Chapter 2
Dynamics in several complex variables

Marco Abate

In this chapter we shall describe the dynamics of holomorphic self-maps of taut manifolds, and in particular the dynamics of holomorphic self-maps of convex and strongly pseudoconvex domains. A main tool in this exploration will be provided by the Kobayashi distance.

Definition 2.0.1. Let $f\colon X\to X$ be a self-map of a set X. Given $k\in\mathbb{N}$, we define the k-th *iterate* f^k of f setting by induction $f^0=\mathrm{id}_X$, $f^1=f$ and $f^k=f\circ f^{k-1}$. Given $x\in X$, the *orbit* of x is the set $\{f^k(x)\mid k\in\mathbb{N}\}$.

Studying the dynamics of a self-map f means studying the asymptotic behavior of the sequence $\{f^k\}$ of iterates of f; in particular, in principle one would like to know the behavior of all orbits. In general this is too an ambitious task; but as we shall see it can be achieved for holomorphic self-maps of taut manifolds, because the normality condition prevents the occurrence of chaotic behavior.

The model theorem for this theory is the famous Wolff-Denjoy theorem (for a proof see, e.g., [3, Theorem 1.3.9]):

Theorem 2.0.2 (Wolff-Denjoy, [149, 52]). *Let $f\in\mathrm{Hol}(\Delta,\Delta)\setminus\{\mathrm{id}_\Delta\}$ be a holomorphic self-map of Δ different from the identity. Assume that f is not an elliptic automorphism. Then the sequence of iterates of f converges, uniformly on compact subsets, to a constant map $\tau\in\overline{\Delta}$.*

Definition 2.0.3. Let $f\in\mathrm{Hol}(\Delta,\Delta)\setminus\{\mathrm{id}_\Delta\}$ be a holomorphic self-map of Δ different from the identity and not an elliptic automorphism. Then the point $\tau\in\overline{\Delta}$ whose existence is asserted by Theorem 2.0.2 is the *Wolff point* of f.

Actually, we can even be slightly more precise, introducing a bit of terminology.

Marco Abate
Dipartimento di Matematica, Università di Pisa, Largo Bruno Pontecorvo 5, 56127 Pisa, Italy, e-mail: abate@dm.unipi.it

Definition 2.0.4. Let $f\colon X\to X$ be a self-map of a set X. A *fixed point* of f is a point $x_0\in X$ such that $f(x_0)=x_0$. We shall denote by $\mathrm{Fix}(f)$ the set of fixed points of f. More generally, we shall say that $x_0\in X$ is *periodic* of *period* $p\geq 1$ if $f^p(x_0)=x_0$ and $f^j(x_0)\neq x_0$ for all $j=1,\ldots,p-1$. We shall say that f is *periodic* of *period* $p\geq 1$ if $f^p=\mathrm{id}_X$, that is if all points are periodic of period p.

Definition 2.0.5. Let $f\colon X\to X$ be a continuous self-map of a topological space X. We shall say that a continuous map $g\colon X\to X$ is a *limit map* of f if there is a subsequence of iterates of f converging to g (uniformly on compact subsets). We shall denote by $\Gamma(f)\subset C^0(X,X)$ the set of limit maps of f. If $\mathrm{id}_X\in\Gamma(f)$ we shall say that f is *pseudoperiodic*.

Example 2.0.6. Let $\gamma_\theta\in\mathrm{Aut}(\Delta)$ be given by $\gamma_\theta(\zeta)=\mathrm{e}^{2\pi\mathrm{i}\theta}\zeta$. It is easy to check that γ_θ is periodic if $\theta\in\mathbb{Q}$, and it is pseudoperiodic (but not periodic) if $\theta\in\mathbb{R}\setminus\mathbb{Q}$.

Definition 2.0.7. Let X and Y be two sets (topological spaces, complex manifolds, etc.). Two self-maps $f\colon X\to X$ and $g\colon Y\to Y$ are *conjugate* if there exists a bijection (homeomorphism, biholomorphism, etc.) $\psi\colon X\to Y$ such that $f=\psi^{-1}\circ g\circ\psi$.

If f and g are conjugate via ψ, we clearly have $f^k=\psi^{-1}\circ g^k\circ\psi$ for all $k\in\mathbb{N}$; therefore f and g share the same dynamical properties.

Example 2.0.8. It is easy to check that any elliptic automorphism of Δ is (biholomorphically) conjugated to one of the automorphisms γ_θ introduced in Example 2.0.6. Therefore an elliptic automorphism of Δ is necessarily periodic or pseudoperiodic.

We can now better specify the content of Theorem 2.0.2 as follows. Take $f\in\mathrm{Hol}(\Delta,\Delta)$ different from the identity. We have two cases: either f has a fixed point $\tau\in\Delta$ or $\mathrm{Fix}(f)=\emptyset$ (notice that, by the Schwarz-Pick lemma and the structure of the automorphisms of Δ, the only holomorphic self-map of Δ with at least two distinct fixed points is the identity). Then:

(a) If $\mathrm{Fix}(f)=\{\tau\}$, then either f is an elliptic automorphism—and hence it is periodic or pseudoperiodc—or the whole sequence of iterates converges to the constant function τ;
(b) if $\mathrm{Fix}(f)=\emptyset$ then there exists a unique point $\tau\in\partial\Delta$ such that the whole sequence of iterates converges to the constant function τ.

So there is a natural dichotomy between self-maps with fixed points and self-maps without fixed points. Our aim is to present a (suitable) generalization of the Wolff-Denjoy theorem to taut manifolds in any (finite) dimension. Even in several variables a natural dichotomy will appear; but it will be slightly different.

2.1 Dynamics in taut manifolds

Let X be a taut manifold. Then the whole family $\mathrm{Hol}(X,X)$ is normal; in particular, if $f\in\mathrm{Hol}(X,X)$ the sequence of iterates $\{f^k\}$ is normal. This suggests to subdivide the study of the dynamics of self-maps of X in three tasks:

(a) to study the dynamics of f when the sequence $\{f^k\}$ is not compactly divergent;
(b) to find conditions on f ensuring that the sequence $\{f^k\}$ is not compactly divergent;
(c) to study the dynamics of f when the sequence $\{f^k\}$ is compactly divergent.

So in several variables the natural dichotomy to consider is between maps having a compactly divergent sequence of iterates and maps whose sequence of iterates is not compactly divergent. If f has a fixed point its sequence of iterates cannot be compactly divergent; so this dichotomy has something to do with the dichotomy discussed in the introduction to this section but, as we shall see, in general they are not the same.

In this subsection we shall discuss tasks (a) and (b). To discuss task (c) we shall need a boundary; we shall limit ourselves to discuss (in the next three subsections) the case of bounded (convex or strongly pseudoconvex) domains in $\mathbb{C}^n$.

An useful notion for our discussion is the following

Definition 2.1.1. A *holomorphic retraction* of a complex manifold X is a holomorphic self-map $\rho \in \mathrm{Hol}(X,X)$ such that $\rho^2 = \rho$. In particular, $\rho(X) = \mathrm{Fix}(\rho)$. The image of a holomorphic retraction is a *holomorphic retract.*

The dynamics of holomorphic retraction is trivial: the iteration stops at the second step. On the other hand, it is easy to understand why holomorphic retractions might be important in holomorphic dynamics. Indeed, assume that the sequence of iterates $\{f^k\}$ converges to a map ρ. Then the subsequence $\{f^{2k}\}$ should converge to the same map; but $f^{2k} = f^k \circ f^k$, and thus $\{f^{2k}\}$ converges to $\rho \circ \rho$ too—and thus $\rho^2 = \rho$, that is ρ is a holomorphic retraction.

In dimension one, a holomorphic retraction must be either the identity or a constant map, because of the open mapping theorem and the identity principle. In several variables there is instead plenty of non-trivial holomorphic retractions.

Example 2.1.2. Let $\mathbb{B}^2$ be the unit Euclidean ball in $\mathbb{C}^2$. The power series

$$1 - \sqrt{1-t} = \sum_{k=1}^{\infty} c_k t^k$$

is converging for $|t| < 1$ and has $c_k > 0$ for all $k \geq 1$. Take $g_k \in \mathrm{Hol}(\mathbb{B}^2, \mathbb{C})$ such that $|g_k(z,w)| \leq c_k$ for all $(z,w) \in \mathbb{B}^2$, and define $\phi \in \mathrm{Hol}(\mathbb{B}^2, \Delta)$ by

$$\phi(z,w) = z + \sum_{k=1}^{\infty} g_k(z,w)\, w^{2k}.$$

Then $\rho(z,w) = \big(\phi(z,w), 0\big)$ always satisfies $\rho^2 = \rho$, and it is neither constant nor the identity.

On the other hand, holomorphic retracts cannot be wild. This has been proven for the first time by Rossi [136]; here we report a clever proof due to H. Cartan [47]:

Lemma 2.1.3. *Let X be a complex manifold, and $\rho\colon X \to X$ a holomorphic retraction of X. Then the image of ρ is a closed submanifold of X.*

Proof. Let $M = \rho(X)$ be the image of ρ, and take $z_0 \in M$. Choose an open neighborhood U of z_0 in X contained in a local chart for X at z_0. Then $V = \rho^{-1}(U) \cap U$ is an open neighborhood of z_0 contained in a local chart such that $\rho(V) \subseteq V$. Therefore without loss of generality we can assume that X is a bounded domain D in $\mathbb{C}^n$.

Set $P = \mathrm{d}\rho_{z_0}\colon \mathbb{C}^n \to \mathbb{C}^n$, and define $\varphi\colon D \to \mathbb{C}^n$ by

$$\varphi = \mathrm{id}_D + (2P - \mathrm{id}_D) \circ (\rho - P) .$$

Since $\mathrm{d}\varphi_{z_0} = \mathrm{id}$, the map φ defines a local chart in a neighborhood of z_0. Now $P^2 = P$ and $\rho^2 = \rho$; hence

$$\begin{aligned}\varphi \circ \rho &= \rho + (2P - \mathrm{id}_D) \circ \rho^2 - (2P - \mathrm{id}_D) \circ P \circ \rho \\ &= P \circ \rho = P + P \circ (2P - \mathrm{id}_D) \circ (\rho - P) = P \circ \varphi .\end{aligned}$$

Therefore in this local chart ρ becomes linear, and M is a submanifold near z_0. By the arbitrariness of z_0, the assertion follows. □

Having the notion of holomorphic retraction, we can immediately explain why holomorphic dynamics is trivial in compact hyperbolic manifolds (for a proof see, e.g., [3, Theorem 2.4.9]):

Theorem 2.1.4 (Kaup, [94]). *Let X be a compact hyperbolic manifold, and $f \in \mathrm{Hol}(X,X)$. Then there is $m \in \mathbb{N}$ such that f^m is a holomorphic retraction.*

So from now on we shall concentrate on non-compact taut manifolds. The basic result describing the dynamics of self-maps whose sequence of iterates is not compactly divergent is the following:

Theorem 2.1.5 (Bedford, [19]; Abate, [2]). *Let X be a taut manifold, and $f \in \mathrm{Hol}(X,X)$. Assume that the sequence $\{f^k\}$ of iterates of f is not compactly divergent. Then there exist a unique holomorphic retraction $\rho \in \Gamma(f)$ onto a submanifold M of X such that every limit map $h \in \Gamma(f)$ is of the form*

$$h = \gamma \circ \rho , \tag{2.1}$$

where γ is an automorphism of M. Furthermore, $\varphi = f|_M \in \mathrm{Aut}(M)$ and $\Gamma(f)$ is isomorphic to a subgroup of $\mathrm{Aut}(M)$, the closure of $\{\varphi^k\}$ in $\mathrm{Aut}(M)$.

