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On Jónsson modules over a commutative ring

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1. Introduction. Let R be a commutative ring with identity, let M be a unitary module over R, and let α be an infinite cardinal. Following the terminology of universal algebra [5], [3], we call M a *Jónsson α-module* over R if $|M|=\alpha$, while $|N|<\alpha$ for each proper submodule N of M. Our attention to this topic was attracted by a recent paper of SHELAH [13], who answered affirmatively the following old question of Kurosh: does there exist a Jónsson ω_1-group — that is, a group G of cardinality ω_1 such that each proper subgroup is countable? Like Shelah, we concentrate primarily on the cases where $\alpha\in\{\omega_0, \omega_1\}$ in this paper, because these are the cases of principal interest within our context.

If I is an ideal of R and if I, considered as an R-module, is a Jónsson α-module, then we refer to I as a *Jónsson α-ideal* of R. By passage to the idealization of R and an R-module M, the theory of Jónsson α-modules is equivalent to the corresponding theory for ideals, but we shall only occasionally make this transition to ideals via idealization.

Section 2 of the paper deals with Jónsson α-modules, Section 3 with Jónsson ω_0-modules, and Section 4 presents some pertinent examples. Corollary 3.2 shows that a finitely generated Jónsson α-module is simple, and hence the set of such modules over a given ring R is easily determined. Theorem 2.4 shows that if the cardinal α is countably inaccessible from below and if R belongs to the class $\mathcal{F}$ of rings over which each (**)-module is finitely generated (see Section 2 for terminology; in particular, $\mathcal{F}$ includes the class of Noetherian rings and the class of finite-dimensional chained rings), then each Jónsson α-module over R is finitely generated, hence simple; in particular, this result applies to Jónsson ω_1-modules over a ring in $\mathcal{F}$. Proposition 2.5 is in this context a useful result; it states that if M is a non-finitely generated

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Jónsson α-module over R, then Ann (M) is a prime ideal and $rM=M$ for each $r \in R-\mathrm{Ann}\,(M)$.

Assume that M is a non-finitely generated Jónsson ω_0-module over the ring R. Theorem 3.1 shows that there exists a maximal ideal Q of R such that Ann (x) is a Q-primary ideal of finite index for each nonzero element x of R; moreover, the powers of Q properly descend and $\bigcap_{i=1}^{\infty} Q^i$ is a prime ideal of R. It follows from Theorem 3.1 that in considering Jónsson ω_0-modules over R, there is no loss of generality in assuming that the module is faithful and R is a quasi-local integral domain. Proposition 3.2 shows that M can be expressed as the union of a strictly ascending sequence of cyclic submodules, and this leads both to a construction of classes of non-finitely generated Jónsson ω_0-modules by means of generators and relations (Theorem 3.5) and to a determination of the isomorphism class of non-finitely generated Jónsson ω_0-modules over a Prüfer domain J (Proposition 3.7 and the paragraph preceding that result).

The examples of Section 4 indicate certain restrictions on what can be said about the structure of a quasi-local domain D such that D admits a non-finitely generated Jónsson ω_0-module. Such a domain D need not be Noetherian, for example, and even for a Noetherian domain D, no restrictions can be placed on the (Krull) dimension of D.

All rings considered in this paper are assumed to be commutative and to contain an identity element; all modules considered are assumed to be unitary.

2. Jónsson modules. If R is a commutative ring with identity and M is a maximal ideal of R such that $|R/M|=\alpha$ is infinite, then R/M is a Jónsson α-module over R. One of our purposes in this section is to attempt to determine the class of rings S such that each Jónsson module over S arises essentially in this way — that is, as S/M for some maximal ideal M of S with infinite residue field.

The main results of this section are Corollary 2.3 and Theorem 2.4. In particular, Theorem 2.4 resolves the question of Jónsson modules over the rings normally encountered in commutative algebra. While the proof of Proposition 2.5 is not difficult, this result is an important tool in the development of Section 3 material.

According to the terminology of [2, Ex. 17, p. 245], the infinite cardinal α is said to be *regular* if $\alpha \neq \sum_{i\in I} \alpha_i$ for each nonempty family $\{\alpha_i\}_{i\in I}$ of cardinals with $|I|<\alpha$ and $\alpha_i<\alpha$ for each i. As noted by Simis [14], this condition is equivalent to the statement that there is no cofinal set of cardinality less than α in the set of ordinals preceding the first ordinal of cardinality α.

Proposition 2.1. *Assume that M is a Jónsson α-module over R, where α is a regular cardinal. If $\{M_i\}_{i\in I}$ is a nonempty family of proper submodules of M, where*

$|I|<\alpha$, *then* $M\neq \sum_{i\in I} M_i$. *In particular, M is indecomposable and M has at most one maximal submodule.*

Proof. Since $|M_i|<\alpha$ for each i and since α is regular, it follows that $|\sum_{i\in I} M_i|< <\alpha$, and hence $\sum_{i\in I} M_i\neq M$. The statements in the second sentence of the proposition follow immediately from the first sentence.

Proposition 2.2. *Assume that each proper ideal of the ring R has cardinality less than* $|R|$. *Then either R is finite or R is a field.*

Proof. We prove that if $|R|=\alpha$ is infinite, then R is a field. Proposition 2.1 shows that R has a unique maximal ideal P. Since $|P|<\alpha$, it follows that $|R/P|=\alpha$; let $\{r_\beta\}_{\beta\in B}$ be a complete set of representatives of the residue classes of P in R. If $x\in P$, then $\{r_\beta x\}\subseteqq P$, so there exist distinct $\beta,\gamma\in B$ so that $r_\beta x=r_\gamma x$. Since $r_\beta-r_\gamma$ is a unit of R, then $x=0$, so $P=(0)$ and R is a field, as asserted.

Corollary 2.3. *Let M be an infinite, finitely generated R-module and let* $\alpha=|M|$. *Then M is a Jónsson module if and only if M is cyclic and* Ann (M) *is a maximal ideal of R such that* $|R/\mathrm{Ann}\,(M)|=\alpha$.

Proof. It's clear that the stated conditions are sufficient for M to be a Jónsson module. Conversely, if M is a Jónsson module and $M=Rm_1+Rm_2+\dots+Rm_n$, then Proposition 2.1 implies that $M=Rm_i$ for some i. Thus, M and $R/\mathrm{Ann}\,(M)$ are isomorphic modules over R and over $R/\mathrm{Ann}\,(M)$, so that $R/\mathrm{Ann}\,(M)$ is a field of cardinality α by Proposition 2.2.

Following the terminology of [1], we call a module M a (**)-module if M cannot be expressed as the union of a strictly ascending sequence $M_1<M_2<\dots<M_n\ \dots$ of submodules; we denote by $\mathscr{F}$ the class of rings R such that each (**)-module over R is finitely generated (clearly a finitely generated module is a (**)-module for any R). Theorems 4.2, 4.7, and 4.10 of [1] show that $\mathscr{F}$ contains the subclasses of Noetherian rings, finite-dimensional chained rings, and W^*-rings; Theorem 6.1 of [10] shows that $\mathscr{F}$ also contains each ring R such that (1) R has Noetherian spectrum, (2) the descending chain condition for prime ideals is satisfied in R, and either (3) each ideal of R is countably generated, or (4) each ideal of R contains a power of its radical.

If α is an infinite cardinal, we say that α *is countably inaccessible from below* if $\alpha\neq\sum_{i\in I}\alpha_i$ for each nonempty countable family $\{\alpha_i\}_{i\in I}$ of cardinals $\alpha_i<\alpha$. According to this terminology, ω_0 is countably accessible from below, while each infinite cardinal with an immediate predecessor (in particular, ω_1) is countably inaccessible from below. The next result deals both with the concept of countable inaccessibility from below and with the class $\mathscr{F}$.

Theorem 2.4. *Assume that R is in the class $\mathscr{F}$ and that the cardinal α is countably inaccessible from below. To within isomoprhism, the set of Jónsson α-modules is $\{R/M_i\}_{i\in I}$, where $\{M_i\}_{i\in I}$ is the set of maximal ideals of R whose associated residue class field has cardinality α.*

Proof. Clearly each R/M_i is a Jónsson α-module over R. Conversely, let L be a Jónsson α-module over R. If $\{L_j\}_{j=1}^{\infty}$ is an ascending sequence of proper submodules of L, then as in the proof of Proposition 2.1, it follows that $L\neq \sum_{j=1}^{\infty} L_j$. Thus, L is a (**)-module over R, and since $R\in\mathscr{F}$, then L is finitely generated. It then follows from Corollary 2.3 that as an R-module, $L\cong R/M_i$ for some $i\in I$.

In our further consideration of Jónsson α-modules, we shall begin in Section 3 to concentrate our attention on the cases where $\alpha=\omega_0$ or $\alpha=\omega_1$. Even for ω_1, Theorem 2.4 resolves the question of Jónsson modules over the rings normally encountered in commutative algebra. Because ω_0 is countably accessible from below, however, Theorem 2.4 does not apply to this case. We know, in fact, that a Jónsson ω_0-module over a principal ideal domain need not be finitely generated; the p-quasicyclic group $Z(p^\infty)$, considered as a Z-module, illustrates this statement. (It is well-known, in fact, that the p-quasicyclic groups are the only Jónsson ω_0-modules over Z [6, Ex. 4, p. 105].)

We conclude Section 2 with a proposition and a corollary that are valid for arbitrary cardinals α. In particular, Proposition 2.5 is used frequently in the rest of this paper.

Proposition 2.5. *Let M be a Jónsson α-module over the ring R.*

(1) *If $r\in R$, then either $rM=M$ or $rM=(0)$.*

(2) Ann (M) *is a prime ideal of R.*

Proof. To prove (1), assume that $rM\neq M$ and let $N=\{m\in M | rm=0\}$. We show that $N=M$. We write rM as $\{rm_i\}_{i\in I}$, where $|I|<\alpha$. If $m\in M$, then $rm=rm_i$ for some i so that $m\in m_i+N$. It follows that $M=\bigcup_{i\in I}(m_i+N)$, and hence $|M|\leqq |I|\cdot|N|$. By hypothesis on M and I, we conclude that $|N|=\alpha$ so that $N=M$ as we wished to prove. It follows from (1) that if $x, y\in R-\mathrm{Ann}\,(M)$, then $M=xM=yM$, and hence $M=xyM$. Thus $xy\notin \mathrm{Ann}\,(M)$, and Ann (M) is prime in R, as asserted.

Corollary 2.6. *Assume that I is a Jónsson α-ideal of the ring R. If $I^2\neq(0)$, then I is a field, and hence I is a direct summand of R.*

Proof. Take $r, s\in I$ such that $rs\neq 0$. Then $rI=I=sI$ by Proposition 2.5, and since $r, s\in I$, then $I=(r)=(s)$. By Corollary 2.3, it follows that I is a simple R-module, so $(rs)=I=I^2$. We conclude that as an ideal of R, I is principal and is generated by an idempotent. Hence I is a direct summand of R and the structure of I as an R-module is the same as its structure as a ring. Consequently, I is a field, as asserted.

It's clear that the converse of Corollary 2.6 is also valid. Namely, if K is an infinite field of cardinality α and if S is a nonzero ring, then K is a Jónsson α-ideal of the ring $S\oplus K$ and $K^2\neq(0)$.

3. Jónsson ω_0-modules. We restrict our consideration in this section to the case where $\alpha=\omega_0$, the first infinite cardinal, and in view of Corollary 2.3, we consider only Jónsson ω_0-modules that are not finitely generated. Such a module M has a particularly simple description: M is not finitely generated, is countably infinite, and each proper submodule of M is finite[3].

Assume that M is a non-finitely generated Jónsson ω_0-module over the ring R. What restrictions are imposed on the structure of R and M? Theorem 3.1 and Proposition 3.2 provide some answers to this question. In particular, these two results allow us to restrict to the case where the module M is faithful and the ring R is a quasilocal integral domain. In the case of a Prüfer domain R, we determine the isomorphism class of non-finitely generated Jónsson ω_0-modules over R.

If N is an R-module, we say that N is a *torsion module* if $\operatorname{Ann}(n)\neq(0)$ for each $n\in N$. On the other hand, the module N is *torsion-free* if $\operatorname{Ann}(n)=(0)$ for each nonzero element $n\in N$. The statement of Theorem 3.1 uses this terminology.

Theorem 3.1. *Let M be a Jónsson ω_0-module over the ring R, where M is not finitely generated. Then M is a torsion R-module, and there exists a maximal ideal Q of R such that the following conditions are satisfied:* (1) $\operatorname{Ann}(x)$ *is a Q-primary ideal of finite index for each* $x\in M-\{0\}$, (2) R/Q *is finite,* (3) *the powers of Q properly descend,* (4) $\bigcap_{i=1}^{\infty} Q^i$ *is a prime ideal, and* (5) *if* $H_i=\{x\in M\mid Q^ix=(0)\}$, *then* $\{H_i\}_{i=1}^{\infty}$ *is a strictly ascending sequence of submodules of M such that* $M=\bigcup_{i=1}^{\infty} H_i$.

Proof. As the first step in the proof, we show that $PM=M$ for each maximal ideal P of R. Thus, if $PM\neq M$, then Proposition 2.5 shows that $PM=(0)$, and hence M is a Jónsson ω_0-module over the field R/P. Since M is indecomposable, M is a one-dimensional vector space over R/P. This implies, however, that M is a cyclic R-module, contradicting the fact that M is not finitely generated. Therefore $PM=M$ for each maximal ideal P of R.

For P_α maximal in R, let M_α be the set of elements x of M such that $P_\alpha\subseteq\sqrt{\operatorname{Ann}(x)}$. Then M_α is a submodule of M since the inclusion $\operatorname{Ann}(x-y)\supseteq$

[3]) We remark that "countably infinite" is redundant in this definition — if M is not finitely generated and each proper submodule of M is finite, then M is countably infinite.

$\supseteq \operatorname{Ann}(x) \cap \operatorname{Ann}(y)$ implies that $\sqrt{\operatorname{Ann}(x-y)} \supseteq \sqrt{\operatorname{Ann}(x)} \cap \sqrt{\operatorname{Ann}(y)}$. We show that M is the direct sum of the family $\{M_\alpha\}$, taken over all maximal ideals P_α of R. If $x \in M - \{0\}$, then $Rx \subset M$ implies that Rx is finite, so $R/\operatorname{Ann}(x)$ is a finite ring. Therefore, $\operatorname{Ann}(x)$ is uniquely expressible as a finite intersection $\bigcap_{i=1}^{n} C_{\alpha_i}$ of primary ideals with distinct (maximal) radicals $P_{\alpha_i} = \sqrt{C_{\alpha_i}}$. If $B_j = \bigcap_{i \neq j} C_{\alpha_i}$ for $1 \leqq j \leqq n$, then no maximal ideal of R contains each B_j, so $R = B_1 + \dots + B_n$, and $1 = b_1 + b_2 + \dots + b_n$ with $b_i \in B_i$ for each i. Then $x = \sum_{j=1}^{n} b_j x$, where $C_{\alpha_i} b_j x = (0)$ for each j, and hence $b_j x \in M_j$. This proves that $M = \sum_\alpha M_\alpha$. The sum is direct, for if $m \in M_\alpha \cap (M_{\alpha_1} + \dots + M_{\alpha_k})$, with $\alpha \neq \alpha_j$ for each j, then $\operatorname{Ann}(m) \supseteq P_\alpha + (P_{\alpha_1} \cap \dots \cap P_{\alpha_k}) = R$, so $m = 0$. Because M is indecomposable, we conclude that $M = M_\alpha$ for some α. Let $Q = P_\alpha$; by definition of M_α, $\operatorname{Ann}(x)$ is a Q-primary ideal of finite index for each $x \in M - \{0\}$; in particular, Q has finite index in R. Let H_i be defined as in the statement of Theorem 3.1. Clearly each H_i is a submodule of M, and $H_i \subseteq H_{i+1}$ for each i. Moreover, for $x \in M$, $\operatorname{Ann}(x)$ contains a power of Q since $R/\operatorname{Ann}(x)$ is finite, so that $x \in H_i$ for some i; that is $H = \bigcup_{i=1}^{\infty} H_i$. Observe that H_i is a proper submodule of M for each i since $M = Q^i M \neq (0)$. Finally, we note that the assumption $H_i = H_{i+1}$ leads to the contradiction that $M = H_i$; it suffices to show that $H_i = H_{i+1}$ implies that $H_{i+1} = H_{i+2}$. Thus, if $x \in H_{i+2}$, then $Qx \subseteq H_{i+1} = H_i$, so $Q^i Q x = (0)$ and $x \in H_{i+1}$, as was to be proved. The fact that $H_i < H_{i+1}$ for each i shows that $Q^i > Q^{i+1}$ for each i; in particular, $Q^i \neq (0)$ for each i so that $\operatorname{Ann}(x) \neq (0)$ for each $x \in M - \{0\}$, and M is a torsion module. The equality $M = \bigcup_{i=1}^{\infty} H_i$ implies that $\bigcap_{i=1}^{\infty} Q^i = \operatorname{Ann}(M)$, and Proposition 2.5 shows that $\operatorname{Ann}(M)$ is prime in R. This completes the proof of Theorem 3.1.

If M is a non-finitely generated Jónsson ω_0-module over R, then replacing R by $R/\operatorname{Ann}(M)$, there is no loss of generality in assuming that M is faithful, and Proposition 2.5 shows that $R/\operatorname{Ann}(M)$ is an integral domain. Under these assumptions on R and M, let Q be as in the statement of Theorem 3.1 It is then possible to consider M as a module over the quasi-local domain R_Q. To wit, for $m \in M$ and $r/s \in R_Q$, we define the product $(r/s) \cdot m$ to be rm_1, where $sm_1 = m$. The product is well-defined, for Proposition 2.5 and Theorem 3.1 show that left multiplication by s induces an R-automorphism of M. It is somewhat lengthy, but routine, to verify that M is an R_Q-module under this definition, and we omit the details. We note that $Rm = R_Q m$ for each $m \in M$; for a proof, we need only show that $R_Q m \subseteq Rm$ — that is, we need to show that if $s \in R - Q$ and if $sm_1 = m$, then $m_1 \in Rm$. This statement follows since Rm is finite and since left multiplication by s induces an injection of Rm into Rm

so that $Rm = sRm$. We conclude that the structure of M as an R_Q-module is essentially the same as the structure of M as an R-module [7, Ex. 2, p. 8]. In particular, M is a Jónsson ω_0-module over R_Q. Thus, in considering non-finitely generated Jónsson ω_0-modules M over a ring R, we are led to consider the case where R is a quasi-local domain and M is faitful. The next result is stated for this hypothesis, and is somewhat analogous to Theorem 3.1.

Proposition 3.2. *Assume that M is a non-finitely generated faithful Jónsson ω_0-module over the quasi-local domain (D, P). For $x \in P - \{0\}$, denote by $M(x)$ the submodule of M consisting of elements annihilated by x. Then $M(x)$ is finite and nonzero, $M(x) < M(x^2) < M(x^3) < \dots$, and $M = \bigcup_{i=1}^{\infty} M(x^i)$. Moreover, if $m_1 \in M(x) - \{0\}$ and if elements $m_2, m_3, \dots \in M$ are chosen successively so that $m_i = xm_{i+1}$ for each i, then $Dm_1 < Dm_2 < \dots$ and $M = \bigcup_{i=1}^{\infty} Dm_i$.*

Proof. Since M is faithful, then $M(x) \neq M$, and hence $M(x)$ is finite. Pick $m \in M - \{0\}$. Since $x \in P = \sqrt{\operatorname{Ann}(m)}$, there exists a positive integer k so that $x^k m = 0$ while $x^{k-1}m \neq 0$. Thus $x^{k-1}m$ is a nonzero element of $M(x)$. For a given i, we assume that $s \in M(x^{i+1}) - M(x^i)$. Then $s \in xM$ implies $s = xt$ for some $t \in M$. Thus $x^{i+2}t = x^{i+1}s = 0$, but $x^{i+1}t = x^i s \neq 0$ so that $t \in M(x^{i+2}) - M(x^{i+1})$. Since $\bigcup_{i=1}^{\infty} M(x^i)$ is an infinite submodule of M, we conclude that $M = \bigcup_{i=1}^{\infty} M(x^i)$.

If $m_1, m_2, \dots$ are as described in the hypothesis of Proposition 3.2, then the proof above shows that $m_{i+1} \in M(x^{i+1}) - M(x^i)$ for each i so that $Dm_i < Dm_{i+1}$ and $M = \bigcup_{i=1}^{\infty} Dm_i$, as asserted.

The next result is a partial converse of Proposition 3.2. The proof of this result is routine and will be omitted.

Proposition 3.3. *Let M be an R-module that can be expressed as the union of an infinite strictly ascending sequence $\{M_i\}_{i=1}^{\infty}$ of finite submodules. The following conditions are equivalent.*

(1) *M is a Jónsson ω_0-module.*
(2) *Each proper submodule of M is contained in some M_i.*
(3) *If $x_i \in M - M_i$ for each i, then $\{x_i\}_{i=1}^{\infty}$ generates M.*

If the notation and hypothesis are as in the statement of Proposition 3.2, if F is a free D-module on the countably infinite set $\{y_i\}_{i=1}^{\infty}$, and if φ is the natural surjection of F onto M induced by the mapping $y_i \to m_i$, then, of course, $M \cong F/\ker \varphi$, where $\ker \varphi$ contains the submodule generated by the set $\{y_i - xy_{i+1}\}_{i=1}^{\infty}$. This observation provided the original motivation for Theorem 3.5. The next result provides some motivation for the hypothesis in the statement of Theorem 3.5.

Proposition 3.4. *Assume that P is a maximal ideal of the ring R such that the powers of P properly descend and such that $P=P^2+tR$ for some $t\in R$. Then $P^i:(t)=P^{i-1}$ for each i.*

Proof. Since P/P^2 is a one-dimensional vector space over R/P, there are no ideals of R strictly between P and P^2. It is known that this implies that $P=P^n+tR$ and that $\{P^j\}_{j=1}^n$ is the set of ideals between P and P^n for each n [7, (38,2)]. If $i>1$ then the inclusion $P^{i-1}\subseteq P^i:(t)$ is clear. Moreover, $t\notin P^i$ implies that $P^i:(t)\subseteq P$. Now $p^{i-2}\subsetneqq P^i:(t)$, for otherwise, $P^{i-1}=P^{i-2}[P^2+(t)]\subseteqq P^i$, contrary to the hypothesis that the powers of P properly descend. We conclude that $P^i:(t)=P^{i-1}$, as asserted.

Theorem 3.5. *Assume that $P=A_1, A_2, A_3, \dots$ is a sequence of ideals of R and $\{t_i\}_{i=2}^\infty$ is a sequence of elements of R such that the following conditions are satisfied: (1) P is a maximal ideal of R and R/P is finite, (2) the powers of P properly descend, and (3) for each $i>1$, $P=A_1+(t_i)$, $A_i\supseteqq P^i$, and $A_i:(t_i)\subseteqq P^{i-1}$. Then there exists a non-finitely generated Jónsson ω_0-module M over R such that* Ann(x) *is P-primary for each $x\in M-\{0\}$.*

Proof. Let F be a free R-module on the set $\{x_i\}_{i=1}^\infty$, let A be the submodule of F generated by $\{A_ix_i\}_{i=1}^\infty\cup\{x_i-t_{i+1}x_{i+1}\}_{i=1}^\infty$, and let $M=F/A$; we prove that M has the required properties. Let $y_i=x_i+A$ for each i. It is clear that $\{y_i\}_{i=1}^\infty$ generates M and that $\langle y_i\rangle\subseteqq\langle y_{i+1}\rangle$ for each i. We prove that the inclusion $\langle y_i\rangle\subseteqq\langle y_{i-1}\rangle$ is proper by establishing the following property of the submodule A: if $a\in A-\{0\}$ and if $a=\sum_{j=1}^k r_jx_j$, where $r_k\neq0$, then $r_k\in P$. For some n, we can write $a=a_1x_1+\dots+a_nx_n+h_2(x_1-t_2)+\dots+h_n(x_{n-1}-t_nx_n)$, where $a_i\in A_i$ and $h_j\in R$. If $k=n$, then $r_k=a_n-h_nt_n\in P$. Otherwise, we obtain a sequence of equations

$$
\begin{aligned}
a_n-h_nt_n &= 0\\
h_n+a_{n-1}-h_{n-1}t_{n-1} &= 0\\
&\ \vdots\\
h_{k+2}+a_{k+1}-h_{k+1}t_{k+1} &= 0.
\end{aligned}
$$

The first equation implies that $h_n\in A_n:(t_n)\subseteqq P^{n-1}$, and hence, from the second equation, $h_{n-1}t_{n-1}=h_n+a_{n-1}\in A_{n-1}$ so that $h_{n-1}\in A_{n-1}:(t_{n-1})\subseteqq P^{n-2}$. Inductively, we obtain $h_{k+1}\in P^k$. If $k>1$, it follows that $r_k=h_{k+1}+a_k-h_kt_k\in P$, and if $k=1$, then $r_k=h_2+a_1$ is also in P. This establishes the assertion concerning A, and hence

$\langle y_i\rangle \neq \langle y_{i+1}\rangle$ for each i. Thus, no finite subset of $\{y_i\}_{i=1}^{\infty}$ generates M, and this implies that M is not finitely generated.

We show next that each $\langle y_i\rangle$ is finite. Since $P \subseteq \operatorname{Ann}(y_1)$ and R/P is finite, the submodule $\langle y_1\rangle$ is finite. Assume that $\langle y_i\rangle$ is finite. To prove that $\langle y_{i+1}\rangle$ is finite, it suffices to prove that $\langle y_{i+1}\rangle/\langle y_i\rangle$ is finite. The annihilator of $\langle y_{i+1}\rangle/\langle y_i\rangle$ contains A_{i+1} and the element t_{i+1}, hence the ideal $A_{i+1}+(t_{i+1})=P$. Therefore $\langle y_{i+1}\rangle/\langle y_i\rangle$ is finite, and $\langle y_{i+1}\rangle$ is finite.

To complete the proof, we show that $y \notin \langle y_i\rangle$ implies that $y_i \in \langle y\rangle$. Choose k so that $y \in \langle y_{k+1}\rangle$, $y \notin \langle y_k\rangle$; thus $k \geqq i$. Then $y=ry_{k+1}$, and since $Py_{k+1} \subseteq \langle y_k\rangle$, it follows that $r \notin P$. Hence $R=A_{k+1}+rR$ and we write $1=q+rs$ for some $q \in A_{k+1}$ and $s \in R$. Then $y_{k+1}=qy_{k+1}+rsy_{k+1}=sy$ and $y_i \in \langle y_{k+1}\rangle \subseteq \langle y\rangle$. This is sufficient to show that each proper submodule of M is finite, for if L is a submodule of M that is contained in no $\langle y_i\rangle$, then L contains $\{y_j\}_1^{\infty}$, and hence $L=M$. It is clear from the construction that $\operatorname{Ann}(x)$ is P-primary for each $x \in M-\{0\}$.

