Complex spherical codes with two inner products

Hiroshi Nozaki, Sho Suda

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Abstract

A finite set $X$ in a complex sphere is called a complex spherical 2-code if the number of inner products between two distinct vectors in $X$ is equal to 2. In this paper, we characterize the tight complex spherical 2-codes by doubly regular tournaments or skew Hadamard matrices. We also give certain maximal 2-codes relating to skew-symmetric $D$-optimal designs. To prove them, we show the smallest embedding dimension of a tournament into a complex sphere by the multiplicity of the smallest or second-smallest eigenvalue of the Seidel matrix.

Key words: complex spherical $s$-code, doubly regular tournament, skew Hadamard matrix, skew-symmetric $D$-optimal design, representable graph, main angle, main eigenvalue, graph spectrum.

1 Introduction

Let $X$ be a finite set of points on the complex unit sphere $\Omega(d)$ in $\mathbb{C}^d$. The angle set $A(X)$ is defined to be

$$A(X) = \{x^*y \mid x, y \in X, x \neq y\},$$

where $x^*$ is the transpose conjugate of a column vector $x$. A finite set $X$ is called a complex spherical $s$-code if $|A(X)| = s$ and $A(X)$ contains an imaginary number. The value $s$ is called the degree of $X$. For $X, X' \subset \Omega(d)$, we say that $X$ is isomorphic to $X'$ if there exists a unitary transformation from $X$ to $X'$. An $s$-code $X \subset \Omega(d)$ is said to be largest if $X$ has the largest possible cardinality in all $s$-codes in $\Omega(d)$. One of major problems on $s$-codes is to classify largest $s$-codes for given $s$ and $d$.

We will survey Euclidean finite sets with only $s$ distances. For $X \subset \mathbb{R}^d$, we define

$$D(X) = \{d(x, y) \mid x, y \in X, x \neq y\},$$

where $d(x, y)$ is the Euclidean distance of $x$ and $y$. A finite set $X$ is called an $s$-distance set if $|D(X)| = s$ holds. We have an upper bound for the size of an $s$-distance set in $\mathbb{R}^d$, namely
$|X| \leq \binom{d+s}{s}$ [2]. Clearly the largest 1-distance set in $\mathbb{R}^d$ is the regular simplex for any $d$. Largest 2-distance sets in $\mathbb{R}^d$ are classified for $d \leq 7$ [9, 11]. Largest $s$-distance sets in $\mathbb{R}^2$ are classified for $s \leq 5$ [10, 19, 20]. The largest 3-distance set in $\mathbb{R}^3$ is the vertex set of the icosahedron [21]. The classification of largest $s$-distance sets is still open for others $(s, d)$. A largest 2-distance set in $\mathbb{R}^8$ is given in [11], and it attains the upper bound.

A spherical $s$-distance set particularly deserves attention because of the connection to association schemes or spherical $t$-designs (see [7, 1] for details). A subset $X$ of $S^{d-1}$ is called a spherical $t$-design if for any polynomial $f$ in $d$ variables of degree at most $t$, the following equality holds:

$$
\frac{1}{|S^{d-1}|} \int_{S^{d-1}} f(x) dx = \frac{1}{|X|} \sum_{x \in X} f(x),
$$

where $|S^{d-1}|$ is the volume of $S^{d-1}$. If a spherical $t$-design $X$ of degree $s$ satisfies $t \geq 2s - 2$, then $X$ has the structure of a $Q$-polynomial association scheme [7]. The size of an $s$-distance set in $S^{d-1}$ is smaller than or equal to $\binom{d+s-1}{s} + \binom{d+s-2}{s-1}$ [7]. An $s$-distance set $X$ is said to be tight if $X$ attains this bound. A tight $s$-distance set becomes a minimal spherical $t$-design and satisfies $t = 2s$ [7]. The classification of tight $s$-distance sets is one of the most interesting problems, and this has been solved except for $s = 2$ [4]. A largest 2-distance set on $S^{d-1}$ is determined for $d \leq 93$ ($d \neq 46, 78$) [13, 5]. A largest 3-distance set on $S^{d-1}$ is determined for $d = 2, 3, 8, 22$ [21, 14].

