Chapter 6
Carleson measures and Toeplitz operators

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In this last chapter we shall describe a completely different application of the Kobayashi distance to complex analysis. To describe the problem we need a few definitions.

Definition 6.0.1. We shall denote by ν the Lebesgue measure in $\mathbb{C}^n$. If $D \subset\subset \mathbb{C}^n$ is a bounded domain and $1 \le p \le \infty$, we shall denote by $L^p(D)$ the usual space of measurable p-integrable complex-valued functions on D, with the norm

$$\|f\|_p = \left[\int_D |f(z)|^p \, d\nu(z)\right]^{1/p}$$

if $1 \le p < \infty$, while $\|f\|_\infty$ will be the essential supremum of $|f|$ in D. Given $\beta \in \mathbb{R}$, we shall also consider the *weighted L^p-spaces* $L^p(D,\beta)$, which are the L^p spaces with respect to the measure $\delta^\beta \nu$, where $\delta \colon D \to \mathbb{R}^+$ is the Euclidean distance from the boundary: $\delta(z) = d(z, \partial D)$. The norm in $L^p(D,\beta)$ is given by

$$\|f\|_{p,\beta} = \left[\int_D |f(z)|^p \delta(z)^\beta \, d\nu(z)\right]^{1/p}$$

for $1 \le p < \infty$, and by $\|f\|_{\beta,\infty} = \|f\delta^\beta\|_\infty$ for $p = \infty$.

Definition 6.0.2. Let $D \subset\subset \mathbb{C}^n$ be a bounded domain in $\mathbb{C}^n$, and $1 \le p \le \infty$. The *Bergman space* $A^p(D)$ is the Banach space $A^p(D) = L^p(D) \cap \mathrm{Hol}(D,\mathbb{C})$ endowed with the norm $\|\cdot\|_p$. More generally, given $\beta \in \mathbb{R}$ the *weighted Bergman space* $A^p(D,\beta)$ is the Banach space $A^p(D,\beta) = L^p(D,\beta) \cap \mathrm{Hol}(D,\mathbb{C})$ endowed with the norm $\|\cdot\|_{p,\beta}$.

The Bergman space $A^2(D)$ is a Hilbert space; this allows us to introduce one of the most studied objects in complex analysis.

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Definition 6.0.3. Let $D \subset\subset \mathbb{C}^n$ be a bounded domain in $\mathbb{C}^n$. The *Bergman projection* is the orthogonal projection $P\colon L^2(D) \to A^2(D)$.

It is a classical fact (see, e.g., [107, Section 1.4] for proofs) that the Bergman projection is an integral operator: it exists a function $K\colon D \times D \to \mathbb{C}$ such that

$$Pf(z) = \int_D K(z,w)f(w)\,\mathrm{d}\nu(w) \tag{6.1}$$

for all $f \in L^2(D)$. It turns out that K is holomorphic in the first argument, $K(w,z) = \overline{K(z,w)}$ for all z, $w \in D$, and it is a *reproducing kernel* for $A^2(D)$ in the sense that

$$f(z) = \int_D K(z,w)f(w)\,\mathrm{d}\nu(w)$$

for all $f \in A^2(D)$.

Definition 6.0.4. Let $D \subset\subset \mathbb{C}^n$ be a bounded domain in $\mathbb{C}^n$. The function $K\colon D \times D \to \mathbb{C}$ satisfying (6.1) is the *Bergman kernel* of D.

Remark 6.0.5. It is not difficult to show (see again, e.g., [107, Section 1.4]) that $K(\cdot,w) \in A^2(D)$ for all $w \in D$, and that

$$\|K(\cdot,w)\|_2^2 = K(w,w) > 0\;.$$

In case $D = \mathbb{B}$ the unit ball, the explicit formula is given in Section 3.3.

A classical result in complex analysis says that in strongly pseudoconvex domains the Bergman projection can be extended to all L^p spaces:

Theorem 6.0.6 (Phong-Stein, [129]). *Let $D \subset\subset \mathbb{C}^n$ be a strongly pseudoconvex domain with C^∞ boundary, and $1 \le p \le \infty$. Then the formula (6.1) defines a continuous operator P from $L^p(D)$ to $A^p(D)$. Furthermore, for any $r > p$ there is $f \in L^p(D)$ such that $Pf \notin A^r(D)$.*

Recently, Čučković and McNeal posed the following question: does there exist a natural operator, somewhat akin to the Bergman projection, mapping $L^p(D)$ into $A^r(D)$ for some $r > p$? To answer this question, they considered Toeplitz operators.

Definition 6.0.7. Let $D \subset\subset \mathbb{C}^n$ be a strongly pseudoconvex domain with C^∞ boundary. Given a measurable function $\psi\colon D \to \mathbb{C}$, the *multiplication operator* of *symbol* ψ is simply defined by $M_\psi(f) = \psi f$. Given $1 \le p \le \infty$, a symbol ψ is *p-admissible* if M_ψ sends $L^p(D)$ into itself; for instance, a $\psi \in L^\infty(D)$ is p-admissible for all p. If ψ is p-admissible, the *Toeplitz operator* $T_\psi\colon L^p(D) \to A^p(D)$ of *symbol* ψ is defined by $T_\psi = P \circ M_\psi$, that is

$$T_\psi(f)(z) = P(\psi f)(z) = \int_D K(z,w)f(w)\psi(w)\,\mathrm{d}\nu(w)\;.$$

Remark 6.0.8. More generally, if A is a Banach algebra, $B \subset A$ is a Banach subspace, $P \colon A \to B$ is a projection and $\psi \in A$, the Toeplitz operator T_ψ of symbol ψ is defined by $T_\psi(f) = P(\psi f)$. Toeplitz operators are a much studied topic in functional analysis; see, e.g., [143].