Assume that (R, P) is a quasi-local domain such that $P=tR$ is principal and R/P is finite. Then the hypothesis of Theorem 3.5 is satisfied for $A_i=P^i$ and $t_i=t$ for each i. In this case, the module M constructed in the proof of Theorem 3.5 is isomorphic to $R[1/t]/R$, and in the case where this module is faithful (that is, where $\bigcap_{i=1}^{\infty} P^i=(0)$), then R is a rank-one discrete valuation ring and $R[1/t]$ is the quotient field of R. The next result determines equivalent conditions in order that the D-module K/D, where D is an integral domain and K is the quotient field of D, should be a Jónsson ω_0-module. The statement of Theorem 3.6 uses the following terminology from [12]. The ring R is said to have the *finite norm property* (FNP) if R/A is finite for each nonzero ideal A of R (such a ring is said to be *residually finite* in [4]).

Theorem 3.6. *Let D be an integral domain with quotient field $K \neq D$. Let D^* be the integral closure of D. Then K/D is a Jónsson ω_0-module over D if and only if the following conditions are satisfied.*

(1) *D has the finite norm property,*

(2) *D^* is a rank-one discrete valutation ring, and*

(3) *D^* is a finite D-module.*

Proof. Assume that K/D is a Jónsson ω_0-module. If d is a nonzero nonunit of D, then Dd^{-1}/D is a proper submodule of K/D, and hence is finite. Since Dd^{-1}/D and D/dD are isomorphic D-modules, it follows that dD has finite norm, and D has the finite norm property. Let $J \neq K$ be an overring of D. Since J/D is finite, J is integral over D; hence $J \subseteq D^*$ and K is the only proper overring of D^*. Therefore D^* is a rank-one valuation ring finitely generated over D, a ring with (FNP), and hence D^* is rank-one discrete with (FNP).

Conversely, assume that conditions (1)—(3) are satisfied, and write V instead of D^*. Assume that π is a generator of the maximal ideal of V. Since V is a finitely generated D-module, the conductor C of D in V is nonzero; say $C=\pi^k V$. We know that $K=\bigcup_{i=1}^{\infty} \pi^{-1}V$, where $\pi^{-1}V<\pi^{-2}V<\ldots$. To prove that K/D is a Jónsson ω_0-module, it suffices to show that $\pi^{-i}V/D$ is finite for each i and that each proper submodule of K/D is a submodule of $\pi^{-i}V/D$ for some i. $\pi^{-i}V/D$ is a finitely generated D-module and π^{i+k} belongs to the annihilator of this module. Since the ring $D/\pi^{i+k}D$ is finite, it follows that $\pi^{-i}V/D$ is finite. To prove that each proper submodule of K/D is contained in some $\pi^{-i}V/D$, it suffices to show that if N is a D-submodule of K such that $N \subsetneqq \pi^{-i}V$ for each i, then $N=K$. Since $K=\bigcup_{i=1}^{\infty} \pi^{-i}D$, it is enough to show that $\pi^{-i}\in N$ for each positive integer i. Choose $n\in N-\pi^{-(i+k)}V$. We write n as $\pi^{-s}u$, where u is a unit of V and $s>i+k$. Then $\pi^{s-i}\in C$ and $\pi^{s-i}u^{-1}n=\pi^{-i}\in Dn\subseteq \subseteq N$. This established Theorem 3.6.

Considerations similar to those in the proof of Theorem 3.6 and in the paragraph preceding that result enable us to determine to within isomorphism the class $\mathscr{C}(J)$ of all non-finitely generated Jónsson ω_0-modules over a Prüfer domain J. In order for $\mathscr{C}(J)$ to be nomepty, we know from Theorem 3.1 that it is necessary that there should exist a maximal ideal M of J such that J/M is finite and the powers of M properly descend. Assume that J has such a maximal ideal and let $\{M_i\}_{i\in I}$ be the family of all such maximal ideals of J. Since J is a Prüfer domain, $P_i=\bigcap_{k=1}^{\infty} M_i^k$ is prime in J and there is no prime of J properly between P_i and M_i [7, Chap. 23]. Moreover, $V_i=(J/P_i)_{(M_i/P_i)}\cong J_{M_i}/P_iJ_{M_i}$ is a rank-one valuation ring with residue field J/M_i, and to within isomorphism. $\mathscr{C}(J)=\bigcup_{i\in I}\mathscr{C}(V_i)$. According to the next result, Proposition 3.7, the unique faithful, non-finitely generated Jónsson ω_0-module over V_i is K_i/V_i, where K_i is the quotient field of V_i, and this in turn yields a determination of $\mathscr{C}(J)$.

Proposition 3.7. *Let V be a rank-one discrete valuation ring with quotient field K and with finite residue field V/P. To within isomorphism, K/V is the unique faithful, non-finitely generated Jónsson ω_0-module over V.*

Proof. Let M be a non-finitely generated faithful Jónsson ω_0-module over V and assume that p generates P. According to Proposition 3.2, M can be expressed as $\bigcup_{i=1}^{\infty} Vx_i$, where $x_i\neq 0$, $px_1=0$, and $px_{i+1}=x_i$ for each i. Noting that the set $\{p^{-i}+V\}_{i+1}^{\infty}$ generates K/V, it is then routine to verify that the mapping $p^{-i}+V\to x_i$ can be extended to a V-module isomorphism of K/V onto M.

Assume that (D, P) is a quasi-local domain that admits a non-finitely generated faithful Jónsson ω_0-module. From Theorem 3.1 and Proposition 3.2, it follows that D/P is finite, that $\bigcap_{i=1}^{\infty} P^i=(0)$, and that (0) can be expressed as the intersection of a strictly decreasing sequence $\{Q_i\}_{i=1}^{\infty}$ of P-primary ideals such that each D/Q_i is finite. Based on considerations up to this point, it seems reasonable to ask if D must be one-dimensional, or Noetherian, or if the residue class rings D/P^i are finite. We present in Section 4 examples that show that each of these questions has a negative answer; moreover, if D is one-dimensional, then D need not be Noetherian, and conversely.

4. Examples. The examples in this section indicate certain limitations on what can be said about the structure of a quasi-local domain (D, P) such that D admits a non-finitely generated faithful Jónsson ω_0-module. In particular, the class of examples included in Example 4.1 is large enough to show that D need not be Noetherian, and that no restriction on the dimension of D is possible.

Example 4.1. *Assume that $(V, M(V))$ and $(W, M(W))$ are independent valuation rings on a field K, that V is rank-one discrete, and that there exists a finite field so that $V=k+M(V)$ and $W=k+M(W)$. Set $D=k+P$, where $P=M(V)\cap M(W)$. Then (D, P) is quasi-local,* $\dim D=\dim W$, *and W/D is a non-finitely generated faithful Jónsson ω_0-module over D.*

Proof. Corollary 5.6 of [8] shows that (D, P) is quasi-local and $\dim D=\dim W$. Let v be a valuation associated with V and choose, by the approximation theorem for independent valuations [7, (22.9)], an element $x\in W-V$ so that $v(x)=-1$. If $d\in D-\{0\}$ and if $v(d)=r\geqq 0$, then $dx^{r+1}\notin D$, so W/D is a faithful D-module. To prove that W/D is a non-finitely generated Jónsson ω_0-module, we show that the sequence $\{(D+Dx^i)/D\}_{i=1}^{\infty}$ of submodules of W/D satisfies the hypothesis and condition (2) of Proposition 3.3. To do so, we prove first the following assertion.

(*) *If $r\in W$, if $s\in W-V$, and if $v(s)<v(r)$, then $r\in D+Ds$.*

To prove (*), consider first the case where s is a unit and r is a nonunit of W. Then $r/s\in M(W)$, and since $v(r/s)>0$, then $r/s\in M(V)$ as well. Hence $r\in Ds$ in this case. On the other hand, if s is a nonunit of W, then we can replace s by the unit $s_1=s+1$ without affecting the hypothesis or the conclusion since $s_1\in W-V$, $v(s)=v(s_1)$ and $D+Ds=D+Ds_1$. Similarly, if r is a unit of W, then $r_1=r-u\in M(W)$ for some nonzero element r of k, and replacing r by r_1 yields the desired conclusion. This establishes (*).

It follows from (*) that $W=\bigcup_{i=1}^{\infty}(D+Dx^i)$ and that $D+Dx^i\subseteqq D+Dx^{i+1}$. The minimum of the v-values of elements of $D+Dx^i$ is $-i$, so $x^{i+1}\notin D+Dx^i$ and the inclusion $D+Dx^i\subseteqq D+Dx^{i+1}$ is proper. Statement (*) also implies that if N is a proper D-submodule of W containing D, then the set of v-values of elements of N is bounded below, and hence $N \subseteqq D+Dx^i$ for some i. Thus, to complete the proof of Example 4.1, we need only show that $(D+Dx^i)/D$ is finite for each i. It is clear, however, that $M(W)\cap(M(V))^i$ is contained in the annihilator of $(D+Dx^i)/D$. As $|V/(M(V))^i|=|k|^i$ is finite, the subring $D/[M(W)\cap(M(V))^i]$ is also finite. Since $(D+Dx^i)/D$ is a finitely generated D-module, we conclude that $(D+Dx^i)/D$ is finite.

If k is a finite field and $\{X_i\}_{i=1}^{\infty}$ is a set of indeterminates over k, then the field $K=k(\{X_i\}_1^{\infty})$ admits independent valuations v, w such that v is rank-one discrete, the valuation ring V of v is of the form $k+M(V)$, and the valuation ring W of w is of the form $k+M(W)$. Example 4.1 shows that W/D, where $D=k+(M(V)\cap M(W))$, is a Jónsson ω_0-module, and $\dim D=\dim W$ can be any positive integer or it can be infinite. Moreover, if W is chosen so that $M(W)$ is unbranched [7, p. 189], then no principal ideal of D is primary for $M(V)\cap M(W)$. Thus the assumption that a quasi-local domain admits a faithful non-finitely generated Jónsson ω_0-module does not imply that the domain is Noetherian, and it imposes no restriction on its dimension. We remark that the approximation theorem for independent valuations can be avoided in the proof of Example 4.1 and that the conclusion concerning W/D remains valid for any quasi-local domain $W=k+M(W)$ with quotient field K such that $W \subsetneqq V$. Using this fact, we see that if W is rank-one nondiscrete, if $B\neq M(W)$ is any $M(W)$-primary ideal and if $J=k+(M(V)\cap B)$, then J admits the non-finitely generated faithful Jónsson ω_0-module $(k+B)/J$, and yet $J/(M(V)\cap B)^n$ is infinite for each $n>1$.

There is an analogue, for generating sets, of the concept of a Jónsson α-module. Namely, we say that a unitary module M over a commutative ring R with identity is a *Jónsson α-generated module* if M has a generating set of cardinality α, no generating set of smaller cardinality, and each proper submodule of M has a generating set of cardinality less than α. We have developed a theory of Jónsson α-generated modules in [11]. This theory contains many similarities, but also some differences, with the theory of Jónsson α-modules. The differences stem frequently from the fact that, by definition, a Jónsson α-generated module is not finitely generated, whereas a Jónsson α-module may be cyclic. In particular, a modification of the proof of [11, Example 3.3] establishes the following result.

Example 4.2. *Assume that D is an integral domain with quotient field K, that (W, M) is a rank-one discrete valuation ring on K containing D, and that $W/M=D/P$ is a finite field, where P is the center of W on D. Then K/W is a Jónsson ω_0-module over D.*

Example 4.2 can be used to show that even in the case of a Noetherian domain D, existence of a non-finitely generated faithful Jónsson ω_0-module over D imposes no restriction on the dimension of D. For example, let k be a finite field, let n be a positive integer, and choose $x_1, x_2, ..., x_n \in Yk[[Y]]$ such that $\{x_i\}_{i=1}^n$ is algebraically independent over k. Then $D=k[x_1, ..., x_n]_{(x_1, ..., x_n)}$ is an n-dimensional regular local ring and $W=k[[Y]]\cap k(x_1, ..., x_n)$ is a rank-one discrete valuation overring of D such that D and W have residue field k. By Example 4.2, $k(x_1, ..., x_n)/W$ is a faithful Jónsson ω_0-module over D.

We remark that, in general, a Noetherian ring R admits a non-finitely generated Jónsson ω_0-module if and only if R contains a maximal ideal M of positive height such that the residue field R/M is finite. This result follows from Theorem 2.7 of [11].

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Simple semimodules over commutative semirings

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The problem of describing all simple medial groupoids (and so all minimal varieties of medial groupoids) is still open, although simple groupoids and minimal varieties are described in various special subclasses (see e.g. [2], [3], [4], [5]; in a yet unpublished paper the authors described all finite simple medial groupoids and all simple commutative medial groupoids). It turns out that for the solution of this problem it is advantageous to have a description of all simple commutative semigroups with two commuting endomorphisms at hand. Now, commutative semigroups with a family of commuting endomorphisms are actually nothing else than semimodules over commutative semirings. For this reason the authors became interested in simple semimodules over commutative semirings. Moreover, the problem of simple semimodules deserves a special attention, and this is why the present paper came to life.

Section 1 contains the basic definitions. In Section 2 we prove that the class of simple semimodules over a commutative semiring can be divided into three subclasses:

(1) two-element semimodules with zero addition;

(2) simple cancellative semimodules;

(3) simple idempotent semimodules.

In Section 3 we describe the two-element semimodules with zero addition and in Section 4 the simple cancellative semimodules (at least in the case when the commutative semiring is finitely generated or, more generally, finitely c-generated). We do not know all simple idempotent semimodules. However, in Section 5 we characterize all simple idempotent semimodules with a zero element o such that $\{o\}$ is a subsemimodule; in particular, all finite simple idempotent semimodules are found. Further, we repeat from [6] the description of simple idempotent semimodules over a commutative semiring with at most two generators. Finally, in Section 6 we give a formula for the number of isomorphism classes of m-element semimodules over the free commutative semiring with n generators (n, m are finite).

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1. Preliminaries

By a commutative semiring we mean an algebra $R=R(+,\ \cdot)$ with two binary operations such that $R(+)$ and $R(\cdot)$ are commutative semigroups and $x(y+z)=$ $=xy+xz$ for all $x, y, z\in R$. Throughout this paper let R be a commutative semiring.

By a (left $R-$) semimodule we mean an algebra $M=M(+, rx)$ with one binary operation $+$ and a family of unary operations $x\mapsto rx$ $(r\in R)$ such that $M(+)$ is a commutative semigroup and

$$r(x+y)=rx+ry, \quad (r+s)x=rx+sx, \quad rs\cdot x=r\cdot sx$$

for all $x, y\in M$ and $r, s\in R$.

A semimodule M is said to be

— trivial if $\mathrm{Card}\,(M)=1$,

— idempotent if it satisfies the identity $x+x=x$ (i.e., if $M(+)$ is a semilattice; in this case we write $x\leqq y$ iff $x=x+y$),

— a semimodule with zero addition if it satisfies the identity $x+y=u+v$,

— cancellative if $x+y=x+z$ implies $y=z$,

— a module if $M(+)$ is a group,

— simple if id_M and $M\times M$ are the only congruences of M.

The semiring R is considered to be also a semimodule over itself. In this case, the subsemimodules of R are called ideals of R.

By a bi-ideal of a semimodule M we mean a non-empty subset I of M such that $M+I\subseteq I$ and $RI\subseteq I$. The equivalence $(I\times I)\cup\mathrm{id}_M$ is then a congruence of M and we denote by M/I the corresponding factor semimodule. If M is simple, then every bi-ideal of M is either at most one-element or equal M.

An element a of a semimodule M is said to be the neutral element (the zero element, resp.) of M if $x+a=x$ ($x+a=a$, resp.) for all $x\in M$. The neutral element is usually denoted by 0 and the zero element by o.

For some results on semimodules with a neutral element over a commutative semiring with a neutral and a unit element see, e.g., [1].

For a semimodule M, put $\mathrm{Ann}\,(M)=\{r\in R;\ rx=ry$ for all $x, y\in M\}$. If $\mathrm{Ann}\,(M)$ is non-empty, then this set is evidently an ideal of R and there exists an element $e\in M$ such that $e=e+e=re=sx$ for all $r\in R$, $s\in\mathrm{Ann}\,(M)$ and $x\in M$; the set $\{e\}$ is a subsemimodule of M.

1.1. Lemma. *Let M be a simple semimodule with $\mathrm{Ann}\,(M)\neq\emptyset$. Then the element e with $sx=e$ for all $s\in\mathrm{Ann}\,(M)$ is either a neutral or a zero element of M.*

Proof. The set $\{e+x;\ x\in M\}$ is a bi-ideal of M containing e, so that it equals either $\{e\}$ or M. In the first case evidently e is a zero element. If $\{e+x;\ x\in M\}=M$ then it is easy to verify that e is a neutral element.

A subsemiring S of R is said to be a closed subsemiring if $b \in S$ whenever $a+b \in S$ for some $a \in S$. Let K be a non-empty subset of R. We shall say that R is c-generated by K if R is the only closed subsemiring of R containing K.

For every non-empty set X there exists the free commutative semiring over X; its elements are the formal sums of elements of the free commutative (multiplicatively denoted) semigroup over X. If R is a free semiring over a set X of cardinality $k \geqq 1$, then the variety of R-semimodules is equivalent to the variety of algebras $A(+, f_1, \ldots, \ldots, f_k)$ such that $A(+)$ is a commutative semigroup and $f_1, \ldots, f_k$ are pairwise commuting endomorphisms of $A(+)$.

Let f be a homomorphism of a semiring S onto a semiring R. Then for any R-semimodule M we can define an S-semimodule structure on M by $sx = f(s) \cdot x$ (for all $s \in S$ and $x \in M$). This correspondence provides an equivalence between the variety of R-semimodules and some subvariety of the variety of S-semimodules. Since every semiring is a homomorphic image of some free semiring, it follows that in order to describe all simple semimodules over arbitrary (commutative) semirings it would suffice to describe all simple semimodules over free (commutative) semirings.

2. The fundamental classification theorem

2.1. Theorem. *Let M be a non-trivial simple semimodule over R. Then exactly one of the following conditions holds:*

(1) *M is a two-element semimodule with zero addition;*

(2) *M is cancellative;*

(3) *M is idempotent.*

Proof. If $\mathrm{Card}\,(M)=2$, then everything is clear. Now we shall assume that $\mathrm{Card}\,(M) \geqq 3$. The rest of the proof will be divided into several lemmas.

2.2. Lemma. *M is not a semimodule with zero addition.*

Proof. Suppose, on the contrary, that there exists an element o such that $x+y=o$ for all $x, y \in M$. We have $ro = r(o+o) = ro+ro = o$ for all $r \in R$. If $r \in R$, then $\mathrm{Ker}\,(L_r)$, where $L_r(x) = rx$ for all $x \in M$, is a congruence of M; since M is simple, it follows that either L_r is injective or $rx = o$ for all $x \in M$. From this it follows that $((M \setminus \{o\}) \times (M \setminus \{o\})) \cup \mathrm{id}_M$ is a congruence of M; since M is simple, $\mathrm{Card}\,(M) \leqq 2$, a contradiction.

A semimodule M is said to be unipotent if $x+x=y+y$ for all $x, y \in M$.

2.3. Lemma. *Suppose that M is unipotent; put $o=x+x$ for all $x \in M$. Then either M is cancellative or $x+o=o$ for all $x \in M$.*

Proof. Put $f(x)=x+x+x$ for all $x\in M$. Then f is an endomorphism of M and we have either $\operatorname{Ker}(f)=M\times M$ or $\operatorname{Ker}(f)=\mathrm{id}_M$. If $\operatorname{Ker}(f)=M\times M$ then $x+o=f(x)=f(o)=o$ for all $x\in M$. Let $\operatorname{Ker}(f)=\mathrm{id}_M$ and $a+c=b+c$ for some $x, b, c\in M$. Then $f(a)=a+o=a+c+c=b+c+c=b+o=f(b)$ and so $a=b$.

2.4. Lemma. *Suppose that M is unipotent. Then M is cancellative.*

Proof. Suppose, on the contrary, that M is not cancellative. Put $o=x+x$ for all $x\in M$. By 2.3, $x+o=o$ for all $x\in M$.

Suppose that $a=b+c\neq o$ for some $a, b, c\in M$. Put $M^*=M\cup\{0\}$ and $I=\{x+a;\ x\in M^*\}\cup\{x+ra;\ x\in M^*,\ r\in R\}$, where $0+a=a$. Then I is a bi-ideal of M containing $\{a, o\}$ and so $I=M$. In particular, $b\in I$ and $c\in I$. We shall consider only the case when $b=x+ra$ and $c=y+sa$ for some $x, y\in M^*$ and $r, s\in R$. (The remaining three cases are similar.) Then $a=b+c=z+ra+sa$ where $z=x+y\in M^*$ and therefore $a=z+r(z+ra+sa)+s(z+ra+sa)=z+rz+sz+r^2a+s^2a+rsa+sra=z+rz+sz+r^2a+s^2a+o=o$, a contradiction.

We have proved that M is a semimodule with zero addition. However, this is in contradiction with 2.2.

2.5. Lemma. *Suppose that M is not unipotent. Then M is either idempotent or cancellative.*

Proof. Put $g(x)=x+x$ for all $x\in M$. Then g is an endomorphism of M; since M is simple and not unipotent, g is injective. From this it follows that M can be embedded into a simple semimodule M' in which the mapping $x\mapsto x+x$ is an automorphism; since subsemimodules of idempotent semimodules are idempotent and subsemimodules of cancellative semimodules are cancellative, it is enough to proceed under the assumption that g is an automorphism of M. Put $M^*=M\cup\{0\}$ and define a binary relation H on M by $(x, y)\in H$ iff $x=u+g^i(y)$ and $y=v+g^j(x)$ for some $u, v\in M^*$ and some integers $i, j\leqq 0$ (if $j<i$ then $x=u+g^i(y)=u+g^{i-1}(y)+g^{i-1}(y)=z_1+g^{i-1}(y)=\ldots=z_{i-j}+g^j(y)$; similarly if $i<j$, and thus we can assume that $i=j$). Obviously, H is an equivalence. Let $x, y, z\in M$, $u, v\in M^*$, $k\leqq 0$, $x=u+g^k(y)$, $y=v+g^k(x)$. Then $z=g^{-k}g^k(z)=w+g^k(z)$ for some $w\in M^*$ and we have $x+z=u+w+g^k(y+z)$ and $y+z=v+w+g^k(x+z)$. Moreover, $rx=rx+g^k(ry)$ and $ry=rv+g^k(rx)$. We have shown that H is a congruence of M.

If $H=\mathrm{id}_M$ then M is idempotent, since $g(x)=x+g^0(x)$ and $x=0+g^{-1}(g(x))$ imply $(x, g(x))\in H$ for all $x\in M$.

Let $H\neq\mathrm{id}_M$, so that $H=M\times M$. Let $a+c=b+c$ for some $a, b, c\in M$. Put $N=\{x\in M;\ a+x=b+x\}$. If $x\in N$, then $g(a+g^{-1}(x))=a+a+x=a+b+x=b+b+x=g(b+g^{-1}(x))$, so that $a+g^{-1}(x)=b+g^{-1}(x)$ and consequently

$g^{-1}(x)\in N$. Now, let $y\in M$. We have $c\in N$, $(c, y)\in H$ and so $y=z+g^k(c)$ for some $z\in M^*$ and $k\leqq 0$. But $g^k(c)\in N$ and so $a+y=a+g^k(c)+z=b+g^k(c)+z=b+y$, i.e., $y\in N$. We have proved $N=M$. In particular, $g(a)=a+a=a+b=b+b=g(b)$, $a=b$, and M is cancellative.

3. Two-element semimodules with zero addition

Denote by $\mathrm{IND}_1(R)$ the set of all subsets I of R with the following properties:

(1) $R+R\subseteqq I$;

(2) $RI\subseteqq I$;

(3) if $r, s\in R\setminus I$ then $rs\in R\setminus I$.

For every $I\in\mathrm{IND}_1(R)$ define a semimodule $Z_{R,I}$ as follows: $Z_{R,I}=\{0, 1\}$; $x+y=0$; if $r\in I$ then $rx=0$; if $r\in R\setminus I$ then $rx=x$.

3.1. Theorem. *The semimodules $Z_{R,I}$ with $I\in\mathrm{IND}_1(R)$ are pairwise non-isomorphic two-element semimodules with zero addition; every two-element semimodule with zero addition is isomorphic to one of them.*

Proof. Easy.

3.2. Proposition. *Let R be a free commutative semiring over a set K of cardinality $\alpha\geqq 1$. Then* $\mathrm{Card}\,(\mathrm{IND}_1(R))=2^\alpha$.

Proof. It is easy to verify that the mapping $I\mapsto I\cap K$ is a one-to-one mapping of $\mathrm{IND}_1(R)$ onto the set of all subsets of K.

It follows that if R is a commutative semiring which can be generated by a set of cardinality $\alpha\geqq 1$ then $1\leqq\mathrm{Card}\,(\mathrm{IND}_1(R))\leqq 2^\alpha$. If R contains a neutral element then $\mathrm{Card}\,(\mathrm{IND}_1(R))=1$.

4. Simple cancellative semimodules

4.1. Lemma. *Let M be a cancellative semimodule. Then there exists a unique (up to isomorphism over M) module N such that M is a subsemimodule of N and $N=\{a-b;\ a, b\in M\}$. Moreover, if M is simple then N is also simple.*

Proof. Define a binary relation H on $M\times M$ by $((a, b), (c, d))\in H$ iff $a+d=$ $=b+c$. Then H is a congruence of the semimodule $M\times M$. Put $N=(M\times M)/H$ and denote by g the corresponding natural homomorphism. We have $g(a, a)=$ $=g(b, b)=0$ for all $a, b\in M$ and 0 is a neutral element of N. Moreover, $g(a, b)+$ $+g(b, a)=0$ and we see that N is a module. The mapping $a\mapsto g(a+a, a)$ is an injective homomorphism of M into N and we can identify any element $a \in M$ with the element $g(a+a, a)$ of N. The rest is easy.

4.2. Lemma. *Let M be a module. Then M is simple iff $\{0\}$ and M are the only submodules of M.*

Proof. Easy.

