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ON A GENERALIZATION OF THE INTERPOLATION METHOD

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In the last ten years considerable progress has been made in the theory of intermediate spaces. The work of J. L. Lions [3] seems to have given impetus to such circumstances and thenceforth several authors have introduced and developed various methods in this theory (cf. Bibliography in [8], [5] or [1]). Especially, for the present, the so-called real interpolation methods due mainly to J. L. Lions and J. Peetre are of most wide application [1], [5], [8], [9]. Immediately after Lions and Peetre presented their original real method [4], much improvements and generalizations thereof have been given by Peetre [6], [7], [8], one of which will be outlined as follows [7]. Let X_0 and X_1 be Banach spaces continuously contained in a linear topological Hausdorff space E and define $K(t, x)=\inf_{x=x_0+x_1}(\|x_0\|_{X_0}+t\|x_1\|_{X_1})$ for $x \in X_0+X_1$, $x_i \in X_i$, $i=0, 1$, and $J(t, x)=\max(\|x\|_{X_0}, t\|x\|_{X_1})$ for $x \in X_0 \cap X_1$. By means of a given function norm Φ [10, p. 442], let us denote by $(X_0, X_1)_{\Phi,K}$ the set of elements $x \in X_0+X_1$ for which $\Phi[K(t, x)]<\infty$ and by $(X_0, X_1)_{\Phi,J}$ the set of elements $x \in X_0+X_1$ for which there exists a strongly measurable function $u(t)$ with values in $X_0 \cap X_1$, such that $x=\int_0^\infty u(t)\frac{dt}{t}$ in X_0+X_1, $\Phi[J(t, u(t))]<\infty$. Obviously these spaces are normed linear spaces provided with their respective norms $\|x\|_{\Phi,K}=\Phi[K(t, x)]$ and $\|x\|_{\Phi,J}=\inf \Phi[J(t, u(t))]$, $x=\int_0^\infty u(t)\frac{dt}{t}$. The correspondences that to any pair (X_0, X_1) of Banach spaces assign $(X_0, X_1)_{\Phi,K}$ and $(X_0, X_1)_{\Phi,J}$ are called K and J-method respectively. The main theorems such as the theorem on the equivalence of two methods, the theorem of reiteration and the interpolation theorem are formulated and proved under some additional assumptions on Φ.

In his lecture [8] given at the University of Pavia, Peetre presented a brief summary of a generalization of the K-method. The purpose of the present paper is to give some supplemental details to this Peetre's lecture and at the same time to make an attempt to generalize the J-method as well under the same circumstances. Let us call the methods so obtained N and M-method in accordance with the K and J-method respectively. The theorem on the equivalence of two methods and the theorem of reiteration will be obtained also in this case. As for the interpolation theorem, we will not deal with it here, because it will now become rather simple in substance in spite of its elucidative complexity.

Section 1 is concerned with preliminary remarks. A pseudonorm is a function satsfying all axioms of the ordinary norm and admitting the value ∞ likewise. By giving two families of pseudonorms $P_0(t, x)$, $P_1(t, x)$, $x \in E$, $0<t<\infty$, we define $N(t, x)=\inf_{x=x_0+x_1}(P_0(t, x_0)+P_1(t, x_1))$, $M(t, x)=\max(P_0(t, x), P_1(t, x))$ and preparatory examinations will be given about P_0, P_1, N and M. In Section 2 the function norm is introduced and intermediate spaces are defined by means of $N(t, x)$ (N-method) and $M(t, x)$ (M-method). The emphasis is upon the theorem on the equivalence of these methods [Theorem 1]. Section 3 is devoted to the theorem of reiteration. To begin with, the notion of the class concerning a normed linear space is defined [Definition 6]. Considerations requisite to formulate the theorem is given and Proposition 9 and 10 are proved, from which the theorem of reiteration [Theorem 2] will now become almost evident.

§ 1. Preliminaries and Pseudonorms.

Let E and F be linear topological Hausdorff spaces. We write $F\subset E$ to mean that F is continuously contained in E, that is, F is a linear subspace of E and the injection of F into E is continuous.

Definition 1. A real valued function P: $E \ni x \rightarrow P(x)$ is a *pseudonorm* on E if it satisfies

1° $0\leq P(x)\leq\infty$ for all $x \in E$,

2° $P(\lambda x)=|\lambda|P(x)$ for all complex number λ and $x \in E$,

3° $P(x+y)\leq P(x)+P(y)$ for all $x, y \in E$,

4° for any sequence $\{x_n\}$ in E, $P(x_n)\rightarrow 0$ $(n\rightarrow\infty)$ implies $x_n\rightarrow 0$ $(n\rightarrow\infty)$ in E.

Remark 1. Adopting the convention $0\cdot\infty=\infty\cdot 0=0$ we may find from 2° and 4° that $P(x)=0$ if and only if $x=0$. Thus the set $F=\{x; x \in E, P(x)<\infty\}$ is a normed linear space equipped with the norm $P(x)$ and continuously contained in E.

Let $P_0(t, x)$ and $P_1(t, x)$ be functions defined for (t, x), $0<t<\infty$, $x \in E$ and assume that for any fixed t, $P_i(t, x)$ $(i=0, 1)$ is a pseudonorm on E and for any fixed x, $P_i(t, x)$ is a locally integrable function of t with respect to the measure $m=\frac{dt}{t}$, i.e.

$$\int_{\lambda}^{\mu} P_i(t, x)\frac{dt}{t}<\infty$$

for any λ, μ, $0<\lambda<\mu<\infty$. Let further $\rho_0(t)$ and $\rho_1(t)$ be non-negative finite valued functions defined for $0<t<\infty$, and locally integrable with respect to m (m-integrable).
We suppose

$$\rho_0(t)\geq\rho_1(t) \quad \text{for all } t,\ 0<t<1, \tag{1.1}$$

$$\rho_0(t)\leq\rho_1(t) \quad \text{for all } t,\ 1<t<\infty, \tag{1.2}$$

and there exist $\gamma>1$, $C_0>0$ such that

$$\rho_i(t)\leq C_0 \quad \text{for all } t,\ \frac{1}{\gamma}\leq t\leq\gamma,\ i=0,\ 1. \tag{1.3}$$

Between P_i and ρ_i, $i=0, 1$, we now assume

$$P_i(t,\ x)\leq\rho_i\left(\frac{t}{s}\right) P_i(s,\ x) \quad \text{for all } 0<s,\ t<\infty,\ x\in E. \tag{1.4}$$

Let us write

$$\alpha(t)=\max(\rho_0(t),\ \rho_1(t)), \quad \beta(t)=\min(\rho_0(t),\ \rho_1(t))$$

and define for $0<t<\infty$, $x\in E$,

$$N(t,\ x)=\inf(P_0(t,\ x_0)+P_1(t,\ x_1)),$$

where the inf is taken for all x_0, $x_1\in E$, $x=x_0+x_1$, and

$$M(t,\ x)=\max(P_0(t,\ x),\ P_1(t,\ x)).$$

It is noted that $M(t, x)$ is strongly m-measurable and locally m-integrable for any $x\in E$ fixed.

We here cite the following example which plays a fundamental rôle in the interpolation theory of Lions and Peetre [4], [7], [9].

