On the maximal controller gain in linear MPC ⋆

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Abstract: The paper addresses the computation of Lipschitz constants for model predictive control (MPC) laws. Such Lipschitz constants are useful to assess the inherent robustness of nominal MPC for disturbed systems. It is shown that a Lipschitz constant can be computed by identifying the maximal controller gain of the MPC. Clearly, given the explicit description of the MPC, this gain can be easily identified. The computation of the explicit MPC may, however, be numerically demanding. The goal of the paper thus is to overestimate the maximal controller gain without using the explicit control law.

Keywords: Model predictive control, linear systems, controller gain, Lipschitz constants, robustness, active set methods

1. INTRODUCTION AND PROBLEM STATEMENT

Model predictive control (MPC) has become a standard tool for the regulation of linear discrete-time systems

$$x(k+1) = A\,x(k) + B\,u(k), \qquad x(0) := x_0 \tag{1}$$

with state and input constraints of the form

$$x(k) \in \mathcal{X} \subset \mathbb{R}^n \;\text{ and }\; u(k) \in \mathcal{U} \subset \mathbb{R}^m \;\text{ for every }\; k \in \mathbb{N}. \tag{2}$$

Compared to other control schemes, it stands out for its ability to consider both constraint satisfaction and performance demands. From a technical point of view, given the current state x_0, MPC relies on the solution of the optimal control problem (OCP)

$$V(x_0) = \min_{X,U} \|x(N)\|_P^2 + \sum_{k=0}^{N-1} \|x(k)\|_Q^2 + \|u(k)\|_R^2 \tag{3}$$

$$\begin{aligned} \text{s.t.} \quad & x(0) = x_0, \\ & x(k+1) = A\,x(k) + B\,u(k) && \forall k \in \mathbb{N}_{[0,N-1]}, \\ & x(k) \in \mathcal{X} && \forall k \in \mathbb{N}_{[0,N-1]}, \\ & u(k) \in \mathcal{U} && \forall k \in \mathbb{N}_{[0,N-1]}, \\ & x(N) \in \mathcal{T}, \end{aligned}$$

where V is the performance index,

$$X := \begin{pmatrix} x(0) \\ \vdots \\ x(N) \end{pmatrix} \in \mathbb{R}^{(N+1)n} \;\text{ and }\; U := \begin{pmatrix} u(0) \\ \vdots \\ u(N-1) \end{pmatrix} \in \mathbb{R}^{Nm}$$

denote the decision variables, $N \in \mathbb{N}$ refers to the prediction horizon, $P, Q \in \mathbb{R}^{n\times n}$ and $R \in \mathbb{R}^{m\times m}$ are weighting matrices, and $\mathcal{T} \subseteq \mathcal{X}$ is a terminal set. Solving (3) for any feasible initial state x_0 yields an optimal input sequence $U^*(x_0)$. Usually, only the optimal control action for the first time step (i.e., $u^*(0)$) is applied to the system and the OCP is solved again at the next sampling instance. We thus obtain the control law $\varrho : \mathcal{F}_N^{\mathcal{T}} \to \mathbb{R}^m$ with

$$\varrho(x) := U^*_{\mathbb{N}_{[1,m]}}(x), \tag{4}$$

where $\mathcal{F}_N^{\mathcal{T}} \subset \mathbb{R}^n$ denotes the set of feasible initial states x_0 for (3) (depending on the choices of N and $\mathcal{T}$). Now, in case system (1) is subject to additive disturbances, the OCP (3) can be extended to take these disturbances explicitly into account. The resulting robust MPC schemes are well-known in theory (see, e.g., Mayne et al. (2005), Mayne et al. (2006), or Raković et al. (2012)) but rarely used in practice. In fact, even for disturbed systems, nominal MPC (which neglects the disturbances in the OCP) is often preferred for its simplicity. This procedure is justifiable to a certain extent since nominal MPC implicitly offers some robustness guarantees (see, e.g., Scokaert et al. (1997), Kerrigan (2000), Limón et al. (2002), Pannocchia et al. (1911), Picasso et al. (2012), or Yu et al. (2014)). In fact, several methods for certifying intrinsic robustness exist. The approaches in Scokaert et al. (1997) and Limón et al. (2002) build on Lipschitz continuity of the performance index V and the control law ϱ. Clearly, Lipschitz continuity allows to bound the effect of disturbances and promotes the identification of robust positively invariant (RPI) sets, which are necessary for guaranteeing robust stability.

In this paper, against the background of intrinsic robustness, we quantify the Lipschitz continuity of the nominal MPC law (4). In particular, we address the computation of Lipschitz constants κ such that

$$\|\varrho(\xi) - \varrho(x)\|_2 \leq \kappa\,\|\xi - x\|_2 \tag{5}$$

for every $\xi, x \in \mathcal{F}_N^{\mathcal{T}}$. Knowing a suitable κ may allow to specify intrinsic robustness guarantees as in (Limón et al., 2002, Cor. 4) for linear MPC. Computing a suitable κ can, in principle, be realized by taking into account the well-known structure of the control law ϱ for linear MPC. In fact, as shown in Bemporad et al. (2002), ϱ is continuous

⋆ This work was partially funded by the German Research Foundation (DFG) under the grants SCHU 2094/2-1 and MO 1086/11-1.

and piecewise affine. More precisely, it can be written as

$$\varrho(x) = \begin{cases} K_1\, x + b_1 & \text{if } x \in \mathcal{P}_1, \\ \vdots & \\ K_p\, x + b_p & \text{if } x \in \mathcal{P}_p \end{cases} \tag{6}$$

based on a finite number p of affine control laws $K_i\, x + b_i$ and convex polytopes $\mathcal{P}_i$. As detailed in Sect. 2.1,

$$\kappa^* := \max_{i \in \mathbb{N}_{[1,p]}} \|K_i\|_2 \tag{7}$$

satisfies (5). In other words, a Lipschitz constant for ϱ on $\mathcal{F}_N^{\mathcal{T}}$ is given by the maximal spectral norm of all controller gains K_i. Clearly, (7) can be easily computed if the explicit control law (6) is at hand. However, it is well-known that the numerical effort for computing (6) can be high even for small-scale systems. In this paper, we thus propose a method for the overestimation of κ^* without computing the explicit MPC law (6). We stress, however, that the method in its current form is not yet capable of reducing the numerical effort compared to the computation of (6).