Proof. Since the sequence $\{f^k\}$ of iterates is not compactly divergent, it must contain a subsequence $\{f^{k_\nu}\}$ converging to $h \in \mathrm{Hol}(X,X)$. We can also assume that $p_\nu = k_{\nu+1} - k_\nu$ and $q_\nu = p_\nu - k_\nu = k_{\nu+1} - 2k_\nu$ tend to $+\infty$ as $\nu \to +\infty$, and that $\{f^{p_\nu}\}$ and $\{f^{q_\nu}\}$ are either converging or compactly divergent. Now we have

$$\lim_{\nu\to\infty} f^{p_\nu}\big(f^{k_\nu}(z)\big) = \lim_{\nu\to\infty} f^{k_{\nu+1}}(z) = h(z)$$

for all $z \in X$; therefore $\{f^{p_\nu}\}$ cannot be compactly divergent, and thus converges to a map $\rho \in \mathrm{Hol}(X,X)$ such that

$$h \circ \rho = \rho \circ h = h . \tag{2.2}$$

Next, for all $z \in X$ we have

$$\lim_{\nu\to\infty} f^{q_\nu}\left(f^{k_\nu}(z)\right) = \lim_{\nu\to\infty} f^{p_\nu}(z) = \rho(z) .$$

Hence neither $\{f^{q_\nu}\}$ can be compactly divergent, and thus converges to a map $g \in \mathrm{Hol}(X,X)$ such that

$$g \circ h = h \circ g = \rho . \tag{2.3}$$

In particular

$$\rho^2 = \rho \circ \rho = g \circ h \circ \rho = g \circ h = \rho ,$$

and ρ is a holomorphic retraction of X onto a submanifold M. Now (2.2) implies $h(X) \subseteq M$. Since $g \circ \rho = \rho \circ g$, we have $g(M) \subseteq M$ and (2.3) yields

$$g \circ h|_M = h \circ g|_M = \mathrm{id}_M ;$$

hence $\gamma = h|_M \in \mathrm{Aut}(M)$ and (2.2) becomes (2.1).

Now, let $\{f^{k'_\nu}\}$ be another subsequence of $\{f^k\}$ converging to a map $h' \in \mathrm{Hol}(X,X)$. Arguing as before, we can assume $s_\nu = k'_\nu - k_\nu$ and $t_\nu = k_{\nu+1} - k'_\nu$ are converging to $+\infty$ as $\nu \to +\infty$, and that $\{f^{s_\nu}\}$ and $\{f^{t_\nu}\}$ converge to holomorphic maps $\alpha \in \mathrm{Hol}(X,X)$, respectively $\beta \in \mathrm{Hol}(X,X)$ such that

$$\alpha \circ h = h \circ \alpha = h' \qquad \text{and} \qquad \beta \circ h' = h' \circ \beta = h . \tag{2.4}$$

Then $h(X) = h'(X)$, and so M does not depend on the particular converging subsequence.

We now show that ρ itself does not depend on the chosen subsequence. Write $h = \gamma_1 \circ \rho_1$, $h' = \gamma_2 \circ \rho_2$, $\alpha = \gamma_3 \circ \rho_3$ and $\beta = \gamma_4 \circ \rho_4$, where ρ_1, ρ_2, ρ_3 and ρ_4 are holomorphic retractions of X onto M, and γ_1, γ_2, γ_3 and γ_4 are automorphisms of M. Then $h \circ h' = h' \circ h$ and $\alpha \circ \beta = \beta \circ \alpha$ together with (2.4) become

$$\begin{gathered}
\gamma_1 \circ \gamma_2 \circ \rho_2 = \gamma_2 \circ \gamma_1 \circ \rho_1 , \\
\gamma_3 \circ \gamma_1 \circ \rho_1 = \gamma_1 \circ \gamma_3 \circ \rho_3 = \gamma_2 \circ \rho_2 , \\
\gamma_4 \circ \gamma_2 \circ \rho_2 = \gamma_2 \circ \gamma_4 \circ \rho_4 = \gamma_1 \circ \rho_1 , \\
\gamma_3 \circ \gamma_4 \circ \rho_4 = \gamma_4 \circ \gamma_3 \circ \rho_3 .
\end{gathered} \tag{2.5}$$

Writing ρ_2 in function of ρ_1 using the first and the second equation in (2.5) we find $\gamma_3 = \gamma_1^{-1} \circ \gamma_2$. Writing ρ_1 in function of ρ_2 using the first and the third equation, we get $\gamma_4 = \gamma_2^{-1} \circ \gamma_1$. Hence $\gamma_3 = \gamma_4^{-1}$ and the fourth equation yields $\rho_3 = \rho_4$. But then, using the second and third equation we obtain

$$\rho_2 = \gamma_3^{-1} \circ \gamma_1^{-1} \circ \gamma_2 \circ \rho_2 = \rho_3 = \rho_4 = \gamma_4^{-1} \circ \gamma_2^{-1} \circ \gamma_1 \circ \rho_1 = \rho_1 ,$$

as claimed.

Next, from $f \circ \rho = \rho \circ f$ it follows immediately that $f(M) \subseteq M$. Put $\varphi = f|_M$; if $f^{p_\nu} \to \rho$ then $f^{p_\nu+1} \to \varphi \circ \rho$, and thus $\varphi \in \mathrm{Aut}(M)$.

Finally, for each limit point $h = \gamma \circ \rho \in \Gamma(f)$ we have $\gamma^{-1} \circ \rho \in \Gamma(f)$. Indeed fix a subsequence $\{f^{p_\nu}\}$ converging to ρ, and a subsequence $\{f^{k_\nu}\}$ converging to h. As usual, we can assume that $p_\nu - k_\nu \to +\infty$ and $f^{p_\nu - k_\nu} \to h_1 = \gamma_1 \circ \rho$ as $\nu \to +\infty$. Then $h \circ h_1 = \rho = h_1 \circ h$, that is $\gamma_1 = \gamma^{-1}$. Hence the association $h = \gamma \circ \rho \mapsto \gamma$ yields an isomorphism between $\Gamma(f)$ and the subgroup of $\mathrm{Aut}(M)$ obtained as closure of $\{\varphi^k\}$. □

Definition 2.1.6. Let X be a taut manifold and $f \in \mathrm{Hol}(X,X)$ such that the sequence $\{f^k\}$ is not compactly divergent. The manifold M whose existence is asserted in the previous theorem is the *limit manifold* of the map f, and its dimension is the *limit multiplicity m_f* of f; finally, the holomorphic retraction is the *limit retraction* of f.

It is also possible to describe precisely the algebraic structure of the group $\Gamma(f)$, because it is compact. This is a consequence of the following theorem (whose proof generalizes an argument due to Całka [43]), that, among other things, says that if a sequence of iterates is not compactly divergent then it does not contain any compactly divergent subsequence, and thus it is relatively compact in $\mathrm{Hol}(X,X)$:

Theorem 2.1.7 (Abate, [4]). *Let X be a taut manifold, and $f \in \mathrm{Hol}(X,X)$. Then the following assertions are equivalent:*

(i) *the sequence of iterates $\{f^k\}$ is not compactly divergent;*
(ii) *the sequence of iterates $\{f^k\}$ does not contain any compactly divergent subsequence;*
(iii) *$\{f^k\}$ is relatively compact in $\mathrm{Hol}(X,X)$;*
(iv) *the orbit of $z \in X$ is relatively compact in X for all $z \in X$;*
(v) *there exists $z_0 \in X$ whose orbit is relatively compact in X.*

Proof. (v)$\Longrightarrow$(ii). Take $H = \{z_0\}$ and $K = \overline{\{f^k(z_0)\}}$. Then H and K are compact and $f^k(H) \cap K \neq \emptyset$ for all $k \in \mathbb{N}$, and so no subsequence of $\{f^k\}$ can be compactly divergent.

(ii)$\Longrightarrow$(iii). Since $\mathrm{Hol}(X,X)$ is a metrizable topological space, if $\{f^k\}$ is not relatively compact then it admits a subsequence $\{f^{k_\nu}\}$ with no converging subsequences. But then, being X taut, $\{f^{k_\nu}\}$ must contain a compactly divergent subsequence, against (ii).

(iii)$\Longrightarrow$(iv). The evaluation map $\mathrm{Hol}(X,X) \times X \to X$ is continuous.

(iv)$\Longrightarrow$(i). Obvious.

(i)$\Longrightarrow$(v). Let M be the limit manifold of f, and let $\varphi = f|_M$. By Theorem 2.1.5 we know that $\varphi \in \mathrm{Aut}(M)$ and that $\mathrm{id}_M \in \Gamma(\varphi)$.

Take $z_0 \in M$; we would like to prove that $C = \{\varphi^k(z_0)\}$ is relatively compact in M (and hence in X). Choose $\varepsilon_0 > 0$ so that $B_M(z_0, \varepsilon_0)$ is relatively compact in M; notice that $\varphi \in \mathrm{Aut}(M)$ implies that $B_M\big(\varphi^k(z_0), \varepsilon_0\big) = \varphi^k\big(B_M(z_0, \varepsilon_0)\big)$ is relatively compact in M for all $k \in \mathbb{N}$. By Lemma 1.2.12 we have

$$\overline{B_M(z_0,\varepsilon_0)} \subseteq B_M\big(B_M(z_0,7\varepsilon_0/8),\varepsilon_0/4\big)\;;$$

hence there are $w_1,\dots,w_r \in B_M(z_0,7\varepsilon_0/8)$ such that

$$\overline{B_M(z_0,\varepsilon_0)}\cap C \subset \bigcup_{j=1}^{r} B_M(w_j,\varepsilon_0/4)\cap C\;,$$

and we can assume that $B_M(w_j,\varepsilon_0/4)\cap C \neq \emptyset$ for $j=1,\dots,r$.

For each $j=1,\dots,r$ choose $k_j \in \mathbb{N}$ so that $\varphi^{k_j}(z_0) \in B_M(w_j,\varepsilon_0/4)$; then

$$B_M(z_0,\varepsilon_0)\cap C \subset \bigcup_{j=1}^{r}\big[B_M\big(\varphi^{k_j}(z_0),\varepsilon_0/2\big)\cap C\big] \tag{2.6}$$

Since $\mathrm{id}_M \in \Gamma(\varphi)$, the set $I=\big\{k\in\mathbb{N}\;\big|\;k_M\big(\varphi^k(z_0),z_0\big)<\varepsilon_0/2\big\}$ is infinite; therefore we can find $k_0 \in \mathbb{N}$ such that

$$k_0 \geq \max\{1,k_1,\dots,k_r\} \quad\text{and}\quad k_M\big(\varphi^{k_0}(z_0),z_0\big)<\varepsilon_0/2\;. \tag{2.7}$$

Put

$$K=\bigcup_{k=1}^{k_0}\overline{B_M\big(\varphi^k(z_0),\varepsilon_0\big)}\;;$$

since, by construction, K is compact, to end the proof it suffices to show that $C\subset K$. Take $h_0 \in I$; since the set I is infinite, it suffices to show that $\varphi^k(z_0)\in K$ for all $0\leq k\leq h_0$.

Assume, by contradiction, that h_0 is the least element of I such that $\{\varphi^k(z_0)\mid 0\leq k\leq h_0\}$ is not contained in K. Clearly, $h_0>k_0$. Moreover, $k_M\big(\varphi^{h_0}(z_0),\varphi^{k_0}(z_0)\big)<\varepsilon_0$ by (2.7); thus

$$k_M\big(\varphi^{h_0-j}(z_0),\varphi^{k_0-j}(z_0)\big)=k_M\big(\varphi^{h_0}(z_0),\varphi^{k_0}(z_0)\big)<\varepsilon_0$$

for every $0\leq j\leq k_0$. In particular,

$$\varphi^j(z_0)\in K \tag{2.8}$$

for every $j=h_0-k_0,\dots,h_0$, and $\varphi^{h_0-k_0}(z_0)\in B_D(z_0,\varepsilon_0)\cap C$. By (2.6) we can find $1\leq l\leq r$ such that $k_M\big(\varphi^{k_l}(z_0),\varphi^{h_0-k_0}(z_0)\big)<\varepsilon_0/2$, and so

$$k_M\big(\varphi^{h_0-k_0-j}(z_0),\varphi^{k_l-j}(z_0)\big)<\varepsilon_0/2 \tag{2.9}$$

for all $0\leq j\leq\min\{k_l,h_0-k_0\}$. In particular, if $k_l\geq h_0-k_0$ then, by (2.6), (2.8) and (2.9) we have $\varphi^j(z_0)\in K$ for all $0\leq j\leq h_0$, against the choice of h_0. So we must have $k_l<h_0-k_0$; set $h_1=h_0-k_0-k_l$. By (2.9) we have $h_1\in I$; therefore, being $h_1<h_0$, we have $\varphi^j(z_0)\in K$ for all $0\leq j\leq h_1$. But (2.8) and (2.9) imply that $\varphi^j(z_0)\in K$ for $h_1\leq j\leq h_0$, and thus we again have a contradiction. □

Corollary 2.1.8 (Abate, [4]). *Let X be a taut manifold, and $f \in \mathrm{Hol}(X,X)$ such that the sequence of iterates is not compactly divergent. Then $\Gamma(f)$ is isomorphic to a compact abelian group $\mathbb{Z}_q \times \mathbb{T}^r$, where $\mathbb{Z}_q$ is the cyclic group of order q and $\mathbb{T}^r$ is the real torus of dimension r.*

Proof. Let M be the limit manifold of f, and put $\varphi = f|_M$. By Theorem 2.1.5, $\Gamma(f)$ is isomorphic to the closed subgroup Γ of $\mathrm{Aut}(M)$ generated by φ. We known that $\mathrm{Aut}(M)$ is a Lie group, by Theorem 1.2.16, and that Γ is compact, by Theorem 2.1.7. Moreover it is abelian, being generated by a single element. It is well known that the compact abelian Lie groups are all of the form $A \times \mathbb{T}^r$, where A is a finite abelian group; to conclude it suffices to notice that A must be cyclic, again because Γ is generated by a single element. □

Definition 2.1.9. Let X be a taut manifold, and $f \in \mathrm{Hol}(X,X)$ such that the sequence of iterates is not compactly divergent. Then the numbers q and r introduced in the last corollary are respectively the *limit period* q_f and the *limit rank* r_f of f.