Then Čučković and McNeal were able to prove the following result:

Theorem 6.0.9 (Čučković-McNeal, [50]). *Let $D \subset\subset \mathbb{C}^n$ be a strongly pseudoconvex domain with C^∞ boundary. If $1 < p < \infty$ and $0 \le \beta < n+1$ are such that*

$$\frac{n+1}{n+1-\beta} < \frac{p}{p-1} \tag{6.2}$$

then the Toeplitz operator T_{δ^β} maps continuously $L^p(D)$ in $A^{p+G}(D)$, where

$$G = \frac{p^2}{\frac{n+1}{\beta} - p} \; .$$

Čučković and McNeal also asked whether the gain G in integrability is optimal; they were able to positively answer to this question only for $n = 1$. The positive answer in higher dimension has been given by Abate, Raissy and Saracco [11], as a corollary of their study of a larger class of Toeplitz operators on strongly pseudoconvex domains. This study, putting into play another important notion in complex analysis, the one of Carleson measure, used as essential tool the Kobayashi distance; in the next couple of sections we shall describe the gist of their results.

6.1 Definitions and results

In this subsection and the next D will always be a bounded strongly pseudoconvex domain with C^∞ boundary. We believe that the results might be generalized to other classes of domains with C^∞ boundary (e.g., finite type domains), and possibly to domains with less smooth boundary, but we will not pursue this subject here.

Let us introduce the main player in this subject.

Definition 6.1.1. Let $D \subset\subset \mathbb{C}^n$ be a strongly pseudoconvex domain with C^∞ boundary, and μ a finite positive Borel measure on D. Then the *Toeplitz operator* T_μ of *symbol* μ is defined by

$$T_\mu(f)(z) = \int_D K(z,w) f(w) \, \mathrm{d}\mu(w) \; ,$$

where K is the Bergman kernel of D.

For instance, if ψ is an admissible symbol then the Toeplitz operator T_ψ defined above is the Toeplitz operator $T_{\psi\nu}$ according to Definition 6.1.1.

In Definition 6.1.1 we did not specify domain and/or range of the Topelitz operator μ because the main point of the theory we are going to discuss is exactly to link properties of the measure μ with domain and range of T_μ.

Toeplitz operators associated to measures have been extensively studied on the unit disc Δ and on the unit ball $\mathbb{B}^n$ (see, e.g., [115], [116], [91], [153] and references therein); but [11] has been one of the first papers studying them in strongly pseudoconvex domains.

The kind of measure we shall be interested in is described in the following

Definition 6.1.2. Let $D \subset\subset \mathbb{C}^n$ be a strongly pseudoconvex domain with C^∞ boundary, A a Banach space of complex-valued functions on D, and $1 \le p \le \infty$. We shall say that a finite positive Borel measure μ on D is a *p-Carleson measure* for A if A embeds continuously into $L^p(\mu)$, that is if there exists $C > 0$ such that

$$\int_D |f(z)|\,\mathrm{d}\mu(z) \le C\|f\|_A^p$$

for all $f \in A$, where $\|\cdot\|_A$ is the norm in A.

Remark 6.1.3. When the inclusion $A \hookrightarrow L^p(\mu)$ is compact, μ is called *vanishing Carleson measure*. Here we shall discuss vanishing Carleson measures only in the remarks.

Carleson measures for the Hardy spaces $H^p(\Delta)$ were introduced by Carleson [46] to solve the famous corona problem. We shall be interested in Carleson measures for the weighted Bergman spaces $A^p(D,\beta)$; they have been studied by many authors when $D = \Delta$ or $D = \mathbb{B}^n$ (see, e.g., [117], [53], [153] and references therein), but more rarely when D is a strongly pseudoconvex domain (see, e.g., [49] and [9]).

The main point here is to give a geometric characterization of which measures are Carleson. To this aim we introduce the following definition, bringing into play the Kobayashi distance.

Definition 6.1.4. Let $D \subset\subset \mathbb{C}^n$ be a strongly pseudoconvex domain with C^∞ boundary, and $\theta > 0$. We shall say that a finite positive Borel measure μ on D is θ-*Carleson* if there exists $r > 0$ and $C_r > 0$ such that

$$\mu\big(B_D(z_0,r)\big) \le C_r \nu\big(B_D(z_0,r)\big)^\theta \tag{6.3}$$

for all $z_0 \in D$. We shall see that if (6.3) holds for some $r > 0$ then it holds for all $r > 0$.

Remark 6.1.5. There is a parallel vanishing notion: we say that μ is *vanishing* θ-*Carleson* if there exists $r > 0$ such that

$$\lim_{z_0 \to \partial D} \frac{\mu\big(B_D(z_0,r)\big)}{\nu\big(B_D(z_0,r)\big)^\theta} = 0\,.$$

For later use, we recall two more definitions.

Definition 6.1.6. Let $D \subset\subset \mathbb{C}^n$ be a strongly pseudoconvex domain with C^∞ boundary. Given $w \in D$, the *normalized Bergman kernel* in w is given by

$$k_w(z) = \frac{K(z,w)}{\sqrt{K(w,w)}} \ .$$

Remark 6.0.5 shows that $k_w \in A^2(D)$ and $\|k_w\|_2 = 1$ for all $w \in D$.

Definition 6.1.7. Let $D \subset\subset \mathbb{C}^n$ be a strongly pseudoconvex domain with C^∞ boundary, and μ a finite positive Borel measure on D. The *Berezin transform* of μ is the function $B\mu\colon D \to \mathbb{R}^+$ defined by

$$B\mu(z) = \int_D |k_z(w)|^2 \, \mathrm{d}\mu(w) \ .$$

Again, part of the theory will describe when the Berezin transform of a measure is actually defined.