The paper is organized as follows. We collect notation and preliminaries in the remainder of this section. In Sec. 2, we address the computation of Lipschitz constants for the predictive controller without investigating the explicit description of the control law. In particular, we present a conjecture that allows to overestimate (7) without knowing (6). Since a complete proof for the conjecture is currently missing, we verify our statement for a number of examples in Sect. 3. Finally, we state conclusions in Sect. 4.

1.1 Notation and Preliminaries

We denote real and natural numbers (excluding 0) by $\mathbb{R}$ and $\mathbb{N}$, respectively. The set $\mathbb{N}_{[1,l]}$ refers to $\{i \in \mathbb{N} \,|\, i \leq l\}$ for some $l \in \mathbb{N}$. For a matrix $E \in \mathbb{R}^{p \times q}$ with $p, q \in \mathbb{N}$, $\|E\|_2$ denotes the matrix 2-norm, which is equivalent to $\sigma_{\max}(E)$, i.e., the largest singular value of E. We further define $\|x\|_P^2 = x^T P x$ for symmetric matrices $P \in \mathbb{R}^{n \times n}$ and vectors $x \in \mathbb{R}^n$. For index sets $\mathcal{I} \in \mathbb{N}_{[1,p]}$ and $\mathcal{J} \in \mathbb{N}_{[1,q]}$, $E_{\mathcal{I},\mathcal{J}}$ refers to the submatrix of E resulting from selecting all rows in $\mathcal{I}$ and columns in $\mathcal{J}$. More precisely, let $i_1 < \cdots < i_p$ and $j_1 < \cdots < j_q$ be the ordered indices in $\mathcal{I}$ and $\mathcal{J}$, respectively. Then,

$$E_{\mathcal{I},\mathcal{J}} := \begin{pmatrix} E_{i_1,j_1} & \dots & E_{i_1,j_q} \\ \vdots & & \vdots \\ E_{i_p,j_1} & \dots & E_{i_p,j_q} \end{pmatrix}.$$

Analogously, $E_{\mathcal{I}}$ results from selecting all lines in $\mathcal{I}$ and keeping all original columns. The cardinality of a set $\mathcal{I}$ is given by $|\mathcal{I}|$. The matrix I_p describes the identity matrix in $\mathbb{R}^{p \times p}$. Finally, throughout the paper we make the following assumption.

Assumption 1. The pair (A, B) is stabilizable. The weighting matrices P and R are positive definite; the matrix Q is positive semi-definite. The pair $(Q^{\frac{1}{2}}, A)$ is detectable. The sets $\mathcal{X}$, $\mathcal{U}$, and $\mathcal{T}$ are convex and compact polytopes containing the origin as an interior point.

2. COMPUTING A LIPSCHITZ CONSTANT

In this section, we propose a method for the computation of a Lipschitz constant κ that satisfies (5). To this end, we first show that κ^* as in (7) is indeed a valid Lipschitz constant for ρ on $\mathcal{F}_N^{\mathcal{T}}$. Afterwards, we introduce an algorithm for the overestimation of κ^* without evaluating the explicit control law (6).

2.1 The maximal controller gain

To show that κ^* satisfies (5), consider two arbitrary states $\xi, x \in \mathcal{F}_N^{\mathcal{T}}$ and define $\Delta x := \xi - x$. Clearly, since ϱ is continuous and since $\mathcal{F}_N^{\mathcal{T}}$ is convex, $\varrho(\xi) = \varrho(x + \Delta x)$ can be expressed as

$$\varrho(x + \Delta x) = \varrho(x) + \int_0^1 J(x + \tau \Delta x)\, \Delta x \, \mathrm{d}\tau,$$

where $J(\xi)$ denotes the Jacobian of ϱ at ξ. The expression on the l.h.s. in (5) can thus be written as

$$\|\varrho(x + \Delta x) - \varrho(x)\|_2 = \left\| \int_0^1 J(x + \tau \Delta x)\, \Delta x \, \mathrm{d}\tau \right\|_2,$$

which can be overestimated by

$$\begin{aligned} \left\| \int_0^1 J(x + \tau \Delta x) \Delta x \, \mathrm{d}\tau \right\|_2 &\leq \int_0^1 \|J(x + \tau \Delta x) \Delta x\, \|_2 \, \mathrm{d}\tau \\ &\leq \|\Delta x\|_2 \int_0^1 \|J(x + \tau \Delta x)\|_2 \, \mathrm{d}\tau \\ &\leq \|\Delta x\|_2 \max_{\xi \in \mathcal{F}_N^{\mathcal{T}}} \|J(\xi)\|_2. \end{aligned}$$

It remains to compute the maximal spectral norm of the Jacobian J on the feasible set $\mathcal{F}_N^{\mathcal{T}}$. Taking (6) into account, we obviously have

$$J(\xi) = \begin{cases} K_1 & \text{if } \xi \in \mathcal{P}_1, \\ \vdots & \\ K_p & \text{if } \xi \in \mathcal{P}_p. \end{cases}$$

for every $\xi \in \mathcal{F}_N^{\mathcal{T}}$. We thus obtain

$$\max_{\xi \in \mathcal{F}_N^{\mathcal{T}}} \|J(\xi)\|_2 = \max_{i \in \mathbb{N}_{[1,p]}} \|K_i\|_2,$$

which shows that κ^* is a Lipschitz constant for ρ on $\mathcal{F}_N^{\mathcal{T}}$.