Example 1. Let X_0, X_1 be normed spaces continuously contained in E and define

$$P_0(t,\ x)=\begin{cases}\|x\|_{X_0} & \text{if } x\in X_0,\\ \infty & \text{if } x\notin X_0,\end{cases}$$

$$P_1(t,\ x)=\begin{cases}t\|x\|_{X_1} & \text{if } x\in X_1,\\ \infty & \text{if } x\notin X_1.\end{cases}$$

Then we may take $\rho_0(t)=1$ and $\rho_1(t)=t$ and so that $\alpha(t)=\max(1, t)$, $\beta(t)=$

$\min(1, t)$. In the present case we write $K(t, x)$ and $J(t, x)$ instead of $N(t, x)$ and $M(t, x)$ respectively:

$$K(t, x)=\begin{cases}\inf\limits_{x=x_0+x_1}(\|x_0\|_{X_0}+t\|x_1\|_{X_1}) & \text{if } x\in X_0+X_1\\ \infty & \text{if } x\notin X_0+X_1,\end{cases}$$

where the inf is taken for all $x_0\in X_0$, $x_1\in X_1$, $x=x_0+x_1$,

$$J(t, x)=\begin{cases}\max(\|x\|_{X_0}, t\|x\|_{X_1}) & \text{if } x\in X_0\cap X_1\\ \infty & \text{if } x\notin X_0\cap X_1.\end{cases}$$

In the sequel we need the following inequalities which we now write down for references:

$$N(t, x)\leq P_i(t, x)\leq M(t, x) \qquad i=0, 1, \tag{1.5}$$

$$N(s, x)\leq\alpha\left(\frac{s}{t}\right)N(t, x), \tag{1.6}$$

$$M(s, x)\leq\alpha\left(\frac{s}{t}\right)M(t, x), \tag{1.7}$$

$$N(s, x)\leq\beta\left(\frac{s}{t}\right)M(t, x). \tag{1.8}$$

Except (1.8) these inequalities are almost evident and (1.8) is seen as follows. If $s>t$ then $\beta\left(\frac{s}{t}\right)=\rho_0\left(\frac{s}{t}\right)$ and $N(s, x)\leq P_0(s, x)\leq\rho_0\left(\frac{s}{t}\right)P_0(t, x)\leq\beta\left(\frac{s}{t}\right)M(t, x)$. The case $s<t$ may similarly be done. When $s=t$ it is enough to note that $N(s, x)\leq P_i(s, x)\leq\rho_i(1)P_i(t, x)\leq\rho_i(1)M(t, x)$ for $i=0, 1$.

PROPOSITION 1. *$N(t, x)$, $M(t, x)$ are pseudonorms on E for each $t>0$.*

PROOF. We only prove the condition 4° for $N(t, x)$, because the rest is nearly plain. Letting $N(t, x_n)\to 0$ $(n\to\infty)$, one may write $x_n=x_{n0}+x_{n1}$ so that $P_0(t, x_{n0})+P_1(t, x_{n1})<\frac{1}{n}$. Then it holds that $x_{n0}\to 0$, $x_{n1}\to 0$ $(n\to\infty)$ in E and consequently $x_n=x_{n0}+x_{n1}\to 0$ $(n\to\infty)$ in E. This completes the proof.

Let us now write

$$E_N=\{x;\ x\in E,\ N(1, x)<\infty\}, \qquad \|x\|_{E_N}=N(1, x),$$

$$E_M=\{x;\ x\in E,\ M(1, x)<\infty\}, \qquad \|x\|_{E_M}=M(1, x),$$

$$A_0=\{x;\ x\in E,\ P_0(1, x)<\infty\}, \qquad \|x\|_{A_0}=P_0(1, x),$$

$$A_1=\{x;\ x \in E,\ P_1(1, x)<\infty\}, \qquad \|x\|_{A_1}=P_1(1, x).$$

These are all normed linear spaces equipped with their respective norms and furthermore in the sense of isometric isomorphism it holds that

$$E_N=A_0+A_1, \quad E_M=A_0\frown A_1.$$

In order to exclude the trivial cases we shall assume henceforth that $A_0 \neq (0)$, $A_1 \neq (0)$. It then follows that there exist $x_0, x_1 \in E$ with $0<P_0(1, x_0)<\infty$ and $0<P_1(1, x_1)<\infty$. Thus by virtue of $0<P_i(1, x_i)\leq \rho_i(t) P_i\left(\frac{1}{t}, x_i\right)$, it now holds that $0<\rho_i(t)<\infty$ and consequently $0<\alpha(t)<\infty$, $0<\beta(t)<\infty$ for $0<t<\infty$. As the result, owing to the inequalities

$$\frac{1}{\rho_i\left(\frac{1}{t}\right)} P_i(1, x)\leq P_i(t, x)\leq \rho_i(t) P_i(1, x) \qquad i=0, 1,$$

$$\frac{1}{\alpha\left(\frac{1}{t}\right)} N(1, x)\leq N(t, x)\leq \alpha(t) N(1, x),$$

$$\frac{1}{\alpha\left(\frac{1}{t}\right)} M(1, x)\leq M(t, x)\leq \alpha(t) M(1, x),$$

one may conclude that $P_i(t, x)$, $N(t, x)$ and $M(t, x)$ are norms equivalent to $\|x\|_{A_i}$, $\|x\|_{E_N}$ and $\|x\|_{E_M}$ respectively.

PROPOSITION 2. *E_N and E_M are Banach spaces if and only if A_0 and A_1 are Banach spaces.*

PROOF. The part "if" is wellknown. The "only if" part is shown as follows. Let E_N and E_M be Banach spaces. Letting $\{x_n\}$ be a Cauchy sequence in A_0, we see $x_n \to x \in E_N (n\to\infty)$, because $\{x_n\}$ is a fortiori a Cauchy sequence in E_N. Thus we may write $x_n-x=y_{n0}+y_{n1}$ with $y_{n0} \in A_0$, $y_{n1} \in A_1$ and $\|y_{n0}\|_{A_0} +\|y_{n1}\|_{A_1}\leq\|x_n-x\|_{E_N}+\frac{1}{n}$. By means of a decomposition $x=x_0+x_1$ with $x_0 \in A_0$ and $x_1 \in A_1$ we now set $x_{n0}=x_0+y_{n0}$, $x_{n1}=x_1+y_{n1}$. Then owing to

$$\|x_{n0}-x_0\|_{A_0}+\|x_{n1}-x_1\|_{A_1}\leq\|x_n-x\|_{E_N}+\frac{1}{n},$$

it results that $x_{n0}\to x_0$ $(n\to\infty)$ in A_0 and $x_{n1}\to x_1 (n\to\infty)$ in A_1. Since $x_{n1}=x_n-x_{n0}$ and since $\{x_n\}$, $\{x_{n0}\}$ are both Cauchy sequences in A_0, it follows that $\{x_{n1}\}$ is also a Cauchy sequence in A_0 and consequently in E_M. By assumption that E_M is a Banach space we may infer $x_{n1} \to x_1 (n \to \infty)$ in E_M and a fortiori in A_0. Therefore $x_n = x_{n0} + x_{n1} \to x_0 + x_1 = x\ (n \to \infty)$ in A_0. Thus A_0 and similarly A_1 are Banach spaces. This completes the proof.

§ 2. Function norms and intermediate spaces.

In this and subsequent sections A_0 and A_1 are assumed to be nontrivial Banach spaces (Proposition 2). Let S_+ be the set of all m-measurable functions $\varphi(t)$, $0 \leq \varphi(t) \leq \infty$, defined for $0 < t < \infty$.

DEFINITION 2. A real valued function Φ: $S_+ \ni \varphi \to \Phi[\varphi] = \Phi[\varphi(t)]$ is called a *function norm* if it satisfies

1° $0 \leq \Phi[\varphi] \leq \infty$,
2° $\Phi[\varphi] = 0$ implies $\varphi(t) = 0$ a.e. $t > 0$,
3° $\Phi[\varphi] < \infty$ implies $\varphi(t) < \infty$ a.e. $t > 0$,
4° $\Phi[\lambda\varphi] = \lambda\Phi[\varphi]$ for all $\lambda \geq 0$,
5° *Riesz-Fischer property* [10, p. 444, Theorem 1]:

$$\varphi(t) \leq \sum_{n=1}^{\infty} \varphi_n(t) \text{ a.e. } t > 0 \text{ implies } \Phi[\varphi] \leq \sum_{n=1}^{\infty} \Phi[\varphi_n].$$

A function norm Φ satisfying the following condition 5* is called a *function norm in the strong sense.*

5* *Strong Riesz-Fischer Property*: for any $\varphi \in S_+$ and for any real valued $m \otimes m$-measurable function $\varphi(t, \lambda)$, $0 \leq \varphi(t, \lambda) \leq \infty$ defined for $0 < t < \infty$, $0 < \lambda < \infty$, it holds that

$$\varphi(t) \leq \int_0^\infty \varphi(t, \lambda) \frac{d\lambda}{\lambda} \text{ a.e. } t > 0 \text{ implies } \Phi[\varphi] \leq \int_0^{*\infty} \Phi[\varphi(t, \lambda)] \frac{d\lambda}{\lambda},$$

where $\int^*$ denotes the upper integral.

