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A RELATION THEORETIC PROOF OF A TRIPLEABILITY THEOREM OVER EXACT CATEGORIES

By

Yasuo KAWAHARA

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§0. Introduction

In [15] and [16], W. Lawvere and F. E. J. Linton characterized categorically the classes of equationally defined algebras introduced by G. D. Birkhoff [4]. As shown in [17, 18], Linton's theorem is a variant of Beck's theorem 3.1 in the case of T-algebras over the category of sets. J. Duskin [7] obtains further variants of Beck's theorem by using equivalence pairs in categories and their effectivities. These concepts are due to A. Grothendiek and initially used in this context by Lawvere [15].

The tripleability theorems are recently applied to the categorical proofs of Stone duality [19], Gelfand and Pontrjagin dualities [23], and other types of dualities of topological algebras [6], [26].

In this note, we give another proof of a tripleability theorem, essentially due to Duskin [7] and Linton [16]. For the proof of the main theorem 3.2, we will not use hom-functors and Yoneda Lemma, but we will use only the elementary theory of categories and the calculation of relations in categories. In § 1, we define exact categories according to M. Barr [2, 3] and review some properties of relations in exact categories. In particular, we show that split forks induce equivalence relations (Theorem 1.4). In § 2, we state a lemma, which is a crux to the proof of the main theorem 3.2, and a relation theoretic version of it. In § 3, we derive from Beck's tripleability theorem 3.1 a variant (Theorem 3.2), essentially due to Duskin and Linton, which seems to be convenient for applications to analysis.

For notations and terms in the elementary theory of categories, we refer to S. Mac Lane's book [21]. The theory of relations in categories is referred to P. A. Grillet [9], Y. Kawahara [11, 12] and A. Klein [14].

§1. Preliminaries

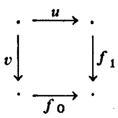
Throughout this note, we denote by fg the composite $\cdot \xrightarrow{f} \cdot \xrightarrow{g} \cdot$ of two arrows in a category. A span is a pair of arrows with common domain. We write $h<f, g>$ for a span $<hf, hg>$. We also say that a span $<f, g>$ is *monic* if $h<f, g>=k<f, g>$

always implies $h=k$. Let C be a category. A parallel pair $<f_0, f_1>: a \to b$ in C is called *an equivalence pair* in C if it is a monic span and

(Q.1) $s<f_0, f_1> = <1_b, 1_b>$ for an arrow s;

(Q.2) $t<f_0, f_1> = <f_1, f_0>$ for an arrow t;

(Q.3) If

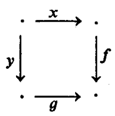


is a pullback in C, then $r<f_0, f_1> = <uf_0, vf_1>$ for an arrow r.

For example, every kernel pair of an arrow is an equivalence pair. An equivalence pair $<f_0, f_1>$ in C is called *effective* if there is an arrow e in C such that e is a coequalizer of $<f_0, f_1>$ and $<f_0, f_1>$ is a kernel pair of e. We say that an arrow is *a regular epi* if it is a coequalizer of some pair of arrows. Following to M. Barr [2, 3], we say that a category C is *exact* if:

(EX.1) C has all finite limits;

(EX.2) Whenever



is a pullback in C and f is a regular epi, so is y;

(EX.3) Every equivalence pair in C is effective.

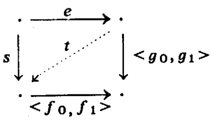
In the rest of the present section, we assume that C is an exact category. Let $Reg(C)$ be the family of all regular epis in C. Then, applying the results of Barr [2; I §2], it follows that $Reg(C)$ is a retractive subcategory of C in the sense of [11]. Hence one can construct the I-category $Rel(C) = Rel(C; Reg(C))$ (in the sense of D. Puppe [24]) of relations in C by the method of Grillet [9], Calenko [5], Klein [14], or Kawahara [11, 12]. We denote the involution and the ordering in the I-category $Rel(C)$ by $*$ and $\subset$, respectively.