2.2 Control law definition based on active constraints

Our goal is to overestimate (7) without computing (6). Nevertheless, we can use techniques for the explicit computation of (6) to identify procedures for the direct overestimation of (7). To this end, first note that not only the control law ϱ is known to be continuous and piecewise affine. In fact, as established in Bemporad et al. (2002), the same holds for the whole optimal input sequence $U^* : \mathcal{F}_N^{\mathcal{T}} \to \mathbb{R}^{Nm}$, which can be written as

$$U^*(x) = \begin{cases} L_1\, x + c_1 & \text{if } x \in \mathcal{P}_1, \\ \vdots & \\ L_p\, x + c_p & \text{if } x \in \mathcal{P}_p. \end{cases} \tag{8}$$

Comparing (4), (6) and (8), we obviously find $K_i = (L_i)_{\mathbb{N}_{[1,m]}}$ (and $b_i = (c_i)_{\mathbb{N}_{[1,m]}}$) for every $i \in \mathbb{N}_{[1,p]}$. In order to overestimate (7), we require a basic understanding of the explicit computation of the matrices L_i. In this context, Assum. 1 guarantees that the OCP (3) can be rewritten as the quadratic program (QP)

$$\begin{aligned} V(x_0) = \min_U \ & \frac{1}{2}\, U^T H U + x_0^T G^T U + \frac{1}{2}\, x_0^T F x_0 \\ \text{s.t.} \quad & E\, U \leq D\, x_0 + d \end{aligned} \tag{9}$$

with a positive definite matrix $H \in \mathbb{R}^{Nm \times Nm}$ and $q \in \mathbb{N}$ constraints, i.e., $d \in \mathbb{R}^q$ (see Bemporad et al. (2002); Pappas et al. (1980)). We refer to (Maciejowski, 2001, Chap. 3) for details on the construction of H, G, F, E, D, and d based on the system matrices A and B, the horizon N, and the constraints $\mathcal{X}$, $\mathcal{U}$, and $\mathcal{T}$. We further stress that the assumed compactness of the polytopes $\mathcal{X}, \mathcal{U}$, and

$\mathcal{T}$ (see Assum. 1) implies $q > Nm$. In other words, the number of constraints in (9) is larger than the number of decision variables. Finally, we assume that the matrix $E \in \mathbb{R}^{q\times Nm}$ has full rank, i.e., $\mathrm{rank}(E) = Nm$. Now, for a given $x_0 \in \mathcal{F}_N^{\mathcal{T}}$, it is well-known that the solution of (9) can be characterized based on the set of active constraints

$$\mathcal{A}(x_0) := \left\{ i \in \mathbb{N}_{[1,q]} \,|\, E_{\{i\}}\, U^*(x_0) = D_{\{i\}}\, x_0 + d_{\{i\}} \right\}.$$

In fact, under the assumption that $E_{\mathcal{A}(x_0)}$ has rank $|\mathcal{A}(x_0)|$, $U^*(x_0)$ can be expressed as

$$U^*(x_0) = L(\mathcal{A}(x_0))x_0 + c(\mathcal{A}(x_0)) \tag{10}$$

with

$$L(\mathcal{A}) := H^{-1}E_{\mathcal{A}}^T(S_{\mathcal{A},\mathcal{A}})^{-1}(D_{\mathcal{A}}+E_{\mathcal{A}}H^{-1}G)-H^{-1}G \tag{11}$$

and

$$c(\mathcal{A}) := H^{-1}E_{\mathcal{A}}^T(S_{\mathcal{A},\mathcal{A}})^{-1}d_{\mathcal{A}},$$

where $S := EH^{-1}E^T$ (see (Bemporad et al., 2002, Sect. 4.1)). Clearly, the condition $\mathrm{rank}(E_{\mathcal{A}(x_0)}) = |\mathcal{A}(x_0)|$ guarantees invertibility of $S_{\mathcal{A}(x_0),\mathcal{A}(x_0)}$. However, a relation similar to (10) even holds in the (degenerated) case that $\mathrm{rank}(E_{\mathcal{A}(x_0)}) < |\mathcal{A}(x_0)|$. In fact, in this case,

$$U^*(x_0) = L(\underline{\mathcal{A}})x_0 + c(\underline{\mathcal{A}})$$

for some subset $\underline{\mathcal{A}} \subset \mathcal{A}(x_0)$ that satisfies $\mathrm{rank}(E_{\underline{\mathcal{A}}}) = |\underline{\mathcal{A}}|$ (see (Bemporad et al., 2002, Section 4.1.1) or (Tøndel et al., 2003, Section II.C)). In summary, we make the following observation.

Lemma 2. Assume the piecewise affine description (8) of the optimizer U^* is known and consider any $i \in \mathbb{N}_{[1,p]}$. Then, there exists a set $\mathcal{A} \subset \mathbb{N}_{[1,q]}$ such that (i)

$$\mathrm{rank}(E_{\mathcal{A}}) = |\mathcal{A}| \leq Nm \tag{12}$$

and (ii) $L_i = L(\mathcal{A})$.

Regarding condition (12), it is interesting to note that $|\mathcal{A}| \leq Nm$ is necessary for $\mathrm{rank}(E_{\mathcal{A}}) = |\mathcal{A}|$ since $E_{\mathcal{A}} \in \mathbb{R}^{|\mathcal{A}|\times Nm}$ (i.e., $\mathrm{rank}(E_{\mathcal{A}}) \leq \min\{|\mathcal{A}|, Nm\}$). Finally, we know from (8) that the affine relation (10) will, in general, not only hold for a single x_0. In this context, it is interesting to note that the active set $\mathcal{A}(x_0)$ also determines the polytopic region $\mathcal{P}(\mathcal{A}(x_0))$ for which we have $\mathcal{A}(\xi) = \mathcal{A}(x_0)$ for every $\xi \in \mathcal{P}(\mathcal{A}(x_0))$. In fact, we obtain $\mathcal{P}(\mathcal{A}) := \{x \in \mathbb{R}^n \,|\, \Phi(\mathcal{A})\, x \leq \beta(\mathcal{A})\}$ with

