Radial Bargmann representation for the Fock space of type B

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Let \( \nu_{\alpha,q} \) be the probability and orthogonality measure for the \( q \)-Meixner-Pollaczek orthogonal polynomials, which has appeared in the work of Bożejko, Ejsmont, and Hasebe [J. Funct. Anal. 269, 1769–1795 (2015)] as the distribution of the \((\alpha,q)\)-Gaussian process (the Gaussian process of type B) over the \((\alpha,q)\)-Fock space (the Fock space of type B). The main purpose of this paper is to find the radial Bargmann representation of \( \nu_{\alpha,q} \). Our main results cover not only the representation of \( q \)-Gaussian distribution by van Leeuwen and Maassen [J. Math. Phys. 36, 4743–4756 (1995)] but also of \( q^2 \)-Gaussian and symmetric free Meixner distributions on \( \mathbb{R} \). In addition, non-trivial commutation relations satisfied by \((\alpha,q)\)-operators are presented. © 2016 AIP Publishing LLC. [http://dx.doi.org/10.1063/1.4939748]

I. INTRODUCTION

Bożejko-Ejsmont-Hasebe\(^1\) considered a deformation of the (algebraic) full Fock space with two parameters \( \alpha \) and \( q \), namely, the \((\alpha,q)\)-Fock space (or the Fock space of type B) \( \mathcal{F}_{\alpha,q}(H) \) over a complex Hilbert space \( H \). The deformation with \( \alpha = 0 \) is equivalent to the \( q \)-deformation by Bożejko-Speicher\(^1\) and Bożejko-Kümmerer-Speicher,\(^15\) and the corresponding \( q \)-Bargmann-Fock space has been constructed by van Leeuwen-Maassen.\(^9\)

For the construction of \( \mathcal{F}_{\alpha,q}(H) \), their starting point is to replace the Coxeter group of type A, that is, symmetric group \( S_n \) for the \( q \)-Fock space by the Coxeter group of type B, \( \Sigma_n := \mathbb{Z}_n^2 \rtimes S_n \) in (A1) of the Appendix. This replacement provides us a more general symmetrization operator on \( H^{\mathbb{Z}_n} \) than that of Ref. 16 to define the \((\alpha,q)\)-inner product \( \langle \cdot,\cdot \rangle_{\alpha,q} \) in (A3). One can define annihilation \( B_{\alpha,q}^*(f) \) and creation \( B_{\alpha,q}^+(f) \) operators acting on \( \mathcal{F}_{\alpha,q}(H) \) and the \((\alpha,q)\)-Gaussian process (the Gaussian process of type B) \( G_{\alpha,q}(f) \) for \( f \in H \) as the sum of them, \( G_{\alpha,q}(f) = B_{\alpha,q}^*(f) + B_{\alpha,q}^+(f) \). It is one of their main interests to find a probability distribution \( \mu_{\alpha,q} \) on \( \mathbb{R} \) of \( G_{\alpha,q}(f) \), \( \|f\|_H = 1 \), with respect to the vacuum state (\( \Omega \cdot \Omega \rangle_{\alpha,q} \), \( \mathcal{F}_{\alpha,q}(H) \) equipped with \( \langle \cdot,\cdot \rangle_{\alpha,q}, B_{\alpha,q}^*(f), \) and \( B_{\alpha,q}^+(f) \)) is a typical example of interacting Fock spaces in the sense of Accardi-Bożejko.\(^1\) It suggests that the theory of orthogonal polynomials plays intrinsic roles in all previous works mentioned above. In fact, the measure \( \mu_{\alpha,q} \) given in Ref. 14 [Theorem 3.3] is derived essentially from the orthogonality measure \( \nu_{\alpha,q} \) associated with the \( q \)-Meixner-Pollaczek orthogonal polynomials \( \{P_n^{(\alpha,q)}(x)\} \) for \( \alpha, q \in (-1, 1) \) given by the recurrence relation,

\[
\begin{align*}
  p_0^{(\alpha,q)}(x) &= 1, \\
  p_1^{(\alpha,q)}(x) &= x, \\
  x p_n^{(\alpha,q)}(x) &= p_{n+1}^{(\alpha,q)}(x) + (1 + \alpha q^{n-1}) [n]_q p_{n-1}^{(\alpha,q)}(x), \quad n \geq 1,
\end{align*}
\]

References:

Bożejko-Ejsmont-Hasebe: Ref. 1
Bożejko-Speicher: Ref. 1
Bożejko-Kümmerer-Speicher: Ref. 15
van Leeuwen-Maassen: Ref. 9
Accardi-Bożejko: Ref. 1

\( \Sigma_n := \mathbb{Z}_n^2 \rtimes S_n \)
\( \mathcal{F}_{\alpha,q}(H) \)
\( G_{\alpha,q}(f) \)
\( \mu_{\alpha,q} \)
\( \mu_{\alpha,q} \)
\( \{P_n^{(\alpha,q)}(x)\} \)
\( \alpha, q \in (-1, 1) \)
where \([n]_q = 1 + q + \cdots + q^{n-1}\) is the \(q\) number. However, the Bargmann representation (measure on \(\mathbb{C}\)) of \(\nu_{\alpha,q}\) has not been obtained yet except the case of \(\alpha = 0\) for \(0 \leq q < 1\),\(^{29}\) for \(q = 1\),\(^{10,9}\) for \(q = 0\),\(^{12}\) and \(t\)-deformed cases of these,\(^ {8,24}\) and for \(q > 1\),\(^{23}\)

Therefore, the main purpose of this paper is to find the radial Bargmann representation of the probability measure \(\nu_{\alpha,q}\) on \(\mathbb{R}\). Our results cover the radial Bargmann representations of \(q\)-Gaussian, symmetric free Meixner (Kesten), and \(q^2\)-Gaussian distributions on \(\mathbb{R}\).

