_a & 0 \\ C_a & I \end{array}\right]. \tag{26}$$

Remark 14. The matrix $A_a - L_a C_a$ is Hurwitz but $A_a - B_a F_a$ has eigenvalues at zero. So the expression $K = Y^{-1}X$ is a left coprime factorization over $\mathscr{H}_\infty$, but the expression $K = \tilde{X}\tilde{Y}^{-1}$ is not a right coprime factorization over $\mathscr{H}_\infty$.

We have the standard generalized Bezout equation

$$\begin{bmatrix} X & Y \\ \tilde{D} & -\tilde{N} \end{bmatrix} \begin{bmatrix} N & \tilde{Y} \\ D & -\tilde{X} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}. \tag{27}$$

We can write the control (12) as

$$u = Qr - F_a \begin{bmatrix} \hat{x} \\ \hat{d} \end{bmatrix} - QM [\, 0 \;\; I \,] \begin{bmatrix} \hat{x} \\ \hat{d} \end{bmatrix}, \tag{28}$$

where the state estimates are given as

$$\begin{bmatrix} \hat{x} \\ \hat{d} \end{bmatrix} = (sI - A_a + L_a C_a)^{-1}(B_a u + L_a y). \tag{29}$$

Hence, using (25) we can write

$$u = Y_Q^{-1}(Qr - X_Q y), \tag{30}$$

with

$$\begin{aligned} X_Q &= X + QM [\, 0 \;\; I \,] (sI - A_a + L_a C_a)^{-1} L_a, \\ Y_Q &= Y + QM [\, 0 \;\; I \,] (sI - A_a + L_a C_a)^{-1} B_a. \end{aligned} \tag{31}$$

Proposition 15. The equivalent compensator K_Q (in the sense that the loop transfer function can be written PK_Q) is

$$K_Q = Y_Q^{-1} X_Q. \tag{32}$$

Remark 16. With this notation

$$X_0 = X, Y_0 = Y \text{ and } K_0 = K. \tag{33}$$

Theorem 17. If Q and M are chosen such that $\tilde{P}(0)Q(0) = I$ and $M = \tilde{P}_d(0)$ then $X_Q(0) = Q(0)$ and $Y_Q(0) = 0$.

Remark 18. It follows that K_Q has an appropriate number of poles at the origin - i.e. the equivalent compensator has integral action. Compare Remark 13.

Proposition 19. The nominal closed-loop sensitivities are given as

$$\begin{aligned} \begin{bmatrix} I & K_Q \\ -P & I \end{bmatrix}^{-1} &= \begin{bmatrix} (I + K_Q P)^{-1} & -(I + K_Q P)^{-1} K_Q \\ (I + P K_Q)^{-1} P & (I + P K_Q)^{-1} \end{bmatrix}, \\ &= \begin{bmatrix} D Y_Q & -D X_Q \\ N Y_Q & I - N X_Q \end{bmatrix}. \end{aligned} \tag{34}$$

4. RELATION WITH THE YOULA PARAMETERIZATION

So far we have considered only full-order observers. Our analysis extends easily to reduced-order observers (e.g. Luenberger, 1966) if we consider them as limiting cases. We illustrate this with one particular structure which leads to the Youla parameterization as a special case.

Let F be a stabilizing state feedback as usual, set $B_d = L$ and $C_d = I$, and let the observer gains in (10) be $L_x = L$ and $L_d = \frac{1}{\delta} I$ with $A - LC$ Hurwitz and $\delta > 0$. Then, the observer has the following structure:

$$\begin{bmatrix} \dot{\hat{x}} \\ \dot{\hat{d}} \end{bmatrix} = \begin{bmatrix} A & L \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{d} \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u + \begin{bmatrix} L \\ \frac{1}{\delta} I \end{bmatrix} (y - C\hat{x} - \hat{d}). \tag{35}$$

We have

$$\begin{aligned} A_a - L_a C_a &= \begin{bmatrix} A & L \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} L \\ \frac{1}{\delta} I \end{bmatrix} [\, C \;\; I \,], \\ &= \begin{bmatrix} A - LC & 0 \\ -\frac{1}{\delta} C & -\frac{1}{\delta} I \end{bmatrix}, \end{aligned} \tag{36}$$

so $A_a - L_a C_a$ is Hurwitz. In the limit as $\delta \to 0$ we obtain the deadbeat disturbance observer

$$\hat{d} = y - C\hat{x}, \tag{37}$$

while the state-estimate becomes

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x}). \tag{38}$$

In this case our control structure corresponds to that of the Youla parameter (with $Q(s)M$ in place of $Q(s)$) and the results of Section 3 can be interpreted as ensuring that a control structure with the Youla parameterization yields offset-free control.