Since each arrow in an exact category C has a canonical factorization [2; I 2.3] as a regular epi followed by a monic and C has products, every span in C has a canonical factorization as a regular epi followed by a monic span. Thus every relation in $Rel(C)$ is represented by a monic span.

1.1. LEMMA. *Let $<f_0, f_1>$ be a monic span in C. Then there exists an arrow t in C such that $<g_0, g_1> = t<f_0, f_1>$ if and only if $g_0^* g_1 \subset f_0^* f_1$ in $Rel(C)$.*

PROOF. Assume that $g_0^* g_1 \subset f_0^* f_1$ in $Rel(C)$. Then, by the definition of relations

[11], there is an arrow s and a regular epi e in C such that $e<g_0, g_1>=s<f_0, f_1>$. If $<f_0, f_1>$ is monic, there is a unique dotted arrow t in the commutative diagram



by the property [2; I 2.6] of regular epis. Hence we have $<g_0, g_1>=t<f_0, f_1>$. The converse follows at once from $g_0^*g_1=f_0^*t^*tf_1 \subset f_0^*f_1$.

1.2. Proposition. *Let e be a regular epi in an exact category C. Then e is a coequalizer of $<d_0, d_1>$ if and only if $d_0^*d_1 \subset ee^*$ and $d_0f=d_1f$ implies $ee^* \subset ff^*$.*

Proof. First assume that e is a coequalizer of $<d_0, d_1>$. It is trivial that $d_0e=d_1e$ and so $d_0^*d_1 \subset ee^*$. For any arrow f with $d_0f=d_1f$, there is a unique arrow g with $f=eg$. Hence we have $ff^*=egg^*e^* \supset ee^*$. Conversely, assume that $d_0^*d_1 \subset ee^*$ and $d_0f=d_1f$ always implies $ee^* \subset ff^*$. It is obvious that $d_0e=d_1e$. Let e be a coequalizer of $<h_0, h_1>$ and assume $d_0f=d_1f$. Then $h_0f=h_1f$ by $h_0^*h_1 \subset ee^* \subset ff^*$ and hence there is a unique arrow g with $f=eg$. This proves that e is a coequalizer of $<d_0, d_1>$.

A relation θ is *an equivalence relation* if it is reflexive ($1 \subset \theta$), symmetric ($\theta^* \subset \theta$) and transitive ($\theta^2 \subset \theta$).

1.3. Proposition. *For any equivalence relation θ in an exact category C, there is a regular epi e such that $\theta=ee^*$.*

Proof. Consider a monic span $<f_0, f_1>$ such that $\theta=f_0^*f_1$. Then it follows from Lemma 1.1 that $<f_0, f_1>$ is an equivalence pair in C. In view of the exactness condition (EX.3), there is a regular epi e such that $<f_0, f_1>$ is a kernel pair of e. Hence we have $\theta=f_0^*f_1=ee^*$.

The proof of the last proposition shows that a parallel pair $<f_0, f_1>$ is an equivalence pair in an exact category C if and only if it is a monic span and $f_0^*f_1$ is an equivalence relation in $Rel(C)$.

1.4. Theorem. *If $x \overset{d_0}{\underset{d_1}{\rightrightarrows}} y \xrightarrow{e} z$ is a split fork in C, then $d_0^*d_1d_1^*d_0=ee^*$ in $Rel(C)$ and hence $d_0^*d_1d_1^*d_0$ is an equivalence relation.*

Proof. By the definition [21; VI § 6] of split forks, there exist two arrows s and t such that $es=td_1$, $td_0=1_y$ and $se=1_z$. From $d_0e=d_1e$, we have $d_0^*d_1 \subset ee^*$ and $d_0^*d_1d_1^*d_0 \subset ee^*ee^*=ee^*$. On the other hand, we have $es=td_1=(td_0)^*td_1=d_0^*t^*td_1 \subset d_0^*d_1$, because $td_0=1_y$ and $t^*t\subset 1_x$, and so $ee^*\subset ess^*e^*\subset d_0^*d_1d_1^*d_0$. Therefore we have $d_0^*d_1d_1^*d_0=ee^*$ and ee^* is clearly an equivalence relation.