It is noted that the condition 5* is stronger than the condition 5° because if $\varphi(t, \lambda)$ is defined by

$$\varphi(t, \lambda) = \begin{cases} 0 & \text{for } 0 < \lambda < 1, \\ \varphi_n(t) & \text{for } e^{n-1} \leq \lambda < e^n,\ n = 1, 2, \cdots, \end{cases}$$

it follows that

$$\int_0^\infty \varphi(t, \lambda) \frac{d\lambda}{\lambda} = \sum_{n=1}^{\infty} \varphi_n(t).$$

In order to state the following definition it will be convenient to prove the next

LEMMA 1. *If $t \to u(t)$ is an E_M-valued strongly m-measurable function defined for $0<t<\infty$, then $t \to M(t, u(t))$ is m-measurable.*

PROOF. When $u(t)$ is a constant function, $M(t, u(t))$ is already known to be m-measurable. Therefore the same is true in case that $u(t)$ is an m-measurable step function. Any E_M-valued strongly m-measurable function $u(t)$ is the limit a.e. in E_M of a sequence $\{u_n(t)\}$ of E_M-valued strongly m-measurable step functions. In consequence $M(t, u(t))$, as the limit a.e. of the sequence $\{M(t, u_n(t))\}$ of m-measurable functions, is itself m-measurable. This completes the proof.

DEFINITION 3. Let Φ be a function norm. We denote by $A_{\Phi,N}$ the set of elements $x \in E$ such that $\Phi[N(t, x)]<\infty$.

It is clear that $A_{\Phi,N}$ is a normed linear space provided with the norm $\|x\|_{\Phi,N}=\Phi[N(t, x)]$. Let us now consider an E_M-valued locally m-integrable function $t \to u(t)$ defined for $0<t<\infty$, i.e. for any λ, μ, $0<\lambda<\mu<\infty$, $u(t)$ is m-integrable (Bochner) on $[\lambda, \mu]$ in E_M[10, p. 217]. We write $\int_0^\infty u(t)\frac{dt}{t}=x$ to mean that there exists $x \in E_N$ such that $\int_\lambda^\mu u(t)\frac{dt}{t} \to x$ $(\lambda \to 0, \mu \to \infty)$ in E_N. It is to be noted that the integral of this type is not the ordinary one (Bochner, Bourbaki etc.). Nevertheless, until otherwise stated, we shall make use of the usual notation, because things will not be confused.

DEFINITION 4. Let Φ be a function norm. We denote by $A_{\Phi,M}$ the set of elements $x \in E$ such that $x=\int_0^\infty u(t)\frac{dt}{t}$, $\Phi[M(t, u(t))]<\infty$ where $t \to u(t)$ is an E_M-valued strongly m-measurable, locally m-integrable function just mentioned.

It is clear that $A_{\Phi,M}$ is a seminormed linear space provided with the seminorm $\|x\|_{\Phi,M}=\inf_u \Phi[M(t, u(t))]$, where inf is taken over all $u(t)$ such that $x=\int_0^\infty u(t)\frac{dt}{t}$.

EXAMPLE 2. In case that $N=K$ and $M=J$ defined in Example 1, we write $A_{\Phi,N}$ and $A_{\Phi,M}$ as $(X_0, X_1)_{\Phi,K}$ and $(X_0, X_1)_{\Phi,J}$ respectively.

PROPOSITION 3. *$\Phi[\beta]<\infty$ implies $E_M \subset A_{\Phi,N}$ for any function norm Φ.*

PROOF. Since $N(t, x) \leq \beta(t)M(1, x)$ by (1.8), it follows that

$$\|x\|_{\Phi,N}=\Phi[N(t, x)] \leq \Phi[\beta]M(1, x).$$

This completes the proof.

PROPOSITION 4. *If* $\varphi \in S_+$ *is given such that* $\varphi(t)>0$ *on a set of positive m-measure contained in* $\left[\frac{1}{\gamma}, \gamma\right]$, *then* $\Phi[\varphi]<\infty$ *implies* $E_M \subset A_{\Phi,M}$ *for any function norm* Φ. *In particular* $\Phi[\beta]<\infty$ *implies* $E_M \subset A_{\Phi,M}$.

PROOF. Letting $T=\left[\frac{1}{\gamma}, \gamma\right] \cap \{t; \varphi(t)>\delta\}$, one obtains $m(T)>0$ for some $\delta>0$. Given $x \in E_M$ let us write $u(t)=\frac{1}{m(T)}\chi_T(t)x$, where $\chi_T(t)$ it the characteristic function of such T. Then it holds that $\int_0^\infty u(t)\frac{dt}{t}=x$ and therefore by (1.7)

$$\|x\|_{\Phi,M} \leq \Phi[M(t, u(t))]=\frac{1}{m(T)}\Phi[\chi_T(t)M(t, x)]$$

$$\leq \frac{1}{m(T)}\Phi[\chi_T(t)\alpha(t)]M(1, x).$$

Consequently, since $\alpha(t) \leq C_0$ on $\left[\frac{1}{\gamma}, \gamma\right]$ by (1.3) and since $\varphi(t)>\delta\chi_T(t)$ a.e., it follows that

$$\|x\|_{\Phi,M} \leq \frac{C_0}{m(T)}\Phi[\chi_T(t)]M(1, x)$$

$$\leq \frac{C_0}{m(T)\delta}\Phi[\varphi]M(1, x).$$

This completes the proof.

Let $f(\lambda)$, $0 \leq f(\lambda) \leq \infty$, $0<\lambda<\infty$, be a locally m-integrable function.

DEFINITION 5. We say that a function norm Φ is of *type* f if the following inequality holds

$$\Phi[\varphi(\lambda t)] \leq f(\lambda)\Phi[\varphi(t)]$$

for all $0<\lambda<\infty$ and for all $\varphi \in S_+$.

According to Peetre [8] we now state the following proposition.

PROPOSITION 5. *Let* Φ *be a function norm of type* f. *Then it holds that*

$$N(s, x)\Phi\left[\frac{1}{\alpha\left(\frac{1}{t}\right)}\right] \leq f(s)\|x\|_{\Phi,N}$$

and

$$\|x\|_{\Phi,N} \leq f\left(\frac{1}{s}\right)\Phi[\beta]M(s, x)$$

for all $x \in E$ *and* s, $0 < s < \infty$.

PROOF. Owing to (1.6): $N(s, x) \leq \alpha\left(\frac{s}{t}\right)N(t, x)$, it holds that

$$N(s, x)\Phi\left[\frac{1}{\alpha\left(\frac{s}{t}\right)}\right] \leq \Phi[N(t, x)].$$

On the other hand, since Φ is of type f, we may infer that

$$\Phi\left[\frac{1}{\alpha\left(\frac{1}{t}\right)}\right] \leq f(s)\Phi\left[\frac{1}{\alpha\left(\frac{s}{t}\right)}\right].$$

Thus the first part of the statement is true. To prove the second, let us take (1.8): $N(t, x) \leq \beta\left(\frac{t}{s}\right)M(s, x)$. It then follows that

$$\Phi[N(t, x)] \leq \Phi\left[\beta\left(\frac{t}{s}\right)\right]M(s, x)$$
$$\leq f\left(\frac{1}{s}\right)\Phi[\beta]M(s, x)$$

as desired. This completes the proof.

For the sake of completeness we also prove the next corollary after Peetre [8].

COROLLARY. *If* Φ *is a function norm of type* f, *then* $A_{\Phi,N}$ *is a Banach space.*

PROOF. It is enough to show that a series $\sum_{k=1}^{\infty} x_k$ in $A_{\Phi,N}$ is convergent in $A_{\Phi,N}$, whenever $\sum_{k=1}^{\infty}\|x_k\|_{\Phi,N} < \infty$. Let us suppose $\sum_{k=1}^{\infty}\|x_k\|_{\Phi,N} < \infty$. Since $f(s) < \infty$ a.e. and $\Phi\left[\frac{1}{\alpha\left(\frac{1}{t}\right)}\right] > 0$, it follows from Proposition 5 that $\sum_{k=1}^{\infty} N(s, x_k) < \infty$ a.e. $s > 0$ and hence everywhere $s > 0$ by (1.6). By the assumption that E_N is a Banach space, there exists $x \in E$ such that $x = \sum_{k=1}^{\infty} x_k$ in E_N. Setting $y_n = \sum_{k=1}^{n} x_k$ it holds that

$$N(t, x - y_n) \leq \sum_{k=n+1}^{\infty} N(t, x_k),$$

and therefore

$$\|x-y_n\|_{\Phi,N}=\Phi[N(t,\ x-y_n)]$$
$$\leq\sum_{k=n+1}^{\infty}\Phi[N(t,\ x_k)]=\sum_{k=n+1}^{\infty}\|x_k\|_{\Phi,N}\to 0\ (n\to\infty).$$

This completes the proof.