When f has a periodic point $z_0 \in X$ of period $p \geq 1$, it is possible to explicitly compute the limit dimension, the limit period and the limit rank of f using the eigenvalues of $\mathrm{d}f^p_{z_0}$. To do so we need to introduce two notions.

Let $m \in \mathbb{N}$ and $\Theta = (\theta_1, \dots, \theta_m) \in [0,1)^m$. Up to a permutation, we can assume that $\theta_1, \dots, \theta_{\nu_0} \in \mathbb{Q}$ and $\theta_{\nu_0+1}, \dots, \theta_m \notin \mathbb{Q}$ for some $0 \leq \nu_0 \leq m$ (where $\nu_0 = 0$ means $\Theta \in (\mathbb{R} \setminus \mathbb{Q})^m$ and $\nu_0 = m$ means $\Theta \in \mathbb{Q}^m$).

Let $q_1 \in \mathbb{N}^*$ be the least positive integer such that $q_1\theta_1, \dots, q_1\theta_{\nu_0} \in \mathbb{N}$; if $\nu_0 = 0$ we put $q_1 = 1$. For $i, j \in \{\nu_0+1, \dots, m\}$ we shall write $i \sim j$ if and only if $\theta_i - \theta_j \in \mathbb{Q}$. Clearly, $\sim$ is an equivalence relation; furthermore if $i \sim j$ then there is a smallest $q_{ij} \in \mathbb{N}^*$ such that $q_{ij}(\theta_i - \theta_j) \in \mathbb{Z}$. Let $q_2 \in \mathbb{N}^*$ be the least common multiple of $\{q_{ij} \mid i \sim j\}$; we put $q_2 = 1$ if $\nu_0 = m$ or $i \not\sim j$ for all pairs (i,j).

Definition 2.1.10. Let $\Theta = (\theta_1, \dots, \theta_m) \in [0,1)^m$. Then the *period* $q(\Theta) \in \mathbb{N}^*$ of Θ is the least common multiple of the numbers q_1 and q_2 introduced above.

Next, for $j = \nu_0+1, \dots, m$ write $\theta'_j = q(\Theta)\theta_j - \lfloor q(\Theta)\theta_j \rfloor$, where $\lfloor s \rfloor$ is the integer part of $s \in \mathbb{R}$. Since

$$\theta'_i = \theta'_j \iff q(\Theta)(\theta_i - \theta_j) \in \mathbb{Z} \iff i \sim j\,,$$

the set $\Theta' = \{\theta'_{\nu_0+1}, \dots, \theta'_m\}$ contains as many elements as the number of $\sim$-equivalence classes. If this number is s, put $\Theta' = \{\theta''_1, \dots, \theta''_s\}$. Write $i \approx j$ if and only if $\theta''_i / \theta''_j \in \mathbb{Q}$ (notice that $0 \notin \Theta'$); clearly $\approx$ is an equivalence relation.

Definition 2.1.11. Let $\Theta = (\theta_1, \dots, \theta_m) \in [0,1)^m$. Then the *rank* $r(\Theta) \in \mathbb{N}$ is the number of $\approx$-equivalence classes. If $\nu_0 = m$ then $r(\Theta) = 0$.

If X is a taut manifold and $f \in \mathrm{Hol}(X,X)$ has a fixed point $z_0 \in X$, Theorem 1.3.10 says that all the eigenvalues of $\mathrm{d}f_{z_0}$ belongs to $\overline{\Delta}$. Then we can prove the following:

Theorem 2.1.12 (Abate, [4]). *Let X be a taut manifold of dimension n, and $f \in \mathrm{Hol}(X,X)$ with a periodic point $z_0 \in X$ of period $p \geq 1$. Let $\lambda_1, \ldots, \lambda_n \in \overline{\Delta}$ be the eigenvalues of $\mathrm{d}(f^p)_{z_0}$, listed accordingly to their multiplicity and so that*

$$|\lambda_1| = \cdots = |\lambda_m| = 1 > |\lambda_{m+1}| \geq \cdots \geq |\lambda_n|$$

for a suitable $0 \leq m \leq n$. For $j = 1, \ldots, m$ write $\lambda_j = \mathrm{e}^{2\pi \mathrm{i}\theta_j}$ with $\theta_j \in [0,1)$, and set $\Theta = (\theta_1, \ldots, \theta_m)$. Then

$$m_f = m\ , \qquad q_f = p \cdot q(\Theta) \qquad \textit{and} \qquad r_f = r(\Theta)\ .$$

Proof. Let us first assume that z_0 is a fixed point, that is $p = 1$. Let M be the limit manifold of f, and $\rho \in \mathrm{Hol}(X,M)$ its limit retraction. As already remarked, by Theorem 1.3.10 the set $\mathrm{sp}(\mathrm{d}f_{z_0})$ of eigenvalues of $\mathrm{d}f_{z_0}$ is contained in $\overline{\Delta}$; furthermore there is a $\mathrm{d}f_{z_0}$-invariant splitting $T_{z_0}X = L_N \oplus L_U$ satisfying the following properties:

(a) $\mathrm{sp}(\mathrm{d}f_{z_0}|_{L_N}) = \mathrm{sp}(\mathrm{d}f_{z_0}) \cap \Delta$ and $\mathrm{sp}(\mathrm{d}f_{z_0}|_{L_U}) = \mathrm{sp}(\mathrm{d}f_{z_0}) \cap \partial\Delta$;
(b) $(\mathrm{d}f_{z_0}|_{L_N})^k \to O$ as $k \to +\infty$;
(c) $\mathrm{d}f_{z_0}|_{L_U}$ is diagonalizable.

Fix a subsequence $\{f^{k_\nu}\}$ converging to ρ; in particular, $(\mathrm{d}f_{z_0})^{k_\nu} \to \mathrm{d}\rho_{z_0}$ as $\nu \to +\infty$. Since the only possible eigenvalues of $\mathrm{d}\rho_{z_0}$ are 0 and 1, properties (b) and (c) imply that $\mathrm{d}\rho_{z_0}|_{L_N} \equiv O$ and $\mathrm{d}\rho_{z_0}|_{L_U} = \mathrm{id}$. In particular, it follows that $L_U = T_{z_0}M$ and $m_f = \dim T_{z_0}M = \dim L_U = m$, as claimed.

Set $\varphi = f|_M \in \mathrm{Aut}(M)$. By Corollary 1.3.11, the map $\gamma \mapsto \mathrm{d}\gamma_{z_0}$ is an isomorphism between the group of automorphisms of M fixing z_0 and a subgroup of linear transformations of $T_{z_0}M$. Therefore, since $\mathrm{d}\varphi_{z_0}$ is diagonalizable by (c), $\Gamma(\varphi)$, and hence $\Gamma(f)$, is isomorphic to the closed subgroup of $\mathbb{T}^m$ generated by $\Lambda = (\lambda_1, \ldots, \lambda_m)$. So we have to prove that this latter subgroup is isomorphic to $\mathbb{Z}_{q(\Theta)} \times \mathbb{T}^{r(\Theta)}$. Since we know beforehand the algebraic structure of this group (it is the product of a cyclic group with a torus), it will suffice to write it as a disjoint union of isomorphic tori; the number of tori will be the limit period of f, and the rank of the tori will be the limit rank of f.

Up to a permutation, we can find integers $0 \leq \nu_0 < \nu_1 < \cdots < \nu_s = m$ such that $\theta_1, \ldots, \theta_{\nu_0} \in \mathbb{Q}$, and the $\sim$-equivalence classes are

$$\{\theta_{\nu_0+1}, \ldots, \theta_{\nu_1}\}, \ldots, \{\theta_{\nu_{s-1}+1}, \ldots, \theta_m\}\ .$$

Then, using the notations introduced for defining $q(\Theta)$ and $r(\Theta)$, we have

$$\Lambda^{q(\Theta)} = (1, \ldots, 1, \mathrm{e}^{2\pi\mathrm{i}\theta''_1}, \ldots, \mathrm{e}^{2\pi\mathrm{i}\theta''_1}, \mathrm{e}^{2\pi\mathrm{i}\theta''_2}, \ldots, \mathrm{e}^{2\pi\mathrm{i}\theta''_2}, \ldots, \mathrm{e}^{2\pi\mathrm{i}\theta''_s}, \ldots, \mathrm{e}^{2\pi\mathrm{i}\theta''_s})\ .$$

This implies that it suffices to show that the subgroup generated by

$$\Lambda_1 = (\mathrm{e}^{2\pi\mathrm{i}\theta''_1}, \ldots, \mathrm{e}^{2\pi\mathrm{i}\theta''_s})$$

in $\mathbb{T}^s$ is isomorphic to $\mathbb{T}^{r(\Theta)}$.

Up to a permutation, we can assume that the $\approx$-equivalence classes are

$$\{\theta''_1,\dots,\theta''_{\mu_1}\},\dots,\{\theta''_{\mu_{r-1}+1},\dots,\theta''_s\}\;,$$

for suitable $1\le\mu_1<\dots<\mu_r=s$, where $r=r(\Theta)$. Now, by definition of $\approx$ we can find natural numbers $p_j\in\mathbb{N}^*$ for $1\le j\le s$ such that

$$\begin{gathered}e^{2\pi i p_1\theta''_1}=\dots=e^{2\pi i p_{\mu_1}\theta''_{\mu_1}}\;,\\ \vdots\\ e^{2\pi i p_{\mu_{r-1}+1}\theta''_{\mu_{r-1}+1}}=\dots=e^{2\pi i p_s\theta''_s}\;,\end{gathered}$$

and no other relations of this kind can be found among $\theta''_1,\dots,\theta''_s$. It follows that $\{\Lambda_1^k\}_{k\in\mathbb{N}}$ is dense in the subgroup of $\mathbb{T}^s$ defined by the equations

$$\lambda_1^{p_1}=\dots=\lambda^{p_{\mu_1}},\dots,\lambda_{\mu_{r-1}+1}^{p_{\mu_{t-1}+1}}=\dots=\lambda_s^{p_s}\;,$$

which is isomorphic to $\mathbb{T}^r$, as claimed.

Now assume that z_0 is periodic of period p, and let ρ_f be the limit retraction of f. Since ρ_f is the unique holomorphic retraction in $\Gamma(f)$, and $\Gamma(f^p)\subseteq\Gamma(f)$, it follows that ρ_f is the limit retraction of f^p too. In particular, the limit manifold of f coincides with the limit manifold of f^p, and hence $m_f=m_{f^p}=m$. Finally, $\Gamma(f)/\Gamma(f^p)\equiv\mathbb{Z}_p$, because $f^j(z_0)\ne z_0$ for $1\le j<p$; hence $\Gamma(f)$ and $\Gamma(f^p)$ have the same connected component at the identity (and hence $r_f=r_{f^p}$), and $q_f=pq_{f^p}$ follows by counting the number of connected components in both groups. □

If $f\in\mathrm{Hol}(X,X)$ has a periodic point then the sequence of iterates is cleraly not compactly divergent. The converse is in general false, as shown by the following example:

Example 2.1.13. Let $D\subset\subset\mathbb{C}^2$ be given by

$$D=\left\{(z,w)\in\mathbb{C}^2\;\middle|\;|z|^2+|w|^2+|w|^{-2}<3\right\}.$$

The domain D is strongly pseudoconvex domain, thus taut, but not simply connected. Given $\theta\in\mathbb{R}$ and $\varepsilon=\pm1$, define $f\in\mathrm{Hol}(D,D)$ by

$$f(z,w)=(z/2,e^{2\pi i\theta}w^\varepsilon)\;.$$

Then the sequence of iterates of f is never compactly divergent, but f has no periodic points as soon as $\theta\notin\mathbb{Q}$. Furthermore, the limit manifold of f is the annulus

$$M=\left\{(0,w)\in\mathbb{C}^2\;\middle|\;|w|^2+|w|^{-2}<3\right\},$$

the limit retraction is $\rho(z,w)=(0,w)$, and suitably choosing ε and θ we can obtain as $\Gamma(f)$ any compact abelian subgroup of $\mathrm{Aut}(M)$.