REMARK. In an exact category, using the result of M. Barr, one can calculate as in the category of sets, "as if" there were elements. If we do this, it is at once clear that $d_0^*d_1d_0^*d_0$ is the kernel pair of e.

The following proposition is essentially due to Linton [18].

1.5. PROPOSITION. *If C is an exact category in which every regular epi splits, then any equivalence pair in C has a split coequalizer.*

PROOF. Suppose $<f_0, f_1>$ is an equivalence pair in C. Then we can take a regular epi e with $f_0^*f_1=ee^*$ by Proposition 1.3 and its section s with $se=1$ by the hypothesis. Since $es\subset esee^*=ee^*=f_0^*f_1$ and $<f_0, f_1>$ is monic, there is an arrow t with $<1, es>=t<f_0, f_1>$ by Lemma 1.1, which completes the proof.

§2. The main lemma

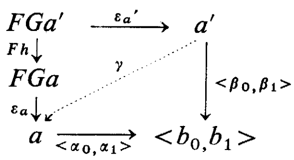
In this section, we will prove the main Lemma 2.1, which is a crux of our proof of the main Theorem 3.2, and state a relation theoretic version of it for the later use.

2.1. LEMMA. *Let $<F, G, \eta, \varepsilon>: X\rightarrow A$ be an adjunction* [21; *Theorem* IV 1.2] *and let the functor G: $A\rightarrow X$ reflect regular epis. If $<\alpha_0, \alpha_1>$ is a monic span in A, then the following conditions are equivalent*:

(i) $<\beta_0, \beta_1>=\gamma<\alpha_0, \alpha_1>$ *for an arrow γ in A.*

(ii) $<G\beta_0, G\beta_1>=h<G\alpha_0, G\alpha_1>$ *for an arrow h in X.*

PROOF. (Compare with H. Herrlich [10; Proposition 2.7] and J. Duskin [7; Lemma 3.4].) Since G reflects regular epis, every component ε_a: $GFa\rightarrow a$ of the counit ε is a regular epi in A by $\eta G\cdot G\varepsilon=1_G$, one of the triangular identities. Let $b_0 \xleftarrow{\alpha_0} a \xrightarrow{\alpha_1} b_1$ be a monic span in A. It is trivial that (i) implies (ii). We now assume (ii) for a span $b_0 \xleftarrow{\beta_0} a' \xrightarrow{\beta_1} b_1$. Then we have $\varepsilon_{a'}\cdot\beta_i=FG\beta_i\cdot\varepsilon_{b_i}$ (Naturality of ε)$=Fh\cdot FG\alpha_i\cdot\varepsilon_{b_i}$ (Assumption)$=Fh\cdot\varepsilon_a\cdot\alpha_i$ (Naturality of ε), where $i=0$, 1. Since $\varepsilon_{a'}$ is a regular epi and $<\alpha_0, \alpha_1>$ is monic, there is a unique arrow γ, which makes the diagram



commute. This completes the proof.

We now note that, under the same assumptions of the above lemma, $G: A \to X$ preserves [21; Theorem V 5.1] and reflects [21; Theorem IV 3.1] monic spans.

In view of Lemma 1.1, the last lemma indicates that, under the suitable conditions, $\beta_0^*\beta_1 \subset \alpha_0^*\alpha_1$ in $Rel(A)$ if and only if $(G\beta_0)^*(G\beta_1) \subset (G\alpha_0)^*(G\alpha_1)$ in $Rel(X)$. To precisely state this fact, we have to recall the notion of I-functors, introduced by D. Puppe [24], and its basic property, initially due to A. Klein [14].