We intend to examine the relationship between $A_{\Phi,N}$ and $A_{\Phi,M}$. To begin with let us prove the following

PROPOSITION 6. *Let Φ be a function norm in the strong sense and of type f. Put $C_f=\int_0^\infty\beta\left(\frac{1}{t}\right)f(t)\frac{dt}{t}$. Then it holds that*

$$\|x\|_{\Phi,N}\leq C_f\|x\|_{\Phi,M}$$

for all $x\in A_{\Phi,M}$. If in particular $C_f<\infty$, it follows that $A_{\Phi,M}\subset A_{\Phi,N}$ and consequently $A_{\Phi,M}$ becomes a normed linear space.

PROOF. Let $x\in A_{\Phi,M}$ and let us write $x=\int_0^\infty u(s)\frac{ds}{s}$ where $s\to u(s)$ is an E_M-valued strongly m-measurable function and for each λ, μ, $0<\lambda<\mu<\infty$, it holds that

$$x_{\lambda\mu}=\int_\lambda^\mu u(s)\frac{ds}{s}\in E_M,\ x_{\lambda\mu}\to x\ (\lambda\to 0,\ \mu\to\infty)\ \text{ in } E_N.$$

Since $N(t,\ y)$ is a norm on E_N equivalent to $\|y\|_{E_N}$, it follows that

$$N(t,\ x)=N(t,\ \lim_{\substack{\lambda\to 0\\ \mu\to\infty}}x_{\lambda\mu})=\lim_{\substack{\lambda\to 0\\ \mu\to\infty}}N(t,\ x_{\lambda\mu})$$
$$\leq\lim_{\substack{\lambda\to 0\\ \mu\to\infty}}\int_\lambda^\mu N(t,\ u(s))\frac{ds}{s}=\int_0^\infty N(t,\ u(s))\frac{ds}{s}.$$

On account of (1.8) and of Lemma 1, we now obtain

$$N(t,\ x)\leq\int_0^\infty\beta\left(\frac{t}{s}\right)M(s,\ u(s))\frac{ds}{s}$$
$$=\int_0^\infty\beta\left(\frac{1}{\lambda}\right)M(t\lambda,\ u(t\lambda))\frac{d\lambda}{\lambda}.$$

The strong Riesz-Fischer property of Φ tells us

$$\Phi[N(t,\ x)]\leq\overset{*}{\int_0^\infty}\beta\left(\frac{1}{\lambda}\right)\Phi[M(t\lambda,\ u(t\lambda))]\frac{d\lambda}{\lambda}$$

which turns out to

$$\Phi[N(t, x)] \leq \int_0^\infty \beta\left(\frac{1}{\lambda}\right) f(\lambda) \frac{d\lambda}{\lambda} \cdot \Phi[M(t, u(t))],$$

because Φ is of type f. Making u vary it now follows that

$$\|x\|_{\Phi, N} \leq C_f \|x\|_{\Phi, M}.$$

The rest of the statement is obvious. This completes the proof.

In order to study further relations it is convenient to provide the next lemma due to [7].

LEMMA 2. *Let $x \in E$ be given and suppose $\beta\left(\frac{1}{t}\right) N(t, x) \to 0$ $(t \to 0$ or $t \to \infty)$. Then we may write $x = \int_0^\infty u(t) \frac{dt}{t}$, that is:*

(i) *$t \to u(t)$ is an E_M-valued strongly m-measurable function defined for $0 < t < \infty$,*

(ii) *for each λ, μ, $0 < \lambda < \mu < \infty$, $u(t)$ is m-integrable (Bochner) on $[\lambda, \mu]$ in E_M and*

$$x_{\lambda\mu} = \int_\lambda^\mu u(t) \frac{dt}{t} \to x \ (\lambda \to 0 \text{ and } \mu \to \infty) \text{ in } E_N.$$

Furthermore $u(t)$ may be taken so that

$$M(t, u(t)) \leq \frac{2C_0^2}{\log \gamma} N(t, x)$$

for all t, $0 < t < \infty$.

PROOF. We assume $N(t, x) > 0$, because the case $N(t, x) = 0$, i.e. $x = 0$, is trivial. For any $\varepsilon > 0$ and $n = 0, \pm 1, \pm 2, \cdots$, we may write $x = x_{n0} + x_{n1}$, x_{n0}, $x_{n1} \in E$, in such a way that

$$P_0(\gamma^n, x_{n0}) + P_1(\gamma^n, x_{n1}) \leq (1+\varepsilon) N(\gamma^n, x).$$

It now follows first that for $n = -1, -2, \cdots$,

$$\begin{aligned} P_0(1, x_{n0}) &\leq (1+\varepsilon)\rho_0(\gamma^{-n}) N(\gamma^n, x) \\ &\leq (1+\varepsilon)\beta(\gamma^{-n}) N(\gamma^n, x) \to 0 \quad (n \to -\infty), \end{aligned}$$

and then next that for $n = 1, 2, \cdots$,

$$P_1(1, x_{n1}) \leq (1+\varepsilon)\rho_1(\gamma^{-n}) N(\gamma^n, x)$$

$$\leq (1+\varepsilon)\beta(\gamma^{-n})N(\gamma^n, x)\to 0 \quad (n\to\infty).$$

Putting $u_n = x_{n+10} - x_{n0} = x_{n1} - x_{n+11}$ for $n=0, \pm 1, \pm 2, \cdots$, we see $u_n \in E_M$, and since $x_{no}\to 0$ $(n\to -\infty)$ in E_N, $x_{n1}\to 0$ $(n\to\infty)$ in E_N, it holds that $u_n\to 0$ $(n\to -\infty$ or $n\to\infty)$ in E_N. Define

$$u(t) = \frac{1}{\log\gamma} u_n \quad \text{for } \gamma^n \leq t < \gamma^{n+1},\ n=0, \pm 1, \pm 2, \cdots,$$

and note that $u(t)$ is an E_M-valued strongly m-measurable function for $0<t<\infty$.

We now show $\int_0^\infty u(t)\frac{dt}{t} = x$. To this end we first observe that for $\gamma^n \leq \mu < \gamma^{n+1}$, $n=1, 2, \cdots$,

$$\int_1^\mu u(t)\frac{dt}{t} = u_0 + \cdots + u_{n-1} + \frac{\log \mu/\gamma^n}{\log\gamma} u_n$$

$$= x_{01} - x_{n1} + \frac{\log\mu/\gamma^n}{\log\gamma} u_n \to x_{01}\,(n\to\infty) \text{ in } E_N.$$

Next we see for $\gamma^{-m-1}\leq\lambda<\gamma^{-m}$, $m=1, 2, \cdots$, that

$$\int_\lambda^1 u(t)\frac{dt}{t} = u_{-1} + \cdots + u_{-m} + \frac{\log\gamma^{-m}/\lambda}{\log\gamma} u_{-m-1}$$

$$= x_{00} - x_{-m0} + \frac{\log\gamma^{-m}/\lambda}{\log\gamma} u_{-m-1} \to x_{00}\,(m\to\infty) \text{ in } E_N.$$

Thus $x = \int_0^\infty u(t)\frac{dt}{t}$ is proved.