A simple graph $G = (V, E)$ is representable in $\mathbb{R}^d$ if there is an embedding $\sigma : V \to \mathbb{R}^d$ such that

$$
d(\sigma(a), \sigma(b)) = \begin{cases} 
\alpha & \text{if } (a, b) \in E, \\
\beta & \text{otherwise},
\end{cases}
$$

for some $\alpha, \beta \in \mathbb{R}$. For a simple graph $G$, Roy [18] gave an explicit expression of the minimal dimension $d$ such that $G$ is representable in $\mathbb{R}^d$ in terms of the multiplicity of the smallest or second-smallest eigenvalue of $A$. This embedding of a graph is useful for the classification of 2-distance sets [9, 11].

Roy and Suda [17] gave the complex analogue of the spherical $s$-distance set theory. Complex spherical $s$-codes are closely related to complex spherical designs or non-symmetric association schemes. In this paper, we consider a complex spherical 2-code $X \subset \Omega(d)$. If $X$ satisfies $A(X) \subset \mathbb{R}$, then the Gram matrix of $X$ is real, and $X$ can be embedded into $\mathbb{R}^d$. We may assume $A(X)$ contains an imaginary number $\alpha$, and $A(X) = \{\alpha, \bar{\alpha}\}$, where $\bar{\alpha}$ is the conjugate of $\alpha$. We have a natural upper bound [17]:

$$
|X| \leq \begin{cases} 
2d + 1 & \text{if } d \text{ is odd,} \\
2d & \text{if } d \text{ is even.}
\end{cases}
$$

A 2-code $X$ is said to be tight if $X$ attains the bound (1.1). This is known as the absolute bound.

A tournament is a directed graph obtained by assigning a direction for each edge in an undirected complete graph. Formally, a tournament is a pair $(V, E)$ such that the vertex set $V$ is a finite set and the edge set $E \subset V \times V$ satisfies $E \cap E^T = \emptyset$ and $E \cup E^T \cup \{(x, x) \mid x \in V\} = V \times V$, where $E^T := \{(x, y) \mid (y, x) \in E\}$. A complex spherical 2-code $X$ has the structure of a tournament $(X, E)$, where $E = \{(x, y) \in X \times X \mid x^*y = \alpha\}$. A tournament $(V, E)$ is representable in $\Omega(d)$ if there exists a mapping $\varphi$ from $V$ to $\Omega(d)$ such that for all
distinct $x, y \in V$,
\[
\varphi(x)^*\varphi(y) = \begin{cases} 
\alpha & \text{if } (x, y) \in E, \\
\overline{\alpha} & \text{if } (y, x) \in E,
\end{cases}
\]
where $\alpha$ is an imaginary number with $\text{Im}(\alpha) > 0$. Such a mapping $\varphi$ is said to be a representation of a tournament. We identify a representation with the image of the representation. Two tournaments $G = (V, E), G' = (V', E')$ are isomorphic if there is a bijection from $V$ to $V'$ such that $(x, y) \in E$ if and only if $(f(x), f(y)) \in E'$. For two tournaments $G$ and $G'$, if $G$ is not isomorphic to $G'$, then a representation of $G$ is not isomorphic to that of $G'$. Let $\text{Rep}(G)$ denote the smallest $d$ such that $G$ is representable in $\Omega(d)$. The Seidel matrix of $G$ is defined to be $\sqrt{-1}(A - A^T)$, where $A$ is the adjacency matrix of $G$. In Section 3, we determine $\text{Rep}(G)$ by the multiplicity of the smallest or second-smallest eigenvalue of the Seidel matrix of $G$.

A tournament $G$ is said to be doubly regular if the number of the neighbors of a vertex does not depend on the choice of the vertex and the number of the common neighbors of a pair of distinct vertices does not depend on the choice of the pair. An $n \times n$ $(\pm 1)$-matrix of $H$ is called a skew Hadamard matrix if $H + H^T = 2I$ and $HH^T = nI$, where $I$ is the identity matrix. Let $X \subset \Omega(d)$ be a 2-code, and $A$ the adjacency matrix of the tournament obtained from $X$. It is known that the existence of a doubly regular tournament of $4d + 3$ vertices is equivalent to that of a skew Hadamard matrix of order $4d + 4$ [16]. In Section 4, we give the following characterizations of tight 2-codes and 2-codes with $n = 2d$ where $d$ is odd.