A functor between I-categories is called *an I-functor* if it preserves the orderings and the involutions of I-categories.

The proof of the following theorem is referred to [14] or [11, 12].

2.2. THEOREM. *Let X and A be two exact categories. If a functor $G: A \to X$ preserves pullbacks and regular epis, then $G: A \to X$ can be uniquely extended to an I-functor $\bar{G}: Rel(A) \to Rel(X)$ so that the square*

$$\begin{array}{ccc} A & \xrightarrow{\subset} & Rel(A) \\ {\scriptstyle G}\downarrow & & \downarrow{\scriptstyle \bar{G}} \\ X & \xrightarrow[\subset]{} & Rel(X) \end{array}$$

commutes.

Combining the above theorem with Lemma 2.1, we have the following:

2.3. COROLLARY. *Let X and A be two exact categories and $<F, G, \eta, \varepsilon>: X \rightharpoonup A$ an adjunction. If $G: A \to X$ preserves and reflects regular epis, then the following statements are valid*:

(a) *For two relations θ and θ' in $Rel(A)$, $\theta' \subset \theta$ in $Rel(A)$ if and only if $\bar{G}\theta' \subset \bar{G}\theta$ in $Rel(X)$.*

(b) *A relation θ is an equivalence relation in $Rel(A)$ if and only if $\bar{G}\theta$ is an equivalence relation in $Rel(X)$.*

PROOF. We can choose a monic span $<\alpha_0, \alpha_1>$ with $\theta = \alpha_0^*\alpha_1$ and a span $<\beta_0, \beta_1>$ with $\theta' = \beta_0^*\beta_1$. Then, since $\bar{G}\theta = (G\alpha_0)^*(G\alpha_1)$ and $\bar{G}\theta' = (G\beta_0)^*(G\beta_1)$, the statement (a) follows from Lemma 1.1 and Lemma 2.1. The statement (b) is a corollary of (a).

§3. Tripleability theorems

For an adjunction $<F, G, \eta, \varepsilon>: X \rightharpoonup A$ in the sense of S. Mac Lane [21], one can define the monad $T = <GF, \eta, G\varepsilon F>$ in the category X and the adjunction $<F^T, G^T, \eta^T, \varepsilon^T>: X \rightharpoonup X^T$, where X^T is the category of T-algebras due to S. Eilenberg and J. C. Moore [8]. Moreover, there is a unique (dotted) functor K, called the comparison functor, in the following diagram so that both the F-square and the G-square commute

$$\begin{array}{ccc} A & \xrightarrow{\;K\;} & X^T \\ {\scriptstyle F}\uparrow\downarrow{\scriptstyle G} & & {\scriptstyle F^T}\uparrow\downarrow{\scriptstyle G^T} \\ X & = & X \end{array} .$$

J. Beck has proved the following theorem characterizing the category of T-algebras up to equivalence. (For the proof, see [21; Exercise VI 7.6].)

3.1. Theorem (J. Beck). *Let* $<F, G, \eta, \varepsilon>: X \rightharpoonup A$ *be an adjunction and* S_G *the set of all those parallel pairs* $<\partial_0, \partial_1>: a \to b$ *in* A *such that* $<G\partial_0, G\partial_1>: Ga \to Gb$ *has a split coequalizer in* X. *Then the comparison functor* $K: A \to X^T$ *is an equivalence of categories if and only if*:

(B.1) *Every parallel pair* $<\partial_0, \partial_1>$ *in* S_G *has a coequalizer*;

(B.2) $G: A \to X$ *preserves and reflects coequalizers for pairs in* S_G.

We are now ready to prove a tripleability theorem which is a variant of the theorems of F. Linton [16, 18] and J. Duskin [7] and will be convenient for applications.