4.3. Lemma. *Let M be a simple cancellative semimodule having a neutral element 0. Then M is a module.*

Proof. Denote by N the set of all $a \in M$ such that $a+b=0$ for some $b \in M$. Then N is a subsemimodule of M and the relation H on M, defined by $(x, y) \in H$ iff $x+N=y+N$, is a congruence of $M(+)$; let us prove that it is a congruence of the semimodule M. For this, it is enough to show that if $x+N=y+N$, $r \in R$ and $a \in N$, then $rx+a \in ry+N$. We have $x+a=y+b$ and $ra+c=0$ for some $b, c \in N$; we have $rx+a=rx+ra+c+a=r(x+a)+c+a=r(y+b)+c+a=ry+rb+c+a \in ry+N$. It follows that H is a congruence of the semimodule M. Since M is simple, either $H=M \times M$ or $H=\mathrm{id}_M$. If $H=M+M$, then $N=M$, M is a module and we are through. Let $H=\mathrm{id}_M$, so that $N=\{0\}$. Put $K=((M \setminus \{0\}) \times (M \setminus \{0\})) \cup \mathrm{id}_M$. Let us prove that K is a congruence of M. Evidently, K is a congruence of $M(+)$. Let $x, y \in M \setminus \{0\}$ and $r \in R$. Since M is simple, the kernel of the endomorphism $x \mapsto rx$ equals either $M \times M$ or id_M; since $r0=0$, it follows that either $rz=0$ for all $z \in M$ or $x \mapsto rx$ is injective; from this it follows that $(rx, ry) \in K$. Since M is simple, it follows that $K=\mathrm{id}_M$ and M contains just two elements; thus M is a module.

4.4. Theorem (The description of simple modules).

(1) *Let f be a homomorphism of the semiring R into a field F such that $F=\{a-b+c \cdot 1;\ a, b \in f(R) \cup \{0\},\ c \in Z\}$ where Z denotes the set of integers. Then F is a simple R-module (if we put $rx=f(r)x$).*

(2) *Every non-trivial simple R-module can be constructed in the way described in* (1).

(3) *Let f and g be homomorphisms of R into fields F and G, resp., such that $F=\{a-b+c \cdot 1;\ a, b \in f(R) \cup \{0\},\ c \in Z\}$ and $G=\{a-b+c \cdot 1;\ a, b \in g(R) \cup \{0\},\ c \in Z\}$. Then the R-semimodules, F, G are isomorphic iff there is a field isomorphism h of F onto G such that $h(f(r))=g(r)$ for all $r \in R$.*

Proof. (1) Evidently, every submodule of the R-module F is an ideal of the field F and we can use 4.2.

(2) Let M be a non-trivial simple R-module. Denote by F the set of endomorphisms of M and define two binary operations on F by $(\varphi+\psi)(x)=\varphi(x)+\psi(x)$ and $(\varphi\psi)(x)=\varphi(\psi(x))$. Evidently, F is a skew field. For every $r \in R$ denote by $f(r)$ the endomorphism $x \mapsto rx$, so that f is a homomorphism of R into F. Let us fix an

element $u \in M \setminus \{0\}$. For every $x \in F$ put $g(x)=x(u)$. It follows from 4.2 that g is an isomorphism of the R-module F onto the R-module M. Put $S=\{a-b+c\cdot 1;\ a, b \in f(R) \cup \{0\},\ c \in Z\}$. Then S is a submodule of F and $g(S) \neq \{0\}$. Consequently $g(S)=M$ and $S=F$. Now it is clear that F is commutative.

(3) Let k be a semimodule isomorphism of F onto G. Put $h(x)=k(x)(k(1))^{-1}$ for all $x \in F$. Then h is a field isomorphism with the desired property.

4.5. Theorem. *Let M be a non-trivial simple cancellative semimodule. Then there exist a field F and a homomorphism f of R into F such that $F=\{a-b+c\cdot 1;\ a, b \in f(R) \cup \{0\},\ c \in Z\}$, where Z denotes the set of integers, M is a subsemimodule of the R-module F and $F=\{a-b;\ a, b \in M\}$. Moreover, $M=F$ if $0 \in M$.*

Proof. Apply 4.1, 4.3 and 4.4.

4.6. Example. Denote by Q the field of rational numbers. Put $R_1=\{x \in Q;\ x>0\}$ and $R_2=\{x \in Q;\ x \geqq 1\}$. Then R_1 and R_2 are commutative semirings. R_1 is a simple cancellative R_1-semimodule, $Q=\{a-b;\ a, b \in R_1\}$; R_2 is a cancellative R_2-semimodule, $Q=\{a-b;\ a, b \in R_2\}$, and R_2 is not simple.

4.7. Theorem. *Let R be finitely generated (or, more generally, finitely c-generated). Then every simple cancellative semimodule is a finite module of prime power order.*

Proof. Let f and F be as in 4.5. Since R is finitely c-generated, F is a finitely generated ring. However, then F is finite. Then evidently $0 \in M$ and $M=F$ by 4.5.

For every prime power p^n (i.e. every prime number p and every positive integer n) denote by $\mathrm{GF}(p^n)$ the finite field with p^n elements. For every prime power p^n and every positive integer m let $S(p, n, m)$ denote the set of ordered m-tuples $(a_1, \dots, a_m)$ of elements of $\mathrm{GF}(p^n)$ such that $GF(p^n)$ is generated as a ring by the set $\{a_1, \dots, a_m, 1\}$ (observe that this set is always non-empty). Define an equivalence $\sim$ on $S(p, n, m)$ by $(a_1, \dots, a_m) \sim (b_1, \dots, b_m)$ iff $b_1=f(a_1), \dots, b_m=f(a_m)$ for some automorphism f of $\mathrm{GF}(p^n)$.

4.8. Lemma. Card $(S(p, n, m)/\sim)=(1/n) \sum_{k|n} \mu(n/k)p^{mk}$, *$\mu$ being the Möbius function.*

Proof. Well known and easy.

4.9. Proposition. *Let R be a free commutative semiring freely generated by a finite set of cardinality $m \geqq 1$. Let p^n be a prime power. Then the number of isomorphism classes of simple modules of order p^n equals $(1/n) \sum_{k|n} \mu(n/k)p^{mk}$.*

Proof. It follows from 4.4 and 4.8.

5. Simple idempotent semimodules

Denote by $\mathrm{IND}_2(R)$ the set of all subsets I of R with the following properties:

(1) $R+I\subseteqq I$;

(2) $RI\subseteqq I$;

(3) if $r, s\in R\setminus I$ then $r+s\in R\setminus I$;

(4) if $r, s\in R\setminus I$ then $rs\in R\setminus I$.

For every $I\in\mathrm{IND}_2(R)$ define a semimodule $X_{R,I}$ as follows: $X_{R,I}=\{0, 1\}$; $0+0=0+1=1+0=0$; $1+1=1$; if $r\in I$ then $rx=0$; if $r\in R\setminus I$ then $rx=x$.

Denote by $\mathrm{IND}_3(R)$ the set of all non-empty subsets I of R with the following properties:

(1) $I+I\subseteqq I$;

(2) $RI\subseteqq I$;

(3) if $r, s\in R$ and $s\notin I$ then $r+s\notin I$;

(4) if $r, s\in R\setminus I$ then $rs\in R\setminus I$.

For every $I\in\mathrm{IND}_3(R)$ define a semimodule $Y_{R,I}$ as follows: $Y_{R,I}=\{0, 1\}$; $0+0=0+1=1+0=0$; $1+1=1$; if $r\in I$ then $rx=1$; if $r\in R\setminus I$ then $rx=x$.

5.1. Theorem. *The semimodules $X_{R,I}$ with $I\in\mathrm{IND}_2(R)$ and the semimodules $Y_{R,I}$ with $I\in\mathrm{IND}_3(R)$ are pairwise non-isomorphic two-element idempotent semimodules; every two-element idempotent semimodule is isomorphic to one of them.*

Proof. Straightforward.

5.2. Proposition. *Let R be a free commutative semiring over a set K of cardinality $\alpha\geqq 1$. Then* $\mathrm{Card}\,(\mathrm{IND}_2(R))=2^\alpha$ *and* $\mathrm{Card}\,(\mathrm{IND}_3(R))=2^\alpha-1$.

Proof. Easy.

5.3. Theorem. *Let M be an idempotent semimodule with a zero element o such that $\{o\}$ is a subsemimodule of M; let* $\mathrm{Card}\,(M)\geqq 3$. *Then M is simple iff the following three conditions are satisfied:*

(1) *$a+b=0$ for all pairs $a, b\in M$ such that $a\neq b$;*

(2) *for every $r\in R$, the mapping $x\mapsto rx$ is either constant (with value o) or a permutation of M;*

(3) *if $x, y\in M\setminus\{o\}$ then $y=rx$ for some $r\in R$.*

Proof. First, let M be simple. For every $a\in M$ denote by K_a the set of all $x\in M$ such that $x\leqq ra$ (i.e. $x=x+ra$) for some $r\in R$. Evidently, K_a is a bi-ideal of M containing o, and so either $K_a=\{o\}$ or $K_a=M$. Put $L=\{a\in M;\ K_a=\{o\}\}$. Evidently, L is a bi-ideal of M, and so either $L=M$ or L contains at most one element.

If $L=M$ then M is a semimodule with zero multiplication; since M is simple, $\mathrm{Card}(M)\leqq 2$, a contradiction. Hence $\mathrm{Card}(L)\leqq 1$. Then evidently $L\subseteq\{o\}$ and so we have proved that if $a\in M\setminus\{o\}$ and $x\in M$ then $x\leqq ra$ for some $r\in R$.

Let $a, b, c\in M$ be such that $a+b\neq o$ and $b+c\neq o$. Then $b\neq o$ and, as we have just proved, there are elements $r, s\in R$ with $b\leqq r(a+b)$ and $b\leqq s(b+c)$. We have $b\leqq ra+rb\leqq rb\leqq rsb+rsc$ and $b\leqq sb+sc\leqq sb\leqq sra+srb$. Consequently, $b\leqq rs(a+b+c)$ and so $a+b+c\neq o$.

Define a relation H on M by $(x, y)\in H$ iff either $x=y$ or $x+y\neq 0$. Using the assertion proved above, it is easy to check that H is a congruence of M. Hence either $H=\mathrm{id}_M$ or $H=M\times M$. We get $H=\mathrm{id}_M$, and (1) is proved.

Let $r\in R$. The mapping $x\mapsto rx$ is an endomorphism of M, so that its kernel equals either id_M or $M\times M$. Hence the mapping $x\mapsto rx$ is either constant (with value o, since $ro=o$) or injective; if it is injective, then it is a permutation of M, since rM is evidently a bi-ideal of M. We have proved (2) and the assertion (3) is similar.

Now, let the conditions (1), (2), (3) be satisfied. Consider a congruence $H\neq\mathrm{id}_M$ of M. Put $L=\{x\in M\setminus\{o\};\ (x, o)\in H\}$. There is a pair $(a, b)\in H$ with $a\neq b$. We have $a+b=o$ and $(a, o)\in H$, $(b, o)\in H$. Hence L is non-empty. It follows from (3) that $L=M\setminus\{o\}$, so that $H=M\times M$.

5.4. Theorem. *Let M be a finite simple idempotent semimodule containing at least three elements. Then M contains a zero element o and $\{o\}$ is a subsemimodule of M (so that M is as in 5.3).*

Proof. Since M is a finite semilattice, it contains a zero element o. Suppose that $\mathrm{Ann}(M)\neq\emptyset$ and the element e with $sx=e$ for all $s\in\mathrm{Ann}(M)$ is a neutral element of M. Then evidently $M\setminus\{e\}$ is a bi-ideal of M, so that it contains at most one element, contradicting $\mathrm{Card}(M)\geqq 3$.

Hence e is either a zero element or $\mathrm{Ann}(M)$ is empty; in both these cases evidently $\{o\}$ is a subsemimodule.

In the rest of this section let R be the free commutative semiring over a set $\{f, g\}$ of cardinality 2. We shall give a list of all simple idempotent R-semimodules in this case. Denote by Z the set of integers and by E the set of real numbers. For every positive integer n denote by $Z_n(+)$ the cyclic group of integers modulo n, and ι_n the natural homomorphism of $Z(+)$ onto $Z_n(+)$. For every pair r, s of integers such that $(r, s)\neq(0, 0)$ denote by $\mathrm{GCD}(r, s)$ the greatest common divisor of r, s. The promised list is the following (denote here by $\wedge$ the binary semimodule operation):

(1) the semimodule U_1 with $U_1=\{0, 1\}$, $0\wedge 1=0$, $f(x)=x$, $g(x)=1$;

(2) the semimodule U_2 with $U_2=\{0, 1\}$, $0\wedge 1=0$, $f(x)=1$, $g(x)=x$;

(3) the semimodule U_3 with $U_3=\{0, 1\}$, $0\wedge 1=0$, $f(x)=g(x)=0$;

(4) the semimodule U_4 with $U_4=\{0, 1\}$, $0\wedge 1=0$, $f(x)=g(x)=1$;

(5) for any positive integer n, the semimodule A_n with $A_n=\{0, 1, \dots, n\}$, $x\wedge y=x$ if $x=y$, $x\wedge y=0$ if $x\neq y$, $f(0)=0, f(1)=2, f(2)=3, \dots, f(n-1)=n\, f(n)=1$, $g(x)=0$;

(6) for any positive integer n, the semimodule B_n with $B_n=\{0, 1, \dots, n\}$, $x\wedge y=x$ if $x=y$, $x\wedge y=0$ if $x\neq y$, $f(x)=0$, $g(0)=0$, $g(1)=2$, $\dots$, $g(n-1)=n$, $g(n)=1$;

(7) for every quadruple $z=(p, q, r, s)$ of integers such that $p, q, r\geqq 1$, $0\leqq s<r$ and $\mathrm{GCD}(r, s)=1$, the semimodule C_z with $C_z=\{0\}\cup\{Z_{rp}\times Z_{rq})/K_z$ where K_z is the subgroup $\{(\iota_{rp}(0), \iota_{rq}(0)), (\iota_{rp}(p), \iota_{rq}(-sq)), (\iota_{rp}(2p), \iota_{rq}(-2sq)), \dots, (\iota_{rp}((r-1)p, \iota_{rq}(-(r-1)sq))\}$, $x\wedge y=x$ if $x=y$, $x\wedge y=0$ if $x\neq y$, $f(0)=g(0)=0$, $f(H)=H+(\iota_{rp}(1), \iota_{rq}(0))$ and $g(H)=H+(\iota_{rp}(0), \iota_{rq}(1))$ for all $H\in(Z_{rp}\times Z_{rq})/K_z$;

(8) for every pair $z=(n, m)$ of positive integers, the semimodule D_z with $D_z=\{0\}\cup(Z\times Z)/K_z$ where K_z is the subgroup of $Z(+)\times Z(+)$ generated by (n, m), $x\wedge y=x$ if $x=y$, $x\wedge y=0$ if $x\neq y$, $f(0)=g(0)=0, f(H)=H+(1, 0)$ and $g(H)=H+(0, 1)$ for all $H\in(Z\times Z)/K_z$;

(9) for every pair r, s of integers such that $\mathrm{GCD}(r, s)=1$ and either $r<0<s$ or $s<0<r$, the semimodule $E_{r,s}$ with $E_{r,s}=Z$, $x\wedge y=\mathrm{Min}(x, y)$, $f(x)=x+r$, $g(x)=x+s$;

(10) for every $u\in\{-1, 1\}$ and every irrational number q such that either $u<0<q$ or $q<0<u$, the semimodule $F_{u,q}$ with $F_{u,q}=E$, $x\wedge y=\mathrm{Min}(x, y), f(x)=x+u$, $g(x)=x+q$;

(11) for every u, q as in (10), every subsemimodule of $F_{u,q}$.

As it is proved in [6], these R-semimodules, together with the trivial R-semimodule, are simple idempotent R-semimodules and every simple idempotent R-semimodule is isomorphic to one of them; the semimodules in (1)—(11) are pairwise non-isomorphic, with the following exception: if M_1 is a subsemimodule of F_{u_1,q_1} and M_2 is a subsemimodule of F_{u_2,q_2}, then $M_1\cong M_2$ iff $u_1=u_2$, $q_1=q_2$ and $M_2=M_1+a$ for some real number a.

6. The number of isomorphism classes of finite simple semimodules

Let R be the free commutative semiring over a set of finite cardinality $n\geqq 1$. For $m\geqq 1$, let $N(n, m)$ denote the number of isomorphism classes of simple R-semimodules having m elements.

Denote by $\alpha(n, k)$ the number of equivalences defined on an n-element set and having exactly k blocks. Denote by $\lambda(n, m)$ the number of isomorphism classes of m-element algebras $A(f_1, \dots f_n)$ with unary operations f_i such that each f_i is a permutation of A, $f_if_j=f_jf_i$ for all i, j and $f_i(x)\neq f_j(x)$ for all $i, j\in\{1, \dots, n\}$, $i\neq j$, $x\in A$, and such that $A(f_1, \dots, f_n)$ contains no proper subalgebra.

The following theorem can be derived from the above results.

6.1. Theorem. (1) $N(n, 1)=1$ *for every* $n\geqq 1$.

(2) $N(n, 2)=2^{n+2}-1$ *for every* $n\geqq 1$.

(3) $N(n, 3)=2\cdot 3^n-2^n$ *for every* $n\geqq 1$.

(4) $$N(n, m) = \sum_{\substack{1\leqq k\leqq n\\ k+1\leqq m}} \alpha(n, k)\,\lambda(k, m-1)+\sum_{\substack{2\leqq t\leqq n\\ t\leqq m}} t\alpha(n, t)\,\lambda(t-1, m-1)$$

for every $n\geqq 1$ *and* $m\geqq 6$ *such that* m *is not a prime power.*

(5) $$N(n, p^m) = \sum_{\substack{1\leqq k\leqq n\\ k+1\leqq p^m}} \alpha(n, k)\,\lambda(k, p^m-1)+\sum_{\substack{2\leqq t\leqq n\\ t\leqq p^m}} t\alpha(n, t)\,\lambda(t-1, p^m-1)+ + (1/m)\sum_{k|m}\mu(m/k)\,p^{nk}$$

for every prime number $p\geqq 2$ *and all integers* $n, m\geqq 1$ *such that* $p^m\geqq 3$.

The values $\lambda(1, m)$ and $\lambda(2, m)$ can be computed as follows:

$$\lambda(1, m) = 1 \quad \text{for every } m \geqq 1;$$

$$\lambda(2, m) = -1+\sum_{\substack{1\leqq k\leqq m\\ k|m}} \varphi(k)\,\varepsilon(m/k) \quad \text{for every } m \geqq 1,$$

where φ denotes Euler's function and $\varepsilon(n)$ is the number of all $i\in\{1, \dots, m\}$ such that i divides n.

As it follows from the results and remarks of this paper, every simple semimodule over a commutative semiring with at most two generators is of cardinality $\leqq 2^{\aleph_0}$. We shall end this paper with the following open problem.

Problem. Let R be a finitely generated (or countable) commutative semiring and let M be a simple R-semimodule. Is it true that $\operatorname{Card}(M)\leqq 2^{\aleph_0}$?

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Mal'cev conditions for regular and weakly regular subalgebras of the square

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1. Introduction. At the beginning of the seventies, Mal'cev conditions characterizing varieties of algebras with regular congruences were given by B. CSÁKÁNY [3], [4], G. GRÄTZER [7], and R. WILLE [16]. Recently, algebras with regular tolerances (=compatible symmetric and reflexive relations) were introduced and a Mal'cev condition for varieties of algebras with regular tolerances was derived by I. CHAJDA [2]. Since the concept of regularity can easily be extended for other sorts of compatible relations we have also varieties of algebras with regular compatible reflexive relations and varieties of algebras with regular quasiorders (=compatible transitive and reflexive relations). The aim of this paper is to show that all the above mentioned varieties form exactly two well-known classes of varieties. Moreover, Mal'cev conditions for these two classes of varieties simplify the Mal'cev characterizations presented in some former papers. In the second part of this paper, analogous results for weakly regular subalgebras of the square are derived.

2. Algebras with regular subalgebras of the square. Throughout this paper, the same symbol stands for an algebra and its base set. Let A be an algebra and let S be a subset of the square $A\times A$. We denote by

$R(S)$ the compatible reflexive relation on A generated by S;

$T(S)$ the tolerance on A generated by S;

$Q(S)$ the quasiorder on A generated by S; and

$[a]S$ the subset $\{x\in A;\ \langle a, x\rangle\in S\}$, where a is some element of A.

Notice that $[a]S$ is called a class of S. The rest of this section is formulated in terms of compatible reflexive relations only; for tolerances, quasiorders, and congruences the Definition and the Lemma below are modified in an evident way.

Definition. We say that an algebra A has regular compatible reflexive relations if any two compatible reflexive relations coincide whenever they have a class in com-

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mon. A variety V of algebras has regular compatible reflexive relations provided each algebra $A \in V$ has this property.

Lemma. *For any algebra A, the following conditions are equivalent:*

(a) *A has regular compatible reflexive relations;*

(b) *For every compatible reflexive relation Ψ on A, $\Psi = R(\{a\} \times [a]\Psi)$ holds for any element a of A;*

(c) *For every compatible reflexive relation Ψ on A and for each element a of A, $\Psi = R(\{a\} \times B)$ holds for some subset $B \subseteqq A$.*

Proof. (a)$\Rightarrow$(b). Apparently, for any compatible reflexive relation Ψ on A, $\{a\} \times [a]\Psi \subseteqq R(\{a\} \times [a]\Psi) \subseteqq \Psi$ hold and thus also $[a](\{a\} \times [a]\Psi) \subseteqq [a]R(\{a\} \times \times [a]\Psi) \subseteqq [a]\Psi$. However, $[a](\{a\} \times [a]\Psi) = [a]\Psi$, which implies $[a]R(\{a\} \times [a]\Psi) = = [a]\Psi$. By applying the hypothesis, the equality $\Psi = R(\{a\} \times [a]\Psi)$ follows.

(b)$\Rightarrow$(a). If Ψ and Φ are two compatible reflexive relations on A with the same class $[a]\Psi = [a]\Phi$ then $\Psi = R(\{a\} \times [a]\Psi) = R(\{a\} \times [a]\Phi) = \Phi$.

(b)$\Rightarrow$(c) is trivial.

(c)$\Rightarrow$(b). It is enough to verify the inclusion $\Psi \subseteqq R(\{a\} \times [a]\Psi)$. By hypothesis, $R(\{a\} \times B) = \Psi$ holds for some B and so we have $\{a\} \times B \subseteqq R(\{a\} \times B) = \Psi$. This yields $B \subseteqq [a]\Psi$ and the conclusion $\Psi = R(\{a\} \times B) \subseteqq R(\{a\} \times [a]\Psi)$ follows.

3. Varieties with regular subalgebras of the square. The main fact we will need about varieties with regular congruences is the following

Theorem 1 (B. Csákány [3]). *For any variety V, the following conditions are equivalent:*

(1) *V has regular congruences;*

(2) *There exist ternary polynomials $p_1, \ldots, p_n$ such that*

$$\big(z = p_i(x, y, z), \quad 1 \leqq i \leqq n\big) \Leftrightarrow x = y.$$

In [5] we announced

Theorem 2. *For any variety V, the following conditions are equivalent:*

(1) *V has regular and permutable congruences;*

(2) *V has regular tolerances;*

(3) *V has regular compatible reflexive relations;*

(4) *There exist ternary polynomials $p_1, \ldots, p_n$ and an $(n+3)$-ary polynomial r such that*

$$x = r(x, y, z, z, \ldots, z), \quad y = r\big(x, y, z, p_1(x, y, z), \ldots, p_n(x, y, z)\big),$$
$$z = p_i(x, x, z) \quad \textit{for} \quad 1 \leqq i \leqq n.$$

Proof. (1)$\Rightarrow$(3) follows directly from the Theorem of H. Werner [11].

(3)⇒(4). Let $F_3(x, y, z)$ be the free algebra in V with free generators x, y, z. The compatible reflexive relation $R(x, y)$ on $F_3(x, y, z)$ is finitely generated, so by Lemma (c) from Section 2 there is a finite subset $\{p_i;\ 1\leqq i\leqq n\}\subseteq F_3(x, y, z)$ with the property $R(x, y)=R(\{z\}\times\{p_i;\ 1\leqq i\leqq n\})=R(\{\langle z, p_i\rangle;\ 1\leqq i\leqq n\})$. Now, condition $\langle x, y\rangle\in R(\{\langle z, p_i\rangle;\ 1\leqq i\leqq n\})$ yields

$$x = \varrho(z, \ldots, z) \quad\text{and}\quad y = \varrho(p_1, \ldots, p_n)$$

for some n-ary algebraic function ϱ over $F_3(x, y, z)$ and thus there are ternary polynomials $p_1, \ldots, p_n$ and an $(n+3)$-ary polynomial r such that

$$x = r(x, y, z, z, \ldots, z) \quad\text{and}\quad y = r\big(x, y, z, p_1(x, y, z), \ldots, p_n(x, y, z)\big).$$

Finally, the identities $z=p_i(x, x, z)$, $1\leqq i\leqq n$, follow immediately from the above equality $R(x, y)=R(\{\langle z, p_i\rangle;\ 1\leqq i\leqq n\})$.

(4)⇒(1). Regularity: Apparently, the ternary polynomials $p_1, \ldots, p_n$ satisfy condition (2) of Theorem 1, i.e., V has regular congruences.

Permutability: It is easily seen that $p(a, b, c):=r\big(c, a, b, p_1(b, a, b), \ldots, \ldots, p_n(b, a, b)\big)$ is the well-known Mal'cev polynomial and thus, by [10], the permutability of V follows.

(1)⇒(2) again by [15].

(2)⇒(1). Similarly as in the proof (3)⇒(4), the formula

$$T(x, y) = T(\{z\}\times\{p_i;\ 1 \leqq i \leqq n\}) \quad\text{for some}\quad \{p_i;\ 1 \leqq i \leqq n\} \subseteq F_3(x, y, z)$$

implies the existence of ternary polynomials $p_1, \ldots, p_n$ and of a $(2n+3)$-ary polynomial t with

$$x = t\big(x, y, z, z, \ldots, z, p_1(x, y, z), \ldots, p_n(x, y, z)\big),$$

$$y = t\big(x, y, z, p_1(x, y, z), \ldots, p_n(x, y, z), z, \ldots, z\big),$$

and

$$z = p_i(x, x, z) \quad\text{for}\quad 1 \leqq i \leqq n.$$

Now, the regularity of V is trivial since any congruence is a tolerance; the permutability of V is entailed by the Mal'cev polynomial

$$p(a, b, c) := t\big(c, a, b, p_1(b, a, b), \ldots, p_n(b, a, b), p_1(c, b, b), \ldots, p_n(c, b, b)\big).$$

In this way, varieties with regular tolerances and also varieties with regular compatible reflexive relations are sufficiently described. For varieties with regular congruences and for varieties with regular quasiorders, the following theorem holds.