$$\Phi(\mathcal{A}) := \begin{pmatrix} E_{\mathcal{I}}H^{-1}E_{\mathcal{A}}^T(S_{\mathcal{A},\mathcal{A}})^{-1} & -I_{|\mathcal{I}|} \\ (S_{\mathcal{A},\mathcal{A}})^{-1} & 0 \end{pmatrix} \begin{pmatrix} D_{\mathcal{A}} + E_{\mathcal{A}}H^{-1}G \\ D_{\mathcal{I}} + E_{\mathcal{I}}H^{-1}G \end{pmatrix}$$

and

$$\beta(\mathcal{A}) := \begin{pmatrix} -E_{\mathcal{I}}H^{-1}E_{\mathcal{A}}^T(S_{\mathcal{A},\mathcal{A}})^{-1} & I_{|\mathcal{I}|} \\ -(S_{\mathcal{A},\mathcal{A}})^{-1} & 0 \end{pmatrix} \begin{pmatrix} d_{\mathcal{A}} \\ d_{\mathcal{I}} \end{pmatrix}$$

where (the set of inactive constraints) $\mathcal{I}$ is defined as $\mathcal{I} := \mathbb{N}_{[1,q]} \setminus \mathcal{A}$.

2.3 Overestimating the maximal controller gain

Lemma 2 contains the central observation for the overestimation of κ^*. In fact, it is easy to see that the following statements holds.

Lemma 3. Assume the piecewise affine description (6) of the control law ϱ is known and let κ^* be defined as in (7). Then, $\kappa^* \leq \ell^*$, where

$$\ell^* := \max_{\mathcal{A}} \|L_{\mathbb{N}_{[1,m]}}(\mathcal{A})\|_2 \quad \text{s.t.} \quad (12) \text{ and } \mathcal{A} \subset \mathbb{N}_{[1,q]}. \tag{13}$$

In other words, an overestimation of κ^* can be identified by analyzing the controller gains for all sets $\mathcal{A}$ that satisfy the rank condition (12). Note that the overestimation is, in general, not tight (i.e., $\kappa^* < \ell^*$) since some sets $\mathcal{A}$ satisfying (12) lead to empty regions $\mathcal{P}(\mathcal{A})$ that do not appear in (6) (and (8)).

Now, while Lemmalinear MPC 3 is a useful starting point for an overestimation of κ^*, the exhaustive analysis of all sets $\mathcal{A}$ satisfying (12) is computationally demanding - also when compared to the effort for the computation of the explicit control law (6). We thus have to identify rules for reducing the size of the feasible set of (13). To this end, we first introduce a modified version of the optimization problem (OP) in (13). As apparent from (11), the matrix $L(\mathcal{A})$ contains the constant term $-H^{-1}G$, which obviously corresponds to the optimizer $U^*(x_0) = -H^{-1}Gx_0$ of the OCP (9) without constraints (or for $\mathcal{A}(x_0) = \emptyset$). Focusing on the term varying with $\mathcal{A}$, we thus consider

$$\Delta L(\mathcal{A}) := L(\mathcal{A}) + H^{-1}G \tag{14}$$

which is given by

$$\Delta L(\mathcal{A}) = H^{-1}E_{\mathcal{A}}^T(S_{\mathcal{A},\mathcal{A}})^{-1}(D_{\mathcal{A}} + E_{\mathcal{A}}H^{-1}G). \tag{15}$$

In analogy to (13), we then introduce the OP

$$\Delta\ell^* := \max_{\mathcal{A}} \|\Delta L_{\mathbb{N}_{[1,m]}}(\mathcal{A})\|_2 \;\; \text{s.t. } (12) \text{ and } \mathcal{A} \subset \mathbb{N}_{[1,q]}. \tag{16}$$

Since we obviously have

$$\|L_{\mathbb{N}_{[1,m]}}(\mathcal{A})\|_2 \leq \|\Delta L_{\mathbb{N}_{[1,m]}}(\mathcal{A})\|_2 + \|H^{-1}_{\mathbb{N}_{[1,m]}}G\|_2,$$

we obtain

$$\kappa^* \leq \ell^* \leq \Delta\ell^* + \kappa_0, \tag{17}$$

where $\kappa_0 := \|H^{-1}_{\mathbb{N}_{[1,m]}}G\|_2$. Based on the following conjecture, we now claim that it is sufficient to only consider sets with $|\mathcal{A}| = Nm$, when solving (16).

Conjecture 4. Let $\mathcal{A}^*$ be the optimizer of (16). Then $|\mathcal{A}^*| = Nm$.

We present a simple algorithm for the computation of $\Delta\ell^*$ that makes use of this conjecture in Sect. 2.4. Observations that may be useful to prove Conj. 4 are discussed in Sect. 2.5. We stress, however, that we currently do not have a complete proof for the conjecture. To substantiate our claim, we thus carry out a numerical verification of Conj. 4 for some examples in Sect. 3.

2.4 Computation of $\Delta\ell^*$ based on Conjecture 4

We can obviously reduce the complexity of OP (16) under the assumption that Conj. 4 holds. In fact, in this case, we have to consider only those sets $\mathcal{A} \subset \mathbb{N}_{[1,q]}$ in (16) for which

$$\mathrm{rank}(E_{\mathcal{A}}) = |\mathcal{A}| = Nm \tag{18}$$

In addition, the cardinality of the universe $\mathbb{N}_{[1,q]}$ can be reduced. In fact, it is easy to see that any $i \in \mathbb{N}_{[1,q]}$, which is not contained in

$$\mathcal{Q} := \{i \in \mathbb{N}_{[1,q]} \,|\, \mathrm{rank}(E_{\{i\}}) = 1\}, \tag{19}$$

cannot be part of a set $\mathcal{A}$ satisfying (12) or (18). The computation of $\Delta\ell^*$ as in (16) can thus be equivalently formulated as

$$\Delta\ell^* = \max_{\mathcal{A}} \|\Delta L_{\mathbb{N}_{[1,m]}}(\mathcal{A})\|_2 \;\; \text{s.t. } (18) \text{ and } \mathcal{A} \subseteq \mathcal{Q}. \tag{20}$$

A solution to (20) can be computed based on simple combinatorial optimization using the following algorithm. Note, however, that the numerical complexity of the algorithm can be high. In fact, we obviously have to consider

$$s := \binom{|\mathcal{Q}|}{Nm} = \frac{|\mathcal{Q}|!}{(Nm)!\,(|\mathcal{Q}| - Nm)!} \tag{21}$$

sets $\mathcal{A}$ in line 2 of Alg. 1. More interestingly is, however, the number of sets $\mathcal{A}$ satisfying the condition in line 3, i.e.,

$$\hat{s} := |\{\mathcal{A} \subseteq \mathcal{Q} \,|\, (18) \text{ holds}\}|\,. \tag{22}$$

We will analyze the numbers s and s^* for some examples in Sect. 3.