It turns out that self-maps without periodic points but whose sequence of iterates is not compactly divergent can exist only when the topology of the manifold is complicated enough. Indeed, using deep results on the actions of real tori on manifolds, it is possible to prove the following

Theorem 2.1.14 (Abate, [4]). *Let X be a taut manifold with finite topological type and such that $H^j(X,\mathbb{Q}) = (0)$ for all odd j. Take $f \in \mathrm{Hol}(X,X)$. Then the sequence of iterates of f is not compactly divergent if and only if f has a periodic point.*

When $X = \Delta$ a consequence of the Wolff-Denjoy theorem is that the sequence of iterates of a self-map $f \in \mathrm{Hol}(\Delta,\Delta)$ is not compactly divergent if and only if f has a fixed point, which is an assumption easier to verify than the existence of periodic points. It turns out that we can generalize this result to convex domains (see also [109] for a different proof):

Theorem 2.1.15 (Abate, [2]). *Let $D \subset\subset \mathbb{C}^n$ be a bounded convex domain. Take $f \in \mathrm{Hol}(D,D)$. Then the sequence of iterates of f is not compactly divergent if and only if f has a fixed point.*

Proof. One direction is obvious; conversely, assume that $\{f^k\}$ is not compactly divergent, and let $\rho\colon D \to M$ be the limit retraction. First of all, note that $k_M = k_D|_{M\times M}$. In fact

$$k_D(z_1,z_2) \le k_M(z_1,z_2) = k_M\big(\rho(z_1),\rho(z_2)\big) \le k_D(z_1,z_2)$$

for every z_1, $z_2 \in M$. In particular, a Kobayashi ball in M is nothing but the intersection of a Kobayashi ball of D with M.

Let $\varphi = f|_M$, and denote by Γ the closed subgroup of $\mathrm{Aut}(M)$ generated by φ; we know, by Corollary 2.1.8, that Γ is compact. Take $z_0 \in M$; then the orbit

$$\Gamma(z_0) = \big\{\gamma(z_0) \bigm| \gamma \in \Gamma\big\}$$

is compact and contained in M. Let

$$\mathscr{C} = \left\{\overline{B_D(w,r)} \,\middle|\, w \in M,\, r > 0 \text{ and } \overline{B_D(w,r)} \supset \Gamma(z_0)\right\}.$$

Every $\overline{B_D(w,r)}$ is compact and convex (by Corollary 1.4.11); therefore, $C = \bigcap \mathscr{C}$ is a not empty compact convex subset of D. We claim that $f(C) \subset C$.

Let $z \in C$; we have to show that $f(z) \in \overline{B_D(w,r)}$ for every $w \in M$ and $r > 0$ such that $\overline{B_D(w,r)} \supset \Gamma(z_0)$. Now, $\overline{B_D(\varphi^{-1}(w),r)} \in \mathscr{C}$: in fact

$$\overline{B_D(\varphi^{-1}(w),r)} \cap M = \varphi^{-1}\big(\overline{B_D(w,r)} \cap M\big) \supset \varphi^{-1}\big(\Gamma(z_0)\big) = \Gamma(z_0).$$

Therefore $z \in \overline{B_D(\varphi^{-1}(w),r)}$ and

$$k_D\big(w,f(z)\big) = k_D\Big(f\big(\varphi^{-1}(w)\big),f(z)\Big) \le k_D\big(\varphi^{-1}(w),z\big) \le r,$$

that is $f(z) \in \overline{B_D(w,r)}$, as we want.

In conclusion, $f(C) \subset C$; by Brouwer's theorem, f must have a fixed point in C. □

The topology of convex domains is particularly simple: indeed, convex domains are topologically contractible, that is they have a point as (continuous) retract of deformation. Using very deep properties of the Kobayashi distance in strongly pseudoconvex domains, outside of the scope of these notes, Huang has been able to generalize Theorem 2.1.15 to topologically contractible strongly pseudoconvex domains:

Theorem 2.1.16 (Huang, [82]). *Let $D \subset\subset \mathbb{C}^n$ be a bounded topologically contractible strongly pseudoconvex C^3 domain. Take $f \in \mathrm{Hol}(D,D)$. Then the sequence of iterates of f is not compactly divergent if and only if f has a fixed point.*

This might suggest that such a statement might be extended to taut manifolds (or at least to taut domains) topologically contractible. Surprisingly, this is not true:

Theorem 2.1.17 (Abate-Heinzner, [7]). *There exists a bounded domain $D \subset\subset \mathbb{C}^8$ which is taut, homeomorphic to $\mathbb{C}^8$ (and hence topologically contractible), pseudoconvex, and strongly pseudoconvex at all points of ∂D but one, where a finite cyclic group acts without fixed points.*

This completes the discussion of tasks (a) and (b). In the next two subsections we shall describe how it is possible to use the Kobayashi distance to deal with task (c).

2.2 Horospheres and the Wolff-Denjoy theorem

When $f \in \mathrm{Hol}(\Delta,\Delta)$ has a fixed point $\zeta_0 \in \Delta$, the Wolff-Denjoy theorem is an easy consequence of the Schwarz-Pick lemma. Indeed if f is an automorphism the statement is clear; if it is not an automorphism, then f is a strict contraction of any Kobayashi ball centered at ζ_0, and thus the orbits must converge to the fixed point ζ_0. When f has no fixed points, this argument fails because there are no f-invariant Kobayashi balls. Wolff had the clever idea of replacing Kobayashi balls by a sort of balls "centered" at points in the boundary, the *horocycles,* and he was able to prove the existence of f-invariant horocycles—and thus to complete the proof of the Wolff-Denjoy theorem.

This is the approach we shall follow to prove a several variable version of the Wolff-Denjoy theorem in strongly pseudoconvex domains, using the Kobayashi distance to define a general notion of multidimensional analogue of the horocycles, the horospheres. This notion, introduced in [2], is behind practically all known generalizations of the Wolff-Denjoy theorem; and it has found other applications as well (see, e.g., the survey paper [6] and other chapters in this book).

Definition 2.2.1. Let $D \subset\subset \mathbb{C}^n$ be a bounded domain. Then the *small horosphere* of *center $x_0 \in \partial D$, radius $R > 0$* and *pole $z_0 \in D$* is the set

$$E_{z_0}(x_0,R) = \left\{ z \in D \;\middle|\; \limsup_{w \to x_0} [k_D(z,w) - k_D(z_0,w)] < \tfrac{1}{2}\log R \right\} ;$$

the *large horosphere* of *center* $x_0 \in \partial D$, *radius* $R > 0$ and *pole* $z_0 \in D$ is the set

$$F_{z_0}(x_0,R) = \left\{ z \in D \;\middle|\; \liminf_{w \to x_0} [k_D(z,w) - k_D(z_0,w)] < \tfrac{1}{2}\log R \right\} .$$

The rationale behind this definition is the following. A Kobayashi ball of center $w \in D$ and radius r is the set of $z \in D$ such that $k_D(z,w) < r$. If we let w go to a point in the boundary $k_D(z,w)$ goes to infinity (at least when D is complete hyperbolic), and so we cannot use it to define subsets of D. We then renormalize $k_D(z,w)$ by subtracting the distance $k_D(z_0,w)$ from a reference point z_0. By the triangular inequality the difference $k_D(z,w) - k_D(z_0,w)$ is bounded by $k_D(z_0,z)$; thus we can consider the liminf and the limsup as w goes to $x_0 \in \partial D$ (in general, the limit does not exist; an exception is given by strongly convex C^3 domains, see [3, Corollary 2.6.48]), and the sublevels provide some sort of balls centered at points in the boundary.

The following lemma contains a few elementary properties of the horospheres, which are an immediate consequence of the definition (see, e.g., [3, Lemmas 2.4.10 and 2.4.11]):

Lemma 2.2.2. *Let $D \subset\subset \mathbb{C}^n$ be a bounded domain of $\mathbb{C}^n$, and choose $z_0 \in D$ and $x \in \partial D$. Then:*

(i) *for every $R > 0$ we have $E_{z_0}(x,R) \subset F_{z_0}(x,R)$;*
(ii) *for every $0 < R_1 < R_2$ we have $E_{z_0}(x,R_1) \subset E_{z_0}(x,R_2)$ and $F_{z_0}(x,R_1) \subset F_{z_0}(x,R_2)$;*
(iii) *for every $R > 1$ we have $B_D(z_0, \frac{1}{2}\log R) \subset E_{z_0}(x,R)$;*
(iv) *for every $R < 1$ we have $F_{z_0}(x,R) \cap B_D(z_0, -\frac{1}{2}\log R) = \emptyset$;*
(v) $\bigcup_{R>0} E_{z_0}(x,R) = \bigcup_{R>0} F_{z_0}(x,R) = D$ *and* $\bigcap_{R>0} E_{z_0}(x,R) = \bigcap_{R>0} F_{z_0}(x,R) = \emptyset$;
(vi) *if $\gamma \in \mathrm{Aut}(D) \cap C^0(\overline{D},\overline{D})$, then for every $R > 0$*

$$\varphi\big(E_{z_0}(x,R)\big) = E_{\varphi(z_0)}\big(\varphi(x),R\big) \qquad \text{and} \qquad \varphi\big(F_{z_0}(x,R)\big) = F_{\varphi(z_0)}\big(\varphi(x),R\big) ;$$

(vii) *if $z_1 \in D$, set*

$$\tfrac{1}{2}\log L = \limsup_{w \to x} \big[k_D(z_1,w) - k_D(z_0,w)\big] .$$

Then for every $R > 0$ we have $E_{z_1}(x,R) \subseteq E_{z_0}(x,LR)$ and $F_{z_1}(x,R) \subseteq F_{z_0}(x,LR)$.

It is also easy to check that the horospheres with pole at the origin in B^n (and thus in Δ) coincide with the classical horospheres:

Lemma 2.2.3. *If $x \in \partial\mathbb{B}^n$ and $R > 0$ then*

$$E_O(x,R) = F_O(x,r) = \left\{ z \in \mathbb{B}^n \;\middle|\; \frac{|1 - \langle z,x\rangle|^2}{1 - \|z\|^2} < R \right\} .$$

Proof. If $z \in \mathbb{B}^n \setminus \{O\}$, let $\gamma_z \colon \mathbb{B}^n \to \mathbb{C}^n$ be given by

$$\gamma_z(w) = \frac{z - P_z(w) - (1-\|z\|^2)^{1/2}\bigl(w - P_z(w)\bigr)}{1 - \langle w, z\rangle}\;, \tag{2.10}$$

where $P_z(w) = \frac{\langle w,z\rangle}{\langle z,z\rangle} z$ is the orthogonal projection on $\mathbb{C}z$; we shall also put $\gamma_O = \mathrm{id}_{B^n}$. It is easy to check that $\gamma_z(z) = O$, that $\gamma_z(\mathbb{B}^n) \subseteq \mathbb{B}^n$ and that $\gamma_z \circ \gamma_z = \mathrm{id}_{\mathbb{B}^n}$; in particular, $\gamma_z \in \mathrm{Aut}(\mathbb{B}^n)$. Furthermore,

$$1 - \|\gamma_z(w)\|^2 = \frac{(1-\|z\|^2)(1-\|w\|^2)}{|1 - \langle w,z\rangle|^2}\;.$$

Therefore for all $w \in \mathbb{B}^n$ we get

$$\begin{aligned} k_{\mathbb{B}^n}(z,w) - k_{\mathbb{B}^n}(O,w) &= k_{\mathbb{B}^n}\bigl(O, \gamma_z(w)\bigr) - k_{\mathbb{B}^n}(O,w) \\ &= \tfrac{1}{2}\log\left(\frac{1+\|\gamma_z(w)\|}{1+\|w\|}\cdot\frac{1-\|w\|}{1-\|\gamma_z(w)\|}\right) \\ &= \log\frac{1+\|\gamma_z(w)\|}{1+\|w\|} + \tfrac{1}{2}\log\frac{|1-\langle w,z\rangle|^2}{1-\|z\|^2}\;. \end{aligned}$$

Letting $w \to x$ we get the assertion, because $\|\gamma_z(x)\| = 1$. □

Thus in $\mathbb{B}^n$ small and large horospheres coincide. Furthermore, the horospheres with pole at the origin are ellipsoids tangent to $\partial\mathbb{B}^n$ in x, because an easy computation yields

$$E_O(x,R) = \left\{ z \in \mathbb{B}^n \;\middle|\; \frac{\|P_x(z) - (1-r)x\|^2}{r^2} + \frac{\|z - P_x(z)\|^2}{r} < 1 \right\}\,,$$

where $r = R/(1+R)$. In particular if $\tau \in \partial\Delta$ we have

$$E_0(\tau,R) = \bigl\{ \zeta \in \Delta \bigm| |\zeta - (1-r)\tau|^2 < r^2 \bigr\}\,,$$

and so a horocycle is an Euclidean disk internally tangent to $\partial\Delta$ in τ.

