ADI-FDTD Modeling of Tellegen media

Ana Grande and José A. Pereda

Abstract— **The magnetoelectric coupling that appears in the constitutive relations of Tellegen media makes formulating unconditionally stable finite-difference time-domain (FDTD) extensions a challenging problem. In this letter, we present an alternating-direction implicit (ADI)-FDTD approach for modeling transient wave propagation in Tellegen media. In the proposed formulation, the discretization of the governing equations is performed introducing weighted average parameters. The numerical features are then carefully examined and the weighted discretization parameters conveniently chosen to yield an unconditionally stable scheme. Moreover, the resulting approach presents second order accuracy and preserves the tridiagonal structure of the data matrices. Finally, the numerical dispersion relation is given in a closed-form. The new method has been validated by means of numerical experiments and has shown good agreement.**

Index Terms— **Alternating-direction implicit finite-difference time-domain method (ADI-FDTD), Tellegen media, bi-isotropic media, unconditionally stable.**

I. Introduction

On the macroscopic level, bi-isotropic (BI) media are characterized by two magnetoelectric-coupling parameters in their constitutive relations. Two subclasses of BI media are chiral and Tellegen media. Chiral media are reciprocal, exhibit an inherent handedness, as well as the well-known phenomena of circular birefringence and dichroism. Tellegen media are nonreciprocal and present a magnetoelectric coupling in-phase with the exciting field [1].

A phenomenological model for Tellegen media consists of particles with permanent coupled electric and magnetic dipole moments. These media were introduced by Tellegen in 1948 [2], when he suggested a gyrator, the corresponding nonreciprocal circuit element. Since then, Tellegen media have drawn considerable attention. In particular, the physical realizability of such materials has been a matter of controversy since they seem to violate Post's constraint [3]–[5].

Several attempts have been made to model Tellegen media in the time domain [6], [7], and good results have also been obtained by using the conventional finite-difference time-domain (FDTD) method [8]–[10]. However, the stability of these approaches has seldom been addressed. Recently, two FDTD-based implicit schemes have been presented for the modeling of monochromatic wave propagation in BI media [11], [12]. These works are based on the locally one-dimensional (LOD)- and the alternating-direction implicit (ADI)-FDTD schemes, respectively. The main advantage of the LOD- and the ADI-FDTD techniques is that these schemes are unconditionally stable, the Courant condition is removed, and the time-step size is only limited by accuracy considerations. However, the stability analysis performed in [11] shows that the proposed formulation is unstable for the case of Tellegen media. Moreover, for Tellegen media, the stability condition of the approach presented in [12] is more restrictive than that of the conventional FDTD method.

This letter presents a successful implementation of an implicit unconditionally stable FDTD scheme for modeling transient wave propagation in Tellegen media. It is based on the ADI approach, and discretization is performed using Yee's mesh. In addition, in order to enable a more general study, the magnetoelectric-coupling terms that appear in the governing equations are discretized by using a weighted average. A stability study is then performed and convenient values of the weight parameters are selected. Thus, the resulting ADI-FDTD scheme for Tellegen media is unconditionally stable and preserves the tridiagonal structure of the data matrices. Moreover, the numerical dispersion relation is given in a closed-form. Finally, the proposed formulation is validated by means of numerical experiments.

II. Differential model

Consider Maxwell's curl equations in the Laplace domain

$$s\vec{D}(\vec{r},s) = \nabla\times\vec{H}(\vec{r},s) \quad (1a)$$
$$s\vec{B}(\vec{r},s) = -\nabla\times\vec{E}(\vec{r},s). \quad (1b)$$

The constitutive relations for Tellegen media are [1]

$$\vec{D}(\vec{r},s) = \epsilon\vec{E}(\vec{r},s)+\bar{\chi}\vec{H}(\vec{r},s) \quad (2a)$$
$$\vec{B}(\vec{r},s) = \mu\vec{H}(\vec{r},s)+\bar{\chi}\vec{E}(\vec{r},s) \quad (2b)$$

with $\bar{\chi}=\chi/c$, being χ the Tellegen parameter. The parameter $\bar{\chi}$ satisfies the condition [1]

$$\bar{\chi}^2 < \mu\epsilon. \quad (3)$$

Substituting (2) into (1) and rearranging terms we obtain

$$\xi s\vec{E}(\vec{r},s) = \mu\nabla\times\vec{H}(\vec{r},s)+\bar{\chi}\nabla\times\vec{E}(\vec{r},s) \quad (4a)$$
$$\xi s\vec{H}(\vec{r},s) = -\epsilon\nabla\times\vec{E}(\vec{r},s)-\bar{\chi}\nabla\times\vec{H}(\vec{r},s) \quad (4b)$$

where $\xi=\epsilon\mu-\bar{\chi}^2$.