We can now state the main results obtained in [11]:

Theorem 6.1.8 (Abate-Raissy-Saracco, [11]). *Let $D \subset\subset \mathbb{C}^n$ be a strongly pseudoconvex domain with C^∞ boundary, $1 < p < r < \infty$ and μ a finite positive Borel measure on D. Then T_μ maps $A^p(D)$ into $A^r(D)$ if and only if μ is a p-Carleson measure for $A^p\big(D, (n+1)(\frac{1}{p} - \frac{1}{r})\big)$.*

Theorem 6.1.9 (Abate-Raissy-Saracco, [11]). *Let $D \subset\subset \mathbb{C}^n$ be a strongly pseudoconvex domain with C^∞ boundary, $1 < p < \infty$ and $\theta \in \big(1 - \frac{1}{n+1}, 2\big)$. Then a finite positive Borel measure μ on D is a p-Carleson measure for $A^p\big(D, (n+1)(\theta - 1)\big)$ if and only if μ is a θ-Carleson measure.*

Theorem 6.1.10 (Abate-Raissy-Saracco, [11]). *Let $D \subset\subset \mathbb{C}^n$ be a strongly pseudoconvex domain with C^∞ boundary, and $\theta > 0$. Then a finite positive Borel measure μ on D is θ-Carleson if and only the Berezin transform $B\mu$ exists and $\delta^{(n+1)(1-\theta)} B\mu \in L^\infty(D)$.*

Remark 6.1.11. This is just a small selection of the results contained in [11]. There one can find statements also for $p = 1$ or $p = \infty$, for other values of θ, and on the mapping properties of Toeplitz operators on weighted Bergman spaces. Furthermore, there it is also shown that T_μ is a compact operator from $A^p(D)$ into $A^r(D)$ if and only if μ is a vanishing p-Carleson measure for $A^p\big(D, (n+1)(\frac{1}{p} - \frac{1}{r})\big)$; that μ is a vanishing p-Carleson measure for $A^p\big(D, (n+1)(\theta - 1)\big)$ if and only if μ is a vanishing θ-Carleson measure; and that μ is a vanishing θ-Carleson measure if and only if $\delta^{(n+1)(1-\theta)}(z) B\mu(z) \to 0$ as $z \to \partial D$.

Remark 6.1.12. The condition "p-Carleson" is independent of any radius $r > 0$, while the condition "θ-Carleson" does not depend on p. Theorem 6.1.9 thus implies that if μ satisfies (6.3) for some $r > 0$ then it satisfies the same condition (with possibly different constants) for all $r > 0$; and that if μ is p-Carleson for $A^p\big(D, (n+1)(\theta - 1)\big)$ for some $1 < p < \infty$ then it is p-Carleson for $A^p\big(D, (n+1)(\theta - 1)\big)$ for all $1 < p < \infty$.

In the next subsection we shall describe the proofs; we end this subsection showing why these results give a positive answer to the question raised by Čučković and McNeal.

Assume that T_{δ^β} maps $L^p(D)$ (and hence $A^p(D)$) into $A^{p+G}(D)$. By Theorem 6.1.8 $\delta^\beta\mu$ must be a p-Carleson measure for $A^p\big(D,(n+1)(\frac{1}{p}-\frac{1}{p+G})\big)$. By Theorem 6.1.9 this can happen if and only if $\delta^\beta\nu$ is a θ-Carleson measure, where

$$\theta = 1+\frac{1}{p}-\frac{1}{p+G}\ ; \tag{6.4}$$

notice that $1\le\theta<2$ because $p>1$ and $G\ge 0$. So we need to understand when $\delta^\beta\nu$ is θ-Carleson. For this we need the following

Lemma 6.1.13. *Let $D\subset\subset\mathbb{C}^n$ be a strongly pseudoconvex domain with C^2 boundary, Then there exists $C>0$ such that for every $z_0\in D$ and $r>0$ one has*

$$\forall z\in B_D(z_0,r)\quad C\mathrm{e}^{2r}\,\delta(z_0)\ge\delta(z)\ge\frac{\mathrm{e}^{-2r}}{C}\,\delta(z_0)\ .$$

Proof. Let us fix $w_0\in D$. Then Theorems 1.5.16 and 1.5.19 yield c_0, $C_0>0$ such that

$$\begin{aligned} c_0-\tfrac{1}{2}\log\delta(z)\ &\le k_D(w_0,z)\le k_D(z_0,z)+k_D(z_0,w_0)\\ &\le r+C_0-\tfrac{1}{2}\log\delta(z_0)\ , \end{aligned}$$

for all $z\in B_D(z_0,r)$, and hence

$$e^{2(c_0-C_0)}\delta(z_0)\le\mathrm{e}^{2r}\delta(z)\ .$$

The left-hand inequality is obtained in the same way reversing the roles of z_0 and z. □

Corollary 6.1.14. *Let $D\subset\subset\mathbb{C}^n$ be a strongly pseudoconvex domain with C^2 boundary, Given $\beta>0$, put $\nu_\beta=\delta^\beta\nu$. Then ν_β is θ-Carleson if and only if $\beta\ge(n+1)(\theta-1)$.*

Proof. Using Lemma 6.1.13 we find that

$$\begin{aligned} \frac{\mathrm{e}^{-2r}}{C}\delta(z_0)^\beta\nu\big(B_D(z_0,r)\big)\ &\le\ \nu_\beta\big(B_D(z_0,r)\big)=\int_{B_D(z_0,r)}\delta(z)^\beta\,\mathrm{d}\nu(z)\\ &\le C\mathrm{e}^{2r}\delta(z_0)^\beta\nu\big(B_D(z_0,r)\big) \end{aligned}$$

for all $z_0\in D$. Therefore ν_β is θ-Carleson if and only if

$$\delta(z_0)^\beta\le C_1\nu\big(B_D(z_0,r)\big)^{\theta-1}$$

for some $C_1>0$. Recalling Theorem 1.5.23 we see that this is equivalent to requiring $\beta\ge(n+1)(\theta-1)$, and we are done. □

In our case, θ is given by (6.4); therefore $\beta \geq (n+1)(\theta-1)$ if and only if

$$\beta \geq (n+1)\left(\frac{1}{p} - \frac{1}{p+G}\right) .$$

Rewriting this in term of G we get

$$G \leq \frac{p^2}{\frac{n+1}{\beta} - p} ,$$

proving that the exponent in Theorem 6.0.9 is the best possible, as claimed. Furthermore, $G > 0$ if and only if

$$\frac{\beta}{n+1} < \frac{1}{p} \Leftrightarrow 1 - \frac{\beta}{n+1} > 1 - \frac{1}{p} \Leftrightarrow \frac{n+1}{n+1-\beta} < \frac{p}{p-1} ,$$

and we have also recovered condition (6.2) of Theorem 6.0.9.