3.2. Theorem. *Let* X *be an exact category and* $<F, G, \eta, \varepsilon>: X \rightharpoonup A$ *an adjunction. Then the comparison functor* $K: A \to X^T$ *is an equivalence of categories if the following conditions are satisfied*:

(EX.1) A *has all finite limits*;

(EX.2*) $G: A \to X$ *preserves and reflects regular epis*;

(EX.3) *Every equivalence pair in* A *is effective.*

Proof. To prove the theorem, it suffices to verify the Beck's tripleability criteria (B.1) and (B.2) in Theorem 3.1. Since G has a left adjoint F, G preserves pullbacks [21; Theorem V 5.1], which shows that the condition (EX.2*) leads to (EX.2). Thus A is an exact category and we can apply Corollary 2.3 to the proof of the present theorem.

(B.1) (Existence of coequalizers) Consider a parallel pair $<\partial_0, \partial_1>: a \to b$ in S_G for which there are commutative squares

$$\begin{array}{ccccc} Gb & \xrightarrow{\;t\;} & Ga & \xrightarrow{\;G\partial_0\;} & Gb \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle G\partial_1} & & \downarrow{\scriptstyle e} \\ z & \xrightarrow[\;s\;]{} & Gb & \xrightarrow[\;e\;]{} & z \end{array}$$

in X such that $se = 1_z$ and $t \cdot G\partial_0 = 1_{Gb}$. Then $\bar{G}(\partial_0^* \partial_1 \partial_1^* \partial_0) = ee^*$ is an equivalence relation in $Rel(X)$ by Theorem 1.4 and so is $\partial_0^* \partial_1 \partial_1^* \partial_0$ by Corollary 2.3 (b). Hence, Proposition 1.3 guarantees the existence of a regular epi γ in A with $\partial_0^* \partial_1 \partial_1^* \partial_0 = \gamma\gamma^*$. We now wish to show that γ is a coequalizer of $<\partial_0, \partial_1>$. Since $\bar{G}(\partial_0^* \partial_1) \subset ee^* = \bar{G}(\gamma\gamma^*)$, it follows from Corollary 2.3 (a) that $\partial_0^* \partial_1 \subset \gamma\gamma^*$. Next assume that $\partial_0 \xi = \partial_1 \xi$. Then we have $\partial_0^* \partial_1 \subset \xi\xi^*$ and so $\gamma\gamma^* = \partial_0^* \partial_1 \partial_1^* \partial_0 \subset \xi\xi^* \xi\xi^* = \xi\xi^*$. Hence, by using Proposition 1.2, γ is a coequalizer of $<\partial_0, \partial_1>$, as desired.

(B.2a) (Preservation of coequalizers) Let $<\partial_0, \partial_1>$ be the same as in (B.1) and λ a coequalizer of $<\partial_0, \partial_1>$. The above argument shows that $\partial_0^*\partial_1\partial_1^*\partial_0=\lambda\lambda^*$ and $(G\lambda)(G\lambda)^*=ee^*$. But $G\lambda$ is a regular epi in X from the hypothesis (EX.2*) and e is a coequalizer of $<G\partial_0, G\partial_1>$ (by [21; Lemma, page 145]); so, again by Proposition 1.2, $G\lambda$ is a coequalizer of $<G\partial_0, G\partial_1>$.

(B.2b) (Reflection of coequalizers) Let $<\partial_0, \partial_1>$ be the same as in (B.1) and $G\lambda$ a coequalizer of $<G\partial_0, G\partial_1>$. Then, by (EX.2*), λ is a regular epi in A. We will prove that λ is a coequalizer of $<\partial_0, \partial_1>$. Since $(G\partial_0)(G\lambda)=(G\partial_1)(G\lambda)$, we have $\partial_0^*\partial_1\subset\lambda\lambda^*$ by Corollary 2.3(a). Next we assume that $\partial_0\xi=\partial_1\xi$. Then we have $\bar{G}(\lambda\lambda^*)\subset\bar{G}(\xi\xi^*)$ by Proposition 1.2, because $(G\partial_0)(G\xi)=(G\partial_1)(G\xi)$, and hence $\lambda\lambda^*\subset\xi\xi^*$ by Corollary 2.3(a). Therefore it follows from Proposition 1.2 that λ is a coequalizer of $<\partial_0, \partial_1>$. This completes the proof of the theorem.