In order to estimate $M(t, u(t))$, let $\gamma^n\leq t<\gamma^{n+1}$, $n=0, \pm 1, \pm 2, \cdots$, and write

$$M(t, u(t)) = \max(P_0(t, u(t)), P_1(t, u(t)))$$

$$= \frac{1}{\log\gamma}\max(P_0(t, u_n), P_1(t, u_n))$$

$$\leq \frac{1}{\log\gamma}\max(P_0(t, x_{n+10}) + P_0(t, x_{n0}), P_1(t, x_{n1}) + P_1(t, x_{n+11})).$$

By means of (1.4) this becomes

$$M(t, u(t)) \leq \frac{1}{\log\gamma}\max\Big(\rho_0\Big(\frac{t}{\gamma^{n+1}}\Big)P_0(\gamma^{n+1}, x_{n+10}) + \rho_0\Big(\frac{t}{\gamma^n}\Big)P_0(\gamma^n, x_{n0}),$$

$$\rho_1\Big(\frac{t}{\gamma^n}\Big)P_1(\gamma^n, x_{n1}) + \rho_1\Big(\frac{t}{\gamma^{n+1}}\Big)P_1(\gamma^{n+1}, x_{n+11})\Big)$$

$$\leq \frac{1+\varepsilon}{\log\gamma}\max\Big(\rho_0\Big(\frac{t}{\gamma^{n+1}}\Big)N(\gamma^{n+1}, x)+\rho_0\Big(\frac{t}{\gamma^n}\Big)N(\gamma^n, x),$$

$$\rho_1\Big(\frac{t}{\gamma^n}\Big)N(\gamma^n, x)+\rho_1\Big(\frac{t}{\gamma^{n+1}}\Big)N(\gamma^{n+1}, x)\Big)$$

$$\leq \frac{1+\varepsilon}{\log\gamma}\max\Big(\rho_0\Big(\frac{t}{\gamma^{n+1}}\Big)\alpha\Big(\frac{\gamma^{n+1}}{t}\Big)+\rho_0\Big(\frac{t}{\gamma^n}\Big)\alpha\Big(\frac{\gamma^n}{t}\Big),$$

$$\rho_1\Big(\frac{t}{\gamma^n}\Big)\alpha\Big(\frac{\gamma^n}{t}\Big)+\rho_1\Big(\frac{t}{\gamma^{n+1}}\Big)\alpha\Big(\frac{\gamma^{n+1}}{t}\Big)\Big)N(t, x).$$

By assumption that $\gamma^n \leq t < \gamma^{n+1}$ and hence $\frac{1}{\gamma} < \frac{\gamma^n}{t} \leq 1 < \frac{\gamma^{n+1}}{t} \leq \gamma$, it follows from (1.3) that $\rho_i\Big(\frac{t}{\gamma^{n+1}}\Big) \leq C_0$, $\rho_i\Big(\frac{t}{\gamma^n}\Big) \leq C_0$, $\alpha\Big(\frac{\gamma^{n+1}}{t}\Big) \leq C_0$, $\alpha\Big(\frac{\gamma^n}{t}\Big) \leq C_0$, and thus

$$M(t, u(t)) \leq \frac{1+\varepsilon}{\log\gamma} 2C_0^2 N(t, x)$$

for each $\varepsilon > 0$. Therefore

$$M(t, u(t)) \leq \frac{2C_0^2}{\log\gamma} N(t, x)$$

as desired. This completes the proof.

PROPOSITION 7. *Let Φ be a function norm of type f and assume $\beta\Big(\frac{1}{t}\Big)f(t) \to 0$ ($t \to 0$ or $t \to \infty$). Then it holds that*

$$\|x\|_{\Phi,M} \leq \frac{2C_0^2}{\log\gamma}\|x\|_{\Phi,N}$$

for all $x \in A_{\Phi,N}$ and hence $A_{\Phi,N} \subset A_{\Phi,M}$.

PROOF. Proposition 5 tells us

$$\beta\Big(\frac{1}{t}\Big)N(t, x) \leq \frac{1}{C}\beta\Big(\frac{1}{t}\Big)f(t)\|x\|_{\Phi,N}$$

for all $0 < t < \infty$, where $C = \Phi\left[\frac{1}{\alpha\left(\frac{1}{t}\right)}\right] > 0$. By assumption, it then follows that $\beta\Big(\frac{1}{t}\Big)N(t, x) \to 0$ ($t \to 0$ or $t \to \infty$) and Lemma 2 may be applicable. Thus $x = \int_0^\infty u(t)\frac{dt}{t}$, $M(t, u(t)) \leq \frac{2C_0^2}{\log\gamma}N(t, x)$, $0 < t < \infty$, and so

$$\|x\|_{\Phi,M}\leq\Phi[M(t,\ u(t))]\leq\frac{2C_0^2}{\log\gamma}\|x\|_{\Phi,N}$$

as desired. This completes the proof.

THEOREM 1. *Let Φ be a function norm in the strong sense and of type f. If*

$$\beta\left(\frac{1}{t}\right)f(t)\rightarrow 0 \quad (t\rightarrow 0 \text{ or } t\rightarrow\infty),$$

$$C_f=\int_0^\infty\beta\left(\frac{1}{t}\right)f(t)\frac{dt}{t}<\infty,$$

then $A_{\Phi,N}=A_{\Phi,M}$ and furthermore it holds that

$$\frac{\log\gamma}{2C_0^2}\|x\|_{\Phi,M}\leq\|x\|_{\Phi,N}\leq C_f\|x\|_{\Phi,M}$$

for all $x\in E$.

PROOF. Clear from Proposition 6 and Proposition 7.

COROLLARY. *Let Φ be a function norm in the strong sense and of type f. If*

$$f(t)\rightarrow 0\ (t\rightarrow 0),\ \frac{f(t)}{t}\rightarrow 0 \quad (t\rightarrow\infty)$$

and if

$$C_f=\int_0^1\frac{f(t)}{t}\,dt+\int_1^\infty\frac{f(t)}{t^2}\,dt<\infty,$$

Then $(X_0,\ X_1)_{\Phi,J}=(X_0,\ X_1)_{\Phi,K}$ and furthermore it holds that

$$\frac{1}{4e}\|x\|_{\Phi,J}\leq\|x\|_{\Phi,K}\leq C_f\|x\|_{\Phi,J}$$

for all $x\in E$

PROOF. To get the last inequality it is enough to take $\gamma=e^{\frac{1}{2}}$ and hence $C_0=e^{\frac{1}{2}}$. The rest is obvious from Theorem 1. This completes the proof.

REMARK 2. In the particular case in question, a closer inspection of the proof of Lemma 2 shows us that the constant $\frac{2C_0^2}{\log\gamma}$ there may be replaced by $\frac{1+\gamma}{\log\gamma}$ and consequently it follows that

$$\frac{1}{\gamma_0}\|x\|_{\Phi,J}\leq\|x\|_{\Phi,K}$$

for all $x\in E$, where γ_0 is the minimum of $\frac{1+\gamma}{\log\gamma}$, $3.5<\gamma_0<3.6$.

EXAMPLE 3. We here give some examples of function norms usually employed. Let $g(t)$ be an m-measurable function such that $0\leq g(t)\leq\infty$, and put for $\varphi\in S_+$

$$\Phi_{g,p}[\varphi]=\begin{cases}\left\{\int_0^\infty\left(\frac{\varphi(t)}{g(t)}\right)^p\frac{dt}{t}\right\}^{\frac{1}{p}} & \text{if } 1\leq p<\infty,\\ \sup\limits_{t>0}\frac{\varphi(t)}{g(t)} & \text{if } p=\infty,\end{cases}$$

where sup is the essential supremum and the convention $\frac{0}{0}=\frac{\infty}{\infty}=\infty\cdot 0=0\cdot\infty=0$ is used. By means of the integral form of Minkowski's inequality:

$$\left\{\int\left(\int f(x,y)\,dx\right)^p dy\right\}^{\frac{1}{p}}\leq\int\left\{\int(f(x,y))^p\,dy\right\}^{\frac{1}{p}}dx,$$

[2, p. 148], it is not difficult to see that $\Phi_{g,p}$ is a function norm in the strong sense. If $g(t)$ satisfies

$$g(st)\leq f(s)\,g(t)\quad\text{for all } 0<s,\ t<\infty,$$

where $f(s)$, $0\leq f(s)<\infty$ a.e., is a measurable function, then $\Phi_{g,p}$ is of type f. By specialization of $g(t)$, the following function norms are offered.

1) $\Phi_{g,p}$ with $g(t)=t^\theta$, $0<\theta<1$. The wellknown interpolation theory of Lions and Peetre is based on this function norm [4], [6], [7], [8].

2) $\Phi_{g,\infty}$ with $g(t)=\max\left(1,\frac{t}{c}\right)$, $c>0$. This is of type $\max(1,\lambda)$ and gives $(X_0, X_1)_{\Phi_{g,\infty},K}=X_0+X_1^c$, where X_1^c is the Banach space X_1 renormed as $\|x\|_{X_1^c}=c\|x\|_{X_1}$. To prove this it is enough to see by (1.6)

$$\|x\|_{\Phi_{g,\infty},K}=\sup_{t>0}\frac{K(t,x)}{\max\left(1,\frac{t}{c}\right)}\leq K(c,x),$$

and the sup is attained by $t=c$.