The organization of this paper will be as follows. In Section II, we shall explain how the \((\alpha,q)\)-Fock space is related to the notion of one-mode interacting Fock spaces and Bargmann representation. In Section III, the radial Bargmann representation of \(\nu_{\alpha,q}\) is constructed explicitly in Theorem 3.11. In Section IV, commutation relations satisfied by one-mode \((\alpha,q)\)-annihilation and creation operators will be treated. In the Appendix, we shall give a minimum reference on the Coxeter group of type \(B\) extracted from Ref. 14.

II. KEY IDEAS AND OUR PURPOSE

Let us point out some of the keys to calculate the distribution of \(G_{\alpha,q}(f)\) in Ref. 14. It is shown that a linear map, \(\Phi: \text{Span}\{f^\otimes n \mid f \in H, n \geq 0\} \to \mathcal{L}^2(\mathbb{R}, \mu_{\alpha,q,f})\) given by \(\Phi(f^\otimes n) = P_{\alpha,f}^{(\alpha,f)H,q}(x)\), is an isometry and a relation under \(\|f\|_{H^1} = 1\),

\[
G_{\alpha,q}(f)f^\otimes n = (B^+_{\alpha,q}(f) + B^-_{\alpha,q}(f))(f^\otimes n)
\]

\[
= f^\otimes(n+1) + (1 + \alpha(f, \overline{f})_H)q^{n+1-1}[n]_q f^\otimes(n-1),
\]

is satisfied where \(\overline{f}\) denotes a self-adjoint involution of \(f \in H\) in (A2). This corresponds to the three terms recursion relation satisfied by \(P_{\alpha,f}^{(\alpha,f)H,q}(x)\) through \(\Phi\). Then, it is proved that \(\mu_{\alpha,q,f} = \nu_{\alpha,f(\overline{f})H,q}\) (see \(\nu_{\alpha,q}\) in (3.3)) in the sense of

\[
\langle \Omega, G_{\alpha,q}(f)^{\alpha,q} \Omega \rangle_{\alpha,q} = \int x^n \mu_{\alpha,q,f} (dx),
\]

(2.1)

where \(\Omega\) denotes the vacuum vector. Therefore, in order to get the Bargmann representation of \(\nu_{\alpha,f(\overline{f})H,q}\), it is enough to consider that of \(\nu_{\alpha,q}\) in the sense of Definition 2.2 given later.

Since the structure mentioned above can be reduced to the one-mode analogue of \((\alpha,q)\)-Fock spaces, let us recall fundamental relationships between one-mode interacting Bargmann-Fock spaces and the theory of orthogonal polynomials of one variable.

Definition 2.1. Let \(\{\omega_n\}_{n=0}^{\infty}\) with \(\omega_0 := 1\) be an infinite sequence of positive real numbers and \(\{\alpha_n\}_{n=0}^{\infty}\) be of real numbers. A one-mode interacting Bargmann-Fock space \(\mathcal{B}\) is defined as \(\bigoplus_{n=0}^{\infty} \Phi_n\) equipped with \(\Phi_n := z^n/\omega_n!,\ [\omega_n!] := \prod_{k=0}^{n} \omega_k\), the inner product \(\langle \Phi_m, \Phi_n \rangle_{\mathcal{B}} = \delta_{m,n}\) for all \(m, n \in \mathbb{N} \cup \{0\}\), operators of creation \(a^+\), annihilation \(a^-\), and conservation \(a^\circ\) defined by

\[
\begin{align*}
\langle a^\circ \Phi_n := \sqrt{\omega_{n+1}} \Phi_{n+1}, & \quad n \geq 0, \\
a^- \Phi_0 = 0, & \quad a^- \Phi_n := \sqrt{\omega_n} \Phi_{n-1}, \quad n \geq 1, \\
a^+ \Phi_n := \alpha_n \Phi_n, & \quad n \geq 0.
\end{align*}
\]

(2.2)

Let \(\{\omega_n\}_{n=0}^{\infty}, \{\alpha_n\}_{n=0}^{\infty}\) be a pair of sequences in Definition 2.1 and define a sequence of monic polynomials \(\{P_n(x)\}\) recurrently by

\[
\begin{align*}
P_0(x) &= 1, & P_1(x) &= x - \alpha_0, \\
x P_n(x) &= P_{n+1}(x) + \omega_n P_{n-1} + \alpha_n P_n(x), & n \geq 1.
\end{align*}
\]

(2.3)

Then, there exists a probability measure \(\mu\) on \(\mathbb{R}\) with finite moments of all orders such that \(\{P_n(x)\}\) is the orthogonal polynomials with \(\langle P_m(x), P_n(x) \rangle_{L^2(\mathbb{R}, \mu)} = \delta_{m,n}[\omega_n]\) for all \(m, n \in \mathbb{N} \cup \{0\}\). (See Refs. 19 and 21, for example.)
It is easy to see that a linear map
\[ U : \mathcal{B} = \bigoplus_{n=0}^{\infty} \mathbb{C} \Phi_n \rightarrow L^2(\mathbb{R}, \mu) \]
defined by \( U(\sqrt{\omega_n} \Phi_n) = P_n(x) \) is an isometry and in addition, \( a^+ + a^- + a^0 = U^* X U \) is satisfied due to (2.2) and (2.3), where \( X \) represents the multiplication operator by \( x \) in \( L^2(\mathbb{R}, \mu) \). This intertwining relation provides a notion of the quantum decomposition of a classical random variable \( X \) and

\[ \langle \Phi_0, (a^+ + a^- + a^0)^n \Phi_0 \rangle = \int x^n \mu(dx). \] (2.4)

Therefore, if \( \omega_n = (1 + \alpha q^{-n})[n]_q \), \( \alpha_n = 0 \), the equality in (2.4) is a one-mode analogue of (2.1).

Now, it is interesting to consider the following moment problem to realize the inner product by the integral.

**Problem 1.** For a given \( \{\omega_n\} \) of \( \mu \), find a probability measure \( \gamma \mu \) satisfying the equality

\[ \int_{\mathbb{C}} z^m \bar{z}^n \gamma \mu(d^2z) = \delta_{m,n} \omega_n! \] (2.5)

for all \( m, n \in \mathbb{N} \cup \{0\} \).