For simplicity, we consider the one-dimensional (1D) problem consisting of plane waves propagating in the z-direction. The characteristic magnetoelectric coupling of Tellegen media

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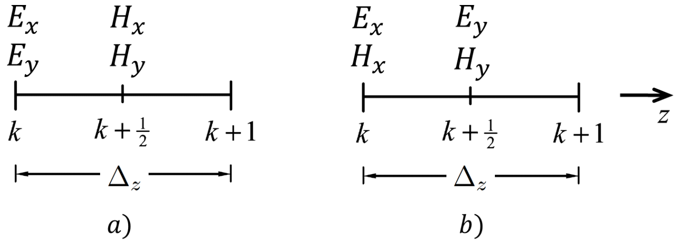

Fig. 1. 1D FDTD meshes for Tellegen media: a) Yee mesh where the electric and the magnetic field components are staggered in space. b) Bi-mesh where we distinguish the "x-nodes" and the "y-nodes."

requires the x and y polarizations to be considered simultaneously. Hence, for the 1D case, (4) reduces to

$$\xi s E_x = -\mu \partial_z H_y - \bar{\chi} \partial_z E_y \quad (5a)$$
$$\xi s E_y = \mu \partial_z H_x + \bar{\chi} \partial_z E_x \quad (5b)$$
$$\xi s H_x = \epsilon \partial_z E_y + \bar{\chi} \partial_z H_y \quad (5c)$$
$$\xi s H_y = -\epsilon \partial_z E_x - \bar{\chi} \partial_z H_x . \quad (5d)$$

III. New ADI-FDTD Scheme for Tellegen Media

In this work, we propose the following ADI-FDTD approximation of (5)

Sub-step 1

$$\frac{E_x^{n+\frac{1}{2}} - E_x^n}{\Delta_t} = -\frac{\mu}{2\xi} \frac{\boldsymbol{\delta}_z}{\Delta_z} H_y^n - \frac{\bar{\chi}}{\xi} \frac{\boldsymbol{\delta}_z}{\Delta_z} \boldsymbol{\mu}_z (\lambda_1 S_t^{-\frac{1}{2}} + \lambda_2^x) E_y^{n+\frac{1}{2}}$$
$$\frac{E_y^{n+\frac{1}{2}} - E_y^n}{\Delta_t} = \frac{\mu}{2\xi} \frac{\boldsymbol{\delta}_z}{\Delta_z} H_x^n + \frac{\bar{\chi}}{\xi} \frac{\boldsymbol{\delta}_z}{\Delta_z} \boldsymbol{\mu}_z (\lambda_1 S_t^{-\frac{1}{2}} + \lambda_2^y) E_x^{n+\frac{1}{2}}$$
$$\frac{H_x^{n+\frac{1}{2}} - H_x^n}{\Delta_t} = \frac{\epsilon}{2\xi} \frac{\boldsymbol{\delta}_z}{\Delta_z} E_y^n + \frac{\bar{\chi}}{\xi} \frac{\boldsymbol{\delta}_z}{\Delta_z} \boldsymbol{\mu}_z (\lambda_1 S_t^{-\frac{1}{2}} + \lambda_2^x) H_y^{n+\frac{1}{2}}$$
$$\frac{H_y^{n+\frac{1}{2}} - H_y^n}{\Delta_t} = -\frac{\epsilon}{2\xi} \frac{\boldsymbol{\delta}_z}{\Delta_z} E_x^n - \frac{\bar{\chi}}{\xi} \frac{\boldsymbol{\delta}_z}{\Delta_z} \boldsymbol{\mu}_z (\lambda_1 S_t^{-\frac{1}{2}} + \lambda_2^y) H_x^{n+\frac{1}{2}}$$