The converse of the last theorem holds under a stronger assumption.

3.3. THEOREM. *If every regular epi in X splits, the converse of Theorem 3.2 holds.*

PROOF. An equivalence of categories preserves and reflects all limits and colimits and consequently it preserves the conditions (EX.1), (EX.2*) and (EX.3). Hence, to prove this theorem, it suffices to show that any adjunction $<F, G, \eta, \varepsilon>: X\rightharpoonup A$ which is isomorphic to $<F^T, G^T, \eta^T, \varepsilon^T>: X\rightharpoonup X^T$ (where $T=<GF, \eta, G\varepsilon F>$ and X^T is the category of T-algebras [21; Chapter VI]) satisfies the conditions (EX.1), (EX.2*) and (EX.3). We recall the following properties [21; Exercise VI 2.2 and Theorem VI 7.1] of such adjunction $<F, G, \eta, \varepsilon>$:

(a) G creates all (small) limits;

(b) G creates coequalizers for parallel pairs in S_G.

Now we will prove the conditions (EX.1), (EX.2*) and (EX.3), separately.

(EX.1) (Finitely completeness) It is obvious from (a).

(EX.2*a) (G reflects regular epis.) Suppose f is an arrow in A with Gf a regular epi and $<\partial_0, \partial_1>$ is a kernel pair of f (which does exist by (EX.1)). Since $<G\partial_0, G\partial_1>$ is a kernel pair of a regular epi Gf, Gf is a coequalizer of $<G\partial_0, G\partial_1>$ and $<\partial_0, \partial_1>\in S_G$ by Proposition 1.5. Hence, by the property (b), f is a coequalizer of $<\partial_0, \partial_1>$.

(EX.3) (Exactness) Let $<\partial_0, \partial_1>$ be an equivalence pair in A. Since G preserves pullbacks, $<G\partial_0, G\partial_1>$ is an equivalence pair in X and, by the exactness condition (EX.3) of X, $<G\partial_0, G\partial_1>$ has a coequalizer e such that $<G\partial_0, G\partial_1>$ is a kernel pair of e. But every epi in X splits, it follows from Proposition 1.5 that $<\partial_0, \partial_1>\in S_G$. By the property (b), $<\partial_0, \partial_1>$ has a unique coequalizer γ with $G\gamma=e$. Since $<G\partial_0, G\partial_1>$ is a kernel pair of $G\gamma$, the property (a) shows that $<\partial_0, \partial_1>$ is a kernel pair of γ. This completes the proof of exactness.

(EX.2*b) (G preserves regular epis.) Let f be a regular epi in A and $<\partial_0, \partial_1>$ a kernel pair of f. As in the proof of (EX.3), there is a coequalizer γ of $<\partial_0, \partial_1>$ with

$G\gamma$ ($=e$) a regular epi in X. But f is also a coequalizer of $<\partial_0, \partial_1>$, so f is isomorphic to γ. This proves that Gf is a regular epi, as desired.

3.4. COROLLARY (Duskin). *A finitely complete category A is tripleable over Set if and only if the following conditions are satisfied*:

(D.1) *There is an object u in A such that the copower $J\cdot u$ of u exists for all (small) sets J;*

(D.2) *An arrow $f: a\to b$ in A is a regular epi if and only if $f_*: A(u, a)\to A(u, b)$ is a surjection;*

(D.3) *Every equivalence pair in A is effective.*

PROOF. By the definition of copowers [21; III § 3], there is a bijection

$$A(J\cdot u, x)\cong Set(J, A(u, x))$$

for each pair of objects $J\in Set$ and $x\in A$. This bijection is natural in J and x. Hence, if the copower $J\cdot u$ of u exists for each set J, the hom-functor $A(u, -): A\to Set$ has a left adjoint $(-)\cdot u: Set\to A$. Conversely, suppose $<F, G, \eta, \varepsilon>: Set\rightharpoonup A$ is an adjunction. Then, putting $u=F$(*one point set*), it is well-known that there are natural isomorphisms $G(x)\cong A(u, x)$ and $F(J)\cong J\cdot u$. Therefore the result follows from Theorem 3.2 and 3.3.