(1) For odd $d$, $X$ is a tight complex 2-code if and only if $A$ is the adjacency matrix of a doubly regular tournament.

(2) For even $d$, $X$ is a tight complex 2-code if and only if $I + A - A^T$ is a skew Hadamard matrix.

(3) For odd $d$, $X$ is a complex 2-code with $n = 2d$ if and only if either $A$ is the adjacency matrix of an induced subgraph of a doubly regular tournament by deleting a vertex, or its Seidel matrix $S$ satisfies that $S^2$ is permutationally similar to
\[
\begin{pmatrix}
  kI + lJ & 0 \\
  0 & kI + lJ
\end{pmatrix},
\]
for some positive integers $k, l$.

We note that the last case in (3) includes skew-symmetric $D$-optimal designs [8, 23]. The table of the number of non-isomorphic tight 2-codes in $\Omega(d)$ for $d \leq 14$ is obtained by a computer calculation based on Theorem 3.2 in [3].

2 Results on main eigenvalues

In this section we give results on main eigenvalues of a Hermitian matrix which will be used later. Let $H$ be a Hermitian matrix of size $n$ with $s$ distinct eigenvalues $\tau_1 < \cdots < \tau_s$. Let $E_i$ be the orthogonal projection matrix onto the eigenspace corresponding to $\tau_i$. The main angle $\beta_i$ of $\tau_i$ is defined to be the value
\[
\beta_i = \frac{1}{\sqrt{n}} \sqrt{(E_i \cdot j)^*(E_i \cdot j)},
\]
where \( j \) is the all-ones vector. It is clear that \( 0 \leq \beta_i \leq 1 \) and \( \sum_{i=1}^{s} \beta_i^2 = 1 \).

Let \( J \) denote the all-ones matrix.

**Lemma 2.1** ([15]). Let \( H \) be a Hermitian matrix of size \( n \) with \( s \) distinct eigenvalues \( \tau_1 < \cdots < \tau_s \). Let \( \beta_i \) be the main angle of \( \tau_i \). Let \( M = H + aJ \), where \( a \) is a complex number. Then

\[
P_M(x) = P_H(x)(1 + a \sum_{i=1}^{s} \frac{n\beta_i^2}{\tau_i - x}),
\]

where \( P_M \) is the characteristic polynomial of matrix \( M \).

An eigenvalue \( \tau_i \) is said to be main if \( \beta_i \neq 0 \).

**Theorem 2.2.** Let \( H \) be a Hermitian matrix of size \( n \), and \( M = H + aJ \), where \( a \) is a real number. Let \( \tau_1 < \tau_2 < \cdots < \tau_r \) be the distinct main eigenvalues of \( H \), and \( \beta_i \) the main angle of \( \tau_i \). Let \( \mu_1 < \mu_2 < \cdots < \mu_s \) be the distinct main eigenvalues of \( M \). Then \( r = s \) holds, and

\[
f(x) = \prod_{i=1}^{r} (\mu_i - x) = \prod_{i=1}^{r} (\tau_i - x)(1 + a \sum_{j=1}^{r} \frac{n\beta_j^2}{\tau_j - x}). \tag{2.1}
\]

Moreover, if \( a > 0 \), then \( \tau_1 < \mu_1 < \tau_2 < \cdots < \tau_r < \mu_r \), and if \( a < 0 \), then \( \mu_1 < \tau_1 < \mu_2 < \cdots < \mu_r < \tau_r \).

**Proof.** By Lemma 2.1, we have the equality

\[
\prod_{i=1}^{s} (\mu_i - x) = \prod_{i=1}^{r} (\tau_i - x)(1 + a \sum_{j=1}^{r} \frac{n\beta_j^2}{\tau_j - x}). \tag{2.2}
\]

By comparing the degrees of the polynomials in both sides, we obtain \( s = r \).