Another domain where we can explicitly compute the horospheres is the polydisc; in this case large and small horospheres are actually different (see, e.g., [3, Proposition 2.4.12]):

Proposition 2.2.4. *Let $x \in \partial\Delta^n$ and $R > 0$. Then*

$$E_O(x,R) = \left\{ z \in \Delta^n \;\middle|\; \max_j \left\{ \frac{|x_j - z_j|^2}{1 - |z_j|^2} \;\middle|\; |x_j| = 1 \right\} < R \right\};$$

$$F_O(x,R) = \left\{ z \in \Delta^n \;\middle|\; \min_j \left\{ \frac{|x_j - z_j|^2}{1 - |z_j|^2} \;\middle|\; |x_j| = 1 \right\} < R \right\}.$$

The key in the proof of the classical Wolff-Denjoy theorem is the

Theorem 2.2.5 (Wolff's lemma, [149]). *Let $f \in \mathrm{Hol}(\Delta,\Delta)$ without fixed points. Then there exists a unique $\tau \in \partial\Delta$ such that*

$$f\big(E_0(\tau,R)\big) \subseteq E_0(\tau,R) \tag{2.11}$$

for all $R > 0$.

Proof. For the uniqueness, assume that (2.11) holds for two distinct points τ, $\tau_1 \in \partial\Delta$. Then we can construct two horocycles, one centered at τ and the other centered at τ_1, tangent to each other at a point of Δ. By (2.11) this point would be a fixed point of f, contradiction.

For the existence, pick a sequence $\{r_\nu\} \subset (0,1)$ with $r_\nu \to 1$, and set $f_\nu = r_\nu f$. Then $f_\nu(\Delta)$ is relatively compact in Δ; by Brouwer's theorem each f_ν has a fixed point $\eta_\nu \in \Delta$. Up to a subsequence, we can assume $\eta_\nu \to \tau \in \overline{\Delta}$. If τ were in Δ, we would have

$$f(\tau) = \lim_{\nu\to\infty} f_\nu(\eta_\nu) = \lim_{\nu\to\infty} \eta_\nu = \tau\,,$$

which is impossible; therefore $\tau \in \partial\Delta$.

Now, by the Schwarz-Pick lemma we have $k_\Delta\big(f_\nu(\zeta),\eta_\nu\big) \le k_\Delta(\zeta,\eta_\nu)$ for all $\zeta \in \Delta$; recalling the formula for the Poincaré distance we get

$$1 - \left|\frac{f_\nu(\zeta)-\eta_\nu}{1-\overline{\eta_\nu}f_\nu(\zeta)}\right|^2 \ge 1 - \left|\frac{\zeta-\eta_\nu}{1-\overline{\eta_\nu}\zeta}\right|^2,$$

or, equivalently,

$$\frac{|1-\overline{\eta_\nu}f_\nu(\zeta)|^2}{1-|f_\nu(\zeta)|^2} \le \frac{|1-\overline{\eta_\nu}\zeta|^2}{1-|\zeta|^2}\,.$$

Taking the limit as $\nu \to \infty$ we get

$$\frac{|1-\overline{\tau}f(\zeta)|^2}{1-|f(\zeta)|^2} \le \frac{|1-\overline{\tau}\zeta|^2}{1-|\zeta|^2}\,,$$

and the assertion follows. □

With this result it is easy to conclude the proof of the Wolff-Denjoy theorem. Indeed, if $f \in \mathrm{Hol}(\Delta,\Delta)$ has no fixed points we already know that the sequence of iterates is compactly divergent, which means that the image of any limit h of a converging subsequence is contained in $\partial\Delta$. By the maximum principle, the map h must be constant; and by Wolff's lemma this constant must be contained in $\overline{E_0(\tau,R)} \cap \partial\Delta = \{\tau\}$. So every converging subsequence of $\{f^k\}$ must converge to the constant τ; and this is equivalent to saying that the whole sequence of iterates converges to the constant map τ.

Remark 2.2.6. Let me make more explicit the final argument used here, because we are going to use it often. Let $D \subset\subset \mathbb{C}^n$ be a bounded domain; in particular, it is (hyperbolic and) relatively compact inside an Euclidean ball $\mathbb{B}$, which is complete hyperbolic and hence taut. Take now $f \in \mathrm{Hol}(D,D)$. Since $\mathrm{Hol}(D,D) \subset \mathrm{Hol}(D,\mathbb{B})$, the sequence of iterates $\{f^k\}$ is normal in $\mathrm{Hol}(D,\mathbb{B})$; but since D is relatively compact in $\mathbb{B}$, it cannot contain subsequences compactly divergent in $\mathbb{B}$. Therefore $\{f^k\}$ is relatively compact in $\mathrm{Hol}(D,\mathbb{B})$; and since the latter is a metrizable topological space, to prove that $\{f^k\}$ converges in $\mathrm{Hol}(D,\mathbb{B})$ it suffices to prove that all converging subsequences of $\{f^k\}$ converge to the same limit (whose image will be contained in $\overline{D}$, clearly).

The proof of the Wolff-Denjoy theorem we described is based on two ingredients: the existence of a f-invariant horocycle, and the fact that a horocycle touches the boundary in exactly one point. To generalize this argument to several variables we need an analogous of Theorem 2.2.5 for our multidimensional horopsheres, and then we need to know how the horospheres touch the boundary.

There exist several multidimensional versions of Wolff's lemma; we shall present three of them (Theorems 2.2.10, 2.4.2 and 2.4.17). To state the first one we need a definition.

Definition 2.2.7. Let $D \subset \mathbb{C}^n$ be a domain in $\mathbb{C}^n$. We say that D has *simple boundary* if every $\varphi \in \mathrm{Hol}(\Delta,\mathbb{C}^n)$ such that $\varphi(\Delta) \subseteq \overline{D}$ and $\varphi(\Delta) \cap \partial D \neq \emptyset$ is constant.

Remark 2.2.8. It is easy to prove (see, e.g., [3, Proposition 2.1.4]) that if D has simple boundary and Y is any complex manifold then every $f \in \mathrm{Hol}(Y,\mathbb{C}^n)$ such that $f(Y) \subseteq \overline{D}$ and $f(Y) \cap \partial D \neq \emptyset$ is constant.

Remark 2.2.9. By the maximum principle, every domain $D \subset \mathbb{C}^n$ admitting a peak function at each point of its boundary is simple. For instance, strongly pseudoconvex domain (Theorem 1.5.18) and (not necessarily smooth) strictly convex domains (Remark 1.4.6) have simple boundary.

Then we are able to prove the following

Theorem 2.2.10 (Abate, [4]). *Let $D \subset\subset \mathbb{C}^n$ be a complete hyperbolic bounded domain with simple boundary, and take $f \in \mathrm{Hol}(D,D)$ with compactly divergent sequence of iterates. Fix $z_0 \in D$. Then there exists $x_0 \in \partial D$ such that*

$$f^p\bigl(E_{z_0}(x_0,R)\bigr) \subseteq F_{z_0}(x_0,R)$$

for all $p \in \mathbb{N}$ and $R > 0$.

Proof. Since D is complete hyperbolic and $\{f^k\}$ is compactly divergent, we have $k_D\bigl(z_0, f^k(z_0)\bigr) \to +\infty$ as $k \to +\infty$. Given $\nu \in \mathbb{N}$, let k_ν be the largest k such that $k_D\bigl(z_0, f^k(z_0)\bigr) \leq \nu$. In particular for every $p > 0$ we have

$$k_D\bigl(z_0, f^{k_\nu}(z_0)\bigr) \leq \nu < k_D\bigl(z_0, f^{k_\nu+p}(z_0)\bigr) \,. \tag{2.12}$$

Since D is bounded, up to a subsequence we can assume that $\{f^{k_\nu}\}$ converges to a holomorphic $h \in \mathrm{Hol}(D,\mathbb{C}^n)$. But $\{f^k\}$ is compactly divergent; therefore $h(D) \subset \partial D$ and so $h \equiv x_0 \in \partial D$, because D has simple boundary (see Remark 2.2.8).

Put $w_\nu = f^{k_\nu}(z_0)$. We have $w_\nu \to x_0$; as a consequence for every $p > 0$ we have $f^p(w_\nu) = f^{k_\nu}\big(f^p(z_0)\big) \to x_0$ and

$$\limsup_{\nu\to+\infty}\big[k_D(z_0,w_\nu) - k_D\big(z_0, f^p(w_\nu)\big)\big] \le 0$$

by (2.12). Take $z \in E_{z_0}(x_0,R)$; then we have

$$\begin{aligned}\liminf_{w\to x_0}\big[k_D\big(f^p(z),w\big) - k_D(z_0,w)\big] &\le \liminf_{\nu\to+\infty}\big[k_D\big(f^p(z),f^p(w_\nu)\big) - k_D\big(z_0,f^p(w_\nu)\big)\big]\\ &\le \liminf_{\nu\to+\infty}\big[k_D(z,w_\nu) - k_D\big(z_0,f^p(w_\nu)\big)\big]\\ &\le \limsup_{\nu\to+\infty}\big[k_D(z,w_\nu) - k_D(z_0,w_\nu)\big]\\ &\qquad + \limsup_{\nu\to+\infty}\big[k_D(z_0,w_\nu) - k_D\big(z_0,f^p(w_\nu)\big)\big]\\ &\le \limsup_{\nu\to+\infty}\big[k_D(z,w_\nu) - k_D(z_0,w_\nu)\big] < \tfrac{1}{2}\log R\,,\end{aligned}$$

that is $f^p(z) \in F_{z_0}(x_0,R)$, and we are done. □

The next step consists in determining how the large horospheres touch the boundary. The main tools here are the boundary estimates proved in Subsection 1.5:

Theorem 2.2.11 (Abate, [2]). *Let $D \subset\subset \mathbb{C}^n$ be a bounded strongly pseudoconvex domain. Then*

$$\overline{E_{z_0}(x_0,R)} \cap \partial D = \overline{F_{z_0}(x_0,R)} \cap \partial D = \{x_0\}$$

for every $z_0 \in D$, $x_0 \in \partial D$ and $R > 0$.

Proof. We begin by proving that x_0 belongs to the closure of $E_{z_0}(x_0,R)$. Let $\varepsilon > 0$ be given by Theorem 1.5.22; then, recalling Theorem 1.5.19, for every z, $w \in D$ with $\|z - x_0\|$, $\|w - x_0\| < \varepsilon$ we have

$$k_D(z,w) - k_D(z_0,w) \le \tfrac{1}{2}\log\left(1 + \frac{\|z-w\|}{d(z,\partial D)}\right) + \tfrac{1}{2}\log\big[d(w,\partial D) + \|z-w\|\big] + K\,,$$

for a suitable constant $K \in \mathbb{R}$ depending only on x_0 and z_0. In particular, as soon as $\|z - x\| < \varepsilon$ we get

$$\limsup_{w\to x}[k_D(z,w) - k_D(z_0,w)] \le \tfrac{1}{2}\log\left(1 + \frac{\|z-x\|}{d(z,\partial D)}\right) + \tfrac{1}{2}\log\|z-x\| + K\,. \quad (2.13)$$

So if we take a sequence $\{z_\nu\} \subset D$ converging to x_0 so that $\{\|z_\nu - x_0\|/d(z_\nu,\partial D)\}$ is bounded (for instance, a sequence converging non-tangentially to x_0), then for every $R > 0$ we have $z_\nu \in E_{z_0}(x_0,R)$ eventually, and thus $x_0 \in \overline{E_{z_0}(x_0,R)}$.