Theorem 3. *For any variety V, the following conditions are equivalent:*

(1) *V has regular congruences;*

(2) *V has regular quasiorders;*

(3) *There exist ternary polynomials* $p_1, \ldots, p_n$ *and* $(n+3)$*-ary polynomials* $r_1, \ldots, r_k$ *such that*

$$x = r_1(x, y, z, z, \ldots, z),$$

$$r_j(x, y, z, p_1(x, y, z), \ldots, p_n(x, y, z)) = r_{j+1}(x, y, z, z, \ldots, z) \quad \textit{for} \quad 1 \leqq j < k,$$

$$y = r_k(x, y, z, p_1(x, y, z), \ldots, p_n(x, y, z)),$$

$$z = p_i(x, x, z) \quad \textit{for} \quad 1 \leqq i \leqq n.$$

Proof. (1)⇒(2). By Theorem 3 of J. HAGEMANN [9; p. 11], varieties with regular congruences are n-permutable for some $n>1$. Then, using Corollary 4 of J. HAGEMANN [9; p. 7], quasiorders coincide with congruences.

(2)⇒(3). The identities (3) are derived from the formula

$$Q(x, y) = Q(\{z\} \times \{p_i;\, 1 \leqq i \leqq n\}) \quad \text{for some} \quad \{p_i;\, 1 \leqq i \leqq n\} \subseteqq F_3(x, y, z)$$

in a similar way as above.

(3)⇒(1). Evidently, the polynomials $p_1, \ldots, p_n$ satisfy condition (2) of Theorem 1, i.e., V has regular congruences.

Remarks. (i) As it was already noted in [13], [14], congruence regularity and congruence permutability are independent conditions.

(ii) The Mal'cev condition from Theorem 2 simplifies the identities given in [1] and [2].

(iii) Part (3) of Theorem 3 is a slightly improved version of [3; p. 188].

4. Varieties with weakly regular subalgebras of the square. Let V be a variety having distinguished nullary operations $c_1, \ldots, c_m$. We say that V has weakly regular congruences with respect to $c_1, \ldots, c_m$ if $[c_i]\Theta = [c_i]\Psi$, $1 \leqq i \leqq m$, imply $\Theta = \Psi$ for any two congruences Θ and Ψ on $A \in V$. Analogously we introduce the concept of varieties with weakly regular tolerances, with weakly regular compatible reflexive relations, etc. This Section contains the variations on theorems of Section 3; the proofs are very similar to those of Section 3, so they can be omitted. For brevity we denote the sequences $c_i, \ldots, c_i$ (n times) and $q_{i1}(x, y), \ldots, q_{in}(x, y)$ by $\vec{c}_i$ and $\vec{q}_{ij}(x, y)$ respectively.

Weakly regular varieties were first investigated by K. FICHTNER; the following theorem is a paraphrase of his result [6; Theorem 1 (I), (IV)].

Theorem 4. *For any variety* V *with nullary operations* $c_1, \ldots, c_m$, *the following conditions are equivalent:*

(1) V *has weakly regular congruences with respect to* $c_1, \ldots, c_m$;

(2) *There exist an integer* $n \geqq 1$ *and binary polynomials* q_{ij}, $1 \leqq i \leqq m$, $1 \leqq j \leqq n$, *such that*

$$(\vec{c}_i = \vec{q}_{ij}(x, y),\ 1 \leqq i \leqq m) \Leftrightarrow x = y.$$

Example. The variety of implicative semilattices (see, e.g., [11], [12] for this concept) has weakly regular congruences with respect to the nullary operation 1: For $n=2$, $q_{11}(x, y)=x*y$, $q_{12}(x, y)=y*x$, we have $(1=x*y=y*x)\Leftrightarrow x=y$.

Theorem 5. *For any variety V with nullary operations $c_1, \ldots, c_m$, the following conditions are equivalent:*

(1) *V has permutable and weakly regular congruences with respect to $c_1, \ldots, c_m$;*

(2) *V has weakly regular tolerances with respect to $c_1, \ldots, c_m$;*

(3) *V has weakly regular compatible reflexive relations with respect to $c_1, \ldots, c_m$;*

(4) *There exist an integer $n\geqq 1$, binary polynomials q_{ij}, $1\leqq i\leqq m$, $1\leqq j\leqq n$, and an $(mn+2)$-ary polynomial w such that*

$$x = w(x, y, \vec{c}_1, \ldots, \vec{c}_m), \quad y = w\big(x, y, \vec{q}_{1j}(x, y), \ldots, \vec{q}_{mj}(x, y)\big),$$
$$\vec{c}_i = \vec{q}_{ij}(x, x) \quad \textit{for} \quad 1 \leqq i \leqq m.$$

Theorem 6. *For any variety V with nullary operations $c_1, \ldots, c_m$, the following conditions are equivalent:*

(1) *V has weakly regular congruences with respect to $c_1, \ldots, c_m$;*

(2) *V has weakly regular quasiorders with respect to $c_1, \ldots, c_m$;*

(3) *There exist integers $n, k\geqq 1$, binary polynomials q_{ij}, $1\leqq i\leqq m$, $1\leqq j\leqq n$, and $(mn+2)$-ary polynomials $w_1, \ldots, w_k$ such that*

$$x = w_1(x, y, \vec{c}_1, \ldots, \vec{c}_m),$$
$$w_h\big(x, y, \vec{q}_{1j}(x, y), \ldots, \vec{q}_{mj}(x, y)\big) = w_{h+1}(x, y, \vec{c}_1, \ldots, \vec{c}_m) \quad \textit{for} \quad 1 \leqq h < k,$$
$$y = w_k\big(x, y, \vec{q}_{1j}(x, y), \ldots, \vec{q}_{mj}(x, y)\big),$$
$$\vec{c}_i = \vec{q}_{ij}(x, x) \quad \textit{for} \quad 1 \leqq i \leqq m.$$

Remarks. (i) The implication (1)⇒(2) is again a direct consequence of Theorem 6 and Corollary 4 from [9].

(ii) Part (3) of our Theorem 6 improves the identities exhibited in [6; Theorem 2].

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Tolerance trivial algebras and varieties

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Tolerances on algebras and varieties were studied in many papers, see e.g. [1], [4], [5], [6] and numerous references there. An importance and suitability of tolerances in algebraic constructions mainly for congruence investigations was shown in [2], [3] and [10]. In particular, the paper [10] uses the concept of tolerance trivial algebra for characterizations of order polynomial completeness of ordered algebras. The aim of this paper is to give necessary and sufficient conditions under which (principal) tolerances and (principal) congruences on a given algebra coincide.

0. Preliminaries. Let $\mathfrak{A}=(A, F)$ be an algebra. A binary relation R on A, i.e., $R \subseteq A \times A$, has the *substitution property* (briefly SP) on $\mathfrak{A}$ if for each n-ary, $f \in F$ we have $\langle f(a_1, \ldots, a_n), f(b_1, \ldots, b_n)\rangle \in R$ whenever $\langle a_i, b_i\rangle \in R$ for $i=1, \ldots, n$. in other words, it is a subalgebra of the direct product $\mathfrak{A} \times \mathfrak{A}$.

A *tolerance* on an algebra $\mathfrak{A}=(A, F)$ is a reflexive and symmetric binary relation on A having SP (on $\mathfrak{A}$). Denote by $\mathrm{LT}(\mathfrak{A})$ the set of all tolerances on $\mathfrak{A}$. Clearly, $LT(\mathfrak{A})$ is a complete lattice with respect to set inclusion [4]. Denote by $\vee$ the join in $LT(\mathfrak{A})$. The meet evidently coincides with set intersection. Let $a, b \in A$. By $T(a, b)$ is denoted the least tolerance of $LT(\mathfrak{A})$ containing the pair $\langle a, b\rangle$. It is called a *principal tolerance (generated by $\langle a, b\rangle$).* The principal congruence generated by $\langle a, b\rangle$ will be denoted by $\Theta(a, b)$.

The following lemma is clear (see e.g. [4]):

Lemma 1. *For every algebra $\mathfrak{A}$ and each $T \in LT(\mathfrak{A})$,*

$$T = \vee \{T(a, b);\ \langle a, b\rangle \in T\}.$$

The next lemma is proved in [1] (see also [2]):

Lemma 2. *Let $\mathfrak{A}=(A, F)$ be an algebra and $a_i, b_i \in A$ for $i=1, \ldots, n$. Then*

$$\langle x, y\rangle \in \vee \{T(a_i, b_i);\ i = 1, \ldots, n\}$$

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if and only if there exists a 2n-ary algebraic function φ *such that*

$$x = \varphi(a_1, \ldots, a_n, b_1, \ldots, b_n), \quad y = \varphi(b_1, \ldots, b_n, a_1, \ldots, a_n).$$

As usual, Con $(\mathfrak{A})$ denotes the congruence lattice of $\mathfrak{A}$. Although every congruence is a tolerance, in general, Con $(\mathfrak{A})$ is not a sublattice of $LT(\mathfrak{A})$ (see Section 3 below).

1. Tolerance trivial varieties. *Definition.* An algebra $\mathfrak{A}$ is *(principal) tolerance trivial* if every (principal) tolerance on $\mathfrak{A}$ is a congruence. A variety $\mathcal{V}$ of algebras is *(principal) tolerance trivial* if each $\mathfrak{A}\in\mathcal{V}$ has this property.

H. Werner [11] proved that for each algebra $\mathfrak{A}$ in a variety $\mathcal{V}$ every reflexive binary relation having SP on $\mathfrak{A}$ is a congruence on $\mathfrak{A}$ if and only if the variety $\mathcal{V}$ is congruence permutable. Hence congruence permutable varieties are tolerance trivial. The following theorem shows that also the converse statement is valid:

Theorem 1. *For a variety* $\mathcal{V}$ *of algebras, the following conditions are equivalent:*

(1) $\mathcal{V}$ *is tolerance trivial;*

(2) $\mathcal{V}$ *is congruence permutable.*

Proof. Taking into account Werner's theorem [11] mentioned above, it remains to prove only (1)$\Rightarrow$(2). Let $\mathcal{V}$ be a variety of algebras and $\mathfrak{A}=F_3(x, y, z)$ the $\mathcal{V}$-free algebra with the set of free generators $\{x, y, z\}$. Since $\langle x, y\rangle\in T(x, y)$, $\langle y, z\rangle\in T(y, z)$ and, by (1), all tolerances are transitive, we obtain $\langle x, z\rangle\in T(x, y)\vee \vee T(y, z)$. By Lemma 2, there exists a 4-ary algebraic function φ over $\mathcal{V}$ such that $x=\varphi(x, y, y, z)$, $z=\varphi(y, x, z, y)$. Since $\mathfrak{A}=F_3(x, y, z)$, there exists a 7-ary polynomial p over $\mathcal{V}$ such that

$$\varphi(x_1, x_2, x_3, x_4) = p(x_1, x_2, x_3, x_4, x, y, z),$$

i.e. $x=p(x, y, y, z, x, y, z)$, $z=p(y, x, z, y, x, y, z)$. Evidently, $t(x, y, z)= =p(x, y, z, y, x, y, z)$ is the Mal'cev polynomial, i.e., $t(x, x, z)=t(z, x, x)=z$, whence $\mathcal{V}$ is congruence permutable.

2. Principal tolerance trivial algebras and varieties. Clearly, every tolerance trivial algebra is also principal tolerance trivial (but not vice versa). However, a characterization of principal tolerance trivial varieties and algebras is more complicated than that of tolerance trivial varieties.

Proposition 1. *If an algebra* $\mathfrak{A}=(A, F)$ *is principal tolerance trivial, then for each* $x, y\in A$ *there exist binary algebraic functions* ψ_1, ψ_2 *such that*

(1) $T(x, y)\supseteq T(\psi_1(x, y), \psi_2(x, y))$;

(2) *if* $\langle x, y\rangle\in\Theta(a, b)$, *then* $\psi_1(x, y)=\psi_1(\psi_1(a, b), \psi_2(a, b))$ *and* $\psi_2(x, y)=\psi_1(\psi_2(a, b), \psi_1(a, b))$.

Proof. If $\langle x, y\rangle \in \Theta(a, b)=T(a, b)$, by Lemma 2 there exists a binary algebraic function φ over $\mathfrak{A}$ such that $x=\varphi(a, b)$, $y=\varphi(b, a)$. Put $\psi_1(x_1, x_2)=\varphi(x_1, x_2)$, $\psi_2(x_1, x_2)=\varphi(x_2, x_1)$. Hence $x=\psi_1(a, b)$, $y=\psi_2(a, b)$ and

$$\psi_1(x, y) = \varphi(x, y) = \psi_1(\psi_1(a, b), \psi_2(a, b)),$$
$$\psi_2(x, y) = \varphi(y, x) = \psi_1(\psi_2(a, b), \quad \psi_1(a, b)),$$

proving (2). Moreover,

$$\langle \psi_1(x, y), \ \psi_2(x, y)\rangle = \langle \varphi(x, y), \ \varphi(y, x)\rangle \in T(x, y),$$

whence (1) is evident.

Now, we give a sufficient condition for principal tolerance triviality in a form closely connected with that of Proposition 1.

Proposition 2. *Let $\mathfrak{A}=(A, F)$ be an algebra such that there exist binary algebraic functions ψ_1, ψ_2 over $\mathfrak{A}$ with*

(1) $T(x, y)=T(\psi_1(x, y), \psi_2(x, y))$;

(2) *if $\langle x, y\rangle \in \Theta(a, b)$, then there exists a binary algebraic function φ over $\mathfrak{A}$ such that $\psi_1(x, y)=\varphi(\psi_1(a, b), \psi_2(a, b))$ and $\psi_2(x, y)=\varphi(\psi_2(a, b), \psi_1(a, b))$.*

Then $\mathfrak{A}$ is principal tolerance trivial.

Proof. Clearly $T(a, b)\subseteqq \Theta(a, b)$ for each $a, b\in A$. Prove the reverse inclusion. Let $\langle x, y\rangle \in \Theta(a, b)$. By (2) and (1), we obtain

$$\langle \psi_1(x, y), \psi_2(x, y)\rangle = \langle \varphi(\psi_1(a, b), \psi_2(a, b)), \varphi(\psi_2(a, b), \psi_1(a, b))\rangle \in$$
$$\in T(\psi_1(a, b), \psi_2(a, b)) = T(a, b),$$

hence, by (1), $T(x, y)=T(\psi_1(x, y), \psi_2(x, y))\subseteqq T(a, b)$, which implies $\langle x, y\rangle \in$ $\in T(a, b)$.

Corollary 1. *The variety of all distributive lattices is principal tolerance trivial but not tolerance trivial.*

Proof. By Theorem 1 (or by [6]), $\mathscr{V}$ is not tolerance trivial. We prove by Proposition 2 that $\mathscr{V}$ is principal tolerance trivial. Put $\psi_1(x, y)=x\wedge y$, $\psi_2(x, y)=x\vee y$. Let $T\in \mathrm{LT}(\mathfrak{A})$ for $\mathfrak{A}\in\mathscr{V}$. If $\langle x, y\rangle\in T$, then also $\langle x, x\vee y\rangle = \langle x\vee x, x\vee y\rangle \in T$ and, analogously, $\langle y, x\vee y\rangle\in T$. Hence

$$\langle x\wedge y, x\vee y\rangle = \langle x\wedge y, (x\vee y)\wedge(x\vee y)\rangle \in T.$$

Conversely, let $\langle x\wedge y, x\vee y\rangle\in T$. Then $\langle x, x\vee y\rangle = \langle x\vee(x\wedge y), x\vee(x\vee y)\rangle\in T$ and, similarly, $\langle y, x\vee y\rangle\in T$, i.e., $\langle x\vee y, y\rangle\in T$. Hence

$$\langle x, y\rangle = \langle x\wedge (x\vee y), (x\vee y)\wedge y\rangle \in T.$$

Accordingly, $T(x, y) = T(x \wedge y, x \vee y)$ is proved, i.e., ψ_1, ψ_2 satisfy (1) of Proposition 2.

By [8], $\langle x, y \rangle \in \Theta(a, b)$ if and only if

$$x \wedge y = [(a \wedge b) \vee (x \wedge y)] \wedge (x \vee y) \quad \text{and} \quad x \vee y = [(a \vee b) \vee (x \wedge y)] \wedge (x \vee y).$$

Putting $\varphi(x_1, x_2) = [x_1 \vee (x \wedge y)] \wedge (x \vee y)$ we have immediately (2) of Proposition 2 which finishes the proof.

Now, we give a characterization of principal tolerance trivial algebras based on a description of $\Theta(a, b)$ by Mal'cev's lemma (see [9]) and that of $T(a, b)$ by Lemma 2:

Theorem 2. *For an algebra $\mathfrak{A} = (A, F)$, the following conditions are equivalent:*

(1) *$\mathfrak{A}$ is principal tolerance trivial;*

(2) *for each $a, b \in A$ and for all unary algebraic functions $\tau_1, \ldots, \tau_n$ over $\mathfrak{A}$, if*

$$\{\tau_i(a), \tau_i(b)\} \cap \{\tau_{i+1}(a), \tau_{i+1}(b)\} \neq \emptyset \quad \textit{for} \quad i = 1, \ldots, n-1,$$

then there exists a binary algebraic function φ over $\mathfrak{A}$ such that $\tau_1(a) = \varphi(a, b)$, $\tau_n(b) = \varphi(b, a)$;

(3) *For each $a, b \in A$ and for all binary algebraic functions φ_1, φ_2 over $\mathfrak{A}$, if $\varphi_1(b, a) = \varphi_2(a, b)$, then there exists a binary algebraic function ψ over $\mathfrak{A}$ such that $\psi(a, b) = \varphi_1(a, b)$, $\psi(b, a) = \varphi_2(b, a)$.*

Proof. (2)$\Rightarrow$(1). Let $a, b \in A$, $\langle x, y \rangle \in \Theta(a, b)$. By Mal'cev's lemma (see [9] or [7]), there exist elements $e_0, \ldots, e_n \in A$ and unary algebraic functions (so called translations) $\tau_1, \ldots, \tau_n$ over $\mathfrak{A}$ such that $\{\tau_i(a), \tau_i(b)\} = \{e_{i-1}, e_i\}$ for $i = 1, \ldots, n$, and either $\{\tau_1(a), \tau_n(b)\} = \{x, y\}$ or $\{\tau_1(b), \tau_n(a)\} = \{x, y\}$. By (2), there exists a binary algebraic function φ over $\mathfrak{A}$ such that $x = \varphi(a, b)$, $y = \varphi(b, a)$, whence $\langle x, y \rangle \in T(a, b)$. The reverse inclusion in evident.

(3)$\Rightarrow$(1). Let $\langle x, y \rangle \in T(a, b)$, $\langle y, z \rangle \in T(a, b)$. By Lemma 2, there exist binary algebraic functions φ_1, φ_2 over $\mathfrak{A}$ such that $\langle x, y \rangle = \langle \varphi_1(a, b), \varphi_1(b, a) \rangle$, $\langle y, z \rangle = \langle \varphi_2(a, b), \varphi_2(b, a) \rangle$. By (3), $\langle x, z \rangle = \langle \psi(a, b), \psi(b, a) \rangle$, whence $\langle x, z \rangle \in T(a, b)$, proving the transitivity of $T(a, b)$, i.e., $T(a, b) = \Theta(a, b)$.

(1)$\Rightarrow$(3). If $\{\varphi_i(a, b), \varphi_i(b, a)\} = \{c_i, c_{i+1}\}$ for $i = 1, 2$, then, by Lemma 2, $\langle c_1, c_2 \rangle \in T(a, b)$, $\langle c_2, c_3 \rangle \in T(a, b)$. Since $T(a, b) = \Theta(a, b)$, also $\langle c_1, c_3 \rangle \in T(a, b)$ and (3) is an easy consequence of Lemma 2.

(1)$\Rightarrow$(2) is analogous to that of (1)$\Rightarrow$(3), only the Mal'cev's lemma is used instead of Lemma 2.

The situation can be essentially simplified for a variety having a *uniform restricted congruence scheme* (for the definition, see [7]) and such principal tolerance trivial varieties can be characterized by a Mal'cev condition:

Theorem 3. *Let $\mathscr{V}$ be a variety of algebras having a uniform restricted congruence scheme $\{p_0, \ldots, p_n; f\}$. The following conditions are equivalent:*

(1) *$\mathscr{V}$ is principal tolerance trivial;*

(2) *There exists a* 6-ary *polynomial q over $\mathscr{V}$ such that*

$$q(x_0, x_1, x_0, x_1, y_0, y_1) = p_0(x_{f(0)}, x_0, x_1, y_0, y_1),$$
$$q(x_1, x_0, x_0, x_1, y_0\, y_1) = p_n(x_{1-f(n)}, x_0, x_1, y_0, y_1).$$

Proof. (1)$\Rightarrow$(2). Let $\{p_0, \ldots, p_n; f\}$ be a restricted congruence scheme satisfied by $\mathscr{V}$. Let $\mathfrak{A}\in\mathscr{V}$ be a $\mathscr{V}$-free algebra generated by the four-element set of free generators $\{x_0, x_1, y_0, y_1\}$. Then $\langle y_0, y_1\rangle\in\Theta(x_0, x_1)$ if and only if (see [7])

$$y_0 = p_0(x_{f(0)}, x_0, x_1, y_0, y_1),$$
$$p_i(x_{1-f(i)}, x_0, x_1, y_0, y_1) = p_{i+1}(x_{f(i+1)}, x_0, x_1, y_0, y_1) \quad \text{for} \quad i = 0, \ldots, n-1,$$
$$y_1 = p_n(x_{1-f(n)}, x_0, x_1, y_0, y_1).$$

According to (1), $\langle y_0, y_1\rangle\in T(x_0, x_1)$, i.e., Lemma 2 yields the existence of a binary algebraic function φ over $\mathscr{V}$ such that $y_0=\varphi(x_0, x_1)$, $y_1=\varphi(x_1, x_0)$. Since $\mathfrak{A}$ is a $\mathscr{V}$-free algebra with generators x_0, x_1, y_0, y_1, there exists a 6-ary polynomial q with

$$\varphi(x, y) = g(x, y, x_0, x_1, y_0, y_1)$$

whence (2) is evident.

The converse implication (2)$\Rightarrow$(1) is a direct consequence of Lemma 2.

3. Tolerance lattices of principal tolerance trivial algebras. It is easy to characterize whether the congruence lattice is a sublattice of the tolerance lattice for a principal tolerance trivial algebra:

Theorem 4. *Let $\mathfrak{A}$ be a principal tolerance trivial algebra. The following conditions are equivalent:*

(1) $\mathrm{Con}(\mathfrak{A})$ *is a sublattice of* $LT(\mathfrak{A})$;

(2) $\mathfrak{A}$ *is tolerance trivial, i.e.,* $\mathrm{Con}(\mathfrak{A})=LT(\mathfrak{A})$.

Proof. (2)$\Rightarrow$(1) is trivial. To prove (1)$\Rightarrow$(2), let $T\in LT(\mathfrak{A})$. By Lemma 1, T is the join of the tolerances $\{T(a, b); \langle a, b\rangle\in T\}$. Since $\mathfrak{A}$ is principal tolerance trivial, T is the join of the congruences $\{\Theta(a, b); \langle a, b\rangle\in T\}$ in $LT(\mathfrak{A})$ and, by (1), also in $\mathrm{Con}(\mathfrak{A})$; thus $T\in\mathrm{Con}(\mathfrak{A})$, proving $\mathrm{Con}(\mathfrak{A})=LT(\mathfrak{A})$.

Corollary 2. *Let $\mathscr{V}$ be a principal tolerance trivial variety. The following conditions are equivalent:*

(1) *For each* $\mathfrak{A}\in\mathscr{V}$, $\mathrm{Con}(\mathfrak{A})$ *is a sublattice of* $LT(\mathfrak{A})$;

(2) *$\mathscr{V}$ is congruence permutable.*

This is a direct consequence of Theorems 1 and 4.

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On classes of ordered algebras and quasiorder distributivity

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0. Introduction. Many kinds of partially ordered algebras have appeared in the literature so far, for example partially ordered groups, rings, fields, etc. In some cases all the fundamental operations were supposed to be monotonic, but in some others there are operations having only special monotonity domains; moreover, some operations may be order reversing (or „antitone") with respect to a (may be the whole) part of their variables. (See FUCHS [5], [6].) There is no doubt that one gets the most general concept, if one imposes no assumption on the monotonity or antitonity domains of the operations. But then it seems to be hopeless to develop such an elegant (or at least approximately so elegant) theory, as the theory of varieties, equational logic, Mal'cev conditions, and so on. The circumstances for obtaining such results become far more advantageous if we require all operations to be monotonic in all of their variables. So we accept the following definition (the exact origin of which is not known for us):

Definition 0.1. By a *partially ordered algebra* (in the sequel simply *ordered algebra*) we mean a triple $\mathfrak{A}=(A; F, \leqq)$, where $(A; F)$ is a universal algebra, $\leqq$ is a partial ordering on A, and all the operations $f\in F$ are monotone with respect to this ordering. (If there is no danger of confusion, we shall simply say „f is monotone".)

Note that, according to this definition, partially ordered algebras are essentially the same as the algebras in the category of partially ordered sets (see FREYD [4], PAREIGIS [9]).

In our work we make an attempt to give a unified theory for these algebras, using such concepts as subalgebras, direct products, homomorphic images, subdirect decompositions, congruences, inequalities, Mal'cev conditions.

1. Basic concepts and facts. In this section we remind the reader of the concepts of homomorphisms, subalgebras, direct and subdirect products, and then we define two kinds of congruences.

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The operations on *subalgebras* and *direct products* are given as usual, the ordering is *restricted* to the subset in question and is understood *componentwise*. It would be possible to define subalgebras such that the ordering on them is obtained by weakening the restricted ordering, but we need not use such subalgebras, so we do not allow them. This agreement will seem to be natural after investigating varieties and their Birkhoff-type characterization, due to S. L. BLOOM [1].

By a *homomorphism* we mean a monotone, operation-preserving map from one algebra to another. A homomorphism $\varphi: \mathfrak{A} \to \mathfrak{B}$ is said to be a *Q-homomorphism,* if the ordering of $\mathfrak{B}$ restricted to $\mathrm{Im}\,\varphi$ cannot be weakened so that φ remains still monotone and the operations on $\mathrm{Im}\,\varphi$ remain still isotone („isotone" is used as a synonym for „monotone"). (It would be possible to describe Q-homomorphisms constructively, but since it is straightforward from the proofof Theorem 1.1 below, it will be omitted.)

$\mathfrak{B}$ is a *homomorphic image* (resp. *Q-homomorphic image*) of $\mathfrak{A}$, if there exists a surjective homomorphism (Q-homomorphism) $\varphi: \mathfrak{A} \to \mathfrak{B}$.