**Definition 2.2.** A measure \( \gamma \mu \) satisfying equality (2.5) is called a Bargmann representation (measure on \( \mathbb{C} \)) of a measure \( \mu \) on \( \mathbb{R} \).

It was proved in Ref. 28 (see also Refs. 8 and 24) that if a measure \( \mu \) admits any Bargmann representation, then it also admits a radial (rotation invariant) Bargmann representation

\[ \gamma \mu(d^2z) = \frac{1}{2\pi} \lambda_{[0,2\pi]}(d\theta) \rho \mu(dr), \ z = re^{i\theta}, \ r \geq 0, \ \theta \in [0,2\pi), \]

where \( \lambda_{[0,2\pi]} \) is the Lebesgue measure on \( [0,2\pi) \). It says that the angular part takes care of orthogonality of (2.5). Therefore, Problem 1 can be transformed into Problem 2.

**Problem 2.** Find a positive radial measure \( \rho \mu \) satisfying

\[ \int_0^{\infty} r^{2n} \rho \mu(dr) = [\omega_n]! \] for all \( m, n \in \mathbb{N} \cup \{0\} \).

**Main purpose.** We shall consider Problem 2 associated with \( \omega_n = (1 + \alpha q^{-n})[n]_q \), \( \alpha_n = 0 \) of \( \nu_{\alpha,q} \) in Section III. Furthermore, commutation relations satisfied by \( a^+, a^- \) acting on \( \mathcal{B} \) associated with \( \omega_n = (1 + \alpha q^{-n})[n]_q \) will be presented in Section IV.

**Remark 2.3.** (1) One can notice that \( \gamma \mu \) and \( \rho \mu \) are determined only by \( [\omega_n]! \). Therefore, it is enough in general for the Bargmann representation in the above sense to consider the symmetric measure \( \mu \) with \( \alpha_n = 0 \) for all \( n \), which implies that \( a^+ \) is a zero operator.

(2) If \( \mu \) is symmetric, then \( \alpha_n = 0 \) for all \( n \) is implied. The converse statement is true if \( \mu \) is determined by its moments.

**III. \( (\alpha, q) \)-BARGMANN REPRESENTATION**

**A. \( q \)-Meixner-Pollaczek polynomials**

Let us recall standard notations from \( q \)-calculus, which can be found in Refs. 20 and 22, for example. The \( q \)-shifted factorials are defined by

\[ (a; q)_0 := 1, \quad (a; q)_k := \prod_{\ell=1}^{k} (1 - a q^{\ell-1}), \ k = 1, 2, \ldots, \infty, \]
and the product of \(q\)-shifted factorials is defined by
\[
(a_1, a_2; q)_k := (a_1; q)_k(a_2; q)_k, \quad k = 1, 2, \ldots, \infty.
\]

**Remark 3.1.** The \(q\)-shifted factorials are a natural extension of the Pochhammer symbol \((\cdot)_n\) because one can see that \(\lim_{q \to 1}[k]_q = k\) implies
\[
\lim_{q \to 1} \left(\frac{q^k; q)_n}{(1 - q)^n}\right) = (k)_n,
\]
where \((k)_0 := 1, (k)_n := k(k + 1) \cdots (k + n - 1), n \geq 1.

As we have mentioned, \(\{P_n^{(α, q)}(x)\}\) for \(α, q \in (-1, 1)\) is the \(q\)-Meixner-Pollaczek polynomials satisfying the recurrence relation,
\[
\begin{align*}
&P_0^{(α, q)}(x) = 1, \quad P_1^{(α, q)}(x) = x, \\
&P_n^{(α, q)}(x) = (\alpha, q; q)_n x P_{n-1}^{(α, q)}(x) + (1 + \alpha q^{n-1})[n]_q P_{n-1}^{(α, q)}(x), \quad n \geq 1.
\end{align*}
\]

It is known in Ref. 22 [14.9.2] and Ref. 14 [page 1781] that the orthogonality measure \(ν_{α, q}\) for such polynomials has the density of the form
\[
\frac{(q, q^2; q)_∞}{2\pi} \sqrt{\frac{1 - q}{4 - (1 - q)x^2}} \left(\frac{g(x, 1; q)g(x, -1; q)g(x, \sqrt{q}; q)g(x, -\sqrt{q}; q)}{g(x, i\gamma; q)g(x, -i\gamma; q)}\right),
\]
supported on the interval \((-2/\sqrt{1 - q}, 2/\sqrt{1 - q})\), where
\[
g(x, b; q) = \prod_{k=0}^{∞} (1 - 4bx(1 - q)^{-1/2}q^k + b^2q^{2k})
\]
and
\[
\gamma = \begin{cases} \sqrt{-α}, & α < 0, \\
i\sqrt{α}, & α \geq 0. \end{cases}
\]

**Example 3.2.** (1) If \(α = 0\), then \(q\)-Meixner-Pollaczek polynomials get back to the \(q\)-Hermite polynomials \(H_n^{(q)}(x)\) whose orthogonality measure is the standard \(q\)-Gaussian measure on \((-2/\sqrt{1 - q}, 2/\sqrt{1 - q})\) given by
\[
ν_q(dx) := \frac{\sqrt{1 - q}}{\pi} \sin θ \prod_{n=1}^{∞} (1 - q^n)^{1 - q^n} e^{2iθ} \pi dx,
\]
where \(x\sqrt{1 - q} = 2\cos θ, θ \in [0, π]\). Furthermore, one can get the standard Gaussian law as \(q \to 1\), the Bernoulli law as \(q \to -1\), and the standard Wigner’s semi-circle law if \(q = 0\). See Refs. 15 and 16.

(2) The measure \(ν_{α, 0}\) is the symmetric free Meixner law.

(3) The measure \(ν_{q, q}\) is the \(q^2\)-Gaussian law scaled by \(\sqrt{1 + q}\).