Sub-step 2

$$\frac{E_x^{n+1} - E_x^{n+\frac{1}{2}}}{\Delta_t} = -\frac{\mu}{2\xi} \frac{\boldsymbol{\delta}_z}{\Delta_z} H_y^{n+1} - \frac{\bar{\chi}}{\xi} \frac{\boldsymbol{\delta}_z}{\Delta_z} \boldsymbol{\mu}_z (\lambda_3^x S_t^{-\frac{1}{2}} + \lambda_4) E_y^{n+1}$$
$$\frac{E_y^{n+1} - E_y^{n+\frac{1}{2}}}{\Delta_t} = \frac{\mu}{2\xi} \frac{\boldsymbol{\delta}_z}{\Delta_z} H_x^{n+1} + \frac{\bar{\chi}}{\xi} \frac{\boldsymbol{\delta}_z}{\Delta_z} \boldsymbol{\mu}_z (\lambda_3^y S_t^{-\frac{1}{2}} + \lambda_4) E_x^{n+1}$$
$$\frac{H_x^{n+1} - H_x^{n+\frac{1}{2}}}{\Delta_t} = \frac{\epsilon}{2\xi} \frac{\boldsymbol{\delta}_z}{\Delta_z} E_y^{n+1} + \frac{\bar{\chi}}{\xi} \frac{\boldsymbol{\delta}_z}{\Delta_z} \boldsymbol{\mu}_z (\lambda_3^x S_t^{-\frac{1}{2}} + \lambda_4) H_y^{n+1}$$
$$\frac{H_y^{n+1} - H_y^{n+\frac{1}{2}}}{\Delta_t} = -\frac{\epsilon}{2\xi} \frac{\boldsymbol{\delta}_z}{\Delta_z} E_x^{n+1} - \frac{\bar{\chi}}{\xi} \frac{\boldsymbol{\delta}_z}{\Delta_z} \boldsymbol{\mu}_z (\lambda_3^y S_t^{-\frac{1}{2}} + \lambda_4) H_x^{n+1} .$$

In these equations, the temporal discretization has been explicitly expressed. The spatial discretization has been performed according to Yee's mesh shown in Fig. 1.a. The spatial central-difference operator $\boldsymbol{\delta}_z$ and spatial average operator $\boldsymbol{\mu}_z$ are defined as

$$\boldsymbol{\delta}_z = S_z^{\frac{1}{2}} - S_z^{-\frac{1}{2}} \quad (8)$$
$$\boldsymbol{\mu}_z = \frac{1}{2}(S_z^{\frac{1}{2}} + S_z^{-\frac{1}{2}}) \quad (9)$$

TABLE I

Weighted-average coefficients for the ADI-FDTD schemes

SCHEMES	λ_1	λ_2^x	λ_2^y	λ_3^x	λ_3^y	λ_4	Λ^2
Backward	1	0	0	0	0	0	1
Forward	0	0	0	0	0	1	Z^2
In-place	0	0.5	0.5	0.5	0.5	0	Z
In-place alternating x-y	0	1	0	0	1	0	Z
In-place alternating y-x	0	0	1	1	0	0	Z

being S_α $(\alpha = t, z)$ the shift operator given by

$$S_t^m F^n = F^{n+m} \quad (10)$$
$$S_z^m F(k) = F(k+m). \quad (11)$$

We have used weighted-average approximations, where λ_i are arbitrary constants that satisfy the constraints $\lambda_1 + \lambda_2^x + \lambda_3^x + \lambda_4 = 1$ and $\lambda_1 + \lambda_2^y + \lambda_3^y + \lambda_4 = 1$. Of all the possible choices for these parameters, we focus on some illustrative choices, which are quoted in Table I.

A. Analysis of the Stability

Adopting the von Neumann method, we obtain the following stability polynomial for the proposed scheme

$$S(Z) = (Z-1)^2 \xi^2 + (Z+1)^2 \epsilon\mu\nu_z^2 - 4A_z^2 \Lambda^2 \bar{\chi}^2 \nu_z^2 \quad (12)$$

where ν_z and A_z are defined as

$$\nu_z = \frac{\Delta_t}{\Delta_z} \sin\left(\frac{\tilde{k}_z \Delta_z}{2}\right) \quad (13)$$
$$A_z = \cos\left(\frac{\tilde{k}_z \Delta_z}{2}\right) \quad (14)$$

and Λ is given by

$$\Lambda = \lambda_1 + (\lambda_2^x + \lambda_3^x)\, Z^{\frac{1}{2}} + \lambda_4 Z. \quad (15)$$

being $\lambda_2^x + \lambda_3^x = \lambda_2^y + \lambda_3^y$. The values of Λ^2 for the schemes analyzed in this work are given in Table I.