Definition 1.1. A binary relation Θ over A will be called an *order-congruence* of $\mathfrak{A}$, if the following hold:

(i) Θ is a congruence on $(A; F)$;

(ii) whenever for some natural numbers n, m and elements $a, b, a_1, \ldots, a_{n-1}, b_1, \ldots, b_{m-1} \in A$ we have

$$a\,\Theta\,a_1 \leqq a_2\,\Theta\,a_3 \leqq \ldots a_n = b\,\Theta\,b_1 \leqq b_2\,\Theta\,b_3 \leqq \ldots b_m = a,$$

we always have also $a\Theta b$. (The sequence of elements of this form is a *Θ-circle* with distinguished elements a, b.)

It is clear, that finitely many Θ-circles (with fixed distinguished elements) can always be unified so that they have common n and common m, moreover, n and m can be required to be equal.

For a homomorphism $\varphi: \mathfrak{A} \to \mathfrak{B}$ let $\mathrm{Ker}\,\varphi$ denote the kernel of φ, i.e. the relation $\{(a, b) \in A^2 \mid a\varphi = b\varphi\}$. The proof of the following theorem can also be found in [3], so here we give only the necessary construction. The theorem justifies Definition 1.1.

Theorem 1.1. *Θ is an order-congruence of $\mathfrak{A}$ iff $\Theta = \mathrm{Ker}\,\varphi$ for some homomorphism $\varphi: \mathfrak{A} \to \mathfrak{B}$ ($\mathfrak{B}$ is an ordered algebra of the same similarity type), or equivalently, $\Theta = \mathrm{Ker}\,\varphi'$ for some surjective Q-homomorphism $\varphi': \mathfrak{A} \to \mathfrak{B}'$.*

Proof. The „if" part is obvious. Assume Θ is an order-congruence, and consider the ordered algebra $(A/\Theta; F, \preceq)$, where $(A/\Theta; F)$ is the corresponding quotient algebra, and

$$[a]\Theta \preceq [b]\Theta \quad \text{iff} \quad a\,\Theta\,a_1 \leqq a_2\,\Theta\,a_3 \leqq \ldots a_n = b \quad \text{for some } n \text{ and } a_1, \ldots, a_{n-1} \in A.$$

Then the natural map $a\mapsto[a]\Theta$ is a surjective Q-homomorphism onto the constructed ordered algebra (which will be usually denoted by $\mathfrak{A}/\Theta$).

However, the order-congruences or, what are the same, the kernels of homomorphisms are not sufficient to reproduce the corresponding homomorphic images in the case when the homomorphisms are surjective, unless they are Q-homomorphisms. But we need also homomorphic images, which are not Q-images. So it is desirable to introduce such relations on the ordered algebras, which enable us to describe all homomorphic images completely. The following definition can be found implicitly in BLOOM [1].

Definition 1.2. Let $\varphi\colon \mathfrak{A}\to\mathfrak{B}$ a homomorphism. By the *directed kernel of* φ we mean the relation

$$\overrightarrow{\mathrm{Ker}}\,\varphi = \{(a, b)\in A^2 | a\varphi \leqq b\varphi \text{ in } \mathfrak{B}\}.$$

The *isomorphisms* are those homomorphisms having a two-sided inverse map, which is also a homomorphism.

It is obvious that knowing $\overrightarrow{\mathrm{Ker}}\,\varphi$ for a surjective homomorphism φ, we can construct — up to isomorphism — the corresponding homomorphic image, thanks to the fact that $\overrightarrow{\mathrm{Ker}}\,\varphi$ determines on $\mathrm{Im}\,\varphi$ the equality, the ordering and the operations, as well.

The directed kernels can be characterized as follows:

Theorem 1.2. *A binary relation Θ over A is the directed kernel for some homomorphism of $\mathfrak{A}$ into some ordered algebra if and only if Θ is a quasiorder compatible with the operations, which extends the ordering of $\mathfrak{A}$ (i.e. $a\leqq b$ implies $a\Theta b$).*

Proof. The „only if" part is trivial; for the converse let us consider the relation $\Phi=\Theta\cap\Theta^{-1}$. It is easily seen, that Φ is an order-congruence; let $[a]\Phi\leqq[b]\Phi$ iff $a\Theta b$. Then $\leqq$ is a (well-defined) partial ordering on A/Φ preserved by the operations of the quotient algebra. Now obviously $\Theta=\overrightarrow{\mathrm{Ker}}\,\eta$ with η the natural map $a\mapsto[a]\Phi$ onto $(A/\Phi;\, F, \leqq)$. (The latter need not be equal to $\mathfrak{A}/\Phi$!)

(Note that Bloom called such quasiorders „admissible preorders".)

Let us denote the ordered algebra constructed in the previous proof by $\mathfrak{A}/\Theta$. We essentially proved also

Theorem 1.3 (Homomorphism Theorem). *If $\varphi\colon\mathfrak{A}\to\mathfrak{B}$ is a surjective homomorphism, then $\mathfrak{A}/\overrightarrow{\mathrm{Ker}}\,\varphi\cong\mathfrak{B}$, an isomorphism is given by $[a]\Phi\mapsto a\varphi$, where Φ denotes the order-congruence $\overrightarrow{\mathrm{Ker}}\,\varphi\cap(\overrightarrow{\mathrm{Ker}}\,\varphi)^{-1}$.*

Next we investigate the connection between order-congruences and directed kernels (in the sequel we refer to the latter simply as *quasiorders*, as they are quasiorders compatible with the operations, which extend the partial order on the algebra).

Proposition 1.4. *The order-congruences are exactly the relations* $\Theta \cap \Theta^{-1}$, *where* Θ *is a quasiorder.*

Proof. We have already seen (proof of Theorem 1.2), that the relations $\Theta \cap \Theta^{-1}$ are all order-congruences. Now if Φ is an order-congruence, then let Θ be the directed kernel of the natural homomorphism of $\mathfrak{A}$ onto the quotient algebra $\mathfrak{A}/\Theta$ (see Theorem 1.1). It is clear that $\Phi = \Theta \cap \Theta^{-1}$.

If Θ is a quasiorder, then $\Theta \cap \Theta^{-1}$ is called the *order-congruence associated with* Θ; cf. BLOOM [1], where it is shown that $\Theta \cap \Theta^{-1}$ is a congruence in the usual sense. The same order-congruence may be associated with *distinct* quasiorders, as trivial examples show. But always there is a smallest among the them: namely, for an order-congruence Φ the Θ in the proof of Proposition 1.4 is the least quasiorder such that $\Phi = \Theta \cap \Theta^{-1}$; call it the *quasiorder associated with* Φ. It can also be defined as the only quasiorder Θ for which the natural map of $\mathfrak{A}$ onto $\mathfrak{A}/\Phi$ is a Q-homomorphism of $\mathfrak{A}$ onto $\mathfrak{A}/\Theta$.

For every binary relation $H \subseteq A^2$ there is a smallest quasiorder Θ on $\mathfrak{A}$ such that $H \subseteq \Theta$; this is *the quasiorder generated by* H (denoted by $\vec{\Theta}(H)$), and is equal to the intersection of all quasiorders containing H. If H consists only of the pair (a, b), then we say that $\vec{\Theta}(H)$ is *the principal quasiorder generated by* (a, b), and denote it by $\vec{\Theta}(a, b)$.

Theorem 1.5. *The quasiorders of an ordered algebra* $\mathfrak{A}$ *form an algebraic lattice under set inclusion with the universal relation of* A *as the unit and the ordering of* $\mathfrak{A}$ *as the zero. The join* $\bigvee_{\gamma \in \Gamma} \Theta_\gamma$ *of the quasiorders* Θ_γ, $\gamma \in \Gamma$, *is given by*

$$a\Big(\bigvee_{\gamma\in\Gamma} \Theta_\gamma\Big)b \quad \text{iff} \quad a\Theta_{\gamma_1}a_1\Theta_{\gamma_2}a_2 \ldots a_{n-1}\Theta_{\gamma_n}b \quad \text{for some elements}$$
$$a_1, \ldots, a_{n-1} \in A \quad \text{and} \quad \gamma_1, \ldots, \gamma_n \in \Gamma.$$

From now on, this lattice is denoted by $\mathrm{Cqu}(\mathfrak{A})$ ("compatible quasiorders").

The straightforward proof of the next theorem will be omitted.

Theorem 1.6 (Second Isomorphism Theorem). *Let* Θ_1, Θ_2 *be quasiorders on* $\mathfrak{A}$ *with* $\Theta_1 \leqq \Theta_2$, *and let* Φ_i *denote the order-congruence associated with* Θ_i, $i = 1, 2$. *Then the relation* $\bar{\Theta}_2$ *on* $\mathfrak{A}/\Theta_1$ *defined by*

$$[a]\Phi_1 \bar{\Theta}_2 [b]\Phi_1 \quad \text{iff} \quad a\Theta_2 b$$

is a quasiorder on $\mathfrak{A}/\Theta_1$ *and* $(\mathfrak{A}/\Theta_1)/\bar{\Theta}_2$ *is isomoprhic* $\mathfrak{A}/\Theta_2$ *via the map*

$$[[a]\Phi_1]\bar{\Phi}_2 \mapsto [a]\Phi_2,$$

where $\bar{\Phi}_2$ *is the order-congruence associated with* $\bar{\Theta}_2$. *Hence, the quasiorder-lattice of* $\mathfrak{A}/\Theta$ *is isomorphic to the interval* $[\Theta)$ *of* Cqu $(\mathfrak{A})$.

The following statement is equally trivial:

Theorem 1.7 (First Isomorphism Theorem). *Given an ordered algebra* $\mathfrak{A}$, *a subalgebra* $\mathfrak{B}$ *of* $\mathfrak{A}$ *and a quasiorder* $\Theta \in \mathrm{Cqu}(\mathfrak{A})$, *define* $[B]=\{a\in A \mid a\Phi b$ for some $b\in B\}$, *where* Φ *is the order-congruence associated with* Θ. *Let* $\bar{\mathfrak{B}}$ *be the subalgebra of* $\mathfrak{A}$ *determined by* $[B]$. *Then the mapping* $[b](\Phi{\restriction}B)\mapsto[b](\Phi{\restriction}[B])$ *is an isomorphism between* $\mathfrak{B}/(\Theta{\restriction}B)$ *and* $\bar{\mathfrak{B}}/(\Theta{\restriction}[B])$. (Here $\restriction$ stands for restriction.)

Now turn back to considering quasiorders generated by given set of pairs of elements.

Proposition 1.8. *For* $c, d, a, b\in A$ $(c, d)\in\vec{\Theta}(a, b)$ *if and only if there exists a natural number n, unary algebraic functions* $q_1(x), \dots, q_n(x)$ *over* $\mathfrak{A}$ *and a sequence* $c=u_1, u_2, \dots, u_{2n}=d$ *of elements of A such that*

(i) $u_{2i}\leqq u_{2i+1}$ for $i=1, \dots, n-1$ *and*

(ii) $u_{2i-1}=q_i(a)$, $u_{2i}=q_i(b)$ *for* $i=1, \dots, n$.

We omit the easy proof. Of course, if $a\leqq b$, then $\vec{\Theta}(a, b)$ is just the ordering of $\mathfrak{A}$, as it follows at once from the definition of $\vec{\Theta}(a, b)$, but it also follows from this proposition. Replacing (a, b) in (ii) by an arbitrary $(v_i, v_i')\in H$, we get the description of $\vec{\Theta}(H)$.

For every $H\subseteq A^2$ let $\Theta_0(H)$ denote the *congruence* on $(A; F)$ generated by H, and for any congruence Θ of $(A; F)$ let $\hat{\Theta}$ denote the smallest order-congruence of $\mathfrak{A}$ containing Θ. Then we can state:

Proposition 1.9. *Let* Θ *be a congruence of* $(A; F)$. *Then for any* $a, b\in A$, $a\hat{\Theta}b$ *if and only if there is a sequence of the form*

$$a\,\Theta\, a_1 \leqq a_2\,\Theta\, a_3 \leqq \dots a_n = b\,\Theta\, b_1 \leqq b_2\,\Theta\, b_3 \leqq \dots b_m = a.$$

Consequently, $\widehat{\Theta_0(H)}$ *is the order-congruence generated by H.*

By means of Proposition 1.9 and the well-known Mal'cev lemma concerning $\Theta_0(H)$ it would be possible to give an explicit description for $\widehat{\Theta_0(H)}$, but we omit this. Obviously, Proposition 1.9 defines also the *join of* (arbitrarily many) *order-congruences*. The formulation and the proof of the analogue of Theorem 1.5 is left to the reader (cf. also [3], Proposition 2.2). The *order-congruence lattice of* $\mathfrak{A}$ is denoted by Con $(\mathfrak{A})$, and the order-congruence generated by (a, b) is $\Theta(a, b)$.

Finally, note that the Second Isomorphism Theorem holds also for order-congruences, but in general the First does not, because the ordering of the congruence classes is defined by means of certain sequences of elements, and it can happen that there is no such sequence between two elements of B *inside* of B, but there is already in $[B]$ (see Theorems 1.1 and 1.7). The corresponding variant of the Homomorphism Theorem is true for Q-epimorphisms (of course, replacing $\overrightarrow{\mathrm{Ker}}$ by Ker).

2. Operators on classes of ordered algebras. Varieties. Classes will always consist of ordered algebras of the same similarity type. Let $\mathbf{I}$, $\mathbf{H}$, $\mathbf{Q}$, $\mathbf{S}$, $\mathbf{P}$ and $\mathbf{P}_S$ be the operators of forming all isomorphic, homomorphic, Q-homomorphic images, subalgebras, direct products and subdirect products, respectively (products of empty families — with the obvious meaning — are also allowed). A class $\mathcal{K}$ is a *variety* (resp. *Q-variety*) provided it is closed under $\mathbf{H}$, $\mathbf{S}$ and $\mathbf{P}$ (resp. $\mathbf{Q}$, $\mathbf{S}$ and $\mathbf{P}$). It is easy to check (cf. [1]) that

Theorem 2.1. *For any class $\mathcal{K}$, $\mathbf{HSP}(\mathcal{K})$ is the smallest variety containing $\mathcal{K}$.*

One would expect an analoguous result for Q-varieties, but it does not hold in general, because the operator inequality $\mathbf{SQ}\leqq\mathbf{QS}$ may be false, as it is seen from very simple counterexamples (see also the remark at the end of the previous section on the First Isomorphism Theorem). The characterization of the Q-variety generated by a class in terms of operators is an open problem yet.

By an *inequality of type τ* we mean a sequence of symbols $f\leqq g$, where f and g are τ-terms. The expression "$f\leqq g$ holds in an algebra $\mathfrak{A}$" (or more generally, in a class $\mathcal{K}$) has the obvious meaning.

There is a Birkhoff-type characterization for varieties (Bloom [1]):

Theorem 2.2. *A class $\mathcal{K}$ is a variety if and only if $\mathcal{K}$ consists exactly of all the algebras satisfying a given set of inequalities.*

For any fixed type τ, the varieties of type τ are in one-to-one correspondence with the *fully invariant quasiorders* (i.e. invariant under all endomoprhisms) of the absolutely free τ-algebra of rank $\aleph_0$. From this fact one can easily conclude Bloom's four rules for the corresponding „inequational logic":

(i) $t\leqq t$;

(ii) $t_1\leqq t_2$ and $t_2\leqq t_3$ imply $t_1\leqq t_3$;

(iii) $t_i\leqq t_i'$, $i=1, ..., n$, imply $f(t_1, ..., t_n)\leqq f(t_1', ..., t_n')$ for any n-ary operation symbol f;

(iv) $t(x_1, ..., x_n)\leqq t'(x_1, ..., x_n)$ implies $t(q_1, ..., q_n)\leqq t'(q_1, ..., q_n)$ for arbitrary terms $q_1, ..., q_n$. (Of course, we are inside of τ).

Now we will consider free algebras over arbitrary posets; they will play an important role in the investigation of Mal'cev-type conditions.

Definition 2.1. Let $\mathscr{K}$ be a class of ordered algebras, $\mathfrak{X}=(X; \leqq)$ a poset, $\mathfrak{F}\in\mathscr{K}$, and let ϱ be a map $X\to F$. $\mathfrak{F}$ is the *free algebra over* $\mathfrak{X}$ *in* $\mathscr{K}$ with the canonical map ϱ, if the following hold:

(i) ϱ is monotone, and $X\varrho$ generates F;

(ii) given any monotone map $\varphi: \mathfrak{X}\to\mathfrak{A}$ into an algebra $\mathfrak{A}\in\mathscr{K}$, there exists a (unique) homomorphism ψ: $\mathfrak{F}\to\mathfrak{A}$ such that $\varrho\psi=\varphi$.

$\mathfrak{F}$ is denoted by $\mathfrak{F}^{\varrho}_{\mathscr{K}}(\mathfrak{X})$, or simply by $\mathfrak{F}_{\mathscr{K}}(\mathfrak{X})$, if we do not want to refer to ϱ explicitly. (Cf. [2] for topological algebras).

Proposition 2.3. *$\mathfrak{F}_{\mathscr{K}}(\mathfrak{X})$ is unique in the sense that always there is an isomorphism ξ between $\mathfrak{F}^{\varrho_1}_{\mathscr{K}}(\mathfrak{X})$ and $\mathfrak{F}^{\varrho_2}_{\mathscr{K}}(\mathfrak{X})$ such that $\varrho_1\xi=\varrho_2$.*

In what follows let us call the **ISP**-closed classes *prevarieties.*

Theorem 2.4. *If $\mathscr{K}$ is a prevariety, then for any poset $\mathfrak{X}$, $\mathfrak{F}^{\varrho}_{\mathscr{K}}(\mathfrak{X})$ exists with some ϱ. ϱ is an order-isomorphism onto a subset of F, provided $\mathscr{K}$ contains a nontrivially ordered member, or $\mathfrak{X}$ is trivially ordered and $\mathscr{K}$ contains an at least two-element member.*

Proof. The existence of $\mathfrak{F}^{\varrho}_{\mathscr{K}}(\mathfrak{X})$ can be seen in the usual way. For the second statement let $x, y\in X$, $x\not\leqq y$, and $a<b$, $a, b\in\mathfrak{A}$, $\mathfrak{A}\in\mathscr{K}$. Then there is a monotone map φ: $\mathfrak{X}\to\mathfrak{A}$ such that $y\varphi=a$, $x\varphi=b$. But then $x\varrho\not\leqq y\varrho$, otherwise with the ψ of (ii) in Definition 2.1 we would get $x\varrho\psi\leqq y\varrho\psi$, i.e. $b\leqq a$, a contradiction. The third statement is obvoius, since in that case we essentially work with usual universal algebras, and the statement simply expresses that ϱ is 1—1.

So, in the two cases mentioned above, we may think $\mathfrak{X}$ to be embedded in $\mathfrak{F}_{\mathscr{K}}(\mathfrak{X})$. If $\mathfrak{X}$ is trivially ordered, then $\mathfrak{F}_{\mathscr{K}}(\mathfrak{X})$ depends only on the cardinality of X. We will freely use such notations as $\mathfrak{F}_{\mathscr{K}}(a, b, c)$, $\mathfrak{F}_{\mathscr{K}}(n)$, etc. if this will result no confusion.

The structure of $\mathfrak{F}_{\mathscr{K}}(\mathfrak{X})$ is given very easily, when $\mathfrak{X}$ is trivially ordered: $p\leqq q$ in $\mathfrak{F}_{\mathscr{K}}(\mathfrak{X})$ (where p, q are terms applied to elements of X) iff the inequality $p\leqq q$ is identically true in $\mathscr{K}$. This remark will be frequently used later on. In the general case we have no satisfactory description yet.

3. Subdirect decompositions. For an ordered algebra $\mathfrak{A}=(A; F, \leqq)$, let $\mathrm{Or}(\mathfrak{A})$ denote the ordering of $\mathfrak{A}$, i.e. the relation $\leqq$. If $\mathfrak{A}$ is a subdirect product of the algebras $\mathfrak{A}_i$, $i\in I$, then

$$\bigwedge_{i\in I} \overrightarrow{\mathrm{Ker}}\,\pi_i = \mathrm{Or}(\mathfrak{A}),$$

where π_i is the i^{th} natural projection. We show that this condition characterizes subdirect decompositions.

Theorem 3.1. *Let* $\mathfrak{A}$ *be an ordered algebra,* $\Theta_i \in \mathrm{Cqu}(\mathfrak{A})$, $i \in I$, *and* $\bigwedge\{\Theta_i | i \in I\} = \mathrm{Or}(\mathfrak{A})$. *Then* $\mathfrak{A}$ *is isomorphic to a subdirect product of the algebras* $\mathfrak{A}/\Theta_i$.

Proof. The map $\psi: a \mapsto ([a]\vec{\Theta}_i)_{i \in I}$, where $\vec{\Theta}_i$ is the order-congruence associated with Θ_i, gives the desired isomorphism (for the definition of $\mathfrak{A}/\Theta_i$ see Theorem 1.2 and the remark after it).

An ordered algebra is *subdirectly irreducible,* if in all its subdirect decompositions some of the projections is in fact an isomorphism, which by the preceding theorem is equivalent to saying that $\mathrm{Or}(\mathfrak{A})$ is completely meet-irreducible in $\mathrm{Cqu}(\mathfrak{A})$, or in other words, $\mathrm{Cqu}(\mathfrak{A})$ contains a smallest nonzero element. $\mathfrak{A}$ is called *simple* (resp. *weakly simple*), provided $\mathrm{Cqu}(\mathfrak{A})$ (resp. $\mathrm{Con}(\mathfrak{A})$) is the two-element chain. A simple algebra is always weakly simple, but not conversely.

The analogue of Birkhoff's subdirect decomposition theorem holds:

Theorem 3.2. *Every ordered algebra is isomorphic to a subdirect product of its subdirectly irreducible quotient algebras.*

Proof. The claim follows from the fact that, $\mathrm{Cqu}(\mathfrak{A})$ being an algebraic lattice, every quasiorder of $\mathfrak{A}$ is the meet of completely meet-irreducible quasiorders, from the definition of the orderings on the quotient algebras, and from the preceding theorem. For a more direct proof, let us consider for every $a, b \in \mathfrak{A}$ with a $a \not\leqq b$ a maximal quasiorder $\psi(a, b)$ not containing (a, b). Then $\bigwedge\{\psi(a, b) | a \not\leqq b\} = \mathrm{Or}(\mathfrak{A})$, and $\psi(a, b) \vee \vec{\Theta}(a, b)$ is the least nonzero element of $\mathrm{Cqu}(\mathfrak{A}/\psi(a, b))$, from which the assertion follows.

Of course, there are several necessary and sufficient conditions on families of quasiorders to determine a *finite direct decomposition.* We formulate only the simplest of them:

Theorem 3.3. *Let* Θ_1, Θ_2 *be quasiorders on* $\mathfrak{A}$, *and let* Φ_1, Φ_2 *be the associated order-congruences. The correspondence*

$$a \mapsto ([a]\Phi_1, [a]\Phi_2)$$

defines an isomorphism between $\mathfrak{A}$ *and the direct product* $\mathfrak{A}/\Theta_1 \times \mathfrak{A}/\Theta_2$ *if and only if the following are satisfied:*

(i) $\Theta_1 \wedge \Theta_2 = \mathrm{Or}(\mathfrak{A})$;

(ii) $\Phi_1 \circ \Phi_2 = \Phi_2 \circ \Phi_1 = \iota$ *(the universal relation).*

Obviously, (i) implies that $\Phi_1 \wedge \Phi_2 = \omega$ (the identity relation), but it is easily seen, that the latter is not sufficient for (i).

4. Conditions in quasiorder-lattices. The analogue of Jónsson's lemma. In this section we investigate the analogues of such properties, as (n-) permutability and distributivity of all congruences on every algebra from a class, having so great importance in the theory of universal algebras.

Proposition 4.1. *Let $\mathcal{K}$ be a prevariety of ordered algebras with a nontrivially ordered member. Then there exist non-permutable order-congruences on an algebra from $\mathcal{K}$.*

Proof. See [3].

We shall not deal with the n-permutability of order-congruences for $n>2$, because the idea of the proof of the next statement carries over easily. (Cf. [7].)

Proposition 4.2. *Under the assumption of the above proposition, the n-permutability of quasiorders does not hold in $\mathcal{K}$.*

Proof. For technical reasons, let $n=2m$. Assume that the quasiorders $\Theta = \bigvee_{i<m} \vec{\Theta}(a_{2i}, a_{2i+1})$ and $\Phi = \bigvee_{i<m} \vec{\Theta}(a_{2i+1}, a_{2i+2})$ of the free algebra $\mathfrak{F} = \mathfrak{F}_{\mathcal{K}}(a_0, \ldots, a_n)$ are n-permutable. Then $(a_0, a_n) \in \underbrace{\Theta \circ \Phi \circ \Theta \circ \ldots \circ \Phi}_{n \text{ times}}$ implies the existence of a sequence $a_0 = b_0 \Phi b_1 \Theta b_2 \Phi b_3 \ldots b_{n-1} \Theta b_n = a_n$. Here $b_i = q_i(a_0, \ldots, a_n)$ for a term q_i. If i is even, then

$$q_i(a_0, a_1, a_1, a_3, a_3, \ldots) \Phi q_i(a_0, a_1, a_2, a_3, a_4, \ldots) \Phi$$
$$\Phi q_{i+1}(a_0, a_1, a_2, a_3, a_4, \ldots) \Phi q_{i+1}(a_0, a_2, a_2, a_4, a_4, \ldots).$$

Consider the endomorphism ξ of $\mathfrak{F}$, which leaves a_0 fixed, and sends a_{2i+1} and a_{2i+2} to a_{2i+2} for every $i<m$. Then $\Phi \leqq \overrightarrow{\operatorname{Ker}}\, \xi$, so

$$q_i(a_0, a_2, a_2, a_4, a_4, \ldots) = q_i(a_0, a_1, a_1, a_3, a_3, \ldots)\xi \leqq$$
$$\leqq q_{i+1}(a_0, a_2, a_2, a_4, a_4, \ldots)\xi = q_{i+1}(a_0, a_2, a_2, a_4, a_4, \ldots).$$

Similarly, for i odd we have

$$q_i(a_1, a_1, a_3, a_3, \ldots) \leqq q_{i+1}(a_1, a_1, a_3, a_3, \ldots).$$

Now let $p_i(x, y, z) = q_i(x, \ldots, x, y, z, \ldots, z)$ with x occurring i times for $1 \leqq i < n$, $p_0(x, y, z) = x$ and $p_n(x, y, z) = z$. Then in $\mathcal{K}$ there hold the following inequalities:

$$(*) \qquad p_i(x, x, z) \leqq p_{i+1}(x, z, z), \quad i = 1, \ldots, n-1.$$

But take elements $a<b$ in some member of $\mathcal{K}$ and compute:

$$b = p_0(b, b, a) \leqq p_1(b, a, a) \leqq p_1(b, b, a) \leqq p_2(b, a, a) \leqq p_2(b, b, a) \leqq \ldots$$
$$\ldots \leqq p_n(b, a, a) = a;$$

this is a contradiction. (Note, that $p_i(b, a, a) \leqq p_i(b, b, a)$ is true by monotonity.)