Algorithm 1. Numerical solution of (20).

1 Define $\mathcal{Q}$ as in (19) and set $\Delta\ell^* \leftarrow 0$.
2 **for every** $\mathcal{A} \subseteq \mathcal{Q}$ with $|\mathcal{A}| = Nm$
3   **if** $\mathrm{rank}(E_{\mathcal{A}}) = Nm$
4   **if** $\Delta\ell^* < \|\Delta L_{\mathbb{N}_{[1,m]}}(\mathcal{A})\|_2$
5   set $\Delta\ell^* \leftarrow \|\Delta L_{\mathbb{N}_{[1,m]}}(\mathcal{A})\|_2$.
6 **return** $\Delta\ell^*$ and terminate.

2.5 Towards a proof of Conjecture 4

Roughly speaking, Conj. 4 states that sets $\mathcal{A} \subset \mathbb{N}_{[1,q]}$ with small cardinality can be neglected in (16) since the optimizer satisfies (18). In order to prove Conj. 4, it is thus useful to study relations between the matrices $\Delta L(\underline{\mathcal{A}})$ and $\Delta L(\mathcal{A})$ for "small" sets $\underline{\mathcal{A}}$ and "large" sets $\mathcal{A}$. The following lemma provides a relation for the case that $\underline{\mathcal{A}}$ is a subset of $\mathcal{A}$.

Lemma 5. Let the sets $\underline{\mathcal{A}}, \mathcal{A} \subset \mathbb{N}_{[1,q]}$ be such that the rank conditions $\mathrm{rank}(E_{\underline{\mathcal{A}}}) = |\underline{\mathcal{A}}|$ and (12) hold and let $\underline{\mathcal{A}} \subseteq \mathcal{A}$. Then

$$\Delta L(\underline{\mathcal{A}}) = W(\underline{\mathcal{A}})\, \Delta L(\mathcal{A}), \tag{23}$$

where

$$W(\underline{\mathcal{A}}) := H^{-1} E_{\underline{\mathcal{A}}}^T (S_{\underline{\mathcal{A}},\underline{\mathcal{A}}})^{-1} E_{\underline{\mathcal{A}}}. \tag{24}$$

Proof. Clearly, evaluating $\Delta L(\underline{\mathcal{A}})$ according to (15) yields

$$\Delta L(\underline{\mathcal{A}}) = H^{-1} E_{\underline{\mathcal{A}}}^T (S_{\underline{\mathcal{A}},\underline{\mathcal{A}}})^{-1} (D_{\underline{\mathcal{A}}} + E_{\underline{\mathcal{A}}} H^{-1} G). \tag{25}$$

Now, multiplying (15) with $E_{\mathcal{A}}$ and taking $S_{\mathcal{A},\mathcal{A}} = E_{\mathcal{A}} H^{-1} E_{\mathcal{A}}^T$ into account, leads to

$$E_{\mathcal{A}} \Delta L(\mathcal{A}) = D_{\mathcal{A}} + E_{\mathcal{A}} H^{-1} G. \tag{26}$$

Since $\underline{\mathcal{A}}$ is a subset of $\mathcal{A}$, there exists an index set $\mathcal{J} \subseteq \mathbb{N}_{[1,|\mathcal{A}|]}$ with $|\mathcal{J}| = |\underline{\mathcal{A}}|$ such that

$$E_{\underline{\mathcal{A}}} = (E_{\mathcal{A}})_{\mathcal{J}} \quad \text{and} \quad D_{\underline{\mathcal{A}}} = (D_{\mathcal{A}})_{\mathcal{J}}. \tag{27}$$

From (26) and (27), we infer

$$E_{\underline{\mathcal{A}}} \Delta L(\mathcal{A}) = D_{\underline{\mathcal{A}}} + E_{\underline{\mathcal{A}}} H^{-1} G.$$

Using this result in (15) proves (23). ■

It is interesting to note that W only depends on the set $\underline{\mathcal{A}}$. The matrix $W(\underline{\mathcal{A}})$ is thus identical for any superset $\mathcal{A}$ of $\underline{\mathcal{A}}$. It is further interesting to study the structure of $W(\underline{\mathcal{A}})$.

Lemma 6. Let the sets $\underline{\mathcal{A}}$ and $\mathcal{A}$ be as in Lem. 5 and let $W(\underline{\mathcal{A}})$ be defined as in (24). Then, there exists a unitary matrix $\Gamma \in \mathbb{R}^{Nm \times Nm}$ and some matrix $\Omega \in \mathbb{R}^{(Nm-|\underline{\mathcal{A}}|)\times|\underline{\mathcal{A}}|}$ such that

$$\Gamma\, W(\underline{\mathcal{A}})\, \Gamma^T = \begin{pmatrix} I_{|\underline{\mathcal{A}}|} & 0 \\ \Omega & 0 \end{pmatrix}. \tag{28}$$