Let \( f(x) \) be the polynomial in (2.2). It is easily shown that for \( a > 0 \),

\[
\begin{align*}
f(\tau_i) &> 0 \text{ if } i \equiv 1 \pmod{2}, \\
f(\tau_i) &< 0 \text{ if } i \equiv 0 \pmod{2}, \\
\lim_{x \to \infty} f(x) &< 0 \text{ if } r \equiv 1 \pmod{2}, \\
\lim_{x \to \infty} f(x) &> 0 \text{ if } r \equiv 0 \pmod{2}.
\end{align*}
\]

This implies that \( \tau_1 < \mu_1 < \tau_2 < \cdots < \tau_r < \mu_r \). By the same manner for \( H = M - aJ \) with \( a < 0 \), we can show \( \mu_1 < \tau_1 < \mu_2 < \cdots < \mu_r < \tau_r \). \( \square \)

### 3 Representations of a tournament

In this section, we determine \( \text{Rep}(G) \) by the multiplicity of the smallest or second-smallest eigenvalue of the Seidel matrix of \( G \). Let \( G = (V,E) \) be a tournament with \( n \) vertices. The **adjacency matrix** \( A \) of \( G \) is the matrix indexed by the vertex set \( V \), with entries given by

\[
A_{xy} = \begin{cases} 
1 & \text{if } (x,y) \in E, \\
0 & \text{otherwise.}
\end{cases}
\]
The Gram matrix of a representation of $G$, with adjacency matrix $A$, can be expressed by
\[ \alpha A + \overline{\alpha} A^T - \tau I, \]
where $\alpha$ is an imaginary number, and $\tau$ is a negative real number. Note that $\tau$ should be the smallest eigenvalue of $\alpha A + \overline{\alpha} A^T$ to minimize the rank. To determine $\text{Rep}(G)$, we will consider $\alpha$ for which the multiplicity of the smallest eigenvalue of $\alpha A + \overline{\alpha} A^T$ is maximum.

**Theorem 3.1.** Let $G$ be a tournament with $n$ vertices, and $A$ the adjacency matrix. Let $\tau_1 < \tau_2 < \cdots < \tau_s$ be the distinct eigenvalues of $S = \sqrt{-1}(A - A^T)$, $\beta_1$ the main angle of $\tau_1$, and $m_l$ the multiplicity of $\tau_l$. Let $\alpha$ be the angle with $\text{Im}(\alpha) > 0$ of the representation of $G$ in $\Omega(\text{Rep}(G))$. Then the following hold.

1. If $\beta_1 = 0$, then $\text{Rep}(G) = n - m_1 - 1$, and $\alpha = (1 - c_1 \sqrt{-1})/(1 + c_1 \tau_1)$, where $c_1 = \sum_{i=2}^s n \beta_i^2/((\tau_i - \tau_1))$.
2. If $\beta_1 \neq 0$, and $m_1 > 1$, then $\text{Rep}(G) = n - m_1$, and $\alpha = -\sqrt{-1}/\tau_1$.
3. If $m_1 = 1$, $\beta_1 = 0$, and $\gamma_2 < 0$, then $\text{Rep}(G) = n - m_2 - 1$, and $\alpha = (1 - c_2 \sqrt{-1})/(1 + c_2 \tau_2)$, where $c_2 = n \beta_2^2/((\tau_2 - \tau_2) + \sum_{i=3}^s n \beta_i^2/((\tau_i - \tau_2))$.
4. Otherwise $\text{Rep}(G) = n - 1$.

**Proof.**

For $\alpha' = a + \sqrt{-1}$ with $a \in \mathbb{R}$, we have
\[ \alpha' A + \overline{\alpha'} A^T = a J + \sqrt{-1}(A - A^T) - a I. \]
The multiplicity of the smallest eigenvalue of $\alpha' A + \overline{\alpha'} A^T$ is equal to that of $M = a J + \sqrt{-1}(A - A^T)$. We would like to find $a \in \mathbb{R}$ such that the multiplicity of the smallest eigenvalue of $M$ is maximum. Let $\tau_1 < \cdots < \tau_s$ be the distinct main eigenvalues of $S$, and $\mu_1 < \cdots < \mu_r$ those of $M$. Let $f(x)$ be the polynomial defined as in Theorem 2.2.