Corollary 6.1.14 provides examples of θ-Carleson measures. A completely different class of examples is provided by Dirac masses distributed along uniformly discrete sequences.

Definition 6.1.15. Let (X,d) be a metric space. A sequence $\Gamma = \{x_j\} \subset X$ is *uniformly discrete* if there exists $\varepsilon > 0$ such that $d(x_j,x_k) \geq \varepsilon$ for all $j \neq k$.

Then it is possible to prove the following result:

Theorem 6.1.16 ([11]). *Let $D \subset\subset \mathbb{C}^n$ be a bounded strongly pseudoconvex domain with C^∞ boundary, considered as a metric space with the Kobayashi distance, and choose $1 - \frac{1}{n+1} < \theta < 2$. Let $\Gamma = \{z_j\}_{j\in\mathbb{N}}$ be a sequence in D. Then Γ is a finite union of uniformly discrete sequences if and only if $\sum_j \delta(z_j)^{(n+1)\theta}\delta_{z_j}$ is a θ-Carleson measure, where δ_{z_j} is the Dirac measure in z_j.*

6.2 Proofs

In this section we shall prove Theorems 6.1.8, 6.1.9 and 6.1.10. To do so we shall need a few technical facts on the Bergman kernel and on the Kobayashi distance. To simplify statements and proofs, let us introduce the following notation.

Definition 6.2.1. Let $D \subset \mathbb{C}^n$ be a domain. Given two non-negative functions f, $g\colon D \to \mathbb{R}^+$ we shall write $f \preceq g$ or $g \succeq f$ to say that there is $C > 0$ such that $f(z) \leq Cg(z)$ for all $z \in D$. The constant C is independent of $z \in D$, but it might depend on other parameters (r, θ, etc.).

The first technical fact we shall need is an integral estimate on the Bergman kernel:

Theorem 6.2.2 ([115], [120], [11]). *Let $D \subset\subset \mathbb{C}^n$ be a strongly pseudoconvex domain with C^∞ boundary. Take $p \geq 1$ and $\beta > -1$. Then*

$$\int_D |K(w,z_0)|^p \delta(w)^\beta \, \mathrm{d}\nu(w) \preceq \begin{cases} \delta(z_0)^{\beta-(n+1)(p-1)} & if -1 < \beta < (n+1)(p-1), \\ |\log \delta(z_0)| & if \beta = (n+1)(p-1), \\ 1 & if \beta > (n+1)(p-1), \end{cases}$$

for all $z_0 \in D$.

In particular, we have the following estimates on the weighted norms of the Bergman kernel and of the normalized Bergman kernel (see, e.g., [11]):

Corollary 6.2.3. *Let $D \subset\subset \mathbb{C}^n$ be a strongly pseudoconvex domain with C^∞ boundary. Take $p > 1$ and $-1 < \beta < (n+1)(p-1)$. Then*

$$\|K(\cdot,z_0)\|_{p,\beta} \preceq \delta(z_0)^{\frac{\beta}{p}-\frac{n+1}{p'}} \quad \text{and} \quad \|k_{z_0}\|_{p,\beta} \preceq \delta(z_0)^{\frac{n+1}{2}+\frac{\beta}{p}-\frac{n+1}{p'}}$$

for all $z_0 \in D$, where $p' > 1$ is the conjugate exponent of p.

We shall also need a statement relating the Bergman kernel with Kobayashi balls.

Lemma 6.2.4 ([115], [9]). *Let $D \subset\subset \mathbb{C}^n$ be a strongly pseudoconvex domain with C^∞ boundary. Given $r > 0$ there is $\delta_r > 0$ such that if $\delta(z_0) < \delta_r$ then*

$$\forall z \in B_D(z_0,r) \qquad \min\{|K(z,z_0)|, |k_{z_0}(z)|^2\} \succeq \delta(z_0)^{-(n+1)} .$$

Remark 6.2.5. Notice that Lemma 6.2.4 implies the well-known estimate

$$K(z_0,z_0) \succeq \delta(z_0)^{-(n+1)} ,$$

which is valid for all $z_0 \in D$.

The next three lemmas involve instead the Kobayashi distance only.

Lemma 6.2.6 ([9]). *Let $D \subset\subset \mathbf{C}^n$ be a strongly pseudoconvex bounded domain with C^2 boundary. Then for every $0 < r < R$ there exist $m \in \mathbb{N}$ and a sequence $\{z_k\} \subset D$ of points such that $D = \bigcup_{k=0}^\infty B_D(z_k,r)$ and no point of D belongs to more than m of the balls $B_D(z_k,R)$.*

Proof. Let $\{B_j\}_{j\in\mathbb{N}}$ be a sequence of Kobayashi balls of radius $r/3$ covering D. We can extract a subsequence $\{\Delta_k = B_D(z_k,r/3)\}_{k\in\mathbb{N}}$ of disjoint balls in the following way: set $\Delta_1 = B_1$. Suppose we have already chosen $\Delta_1,\ldots,\Delta_l$. We define Δ_{l+1} as the first ball in the sequence $\{B_j\}$ which is disjoint from $\Delta_1 \cup \cdots \cup \Delta_l$. In particular, by construction every B_j must intersect at least one Δ_k.

We now claim that $\{B_D(z_k,r)\}_{k\in\mathbb{N}}$ is a covering of D. Indeed, let $z \in D$. Since $\{B_j\}_{j\in\mathbb{N}}$ is a covering of D, there is $j_0 \in \mathbb{N}$ so that $z \in B_{j_0}$. As remarked above, we get $k_0 \in \mathbb{N}$ so that $B_{j_0} \cap \Delta_{k_0} \neq \emptyset$. Take $w \in B_{j_0} \cap \Delta_{k_0}$. Then

$$k_D(z,z_{k_0}) \le k_D(z,w) + k_D(w,z_{k_0}) < r\ ,$$

and $z \in B_D(z_{k_0}, r)$.