This means that in nontrivially ordered prevarieties the description of the join of quasiorders cannot be reduced so that we take sequences of elements of a fixed length.

Corollary 4.3. *Let $\mathscr{K}$ be a prevariety of universal algebras, $n\geqq 2$ a natural number, and suppose that all compatible quasiorders on algebras in $\mathscr{K}$ are n-permutable. Then all these quasiorders are congruences (i.e. are also symmetric).*

Proof. Endow all $\mathscr{K}$-algebras with trivial order. Considering a pair $(a, b)\in \in\varrho\in \mathrm{Cqu}(\mathfrak{A})$, $\mathfrak{A}\in\mathscr{K}$, and defining the $p_i(x, y, z)$ as above, we can compute by $(*)$ (which gives now equations!) and the compatibility of ϱ:

$$b = p_0(b, b, a) = p_1(b, a, a)\,\varrho\, p_1(b, b, a) = p_2(b, a, a)\,\varrho\, p_2(b, b, a) = \\ \ldots = p_n(b, a, a) = a, \quad \text{from which } (b, a)\in\varrho \quad \text{follows.}$$

Fortunately, besides the negative phenomena mentioned so far, there are positive facts, too. The concept of *quasiorder distributivity* of all algebras in a prevariety is already useful. The significance of quasiorder distributivity is seen from the next two statements. All algebras are ordered algebras of a fixed type. We follow JÓNSSON's [8] original proofs mutatis mutandis, keeping also his notations.

Lemma 4.4. *If $\mathfrak{A}$ is a subalgebra of $\prod(\mathfrak{C}_i|i\in I)$,* Cqu$(\mathfrak{A})$ *is distributive and $\mathfrak{A}/\varphi$ is subdirectly irreducible, where $\varphi\in$ Cqu$(\mathfrak{A})$, then there exists an ultrafilter U over I such that $U^\wedge\restriction A\leqq\varphi$.* (For any filter V over I, $V^\wedge$ denotes the relation defined by $xV^\wedge y$ iff $\{i|x(i)\leqq y(i)\}\in V$.)

Proof. Obviously, the $V^\wedge$ are always quasiorders on $\mathfrak{C}=\prod(\mathfrak{C}_i|i\in I)$. Write $J^\wedge$ instead of $V^\wedge$, if V is the principal filter generated by a subset J of I. Let $D=\{J|J\subseteqq I$ and $J^\wedge\restriction A\leqq\varphi\}$, and let U be a maximal filter contained in D (Zorn's lemma applies since $I\in D$). Then $U^\wedge=\bigcup(J^\wedge|J\in U)$, so $U^\wedge\restriction A\leqq\varphi$. We show, that U is an ultrafilter. For every $J, K\subseteqq I$

(1) $\qquad I\supseteqq J\supseteqq K$ and $K\in D$ implies $J\in D$,

and $(J\cup K)^\wedge\restriction A=(J^\wedge\restriction A)\cap(K^\wedge\restriction A)$, so by distributivity

(2) $\qquad \varphi=\varphi\vee((J\cup K)^\wedge\restriction A)=(\varphi\vee(J^\wedge\restriction A))\cap(\varphi\vee(K^\wedge\restriction A))$ if $J\cup K\in D$.

But φ is meet-irreducible, so $\varphi\vee(J^\wedge\restriction A)=\varphi$ or $\varphi\vee(K^\wedge\restriction A)=\varphi$, i.e.

(3) $\qquad J\cup K\in D$ implies $J\in D$ or $K\in D$.

If U were not an ultrafilter, then we would have $J\notin U$ and $I\setminus J\notin U$ for some $J\subseteqq I$. Then by (1) and the maximality of U there exist sets K', $K''\in U$ such that $J\cap K'\notin D$ and $(I\setminus J)\cap K''\notin D$. However, $K=K'\cap K''\in U$, so $K\in D$, and $K=(J\cap K)\cup$

$\cup((I\setminus J)\cap K)$. But this contradicts (3), since the members of the latter union do not belong to D by (1).

Lemma 4.5. (Jónsson-lemma). *If $\mathscr{K}$ is a class of ordered algebras, $\mathscr{V}$ is the variety generated by $\mathscr{K}$, and all the* $\mathrm{Cqu}(\mathfrak{A})$, $\mathfrak{A}\in\mathscr{V}$, *are distributive, then all subdirectly irreducible members of $\mathscr{V}$ belong to* $\mathbf{HSP}_U(\mathscr{K})$, *where $\mathbf{P}_U$ denotes the model-theoretic operator of forming ultraproducts. Consequently,* $\mathscr{V}=\mathbf{IP}_S\mathbf{HSP}_U(\mathscr{K})$.

Proof. Every algebra in $\mathscr{V}$ is of the form $\mathfrak{A}/\varphi$, where $\mathfrak{A}$ is a subalgebra of a direct product $\prod\{\mathfrak{C}_i|i\in I\}$, $\mathfrak{C}_i\in\mathscr{K}$, and $\varphi\in\mathrm{Cqu}(\mathfrak{A})$. If $\mathfrak{A}/\varphi$ is subdirectly irreducible, then $U^\wedge\restriction A\leqq\varphi$ for a suitable ultrafilter U over I by the preceding lemma. Therefore, $\mathfrak{A}/\varphi$ is a homomorphic image of $\mathfrak{A}(U^\wedge\restriction A)$, and the latter is obviously a subalgebra of $(\prod\{\mathfrak{C}_i|i\in I\})/U^\wedge$.

It remains to show, that $(\prod\{\mathfrak{C}_i|i\in I\})/U^\wedge$ is an ultraproduct of members of $\mathscr{K}$. We point out, that this is just the ultraproduct of the $\mathfrak{C}_i$ over the ultrafilter U. Indeed, let $[f]U$ denote the equivalence class modulo U of any function $f\in\prod C_i$ according to the definition of ultraproduct, and let Θ be the order-congruence associated with $U^\wedge$, i.e. $\Theta=U^\wedge\cap(U^\wedge)^{-1}$. Now $[f]\Theta=[g]\Theta$ means that $\{i|f(i)\leqq g(i)\}\in U$ and $\{i|g(i)\leqq f(i)\}\in U$, which is equivalent to $\{i|f(i)=g(i)\}\in U$, i.e. $[f]U=[g]U$. From this it follows at once, that the operations are also the same. Let $[f]\Theta\leqq[g]\Theta$, then $fU^\wedge g$ (see the proof of Theorem 1.2), which means $\{i|f(i)\leqq g(i)\}\in U$. But this expresses just the fact that $[f]U\leqq[g]U$ in the ultraproduct.

Let us mention, that many results of Jónsson's fundamental paper [8] on congruence distributivity can be reformulated and proved for ordered varieties, using quasiorders instead of congruences. To work with order-congruences is generally more difficult, although not always: for example, the authors succeeded in characterizing order-congruence distributivity of prevarieties in [3] by Mal'cev-type conditions, while for quasiorder distributivity there is no such result yet; there is only a criterion in terms of weak Mal'cev conditions (see below).

5. Characterization of quasiorder-distributivity. Some examples. Now we intend to characterize the distributivity of quasiorders in a prevariety by a (weak) Mal'cev condition. This characterization will enable us to present some nontrivial examples, too.

Theorem 5.1. *Let $\mathscr{K}$ be a class of ordered algebras closed under* **I**, **S** *and* **P** *(i.e. a prevariety). Then the following two conditions are equivalent:*

(i) $\mathrm{Cqu}(\mathfrak{A})$, *the lattice of quasiorders of $\mathfrak{A}$, is distributive for any member $\mathfrak{A}$ of $\mathscr{K}$;*

(ii) *For any even integer $n\geqq 2$ there exists a positive multiple k of $n/2$ such that $U(n, k)$ holds in $\mathscr{K}$, where $U(n, k)$ is a (strong) Mal'cev condition defined as follows ($(x_0, x_1, \ldots, x_n)$ is denoted by* $\mathbf{x}$ *and $n/2$ by m):*

"There exist ((n+1)-ary and (n+2)-ary) terms

$$p_0(\mathbf{x}), p_1(\mathbf{x}), \dots, p_k(\mathbf{x}),$$
$$q_j^i(t, \mathbf{x}) \quad \textit{for} \quad 1 \leqq i,\ j \leqq k,$$
$$r_j^i(t, \mathbf{x}) \quad \textit{for} \quad 1 \leqq i \leqq k,\ i \textit{ odd, and } 0 \leqq j \leqq k-1, \textit{ and}$$
$$s_j^i(t, \mathbf{x}) \quad \textit{for} \quad 1 \leqq i,\ j \leqq k,\ i \textit{ even,}$$

such that the following inequalities and identities hold:

$$p_0(\mathbf{x}) = x_0, \quad p_k(\mathbf{x}) = x_n,$$
$$p_{i-1}(\mathbf{x}) = q_1^i(x_0, \mathbf{x}), \quad p_i(\mathbf{x}) = q_k^i(x_n, \mathbf{x}) \quad \textit{for} \quad 1 \leqq i \leqq k,$$
$$q_l^i(x_n, \mathbf{x}) \leqq q_{l+1}^i(x_0, \mathbf{x}) \quad \textit{for} \quad 1 \leqq i \leqq k, \quad 1 \leqq l < k,$$
$$p_{i-1}(\mathbf{x}) = r_0^i(x_0, \mathbf{x}), \quad p_i(\mathbf{x}) = r_{k-1}^i(x_{n-1}, \mathbf{x}) \quad \textit{for } i \textit{ odd}, \quad 1 \leqq i \leqq k,$$
$$r_l^i(x_{2j+1}, \mathbf{x}) \leqq r_{l+1}^i(x_{2j+2}, \mathbf{x}) \quad \textit{for } i \textit{ odd}, \quad 1 \leqq i \leqq k,\ 0 \leqq j < m,\ 0 \leqq l < k-1,$$

$j \equiv l(m)$, where $+$ is understood modulo n so that $0 \leqq 2j+2 < n$,

$$p_{i-1}(\mathbf{x}) = s_1^i(x_1, \mathbf{x}), \quad p_i(\mathbf{x}) = s_k^i(x_n, \mathbf{x}) \quad \textit{for } i \textit{ even}, \quad 1 < i \leqq k,$$
$$s^i(x_{2j}, \mathbf{x}) \leqq s_{l+1}^i(x_{2j+1}, \mathbf{x}) \quad \textit{for } i \textit{ even}, \quad 1 < i \leqq k,\ 0 < j \leqq m,\ 1 \leqq l < k,\ j \equiv l(m),$$

where $+$ is understood modulo n so that $0 < 2j+1 \leqq n$."

Proof. Suppose (i) holds, n is an even positive integer, and consider the quasi-orders $\alpha = \overrightarrow{\Theta}(x_0, x_n)$, $\beta = \overrightarrow{\Theta}(\{(x_0, x_1), (x_2, x_3), \dots, (x_{n-2}, x_{n-1})\})$, $\gamma = \overrightarrow{\Theta}(\{(x_1, x_2), (x_3, x_4), \dots, (x_{n-1}, x_n)\})$ on the free algebra $\mathfrak{F} = \mathfrak{F}_{\mathscr{K}}(n+1)$ freely generated by $\{x_0, x_1, \dots, x_n\}$. Since $(x_0, x_n) \in \alpha \wedge (\beta \vee \gamma)$, we have $(x_0, x_n) \in (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$ as well. Therefore, $x_0 = p_0 \alpha \wedge \beta p_1 \alpha \wedge \gamma p_2 \alpha \wedge \beta p_3 \alpha \wedge \gamma \dots p_k = x_n$ holds for some multiple k of m and elements $p_i = p_i(\mathbf{x})$ of $\mathfrak{F}$. Since $(p_{i-1}, p_i) \in \alpha = \overrightarrow{\Theta}(x_0, x_n)$, by Proposition 1.8 there are unary algebraic functions $\hat{q}_l^i(t)$ on $\mathfrak{F}$, which can be considered as $(n+2)$-ary terms $q_l^i(t, \mathbf{x})$ $1 \leqq l \leqq k_i$, such that $q_l^i(x_0, \mathbf{x}) = p_{i-1}(\mathbf{x})$, $q_k^i(x_n, \mathbf{x}) = p_i(\mathbf{x})$ and $q_l^i(x_n, \mathbf{x}) \leqq q_{l+1}^i(x_0, \mathbf{x})$ for $1 \leqq l < k_i$. Both k and k_i can be enlarged by repeating the last terms, whence they can be assumed to be equal. Now all the identities and inequalities involving some q_l^i hold for the generators of $\mathfrak{F}$, therefore they hold throughout $\mathscr{K}$. The case of the r_l^i and s_l^i is a little bit more complicated from technical point of view, but can be handled similarly, while $p_0(\mathbf{x}) = x_0$ and $p_k(\mathbf{x}) = x_n$ are evidently true in $\mathscr{K}$.

Conversely, let (ii) be satisfied. Assume $\mathfrak{A} \in \mathscr{K}$, $\alpha, \beta, \gamma \in \mathrm{Cqu}(\mathfrak{A})$ and $(a, b) \in \alpha \wedge (\beta \vee \gamma)$; then $(a, b) \in (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$ has to be shown. From the assumption we obtain a sequence of the form $a = a_0 \beta a_1 \gamma a_2 \beta a_3 \gamma \dots \beta a_{n-1} \gamma a_n = b$ for some even n; moreover $a_0 \alpha a_n$. Let k be such a multiple of $n/2$ for which $U(n, k)$ holds in $\mathscr{K}$. It is sufficient to show that for $p_0(a_0, \ldots, a_n) = p_0(\mathbf{a})$ (notation!), $p_1(\mathbf{a}), \ldots, p_k(\mathbf{a})$ we have

$$a_0 = p_0(\mathbf{a})\ \alpha\wedge\beta\ p_1(\mathbf{a})\ \alpha\wedge\gamma\ p_2(\mathbf{a})\ \alpha\wedge\beta\ p_3(\mathbf{a})\ \alpha\wedge\gamma \ldots p_k(\mathbf{a}) = a_n.$$

Indeed, $p_{i-1}(\mathbf{a}) = q_1^i(a_0, \mathbf{a})\alpha q_1^i(a_n, \mathbf{a}) \leqq q_2^i(a_0, \mathbf{a})\alpha q_2^i(a_n, \mathbf{a}) \leqq q_3^i(a_0, \mathbf{a})\alpha \ldots \leqq$ $\leqq q_k^i(a_0, \mathbf{a})\alpha q_k^i(a_n, \mathbf{a}) = p_i(\mathbf{a})$ yields $(p_{i-1}(\mathbf{a}), p_i(\mathbf{a}))\in\alpha$, for i odd $p_{i-1}(\mathbf{a}) =$ $= r_0^i(a_0, \mathbf{a})\beta r_0^i(a_1, \mathbf{a}) \leqq r_1^i(a_2, \mathbf{a})\beta r_1^i(a_3, \mathbf{a}) \leqq \ldots \leqq r_{k-1}^i(a_{n-2}, \mathbf{a})\beta r_{k-1}^i(a_{n-1}, \mathbf{a}) = p_i(\mathbf{a})$ implies $(p_{i-1}(\mathbf{a}), p_i(\mathbf{a}))\in\beta$, while $(p_{i-1}(\mathbf{a}), p_i(\mathbf{a}))\in\gamma$ for i even follows similarly.

Before formulating a corollary to this theorem, two relevant remarks will be made. Firstly, the theorem is obviously applicable for any class $\mathscr{K}$ of ordered algebras, containing all free algebras $\mathfrak{F}_{\mathscr{K}}(X)$ for finite unordered X. Secondly, any universal algebra can be considered as a trivially ordered algebra. Thus the theorem also holds for certain classes (including varieties and prevarieties) of universal algebras. In this case Cqu $(\mathfrak{A})$ is the lattice of all compatible, reflexive and transitive binary relations of $\mathfrak{A}$, and the inequalities in $U(n, k)$ simply turn into identities.

Corollary 5.2. *Let $\mathscr{K}$ be a class as in Theorem* 5.1, *and let there exist a ternary term $u(x, y, z)$ for which the identities $u(x, x, y) = u(x, y, x) = u(y, x, x) = x$ hold throughout $\mathscr{K}$ (i.e. u induces a majority function on the members of $\mathscr{K}$). Then* Cqu$(\mathfrak{A})$ *is distributive for any $\mathfrak{A}$ in $\mathscr{K}$.*

Proof. It is sufficient to show that $U(n, n)$ holds in $\mathscr{K}$ for any even n. Let us agree that all the terms p, q, r, s, h, g (with indices) contain at least the variables $x_0, x_1, \ldots, x_n$, but, for the sake of brevity, these common variables will not be indicated. First we define $p_0, \ldots, p_n$ and $h_0(t), \ldots, h_n(t)$ by induction:

$$h_0(t) = t, \quad p_0 = h_0(x_0),$$
$$h_i(t) = u\big(p_{i-1}, x_n, h_{i-1}(t)\big), \quad p_i = h_i(x_i).$$

The terms $g_1(t), \ldots, g_n(t)$ are determined by

$$g_1(t) = h_1(t), \quad g_{i+1}(t) = u\big(g_i(t), x_n, h_{i-1}(x_i)\big).$$

For $1 \leqq i \leqq n$ set $q_1^i(t) = q_2^i(t) = \ldots = q_{n-1}^i(t) = p_{i-1}$ (so in fact these terms do not depend on t) and $q_n^i(t) = u\big(p_{i-1}, g_i(t), h_{i-1}(x_i)\big)$. For i odd, $1 \leqq i < n$, let $j = (i-1)/2$,

$$r_0^i(t) = \ldots = r_{j-1}^i(t) = p_{i-1},$$
$$r_j^i(t) = u\big(p_{i-1}, x_n, h_{i-1}(t)\big), \quad \text{and} \quad r_{j+1}^i(t) = \ldots = r_{n-1}^i(t) = p_i.$$

For i even, $1 < i \leqq n$, set $j = i/2$,

$$s_1^i(t) = \ldots = s_{j-1}^i(t) = p_{i-1},$$
$$s_j^i(t) = u\big(p_{i-1}, x_n, h_{i-1}(t)\big), \quad \textit{and} \quad s_{j+1}^i(t) = \ldots = s_n^i(t) = p_i.$$

A trivial induction shows that $h_i(x_n)=x_n$ $(0\leqq i\leqq n)$, $g_i(x_n)=x_n$ and $g_i(x_0)=p_{i-1}$ $(1\leqq i\leqq n)$. Thus it is not difficult to check that the terms p, q, r, s (with the corresponding indices) satisfy the identities and inequalities required in $U(n, n)$.

We note that it is possible to state and prove an analogous general theorem which „translates" every lattice identity holding in all quasiorder-lattices of members in a prevariety, similarly as it was done in [3] for order-congruence lattices. This is straighforward enough, so we omit it.

To conclude our paper, we present some examples. Since lattices are ordered algebras with their natural orderings and $u(x, y, z)=(x\wedge y)\vee(x\wedge z)\vee(y\wedge z)$ induces a majority function on any lattice, Cqu $(\mathscr{L})$ is distributive for any lattice $\mathscr{L}$. To give another example which is far from lattice orders, set $\mathfrak{A}=(A; u, \leqq)$ where $A=\{a, b, c\}$, u is a ternary majority function such that $u(x, y, z)=c$ provided $\{x, y, z\}=\{a, b, c\}$, and $a<c$, $b<c$ are the only comparable pairs of distinct elements in $(A, \leqq)$. Then $\mathfrak{A}$ is an ordered algebra, and any member of $\mathbf{HSP}(\mathfrak{A})$ is quasiorder distributive by corollary 5.2.

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Toleranzrelationen als Galoisverbindungen

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Über Toleranzrelationen auf Verbänden ist schon einiges geschrieben worden, siehe etwa den Überblicksartikel [6] von Ivan Chajda. Ein jüngst verfaßter Aufsatz [8] von Gábor Czédli gibt mir einen Anlaß, die bereits bekannten Ergebnisse um den im Titel genannten Aspekt zu ergänzen. Ich weiß zwar nicht, ob durch Interpretation der Toleranzrelationen als gewisse Galoisverbindungen zwischen Idealverband und Filterverband erstere oder letztere besser verstanden werden; auf jeden Fall bleibt — wie eigentlich immer bei Galoisverbindungen — nicht viel zu beweisen.

Zur Erinnerung sei gesagt, daß mit einer Toleranzrelation ξ auf einem Verband L schlicht ein reflexiver und symmetrischer Unterverband von $L\times L$ gemeint ist. Bezüglich Inklusion geordnet bilden die Toleranzrelationen auf L einen Verband $\Xi(L)$, der in [3] näher betrachtet wurde. Eine Toleranzrelation ξ wird am besten durch ihre Blockstruktur verstanden; jede maximale Teilmenge B von L paarweise modulo ξ toleranter Elemente (also $B\times B\subseteq\xi$) heißt ein Block von ξ. Die von allen ξ-Blöcken erzeugten unteren Abschnitte und oberen Abschnitte bilden jeweils zueinander antiisomorphe Verbände von Idealen und Filtern (vgl. [8]). In der Tat wird diese Antiisomorphie durch eine Galoisverbindung zwischen Idealverband $\mathscr{I}(L)$ und Filterverband $\mathscr{F}(L)$ induziert. Es ist dann unschwer zu erkennen, daß der Toleranzverband $\Xi(L)$ isomorph ist zu einem gewissen Hauptfilter in dem Tensorprodukt von $\mathscr{I}(L)$ und $\mathscr{F}(L)$. Vielleicht bedarf noch das Tensorprodukt $M\otimes N$ vollständiger Verbände M und N einer Erklärung: Eine Galoisverbindung (σ, τ) zwischen M und N besteht aus (einander eindeutig bestimmenden) Abbildungen $\sigma\colon M\to N$ und $\tau\colon N\to M$, für die $y\leqq x\sigma$ mit $x\leqq y\tau$ gleichbedeutend ist. $M\otimes N$ besteht aus allen Galoisverbindungen (σ, τ) zwischen M und N, vertreten durch die Komponenten σ. $M\otimes N$ ist bezüglich der punktweisen Ordnung ein vollständiger Verband, der sich bekanntlich als Verband bestimmter unterer Abschnitte in $M\times N$ darstellen läßt (eine Menge A heißt unterer Abschnitt, wenn mit jedem $a\in A$ auch jedes $x\leqq a$ zu A gehört). Hier sind natürlich nur algebraische Verbände von Interesse:

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Hilfssatz. *Es seien M und N algebraische Verbände, sowie S und T die zugehörigen Halbverbände der kompakten Elemente. Das Tensorprodukt $M \otimes N$ ist isomorph zu dem (bzgl. Inklusion) geordneten System aller unteren Abschnitte H von $S \times T$, die den nachfolgenden Bedingungen genügen:*

(i) $S \times \{0\}, \quad \{0\} \times T \subseteq H,$

(ii) $(u \wedge x, v \vee y), \ (u \vee x, v \wedge y) \in H$ *falls* $(u, v), \ (x, y) \in H.$

Beweis. Wie mit Lemma 1.1 aus [1] gezeigt wurde, läßt sich $M \otimes N$ vermöge $(\sigma, \tau) \to \{(x, y) \in M \times N \mid y \leqq x\sigma\}$ identifizieren mit dem System der Scott-abgeschlossenen unteren Abschnitte G von $M \times N$, die $(0, 1)$, $(1, 0)$ und mit (u, v), (x, y) auch $(u \wedge x, v \vee y)$, $(u \vee x, v \wedge y)$ enthalten. Es ist klar, daß für jede Menge G mit diesen Eigenschaften die Menge $H = G \cap S \times T$ ein unterer Abschnitt von $S \times T$ ist, der (i) und (ii) genügt. Umgekehrt läßt sich jeder solchen Menge H der Scott-Abschluß $G = \overline{H}$ in $M \times N$ zuordnen; $\overline{H}$ besteht genau aus allen gerichteten Suprema (in $M \times N$) von Elementen aus H. Übliches Hantieren mit algebraischen Verbänden (vgl. [10]) führt hier zur Einsicht, daß $G \leftrightarrow H$ die gewünschte Isomorphie vermittelt.

Standardbeispiele algebraischer Verbände sind Idealverbände $\mathscr{I}(L)$ (bzw. Filterverbände $\mathscr{F}(L)$) irgendwelcher Verbände L. Die folgende Vereinbarung mag sich hier als sinnvoll erweisen: Die leere Menge zählt zu $\mathscr{I}(L)$ (bzw. zu $\mathscr{F}(L)$) genau dann, wenn L kein kleinstes (bzw. größtes) Element besitzt. Die Dedekind—MacNeille-Vervollständigung eines Verbandes L wird bekanntlich mittels einer Galoisverbindung $(\uparrow, \downarrow)$ zwischen $\mathscr{I}(L)$ und $\mathscr{F}(L)$ hergestellt; dabei werden einem Ideal I und einem Filter F von L der Filter $I^{\uparrow}$ der oberen Schranken von I und das Ideal $F^{\downarrow}$ der unteren Schranken von F zugeordnet. Eine beliebige Galoisverbindung (σ, τ) zwischen $\mathscr{I}(L)$ und $\mathscr{F}(L)$ werde tolerant genannt, wenn I^{σ} stets $I^{\uparrow}$ umfaßt (d. h. immer $F^{\downarrow} \subseteq F^{\tau}$ gilt). Die toleranten Galoisverbindungen bilden somit in dem Tensorprodukt $\mathscr{I}(L) \otimes \mathscr{F}(L)$ gerade den von $(\uparrow, \downarrow)$ erzeugten Hauptfilter.

Satz. *Für jeden Verband L sind der Toleranzverband $\Xi(L)$ und der Verband der toleranten Galoisverbindungen zwischen $\mathscr{I}(L)$ und $\mathscr{F}(L)$ isomorph.*

Beweis. Die kompakten Elemente von $\mathscr{I}(L)$ und $\mathscr{F}(L)$ außer der leeren Menge bilden einen Verband, der mit L bzw. dem zu L dualen Verband identifiziert werden kann. Offenbar ist eine Galoisverbindung (σ, τ) zwischen $\mathscr{I}(L)$ und $\mathscr{F}(L)$ genau dann tolerant, wenn für jedes Hauptideal $(x]$ der Filter $(x]^{\sigma}$ jeweils x enthält, d. h. wenn die Menge $\gamma = \{(x, y) \in L \times L \mid y \in (x]^{\sigma}\}$ eine reflexive Relation ist. Der Verband der toleranten Galoisverbindungen ist daher aufgrund des Hilfssatzes isomorph zum Verband $\Gamma(L)$ aller reflexiven Unterverbände γ von $L \times L$, für die mit $w \leqq x$, $(x, y) \in \gamma$, $y \leqq z$ stets $(w, z) \in \gamma$ gilt. Gemäß [2] ist vermöge $\gamma \to \gamma \cap \gamma^{-1}$ der Verband $\Gamma(L)$ isomorph zum Toleranzverband $\Xi(L)$.