(4) If \(α = -q^{2β}\) as suggested in Remark 3.1, then the measure \(ν_{-q^{2β}, q}\) under a certain scaling converges to the classical symmetric Meixner law as \(q \uparrow 1\),
\[
\frac{2^{2β}}{2πΓ(2β)}|Γ(β + ix)|^2 dx, \quad x \in \mathbb{R}.
\]

**See also Ref. 22 [14.9.15].**

**B. Problem**

For \(α, q \in (-1, 1)\), we would like to know when there exists a radial measure \(ρ_{να, q}\) satisfying
\[
\int_0^{∞} r^{2k} ρ_{να, q}(dr) = (-α; q)_k[k]_q, \quad k \in \mathbb{N} ∪ \{0\}.
\]
Here, \([k]_q!\) denotes the \(q\)-factorials defined by

\[
[k]_q! : = 1, \quad [k]_q! : = \prod_{\ell = 1}^{k} [\ell]_q = \frac{(q; q)_k}{(1 - q)^k}, \quad k \geq 1.
\]

It is easy to get the inequality for \(\alpha, q \in (-1, 1)\),

\[
|(-\alpha; q)_k[k]_q!| \leq \left(\frac{4}{1 - |q|}\right)^k, \quad k \in \mathbb{N} \cup \{0\}.
\]

Due to Carleman criterion for the moment problem, this inequality implies that a radial measure \(\rho_{\nu, a, q}\) is determined uniquely by the sequence \(\{(\alpha; q)_k[k]_q!\} \).

We shall follow the procedure below to construct \(\rho_{\nu, a, q}\) in (3.5).

1. Recall the radial part of the \(q\)-Gaussian measure on \(\mathbb{C}\) \((q\)-Bargmann measure\), \(\rho_{\nu q} = \rho_{\nu_0, q}\), obtained in Ref. 29,

\[
\int_0^\infty r^{2k} \rho_{\nu q}(dr) = [k]_q!.
\]

2. Find a radial (possibly signed) measure \(\rho_{a, q}\) having the moment \((-\alpha; q)_k\).

3. Compute the multiplicative (Mellin) convolution \(\rho_{\nu q} \circ \rho_{a, q}\) to get \(\rho_{\nu a, q}\).

**Remark 3.3.** It is known\(^{29}\) that a radial measure \(\rho_{\nu q}\) in (3.7) does not exist for \(q < 0\). However, one can see that the positivity assumption on \(q\) can be relaxed for \(\rho_{\nu a, q}\) if \(\alpha = q\). It will be discussed right after the proof of Proposition 3.6 and in Proposition 3.7.

### C. Construction of \((\alpha, q)\)-radial measures

**Lemma 3.4.** Suppose that \(\alpha \in (-1, 1)\) and \(q \in [0, 1)\). Let

\[
\rho_{\alpha, q} := (-\alpha; q)_\infty \sum_{n=0}^{\infty} (-\alpha)^n \left(\frac{q}{q}\right)_n \delta_{\alpha^n/2},
\]

which is a signed measure. Then, we have

\[
\int_0^\infty r^{2k} \rho_{\alpha, q}(dr) = (-\alpha; q)_k, \quad k \in \mathbb{N} \cup \{0\}.
\]

In particular, if taking \(\alpha = -q\), then one can see \(\rho_{\nu q} = D_{(1-q)^{-1/2}}(\rho_{-q, q})\), namely,

\[
\int_0^\infty r^{2k} D_{(1-q)^{-1/2}}(\rho_{-q, q})(dr) = \frac{(q; q)_k}{(1 - q)^k} = [k]_q!.
\]

where \(D_t(\lambda)\) is the push-forward of \(\lambda\) by the map \(x \mapsto tx\) for a measure \(\lambda\) on \(\mathbb{R}\).

**Proof.** First, we have

\[
\int_0^\infty r^{2k} \rho_{\alpha, q}(dr) = (-\alpha; q)_\infty \sum_{n=0}^{\infty} (-\alpha q^k)^n \left(\frac{q}{q}\right)_n.
\]

Since Euler’s formula (see Ref. 20 [1.3.15]),

\[
\frac{1}{(\alpha; q)_\infty} = \sum_{n=0}^{\infty} a^n \left(\frac{q}{q}\right)_n,
\]

is known, we replace \(a\) by \(-\alpha q^k\) in (3.8) to obtain

\[
\int_0^\infty r^{2k} \rho_{\alpha, q}(dr) = \frac{(-\alpha; q)_\infty}{(-\alpha q^k; q)_\infty} = (-\alpha; q)_k.
\]

The proof is complete. \(\square\)
Remark 3.5. (1) The last equality in proof is due to the formula
\[(a; q)_k = \frac{(a; q)_\infty}{(aq^k; q)_\infty}.
\]
See Ref. 20 [1.2.30], for example.

(2) Euler’s formula is considered as the \(q\)-analogue of exponential function \(e^a\) due to
\[\lim_{q \to 1} \frac{1}{((1 - q)a; q)_n} = e^a.
\]
Let
\[\begin{bmatrix} n \\ \ell \end{bmatrix}_q := \frac{[n]_q!}{[\ell]_q! [n - \ell]_q!} = \frac{(q; q)_n}{(q; q)_\ell (q; q)_{n - \ell}}\]
be the \(q\)-binomial coefficients and \(h_n(z | q)\) be the Rogers-Szegö polynomials\(^\text{20,27}\) defined by
\[h_n(z | q) = \sum_{\ell=0}^n \frac{[n]_q!}{[\ell]_q! [n - \ell]_q!} z^{\ell}.
\]

Proposition 3.6. Suppose that \(\alpha \in (-1, 1) \text{ and } q \in [0, 1)\). Let
\[\nu_{\alpha, q} := \begin{cases} (-\alpha, q; q)_\infty \sum_{n=0}^\infty \frac{q^n}{(q; q)_n} h_n(-\alpha q^{-1} | q) \delta_{(1-q)^{-1/2} q^n/2}, & q > 0, \\ -\alpha \delta_0 + (1 + \alpha) \delta_1, & q = 0, \end{cases}
\]
which is a signed measure in general. Then, we have
\[\int_0^\infty r^{2k} \nu_{\alpha, q}(dr) = \frac{(-\alpha, q; q)_k}{(1 - q)_k} = (-\alpha; q)_k [k]_q!, \quad k \in \mathbb{N} \cup \{0\}.
\]  