Computing the roots of the stability polynomial (12) for the schemes detailed in Table I we find:

- *Backward scheme*: is unstable.
- *Forward scheme*: is unconditionally stable. However, this approach is ruled out because the coupling of the equations makes implementation impossible.
- *In-place* and *In-place alternating schemes*: these three methods have the same stability polynomial $S(Z)$. The roots Z_i of $S(Z)$ can be written as

$$Z_i = \frac{\xi^2 - \epsilon\mu\nu_z^2 + 2A_z^2\bar{\chi}^2\nu_z^2}{\xi^2 + \epsilon\mu\nu_z^2} \pm 2j \frac{\sqrt{(\epsilon\mu - A_z^2\bar{\chi}^2)\left(\xi^2 + A_z^2\bar{\chi}^2\nu_z^2\right)\nu_z^2}}{\xi^2 + \epsilon\mu\nu_z^2}. \quad (16)$$

These roots verify $|Z_i| \leqslant 1$ provided

$$\bar{\chi}^2 < \epsilon\mu. \quad (17)$$

This means that these schemes are unconditionally stable, since (17) is just the physical constraint of the parameters of Tellegen media given in (3). Therefore, and for simplicity, we hereafter focus on studying and implementing of the *in-place alternating y-x* scheme.

B. Numerical Dispersion Relation

Equating (12) to zero and letting $Z = \exp(j\omega\Delta_t)$ we obtain the dispersion relation for the three *in-place* schemes

$$\tan^2\left(\frac{\omega\Delta_t}{2}\right)\frac{\left(\epsilon\mu-\bar{\chi}^2\right)^2}{\epsilon\mu-\bar{\chi}^2\frac{\cos^2\left(\frac{\tilde{k}_z\Delta_z}{2}\right)}{\cos^2\left(\frac{\omega\Delta_t}{2}\right)}}=\frac{\Delta_t^2}{\Delta_z^2}\sin^2\left(\frac{\tilde{k}_z\Delta_z}{2}\right). \quad (18)$$

C. Local Truncation Error

For ADI schemes the local truncation error presents additional terms that depend on the spatial derivatives of the fields with respect to two or three different spatial coordinates [13]. Hence, for the present 1D formulation, those error terms vanish. However, in a further extension of this scheme to 2D and 3D, those error terms could play a critical role, especially in problems where fields present strong spatial variations, e.g. around singularities associated with corners or near-field sources. To overcome this limitation several modifications have been proposed [14].

D. Implementation of the ADI Scheme for Tellegen Media

For the *in-place alternating* y-x scheme, the ADI general equations given at the beginning of section IV reduce to

Sub-step 1

$$E_x^{n+\frac{1}{2}} = E_x^n - \frac{\mu}{2\xi}\frac{\Delta_t}{\Delta_z}\boldsymbol{\delta}_z H_y^n \quad (19a)$$

$$E_y^{n+\frac{1}{2}} = E_y^n + \frac{\mu}{2\xi}\frac{\Delta_t}{\Delta_z}\boldsymbol{\delta}_z H_x^n + \frac{\bar{\chi}}{\xi}\frac{\Delta_t}{\Delta_z}\boldsymbol{\delta}_z\boldsymbol{\mu}_z E_x^{n+\frac{1}{2}} \quad (19b)$$

$$H_x^{n+\frac{1}{2}} = H_x^n + \frac{\epsilon}{2\xi}\frac{\Delta_t}{\Delta_z}\boldsymbol{\delta}_z E_y^n \quad (19c)$$

$$H_y^{n+\frac{1}{2}} = H_y^n - \frac{\epsilon}{\xi}\frac{\Delta_t}{2\Delta_z}\boldsymbol{\delta}_z E_x^n - \frac{\bar{\chi}}{\xi}\frac{\Delta_t}{\Delta_z}\boldsymbol{\delta}_z\boldsymbol{\mu}_z H_x^{n+\frac{1}{2}} \quad (19d)$$

Sub-step 2

$$E_x^{n+1} = E_x^{n+\frac{1}{2}} - \frac{\mu}{\xi}\frac{\Delta_t}{2\Delta_z}\boldsymbol{\delta}_z H_y^{n+1} - \frac{\bar{\chi}}{\xi}\frac{\Delta_t}{\Delta_z}\boldsymbol{\delta}_z\boldsymbol{\mu}_z E_y^{n+\frac{1}{2}} \quad (20a)$$

$$E_y^{n+1} = E_y^{n+\frac{1}{2}} + \frac{\mu}{\xi}\frac{\Delta_t}{2\Delta_z}\boldsymbol{\delta}_z H_x^{n+1} \quad (20b)$$

$$H_x^{n+1} = H_x^{n+\frac{1}{2}} + \frac{\epsilon}{2\xi}\frac{\Delta_t}{\Delta_z}\boldsymbol{\delta}_z E_y^{n+1} + \frac{\bar{\chi}}{\xi}\frac{\Delta_t}{\Delta_z}\boldsymbol{\delta}_z\boldsymbol{\mu}_z H_y^{n+\frac{1}{2}} \quad (20c)$$

$$H_y^{n+1} = H_y^{n+\frac{1}{2}} - \frac{\epsilon}{2\xi}\frac{\Delta_t}{\Delta_z}\boldsymbol{\delta}_z E_x^{n+1}. \quad (20d)$$