1. By $\beta_1 = 0$, we have $\tau_1 < \tau_2$. We would like to find $a \in \mathbb{R}$ such that $\mu_1 = \tau_1$. For such $a$, the multiplicity of the smallest eigenvalue $\tau_1$ of $M$ is maximum, and equal to $m_1 + 1$. By Theorem 2.2, $\mu_1 = \tau_1$ if and only if $f(\tau_1) = 0$, namely, $a = -1/c_1$. Therefore $\text{Rep}(G) = n - m_1 - 1$ for $a = -1/c_1$. By rescaling the diagonal entries of $\alpha' A + \overline{\alpha'} A^T - (\tau_1 - a) I$ to 1, we obtain $\alpha = (1 - c_1 \sqrt{-1})/(1 + c_1 \tau_1)$.

2. Since $\beta_1 \neq 0$, we have $\tau_1 = \tau_2 \neq \mu_1$ by Theorem 2.2. Therefore, if $a \neq 0$, the multiplicity of the smallest eigenvalue of $M$ is at most $m_1 - 1$. Thus, for $a = 0$, the multiplicity of the smallest eigenvalue of $M$ is maximum, and equal to $m_1$. Hence $\text{Rep}(G) = n - m_1$, and $\alpha = -\sqrt{-1}/\tau_1$.

3. By $c_2 < 0$, we have $\beta_1 > 0$ and $\tau_1$ is a main eigenvalue. We would like to find $a \in \mathbb{R}$ such that $\mu_1 = \tau_2$. For such $a$, the multiplicity of the smallest eigenvalue $\tau_2$ of $M$ is maximal, and it is $m_2 + 1$. By Theorem 2.2, $\mu_1 = \tau_2$ if and only if $f(\tau_2) = 0$ and $a > 0$, namely, $a = -1/c_2$ and $c_2 < 0$. Therefore we obtain $\text{Rep}(G) = n - m_2 - 1$, and $\alpha = (1 - c_2 \sqrt{-1})/(1 + c_2 \tau_2)$.

4. If $a = 0$ and $m_1 = 1$, then the multiplicity of the smallest eigenvalue of $M$ is clearly 1. Suppose $\beta_1 \neq 0$, $m_1 = 1$, $\beta_2 = 0$, and $c_2 \geq 0$. If $a > 0$ holds, then $\mu_1 < \tau_2$ by $f(\tau_2) < 0$ and $\lim_{x \to -\infty} f(x) > 0$. If $a < 0$ holds, then $\mu_1 < \tau_1$ by Theorem 2.2. The multiplicity of the smallest eigenvalue $\mu_1$ of $M$ is 1.

Suppose $\beta_1 \neq 0$, $m_1 = 1$, $\beta_2 \neq 0$. Then for any $a \neq 0$, the multiplicity of the smallest eigenvalue $\mu_1$ of $M$ is 1 by Theorem 2.2.

From the above facts, $\text{Rep}(G) = n - 1$ follows. \( \square \)
Note that the conditions (1)–(4) in Theorem 3.1 are disjoint. A tournament which satisfies the condition \( i \) in Theorem 3.1 is said to be of Type \( i \) for \( i = 1, \ldots, 4 \). There is a tournament of each type. Lemmas 4.3, 4.4, and Remark 4.9 give examples of Type (1), (2), and (3), respectively.

## 4 Tight complex spherical 2-codes

In this section, we give bounds on complex spherical 2-codes. We also characterize the tight 2-codes and 2-codes in \( \Omega(d) \) with \( n = 2d \) vertices, where \( d \) is odd in terms of doubly regular tournaments, skew Hadamard matrix and some skew symmetric \((0, \pm 1)\)-matrices including skew-symmetric \( D \)-optimal designs as an application of Theorem 3.1.

Let \( X \) be a finite subset in \( \Omega(d) \) of size \( n \) with degree 2, and let \( A \) be the adjacency matrix of \( X \). Example 6.3 in [17] shows that the following are equivalent:

1. \( |X| = 2d + 1 \).
2. \( \{I, A, J - A - I\} \) forms the set of adjacency matrices of a non-symmetric association scheme of class 2.

### Theorem 4.1

Let \( X \) be a finite subset in \( \Omega(d) \) of size \( n \) with degree 2, and let \( A \) be the adjacency matrix of \( X \). If \( d \) is odd, \( |X| \leq 2d + 1 \) holds. Equality holds if and only if \( A \) is the adjacency matrix of a doubly regular tournament.