Proposition 20. Suppose a control structure is implemented with the Youla parameterization (with $Q(s)M$ in place of $Q(s)$) as

$$u = -F\hat{x} + Q(s)r - Q(s)M(y - C\hat{x}), \tag{39}$$

with $\hat{x}$ obtained from (38), with $Q(s)$ stable and both $A - BF$ and $A - LC$ Hurwitz. Then offset-free control is achieved provided

$$-C(A - BF)^{-1} B Q(0) = I, \tag{40}$$

and

$$M = I - C(A - BF)^{-1} L. \tag{41}$$

Remark 21. Proposition 20 is a natural generalization of internal model control tuning rules to the Youla parameterization. Specifically, if the plant is open-loop stable we may set $F = 0$ and $L = 0$ and the structure reduces to standard internal model control (as is well-known). In this case (40) and (41) become

$$-CA^{-1} B Q(0) = I, \text{ i.e. } P(0)Q(0) = I, \tag{42}$$

and

$$M = I, \tag{43}$$

respectively.

5. ILLUSTRATIVE EXAMPLE

To illustrate the method, consider a simple single-input single-output example with delay and a pole at the origin. Suppose the plant model is

$$P(s) = \frac{g}{s(s + a)} e^{-s\tau}, \tag{44}$$

with g, a and τ all positve.

Remark 22. The delay takes us outside the scope of the development in the previous two sections. Nevertheless it is straightforward to include it; in the following we will consider it as an output delay, and replace the matrix C with $Ce^{-s\tau}$ where appropriate.

It is, of course, straightforward to design a compensator for such a plant model using classical methods. But it is well-known that attempting to design a compensator using the internal model control tuning rules for open-loop stable plants would fail because of the pole at the origin. Specifically, suppose the desired closed-loop setpoint response is

$$T_{ry}(s) = \frac{1}{(\lambda s+1)^2}e^{-s\tau}. \tag{45}$$

with $\lambda > 0$. We obtain the IMC transfer function

$$Q_{IMC}(s) = \frac{s(s+a)}{g(\lambda s+1)^2}. \tag{46}$$

The corresponding compensator is

$$\begin{aligned} K_{IMC}(s) &= \frac{Q_{IMC}(s)}{1-P(s)Q_{IMC}(s)}, \\ &= \frac{s(s+a)}{g\left((\lambda s+1)^2 - e^{-s\tau}\right)}. \end{aligned} \tag{47}$$

We find

$$\lim_{s\to 0} K_{IMC}(s) = \frac{a}{g(2\lambda+\tau)}, \tag{48}$$

and hence the compensator does not include integral action. As a result, offset-free control is not guaranteed with input disturbances. Furthermore, the internal model control structure is not internally stable with integrating or unstable plant models.

By contrast, using the proposed methodology, we can design and implement a compensator as follows. First represent the plant model in state space as

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y(t) &= Cx(t-\tau), \end{aligned} \tag{49}$$

with

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -a \end{bmatrix}, B = \begin{bmatrix} 0 \\ g \end{bmatrix} \text{ and } C = [\,1 \;\; 0\,]. \tag{50}$$

As we have chosen to model the delay at the output the stabilizing state feedback F can be chosen simply by ensuring $A - BF$ is Hurwitz. For example, the choice

$$F = \frac{1}{g}\left[\, a^2 \;\; a \,\right] \tag{51}$$

puts both eigenvalues of $A - BF$ at $-a$. In this case the choice

$$Q(s) = \frac{1}{g}\frac{(s+a)^2}{(\lambda s+1)^2}, \tag{52}$$

achieves the appropriate setpoint response.

Let the observer take the form similar to (35) but taking into account the delay at the output:

$$\begin{aligned} \begin{bmatrix} \dot{\hat{x}}(t) \\ \dot{\hat{d}}(t) \end{bmatrix} &= \begin{bmatrix} A & L \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{x}(t) \\ \hat{d}(t) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u(t) \\ &+ \begin{bmatrix} L \\ \frac{1}{\delta} I \end{bmatrix} \left(y(t) - C\hat{x}(t-\tau) - \hat{d}(t) \right), \end{aligned} \tag{53}$$

with $\delta > 0$. If

$$L = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix}, \tag{54}$$

then stability of the observer can be assured by the Nyquist criterion for the loop transfer function

$$\frac{1}{s}\left(l_1 + \frac{l_2}{s+a} \right) e^{-s\tau}. \tag{55}$$

For example, the choice

$$L = \begin{bmatrix} l_1 \\ 0 \end{bmatrix} \text{ with } l_1 < \frac{\pi}{2\tau}, \tag{56}$$

guarantees stability. Finally we choose

$$M = 1 - C(A-BF)^{-1}L. \tag{57}$$

As an example, suppose

$$P(s) = \frac{1}{s(s+2)}e^{-s/2} \tag{58}$$

and we choose

$$\lambda = 1, F = [\,4 \;\; 2\,] \text{ and } L = \begin{bmatrix} \pi/4 \\ 0 \end{bmatrix}. \tag{59}$$