3) $\Phi_{g,1}$ with $g(t)=\min\left(1,\frac{t}{c}\right)$, $c>0$. This is of type $\max(1,\lambda)$ and gives $(X_0, X_1)_{\Phi_{g,1},J}=X_0\cap X_1^c$, the latter half of which is seen as follows. Letting $x\in(X_0, X_1)_{\Phi_{g,1},J}$ and writing $x=\int_0^\infty u(t)\frac{dt}{t}$ where $t\to u(t)$ is an $X_0\cap X_1$-valued func-

tion locally m-integrable in $X_0 \cap X_1$, we get

$$\int_0^\infty J(c, u(t))\frac{dt}{t} \leq \int_0^\infty \max\left(1, \frac{c}{t}\right) J(t, u(t))\frac{dt}{t}$$

$$= \Phi_{g,1}[J(t, u(t))] < \infty.$$

Hence it follows that $\int_0^\infty \|u(t)\|_{X_0}\frac{dt}{t} < \infty$ and $\int_0^\infty \|u(t)\|_{X_1}\frac{dt}{t} < \infty$. Consequently we may infer that $u(t)$ is m-integrable both in X_0, X_1 and so $x = \int_0^\infty u(t)\frac{dt}{t} \in X_0 \cap X_1$. Since

$$\|x\|_{X_0} \leq \int_0^\infty \|u(t)\|_{X_0}\frac{dt}{t}, \quad \|x\|_{X_1} \leq \int_0^\infty \|u(t)\|_{X_1}\frac{dt}{t},$$

it holds that

$$J(c, x) = \max(\|x\|_{X_0}, c\|x\|_{X_1})$$

$$\leq \int_0^\infty J(c, u(t))\frac{dt}{t} \leq \Phi_{g,1}[J(t, u(t))].$$

This proves $J(c, x) \leq \|x\|_{\Phi_{g,1},J}$ and $X_0 \cap X_1^c \supset (X_0, X_1)_{\Phi_{g,1},J}$. Conversely now let $x \in X_0 \cap X_1^c$ and take for any $\varepsilon > 0$

$$\varphi(t) = \begin{cases} \dfrac{1}{\varepsilon} & \text{for } c \leq t \leq ce^{\varepsilon} \\ 0 & \text{otherwise.} \end{cases}$$

Then $u(t) = \varphi(t)x$ satisfies $\int_0^\infty u(t)\frac{dt}{t} = x$. Thus it follows that

$$\|x\|_{\Phi_{g,1},J} \leq \int_0^\infty \max\left(1, \frac{c}{t}\right) J(t, u(t))\frac{dt}{t}$$

$$= \frac{1}{\varepsilon}\int_c^{ce^{\varepsilon}} J(t, x)\frac{dt}{t} \leq \frac{1}{\varepsilon}\int_c^{ce^{\varepsilon}} \max\left(1, \frac{t}{c}\right) J(c, x)\frac{dt}{t}$$

$$= \frac{1}{c\varepsilon}\int_c^{ce^{\varepsilon}} dt \cdot J(c, x) = \frac{e^{\varepsilon}-1}{\varepsilon} J(c, x).$$

Therefore $\|x\|_{\Phi_{g,1},J} \leq J(c, x)$ and $X_0 \cap X_1^c \subset (X_0, X_1)_{\Phi_{g,1},J}$ is obtained. We have thus proved $X_0 \cap X_1^c = (X_0, X_1)_{\Phi_{g,1},J}$ with $\|x\|_{\Phi_{g,1},J} = J(c, x)$. This completes the proof.

3. Reiteration

Let $g(t)$, $0\leq g(t)\leq\infty$, $0<t<\infty$ be an m-measurable function and assume that it is finite valued on a set of positive measure, i.e. the trivial case $g(t)=\infty$ a.e. is excluded.

DEFINITION 6. A normed linear space $X\subset E$ is said to be of

1° *class* $\mathscr{C}_{g,N}$(i.e. $X\in\mathscr{C}_{g,N}$) if $x\in X$ implies $N(t,x)\leq Cg(t)\|x\|_X$ a.e. $t>0$, where C is a constant independent of x;

2° *class* $\mathscr{C}_{g,M}$(i.e. $X\in\mathscr{C}_{g,M}$) if $x\in E_M$ implies $\|x\|_X\leq Dg\left(\frac{1}{t}\right)M(t,x)$ a.e. t>0, where D is a constant independent of x;

3° *class* $\mathscr{C}_g$(i.e. $X\in\mathscr{C}_g$) if $X\in\mathscr{C}_{g,N}\cap\mathscr{C}_{g,M}$.

REMARK 3. It follows from the definition that $X\subset E_N$ if $X\in\mathscr{C}_{g,N}$ and that $E_M\subset X$ if $X\in\mathscr{C}_{g,M}$. It is also noted that E_N and E_M are of class $\mathscr{C}_\alpha$, $E_N\in\mathscr{C}_{\beta,N}$, $E_M\in\mathscr{C}_{\beta,M}$, $A_0\in\mathscr{C}_{\rho_0}$ and $A_1\in\mathscr{C}_{\rho_1}$.

PROPOSITION 8. *Let X, $X\subset E$, be a Banach space and let $g(t)$, $0\leq g(t)\leq\infty$, be a locally m-integrable function on $0<t<\infty$. Then it holds that*

(i) *$X\in\mathscr{C}_{g,N}$ if and only if $X\subset A_{\Phi_{g,\infty},N}$,*

(ii) *if $g(t)$ satisfies $g(st)\leq f(s)g(t)$ for all $0<s$, $t<\infty$ and for a given locally m-integrable function $f(s)$ on $0<s<\infty$, then $X\in\mathscr{C}_{g,M}$ if and only if $A_{\Phi_{\tilde{g},1},M}\subset X$, where $\tilde{g}(t)=\dfrac{1}{g(1/t)}$,*

(iii) *under the same condition on $g(t)$ in* (ii), *$X\in\mathscr{C}_g$ if and only if $A_{\Phi_{\tilde{g},1},M}\subset X\subset A_{\Phi_{g,\infty},N}$.*

PROOF. (i) Suppose $X\in\mathscr{C}_{g,N}$ and take any $x\in X$, then

$$\|x\|_{\Phi_{g,\infty},N}=\sup_{t>0}\frac{N(t,x)}{g(t)}\leq C\|x\|_X,$$

and $X\subset A_{\Phi_{g,\infty},N}$ is obtained. Conversely, since clearly $A_{\Phi_{g,\infty},N}\in\mathscr{C}_{g,N}$, it follows that $X\in\mathscr{C}_{g,N}$ if $X\subset A_{\Phi_{g,\infty},N}$. This proves (i).

(ii) Let $X\in\mathscr{C}_{g,M}$ and let $x\in A_{\Phi_{\tilde{g},1},M}$. By means of a locally m-integrable E_M-valued function $u(t)$, $\Phi_{\tilde{g},1}[M(t,u(t))]<\infty$, we write $x=\int_0^\infty u(t)\frac{dt}{t}$. Take a continuous function $\varphi(t)$ with the support in $\left]\frac{1}{\gamma},\gamma\right[$ such that $\varphi(t)\geq 0,\int_0^\infty\varphi(t)\frac{dt}{t}=1$, and put

$$v(t)=\int_0^\infty u\left(\frac{t}{s}\right)\varphi(s)\frac{ds}{s}.$$

Then clearly $v(t)$ is a continuous E_M-valued and hence, owing to Remark 3, a continuous X-valued function. It also holds that $x=\int_0^\infty v(t)\frac{dt}{t}$ and for each $s>0$ one obtains

$$\|v(s)\|_X \leq Dg\left(\frac{1}{t}\right)M(t, v(s)) \quad \text{a.e. } t>0.$$

Therefore letting $\{r_n\}$ be an arrangement of all positive rational numbers, one finds, for each n, a set $T_n\subset]0, \infty[$, $m(T_n)=0$, such that

$$\|v(r_n)\|_X \leq Dg\left(\frac{1}{t}\right)M(t, v(r_n)) \quad \text{for all } t,\ 0<t\notin T_n. \tag{3.1}$$