Proof. First of all, it is easy to show (3.10) for the case \(q = 0\). Therefore, let us assume \(q > 0\). One can compute the multiplicative (Mellin) convolution \(\circ\) to get \(\nu_{\alpha, q}\) as follows:
\[\nu_{\alpha, q} = \nu_{\alpha, q} \circ D_{(1-q)^{-1/2} (p_{-q, q})} = (-\alpha, q; q)_\infty \sum_{n=0}^\infty \left( \sum_{\ell=0}^n \frac{(-\alpha)^{q^{n-\ell}}}{(q; q)_\ell (q; q)_{n-\ell}} \right) \delta_{(1-q)^{-1/2} q^n/2} = (-\alpha, q; q)_\infty \sum_{n=0}^\infty \frac{q^n}{(q; q)_n} h_n(-\alpha q^{-1} | q) \delta_{(1-q)^{-1/2} q^n/2}.
\]
On the other hand, by Lemma 3.4, we have
\[\int_0^\infty r^{2k} D_{(1-q)^{-1/2} (p_{-q, q})}(dr) = \frac{(q; q)_k}{(1 - q)_k} = [k]_q!.
\]
Therefore, one can get
\[\int_0^\infty r^{2k} \nu_{\alpha, q}(dr) = \int_0^\infty r^{2k} \nu_{\alpha, q}(dr) \int_0^\infty r^{2k} D_{(1-q)^{-1/2} (p_{-q, q})}(dr) = (-\alpha; q)_k [k]_q!, \quad k \in \mathbb{N} \cup \{0\}.
\]
Proposition 3.7. Suppose \(-1 < \alpha = q < 0\). We define

\[
\rho_{\nu,q} := D_{(1+q)^{1/2}}(\rho_{\nu,q}) = (q^2; q^2)_{\infty} \sum_{n=0}^{\infty} \frac{q^{2n}}{(q^2; q^2)_n} \delta_{(1-q^{1/2})^{-1}(-q)^n}.
\]  \hspace{1cm} (3.11)

Then, one can see

\[
\int_0^{2\pi} r^{2k} \rho_{\nu,q}(dr) = (1 + q)^k [k]_q! = (-q; q)_k [k]_q!.
\]

Proof. One can see by direct computations

\[
(-q; q)_k [k]_q! = \left\{ \prod_{\ell=1}^k (1 - (1-q)q^{\ell-1}) \right\} \left\{ \prod_{\ell=1}^k \frac{1 - q^{2\ell}}{1 - q} \right\}
= (1 + q)^k \prod_{\ell=1}^k \frac{1 - q^{2\ell}}{1 - q}
= (1 + q)^k [k]_q!.
\]

Thus, \(\rho_{\nu,q}\) can be defined as the radial measure for \(q^2\)-Gaussian measure on \(\mathbb{C}\) scaled by \((1 + q)^{1/2}\), namely, \(\rho_{\nu,q} = D_{(1+q)^{1/2}}(\rho_{\nu,q})\).

Remark 3.8. If we use the fact that \(h_n(-1 \mid q) = 0\) for odd \(n \geq 1\) (see proof of Lemma 3.9), we can extend definition (3.9) to the case \(-1 < \alpha = q < 0\). This will give an alternative way to define \(\rho_{\nu,q}\) for \(-1 < q < 0\), but both definitions give the same measure.

We need some properties of the Rogers-Szegö polynomials to know when the measure \(\rho_{\nu,\alpha,q}\) becomes positive.

Lemma 3.9 (Ref. 25). Suppose that \(q \in (-1,1)\).

1. If \(n \geq 0\) is odd, then \(h_n(x \mid q) \geq 0\) if and only if \(x \geq -1\). Moreover, the point \(x = -1\) is the unique zero of \(h_n(x \mid q)\) on \(\mathbb{R}\).
2. If \(n \geq 0\) is even, then \(h_n(x \mid q) > 0\) for all \(x \in \mathbb{R}\).

Proof. It is known that all the zeros of \(h_n(z \mid q)\) lie on the unit circle \(|z| = 1\). See Ref. 25 or Ref. 27 [Theorem 1.6.11]. Note that the result was obtained for \(q \in [0,1)\), but the proof can be extended to \(q \in (-1,1)\) without any modifications.

By definition, one can see

\[
\left\lfloor \frac{n}{\ell} \right\rfloor_q = \frac{(1 - q^{n-\ell+1})(1 - q^{n-\ell+2}) \cdots (1 - q^n)}{(1 - q)(1 - q^2) \cdots (1 - q^\ell)} > 0,
\]
and hence, \(h_n(1 \mid q) > 0\) for all \(n \geq 0\). Thus, \(h_n(x \mid q) \neq 0\) for \(x \in \mathbb{R} \setminus \{-1\}\). It then suffices to show that \(h_n(-1 \mid q) > 0\) for all even \(n \geq 0\) and \(h_n(-1 \mid q) = 0\) for all odd \(n \geq 1\). The recurrence relation for the Rogers-Szegö polynomials is known to be

\[
h_{n+1}(z \mid q) = (z + 1)h_n(z \mid q) - (1 - q^n)zh_{n-1}(z \mid q), \quad n \geq 1.
\]  \hspace{1cm} (3.12)

See Ref. 27 [1.6.76] (note that formula (1.6.76) has an error of a sign). It is easy to see that \(h_0(-1 \mid q) = 1 > 0, h_1(-1 \mid q) = 0\), so by induction and (3.12), one can show \(h_n(-1 \mid q) > 0\) for all even \(n \geq 0\) and \(h_n(-1 \mid q) = 0\) for all odd \(n \geq 1\).

We need the following lemma in proof of Theorem 3.11 for the non-existence part of a radial Bargmann measure.