To conclude the proof, we have to show that x_0 is the only boundary point belonging to the closure of $F_{z_0}(x_0,R)$. Suppose, by contradiction, that there exists $y \in \partial D \cap \overline{F_{z_0}(x_0,R)}$ with $y \neq x_0$; then we can find a sequence $\{z_\mu\} \subset F_{z_0}(x_0,R)$ with $z_\mu \to y$.

Theorem 1.5.21 provides us with $\varepsilon > 0$ and $K \in \mathbb{R}$ associated to the pair (x_0, y); we may assume $\|z_\mu - y\| < \varepsilon$ for all $\mu \in \mathbb{N}$. Since $z_\mu \in F_{z_0}(x_0, R)$, we have

$$\liminf_{w \to x} \big[k_D(z_\mu, w) - k_D(z_0, w)\big] < \tfrac{1}{2} \log R$$

for every $\mu \in \mathbb{N}$; therefore for each $\mu \in \mathbb{N}$ we can find a sequence $\{w_{\mu\nu}\} \subset D$ such that $\lim\limits_{\nu \to \infty} w_{\mu\nu} = x_0$ and

$$\lim_{\nu \to \infty} \big[k_D(z_\mu, w_{\mu\nu}) - k_D(z_0, w_{\mu\nu})\big] < \tfrac{1}{2} \log R \,.$$

Moreover, we can assume $\|w_{\mu\nu} - x\| < \varepsilon$ and $k_D(z_\mu, w_{\mu\nu}) - k_D(z_0, w_{\mu\nu}) < \frac{1}{2} \log R$ for all μ, $\nu \in \mathbb{N}$.

By Theorem 1.5.21 for all μ, $\nu \in \mathbb{N}$ we have

$$\begin{aligned} \tfrac{1}{2} \log R &> k_D(z_\mu, w_{\mu\nu}) - k_D(z_0, w_{\mu\nu}) \\ &\geq -\tfrac{1}{2} \log d(z_\mu, \partial D) - \tfrac{1}{2} \log d(w_{\mu\nu}, \partial D) - k_D(z_0, w_{\mu\nu}) - K \,. \end{aligned}$$

On the other hand, Theorem 1.5.16 yields $c_1 > 0$ (independent of $w_{\mu\nu}$) such that

$$k_D(z_0, w_{\mu\nu}) \leq c_1 - \tfrac{1}{2} \log d(w_{\mu\nu}, \partial D)$$

for every μ, $\nu \in \mathbb{N}$. Therefore

$$\tfrac{1}{2} \log R > -\tfrac{1}{2} \log d(z_\mu, \partial D) - K - c_1$$

for every $\mu \in \mathbb{N}$, and, letting μ go to infinity, we get a contradiction. □

We are then able to prove a Wolff-Denjoy theorem for strongly pseudoconvex domains:

Theorem 2.2.12 (Abate, [4]). *Let $D \subset\subset \mathbb{C}^n$ be a strongly pseudoconvex C^2 domain. Take $f \in \mathrm{Hol}(D, D)$ with compactly divergent sequence of iterates. Then $\{f^k\}$ converges to a constant map $x_0 \in \partial D$.*

Proof. Fix $z_0 \in D$, and let $x_0 \in \partial D$ be given by Theorem 2.2.10. Since D is bounded, it suffices to prove that every subsequence of $\{f^k\}$ converging in $\mathrm{Hol}(D, \mathbb{C}^n)$ actually converges to the constant map x_0.

Let $h \in \mathrm{Hol}(D, \mathbb{C}^n)$ be the limit of a subsequence of iterates. Since $\{f^k\}$ is compactly divergent, we must have $h(D) \subset \partial D$. Hence Theorem 2.2.10 implies that

$$h\big(E_{z_0}(x_0, R)\big) \subseteq \overline{F_{z_0}(x_0, R)} \cap \partial D$$

for any $R > 0$; since (Theorem 2.2.11) $\overline{F_{z_0}(x_0, R)} \cap \partial D = \{x_0\}$ we get $h \equiv x_0$, and we are done. □

Remark 2.2.13. The proof of Theorem 2.2.12 shows that we can get such a statement in any complete hyperbolic domain with simple boundary satisfying Theorem 2.2.11; and the proof of the latter theorem shows that what is actually needed are suitable estimates on the boundary behavior of the Kobayashi distance. Using this remark, it is possible to extend Theorem 2.2.12 to some classes of weakly pseudoconvex domains; see, e.g., Ren-Zhang [133] and Khanh-Thu [97].

2.3 Strictly convex domains

The proof of Theorem 2.2.12 described in the previous subsection depends in an essential way on the fact that the boundary of the domain D is of class at least C^2. Recently, Budzyńska [39] (see also [40]) found a way to prove Theorem 2.2.12 in *strictly convex* domains without any assumption on the smoothness of the boundary; in this subsection we shall describe a simplified approach due to Abate and Raissy [8].

The result which is going to replace Theorem 2.2.11 is the following:

Proposition 2.3.1. *Let $D \subset \mathbb{C}^n$ be a hyperbolic convex domain, $z_0 \in D$, $R > 0$ and $x \in \partial D$. Then we have $[x,z] \subset \overline{F_{z_0}(x,R)}$ for all $z \in \overline{F_{z_0}(x,R)}$. Furthermore,*

$$x \in \bigcap_{R>0} \overline{F_{z_0}(x,R)} \subseteq \mathrm{ch}(x) . \tag{2.14}$$

In particular, if x is a strictly convex point then $\bigcap_{R>0} \overline{F_{z_0}(x,R)} = \{x\}$.

Proof. Given $z \in F_{z_0}(x,R)$, choose a sequence $\{w_\nu\} \subset D$ converging to x and such that the limit of $k_D(z,w_\nu) - k_D(z_0,w_\nu)$ exists and is less than $\frac{1}{2}\log R$. Given $0 < s < 1$, let $h_\nu^s \colon D \to D$ be defined by

$$h_\nu^s(w) = sw + (1-s)w_\nu$$

for every $w \in D$; then $h_\nu^s(w_\nu) = w_\nu$. In particular,

$$\limsup_{\nu\to+\infty}\bigl[k_D\bigl(h_\nu^s(z),w_\nu\bigr) - k_D(z_0,w_\nu)\bigr] \le \lim_{\nu\to+\infty}\bigl[k_D(z,w_\nu) - k_D(z_0,w_\nu)\bigr] < \tfrac{1}{2}\log R .$$

Furthermore we have

$$\bigl|k_D\bigl(sz+(1-s)x,w_\nu\bigr) - k_D\bigl(h_\nu^s(z),w_\nu\bigr)\bigr| \le k_D\bigl(sz+(1-s)w_\nu, sz+(1-s)x\bigr) \to 0$$

as $\nu \to +\infty$. Therefore

$$\begin{aligned}
\liminf_{w\to x}\big[k_D\left(sz+(1-s)x,w\right)-k_D(z_0,w)\big] \\
&\le \limsup_{\nu\to+\infty}\big[k_D\big(sz+(1-s)x,w_\nu\big)-k_D(z_0,w_\nu)\big] \\
&\le \limsup_{\nu\to+\infty}\big[k_D\big(h^s_\nu(z),w_\nu\big)-k_D(z_0,w_\nu)\big] \\
&\qquad+\lim_{\nu\to+\infty}\big[k_D\big(sz+(1-s)x,w_\nu\big)-k_D\big(h^s_\nu(z),w_\nu\big)\big] \\
&< \tfrac{1}{2}\log R\,,
\end{aligned}$$

and thus $sz+(1-s)x\in F_{z_0}(x,R)$. Letting $s\to 0$ we also get $x\in\overline{F_{z_0}(x,R)}$, and we have proved the first assertion for $z\in F_{z_0}(x,R)$. If $z\in\partial F_{z_0}(x,R)$, it suffices to apply what we have just proved to a sequence in $F_{z_0}(x,R)$ approaching z.

In particular we have thus shown that $x\in\bigcap_{R>0}\overline{F_{z_0}(x,R)}$. Moreover this intersection is contained in ∂D, by Lemma 2.2.2. Take $y\in\bigcap_{R>0}\overline{F_{z_0}(x,R)}$ different from x. Then the whole segment $[x,y]$ must be contained in the intersection, and thus in ∂D; hence $y\in\mathrm{ch}(x)$, and we are done. □

We can now prove a Wolff-Denjoy theorem in strictly convex domains without any assumption on the regularity of the boundary:

Theorem 2.3.2 (Budzyńska, [39]; Abate-Raissy, [8]). *Let $D\subset\subset\mathbb{C}^n$ be a bounded strictly convex domain, and take $f\in\mathrm{Hol}(D,D)$ without fixed points. Then the sequence of iterates $\{f^k\}$ converges to a constant map $x\in\partial D$.*

Proof. Fix $z_0\in D$, and let $x\in\partial D$ be given by Theorem 2.2.10, that can be applied because strictly convex domains are complete hyperbolic (by Proposition 1.4.8) and have simple boundary (by Remark 2.2.9). So, since D is bounded, it suffices to prove that every converging subsequence of $\{f^k\}$ converges to the constant map x.

Assume that $\{f^{k_\nu}\}$ converges to a holomorphic map $h\in\mathrm{Hol}(D,\mathbb{C}^n)$. Clearly, $h(D)\subset\overline{D}$; since the sequence of iterates is compactly divergent (Theorem 2.1.15), we have $h(D)\subset\partial D$; since D has simple boundary, it follows that $h\equiv y\in\partial D$. So we have to prove that $y=x$.

Take $R>0$, and choose $z\in E_{z_0}(x,R)$. Then Theorem 2.2.10 yields $y=h(z)\in\overline{F_{z_0}(x,R)}\cap\partial D$. Since this holds for all $R>0$ we get $y\in\bigcap_{R>0}\overline{F_{z_0}(x,R)}$, and Proposition 2.3.1 yields the assertion. □

2.4 Weakly convex domains

The approach leading to Theorem 2.3.2 actually yields results for weakly convex domains too, even though we cannot expect in general the convergence to a constant map.

Example 2.4.1. Let $f\in\mathrm{Hol}(\Delta^2,\Delta^2)$ be given by

$$f(z,w) = \left(\frac{z+1/2}{1+z/2}, w\right) .$$

Then it is easy to check that the sequence of iterates of f converges to the non-constant map $h(z,w) = (1,w)$.

The first observation is that we have a version of Theorem 2.2.10 valid in all convex domains, without the requirement of simple boundary:

Theorem 2.4.2 ([2]). *Let $D \subset\subset \mathbb{C}^n$ be a bounded convex domain, and take a map $f \in \mathrm{Hol}(D,D)$ without fixed points. Then there exists $x \in \partial D$ such that*

$$f^k\big(E_{z_0}(x,R)\big) \subset F_{z_0}(x,R)$$

for every $z_0 \in D$, $R > 0$ and $k \in \mathbb{N}$.

Proof. Without loss of generality we can assume that $O \in D$. For $\nu > 0$ let $f_\nu \in \mathrm{Hol}(D,D)$ be given by

$$f_\nu(z) = \left(1 - \frac{1}{\nu}\right) f(z) ;$$

then $f_\nu(D)$ is relatively compact in D and $f_\nu \to f$ as $\nu \to +\infty$. By Brouwer's theorem, every f_ν has a fixed point $w_\nu \in D$. Up to a subsequence, we may assume that $\{w_\nu\}$ converges to a point $x \in \overline{D}$. If $x \in D$, then

$$f(x) = \lim_{\nu\to\infty} f_\nu(w_\nu) = \lim_{\nu\to\infty} w_\nu = x ,$$

impossible; therefore $x \in \partial D$.

Now fix $z \in E_{z_0}(x,R)$ and $k \in \mathbb{N}$. We have

$$\big|k_D\big(f_\nu^k(z), w_\nu\big) - k_D\big(f^k(z), w_\nu\big)\big| \le k_D\big(f_\nu^k(z), f^k(z)\big) \longrightarrow 0$$

as $\nu \to +\infty$. Since w_ν is a fixed point of f_ν^k for every $k \in \mathbb{N}$, we then get

$$\begin{aligned}
\liminf_{w\to x}\big[k_D(f^k(z),w) - k_D(z_0,w)\big] &\le \liminf_{\nu\to+\infty}\big[k_D(f^k(z),w_\nu) - k_D(z_0,w_\nu)\big] \\
&\le \limsup_{\nu\to+\infty}\big[k_D\big(f_\nu^k(z),w_\nu\big) - k_D(z_0,w_\nu)\big] \\
&\quad + \lim_{\nu\to+\infty}\big[k_D\big(f^k(z),w_\nu\big) - k_D\big(f_\nu^k(z),w_\nu\big)\big] \\
&\le \limsup_{\nu\to+\infty}\big[k_D\big(z,w_\nu\big) - k_D(z_0,w_\nu)\big] \\
&\le \limsup_{w\to x}\big[k_D\big(z,w\big) - k_D(z_0,w)\big] < \tfrac{1}{2}\log R ,
\end{aligned}$$

and $f^k(z) \in F_{z_0}(x,R)$. □

When D has C^2 boundary this is enough to get a sensible Wolff-Denjoy theorem, because of the following result:

Proposition 2.4.3 ([8]). *Let $D \subset\subset \mathbb{C}^n$ be a bounded convex domain with C^2 boundary, and $x \in \partial D$. Then for every $z_0 \in D$ and $R > 0$ we have*

$$\overline{F_{z_0}(x,R)} \cap \partial D \subseteq \mathrm{Ch}(x) .$$

In particular, if x is a strictly $\mathbb{C}$-linearly convex point then $\overline{F_{z_0}(x,R)} \cap \partial D = \{x\}$.