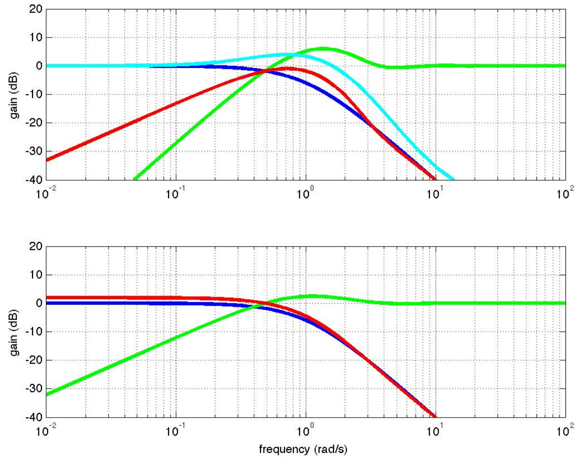

Fig. 4. The upper figure shows nominal closed-loop sensitivities for the proposed control structure: $|\tilde{P}Q|$ (blue), $|DY_Q|$ (green); $|1-DY_Q|$ (cyan); $|PDY_Q|$ (red). The lower figure shows nominal closed-loop sensitivities for IMC with open-loop stable tuning: $|PQ_{IMC}|$ (blue), $|1-PQ_{IMC}|$ (green); $|P(1-PQ_{IMC})|$ (red).

Fig. 4 (upper) shows the following nominal gains: gain from setpoint to output $|\tilde{P}Q|$ (blue); sensitivity $|DY_Q|$ (green); complementary sensitivity $|1-DY_Q|$ (cyan); gain from input disturbance to output $|PDY_Q|$ (red). Note in particular that the sensitivity $|DY_Q|$ rolls off at -40dB/decade as the frequency reduces to zero and the gain from input disturbance to output rolls off at -20dB/decade as the frequency reduces to zero; these indicate that the controller includes integral action. By contrast Fig. 4 (lower) shows the following nominal sensitivities when internal model control is used with standard open-loop tuning: complementary sensitivity $|PQ_{IMC}|$ which is also the gain from setpoint to output sensitivity (blue); sensitivity $|1-PQ_{IMC}|$ (green); gain from input disturbance to output $|PDY_Q|$ (red). The sensitivity only rolls off at -20dB/decade as the frequency reduces to zero; correspondingly the gain from input disturbance to output does not attenuate at low frequencies.

Fig. 5 shows the corresponding time responses from simulations. The upper plot shows the nominal responses of the proposed controller while the lower figure shows the nominal response of IMC with open-loop tuning. In both figures the

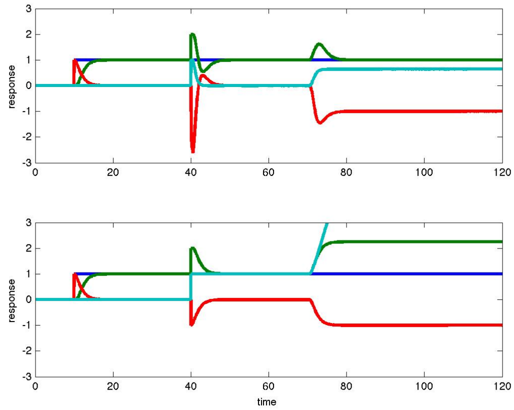

Fig. 5. The upper figure shows the nominal closed-loop time response for the proposed controller and the lower figure for IMC with open-loop stable tuning. Both figures show: setpoint (blue); output (green); input (red); disturbance estimate (cyan).

setpoint (blue), plant output (green), plant input (red) and disturbance estimate (cyan) are shown. There is a unit step in the setpoint at time $t = 10$s, a unit step output disturbance at time $t = 40$s and a unit step input disturbance at time $t = 70$s. For the proposed controller all signals are bounded and offset-free action is achieved. Similar results are obtained when there is model mismatch. By contrast for IMC offset-free performance is not achieved when there is an input disturbance and the disturbance estimate grows without bound.

6. CONCLUSIONS

We have proposed a control structure that allows the intuitive tuning procedure of IMC to be extended to unstable and integrating plants. In particular we estimate both states and disturbances using an augmented model; we provide simple rules for disturbance estimate feedback that guarantee both nominal stability and offset-free control action, even in the face of plant-model match. The control structure is closely related to the Youla paramterization, which emerges as a special case with a reduced order observer; in this sense we have provided a simple tuning procedure to guarantee offset-free control action using the Youla parameterization.