Lemma 3.10. Let \(\mu\) be a signed measure on \(\mathbb{R}\) with compact support and let \(\nu\) be a non-negative measure on \(\mathbb{R}\). If \(\mu\) and \(\nu\) have the same finite moments of all orders, then \(\mu = \nu\).
Proof. We denote by $m_n$ the moments of $\mu$ (and $\nu$ by assumption). Since $\mu$ is compactly supported, say on $[-R, R]$, 

$$|m_n| = \left| \int_{[-R, R]} x^n \mu(dx) \right| \leq \|\mu\| R^n, \quad n \in \mathbb{N} \cup \{0\},$$

where $\|\mu\|$ denotes the total variation of $\mu$. Therefore, $\nu$ is also supported on $[-R, R]$. By Weierstrass’ approximation, we have 

$$\int_{[-R, R]} f(x) \mu(dx) = \int_{[-R, R]} f(x) \nu(dx) \quad (3.13)$$

for all $f \in C([-R, R])$. This implies that $\mu = \nu$ since, if we use the Hahn decomposition $\mu = \mu_+ - \mu_-$, then (3.13) implies

$$\int_{[-R, R]} f(x) \mu_+(dx) = \int_{[-R, R]} f(x) (\nu + \mu_-)(dx),$$

and hence, $\mu_+ = \nu + \mu_-$ as non-negative finite measures. \hfill \Box

In summary, the complete answer to the unique existence of a radial Bargmann representation of $\nu_{\alpha, q}$ is stated as follows.

**Theorem 3.11.** Suppose that $\alpha, q \in (-1, 1)$, The probability measure $\nu_{\alpha, q}$ has a radial Bargmann representation if and only if either (i) $q \geq 0$ and $\alpha \leq q$ or (ii) $\alpha = q \neq 0$.

In fact, the radial measure is given uniquely by

$$\rho_{\nu_{\alpha, q}} = \begin{cases} -\alpha \delta_0 + (1 + \alpha) \delta_1, & (\alpha \leq q, 0), \\ (-\alpha, q; q)_{\alpha} \sum_{n=0}^{\infty} \frac{q^n}{(q; q)_n} h_n(-\alpha q^{-1} | q) \delta_{(1-q)^{1/2} q^n/2}, & (q > 0, \alpha < q), \\ (q^2; q^2)_{\alpha} \sum_{n=0}^{\infty} \frac{q^{2n}}{(q^2; q^2)_n} \delta_{(1-q)^{-1/2} q^n}, & (\alpha = q \neq 0). \end{cases}$$

Proof. 1. Existence and uniqueness. If $q \in [0, 1)$ and $\alpha \leq q$, then by Proposition 3.6 and Lemma 3.9, the signed measure $\rho_{\nu_{\alpha, q}}$ is in fact a non-negative measure and becomes the radial part of a Bargmann measure. The case $\alpha = q < 0$ was discussed in Proposition 3.7. Due to Carleman criterion for the moment problem, the inequality given in (3.6) guarantees the uniqueness of $\rho_{\nu_{\alpha, q}}$ for these cases.

2. Non-existence. (1) If $q \in (0, 1)$ and $\alpha > q$, then $\rho_{\nu_{\alpha, q}}$ is not a non-negative measure and is really a signed measure since $h_n(-\alpha q^{-1} | q) < 0$ for odd integers $n \geq 0$ and $q > 0$ from Lemma 3.9. By Lemma 3.10, if a radial Bargmann measure exists, then it must be equal to the signed measure $\nu_{\alpha, q}$. This is a contradiction. Thus, a radial Bargmann measure does not exist.

(2) If $q = 0$ and $\alpha > q = 0$, then by (3.9), $\nu_{\alpha, 0}$ is really a signed measure, and hence, by the same argument as above, a radial Bargmann measure does not exist.

(3) Let

$$\beta_k(\alpha, q) := (-\alpha; q)_{k} k!, \quad k \geq 0, \alpha, q \in (-1, 1).$$

Given $q < 0$ and $\alpha \neq q$, suppose that there exists a radial part of a Bargmann measure, $\rho$. Let $\rho^2$ be the push-forward of $\rho$ by the map $x \mapsto x^2$. Then,

$$\beta_k(\alpha, q) = \int_{0}^{\infty} x^k \rho^2(dx) = \int_{0}^{\infty} x^{2k} \rho(dx). \quad (3.14)$$

By the way, by Proposition 3.6, it holds that $\beta_k(\alpha, q') = \int_{0}^{\infty} x^{2k} \rho_{\nu_{\alpha, q'}}(dx)$ for any $q' > 0$, that is,

$$\beta_k(\alpha, q') = (-\alpha, q'; q')_{\alpha} \sum_{n=0}^{\infty} \frac{(q')^n}{(q'; q')_n} h_n(-\alpha(q')^{-1} | q') \frac{(q')^{kn}}{(1-q')^k}, \quad q' > 0, \quad (3.15)$$

which is true even for $q' = q$ by analytic continuation.
Now let us consider the signed measure

\[ \mu := (-\alpha, q; q)_0 \sum_{n=0}^{\infty} \frac{q^n}{(q; q)_n} h_n(-\alpha q^{-1} | q) \delta_{(1-q)^{-1} q^n}, \quad \alpha \neq q < 0, \]

supported on the points \( \frac{q^n}{1-q} \) for \( n = 0, 1, 2, 3, \ldots \). Then, by (3.15) for \( q' = q \) and by (3.14),

\[ \int x^k \mu(dx) = \beta_k(\alpha, q) = \int_0^\infty x^k \rho^2(dx), \quad k \in \mathbb{N} \cup \{0\}. \]

By Lemma 3.10, the signed measure \( \mu \) and the probability measure \( \rho^2 \) should be equal. However, the support of \( \mu \) is not contained in \([0, \infty)\), and hence, \( \mu \) cannot be equal to \( \rho^2 \). This is a contradiction. \( \square \)

**Example 3.12.** (1) The radial measure \( \rho_{\nu,0,q} \) for \( q \in [0, 1) \) is of the \( q \)-Bargmann.\(^{29}\)

(2) The radial measure \( \rho_{\nu,q,q} \) for \( q \in (-1, 1) \) is of the \( q^2 \)-Bargmann.