**Proof.** The absolute bound (1.1) shows that \( |X| \leq 2d + 1 \) holds. Example 6.3 in [17] shows that equality holds if and only if \( \{I, A, J - A - I\} \) forms the set of adjacency matrices of a non-symmetric association scheme of class 2. The latter condition is equivalent to the condition that \( A \) is the adjacency matrix of a doubly regular tournament. \( \square \)

To prove Theorems 4.7, 4.8, we need the following lemmas.

### Lemma 4.2

There exists no tournament \( A \) of Type (1) with \( n = 2d \) vertices and the spectrum \( \{(-\theta)^{d-1}, 0^2, (\theta)^{d-1}\} \) where \( 0 < \theta \).

**Proof.** Suppose that there exists such a tournament with Seidel matrix \( S \). It holds that \( Sj = 0 \) because \( \beta_1 = \beta_3 = 0 \) and the remaining eigenvalues are all 0. However it does not happen because \( n = 2d \). \( \square \)

### Lemma 4.3

Let \( d \) be an integer at least 3. Let \( A \) be the adjacency matrix of a tournament of Type (1) with \( n = 2d \) vertices and the spectrum \( \{(-\theta)^{d-1}, (-\phi)^{1}, (\phi)^{1}, (\theta)^{d-1}\} \) where \( 0 < \phi < \theta \). Then \( d \) is odd and \( A \) is the adjacency matrix of an induced subgraph of a doubly regular tournament by deleting a vertex.

**Proof.** Since the entries of \( S^2 \) are integers, the eigenvalues of \( S^2 \) are algebraic integers. Therefore \( \theta^2 \) and \( \phi^2 \) are integer because their multiplicities \( 2d - 2 \) and 2 are different. From taking the trace of \( S^2 \), it follows that the possibility of \( (\theta^2, \phi^2) \) is \((2d + 1, 1)\) or \((2d, d)\).

For the first case, \( A \) is the adjacency matrix of an induced subgraph of a doubly regular tournament by deleting a vertex [15, Theorem 1.1]. Thus \( n + 1 = 2d + 1 \) must be congruent to 3 modulo 4, which implies that \( d \) is odd.

For the second case, consider \( \theta^2 I - S^2 \). Since \( \theta^2 I - S^2 \) is positive semidefinite and the diagonal entries are all 1, the absolute value of an off-diagonal entry of this matrix must be at
most 1. In fact they must be zero because the size of the matrix $\theta^2 I - S^2$ is even. Therefore $S^2 = (\theta^2 - 1)I$, which contradicts the fact that $S^2$ has the other eigenvalue $\phi^2$. \hfill \Box

**Lemma 4.4.** Let $A$ be the adjacency matrix of a tournament of Type (2) with $n = 2d$ vertices and the spectrum $\{(-\theta)^d, (\theta)^d\}$ where $0 < \theta$. Then $d$ is even and $I + A - A^T$ is a skew Hadamard matrix.

**Proof.** The fact that $I + A - A^T$ is a skew Hadamard matrix follows from direct calculation, and thus $d$ must be even. \hfill \Box

**Lemma 4.5.** Let $A$ be the adjacency matrix of a tournament of Type (3) with the spectrum $\{(-\theta)^d, (-\phi)^d, (\phi)^{d-1}, (\theta)^1\}$ where $0 < \phi < \theta$. Then $d$ is odd and the Seidel matrix $S$ satisfies that $S^2$ is permutationally similar to

\[
\begin{pmatrix}
  kI + lJ & 0 \\
  0 & kI + lJ
\end{pmatrix},
\]

for some positive integers $k, l$.

**Proof.** By the condition of Type (3), $\beta_2 = \beta_4 = 0$ and $\beta_1 = \beta_3 = 1/\sqrt{2}$ hold. Consider the eigenspaces of $S^2 - \phi^2 I$. The main angle condition of $S$ implies that the all-ones vector is an eigenvector of $S^2 - \phi^2 I$ corresponding to the eigenvalue $\theta^2 - \phi^2$. Since the multiplicity of $\theta^2 - \phi^2$ is two, let $x$ be the remaining normalized real eigenvector orthogonal to $j$. Then it holds that

\[
S^2 = \phi^2 I + (\theta^2 - \phi^2)((1/n)J + xx^T).
\]