Proof. Assume that the matrix $E_{\underline{\mathcal{A}}}$ can be written as

$$E_{\underline{\mathcal{A}}} = Y\, \Sigma\, Z, \tag{29}$$

where $Y \in \mathbb{R}^{|\underline{\mathcal{A}}|\times|\underline{\mathcal{A}}|}$ and $Z \in \mathbb{R}^{Nm\times Nm}$ are unitary matrices and where $\Sigma \in \mathbb{R}^{|\underline{\mathcal{A}}|\times Nm}$ offers the structure

$$\Sigma = \begin{pmatrix} \sigma_1 & & 0 & 0 & & 0 \\ & \ddots & & & \ddots & \\ 0 & & \sigma_{|\underline{\mathcal{A}}|} & 0 & & 0 \end{pmatrix}$$

with $\sigma_1 \geq \ldots \sigma_{|\underline{\mathcal{A}}|} > 0$. We then obtain $Y\,Y^T = I_{|\underline{\mathcal{A}}|}$ and $Z\,Z^T = I_{Nm}$. Moreover, with $\mathcal{J}_1 := \mathbb{N}_{[1,|\underline{\mathcal{A}}|]}$ and $\mathcal{J}_2 := \mathbb{N}_{[|\underline{\mathcal{A}}|+1,Nm]}$, we find $\Sigma = (\Sigma_{\mathcal{J}_1,\mathcal{J}_1}\ 0)$, $\det(\Sigma_{\mathcal{J}_1,\mathcal{J}_1}) > 0$, and $(\Sigma_{\mathcal{J}_1,\mathcal{J}_1})^{-1}\Sigma = \big(I_{|\underline{\mathcal{A}}|}\ 0\big)$. We thus have

$$\begin{pmatrix} (\Sigma_{\mathcal{J}_1,\mathcal{J}_1})^{-1} Y^T Y \Sigma \\ (I_{Nm})_{\mathcal{J}_2} \end{pmatrix} = I_{Nm}$$

and consequently

$$Z = \begin{pmatrix} (\Sigma_{\mathcal{J}_1,\mathcal{J}_1})^{-1} Y^T Y \Sigma \\ (I_{Nm})_{\mathcal{J}_2} \end{pmatrix} Z = \begin{pmatrix} (\Sigma_{\mathcal{J}_1,\mathcal{J}_1})^{-1} Y^T E_{\underline{\mathcal{A}}} \\ Z_{\mathcal{J}_2} \end{pmatrix}. \tag{30}$$

Now, evaluating $Z\,W(\underline{\mathcal{A}})Z^T$ yields

$$\begin{aligned} ZW(\underline{\mathcal{A}})Z^T &= ZH^{-1}E_{\underline{\mathcal{A}}}^T (S_{\underline{\mathcal{A}},\underline{\mathcal{A}}})^{-1} E_{\underline{\mathcal{A}}} Z^T \\ &= \begin{pmatrix} (\Sigma_{\mathcal{J}_1,\mathcal{J}_1})^{-1} Y^T E_{\underline{\mathcal{A}}} \\ Z_{\mathcal{J}_2} \end{pmatrix} H^{-1} E_{\underline{\mathcal{A}}}^T (S_{\underline{\mathcal{A}},\underline{\mathcal{A}}})^{-1} Y \Sigma \\ &= \begin{pmatrix} I_{|\underline{\mathcal{A}}|} & 0 \\ Z_{\mathcal{J}_2} H^{-1} E_{\underline{\mathcal{A}}}^T (S_{\underline{\mathcal{A}},\underline{\mathcal{A}}})^{-1} Y \Sigma_{\mathcal{J}_1,\mathcal{J}_1} & 0 \end{pmatrix}, \end{aligned} \tag{31}$$

where we used (24), (30), and $E_{\underline{\mathcal{A}}} Z^T = Y\Sigma$. At this point, we note that the decomposition in (29) can always be achieved by evaluating a singular value decomposition (SVD) of $E_{\underline{\mathcal{A}}}$. In fact, σ_1 through $\sigma_{|\underline{\mathcal{A}}|}$ refer to the $|\underline{\mathcal{A}}|$ non-zero singular values of $E_{\underline{\mathcal{A}}}$. This completes the proof, since the r.h.s. in (28) and (31) have the same structure. As a consequence, (28) holds with $\Gamma = Z$ and $\Omega = Z_{\mathcal{J}_2} H^{-1} E_{\underline{\mathcal{A}}}^T (S_{\underline{\mathcal{A}},\underline{\mathcal{A}}})^{-1} Y \Sigma_{\mathcal{J}_1,\mathcal{J}_1}$. ■

Clearly, since Γ in Lem. 6 is unitary, (28) describes a similarity transformation. As apparent from the r.h.s. in (28), the matrix $W(\underline{\mathcal{A}})$ thus has $|\underline{\mathcal{A}}|$ eigenvalues of value 1 and $Nm - |\underline{\mathcal{A}}|$ eigenvalues of value 0. While this observation is interesting, it is of minor relevance for proving Conj. 4. In fact, having information on the singular values of $W(\underline{\mathcal{A}})$ would be more constructive. To see this, note that (23) implies

$$\begin{aligned} \|\Delta L(\underline{\mathcal{A}})\|_2 &\leq \|W(\underline{\mathcal{A}})\|_2 \|\Delta L(\mathcal{A})\|_2 \\ &= \sigma_{\max}\left(W(\underline{\mathcal{A}})\right) \|\Delta L(\mathcal{A})\|_2 \end{aligned}$$

Thus, having

$$\sigma_{\max}\left(W(\underline{\mathcal{A}})\right) \leq 1 \tag{32}$$

would immediately allow to prove Conj. 4 for the special case $N = 1$, in which $\Delta L_{\mathbb{N}_{[1,m]}}(\mathcal{A}) = \Delta L(\mathcal{A})$. Unfortunately, the observation that $W(\underline{\mathcal{A}})$ has only eigenvalues in the set $\{0, 1\}$ does not imply (32). Actually, based on the structure in (28), it is straightforward to prove

$$\sigma_{\max}\left(W(\underline{\mathcal{A}})\right) \geq 1.$$

Studying some sets $\underline{\mathcal{A}} \subset \mathbb{N}_{[1,q]}$ for the examples discussed in the next sections shows that the special case $\sigma_{\max}\left(W(\underline{\mathcal{A}})\right) = 1$ occurs, but not always.