Since the discretization of (5) is performed according to Yee's mesh shown in Fig. 1.a, equations (19a), (19b), (20a) and (20b) are evaluated at the spatial position $z = k\Delta_z$, while (19c), (19d), (20c) and (20d) are computed at $z = (k+\frac{1}{2})\Delta_z$.

The implementation of (19) and (20) comprises the following calculations:

Sub-step 1

1) $\vec{E}^{n+\frac{1}{2}}$ and $\vec{H}^{n+\frac{1}{2}}$ are explicitly updated by using (19).

Sub-step 2

1) Equation (20d) is substituted into (20a), and E_x^{n+1} is implicitly computed (tridiagonal matrix).
2) Equation (20b) is substituted into (20c), and H_x^{n+1} is implicitly computed (tridiagonal matrix).
3) E_y^{n+1} is explicitly updated by using (20b).
4) H_y^{n+1} is explicitly computed by means of (20d).

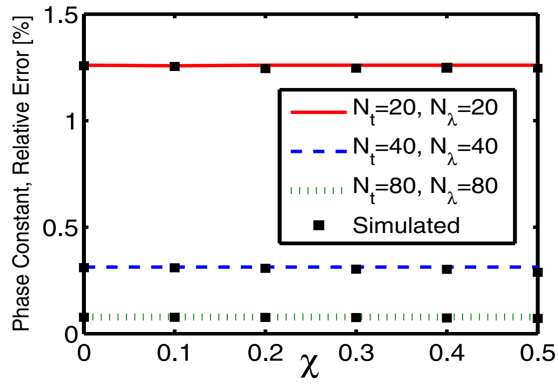

Fig. 2. Relative error for the phase constant for several values of N_λ and N_t. Second-order accuracy of the scheme: when both resolutions are multiplied by 2, the relative error is divided by 4.

E. Bi-Mesh ADI-FDTD Scheme for Tellegen Media

The so-called "bi-mesh" shown in Fig. 1.b was introduced to facilitate the modeling of bi-isotropic media [6], [9], whose peculiar constitutive relations relate the electric and the magnetic fields in the same point and at the same instant. Thus, in the bi-mesh all the x-components are placed within the same point, which we call the "x-node." Similarly, the y-components are positioned in the "y-node." The proposed ADI-FDTD scheme can easily be adapted to the bi-mesh, by substituting the spatial average operator $\boldsymbol{\mu}_z$ that appears in (19) and (20) by a factor of 1. Hence, equations (19a), (19c), (20a) and (20c) would be evaluated at the spatial position $z = k\Delta_z$, while (19b), (19d), (20b) and (20d) would be computed at $z = (k+\frac{1}{2})\Delta_z$. The stability polynomial and the numerical dispersion relation can also be obtained replacing the factor A_z which appears in (12) and (18) by a factor of 1. The resulting scheme is unconditionally stable provided that (17). The implementation procedure follows the same steps as described above.

IV. Numerical Results

To validate the proposed ADI-FDTD scheme, we have computed the phase constants of electromagnetic pulses travelling in Tellegen media with ϵ_r= 4, μ_r= 1.2. The spatial resolution was defined as $N_\lambda = \lambda_0/\Delta_z$, with λ_0 being the wavelength in the Tellegen medium at the center frequency of the pulse f_0= 9 GHz. Analogously, the temporal resolution was given by $N_t = T_0/\Delta_t$, with $T_0 = 1/f_0$. Fig. 2 shows the relative error for the phase constant as a function of the Tellegen parameter for several values of N_λ and N_t. When both resolutions are multiplied by 2, the relative error is divided by 4, which highlights the second-order accuracy of the method.