To conclude the proof we have to show that there is $m = m_r \in \mathbb{N}$ so that each point $z \in D$ belongs to at most m of the balls $B_D(z_k, R)$. Put $R_1 = R + r/3$. Since $z \in B_D(z_k, R)$ is equivalent to $z_k \in B_D(z, R)$, we have that $z \in B_D(z_k, R)$ implies $B_D(z_k, r/3) \subseteq B_D(z, R_1)$. Furthermore, Theorem 1.5.23 and Lemma 6.1.13 yield

$$\nu\big(B_D(z_k, r/3)\big) \succeq \delta(z_k)^{n+1} \succeq \delta(z)^{n+1}$$

when $z_k \in B_D(z, R)$. Therefore, since the balls $B_D(z_k, r/3)$ are pairwise disjoint, using again Theorem 1.5.23 we get

$$\mathrm{card}\{k \in \mathbb{N} \mid z \in B_D(z_k, R)\} \le \frac{\nu\big(B_D(z, R_1)\big)}{\nu\big(B_D(z_k, r/3)\big)} \preceq 1\ ,$$

and we are done. □

Lemma 6.2.7 ([9]). *Let $D \subset\subset \mathbf{C}^n$ be a strongly pseudoconvex bounded domain with C^2 boundary, and $r > 0$. Then*

$$\chi(z_0) \preceq \frac{1}{\nu\big(B_D(z_0, r)\big)} \int_{B_D(z_0,r)} \chi\, \mathrm{d}\nu$$

for all $z_0 \in D$ and all non-negative plurisubharmonic functions $\chi\colon D \to \mathbf{R}^+$.

Proof. Let us first prove the statement when D is an Euclidean ball $\mathbb{B}$ of radius $R > 0$. Without loss of generality we can assume that B is centered at the origin. Fix $z_0 \in \mathbb{B}$, let $\gamma_{z_0/R} \in \mathrm{Aut}(\mathbb{B}^n)$ be given by (2.10), and let $\Phi_{z_0}\colon \mathbb{B}^n \to \mathbb{B}$ be defined by $\Phi_{z_0} = R\gamma_{z_0/R}$; in particular, Φ_{z_0} is a biholomorphism with $\Phi_{z_0}(O) = z_0$, and thus $\Phi_{z_0}\big(B_{\mathbb{B}^n}(O,\hat r)\big) = B_{\mathbb{B}}(z_0, \hat r)$. Furthermore (see [138, Theorem 2.2.6])

$$|\mathrm{Jac}_{\mathbb{R}} \Phi_{z_0}(z)| = R^{2n}\left(\frac{R^2 - \|z_0\|^2}{|R - \langle z, z_0\rangle|^2}\right)^{n+1} \ge \frac{R^{n-1}}{4^{n+1}}\, d(z_0, \partial\mathbb{B})^{n+1}\ ,$$

where $\mathrm{Jac}_{\mathbb{R}}\Phi_{z_0}$ denotes the (real) Jacobian determinant of Φ_{z_0}. It follows that

$$\begin{aligned}\int_{B_{\mathbb{B}}(z_0,r)} \chi\, \mathrm{d}\nu &= \int_{B_{\mathbb{B}^n}(O,r)} (\chi \circ \Phi_{z_0}) |\mathrm{Jac}_{\mathbb{R}}\, \Phi_{z_0}|\, \mathrm{d}\nu \\ &\ge \frac{R^{n-1}}{4^{n+1}}\, d(z_0, \partial\mathbb{B})^{n+1} \int_{B_{\mathbb{B}^n}(O,r)} (\chi \circ \Phi_{z_0})\, \mathrm{d}\nu\ .\end{aligned}$$

Using [138, 1.4.3 and 1.4.7.(1)] we obtain

$$\int_{B_{\mathbb{B}^n}(O,r)} (\chi \circ \Phi_{z_0})\, \mathrm{d}\nu = 2n \int_{\partial\mathbb{B}^n} \mathrm{d}\sigma(x) \frac{1}{2\pi} \int_0^{\tanh r} \int_0^{2\pi} \chi \circ \Phi_{z_0}(t e^{\mathrm{i}\theta} x) t^{2n-1} \mathrm{d}t\, \mathrm{d}\theta\ ,$$

where σ is the area measure on $\partial\mathbb{B}^n$ normalized so that $\sigma(\partial\mathbb{B}^n) = 1$. Now, $\zeta \mapsto \chi \circ \Phi_{z_0}(\zeta x)$ is subharmonic on $(\tanh r)\Delta = \{|\zeta| < \tanh r\} \subset \mathbb{C}$ for any $x \in \partial\mathbb{B}^n$, since Φ_{z_0} is holomorphic and χ is plurisubharmonic. Therefore [81, Theorem 1.6.3] yields

$$\frac{1}{2\pi}\int_0^{\tanh r}\int_0^{2\pi} \chi \circ \Phi_{z_0}(te^{i\theta}x)t^{2n-1}\mathrm{d}t\,\mathrm{d}\theta \geq \chi(z_0)\int_0^{\tanh r} t^{2n-1}\,\mathrm{d}t = \frac{1}{2n}(\tanh r)^{2n}\chi(z_0)\,.$$

So

$$\int_{B_{\mathbb{B}^n}(O,r)} (\chi \circ \Phi_{z_0})\,\mathrm{d}\nu \geq (\tanh r)^{2n}\chi(z_0)\,,$$

and the assertion follows from Theorem 1.5.23.

Now let D be a generic strongly pseudoconvex domain. Since D has C^2 boundary, there exists a radius $\varepsilon > 0$ such that for every $x \in \partial D$ the euclidean ball $\mathbb{B}_x(\varepsilon)$ of radius ε internally tangent to ∂D at x is contained in D.

Let $z_0 \in D$. If $\delta(z_0) < \varepsilon$, let $x \in \partial D$ be such that $\delta(z_0) = \|z_0 - x\|$; in particular, z_0 belongs to the ball $\mathbb{B}_x(\varepsilon) \subset D$. If $\delta(z_0) \geq \varepsilon$, let $\mathbb{B} \subset D$ be the Euclidean ball of center z_0 and radius $\delta(z_0)$. In both cases we have $\delta(z_0) = d(z_0, \partial\mathbb{B})$; moreover, the decreasing property of the Kobayashi distance yields $B_D(z_0,r) \supseteq B_{\mathbb{B}}(z_0,r)$ for all $r > 0$.