Finally, we will derive Linton's Theorem [16, 18] as an application of the above arguments.

3.5. THEOREM (Linton). *Let X be a (small) complete and well-powered exact category in which every regular epi splits, and let $<F, G, \eta, \varepsilon>: X\rightharpoonup A$ be an adjunction. Then the comparison functor $K: A\to X^T$ is an equivalence of categories if and only if the following conditions are satisfied*:

(L.1) *A has all (small) limits and coequalizers;*

(L.2) *$G: A\to X$ preserves and reflects regular epis;*

(L.3) *$G: A\to X$ reflects kernel pairs.*

PROOF. First assume that the conditions (L.1), (L.2) and (L.3) are satisfied. Then it is trivial that (L.1) and (L.2) imply (EX.1) and (EX.2*) in Theorem 3.2. To prove (EX.3) for A, we assume $<f_0, f_1>$ is an equivalence pair in A. Since G preserves all limits, $<Gf_0, Gf_1>$ is also an equivalence pair in X. By the exactness (EX.3) for X, $<Gf_0, Gf_1>$ is a kernel pair in X and, by the assumption (L.3), so is $<f_0, f_1>$. On the other hand, the pair $<f_0, f_1>$ has a coequalizer by (L.1). Hence it is effective, which shows (EX.3) for A. By Theorem 3.2, the comparison functor K is an equivalence of categories.

Conversely, suppose that the comparison functor $K: A\to X^T$ is an equivalence of categories. Then, by Theorem 3.3, the conditions (EX.1), (EX.2*) and (EX.3) for A are satisfied. We will show the conditions (L.1), (L.2) and (L.3), separately.

(L.1a) (Completeness) It follows from the property (a) in the proof of Theorem 3.3.

(L.1b) (Existence of coequalizers) Let $<\partial_0, \partial_1>$ be a parallel pair in A. Then, since X is well-powered, it turns out from Corollary 2.3(a) that the set $\{\xi\xi^*; \partial_0\xi=\partial_1\xi\}$ of all equivalence relations $\xi\xi^*$ with $\partial_0\xi=\partial_1\xi$ forms a non-empty small set and there is the greatest lower bound θ of the set $\{\xi\xi^*; \partial_0\xi=\partial_1\xi\}$ (because of the completeness of A). But the greatest lower bound of a set of equivalence relations is also an equivalence relation; so there is a regular epi γ in A with $\gamma\gamma^*=\theta$ (by (EX.3) and Proposition 1.3) and one can verify, using Proposition 1.2, that γ is a coequalizer of $<\partial_0, \partial_1>$. Hence every parallel pair of arrows in A has a coequalizer.

(L.2) This is identical with (EX.2*).

(L.3) Assume $<f_0, f_1>$ is a parallel pair in A such that $<Gf_0, Gf_1>$ is a kernel pair in X. Since kernel pairs are monic spans and G is faithful [21; Theorem IV 3.1], $<f_0, f_1>$ is a monic span. The relation $\bar{G}(f_0^* f_1)$ is an equivalence relation in $Rel(X)$ and hence, applying Corollary 3.3(b), $f_0^* f_1$ is an equivalence relation in $Rel(A)$. Therefore $<f_0, f_1>$ is an equivalence pair in A (by Lemma 1.1) and, in view of the exactness (EX.3) for A, it is a kernel pair. Hence we have completed the proof of the theorem.

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Department of Applied Mathematics
Kyushu Institute of Technology