Setting $T=\overline{\lim_{n\to\infty}}T_n=\bigcap_{k=1}^\infty\bigcup_{n=k}^\infty T_n$, we see $m(T)=0$ and for every t, $0<t\notin T$, it holds that $t\notin T_n$ except for finite numbers of n. Hence, for this t, we may select a subsequence $\{n_i\}$ in such a way that $t\notin T_{n_i}$ and $r_{n_i}\to t\,(i\to\infty)$. Putting, as we may, $n=n_i$ in (3.1) and letting $i\to\infty$, we may find

$$\|v(t)\|_X \leq Dg\left(\frac{1}{t}\right)M(t, v(t))$$

for all t, $0<t\notin T$. In consequence it results that

$$\begin{aligned}
\int_0^\infty \|v(t)\|_X\frac{dt}{t} &\leq D\int_0^\infty g\left(\frac{1}{t}\right)M(t, v(t))\frac{dt}{t}\\
&= D\int_0^\infty g\left(\frac{1}{t}\right)M\left(t, \int_0^\infty u\left(\frac{t}{s}\right)\varphi(s)\frac{ds}{s}\right)\frac{dt}{t}\\
&\leq D\int_0^\infty g\left(\frac{1}{t}\right)\frac{dt}{t}\int_0^\infty M\left(t, u\left(\frac{t}{s}\right)\right)\varphi(s)\frac{ds}{s}\\
&\leq D\int_0^\infty \varphi(s)\frac{ds}{s}\int_0^\infty f\left(\frac{1}{s}\right)g\left(\frac{s}{t}\right)\alpha(s)M\left(\frac{t}{s}, u\left(\frac{t}{s}\right)\right)\frac{dt}{t}\\
&= D\int_{\frac{1}{\gamma}}^\gamma \varphi(s)f\left(\frac{1}{s}\right)\alpha(s)\frac{ds}{s}\int_0^\infty g\left(\frac{1}{t}\right)M(t, u(t))\frac{dt}{t}\\
&\leq DC_0C'\Phi_{\tilde{g},1}[M(t, u(t))]<\infty,
\end{aligned}$$

where we put $C'=\|\varphi\|_\infty\int_{\frac{1}{\gamma}}^\gamma f(s)\frac{ds}{s}<\infty$. Thus $v(t)$ is m-integrable in X and therefore $y_{\lambda\mu}=\int_\lambda^\mu v(t)\frac{dt}{t}\to y$ ($\lambda\to 0$ and $\mu\to\infty$) in X and hence in E. It follows that $x=y\in X$ together with $\|x\|_X\leq DC_0C'\|x\|_{\Phi_{\tilde{g},1},M}$. This proves $X\supset A_{\Phi_{\tilde{g},1},M}$. Conversely now we must show that any $X \supset A_{\Phi_{\tilde{g},1},M}$ is of class $\mathscr{C}_{g,M}$. To do this it is enough to prove $A_{\Phi_{\tilde{g},1},M} \in \mathscr{C}_{g,M}$. Let $x \in E_M$ and take any m-measurable function $\psi(t) \geq 0$ with $\int_0^\infty \psi(t) \frac{dt}{t} = 1$. Setting $u(t) = \psi(t)x$, it holds that $x = \int_0^\infty u(t) \frac{dt}{t}$ and further that

$$\|x\|_{\Phi_{\tilde{g},1},M} \leq \Phi_{\tilde{g},1}[M(t, u(t))]$$
$$= \int_0^\infty g\left(\frac{1}{t}\right) \psi(t) M(t, x) \frac{dt}{t}.$$

Since this is true for any $\psi(t)$ described above, it follows that

$$\|x\|_{\Phi_{\tilde{g},1},M} \leq g\left(\frac{1}{t}\right) M(t, x) \quad \text{a.e. } t > 0.$$

This proves $A_{\Phi_{\tilde{g},1},M} \in \mathscr{C}_{g,M}$ as desired.

(iii) is clear from (i) and (ii).

This completes the proof.

In order to formulate the reiteration theorem, let us suppose first that we are given an m-measurable function $h(t)$, $0 \leq h(t) \leq \infty$, $h(t) \not\equiv 0$, $h(t) \not\equiv \infty$, and next that there exist numbers ω, $\omega' > 0$ such that

$$h(st) \leq \omega h(s) h(t), \quad h(s) h(t) \leq \omega' h(st)$$

for all $0 < s$, $t < \infty$. From these conditions it follows that $0 < h(t) < \infty$ for all $0 < t < \infty$.

EXAMPLE 4. The following are examples of such functions.

1) $h(t) = t^\theta$, $0 < \theta < 1$ with $\omega = \omega' = 1$, is the most important in practice [4], [7].

2) Any measurable function $h(t)$, $\frac{1}{c} \leq h(t) \leq c$, $0 < t < \infty$ with $\omega = \omega' = c^3$.

3) The product of any two such functions.

By means of this h, a given function norm Φ and a given real number $\sigma \neq 0$, let us define

$$\Omega[\varphi] = \Phi[h(t) \varphi(t^\sigma)], \quad \varphi \in S_+,$$

and therefore

$$\Phi[\psi] = \Omega\left[\frac{\psi(t^{\frac{1}{\sigma}})}{h(t^{\frac{1}{\sigma}})}\right], \quad \psi \in S_+.$$

Then, since Φ is a function norm, Ω is clearly a function norm and the converse is also true. If Φ is of type f, it follows that

$$\Omega[\varphi(\lambda t)]=\Phi[h(t)\varphi(\lambda t^{\sigma})]$$
$$\leq F(\lambda)\Phi[h(\lambda^{\frac{1}{\sigma}}t)\varphi((\lambda^{\frac{1}{\sigma}}t)^{\sigma})],$$

where $F(\lambda)=\min\left(\omega h(\lambda^{-\frac{1}{\sigma}}), \frac{\omega'}{h(\lambda^{\frac{1}{\sigma}})}\right)$, because $h(t)\leq\omega h(\lambda^{-\frac{1}{\sigma}})h(\lambda^{\frac{1}{\sigma}}t)$ and $h(t)h(\lambda^{\frac{1}{\sigma}})\leq\omega' h(\lambda^{\frac{1}{\sigma}}t)$. Thus we get

$$\Omega[\varphi(\lambda t)]\leq f(\lambda^{\frac{1}{\sigma}})F(\lambda)\Phi[h(t)\varphi(t^{\sigma})],$$

i.e. Ω is of type $f(\lambda^{\frac{1}{\sigma}})F(\lambda)$. On the other hand if Ω is of type f_1, we may infer in the same way that Φ is of type $F(\lambda^{-\sigma})f_1(\lambda^{\sigma})$ and therefore in case $f_1(\lambda)=f(\lambda^{\frac{1}{\sigma}})F(\lambda)$, Φ is of type $F(\lambda^{-\sigma})F(\lambda^{\sigma})f(\lambda)\leq\omega\omega' f(\lambda)$. We finally note that Ω satisfies the strong Riesz-Fischer property if and only if so does Φ. Letting $h_0(t)=h(t)$ and $h_1(t)=t^{\sigma}h(t)$, we begin with the following proposition.

PROPOSITION 9. *Let Φ be a function norm of type f. Let X_0, X_1 be Banach spaces $\subset E$ and assume that $X_0\in\mathscr{C}_{h_0,N}$, $X_1\in\mathscr{C}_{h_1,N}$. Then it holds that*

$$(X_0, X_1)_{\Omega,K}\subset A_{\Phi,N}.$$

PROOF. By the definition of $X_i\in\mathscr{C}_{h_i,N}$ $(i=0, 1)$ it holds that

$$N(t, x_i)\leq C_i h_i(t)\|x_i\|_{X_i} \quad \text{a.e. } t>0$$

for all $x_i\in X_i$. Putting $x=x_0+x_1$, we may write

$$N(t, x)\leq N(t, x_0)+N(t, x_1)$$
$$\leq C_0 h(t)\left(\|x_0\|_{X_0}+\frac{C_1}{C_0}t^{\sigma}\|x_1\|_{X_1}\right).$$

Hence

$$N(t, x)\leq C_0 h(t)K\left(\frac{C_1}{C_0}t^{\sigma}, x\right)$$
$$\leq C_0\omega h\left(\left(\frac{C_1}{C_0}\right)^{-\frac{1}{\sigma}}\right)h\left(\left(\frac{C_1}{C_0}\right)^{\frac{1}{\sigma}}t\right)K\left(\left(\left(\frac{C_1}{C_0}\right)^{\frac{1}{\sigma}}t\right)^{\sigma}, x\right),$$

and therefore we get

$$\|x\|_{\Phi,N}\leq C_0\omega h\left(\left(\frac{C_1}{C_0}\right)^{-\frac{1}{\sigma}}\right)f\left(\left(\frac{C_1}{C_0}\right)^{\frac{1}{\sigma}}\right)\Phi[h(t)K(t^{\sigma}, x)]$$

$$=C_0\,\omega h\Big(\Big(\frac{C_1}{C_0}\Big)^{-\frac{1}{\sigma}}\Big)f\Big(\Big(\frac{C_1}{C_0}\Big)^{\frac{1}{\sigma}}\Big)\|x\|_{\Omega,K}.$$

This completes the proof.