3. NUMERICAL BENCHMARK

3.1 Setup

Section 2.5 provides some observations that might be useful to prove Conj. 4 but a complete proof is missing. In order to substantiate our claim, we use the following algorithm to verify Conj. 4 for the five numerical examples in Tab. 1.

Algorithm 2. Numerical verification of Conj. 4.

Define $\mathcal{Q}$ as in (19) and set `foundCE`←**false**.
Compute $\Delta\ell^*$ using Alg. 1.
for every $\mathcal{A} \subseteq \mathcal{Q}$ with $1 \leq |\mathcal{A}| < Nm$
  if $\mathrm{rank}(E_{\mathcal{A}}) = |\mathcal{A}|$
    if $\|\Delta L_{\mathbb{N}_{[1,m]}}(\mathcal{A})\|_2 > \Delta\ell^*$
      set `foundCE`←**true** and **break**.
return `foundCE` and terminate.

Table 1. Numerical examples from the literature.

Exmp.	A	B	$\mathcal{X}$	$\mathcal{U}$	Q	R	N	Reference
1	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0.5 \\ 1.0 \end{pmatrix}$	$\|x_1\| \leq 25$ $\|x_2\| \leq 5$	$\|u\| \leq 1$	I_2	0.1	4	(Gutman and Cwikel, 1987, Eqs. (2.8)–(2.9))
2	$\begin{pmatrix} 1 & 1/2 & 1/8 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1/48 \\ 1/8 \\ 1/2 \end{pmatrix}$	$\|x_1\| \leq 20$ $\|x_2\| \leq 3$ $\|x_3\| \leq 1$	$\|u\| \leq 0.5$	I_3	1	3	(Gutman and Cwikel, 1987, Rem. 4.8)
3	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 2 \\ 4 \end{pmatrix}$	$\|x_1\| \leq 5$ $\|x_2\| \leq 5$	$\|u\| \leq 1$	I_2	4.5	3	(Schulze Darup and Cannon, 2016, Exmp. 3)
4	$\begin{pmatrix} 1.1 & 0.2 \\ -0.2 & 1.1 \end{pmatrix}$	$\begin{pmatrix} 0.5 & 0 \\ 0 & 0.2 \end{pmatrix}$	$\|x_1\| \leq 5$ $\|x_2\| \leq 5$	$\|u_1\| \leq 1$ $\|u_2\| \leq 1$	I_2	$10\, I_2$	2	(Schulze Darup, 2014, Exmp. 2.26)
5	$\begin{pmatrix} 0.7969 & -0.2247 \\ 0.1798 & 0.9767 \end{pmatrix}$	$\begin{pmatrix} 0.1271 \\ 0.0132 \end{pmatrix}$	$\|x_1\| \leq 4$ $\|x_2\| \leq 4$	$\|u\| \leq 1$	I_2	0.1	4	(Bemporad and Filippi, 2003, Exmp. 6.1)

Obviously, in Alg. 2, we again use the set $\mathcal{Q}$ from (19) as the universe for the sets $\mathcal{A}$. This is admissible since condition (12) can only be satisfied if $\mathcal{A} \subseteq \mathcal{Q}$. Moreover, we can neglect the choice $\mathcal{A} = \emptyset$ for which we obtain $\|\Delta L_{\mathbb{N}_{[1,m]}}(\mathcal{A})\|_2 = 0$.

In order to apply Alg. 2 (and Alg. 1) to the examples in Tab. 1, we need to specify the terminal weighting P and the terminal set $\mathcal{T}$ in the OCP (3). In fact, in Tab. 1, we only list the system matrices A and B, the constraints $\mathcal{X}$ and $\mathcal{U}$, the weighting matrices Q and R, the prediction horizon N, and the origin of the example. Now, for every example, P is chosen as the solution of the discrete-time algebraic Riccati equation. Regarding the set $\mathcal{T}$, we consider two different choices. First, we simply set $\mathcal{T} = \mathcal{X}$. Second, we use a stabilizing terminal set as specified in Mayne et al. (2000). For the computation of such a terminal set, we exploit the construction rules from Gilbert and Tan (1991). More precisely, we evaluate the linear quadratic regulator (LQR) gain K^* (based on P), define

$$\mathcal{X}^* := \{x \in \mathcal{X} \,|\, K^* x \in \mathcal{U}\},$$

and compute

$$\mathcal{S} = \{x \in \mathbb{R}^n \,|\, (A + BK^*)^k x \in \mathcal{X}^*, \, \forall k \in \mathbb{N}\}.$$

The second choice for the terminal set then results in $\mathcal{T} = \mathcal{S}$. We stress that $\mathcal{S}$ is a polytope for polytopic sets $\mathcal{X}$ and $\mathcal{U}$ as in Assum. 1. We further stress that the resulting control laws for $\mathcal{T} = \mathcal{X}$ and $\mathcal{T} = \mathcal{S}$ differ in terms of the domain $\mathcal{F}_N^{\mathcal{T}}$ and the specified control actions. Both control laws are, however, guaranteed to be identical in $\mathcal{S} \subseteq \mathcal{F}_N^{\mathcal{S}} \subseteq \mathcal{F}_N^{\mathcal{X}}$.