Da das Tensorprodukt distributiver algebraischer Verbände wieder distributiv ist (siehe [1] oder [12]), folgt aus dem voranstehenden Satz sofort, daß Toleranzverbände distributiver Verbände stets distributiv sind (siehe [7], vgl. [4]).

Für endliche Verbände L ist der obige Satz schon als Lemma 3 in [2] erwähnt worden: Der Toleranzverband $\Xi(L)$ eines endlichen Verbandes L stimmt mit dem Verband der verbindungstreuen Subjektionen (im Sinne von [13], [14], [15]), d. h. der absteigenden residuierten Abbildungen (im Sinne von [5]) überein. Für eine gegebene Toleranzrelation ξ wird dabei durch die zugehörige verbindungstreue Subjektion σ jedes Element x abgebildet auf das kleinste Element $x\sigma$, das zu x tolerant modulo ξ ist. Allgemeiner ergibt sich hier für einen beliebigen Verband L: Die zu einer Tolreanzrelation ξ auf L gehörige Galoisverbindung (σ, τ) ordnet einem Ideal I den größten Filter $F=I^\sigma$ (bzw. einem Filter F das größte Ideal $I=F^\tau$) zu, so daß $I\cap F$ in einem Block von ξ enthalten ist. Umgekehrt liefert eine tolerante Galoisverbindung (σ, τ) vermöge $\{I\cap F|I=F^\tau,\ F=I^\sigma,\ I\cap F\neq\emptyset\}$ die Blöcke der zugehörigen Toleranzrelation ξ auf L. Rudimente dieser Beobachtung finden sich auch schon in [8] Theorem 2. Es mögen $\mathscr{I}^{\sigma\tau}(L)$ und $\mathscr{F}^{\tau\sigma}(L)$ die zueinander antiisomorphen Verbände aller Ideale der Form $I^{\sigma\tau}$ bzw. aller Filter der Form $F^{\tau\sigma}$ bezeichnen. Die voranstehende Beobachtung läßt sich dann auch wie folgt formulieren (und umfaßt somit [8] Theorem 1): Das System L/ξ aller Blöcke von ξ kann mit dem Unterverband $\{I\in\mathscr{I}^{\sigma\tau}(L)|\,|I\cap I^\sigma\neq\emptyset\}$ von $\mathscr{I}^{\sigma\tau}(L)$ identifiziert werden; dieser Unterverband ist vermöge σ antiisomorph zu $\{F\in\mathscr{F}^{\tau\sigma}(L)|F^\tau\cap F\neq\emptyset\}$. Der sogenannte Faktorverband L/ξ von L modulo ξ erbt seine Verbandsstruktur also von dem vollständigen Verband $\mathscr{I}^{\sigma\tau}(L)$, wobei (σ, τ) die zugehörige Galoisverbindung ist. Die Art der Einbettung von L/ξ in $\mathscr{I}^{\sigma\tau}(L)$ ist auch schnell geklärt: Die Ideale $[x)^\tau$ und die Filter $(x]^\sigma$ liegen infimumdicht in $\mathscr{I}^{\sigma\tau}(L)$ bzw. $\mathscr{F}^{\tau\sigma}(L)$. Somit ist L/ξ supremum- und infimumdicht in $\mathscr{I}^{\sigma\tau}(L)$, d. h. $\mathscr{I}^{\sigma\tau}(L)$ ist die Dedekind—MacNeille-Vervollständigung von L/ξ.

Jeder endliche Verband kommt als Faktorverband eines endlichen distributiven Verbandes modulo einer Toleranzrelation vor, siehe [8] Theorem 3. Diese Tatsache leitet sich auch schon aus [11] Satz 7.2 ab: Jeder endliche Verband ist isomorph zum Skelett eines endlichen distributiven Verbandes. Das Skelett eines modularen Verbandes L endlicher Länge ist nämlich das Bild einer gewissen verbindungstreuen Subjektion auf L (siehe [11] Lemma 6.1), also der Faktorverband von L modulo einer kanonischen Toleranzrelation (vgl. [3] Theorem 3.1). Ich weiß allerdings nicht, ob auch im unendlichen Fall jeder Verband als Faktorverband eines distributiven Verbandes modulo einer Toleranzrelation auftritt.

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E-unitary covers and varieties of inverse semigroups

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1. Introduction and summary

E-unitary inverse semigroups have attracted considerable attention as a result of the remarkable work of McAlister [5], [6] concerning their structure and properties. He proved, inter alia, that every inverse semigroup S has an E-unitary cover, in the sense that there exists an E-unitary inverse semigroup P and an idempotent separating homomorphism of P onto S. Various properties and constructions of E-unitary covers were further established by McAlister and Reilly [7]. On the other hand, the lattice of varieties of inverse semigroups as algebras with a binary and a unary operation has been the focus of extensive investigations by several researchers; we mention only Kleĭman [3], [4].

The purpose of this note is to establish some surprising relationships between the two areas of research discussed above, viz., E-unitary covers and varieties of inverse semigroups. The main points of our consideration are: (i) which varieties admit E-unitary covers for their members, (ii) for a given variety of groups $\mathcal{U}$, which varieties of inverse semigroups $\mathcal{V}$ have E-unitary covers over $\mathcal{U}$, in the sense that every member S of $\mathcal{V}$ has an E-unitary cover P such that $P/\sigma\in\mathcal{U}$. The class $\mathcal{E}$ of all E-unitary inverse semigroups plays an important role in our investigation.

The content of the paper is briefly as follows. Some preliminary material is discussed in Section 2 in order to establish the notation and terminology. Several characterizations of varieties with E-unitary covers are established in Section 3. This is followed, in Section 4, by a description of subhomomorphisms in terms of homomorphisms of inverse semigroups, a result needed in the next section. The principal result of the paper, proved in Section 5 along with some consequences, provides several criteria for the existence of an E-unitary cover of an inverse semigroup S over

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a group variety $\mathcal{U}$. All varieties of inverse semigroups having E-unitary covers over a fixed group variety $\mathcal{U}$ are described in Section 6 in several ways. The relation ν_3 defined on the lattice of varieties of inverse semigroups by: $\mathcal{U}\nu_3\mathcal{V}$ if $\mathcal{U}\cap\mathcal{E}=\mathcal{V}\cap\mathcal{E}$ is discussed briefly in Section 7.

2. Preliminaries

We will follow the notation and terminology of Howie [2]. For background concerning inverse semigroups, we also refer the reader to this book.

Let S be an inverse semigroup. Then S is *E-unitary* if it satisfies the implication $xy=y\Rightarrow x^2=x$. The semilattice of idempotents of S will be denoted by E_S, the least group congruence by σ, the universal congruence by ω. The closure of a nonempty set A of S will be denoted by $A\omega$. An inverse semigroup P is an *E-unitary cover* of S if P is E-unitary and there is an idempotent separating homomorphism of P onto S; if $P/\sigma\cong G$ then P is an *E-unitary cover of S over G*.

Let ϱ be a congruence on S. The set

$$\ker\varrho = \{s\in S \mid s\varrho e \quad \text{for some} \quad e\in E_S\}$$

is the *kernel* of ϱ, $\operatorname{tr}\varrho=\varrho|_{E_S}$ is the *trace* of ϱ. The least congruence on S with the same trace as ϱ will be denoted by $\varrho_{\min}$. For a full discussion of these concepts, see Petrich [9]. The natural homomorphism $S\to S/\varrho$ will be denoted by $\varrho^\natural$. If $\varphi: S\to T$ is a homomorphism, we will denote by *ker* φ the kernel of the congruence on S induced by φ.

For any nonempty set X, we will denote the free inverse semigroup on X by I_X and the free group on X by G_X. The variety of all inverse semigroups will be denoted by $\mathcal{I}$, that of all groups by $\mathcal{G}$ and the lattice of all varieties of inverse semigroups by $\mathcal{L}(\mathcal{I})$. The variety generated by the semigroup S will be denoted by $\langle S\rangle$.

For a countably infinite set X and any $\mathcal{V}\in\mathcal{L}(\mathcal{I})$, let $\varrho(\mathcal{V})$ denote the fully invariant congruence on I_X corresponding to $\mathcal{V}$.

3. Varieties with E-unitary covers

The principal result here gives several characterizations of the varieties of inverse semigroups which have E-unitary covers. These characterizations involve free objects, E-unitary inverse semigroups and the kernel of the corresponding fully invariant congruence on the free object.

We start with a simple useful result.

Lemma 3.1. *Let ϱ be a congruence on an inverse semigroup S. Then S/ϱ is E-unitary if and only if* $\ker \varrho$ *is closed.*

Proof. Suppose that S/ϱ is E-unitary and let $a \in (\ker \varrho)\omega$. Then $ea \in \ker \varrho$ for some $e \in E_S$ and thus $ea\,\varrho\,(ea)^2$ which implies that $a\,\varrho\,a^2$ since S/ϱ is E-unitary. But then $a \in \ker \varrho$ and thus $\ker \varrho$ is closed.

Conversely, assume that $\ker \varrho$ is closed, and let $xy\,\varrho\,x$. Then $(x^{-1}x)y\,\varrho\,x^{-1}x$ so that $y \in (\ker \varrho)\omega = \ker \varrho$ and thus $y^2\varrho y$. Hence S/ϱ is E-unitary.

The following concept is basic for our considerations.

Definition 3.2. A variety $\mathcal{V}$ of inverse semigroups *has E-unitary covers* if, for every $S \in \mathcal{V}$, there is an E-unitary cover of S in $\mathcal{V}$.

We can now establish the first highlight of the paper.

Theorem 3.3. *The following conditions on a variety $\mathcal{V}$ of inverse semigroups are equivalent.*

(i) *$\mathcal{V}$ has E-unitary covers.*

(ii) *The free objects in $\mathcal{V}$ are E-unitary.*

(iii) *$\mathcal{V}$ is generated by its E-unitary members.*

(iv) $\ker \varrho(\mathcal{V})$ *is closed.*

Proof. (i) implies (ii). Let F be a $\mathcal{V}$-free inverse semigroup and S be an E-unitary cover for F in $\mathcal{V}$. There is an (idempotent separating) epimorphism $\varphi\colon S \to F$. Let $X \subseteq F$ be a set of $\mathcal{V}$-free generators of F, and let T be a cross section of the congruence on S induced by φ. Define a bijection $\psi\colon X \to T$ by $x\psi = t$ if $t \in T$ and $t\varphi = x$. Then ψ extends uniquely to a homomorphism ψ of F into S. For any $x \in X$, we have $x\psi\varphi = x$ so that $\psi\varphi$ is an endomorphism on F which restricts to the identity on X. Since X is a set of $\mathcal{V}$-free generators of F it follows that $\psi\varphi$ is the identity map on F. But then ψ is one-to-one and thus a monomorphism of F into S. Since S is E-unitary, so also is $F\psi$. Since ψ is a monomorphism, it follows that F is E-unitary.

(ii) implies (iii) trivially.

(iii) implies (i). Let $S \in \mathcal{V}$. By the general theory of varieties and the hypothesis, there exist E-unitary inverse semigroups T_α in $\mathcal{V}$, an inverse semigroup T which is a subdirect product of T_α's and an epimorphism $\varphi\colon T \to S$. Let ϱ be the congruence on T induced by φ. Letting $\varrho_{\min}$ be the least congruence on T with the same trace as ϱ, we obtain the following diagram of epimorphisms:

$$\begin{tikzcd} T \arrow[r, "\varphi"] \arrow[d, "\varrho_{\min}^{\natural}"'] & S \\ T/\varrho_{\min} \arrow[r, "\tau"] & T/\varrho \arrow[u, "\psi"'] \end{tikzcd}$$

where $\tau: t\varrho_{\min} \to t\varrho$ $(t \in T)$, and ψ is an isomorphism. Since ϱ and $\varrho_{\min}$ have the same trace, τ is one-to-one on idempotents, that is to say, it is idempotent separating. In view of ([10], Theorem 4.2), $a\, \varrho_{\min}\, b$ if and only if $ae=be$ and $e\, \varrho\, a^{-1}a\, \varrho\, b^{-1}b$ for some $e \in E_S$. Thus $\sigma \supseteq \varrho_{\min}$. This together with the fact that T is E-unitary implies

$$\ker \varrho_{\min} \subseteq \ker \sigma = E_T$$

and thus $\ker \varrho_{\min} = E_T = E_T\omega$. This implies by Lemma 3.1 that $T/\varrho_{\min}$ is E-unitary. Since $T/\varrho_{\min} \in \mathscr{V}$, we have proved that S has an E-unitary cover in $\mathscr{V}$.

The equivalence of items (ii) and (iv) follows by Lemma 3.1.

Remark. Part of Theorem 3.3 has been obtained independently by F. Pastijn [8].

4. Subhomomorphisms

The results proved in this section contain a description of subhomomorphisms in terms of homomorphisms and will be used in the construction of subdirect products which in turn will be needed in a construction of E-unitary covers.

We start with a concept which will prove quite useful.

Definition 4.1. Let S and T be inverse semigroups. Then a mapping $\varphi: S \to 2^T$ is a *subhomomorphism* of S into T if, for all $s, t \in S$,

(i) $s\varphi \neq \emptyset$;

(ii) $(s\varphi)(t\varphi) \subseteq (st)\varphi$;

(iii) $s^{-1}\varphi = (s\varphi)^{-1}$,

where, for any subset A of T, $A^{-1} = \{a^{-1} | a \in A\}$.

From (ii) and (iii) it follows that $S\varphi = \cup \{s\varphi : s \in S\}$ is an inverse subsemigroup of T and φ is said to be *surjective*, if $S\varphi = T$.

If T is a group, then the subhomomorphism φ above is *unitary* if for any $s \in S$, $1 \in s\varphi$ implies $s \in E_S$.

The following result will be needed.

Proposition 4.2. [7] *Let S and T be inverse semigroups and let φ be a (surjective) subhomomorphism of S into T. Then*

$$\Pi(S, T, \varphi) = \{(s, t) \in S \times T \,|\, t \in s\varphi\}$$

is an inverse semigroup (which is a subdirect product of S and T).

Conversely, suppose that V is an inverse semigroup which is a subdirect product of S and T and let ψ be the induced monomorphism of V into $S\times T$. Then φ defined by

$$s\varphi = \{t\in T \mid (s, t)\in V\psi\}$$

is a surjective subhomomorphism of S into T. Furthermore, $V\psi=\Pi(S, T, \varphi)$.

Theorem 4.3. *Let R, S and T be inverse semigroups. Let $\alpha: R\to S$ be an epimorphism and $\beta: R\to T$ a homomorphism. Then $\varphi=\alpha^{-1}\beta$ is a subhomomorphism of S into T and every such subhomomorphism is obtained in this way. If, in addition, T is a group, then φ is unitary if and only if* $\ker\beta\subseteqq\ker\alpha$.

Proof. (i) It is clear that $s\varphi\neq\emptyset$ $(s\in S)$, since α is an epimorphism.

(ii) Let $x\in s\varphi, y\in t\varphi$. Then there exist $x', y'\in R$ with $x'\alpha=s$, $x'\beta=x$, $y'\alpha=t$, $y'\beta=y$. Hence $(x'y')\alpha=st$ while $(x'y')\beta=xy$ and $xy\in(st)\varphi$. Therefore $(s\varphi)(t\varphi)\subseteqq\subseteqq(st)\varphi$.

(iii) With x, x' as in (ii), $(x')^{-1}\alpha=s^{-1}$, $(x')^{-1}\beta=x^{-1}$. Hence $x^{-1}\in s^{-1}\varphi$, $(s\varphi)^{-1}\subseteqq s^{-1}\varphi$ and conversely. Thus φ is a subhomomorphism.

Conversely, if φ is a subhomomorphism of S into T, let $R=\Pi(S, T, \varphi)$. Let $\alpha:(s, t)\to s$ and $\beta:(s, t)\to t$ be the projections of R onto S and onto T, respectively. Now, $(s, t)\in R$ if and only if $t\in s\varphi$ while $t\in s\alpha^{-1}\beta$ if and only if $(s, t)\in R$ which gives $\varphi=\alpha^{-1}\beta$.

Let T be a group, φ be unitary and $r\in\ker\beta$. Then $r\beta=1$ and $1\in(r\alpha)\varphi$. Since φ is unitary, $r\alpha\in E_S$, $r\in\ker\alpha$ and so $\ker\beta\subseteqq\ker\alpha$. Conversely, if this inclusion holds and $1\in s\varphi$, then for some $r\in R, r\alpha=s$ and $r\beta=1$. Hence $r\in\ker\beta\subseteqq\subseteqq\ker\alpha$ so that $s^2=s$ and φ is unitary.

The usefulness of Theorem 4.3 lies in the fact that by choosing R appropriately, for example to be a free inverse semigroup, it is possible to generate subhomomorphisms. This technique will be used in the next section.

In fact, in order to obtain all subhomomorphisms it suffices to let R range over all free inverse semigroups, as we now show.

Proposition 4.4. *Let $\theta: S\to T$ be a subhomomorphism of the inverse semigroup S into the inverse semigroup T. Then there exist a free inverse semigroup F, an epimorphism $\alpha: F\to S$, and a homomorphism $\beta: F\to T$ with $\theta=\alpha^{-1}\beta$.*

Proof. By Theorem 4.3, there exist an inverse semigroup R, an epimorphism $\gamma: R\to S$ and a homomorphism $\delta: R\to T$ with $\theta=\gamma^{-1}\delta$. Let I_R be the free inverse semigroup on the set R and let $\pi: I_R\to R$ be the homomorphism defined by the identity mapping on the set of generators R. Let $\alpha=\pi\gamma, \beta=\pi\delta$ and let $x\in S$.

If $y \in x\theta$, then $x=z\gamma$, $y=z\delta$, for some $z \in R$ and so, considering z as a generator of I_R, we have $x=z\pi\gamma=z\alpha$, $y=z\pi\delta=z\beta$ and so $y \in x\alpha^{-1}\beta$. Conversely, if $y \in x\alpha^{-1}\beta$, then $x=z\alpha=(z\pi)\gamma$, $y=z\beta=(z\pi)\delta$, for some $z \in I_R$, and so $y \in x\gamma^{-1}\delta=\theta$. Therefore $\theta=\alpha^{-1}\beta$.

5. *E*-unitary covers over a group variety

The question that we now wish to consider is the following: for a given inverse semigroup S, or variety of inverse semigroups $\mathscr{V}$, and a given group variety $\mathscr{U}$, when will S or every member of $\mathscr{V}$ possess an E-unitary cover over some member of $\mathscr{U}$?

For the purposes of the following discussion, we consider inverse semigroups and groups as algebras in the variety of unary semigroups, that is as algebras with a binary operation $((x, y)\to xy)$ and unary operation $(x\to x^{-1})$.

Notation 5.1. Let X be a countably infinite set. We denote the free unary semigroup on X by U_X.

Any law in a unary semigroup is of the form $u=v$, for some $u, v \in U_X$. A construction for U_X was recently given by CLIFFORD [1].

For each set X, there exist fully invariant congruences $\varkappa, \lambda$ on U_X such that I_X and G_X are isomorphic to $U_X/\varkappa$ and U_X/λ, respectively, since I_X and G_X are free objects in their respective varieties. We will identify I_X and G_X with $U_X/\varkappa$ and U_X/λ, respectively.

Notation 5.2. Let X be any countably infinite set. For any variety of inverse semigroups $\mathscr{V}$, let $K_{\mathscr{V}}=\ker \varrho(\mathscr{V})$ and for any variety of groups $\mathscr{U}$, let $N_{\mathscr{U}}$ denote the corresponding fully invariant subgroup of G_X.

Definition 5.3. Let $\mathscr{U}$ be a variety of groups, S an inverse semigroup and $\mathscr{V}$ a variety of inverse semigroups. We will say that S (respectively, $\mathscr{V}$) *has E-unitary covers over* $\mathscr{U}$ if (for every $S \in \mathscr{V}$) there is a group $G \in \mathscr{U}$ for which there is an E-unitary cover of S over G.

It follows that $\mathscr{V}$ has E-unitary covers if and only if it has E-unitary covers over $\mathscr{V}\cap\mathscr{G}$.

Recall that an inverse monoid S with a group of units G is called *factorizable* if for each $s \in S$, there exists $g \in G$ such that $s \leqq g$. We will need the following results.

Theorem 5.4. [7] *Let G be a group and let S be an inverse semigroup. Let F be a factorizable inverse monoid with group of units G which contains S as an* *inverse subsemigroup. Suppose that, for each $g \in G$, there exists $s \in S$ such that $s \leqq g$. Then*

$$\{(s, g) \in S \times G | s \leqq g\}$$

is an E-unitary cover of S over G. Conversely, each E-unitary cover is isomorphic to a cover obtained in this way.

Proposition 5.5. [7] *Let S be an inverse semigroup and let G be a group. Suppose that φ is a surjective unitary subhomomorphism of S into G. Then $\Pi(S, G, \varphi)$ is an E-unitary cover of S over G. Conversely, let P be an E-unitary cover of S over G with associated homomorphisms $\alpha: P \to S, \beta: P \to G$ and let $\psi: P \to S \times G$ be the induced monomorphism. Then φ defined by*

$$s\varphi = \{g \in G | (s, g) \in P\psi\}$$

is a surjective unitary subhomomorphism of S into G and $P \cong \Pi(S, G, \varphi)$.

We are now ready for one of the main results of the paper.

Theorem 5.6. *Let S be an inverse semigroup, $\mathcal{U}$ be a variety of groups and X be a countably infinite set. The following are equivalent.*

(i) *S has an E-unitary cover over $\mathcal{U}$.*

(ii) *If $u^2 = u$ is a law in $\mathcal{U}$, then it is also a law in S.*

(iii) *For all homomorphisms $\alpha: I_X \to S, K_{\mathcal{U}} \subseteq \ker \alpha$.*

Proof. (i) implies (ii). Let $G \in \mathcal{U}$ and P be an E-unitary cover of S over G. By Theorem 5.4, P is isomorphic to an inverse subsemigroup of a factorizable inverse monoid F with group of units G. Let $u^2 = u$ be a law in $\mathcal{U}$, say $u = u(x_1, \ldots, x_n)$. Let $s_1, \ldots, s_n \in S$. Since F is factorizable, there exist $g_1, \ldots, g_n \in G$, with $s_i \leqq g_i$ $(i = 1, \ldots, n)$. Then

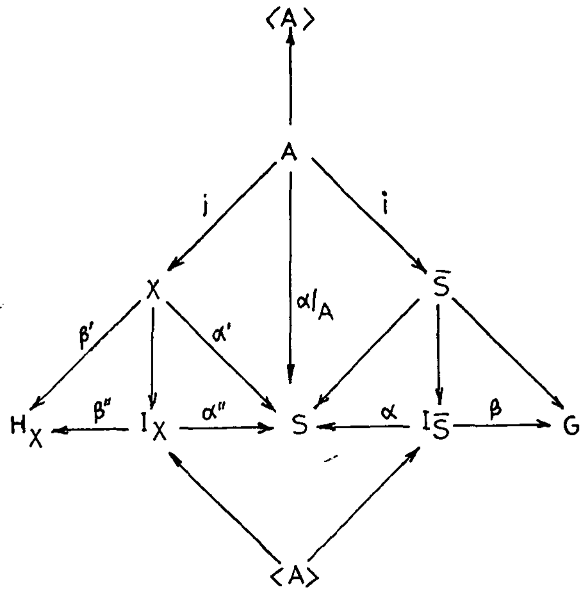
$$u(s_1, \ldots, s_n) \leqq u(g_1, \ldots, g_n)$$

where $u(g_1, \ldots, g_n)$ is the identity of G, since $G \in \mathcal{U}$ and $u^2 = u$ is a law in $\mathcal{U}$. Hence $u(s_1, \ldots, s_n)$ is an idempotent and $u^2 = u$ is a law in S.

(ii) implies (iii). Let $u \in U_X$ be such that $u\varkappa \in K_{\mathcal{U}}$. Then $u\lambda \in N_{\mathcal{U}}$ so that $u^2 = u$ is a law in $\mathcal{U}$ and so, by assumption, also in S. Hence, for any homomorphism $\beta: U_X \to S$, we have $u^2\beta = u\beta$. In particular, for any $\alpha: I_X \to S$, $u^2(\varkappa^{\natural}\alpha) = u(\varkappa^{\natural}\alpha)$ or $(u^2\varkappa, u\varkappa) \in \alpha \circ \alpha^{-1}$. Hence $u\varkappa \in \ker \alpha$.

(iii) implies (i). Let $\alpha: I_S \to S$ be the homomorphism defined on the generators of I_S by $s \to s$, let G be the free group in $\mathcal{U}$ on the set of generators S and let $\beta: I_S \to G$ be the natural homomorphism. By Theorem 4.3, $\theta = \alpha^{-1}\beta$ is a subhomomorphism of S into G. Since β is surjective so also is θ.

We next show that $\ker \beta \subseteq \ker \alpha$. The following diagram illustrates the proof.



Since it will help to clarify the discussion, we will denote by $\bar{S}$ the underlying set of S.

Let $a \in \ker \beta$. Then there exists a finite subset $A = \{x_1, \ldots, x_n\}$ of $\bar{S}$ such that a is contained in the inverse subsemigroup $\langle A \rangle$ of $I_{\bar{S}}$ generated by A. Let us identify A with a subset of X and extend $\alpha|_A$ arbitrarily to a mapping $\alpha' : X \to S$. Let $\alpha'' : I_X \to S$ be the unique extension of α' to a homomorphism of I_X into S. Then $\alpha''|_{\langle A \rangle} = \alpha|_{\langle A \rangle}$.

Let H_X be the relatively free group in $\mathcal{U}$ on the set X and let $\beta' : X \to H_X$ embed X identically. Let $\beta'' : I_X \to H_X$ be the unique extension of β' to a homomorphism of I_X into H_X. Then $\ker \beta'' = K_{\mathcal{U}}$. Furthermore, since $\beta'|_A = \beta|_A$ we have $\beta''|_{\langle A \rangle} = \beta|_{\langle A \rangle}$. Since $a \in \ker \beta$, we have $a \in \ker \beta'' = K_{\mathcal{U}}$. Hence, by (iii), $a \in \ker \alpha''$ and so $a \in \ker \alpha$. Thus $\ker \beta \subseteq \ker \alpha$.

Hence by Theorem 4.3, θ is a unitary subhomomorphism and by Proposition 5.5, there exists an E-unitary cover of S over G.