PROPOSITION 10. *Let Φ be a function norm in the strong sense and of type f where f is locally m-integrable. Let X_0, X_1 be Banach spaces $\subset E$ and assume that $X_0 \in \mathscr{C}_{h_0,M}$, $X_1 \in \mathscr{C}_{h_1,M}$. Then it holds that*

$$(X_0,\ X_1)_{\Omega,J}\supset A_{\Phi,M}.$$

PROOF. Take any $x\in A_{\Phi,M}$ and by means of a locally m-integrable E_M-valued function $u(t)$, $\Phi[M(t,\ u(t))]<\infty$, let us write $x=\int_0^\infty u(t)\frac{dt}{t}$. As in the proof (ii) of Proposition 8, we take a continuous function $\varphi(t)$ with the support in $\left]\frac{1}{\gamma},\ \gamma\right[$ such that $\varphi(t)\geq 0$, $\int_0^\infty \varphi(t)\frac{dt}{t}=1$, and put

$$v(t)=\int_0^\infty u\Big(\frac{t}{s}\Big)\varphi(s)\frac{ds}{s}.$$

The continuous E_M-valued and hence the continuous X_i-valued $(i=0,1)$ function $v(t)$ satisfies $x=\int_0^\infty v(t)\frac{dt}{t}$. It is also already known that

$$\|v(t)\|_{X_i}\leq D_i h_i\Big(\frac{1}{t}\Big)M(t,\ v(t))\quad \text{a.e. } t>0,\ i=0,\ 1.$$

This proves

$$\frac{1}{D_0 h\Big(\frac{1}{t}\Big)}\max\Big(\|v(t)\|_{X_0}, \frac{D_0}{D_1}t^\sigma\|v(t)\|_{X_1}\Big)\leq M(t,\ v(t))$$

and therefore, since $h(t)h\Big(\frac{1}{t}\Big)\leq\omega' h(1)$, it follows that

$$h(t)J\Big(\frac{D_0}{D_1}t^\sigma,\ v(t)\Big)\leq\omega' D_0 h(1)M(t,\ v(t))\quad \text{a.e. } t>0. \tag{3.2}$$

Define $w(t)=\frac{1}{\sigma}v\Big(\Big(\frac{D_1}{D_0}t\Big)^{\frac{1}{\sigma}}\Big)$ and let us see $x=\int_0^\infty w(t)\frac{dt}{t}$. Then we get

$$\|x\|_{\Omega,J}\leq\Omega[J(t,\ w(t))]=\Phi[h(t)J(t^\sigma,\ w(t^\sigma))]$$

$$\leq f\Big(\Big(\frac{D_1}{D_0}\Big)^{\frac{1}{\sigma}}\Big)\Phi\Big[h\Big(\Big(\frac{D_0}{D_1}\Big)^{\frac{1}{\sigma}}t\Big)J\Big(\frac{D_0}{D_1}t^\sigma,\ w\Big(\frac{D_0}{D_1}t^\sigma\Big)\Big)\Big]$$

$$\leq \omega f\Big(\Big(\frac{D_1}{D_0}\Big)^{\frac{1}{\sigma}}\Big) h\Big(\Big(\frac{D_0}{D_1}\Big)^{\frac{1}{\sigma}}\Big) \Phi\Big[h(t) J\Big(\frac{D_0}{D_1} t^{\sigma}, \frac{1}{\sigma} v(t)\Big)\Big].$$

By virtue of (3.2) one obtains

$$\|x\|_{\Omega,J} \leq \frac{\omega\omega'}{\sigma} D_0 h(1) f\Big(\Big(\frac{D_1}{D_0}\Big)^{\frac{1}{\sigma}}\Big) h\Big(\Big(\frac{D_0}{D_1}\Big)^{\frac{1}{\sigma}}\Big) \Phi[M(t, v(t))].$$

By means of the strong Riesz-Fischer property, let us make an estimate of $\Phi[M(t, v(t))]$ by $\Phi[M(t, u(t))]$ as follows.

$$\begin{aligned}\Phi[M(t, v(t))] &= \Phi\Big[M\Big(t, \int_0^\infty u\Big(\frac{t}{s}\Big)\varphi(s)\frac{ds}{s}\Big)\Big] \\ &\leq \Phi\Big[\int_0^\infty M\Big(t, u\Big(\frac{t}{s}\Big)\Big)\varphi(s)\frac{ds}{s}\Big] \\ &\overset{*}{\leq} \int_0^\infty \varphi(s)\,\Phi\Big[M\Big(t, u\Big(\frac{t}{s}\Big)\Big)\Big]\frac{ds}{s} \\ &\overset{*}{\leq} \int_0^\infty \varphi(s)\alpha(s)\,\Phi\Big[M\Big(\frac{t}{s}, u\Big(\frac{t}{s}\Big)\Big)\Big]\frac{ds}{s} \\ &\leq C'\Phi[M(t, u(t))],\end{aligned}$$

where we write $C' = \int_{\frac{1}{\gamma}}^{\gamma} \varphi(s)\alpha(s) f\Big(\frac{1}{s}\Big)\frac{ds}{s} < \infty$. Thus we get

$$\|x\|_{\Omega,J} \leq C''\Phi[M(t, u(t))]$$

and consequently

$$\|x\|_{\Omega,J} \leq C''\|x\|_{\Phi,M}$$

with $C'' = \frac{\omega\omega'}{\sigma} D_0 h(1) f\Big(\Big(\frac{D_1}{D_0}\Big)^{\frac{1}{\sigma}}\Big) h\Big(\Big(\frac{D_0}{D_1}\Big)^{\frac{1}{\sigma}}\Big) C'$. This completes the proof.

We now state the theorem of reiteration as follows.

THEOREM 2. *Let Φ be a function norm in the strong sense and of type f, and let $h(t)$ be an m-measurable function such that $0 < h(t) < \infty$ and $\frac{1}{\omega} h(s)h(t) \leq h(st) \leq \omega h(s)h(t)$ for all $0 < s, t < \infty$, where $\omega > 0$ is a given constant. Given $\sigma \neq 0$ real, put $h_0(t) = h(t)$, $h_1(t) = t^{\sigma}h(t)$ and*

$$\Omega[\varphi] = \Phi[h(t)\varphi(t^{\sigma})], \ \varphi \in S_+.$$

Suppose that f is locally m-integrable and satisfies

$$\beta\left(\frac{1}{t}\right)f(t)\to 0 \quad (t\to 0 \text{ or } t\to\infty),$$

$$\int_0^\infty \min(1, t^{-\sigma})h(t^{-1})f(t)\frac{dt}{t}<\infty.$$

If X_0, X_1 are Banach spaces continuously contained in E and if $X_0 \in \mathscr{C}_{h_0}$, $X_1 \in \mathscr{C}_{h_1}$, then it holds that

$$A_{\Phi,N}=A_{\Phi,M}=(X_0, X_1)_{\Omega,K}=(X_0, X_1)_{\Omega,J}.$$

Proof. Proposition 7, 9 and 10 tell us

$$(X_0, X_1)_{\Omega,K}\subset A_{\Phi,N}\subset A_{\Phi,M}\subset (X_0, X_1)_{\Omega,J},$$

and Proposition 6 applied to Ω, J, K gives $(X_0, X_1)_{\Omega,J}\subset (X_0, X_1)_{\Omega,K}$. This completes the proof.

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