To simplify subsequent statements, let us introduce a definition.

Definition 2.4.4. Let $D \subset \mathbb{C}^n$ be a hyperbolic convex domain, and $f \in \mathrm{Hol}(D,D)$ without fixed points. The *target set* of f is defined as

$$T(f) = \bigcup_h h(D) \subseteq \partial D ,$$

where the union is taken with respect to all the holomorphic maps $h \in \mathrm{Hol}(D,\mathbb{C}^n)$ obtained as limit of a subsequence of iterates of f. We have $T(f) \subseteq \partial D$ because the sequence of iterates $\{f^k\}$ is compactly divergent.

As a consequence of Proposition 2.4.3 we get:

Corollary 2.4.5 ([8]). *Let $D \subset\subset \mathbb{C}^n$ be a C^2 bounded convex domain, and $f \in \mathrm{Hol}(D,D)$ without fixed points. Then there exists $x_0 \in \partial D$ such that*

$$T(f) \subseteq \mathrm{Ch}(x_0) .$$

In particular, if D is strictly $\mathbb{C}$-linearly convex then the sequence of iterates $\{f^k\}$ converges to the constant map x_0.

Proof. Let $x_0 \in \partial D$ be given by Theorem 2.4.2, and fix $z_0 \in D$. Given $z \in D$, choose $R > 0$ such that $z \in E_{z_0}(x_0,R)$. If $h \in \mathrm{Hol}(D,\mathbb{C}^n)$ is the limit of a subsequence of iterates then Theorem 2.4.2 and Proposition 2.4.3 yield

$$h(z) \in \overline{Fz_0(x,R)} \cap \partial D \subset \mathrm{Ch}(x_0) ,$$

and we are done. □

Remark 2.4.6. Zimmer [156] has proved Corollary 2.4.5 for bounded convex domains with $C^{1,\alpha}$ boundary. We conjecture that it should hold for strictly $\mathbb{C}$-linearly convex domains without smoothness assumptions on the boundary.

Let us now drop any smothness or strict convexity condition on the boundary. In this general context, an useful result is the following:

Lemma 2.4.7. *Let $D \subset \mathbb{C}^n$ be a convex domain. Then for every connected complex manifold X and every holomorphic map $h\colon X \to \mathbb{C}^n$ such that $h(X) \subset \overline{D}$ and $h(X) \cap \partial D \neq \emptyset$ we have*

$$h(X) \subseteq \bigcap_{z\in X} \mathrm{Ch}\big(h(z)\big) \subseteq \partial D .$$

Proof. Take $x_0 = h(z_0) \in h(X) \cap \partial D$, and let ψ be the weak peak function associated to a complex supporting functional at x_0. Then $\psi \circ h$ is a holomorphic function with modulus bounded by 1 and such that $\psi \circ h(z_0) = 1$; by the maximum principle we have $\psi \circ h \equiv 1$, and hence $L \circ h \equiv L(x_0)$. In particular, $h(X) \subseteq \partial D$.

Since this holds for all complex supporting hyperplanes at x_0 we have shown that $h(X) \subseteq \mathrm{Ch}\big(h(z_0)\big)$; but since we know that $h(X) \subseteq \partial D$ we can repeat the argument for any $z_0 \in X$, and we are done. □

We can then prove a weak Wolff-Denjoy theorem:

Proposition 2.4.8. *Let $D \subset\subset \mathbb{C}^n$ be a bounded convex domain, and $f \in \mathrm{Hol}(D,D)$ without fixed points. Then there exists $x \in \partial D$ such that for any $z_0 \in D$ we have*

$$T(f) \subseteq \bigcap_{R>0} \mathrm{Ch}\big(\overline{F_{z_0}(x,R)} \cap \partial D\big) . \tag{2.15}$$

Proof. Let $x \in \partial D$ be given by Theorem 2.4.2. Choose $z_0 \in D$ and $R > 0$, and take $z \in E_{z_0}(x,R)$. Let $h \in \mathrm{Hol}(D,\mathbb{C}^n)$ be obtained as limit of a subsequence of iterates of f. Arguing as usual we know that $h(D) \subseteq \partial D$; therefore Theorem 2.4.2 yields $h(z) \in \overline{F_{z_0}(x,R)} \cap \partial D$. Then Lemma 2.4.7 yields

$$h(D) \subseteq \mathrm{Ch}\big(h(z)\big) \subseteq \mathrm{Ch}\big(\overline{F_{z_0}(x,R)} \cap \partial D\big) .$$

Since z_0 and R are arbitrary, we get the assertion. □

Remark 2.4.9. Using Lemma 2.2.2 it is easy to check that the intersection in (2.15) is independent of the choice of $z_0 \in D$.

Unfortunately, large horospheres can be too large. For instance, take $(\tau_1, \tau_2) \in \partial\Delta \times \partial\Delta$. Then Proposition 2.2.4 says that the horosphere of center (τ_1, τ_2) in the bidisk are given by

$$F_O\big((\tau_1,\tau_2),R\big) = E_0(\tau_1,R) \times \Delta \cup \Delta \times E_0(\tau_2,R) ,$$

where $E_0(\tau,R)$ is the horocycle of center $\tau \in \partial\Delta$ and radius $R > 0$ in the unit disk Δ, and a not difficult computation shows that

$$\mathrm{Ch}\big(\overline{F_O\big((\tau_1,\tau_2),R\big)} \cap \partial\Delta^2\big) = \partial\Delta^2 ,$$

making the statement of Proposition 2.4.8 irrelevant. So to get an effective statement we need to replace large horospheres with smaller sets.

Small horospheres might be too small; as shown by Frosini [63], there are holomorphic self-maps of the polydisk with no invariant small horospheres. We thus need another kind of horospheres, defined by Kapeluszny, Kuczumow and Reich [89], and studied in detail by Budzyńska [39]. To introduce them we begin with a definition:

Definition 2.4.10. Let $D \subset\subset \mathbb{C}^n$ be a bounded domain, and $z_0 \in D$. A sequence $\mathbf{x} = \{x_\nu\} \subset D$ converging to $x \in \partial D$ is a *horosphere sequence* at x if the limit of $k_D(z,x_\nu) - k_D(z_0,x_\nu)$ as $\nu \to +\infty$ exists for all $z \in D$.

Remark 2.4.11. It is easy to see that the notion of horosphere sequence does not depend on the point z_0.

Horosphere sequences always exist. This follows from a topological lemma:

Lemma 2.4.12 ([132]). *Let (X,d) be a separable metric space, and for each $\nu \in \mathbb{N}$ let $a_\nu\colon X \to \mathbb{R}$ be a 1-Lipschitz map, i.e., $|a_\nu(x) - a_\nu(y)| \le d(x,y)$ for all x, $y \in X$. If for each $x \in X$ the sequence $\{a_\nu(x)\}$ is bounded, then there exists a subsequence $\{a_{\nu_j}\}$ of $\{a_\nu\}$ such that* $\lim_{j\to\infty} a_{\nu_j}(x)$ *exists for each $x \in X$.*

Proof. Take a countable sequence $\{x_j\}_{j\in\mathbb{N}} \subset X$ dense in X. Clearly, the sequence $\{a_\nu(x_0)\} \subset \mathbb{R}$ admits a convergent subsequence $\{a_{\nu,0}(x_0)\}$. Analogously, the sequence $\{a_{\nu,0}(x_1)\}$ admits a convergent subsequence $\{a_{\nu,1}(x_1)\}$. Proceeding in this way, we get a countable family of subsequences $\{a_{\nu,k}\}$ of the sequence $\{a_\nu\}$ such that for each $k \in \mathbb{N}$ the limit $\lim_{\nu\to\infty} a_{\nu,k}(x_j)$ exists for $j = 0,\dots,k$. We claim that setting $a_{\nu_j} = a_{j,j}$ the subsequence $\{a_{\nu_j}\}$ is as desired. Indeed, given $x \in X$ and $\varepsilon > 0$ we can find x_h such that $d(x,x_h) < \varepsilon/2$, and then we have

$$
\begin{aligned}
0 &\le \limsup_{j\to\infty} a_{\nu_j}(x) - \liminf_{j\to\infty} a_{\nu_j}(x) \\
&= \bigl[\limsup_{j\to\infty}\bigl(a_{\nu_j}(x) - a_{\nu_j}(x_h)\bigr) + \lim_{j\to\infty} a_{\nu_j}(x_h)\bigr] \\
&\qquad - \bigl[\liminf_{j\to\infty}\bigl(a_{\nu_j}(x) - a_{\nu_j}(x_h)\bigr) + \lim_{j\to\infty} a_{\nu_j}(x_h)\bigr] \\
&\le 2d(x,x_h) < \varepsilon\,.
\end{aligned}
$$

Since ε was arbitrary, it follows that the limit $\lim_{j\to\infty} a_{\nu_j}(x)$ exists, as required. □

Then:

Proposition 2.4.13 ([40]). *Let $D \subset\subset \mathbb{C}^n$ be a bounded convex domain, and $x \in \partial D$. Then every sequence $\{x_\nu\} \subset D$ converging to x contains a subsequence which is a horosphere sequence at x.*

Proof. Let $X = D \times D$ be endowed with the distance

$$
d\bigl((z_1,w_1),(z_2,w_2)\bigr) = k_D(z_1,z_2) + k_D(w_1,w_2)
$$

for all z_1, z_2, w_1, $w_2 \in D$.

Define $a_\nu\colon X \to \mathbb{R}$ by setting $a_\nu(z,w) = k_D(w,x_\nu) - k_D(z,x_\nu)$. The triangular inequality shows that each a_ν is 1-Lipschitz, and for each $(z,w) \in X$ the sequence $\{a_\nu(z,w)\}$ is bounded by $k_D(z,w)$. Lemma 2.4.12 then yields a subsequence $\{x_{\nu_j}\}$ such that $\lim_{j\to\infty} a_{\nu_j}(z,w)$ exists for all z, $w \in D$, and this exactly means that $\{x_{\nu_j}\}$ is a horosphere sequence. □

We can now introduce a new kind of horospheres.

Definition 2.4.14. Let $D \subset\subset \mathbb{C}^n$ be a bounded convex domain. Given $z_0 \in D$, let $\mathbf{x} = \{x_\nu\}$ be a horosphere sequence at $x \in \partial D$, and take $R > 0$. Then the *sequence horosphere* $G_{z_0}(x,R,\mathbf{x})$ is defined as

$$G_{z_0}(x,R,\mathbf{x}) = \left\{ z \in D \;\Big|\; \lim_{\nu\to+\infty} \left[k_D(z,x_\nu) - k_D(z_0,x_\nu) \right] < \tfrac{1}{2}\log R \right\} .$$

The basic properties of sequence horospheres are contained in the following:

Proposition 2.4.15 ([89, 39, 40]). *Let $D \subset\subset \mathbb{C}^n$ be a bounded convex domain. Fix $z_0 \in D$, and let $\mathbf{x} = \{x_\nu\} \subset D$ be a horosphere sequence at $x \in \partial D$. Then:*

(i) $E_{z_0}(x,R) \subseteq G_{z_0}(x,R,\mathbf{x}) \subseteq F_{z_0}(x,R)$ *for all* $R > 0$*;*
(ii) $G_{z_0}(x,R,\mathbf{x})$ *is nonempty and convex for all* $R > 0$*;*
(iii) $\overline{G_{z_0}(x,R_1,\mathbf{x})} \cap D \subset G_{z_0}(x,R_2,\mathbf{x})$ *for all* $0 < R_1 < R_2$*;*
(iv) $B_D(z_0, \frac{1}{2}\log R) \subset G_{z_0}(x,R,\mathbf{x})$ *for all* $R > 1$*;*
(v) $B_D(z_0, -\frac{1}{2}\log R) \cap G_{z_0}(x,R,\mathbf{x}) = \emptyset$ *for all* $0 < R < 1$*;*
(vi) $\bigcup_{R>0} G_{z_0}(x,R,\mathbf{x}) = D$ *and* $\bigcap_{R>0} G_{z_0}(x,R,\mathbf{x}) = \emptyset$*.*

Remark 2.4.16. If $\mathbf{x}$ is a horosphere sequence at $x \in \partial D$ then it is not difficult to check that the family $\{G_z(x,1,\mathbf{x})\}_{z\in D}$ and the family $\{G_{z_0}(x,R,\mathbf{x})\}_{R>0}$ with $z_0 \in D$ given, coincide.