Comparing the diagonal entries, we observe that $n - 1 = \phi^2 + (\theta^2 - \phi^2)(1/n + x_i^2)$ for each $i$, where $x_i$ is the $i$-th entry of $x$. This implies that $x_i$ is independent of the choice of $i$. Since the vector $x$ is normalized, we obtain $x_i = \pm 1/\sqrt{n}$. The assumption that $x$ is orthogonal to the all-ones vector shows that each $\pm 1/\sqrt{n}$ appears in the entries of $x$ exactly same times. After some permutation of entries, we may assume that the first half entries of $x$ are $1/\sqrt{n}$ which means $S^2$ has the form

\[
S^2 = \begin{pmatrix}
  \phi^2 I + \frac{2(\theta^2 - \phi^2)}{n} J & 0 \\
  0 & \phi^2 I + \frac{2(\theta^2 - \phi^2)}{n} J
\end{pmatrix}.
\]

Since a vector $S(j + \sqrt{n}x)$ is written as a linear combination of $j, x$ and $S = \sqrt{-I}(2A - J + I)$, we have

\[
A \begin{pmatrix}
  j \\
  0
\end{pmatrix} = \begin{pmatrix}
  aj \\
  b_j
\end{pmatrix}
\]

for some $a, b$. Letting $A_1$ be the principal submatrix of $A$ lying the first $d$ rows and columns, then $A_1j = aj$, namely $A_1$ is the adjacency matrix of a regular tournament of order $d$. This implies $d$ must be odd. \hfill \Box

**Lemma 4.6.** Let $X$ be a finite subset in $\Omega(d)$ with degree 2 and size $n = 2d$. The possibilities of the spectrum of $S = \sqrt{-I}(A - A^T)$ are as follows:

(i) $X$ is of Type (1) with the spectrum $\{(-\theta)^{d-1}, 0^2, (\theta)^{d-1}\}$. 


(ii) $X$ is of Type (1) with the spectrum $\{(-\theta)^{d-1}, (-\phi)^1, (\phi)^1, (\theta)^{d-1}\}$ with $0 < \phi < \theta$.

(iii) $X$ is of Type (2) with the spectrum $\{(-\theta)^d, (\theta)^d\}$.

(iv) $X$ is of Type (3) with the spectrum $\{(-\theta)^1, (-\phi)^{d-1}, (\phi)^1, (\theta)^1\}$ with $0 < \phi < \theta$.

**Proof.** Follows from Theorem 3.1. 

**Theorem 4.7.** Let $X$ be a finite subset of $\Omega(d)$ of size $n$ with degree 2, and let $A$ be the adjacency matrix of $X$. If $d$ is even, $|X| \leq 2d$ holds. Equality holds if and only if $I + A - A^T$ is a skew Hadamard matrix.

**Proof.** A necessary condition for the existence of doubly regular tournaments is $|X| \equiv 3 \pmod{4}$, namely $d$ is odd. Therefore if $d$ is even then $|X| < 2d + 1$, that is, $|X| \leq 2d$ holds.

Let $H$ be a skew Hadamard matrix of size $n$. Then $n$ must be a multiple of 4. Define $S = \sqrt{-1}(H-I)$ and $A = \frac{1}{2}(-\sqrt{-1}S+J-I)$. Then the spectrum of $S$ is $\{(-\sqrt{n-1})^{n/2}, (\sqrt{n-1})^{n/2}\}$. Thus $A$ is of Type (2) and the minimum embedding dimension is $d = n/2$. Therefore $n = 2d$.

Let $X$ be a finite subset of $\Omega(d)$ with degree 2 and size $n = 2d$. First we consider the case $d = 2$. In this case, the classification of tournaments of order 4 is given [12] and the list of $A$ are

\[
\begin{align*}
\text{(a)} & \quad \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ with } \text{Rep}(G) = 3, \\
\text{(b)} & \quad \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \text{ with } \text{Rep}(G) = 2, \\
\text{(c)} & \quad \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \text{ with } \text{Rep}(G) = 3, \\
\text{(d)} & \quad \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ with } \text{Rep}(G) = 2. 
\end{align*}
\]

The tournaments (b) and (d) satisfy $n = 2d$, and in these cases, $I + A - A^T$ is a skew Hadamard matrix.

Next we consider the case where $d \geq 4$. By Lemmas 4.2-4.6 and the assumption that $d$ is even, $I + A - A^T$ is a skew Hadamard matrix as desired.