(3) \( \lim_{q \uparrow 1} \rho_{\nu,q,q} \) is of the classical Bargmann.\(^{10,9}\)

(4) Consider \( \alpha = -q^2 \beta, \beta > 0 \). This choice of \( \alpha \) is suggested by (3.1) in Remark 3.1. In fact, one can see

\[ \lim_{q \uparrow 1} \frac{(1 - q^{2\beta+n-1})[n]q}{4(1 - q)} = \frac{1}{4} (n + 2\beta - 1)n. \]

This limit sequence is the Jacobi sequence of the symmetric Meixner distribution in (3.4), so that \( \rho_{\nu,q,q} \) under suitable scaling converges weakly as \( q \uparrow 1 \) to the radial measure with the density,

\[ \frac{2\pi r}{I(2\beta)} \int_0^\infty h(r, t/4) e^{-t/2\beta^2} dt, \]

where

\[ h(r, t) = \frac{1}{\pi t} \exp \left( -\frac{r^2}{t} \right), \quad r \in \mathbb{R}, \quad t > 0. \]

This is an integral representation of the radial density for the Bessel kernel measure, which can be also represented by the modified Bessel function.\(^{3,7}\)

(5) \( \rho_{\nu,0,0} \) for \( \alpha \in (-1, 0] \) is the radial measure for the symmetric free Meixner distribution. See Remark 3.13.

**Remark 3.13.** Let \( \mu_t \) be a \( t \)-deformed probability measure of a probability measure \( \mu \) on \( \mathbb{R} \) defined through the Cauchy transform \( G_\mu \) of \( \mu \),

\[ \frac{1}{G_\mu(t)} := \frac{t}{G_\mu(z)} + (1 - t)z, \quad t \geq 0, \]

examined by Bożejko-Wysoczanski.\(^{17,18}\) Krystek-Wojakowski\(^{24}\) discussed the radial Bargmann representation of a \( t \)-deformed probability measure \( \mu_t \), \( t \)-Bargmann representation for short, and obtained necessary and sufficient condition for the admissibility of the representation. The \( t \)-Bargmann representation of the Kesten measure \( \kappa_t \), has the form

\[ \rho_{\kappa_t} = \left( 1 - \frac{1}{t} \right) \delta_0 + \frac{1}{t} \delta_{\sqrt{t}}, \quad t \geq 1. \]

In Ref. 8, the \( t \)-Bargmann representation of a symmetric free Meixner law \( \varphi_{s,t} \), with two positive parameters \( s, t \) is treated and is admitted if and only if \( t \geq 1 \). In fact, one can see \( \rho_{\varphi_{s,t}} = D_s(\rho_{\kappa_t}) \) and hence,

\[ \rho_{\varphi_{(-s^{-1})/t,0}} = \rho_{\varphi_{1/\sqrt{t},t}} = D_{1/\sqrt{t}}(\rho_{\kappa_t}), \quad t \geq 1. \]

Therefore, case (5) in Example 3.12 can be viewed as a \( t \)-Bargmann representation, too.
Furthermore, let us state the $t$-deformed version of Theorem 3.11 for $q \neq 0$ without proof.

**Proposition 3.14.** The $t$-deformed version of $\nu_{\alpha,q}$ for $q \neq 0$ is given by

$$\left(1 - \frac{1}{t}\right) \delta_0 + \frac{1}{t} \nu_{\alpha,q}, \quad t \geq 1.$$ 

**Remark 3.15.** The $t$-Bargmann representation of $\nu_q$ is treated in Ref. 24 for $q = 1$ and Ref. 8 for $0 \leq q < 1$.

Before closing this section, let us give a short remark about relations with the free infinite divisibility. Many of the particular examples have so far suggested that the free infinite divisibility of a probability measure implies the existence of a radial Bargmann representation. The converse is not true in general because the Askey-Wimp-Kerov distribution (Ref. 26 [Theorem 3.2]), $\{\nu_{\alpha,0} | \alpha \in (-1,1)\}$ (Ref. 26 [Theorem 3.2]), $\{\nu_q | -1 < q < 1\}$ (Refs. 3 and 4 [Example 3.11] for the free infinite divisibility), and $\{\nu_{q,q} | q \in (-1,1)\}$ (all measures in this family are freely infinitely divisible since they are $q^2$-Gaussians).

**Remark 4.2.** It is easy to check that the $\alpha$-deformed Jackson derivative is equivalently defined as

$$(D_{\alpha,q} f)(z) = (D_q f)(z) + \alpha(D_{1/q} f)(q^2 z), \quad q \neq 0.$$ 

For example, if $f(z) = z^n$, $(D_{\alpha,q} f)(z) = (1 + \alpha q^{n-1})[n]_q z^{n-1}$ holds. In fact, the $\alpha$-deformed Jackson derivative is an analogue of the operator in Ref. 14 [Theorem 2.5].
Then, one can realize one-mode analogue of \((a,q)\)-operators on an appropriate domain of the one-mode interacting Bargmann-Fock space \(\mathcal{B}\), with \(\omega_n = (1 + q^{a_n-1})[n]_q\) and \(\alpha_n = 0\) by

\[
a^+ := Z, \ a^- := D_{a,q}, \text{ and } \Phi_n := \frac{z^n}{\sqrt{[\omega_n]!}}.
\]

In fact, it is easy to check that

\[
\begin{align*}
\alpha \Phi_n &= \sqrt{\omega_{n+1}} \Phi_{n+1}, \\
\alpha \Phi_n &= \sqrt{\omega_{n-1}} \Phi_{n-1}
\end{align*}
\]

hold and the \(q\)-commutation relation, one-mode analogue of (A4),

\[
[a^-, a^+)_q := (a^- a^+ - q a^+ a^-)\Phi_n
\]

is satisfied. Let us put \(M_{a,q} = I + \alpha q^{2N}\) and then one can get the expression

\[
M_{a,q} = (1 + \alpha)I - \alpha(1 - q^2)ZD_q z,
\]

due to \((ZD_q z)\Phi_n = [n]_q, \Phi_n\).