3.2 Analysis

We applied Alg. 2 to the five examples in Tab. 1 (with $\mathcal{T} = \mathcal{X}$ as well as $\mathcal{T} = \mathcal{S}$ in each case) without finding a counterexample for Conj. 4. Thus, for these examples, Alg. 1 can be used to solve (16). Numerical results for $\Delta\ell^\star$ are listed in Tab. 2. In addition, we provide information on the number q of constraints of the condensed QP (9), the number $\hat{q}$ of constraints that are required to describe the terminal set $\mathcal{T}$, the cardinality of the set $\mathcal{Q}$ as in (19), the numbers s and $\hat{s}$ as defined in (21) and (22), and the values $\kappa_0 = \|H_{\mathbb{N}_{[1,m]}}^{-1} G\|_2$ and $\Delta\ell^* + \kappa_0$. To allow a comparison with the straightforward approach via explicit MPC (EMPC), we also list the number of polytopes p in (6) and κ^* as in (7). We obviously have $\kappa^* \leq \Delta\ell^* + \kappa_0$ for every example as predicted by (17). However, the transformation (14) obviously introduces some (additional) conservatism to the overestimation. For analysis purpose, it thus makes sense to compare $\Delta\ell^*$ with

$$\Delta\kappa^* := \max_{i \in \mathbb{N}_{[1,p]}} \|K_i - K^*\|_2, \tag{33}$$

where K^* is the LQR gain from Sect. 3.1. Finally, let ξ_i be the Chebyshev center of the polytope $\mathcal{P}_i$. Then, it is also interesting to study the number

$$\hat{p} := \left|\{i \in \mathbb{N}_{[1,p]} \,|\, |\mathcal{A}(\xi_i)| = Nm\}\right|,$$

i.e., the number of polytopes (in the domain of the control law) in which Nm constraints are active.

The numerical results in Tab. 2 offer some interesting observations. First note that, due to the box constraints in Tab. 1, the numbers q and $\hat{q}$ are linked via

$$q = 2\, N\, (n + m) + \hat{q}.$$

Thereby, the number $\hat{q}$ evaluates to $2\, n$ for $\mathcal{T} = \mathcal{X}$. For the choice $\mathcal{T} = \mathcal{S}$, $\hat{q}$ depends on the number of hyperplanes required to describe $\mathcal{S}$. It can further be seen from Tab. 2 that the numerical effort for running Alg. 1 is indeed significant (even for the short prediction horizons considered in Tab. 1). In fact, we find

$$s > \hat{s} \gg p > \hat{p}$$

for every example. In other words, the number s of sets with cardinality Nm is, as expected, high. More interestingly, the number $\hat{s}$ of sets that additionally satisfy the rank condition in (18) is also high, especially compared to the number p of polytopes describing the explicit control law (6).

When comparing the the computed Lipschitz constants $\Delta\ell^*$ with the maximal gain $\Delta\kappa^*$ of the transformed controller, we can distinguish different cases. In fact, we have $\Delta\ell^* = \Delta\kappa^*$ for the third and fourth example, $\Delta\ell^* > \Delta\kappa^*$ for the first example, and $\Delta\ell^* \gg \Delta\kappa^*$ for the second and fifth example. It is further interesting that $\Delta\ell^*$ and $\Delta\kappa^*$ do not change with the terminal set $\mathcal{T}$ for the first three examples, while they do change for the last two examples. Finally, it is remarkable that the relation between κ_0, $\Delta\kappa^*$, and κ^* is also different for most examples. We find, for example, $\kappa_0 = \Delta\kappa^* = \kappa^*$ for the first example, $\kappa_0 < \Delta\kappa^* < \kappa^*$ for the second example, and $\Delta\kappa^* > \kappa^* > \kappa_0$ for the third example. In fact, the examples were selected in order to show that Conj. 4 holds independently of the relationship between κ_0, $\Delta\kappa^*$, and κ^*.

Table 2. Analysis of the proposed method for computing Lipschitz constants.

		Analysis of proposed method								Comparison with EMPC			
Exmp.	$\mathcal{T}$	q	$\hat{q}$	$\lvert\mathcal{Q}\rvert$	s	$\hat{s}$	$\Delta\ell^*$	κ_0	$\Delta\ell^* + \kappa_0$	p	$\hat{p}$	$\Delta\kappa^*$	κ^*
1	$\mathcal{X}$	28	4	24	10626	4112	1.5640	1.4121	2.9761	65	28	1.4121	1.4121
1	$\mathcal{S}$	28	4	24	10626	4240	1.5640	1.4121	2.9761	43	18	1.4121	1.4121
2	$\mathcal{X}$	30	6	24	2024	1104	51.8065	3.5420	55.3485	111	62	5.8930	8.9443
2	$\mathcal{S}$	38	14	32	4960	3616	51.8065	3.5420	55.3485	89	50	5.8930	8.9443
3	$\mathcal{X}$	22	4	18	816	416	0.5472	0.2222	0.7694	15	6	0.5472	0.5000
3	$\mathcal{S}$	24	6	20	1140	656	0.5472	0.2222	0.7694	15	6	0.5472	0.5000
4	$\mathcal{X}$	20	4	16	1820	528	31.2767	0.9246	32.2013	33	6	31.2767	31.2500
4	$\mathcal{S}$	26	10	22	7315	3104	212.5562	0.9246	213.4808	73	22	212.5562	212.6942
5	$\mathcal{X}$	28	4	24	10626	4288	73.6632	2.4871	76.1503	29	4	2.4871	3.1401
5	$\mathcal{S}$	34	10	30	27405	14080	73.6632	2.4871	76.1503	89	35	37.0267	38.7305

4. CONCLUSION

We proposed a method for computing Lipschitz constants for linear MPC without evaluating the explicit control law. The approach is based on a reformulation of the problem of interest and a conjecture characterizing the solution of the transformed problem. A procedure implementing the conjecture was successfully applied to compute Lipschitz constants for five numerical examples.

Although we provided some observations that may allow to prove the conjecture, a complete proof is currently missing. To compensate this gap, we numerically verified the statement for the five analyzed examples. While the conjecture withstand this verification, the analysis showed that the numerical effort for computing a Lipschitz constant based on the proposed procedure is high. In particular, the method (in its current form) does not provide the desired alternative to the explicit computation of the control law. Future research thus has to address novel strategies for solving the formulated task. Abstractly speaking, we tried to identify Lipschitz constants by analyzing sets of active constraints with large cardinalities. It could, however, make sense to address the problem starting from active sets with small cardinalities. In this context, we might be able to combine our results from Sect. 2.5 with the techniques proposed in Gupta et al. (2011).

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