From a practical point of view, we would like to extend the methodology to address control design and implementation for large scale plants (as typically encountered, for example, in process control applications). To demonstrate the method for such applications requires some further work. While it is encouraging that we can demonstrate the method on a simple example with delays and integrators, application of the design methodology to large scale plants with these features is beyond the scope of this paper. Similarly consideration of actuator constraints, which are likely to be encountered in practical applications, is beyond the scope of this paper.

Further extensions and generalizations are also possible. As examples: it is possible to include more general disturbance dynamics (such as sinusoids) in (10); in a more generalized framework we would distinguish measured variables from controlled variables; although M in the proposed structure is a static gain matrix, it is straightforward to allow it to be a stable transfer function matrix; similarly it is straightforward to include a precompensator on the setpoint r.

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Appendix A. PROOFS

A.1 Proof of Theorem 8

Define the state estimate error

$$\varepsilon = x - \hat{x}, \tag{A.1}$$

and let the state-space description for the path from $\hat{d}$ and r to $\tilde{u}$ be

$$\begin{aligned} \dot{z} &= A_Q z + B_Q (r - M\hat{d}) \\ \tilde{u} &= C_Q z + D_Q (r - M\hat{d}). \end{aligned} \tag{A.2}$$

Since $Q(s)$ is assumed stable, A_Q must be Hurwitz. Then in closed loop we find

$$\begin{bmatrix} \dot{x} \\ \dot{z} \\ \dot{\varepsilon} \\ -\dot{\hat{d}} \end{bmatrix} = A_{CL} \begin{bmatrix} x \\ z \\ \varepsilon \\ -\hat{d} \end{bmatrix} + \begin{bmatrix} BD_Q \\ B_Q \\ 0 \\ 0 \end{bmatrix} r + \begin{bmatrix} \bar{B}_d \\ 0 \\ \bar{B}_d - L_x \bar{C}_d \\ -L_d \bar{C}_d \end{bmatrix} d \tag{A.3}$$

with

$$A_{CL} = \begin{bmatrix} A - BF & BC_Q & BF & BD_Q M \\ & A_Q & & B_Q M \\ & & A - L_x C & B_d - L_x C_d \\ & & -L_d C & -L_d C_d \end{bmatrix} \tag{A.4}$$

The matrix A_{CL} is block triangular with diagonal block entries Hurwitz. $A - BF$ and A_Q are Hurwitz by assumption. Furthermore, the last 2×2 block matrix is equal to $A_a - L_a C_a$, which is also Hurwitz by assumption. Hence A_{CL} itself is Hurwitz. □

A.2 Proof of Lemma 11

We can write $A_a - L_a C_a$ in (11) as

$$A_{obs} = \begin{bmatrix} I & \\ & L_d \end{bmatrix} \begin{bmatrix} A - L_x C & B_d - L_x C_d \\ -C & -C_d \end{bmatrix}. \tag{A.5}$$

Since A_{obs} is Hurwitz, A_{obs}^{-1} exists and thus, since L_d is square, L_d^{-1} exists. □

A.3 Proof of Theorem 12

Let $(\hat{x}, \hat{d})$ be the equilibrium variables reached by the observer, which satisfy:

$$\begin{aligned} 0 &= A\hat{x} + Bu + B_d \hat{d} + L_x (y - C\hat{x} - C_d \hat{d}), \\ 0 &= L_d (y - C\hat{x} - C_d \hat{d}). \end{aligned} \tag{A.6}$$

Thus, from (A.6) and recalling that $u = \tilde{u} - F\hat{x}$ (12), we obtain:

$$\begin{aligned} 0 &= (A - BF)\hat{x} + B\tilde{u} + B_d \hat{d}, \\ y &= C\hat{x} + C_d \hat{d}. \end{aligned} \tag{A.7}$$

Since $A - BF$ is Hurwitz, we can write this

$$y = -C(A - BF)^{-1} B\tilde{u} + \left(C_d - C(A - BF)^{-1} B_d\right) \hat{d}. \tag{A.8}$$

Then we can write this

$$y = \tilde{P}(0)\tilde{u} + \tilde{P}_d(0)\hat{d}, \tag{A.9}$$

with $\tilde{P}(s)$ given in (15) and $\tilde{P}_d(s)$ given in (17). From (12) the steady state value of $\tilde{u}$ is

$$\tilde{u} = Q(0)r - Q(0)M\hat{d}. \tag{A.10}$$

Hence

$$y = \tilde{P}(0)Q(0)r + \left(\tilde{P}_d(0) - \tilde{P}(0)Q(0)M\right) \hat{d}. \tag{A.11}$$

From Assumption 9 and $M = \tilde{P}_d(0)$, we obtain offset-free behavior in steady state. □