Then we have the following version of Theorem 2.2.5:

Theorem 2.4.17 ([39, 8]). *Let $D \subset\subset \mathbb{C}^n$ be a convex domain, and let $f \in \mathrm{Hol}(D,D)$ without fixed points. Then there exists $x \in \partial D$ and a horosphere sequence $\mathbf{x}$ at x such that*

$$f\big(G_{z_0}(x,R,\mathbf{x})\big) \subseteq G_{z_0}(x,R,\mathbf{x})$$

for every $z_0 \in D$ and $R > 0$.

Proof. As in the proof of Theorem 2.4.2, for $\nu > 0$ put $f_\nu = (1-1/\nu)f \in \mathrm{Hol}(D,D)$; then $f_\nu \to f$ as $\nu \to +\infty$, each f_ν has a fixed point $x_\nu \in D$, and up to a subsequence we can assume that $x_\nu \to x \in \partial D$. Furthermore, by Proposition 2.4.13 up to a subsequence we can also assume that $\mathbf{x} = \{x_\nu\}$ is a horosphere sequence at x.

Now, for every $z \in D$ we have

$$\left| k_D\big(f(z),x_\nu\big) - k_D\big(f_\nu(z),x_\nu\big) \right| \le k_D\big(f_\nu(z),f(z)\big) \to 0$$

as $\nu \to +\infty$. Therefore if $z \in G_{z_0}(x,R,\mathbf{x})$ we get

$$\begin{aligned}
\lim_{\nu\to+\infty} \big[k_D\big(f(z),x_\nu\big) \;-\; k_D(z_0,x_\nu)\big] \\
&\le \lim_{\nu\to+\infty} \big[k_D\big(f_\nu(z),x_\nu\big) - k_D(z_0,x_\nu)\big] \\
&\qquad + \limsup_{\nu\to+\infty} \big[k_D\big(f(z),x_\nu\big) - k_D\big(f_\nu(z),x_\nu\big)\big] \\
&\le \lim_{\nu\to+\infty} \big[k_D(z,x_\nu) - k_D(z_0,x_\nu)\big] < \tfrac{1}{2}\log R
\end{aligned}$$

because $f_\nu(x_\nu)=x_\nu$ for all $\nu\in\mathbb{N}$, and we are done. □

Putting everything together we can prove the following Wolff-Denjoy theorem for (not necessarily strictly or smooth) convex domains:

Theorem 2.4.18 ([8]). *Let $D\subset\subset\mathbb{C}^n$ be a bounded convex domain, and $f\in\mathrm{Hol}(D,D)$ without fixed points. Then there exist $x\in\partial D$ and a horosphere sequence $\mathbf{x}$ at x such that for any $z_0\in D$ we have*

$$T(f)\subseteq\bigcap_{z\in D}\mathrm{Ch}\big(\overline{G_z(x,1,\mathbf{x})}\cap\partial D\big)=\bigcap_{R>0}\mathrm{Ch}\big(\overline{G_{z_0}(x,R,\mathbf{x})}\cap\partial D\big)\;.$$

Proof. The equality of the intersections is a consequence of Remark 2.4.16. Then the assertion follows from Theorem 2.4.17 and Lemma 2.4.7 as in the proof of Proposition 2.4.8. □

To show that this statement is actually better than Proposition 2.4.8 let us consider the case of the polydisc.

Lemma 2.4.19. *Let $\mathbf{x}=\{x_\nu\}\subset\Delta^n$ be a horosphere sequence converging to $\xi=(\xi_1,\dots,\xi_n)\in\partial\Delta^n$. Then for every $1\le j\le n$ such that $|\xi_j|=1$ the limit*

$$\alpha_j:=\lim_{\nu\to+\infty}\min_h\left\{\frac{1-|(x_\nu)_h|^2}{1-|(x_\nu)_j|^2}\right\}\le 1 \tag{2.16}$$

exists, and we have

$$G_O(\xi,R,\mathbf{x})=\left\{z\in\Delta^n\;\middle|\;\max_j\left\{\alpha_j\frac{|\xi_j-z_j|^2}{1-|z_j|^2}\;\middle|\;|\xi_j|=1\right\}<R\right\}=\prod_{j=1}^n E_j\;,$$

where

$$E_j=\begin{cases}\Delta & \text{if } |\xi_j|<1,\\ E_0(\xi_j,R/\alpha_j) & \text{if } |\xi_j|=1.\end{cases}$$

Proof. Given $z=(z_1,\dots,z_n)\in\Delta^n$, let $\gamma_z\in\mathrm{Aut}(\Delta^n)$ be defined by

$$\gamma_z(w)=\left(\frac{w_1-z_1}{1-\overline{z_1}w_1},\dots,\frac{w_n-z_n}{1-\overline{z_n}w_n}\right)\;,$$

so that $\gamma_z(z)=O$. Then

$$k_{\Delta^n}(z,x_\nu)-k_{\Delta^n}(O,x_\nu)=k_{\Delta^n}\big(O,\gamma_z(x_\nu)\big)-k_{\Delta^n}(O,x_\nu)\;.$$

Now, writing $|||z|||=\max_j\{|z_j|\}$ we have

$$k_{\Delta^n}(O,z)=\max_j\{k_\Delta(0,z_j)\}=\max_j\left\{\tfrac12\log\frac{1+|z_j|}{1-|z_j|}\right\}=\tfrac12\log\frac{1+|||z|||}{1-|||z|||}\;,$$

and hence

$$k_{\Delta^n}(z,x_\nu)-k_{\Delta^n}(O,x_\nu)=\log\left(\frac{1+|||\gamma_z(x_\nu)|||}{1+|||x_\nu|||}\right)+\tfrac{1}{2}\log\left(\frac{1-|||x_\nu|||^2}{1-|||\gamma_z(x_\nu)|||^2}\right).$$

Since $|||\gamma_z(\xi)|||=|||\xi|||=1$, we just have to study the behavior of the second term, that we know has a limit as $\nu\to+\infty$ because $\mathbf{x}$ is a horosphere sequence. Now

$$1-|||x_\nu|||^2=\min_h\{1-|(x_\nu)_h|^2\};$$
$$1-|||\gamma_z(x_\nu)|||^2=\min_j\left\{\frac{1-|z_j|^2}{|1-\overline{z_j}(x_\nu)_j|^2}(1-|(x_\nu)_j|^2)\right\}.$$

Therefore

$$\frac{1-|||x_\nu|||^2}{1-|||\gamma_z(x_\nu)|||^2}=\max_j\min_h\left\{\frac{1-|(x_\nu)_h|^2}{1-|(x_\nu)_j|^2}\cdot\frac{|1-\overline{z_j}(x_\nu)_j|^2}{1-|z_j|^2}\right\}.$$

Taking the limit as $\nu\to+\infty$ we get

$$\lim_{\nu\to+\infty}\frac{1-|||x_\nu|||^2}{1-|||\gamma_z(x_\nu)|||^2}=\max_j\left\{\frac{|1-z_j\overline{\xi_j}|^2}{1-|z_j|^2}\lim_{\nu\to+\infty}\min_h\left\{\frac{1-|(x_\nu)_h|^2}{1-|(x_\nu)_j|^2}\right\}\right\}. \quad (2.17)$$

In particular, we have shown that the limit in (2.16) exists, and it is bounded by 1 (it suffices to take $h=j$). Furthermore, if $|\xi_j|<1$ then $\alpha_j=0$; so (2.17) becomes

$$\lim_{\nu\to+\infty}\frac{1-|||x_\nu|||^2}{1-|||\gamma_z(x_\nu)|||^2}=\max\left\{\alpha_j\frac{|1-z_j\overline{\xi_j}|^2}{1-|z_j|^2}\;\middle|\;|\xi_j|=1\right\},$$

and the lemma follows. □

Now, a not too difficult computation shows that

$$\mathrm{Ch}(\xi)=\bigcap_{|\xi_j|=1}\{\eta\in\partial\Delta^n\mid\eta_j=\xi_j\}$$

for all $\xi\in\partial\Delta^n$. As a consequence,

$$\mathrm{Ch}\big(\overline{G_O(\xi,R,\mathbf{x})}\cap\partial\Delta^n\big)=\bigcup_{j=1}^n\overline{\Delta}\times\cdots\times C_j(\xi)\times\cdots\times\overline{\Delta},$$

where

$$C_j(\xi)=\begin{cases}\{\xi_j\} & \text{if } |\xi_j|=1,\\ \partial\Delta & \text{if } |\xi_j|<1.\end{cases}$$

Notice that the right-hand sides do not depend either on R or on the horosphere sequence $\mathbf{x}$, but only on ξ.

So Theorem 2.4.18 in the polydisc assumes the following form:

Corollary 2.4.20. *Let $f \in \mathrm{Hol}(\Delta^n, \Delta^n)$ be without fixed points. Then there exists $\xi \in \partial\Delta^n$ such that*

$$T(f) \subseteq \bigcup_{j=1}^{n} \overline{\Delta} \times \cdots \times C_j(\xi) \times \cdots \times \overline{\Delta} \,. \tag{2.18}$$

Roughly speaking, this is the best one can do, in the sense that while it might be true (for instance in the bidisk; see Theorem 2.4.21 below) that the image of a limit point of the sequence of iterates of f is always contained in just one of the sets appearing in the right-hand side of (2.18), it is impossible to determine a priori in which one it is contained on the basis of the point ξ only; it is necessary to know something more about the map f. Indeed, Hervé has proved the following:

Theorem 2.4.21 (Hervé, [80]). *Let $F = (f, g) \colon \Delta^2 \to \Delta^2$ be a holomorphic self-map of the bidisc, and write $f_w = f(\cdot, w)$ and $g_z = g(z, \cdot)$. Assume that F has no fixed points in Δ^2. Then one and only one of the following cases occurs:*

(i) *if $g(z,w) \equiv w$ (respectively, $f(z,w) \equiv z$) then the sequence of iterates of F converges uniformly on compact sets to $h(z,w) = (\sigma, w)$, where σ is the common Wolff point of the f_w's (respectively, to $h(z,w) = (z, \tau)$, where τ is the common Wolff point of the g_z's);*
(ii) *if $\mathrm{Fix}(f_w) = \emptyset$ for all $w \in \Delta$ and $\mathrm{Fix}(g_z) = \{y(z)\} \subset \Delta$ for all $z \in \Delta$ (respectively, if $\mathrm{Fix}(f_w) = \{x(w)\}$ and $\mathrm{Fix}(g_z) = \emptyset$) then $T(f) \subseteq \{\sigma\} \times \overline{\Delta}$, where $\sigma \in \partial\Delta$ is the common Wolff point of the f_w's (respectively, $T(f) \subseteq \overline{\Delta} \times \{\tau\}$, where τ is the common Wolff point of the g_z's);*
(iii) *if $\mathrm{Fix}(f_w) = \emptyset$ for all $w \in \Delta$ and $\mathrm{Fix}(g_z) = \emptyset$ for all $z \in \Delta$ then either $T(f) \subseteq \{\sigma\} \times \overline{\Delta}$ or $T(f) \subseteq \overline{\Delta} \times \{\tau\}$, where $\sigma \in \partial\Delta$ is the common Wolff point of the f_w's, and $\tau \in \partial\Delta$ is the common Wolff point of the g_z;*
(iv) *if $\mathrm{Fix}(f_w) = \{x(w)\} \subset \Delta$ for all $w \in \Delta$ and $\mathrm{Fix}(g_z) = \{y(z)\} \subset \Delta$ for all $z \in \Delta$ then there are σ, $\tau \in \partial D$ such that the sequence of iterates converges to the constant map (σ, τ).*

All four cases can occur: see [80] for the relevant examples.

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