**Theorem 4.8.** Let $d$ be an odd integer at least 3. Let $X$ be a finite subset of $\Omega(d)$ of size $n$ with degree 2, and let $A$ be the adjacency matrix of the tournament obtained from $X$. The finite subset $X$ has the size $n = 2d$ if and only if one of the following occurs:

(i) $A$ is the adjacency matrix of an induced subgraph of a doubly regular tournament by deleting a vertex.

(ii) the Seidel matrix $S$ satisfies that $S^2$ is permutationally similar to

\[
\begin{pmatrix} kI+lJ & 0 \\ 0 & kI+lJ \end{pmatrix},
\]

for some positive integers $k,l$. 

\[ \quad (4.2) \]
Proof. Let $A$ be the adjacency matrix of an induced subgraph of a doubly regular tournament by deleting a vertex. From Theorem 1.1 and Remark 2.8 in [15] $A$ is of Type (1) and the minimum embedding dimension is $d = n/2$. Therefore $n = 2d$.

Let $S$ be the Seidel matrix which satisfies (4.2). By the block form of $S^2$, the eigenvalues $S^2$ are $k + ld, k$ with multiplicities $2, 2d - 2$ respectively. Thus the eigenvalues of $S$ are $\pm \sqrt{k + ld}, \pm \sqrt{k}$ with multiplicities $1, d - 1$ respectively. The eigenvectors of $S^2$ corresponding to $k + ld$ are the all-ones vector and the $(\pm 1)$-vector with the first $d$ entries equal to $1$ and the last $d$ entries equal to $-1$. This implies that main angles of $S$ corresponding to $\pm \sqrt{k}$ are $0$. Thus the adjacency matrix of $S$ is of Type (3) and the minimum embedding dimension $d = n/2$. Therefore $n = 2d$.

Let $X$ be a finite subset in $\Omega(d)$ with degree 2 and size $n = 2d$. By Lemmas 4.2–4.6 and the assumption that $d$ is odd, either $A$ is the adjacency matrix of an induced subgraph of a doubly regular tournament by deleting a vertex or the Seidel matrix $S$ satisfies that $S^2$ is permutaionally similar to (4.2) as desired.

\begin{remark}

Chadjipantelis and Kounias [6, Theorem] showed that supplementary difference sets construct $(\pm 1)$-matrix $S$ satisfying (4.2).

For the Seidel matrix $S$ satisfying (4.2) with $(k, l) = (n - 3, 2)$, $\sqrt{-1} S + I$ is known as the $D$-optimal designs [8, 23]. Let $A_1, A_2$ be the adjacency matrices of doubly regular tournaments of same order. Then a tournament of the adjacency matrix

\[
\begin{pmatrix}
A_1 & J \\
0 & A_2
\end{pmatrix}
\]

satisfies (4.2) for $(k, l) = (d, d - 1)$. For $d = 2$, this example corresponds to a skew $D$-optimal design.

When $d$ is odd, the number of tight 2-codes in $\Omega(d)$ is equal to that of doubly regular tournaments of order $2d + 1$. When $d$ is even, the number of tight 2-codes in $\Omega(d)$ is that of tournaments in the switching classe of the tournament obtained by adding one vertex with no outward edges and all possible inward edges to a doubly regular tournament. If we use a computer, the number of non-isomorphic tournaments in a switching class can be calculated by Theorem 3.2 in [3]. Therefore if doubly regular tournaments are classified, then we can determine the number of tight 2-codes. Doubly regular tournaments have been classified for order at most 27 [22], and we can find the catalogue in [12]. Note that non-isomorphic doubly regular tournaments may be in the same switching class. By using a computer calculation based on Theorem 3.2 in [3], we can give the number of tight 2-codes as Table 1.

<table>
<thead>
<tr>
<th>$d$</th>
<th>1 2 3 4 5 6 7 8 9 10 11 12 13 14</th>
</tr>
</thead>
<tbody>
<tr>
<td>$</td>
<td>X</td>
</tr>
<tr>
<td>$#$</td>
<td>1 2 1 4 1 8 2 240 2 8956 37 11339044 722 9897616700</td>
</tr>
</tbody>
</table>

Table 1: Tight complex 2-code $X$ in $\Omega(d)$

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