Let χ be a non-negative plurisubharmonic function. Then Theorem 1.5.23 and the assertion for a ball imply

$$\begin{aligned}\int_{B_D(z_0,r)} \chi\, d\nu \geq \int_{B_{\mathbb{B}}(z_0,r)} \chi\, d\nu &\succeq \nu\big(B_B(z_0,r)\big)\chi(z_0)\\ &\succeq d(z_0,\partial\mathbb{B})^{n+1}\chi(z_0) = \delta(z_0)^{n+1}\chi(z_0)\\ &\succeq \nu\big(B_D(z_0,r)\big)\chi(z_0)\,,\end{aligned}$$

and we are done. □

Lemma 6.2.8 ([9]). *Let $D \subset\subset \mathbf{C}^n$ be a strongly pseudoconvex bounded domain with C^2 boundary. Given $0 < r < R$ we have*

$$\forall z_0 \in D \quad \forall z \in B_D(z_0,r) \qquad \chi(z) \preceq \frac{1}{\nu\big(B_D(z_0,r)\big)}\int_{B_D(z_0,R)} \chi\,\mathrm{d}\nu$$

for every nonnegative plurisubharmonic function $\chi\colon D \to \mathbb{R}^+$.

Proof. Let $r_1 = R - r$; by the triangle inequality, $z \in B_D(z_0,r)$ yields $B_D(z,r_1) \subseteq B_D(z_0,R)$. Lemma 6.2.7 then implies

$$\begin{aligned}\chi(z) &\preceq \frac{1}{\nu(B_D(z,r_1))}\int_{B_D(z,r_1)} \chi\,\mathrm{d}\nu\\ &\leq \frac{1}{\nu(B_D(z,r_1))}\int_{B_D(z_0,R)} \chi\,\mathrm{d}\nu = \frac{\nu(B_D(z_0,r))}{\nu(B_D(z,r_1))}\cdot\frac{1}{\nu(B_D(z_0,r))}\int_{B_D(z_0,R)} \chi\,\mathrm{d}\nu\end{aligned}$$

for all $z \in B_D(z_0,r)$. Now Theorem 1.5.23 and Lemma 6.1.13 yield

$$\frac{\nu(B_D(z_0,r))}{\nu(B_D(z,r_1))} \preceq 1$$

for all $z \in B_D(z_0,r)$, and so

$$\chi(z) \preceq \frac{1}{\nu\big(B_D(z_0,r)\big)} \int_{B_D(z_0,R)} \chi \, d\nu$$

as claimed. ◻

Finally, the linking between the Berezin transform and Toeplitz operators is given by the following

Lemma 6.2.9. *Let μ be a finite positive Borel measure on a bounded domain $D \subset\subset \mathbb{C}^n$. Then*

$$B\mu(z) = \int_D (T_\mu k_z)(w)\overline{k_z(w)} \, d\nu(w) \tag{6.5}$$

for all $z \in D$.

Proof. Indeed using Fubini's theorem and the reproducing property of the Bergman kernel we have

$$\begin{aligned}
B\mu(z) &= \int_D \frac{|K(x,z)|^2}{K(z,z)} \, d\mu(x) \\
&= \int_D \frac{K(x,z)}{K(z,z)} K(z,x) \, d\mu(x) \\
&= \int_D \frac{K(x,z)}{K(z,z)} \left(\int_D K(w,x)K(z,w) \, d\nu(w) \right) d\mu(x) \\
&= \int_D \left(\int_D \frac{K(x,z)}{\sqrt{K(z,z)}} K(w,x) \, d\mu(x) \right) \frac{\overline{K(w,z)}}{\sqrt{K(z,z)}} \, d\nu(w) \\
&= \int_D \left(\int_D K(w,x)k_z(x) \, d\mu(x) \right) \overline{k_z(w)} \, d\nu(w) \\
&= \int_D (T_\mu k_z)(w)\overline{k_z(w)} \, d\nu(w) \, ,
\end{aligned}$$

as claimed. ◻

We can now prove Theorems 6.1.8, 6.1.9 and 6.1.10.

Proof (of Theorem 6.1.9). Assume that μ is a p-Carleson measure for $A^p\big(D,(n+1)(\theta-1)\big)$, and fix $r > 0$; we need to prove that $\mu\big(B_D(z_0,r)\big) \preceq \nu\big(B_D(z_0,r)\big)^\theta$ for all $z_0 \in D$.

First of all, it suffices to prove the assertion for z_0 close to the boundary, because both μ and ν are finite measures. So we can assume $\delta(z_0) < \delta_r$, where δ_r is given by Lemma 6.2.4. Since, by Corollary 6.2.3, $k_{z_0}^2 \in A^p\big(D,(n+1)(\theta-1)\big)$, we have

$$\begin{aligned}\frac{1}{\delta(z_0)^{(n+1)p}}\mu\big(B_D(z_0,r)\big) &\preceq \int_{B_D(z_0,r)}|k_{z_0}(w)|^{2p}\,\mathrm{d}\mu(w) \le \int_D |k_{z_0}(w)|^{2p}\,\mathrm{d}\mu(w)\\ &\preceq \int_D |k_{z_0}(w)|^{2p}\delta(w)^{(n+1)(\theta-1)}\,\mathrm{d}\nu(w)\\ &\preceq \delta(z_0)^{(n+1)p}\int_D |K(w,z_0)|^{2p}\delta(w)^{(n+1)(\theta-1)}\,\mathrm{d}\nu(w)\\ &\preceq \delta(z_0)^{(n+1)(\theta-p)}\end{aligned}$$

by Theorem 6.2.2, that we can apply because $1-\frac{1}{n+1}<\theta<2$. Recalling Theorem 1.5.23 we see that μ is θ-Carleson.