Therefore, one can obtain the following,

**Theorem 4.3.** Suppose \(\alpha \in (-1, 1)\) and \(q \in (-1, 1)\). Then, the following are satisfied.

1. \(\left[a^-, a^+\right] = M_{a,q}, \ \left[a^-, M_{a,q}\right] = (1 - q^2)a^-, \ \left[M_{a,q}, a^+\right] = (1 - q^2)a^-.
2. \(M_{a,q}(1 + \alpha)I - \alpha(1 - q^2)ZD_q z.
3. In particular, if \(\alpha = q\), then one can obtain a more refined relation, \(\left[a^-, a^+\right] = (1 + q)I.

**Example 4.4.**

1. \(\alpha = 0\) implies \(\left[a^-, a^+\right] = I\). Hence, \(M_{0,q} = I\) commutes with both \(a^+\) and \(a^-\).

\[
\left[a^-, M_{0,q}\right] = \left[M_{0,q}, a^+\right] = 0.
\]

Therefore, the case \(\alpha \neq 0\) provides non-trivial commutation relations.

2. If \(\alpha = -q^{2\beta}\) for \(\beta > 0\), then the limiting case of the scaled operator is obtained as

\[
\lim_{q^{\uparrow}} \frac{M_{\alpha,q}^{2\beta,q}}{1 - q^{2}} = \lim_{q^{\uparrow}} \frac{I - q^{2}\beta} {1 - q^{2}} = N + \beta.
\]

Moreover, let us consider the scaled operators,

\[
A^\pm := \lim_{q^{\uparrow}} \frac{a^\pm}{\sqrt{1 - q^{2}}}.
\]

Then, one can get

\[
\left[A^-, A^+\right] = N + \beta
\]

and hence,

\[
\left[A^-, N\right] = A^-, \left[N, A^+\right] = A^+.
\]

It should be noted that these are the commutation relations for the classical Meixner-Pollaczek polynomials with respect to the symmetric Meixner distribution in (3.4). See Ref. 6.

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APPENDIX: ON A COXETER GROUP OF TYPE B

Let $\Sigma_n$ be the set of bijections $\sigma$ of the $2n$ points $\{\pm 1, \pm 2, \ldots, \pm n\}$ with $\sigma(-k) = -\sigma(k)$. Equipped with the composition operation as a product, $\Sigma_n$ becomes what is called a Coxeter group of type B. It is generated by $\pi_0 := (1,-1)$ and $\pi_i := (i, i+1)$, $1 \leq i \leq n-1$, which satisfy the generalized braid relations

\[ \begin{aligned}
\pi_i^2 &= e, & 0 \leq i \leq n-1, \\
(\pi_0 \pi_1)^4 &= (\pi_1 \overline{\pi_0})^3 = e, & 1 \leq i \leq n-1, \\
(\pi_i \pi_j)^2 &= e, & |i - j| \geq 2, 0 \leq i, j \leq n-1.
\end{aligned} \quad (A1) \]

An element $\sigma \in \Sigma_n$ expresses an irreducible form

$$ \sigma = \pi_{i_1} \cdots \pi_{i_k}, \quad 0 \leq i_1, \ldots, i_k \leq n-1, $$

and in this case,

$$ \ell_1(\sigma) := \text{the number of } \pi_0 \text{ in } \sigma, $$

$$ \ell_2(\sigma) := \text{the number of } \pi_i, \quad 1 \leq i \leq n-1, \text{ in } \sigma $$

are well defined. Let $H$ be a separable Hilbert space. For a given self-adjoint involution $f \mapsto \overline{f}$ for $f \in H$, an action of $\Sigma_n$ on $H^\otimes n$ is defined by

\[ \begin{aligned}
\pi_0(f_1 \otimes \cdots \otimes f_n) &= \overline{f_1} \otimes f_2 \otimes \cdots \otimes f_n, & n \geq 1, \\
\pi_i(f_1 \otimes \cdots \otimes f_n) &= f_1 \otimes \cdots \otimes f_{i-1} \otimes f_{i+1} \otimes f_i \otimes f_{i+2} \otimes \cdots \otimes f_n, & n \geq 2, 1 \leq i \leq n-1.
\end{aligned} \quad (A2) \]

The $(\alpha, q)$-inner product on the full Fock space $\mathcal{F}(H)$ is defined by

$$ \langle f \otimes \cdots \otimes f_m, g_1 \otimes \cdots \otimes g_n \rangle_{\alpha,q} := \delta_{m,n} \sum_{\sigma \in \Sigma_n} a_{m,n}^{\ell_1(\sigma)} q^{\ell_2(\sigma)} \prod_{j=1}^n \langle f_j, g_{\sigma(j)} \rangle_H, \quad \alpha, q \in (-1,1), $$

(A3)

with conventions $0^0 = 1$ and $g_{-k} = \overline{g_k}$, $k = 1, 2, \ldots, n$. For example, if one may define the involution as $\overline{f} := -f$, then $g_{-k} = -g_k$. Equipped with this inner product, the full Fock space $\mathcal{F}(H)$ is denoted by $\mathcal{F}_{\alpha,q}(H)$ to emphasize on the dependence of the inner product on $\alpha, q$.

The $(\alpha, q)$-creation operator $B^+_{\alpha,q}(f)$ is the usual left creation operator on the full Fock space, and the $(\alpha, q)$-annihilation operator $B^-_{\alpha,q}(f)$ is its adjoint with respect to the inner product $\langle \cdot, \cdot \rangle_{\alpha,q}$. They satisfy the commutation relation

$$ B^-_{\alpha,q}(f) B^+_{\alpha,q}(g) - q B^+_{\alpha,q}(g) B^-_{\alpha,q}(f) = \langle f, g \rangle_H + q^{2n} \langle f, g \rangle_H, \quad f, g \in H. $$

(A4)

The readers can consult Ref. 14 for details.