In a second simulation, and in order to show the characteristic behavior of waves propagating in Tellegen media, we have considered a band-limited pulse impinging on the interface between air and a Tellegen medium with $\chi = 0.6$, $\epsilon_r = 3.5$,

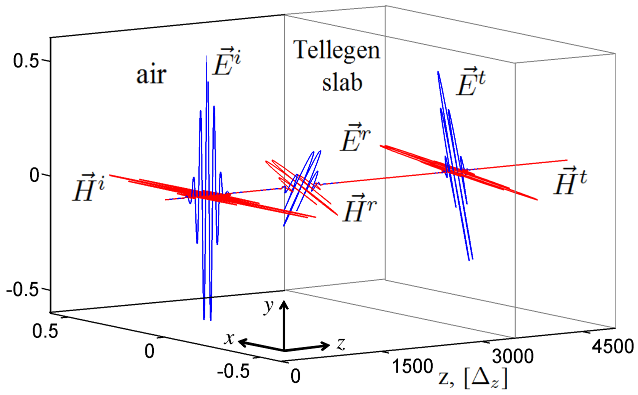

Fig. 3. Incidence of a y-polarized pulse travelling in the $(+z)$-direction at the interface between air and a Tellegen medium. $\vec{E}^r$ is the reflected field which is rotated an angle ϕ_R. $\vec{E}^t$ is the transmitted field which is rotated ϕ_T. $\vec{E}^i$ is the incident pulse travelling in the $(-z)$-direction. The $\vec{E}^t$ and $\vec{H}^t$ fields are not orthogonal.

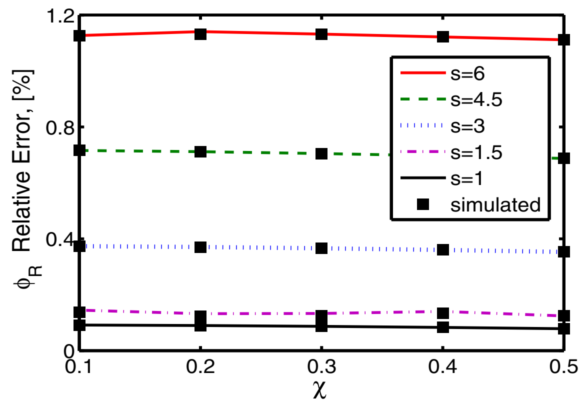

Fig. 4. Relative error of the angle ϕ_R as a function of χ for several values of the stability factor s, with $N_z = 80$.

$\mu_r = 1.2$. The excitation pulse was linearly polarized in the y-direction with a center frequency of $f_0 = 9$ GHz. It was applied at point $z = 2800\Delta_z$. The discretization parameters were $N_\lambda = 50$ and $s = 4$, being $s = c_0\Delta_t/\Delta_z$. The resulting temporal resolution was $N_t = 24.5$. Fig. 3 shows a snapshot of the simulation at $t = 950\Delta_t$. The incident pulse, travelling in the $(+z)$-direction, reflects back into the air region. Due to the non-null crosspolarized reflection coefficient, the polarization of the reflected field ($\vec{E}^r$) is rotated an angle of $\phi_R = 27.553^o$. The polarization of the transmitted field ($\vec{E}^t$) is rotated an angle of $\phi_T = -10.752^o$. The incident pulse, travelling in the $(-z)$-direction ($\vec{E}^i$) is also shown in Fig. 3. The nonorthogonality of the electric- and magnetic-field vectors ($\vec{E}^t$, $\vec{H}^t$) in the Tellegen medium can be seen in Fig. 3.

Finally, Fig. 4, depicts the relative error of the angle ϕ_R as a function of χ for several values of the stability factor s. The spatial resolution considered was $N_z = 80$. The rotation of the reflected field was computed considering [1]

$$\phi_R = a\tan\left(\frac{\tilde{R}_{cr}}{\tilde{R}_{co}}\right) \tag{21}$$

with $\tilde{R}_{co}$ and $\tilde{R}_{cr}$ being the numerically-computed moduli of the co- and crosspolarized reflection coefficients, respectively.

V. Conclusions

Due to the magnetoelectric coupling, developing of implicit unconditionally stable FDTD extensions for Tellegen media is a challenging problem. This letter presents a successful ADI-FDTD technique for the modeling of transient wave propagation in Tellegen media. The proposed formulation is unconditionally stable, second order accurate, and preserves the tridiagonal structure of the data matrices. The numerical dispersion relation has been given in closed-form. The new scheme has been validated by means of numerical results. The proposed formulation may provide the basis for the transient unconditionally stable FDTD modeling of general biisotropic media [1], and also of the recently introduced perfect electromagnetic conductors (PEMC) [15], which are a special limit class of Tellegen media.

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