Conversely, assume that μ is θ-Carleson for some $r>0$, and let $\{z_k\}$ be the sequence given by Lemma 6.2.6. Take $f\in A^p\big(D,(n+1)(\theta-1)\big)$. First of all

$$\int_D |f(z)|^p\,\mathrm{d}\mu(z) \le \sum_{k\in\mathbb{N}}\int_{B_D(z_k,r)}|f(z)|^p\,\mathrm{d}\mu(z)\;.$$

Choose $R>r$. Since $|f|^p$ is plurisubharmonic, by Lemma 6.2.8 we get

$$\begin{aligned}\int_{B_D(z_k,r)}|f(z)|^p\,\mathrm{d}\mu(z) &\preceq \frac{1}{\nu\big(B_D(z_k,r)\big)}\int_{B_D(z_k,r)}\left[\int_{B_D(z_k,R)}|f(w)|^p\,\mathrm{d}\nu(w)\right]\mathrm{d}\mu(z)\\ &\preceq \nu\big(B_D(z_k,r)\big)^{\theta-1}\int_{B_D(z_k,R)}|f(w)|^p\,\mathrm{d}\nu(w)\end{aligned}$$

because μ is θ-Carleson. Recalling Theorem 1.5.23 and Lemma 6.1.13 we get

$$\begin{aligned}\int_{B_D(z_k,r)}|f(z)|^p\,\mathrm{d}\mu(z) &\preceq \delta(z_k)^{(n+1)(\theta-1)}\int_{B_D(z_k,R)}|f(w)|^p\,\mathrm{d}\nu(w)\\ &\preceq \int_{B_D(z_k,R)}|f(w)|^p\delta(w)^{(n+1)(\theta-1)}\,\mathrm{d}\nu(w)\;.\end{aligned}$$

Since, by Lemma 6.2.6, there is $m\in\mathbb{N}$ such that at most m of the balls $B_D(z_k,R)$ intersect, we get

$$\int_D |f(z)|^p\,\mathrm{d}\mu(z) \preceq \int_D |f(w)|^p\delta(w)^{(n+1)(\theta-1)}\,\mathrm{d}\nu(w)\;,$$

and so we have proved that μ is p-Carleson for $A^p\big(D,(n+1)(\theta-1)\big)$. □

We explicitly remark that the proof of the implication "θ-Carleson implies p-Carleson for $A^p\big(D,(n+1)(\theta-1)\big)$" works for all $\theta>0$, and actually gives the following

Corollary 6.2.10. *Let $D\subset\subset\mathbb{C}^n$ be a bounded strongly pseudoconvex domain with C^2 boundary, $\theta>0$, and μ a θ-Carleson measure on D. Then*

$$\int_D \chi(z)\,d\mu(z) \preceq \int_D \chi(w)\delta(w)^{(n+1)(\theta-1)}\,\mathrm{d}\nu(w)$$

for all nonnegative plurisubharmonic functions $\chi\colon D\to\mathbb{R}^+$ such that $\chi\in L^p\big(D,(n+1)(\theta-1)\big)$.

Now we prove the equivalence between θ-Carleson and the condition on the Berezin transform.

Proof (of Theorem 6.1.10). Let us first assume that μ is θ-Carleson. By Theorem 6.1.9 we know that μ is 2-Carleson for $A^2\big(D,(n+1)(\theta-1)\big)$. Fix $z_0\in D$. Then Corollary 6.2.3 yields

$$B\mu(z_0)=\int_D |k_{z_0}(w)|^2\,\mathrm{d}\mu(w)\preceq \|k_{z_0}\|^2_{2,(n+1)(\theta-1)}\preceq \delta(z_0)^{(n+1)(\theta-1)}\,,$$

as required.

Conversely, assume that $\delta^{(n+1)(1-\theta)}B\mu\in L^\infty(D)$, and fix $r>0$. Then Lemma 6.2.4 yields

$$\begin{aligned}\delta(z_0)^{(n+1)(\theta-1)}\succeq B\mu(z_0)&=\int_D |k_{z_0}(w)|^2\,\mathrm{d}\mu(w)\geq \int_{B_D(z_0,r)}|k_{z_0}(w)|^2\,\mathrm{d}\mu(w)\\&\succeq \frac{1}{\delta(z_0)^{n+1}}\mu\big(B_D(z_0,r)\big)\end{aligned}$$

as soon as $\delta(z_0)<\delta_r$, where $\delta_r>0$ is given by Lemma 6.2.4. Recalling Theorem 1.5.23 we get

$$\mu\big(B_D(z_0,r)\big)\preceq\delta(z_0)^{(n+1)\theta}\preceq\nu\big(B_D(z_0,r)\big)^\theta\,,$$

and the assertion follows when $\delta(z_0)<\delta_r$. When $\delta(z_0)\geq\delta_r$ we have

$$\mu\big(B_D(z_0,r)\big)\leq\mu(D)\preceq\delta_r^{(n+1)\theta}\leq\delta(z_0)^{(n+1)\theta}\preceq\nu\big(B_D(z_0,r)\big)^\theta$$

because μ is a finite measure, and we are done. □

For the last proof we need a final

Lemma 6.2.11. *Let $D\subset\subset\mathbb{C}^n$ be a bounded stongly pseudoconvex domain with C^2 boundary, and θ, $\eta\in\mathbb{R}$. Then a finite positive Borel measure μ is θ-Carleson if and only if $\delta^\eta\mu$ is $(\theta+\frac{\eta}{n+1})$-Carleson.*

Proof. Assume μ is θ-Carleson, set $\mu_\eta=\delta^\eta\mu$, and choose $r>0$. Then Theorem 1.5.23 and Lemma 6.1.13 yield

$$\begin{aligned}\mu_\eta\big(B_D(z_0,r)\big)&=\int_{B_D(z_0,r)}\delta(w)^\eta\,d\mu(w)\preceq\delta(z_0)^\eta\mu\big(B_D(z_0,r)\big)\\&\preceq\delta(z_0)^\eta\nu\big(B_D(z_0,r)\big)^\theta\preceq\nu\big(B_D(z_0,r)\big)^{\theta+\frac{\eta}{n+1}}\,,\end{aligned}$$

and so μ_η is $\big(\theta+\frac{\eta}{n+1}\big)$-Carleson. Since $\mu=(\mu_\eta)_{-\eta}$, the converse follows too. □

And at last we have reached the

Proof (of Theorem 6.1.8). Let us assume that T_μ maps $A^p(D)$ continuously into $A^r(D)$, and let r' be the conjugate exponent of r. Since, by Corollary 6.2.3, $k_{z_0} \in A^q(D)$ for all $q > 1$, applying Hölder estimate to (6.5) and using twice Corollary 6.2.3 we get

$$B\mu(z_0) \le \|T_\mu k_{z_0}\|_r \|k_{z_0}\|_{r'} \preceq \|k_{z_0}\|_p \|k_{z_0}\|_{r'} \\ \preceq \delta(z_0)^{(n+1)(1-\frac{1}{p'}-\frac{1}{r})} = \delta(z_0)^{(n+1)(\frac{1}{p}-\frac{1}{r})} ,$$

where p' is the conjugate exponent of p. By Theorem 6.1.10 it follows that μ is $\left(1+\frac{1}{p}-\frac{1}{r}\right)$-Carleson, and Theorem 6.1.9 yields that μ is p-Carleson for $A^p\big(D,(n+1)(\frac{1}{p}-\frac{1}{r})\big)$ as claimed.

Conversely, assume that μ is p-Carleson for $A^p\big(D,(n+1)(\frac{1}{p}-\frac{1}{r})\big)$; we must prove that T_μ maps continuously $A^p(D)$ into $A^r(D)$. Put $\theta = 1+\frac{1}{p}-\frac{1}{r}$. Choose $s \in (p,r)$ such that

$$\frac{\theta}{p'} < \frac{1}{s'} < \frac{\theta}{p'} + \frac{1}{(n+1)r} , \tag{6.6}$$

where s' be its conjugate exponent of s; this can be done because $p' \ge s' \ge r'$ and

$$\frac{\theta}{p'} < \frac{1}{r'} .$$

Take $f \in A^p(D)$; since $|K(z,\cdot)|^{p'/s'}$ is plurisubharmonic and belongs to $L^p\big(D,(n+1)(\theta-1)\big)$ by Theorem 6.2.2, applying the Hölder inequality, Corollary 6.2.10 and Theorem 6.2.2 (recalling that $\theta < p'/s'$) we get

$$\begin{aligned} |T_\mu f(z)| &\le \int_D |K(z,w)||f(w)|\,\mathrm{d}\mu(w) \\ &\le \left[\int_D |K(z,w)|^{p/s}|f(w)|^p\,\mathrm{d}\mu(w)\right]^{1/p}\left[\int_D |K(z,w)|^{p'/s'}\,\mathrm{d}\mu(w)\right]^{1/p'} \\ &\preceq \left[\int_D |K(z,w)|^{p/s}|f(w)|^p\,\mathrm{d}\mu(w)\right]^{1/p} \\ &\qquad \times \left[\int_D |K(z,w)|^{p'/s'}\delta(w)^{(n+1)(\theta-1)}\,\mathrm{d}\nu(w)\right]^{1/p'} \\ &\preceq \left[\int_D |K(z,w)|^{p/s}|f(w)|^p\,\mathrm{d}\mu(w)\right]^{1/p}\delta(z)^{(n+1)\frac{1}{p'}(\theta-\frac{p'}{s'})} . \end{aligned}$$

Applying the classical Minkowski integral inequality (see, e.g., [57, 6.19] for a proof)

$$\left[\int_D\left[\int_D |F(z,w)|^p\,\mathrm{d}\mu(w)\right]^{r/p}\mathrm{d}\nu(z)\right]^{1/r} \le \left[\int_D\left[\int_D |F(z,w)|^r\,\mathrm{d}\nu(z)\right]^{p/r}\mathrm{d}\mu(w)\right]^{1/p}$$

we get

$$\|T_\mu f\|_r^p \preceq \left[\int_D \left[\int_D |K(z,w)^{p/s}|f(w)|^p \delta(z)^{(n+1)\frac{p}{p'}(\theta-\frac{p'}{s'})}\,\mathrm{d}\mu(w)\right]^{r/p} \mathrm{d}\nu(z)\right]^{p/r}$$
$$\leq \int_D |f(w)|^p \left[\int_D |K(z,w)|^{r/s} \delta(z)^{(n+1)\frac{r}{p'}(\theta-\frac{p'}{s'})}\,\mathrm{d}\nu(z)\right]^{p/r} \mathrm{d}\mu(w)\,.$$

To estimate the integral between square brackets we need to know that

$$-1 < (n+1)\frac{r}{p'}\left(\theta - \frac{p'}{s'}\right) < (n+1)\left(\frac{r}{s} - 1\right)\,.$$

The left-hand inequality is equivalent to the right-hand inequality in (6.6), and thus it is satisfied by assumption. The right-hand inequality is equivalent to

$$\frac{\theta}{p'} - \frac{1}{s'} < \frac{1}{s} - \frac{1}{r} \iff \frac{\theta}{p'} < 1 - \frac{1}{r}\,.$$

Recalling the definition of θ we see that this is equivalent to

$$\frac{1}{p'}\left(1 + \frac{1}{p} - \frac{1}{r}\right) < 1 - \frac{1}{r} \iff \frac{1}{p'} < 1 - \frac{1}{r}\,,$$

which is true because $p < r$. So we can apply Theorem 6.2.2 and we get

$$\|T_\mu f\|_r^p \preceq \int_D |f(w)|^p \delta(w)^{(n+1)p\left[\frac{1}{p'}(\theta-1)+\frac{1}{r}-\frac{1}{p}\right]} \mathrm{d}\mu(w)$$
$$= \int_D |f(w)|^p \delta(w)^{-(n+1)(\theta-1)}\,\mathrm{d}\mu(w)$$
$$\preceq \|f\|_p^p\,,$$

where in the last step we applied Theorem 6.1.9 to $\delta^{-(n+1)(\theta-1)}\mu$, which is 1-Carleson (Lemma 6.2.11) and hence p-Carleson for $A^p(D)$, and we are done. □

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