Theorem 5.6 has an obvious analogue for any variety of inverse semigroups $\mathcal{V}$, obtained by letting S range over $\mathcal{V}$.

Corollary 5.7. *Let $\mathcal{V}$ be a variety of inverse semigroups and $\mathcal{U}$ be a variety of groups. The following are equivalent.*

(i) *$\mathcal{V}$ has E-unitary covers over $\mathcal{U}$.*

(ii) *If* $u^2=u$ *is a law in* $\mathscr{U}$, *then it is also a law in* $\mathscr{V}$.
(iii) $K_{\mathscr{U}} \subseteq K_{\mathscr{V}}$.

Corollary 5.8. *Let* S *be an inverse semigroup and* $\mathscr{U}$ *be a group variety. If* S *has an E-unitary cover over* $\mathscr{U}$, *then* $\langle S \rangle$ *has E-unitary covers over* $\mathscr{U}$.

Proof. Let $u^2=u$ be a law in $\mathscr{U}$. By Theorem 5.6 (ii), $u^2=u$ is also a law in S. But then $u^2=u$ is also a law in $\langle S \rangle$, and the desired conclusion follows from Corollary 5.7.

As an application of the above theory, we now produce a variety of inverse semigroups which has E-unitary covers over almost all varieties of groups, but which does not itself have E-unitary covers.

Proposition 5.9 . *Let* B_2 *denote the 5-element Brandt semigroup with 3 idempotents. Then* $\langle B_2 \rangle$ *has E-unitary covers over any nontrivial group variety.*

Proof. Let I_1 denote the free inverse semigroup on one generator. It follows from [9] that, for each integer $n>1$, there is a congruence ϱ_n on I_1 such that $P_n = I_1/\varrho_n$ is an ideal extension of the cyclic group Z_n of order n by B_2 which is E-unitary. Furthermore, the projection of P_n onto B_2 is idempotent separating, since the ideal is a group. Hence each P_n is an E-unitary cover for B_2. Now $\mathscr{G} \cap \langle P_n \rangle$ is simply the variety $\mathscr{A}_n$ of abelian groups of exponent n. Thus B_2 and so, by Corollary 5.8, $\langle B_2 \rangle$ has E-unitary covers over each variety $\mathscr{A}_n$ $(n>1)$, of abelian groups of exponent n, and so over every nontrivial variety of groups.

We shall now see how the equivalence of (iv) and (i) in Theorem 3.3 can be used to establish that varieties have E-unitary covers.

In $\mathscr{L}(\mathscr{I})$, the various varieties generated by groups, semilattices and Brandt semigroups constitute an ideal isomorphic to the product of $\mathscr{L}(\mathscr{G})$ and a three element chain. (See KLEĬMAN [3].) Following [9], we will call any semigroup in any of these varieties a *strict inverse semigroup*. Each variety of strict inverse semigroups which is not a variety of groups and semilattices of groups is generated by a single Brandt semigroup. Moreover, if $\mathscr{V} = \langle B \rangle$ where $B = \mathscr{M}^0(I, G, I; \Delta)$, then $\mathscr{V} = \langle G \rangle \vee \langle B_2 \rangle$ where $\langle G \rangle$ is now a variety of groups. Similarly, any variety $\mathscr{V}$ of semilattices of groups which is not a variety of groups is of the form $\mathscr{U} \vee \mathscr{S}$, where $\mathscr{U}$ is a variety of groups and $\mathscr{S}$ is the variety of semilattices. For more details on this subject, see KLEĬMAN [3]:

Proposition 5.10. *If* $\mathscr{V}$ *is a variety of strict inverse semigroups containing nontrivial groups, then* $\mathscr{V}$ *has E-unitary covers.*

Proof. First let $\mathscr{V} = \mathscr{U} \vee \langle B_2 \rangle$, where $\mathscr{U}$ is a nontrivial variety of groups and let $S \in \mathscr{V}$. By the general theory of varieties, there exist T, A, B where $A \in \mathscr{U}$,

$B \in \langle B_2 \rangle$ and $T \subseteq A \times B$ is a subdirect product of A and B together with an epimorphism φ of T onto S. Since $\mathcal{U}$ is nontrivial, by Proposition 5.9 there exists an E-unitary cover P, say, of B over $\mathcal{U}$. Then $P \in \mathcal{U} \vee \langle B_2 \rangle = \mathcal{V}$ by ([7], Corollary 1.8). Let $\alpha: P \to B$ be an idempotent separating epimorphism and let $T' = \{(a, p) | (a, p\alpha) \in T\} \subseteq A \times P$. Since A is a group and P is E-unitary, $A \times P$ is E-unitary. Hence T' is also E-unitary. Moreover, $T' \in \mathcal{V}$ and $(a, p) \to (a, p\alpha)\varphi$ is an epimorphism of T' onto S. By Theorem 3.3 (iv), $\mathcal{V}$ has E-unitary covers (over $\mathcal{U}$). A similar argument will show that any variety of semilattices of groups has E-unitary covers and clearly varieties of groups do also.

Remark 5.11. The arguments of Proposition 5.10 would also apply to any variety of the form $\mathcal{U} \vee \langle B_2^1 \rangle$, where $\mathcal{U}$ is a non-trivial variety of groups.

6. The Malcev product

For any group variety $\mathcal{U}$ we will now characterize the class of all inverse semigroups $\mathcal{V}$ which have E-unitary covers over $\mathcal{U}$. It will turn out that the variety generated by the Malcev product $\mathcal{S} \circ \mathcal{U}$, where $\mathcal{S}$ denotes the variety of semilattices, is the greatest variety of inverse semigroups having E-unitary covers over $\mathcal{U}$. The variety generated by $\mathcal{S} \circ \mathcal{U}$ will be characterized in several ways.

Notation 6.1. We will denote by $\mathcal{S}$ the variety of all semilattices. For any variety of groups $\mathcal{U}$,

$$\mathcal{S} \circ \mathcal{U} = \{P \in \mathcal{I} | P \text{ is } E\text{-unitary and } P/\sigma \in \mathcal{U}\}$$

is the *Malcev product* of $\mathcal{S}$ and $\mathcal{U}$. For any family of laws $u_\alpha = v_\alpha$, $\alpha \in A$, we write $\langle u_\alpha = v_\alpha | \alpha \in A \rangle$ for the variety of inverse semigroups determined by these laws.

Another highlight of the paper can now be established.

Theorem 6.2. *The following statements are valid for any group variety $\mathcal{U}$.*

(i) $\langle \mathcal{S} \circ \mathcal{U} \rangle = \langle u^2 = u | u^2$ *is a law in* $\mathcal{U} \rangle$.

(ii) $\langle \mathcal{S} \circ \mathcal{U} \rangle = \{S \in \mathcal{I} | S$ *has an E-unitary cover over* $\mathcal{U}\}$.

(iii) $\langle \mathcal{S} \circ \mathcal{U} \rangle$ *is the largest variety of inverse semigroups with E-unitary covers over* $\mathcal{U}$.

(iv) $\mathcal{U}$ *is the smallest variety of groups over which* $\langle \mathcal{S} \circ \mathcal{U} \rangle$ *has E-unitary covers.*

Proof. (i) Let $\mathcal{V} = \langle \mathcal{S} \circ \mathcal{U} \rangle$ and $\mathcal{W} = \langle u^2 = u | u^2 = u$ is a law in $\mathcal{U} \rangle$. First let $S \in \mathcal{S} \circ \mathcal{U}$ and let $u^2 = u$ be a law in $\mathcal{U}$. By the definition of $\mathcal{S} \circ \mathcal{U}$, we have

$S/\sigma \in \mathcal{U}$ and thus $u^2=u$ is a law in S/σ. Hence, for any substitution $\bar{u}$ of u in S, it follows that $\bar{u}^2 \sigma \bar{u}$, whence $\bar{u} \in \ker \sigma = E_S$. Thus $u^2=u$ is a law in S. Consequently, $S \in \mathcal{W}$ and thus $\mathcal{S} \circ \mathcal{U} \subseteqq \mathcal{W}$. But then also $\mathcal{V} = \langle \mathcal{S} \circ \mathcal{U} \rangle \subseteqq \mathcal{W}$.

Conversely, let $S \in \mathcal{W}$. Then by Theorem 5.6, S has an E-unitary cover P over G for some $G \in \mathcal{U}$. It follows that $P \in \mathcal{S} \circ \mathcal{U}$ and hence $S \in \langle \mathcal{S} \circ \mathcal{U} \rangle = \mathcal{V}$. Therefore $\mathcal{W} \subseteqq \mathcal{V}$ and equality prevails.

(ii) This is a direct consequence of part (i) and Theorem 5.6.

(iii) This is an obvious consequence of part (ii).

(iv) Let $\mathcal{V}$ be a variety of groups over which $\langle \mathcal{S} \circ \mathcal{U} \rangle$ has E-unitary covers, and let $G \in \mathcal{U}$. Then $G \in \langle \mathcal{S} \circ \mathcal{U} \rangle$ and hence has an E-unitary cover P over $\mathcal{V}$. Now, P being an E-unitary cover of a group must itself be a group. Since G is a homomorphic image of P, we obtain that $G \in \mathcal{V}$. Consequently $\mathcal{U} \subseteqq \mathcal{V}$, as required.

An interesting property of the varieties $\mathcal{V}$ between $\mathcal{U}$ and $\mathcal{S} \circ \mathcal{U}$ is provided by the next result.

Proposition 6.3. *For any variety of groups $\mathcal{U}$ and any variety $\mathcal{V}$ of inverse semigroups, the following holds:*

$$\ker \varrho(\mathcal{U}) = \ker \varrho(\mathcal{V}) \Leftrightarrow \mathcal{U} \subseteqq \mathcal{V} \subseteqq \langle \mathcal{S} \circ \mathcal{U} \rangle.$$

Proof. First assume that $\ker \varrho(\mathcal{U}) = \ker \varrho(\mathcal{V})$. This means that $w^2=w$ is a law in $\mathcal{U}$ if and only if $w^2=w$ is a law in $\mathcal{V}$. It follows from Theorem 6.2 (i) that $\mathcal{V} \subseteqq \langle \mathcal{S} \circ \mathcal{U} \rangle$. Since $\mathcal{U}$ is a group variety, $\operatorname{tr} \varrho(\mathcal{U}) = \omega$ and thus $\operatorname{tr} \varrho(\mathcal{U}) \supseteqq \operatorname{tr} \varrho(\mathcal{V})$. This together with the hypothesis that $\ker \varrho(\mathcal{U}) = \ker \varrho(\mathcal{V})$ implies that $\varrho(\mathcal{U}) \supseteqq \varrho(\mathcal{V})$ and thus $\mathcal{U} \subseteqq \mathcal{V}$.

Conversely, assume that $\mathcal{U} \subseteqq \mathcal{V} \subseteqq \langle \mathcal{S} \circ \mathcal{U} \rangle$. The first inclusion implies $\varrho(\mathcal{U}) \supseteqq \varrho(\mathcal{V})$ and thus $\ker \varrho(\mathcal{U}) \supseteqq \ker \varrho(\mathcal{V})$. The second inclusion implies $\ker \varrho(\mathcal{U}) \subseteqq \ker \varrho(\mathcal{V})$ by Theorem 6.2 (i), as above. Therefore $\ker \varrho(\mathcal{U}) = \ker \varrho(\mathcal{V})$.

7. An equivalence relation on $\mathcal{L}(\mathcal{I})$

We introduce a relation on $\mathcal{L}(\mathcal{I})$ which relates any two varieties if they have the same E-unitary members and consider some associated properties.

In order to put the relation we are introducing into the proper perspective, we include two known relations v_1 and v_2 in our scheme. For any $\mathcal{U}, \mathcal{V} \in \mathcal{L}(\mathcal{I})$, let

$$\mathcal{U} v_1 \mathcal{V} \Leftrightarrow \mathcal{U} \cap \mathcal{A} = \mathcal{V} \cap \mathcal{A}, \quad \mathcal{U} v_2 \mathcal{V} \Leftrightarrow \mathcal{U} \cap \mathcal{G} = \mathcal{V} \cap \mathcal{G}, \quad \mathcal{U} v_3 \mathcal{V} \Leftrightarrow \mathcal{U} \cap \mathcal{E} = \mathcal{V} \cap \mathcal{E}.$$

Here $\mathscr{A}$, $\mathscr{G}$, and $\mathscr{E}$ denote the classes of all antigroups (fundamental inverse semigroups), groups and E-unitary inverse semigroups. The relations ν_1 and ν_2 were introduced by KLEĬMAN [3], who showed that they are congruences. He defined ν_1 as follows: $\mathscr{U}\nu_1\mathscr{V}\Leftrightarrow\mathscr{U}\vee\mathscr{G}=\mathscr{V}\vee\mathscr{G}$, and then proved the above equivalence. The relation ν_3 is new and the subject of our study in this section.

We can say that $\mathscr{U}\nu_3\mathscr{V}$ precisely when $\mathscr{U}$ and $\mathscr{V}$ have the same E-unitary members.

Proposition 7.1. $\nu_1\cap\nu_2\subseteqq\nu_3\subseteqq\nu_2$.

Proof. Let $\mathscr{U}(\nu_1\cap\nu_2)\mathscr{V}$ and $S\in\mathscr{U}\cap\mathscr{E}$. Since S is E-unitary, $\mathscr{H}\cap\sigma=\varepsilon$, the equality relation. Hence $\mu\cap\sigma=\varepsilon$ and thus S is a subdirect product of S/μ and S/σ. Here $S/\mu\in\mathscr{U}\cap\mathscr{A}$ and $S/\sigma\in\mathscr{U}\cap\mathscr{G}$. Since $\mathscr{U}\nu_1\mathscr{V}$, we have $S/\mu\in\mathscr{V}\cap\mathscr{A}$, and since $\mathscr{U}\nu_2\mathscr{V}$, we get $S/\sigma\in\mathscr{V}\cap\mathscr{G}$. But then $S\in(\mathscr{V}\cap\mathscr{A})\vee(\mathscr{V}\cap\mathscr{G})\subseteqq\mathscr{V}$, which proves that $\mathscr{U}\cap\mathscr{E}\subseteqq\mathscr{V}\cap\mathscr{E}$. By symmetry, we conclude that $\mathscr{U}\nu_3\mathscr{V}$. This proves that $\nu_1\cap\nu_2\subseteqq\nu_3$. If $\mathscr{U}\cap\mathscr{E}=\mathscr{V}\cap\mathscr{E}$, then intersecting by $\mathscr{G}$, we get $\mathscr{U}\cap\mathscr{G}=\mathscr{V}\cap\mathscr{G}$. Hence $\nu_3\subseteqq\nu_2$.

Remark 7.2. It should be noted that ν_3 is not a congruence on $\mathscr{L}(\mathscr{I})$. If $\mathscr{W}=\langle B_2\rangle$, $\mathscr{W}'=\langle B_2^1\rangle$, then $\mathscr{W}\nu_3\mathscr{W}'$. However, $(\mathscr{W}\vee\mathscr{G})\cap\mathscr{E}\subset(\mathscr{W}'\vee\mathscr{G})\cap\mathscr{E}$.

Proposition 5.9 shows that, in general, for a given variety of inverse semigroups $\mathscr{V}$, there is no minimum variety $\mathscr{U}$ of groups such that $\mathscr{V}$ has E-unitary covers over $\mathscr{U}$. This may be contrasted with the next result.

Proposition 7.3. *The following statements are true for any variety of inverse semigroups* $\mathscr{V}$.

(i) $\langle\mathscr{V}\cap\mathscr{E}\rangle$ *is the smallest member of the* ν_3*-class containing* $\mathscr{V}$.

(ii) $\langle\mathscr{V}\cap\mathscr{E}\rangle$ *is the largest variety contained in* $\mathscr{V}$ *having E-unitary covers.*

(iii) $\langle\mathscr{V}\cap\mathscr{E}\rangle=\{S\in\mathscr{I}\mid S$ *has an E-unitary cover in* $\mathscr{V}\}$.

Proof. (i) First note that

$$\langle\mathscr{V}\cap\mathscr{E}\rangle\cap\mathscr{E}\subseteqq\mathscr{V}\cap\mathscr{E}\subseteqq\langle\mathscr{V}\cap\mathscr{E}\rangle\cap\mathscr{E}$$

which shows that $\langle\mathscr{V}\cap\mathscr{E}\rangle\nu_3\mathscr{V}$. Now let $\mathscr{W}\nu_3\mathscr{V}$. Then $\mathscr{W}\cap\mathscr{E}=\mathscr{V}\cap\mathscr{E}$ which implies that $\langle\mathscr{V}\cap\mathscr{E}\rangle=\langle\mathscr{W}\cap\mathscr{E}\rangle\subseteqq\mathscr{W}$, as required.

(ii) Since $\langle\mathscr{V}\cap\mathscr{E}\rangle$ is generated by E-unitary inverse semigroups, it has E-unitary covers by Theorem 3.3. Let $\mathscr{W}$ be a variety of inverse semigroups contained in $\mathscr{V}$ and having E-unitary covers. Again by Theorem 3.3, we get $\mathscr{W}=\langle\mathscr{W}\cap\mathscr{E}\rangle$. Since also $\langle\mathscr{W}\cap\mathscr{E}\rangle\subseteqq\langle\mathscr{V}\cap\mathscr{E}\rangle$, we conclude that $\mathscr{W}\subseteqq\langle\mathscr{V}\cap\mathscr{E}\rangle$, as required.

(iii) We have already observed that every S in $\langle\mathscr{V}\cap\mathscr{E}\rangle$ has an E-unitary cover in $\langle\mathscr{V}\cap\mathscr{E}\rangle$ and thus in $\mathscr{V}$. Conversely, let S have an E-unitary cover

P in $\mathcal{V}$. Hence $P\in\mathcal{V}\cap\mathcal{E}\subseteq\langle\mathcal{V}\cap\mathcal{E}\rangle$ and S is a homomorphic image of P so that $S\in\langle\mathcal{V}\cap\mathcal{E}\rangle$.

It can be verified easily that any group variety $\mathcal{U}$ alone constitutes a ν_3-class. If $\mathcal{V}$ is a variety of inverse semigroups contained in $\langle x^n=x^{n+1}\rangle$, then no S in $\mathcal{V}$ which is not a semilattice is E-unitary since $a^n=a^na$ and $a^2\neq a$ for any non-idempotent element a in S. In view of this and the results of KLEĬMAN [3], we conclude that the join of all varieties ν_3-equivalent to $\mathcal{S}$ is equal to $\mathcal{I}$.

Some additional information about $\langle\mathcal{S}\circ\mathcal{U}\rangle$ is provided by the following statement.

Proposition 7.4. *For any group variety $\mathcal{U}$, we have*

$$\langle\mathcal{S}\circ\mathcal{U}\rangle\cap\mathcal{G}=\mathcal{U},\ \langle\mathcal{S}\circ\mathcal{U}\rangle\cap\mathcal{E}=\mathcal{S}\circ\mathcal{U}.$$

Proof. Let $G\in\langle\mathcal{S}\circ\mathcal{U}\rangle\cap\mathcal{G}$ and let $u^2=u$ be a law in $\mathcal{U}$. By Theorem 5.6, $u^2=u$ is also a law in G, and thus $G\in\mathcal{U}$ since every law in $\mathcal{U}$, except $xx^{-1}=yy^{-1}$, can be written in the form $u^2=u$. Consequently, $\langle\mathcal{S}\circ\mathcal{U}\rangle\cap\mathcal{G}\subseteq\mathcal{U}$; the opposite inclusion is obvious.

Let $S\in\langle\mathcal{S}\circ\mathcal{U}\rangle\cap\mathcal{E}$. Then $S/\sigma\in\langle\mathcal{S}\circ\mathcal{U}\rangle\cap\mathcal{G}=\mathcal{U}$ by the first formula. Since S is E-unitary, we obtain that $S\in\mathcal{S}\circ\mathcal{U}$. Therefore $\langle\mathcal{S}\circ\mathcal{U}\rangle\cap\mathcal{E}\subseteq\mathcal{S}\circ\mathcal{U}$; the opposite inclusion is trivial.

In connection with the congruences ν_1 and ν_2, and Theorem 3.3, the next proposition seems to be of some interest. For it, we need a known result.

Lemma 7.5. [3] *For any variety of inverse semigroups $\mathcal{V}$, the minimum element of $\mathcal{V}(\nu_1\cap\nu_2)$ is $(\mathcal{V}\cap\mathcal{G})\vee\langle\mathcal{V}\cap\mathcal{A}\rangle$.*

Proposition 7.6. *Let $\mathcal{V}$ be a variety of inverse semigroups. Consider the following conditions on $\mathcal{V}$.*

(i)—(iv) *The conditions of Theorem* 3.3.

(v) *For every $S\in\mathcal{V}$, there exists $G\in\mathcal{V}\cap\mathcal{G}$, an inverse semigroup T which is a subdirect product of S/μ and G, and an idempotent separating epimorphism $\varphi\colon T\to S$.*

(vi) *$\mathcal{V}$ is the minimum element of its $\nu_1\cap\nu_2$-class.*

Then (i) *implies* (v) *and* (v) *implies* (vi).

Proof. (i) implies (v). Let $S, T\in\mathcal{V}$ where T is an E-unitary cover of S. Then T is a subdirect product of T/μ and T/σ since $\mu\cap\sigma=\varepsilon$. Since T is an E-unitary cover of S it follows that $T/\mu\cong S/\mu$, so that T is a subdirect product of S/μ and S/σ, where the latter is in $\mathcal{V}\cap\mathcal{G}$.

(v) implies (vi). Let the notation be as in part (v). Then $S \in \langle S/\mu \times G \rangle \subseteqq \subseteqq \langle \mathscr{V} \cap \mathscr{A} \rangle \vee (\mathscr{V} \cap \mathscr{G})$ which proves that $\mathscr{V} \subseteqq \langle \mathscr{V} \cap \mathscr{A} \rangle \vee (\mathscr{V} \cap \mathscr{G})$; the opposite inclusion is trivial. By Lemma 7.5, we have that $\mathscr{V}$ is the minimum element of its $v_1 \cap v_2$-class.

The first implication in the above proposition cannot be reversed. For example, the variety $\mathscr{V} = \langle x^3 = x^2 \rangle$ of inverse semigroups satisfies part (v) but not part (i). We have no counterexample for the converse of the second implication.

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Separation of the radical in ring varieties

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Varieties of associative rings in which the Jacobson radical of every member is 1) nil, 2) nilpotent, or 3) a direct summand were studied in [1]. Varieties satisfying 1) or 2) were described there; the same varieties were characterized independently by the author in [2]. As to condition 3), Theorem 19 from [1] states that varieties in which the Jacobson radical of every finitely generated ring is a direct summand may be given by a finite set of two-variable identities. However these identities cannot be found by the method from [1], and the problem of exact description of varieties satisfying condition 3) remained open. This note is devoted to solve that problem.

Theorem. *The following conditions on an associative ring variety $\mathfrak{X}$ are equivalent:*

(a) *the Jacobson radical of every member is a direct summand;*

(b) *the Jacobson radical of every finitely generated member is a direct summand;*

(c) *$\mathfrak{X}$ is generated by a finite (possibly empty) set of finite fields and by a nil-ring of restricted index;*

(d) *the identities*

$$X^k Y = YX^k = X^k Y^n \tag{*}$$

hold in $\mathfrak{X}$ for some natural numbers $k \geqq 1$ and $n \neq 1$.

Proof. (a)$\to$(b) obviously.

(b)$\to$(c). We consider for every prime number p the variety $\mathfrak{A}_p$ given by the identities $XY-YX=pX=0$. There are finitely generated rings in $\mathfrak{A}_p$ in which the radical is not a direct summand, for example, the ring S_p of all 2×2 matrices of the form $\begin{pmatrix}\alpha & \beta\\ 0 & \alpha\end{pmatrix}$ is such where α and β run through the p-element field. Hence

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$\mathfrak{X}$ does not contain $\mathfrak{A}_p$ for any p, and some identity $X^m=X^n$, $m<n$, holds in $\mathfrak{X}$ by the main theorem from [2]. $\mathfrak{X}$ is generated by its finite rings ([2], Corollary 1) and therefore by its finite subdirectly irreducible rings. If R is such a ring then we get by (b) that either $R=J(R)$ (and R is nilpotent) or $J(R)=0$ (and R is simple). A finite simple ring is either a finite field or the ring of all $r\times r$ matrices over a finite field ($r>1$). However, rings of the second type cannot be contained in $\mathfrak{X}$ since every such ring contains the ring S_p for some p as subring. We see that the variety is generated by its finite nilpotent rings and finite fields. It remains to note that only a finite number of finite fields may be contained in $\mathfrak{X}$ and all finite nilpotent rings from $\mathfrak{X}$ satisfy the identity $X^m=0$ (in view of the identity $X^m=X^n$ holding in $\mathfrak{X}$). Thus, the direct sum of all finite nilpotent rings from $\mathfrak{X}$ is the required nilring of restricted index.

(c)$\to$(d). Let $N, F_1, \ldots, F_s$ be rings generating $\mathfrak{X}$ where the identity $X^k=0$ holds in N, and $F_1, \ldots, F_s$ are finite fields. If F_i consists of m_i elements and $n=(m_1-1)\ldots(m_s-1)+1$, then the identity $X^n=X$ holds in every field F_i. We see that the identities (*) hold in all rings generating $\mathfrak{X}$, hence they hold in all rings from $\mathfrak{X}$.

(d)$\to$(a). Let R be a ring satisfying (*). It is easy to see that $J(R)$ is nil and the idempotents of R lie in its center. Further, since an arbitrary ring of $r\times r$ matrices over a field ($r>1$) does not satisfy (*) a standard application of Kaplansky's theorem about primitive *PI*-rings shows that $R/J(R)$ is a subdirect sum of finite fields and satisfies therefore the identity $X^n=X$. Denote by E the ideal of R generated by all idempotents of R. Let $y=\sum_{i=1}^{m} r_i e_i \in J(R)\cap E$, where $r_i\in R$, e_i are idempotents. Let us consider the element

$$e=\sum_{i=1}^{m} e_i-\sum_{1\le i<j\le m} e_ie_j+\sum_{1\le i<j<s\le m} e_ie_je_s-\ldots+(-1)^{m+1}e_1\ldots e_m.$$

It can be immediately verified that $e^2=e$ and $e_ie=e_i$ for any i. Thus, $y=ye=$ $=ye^k=y^ne^k=y^{2n-1}e^k=\ldots=0$. On the other hand, the image of the element x^{n-1} in the ring $R/J(R)$ is an idempotent for every $x\in R$. We lift it to an idempotent e_x of the ring R; then $x-xe_x\in J(R)$ and $x=xe_x+(x-xe_x)\in E+J(R)$. We see that R is a direct sum of the ideals $J(R)$ and E.

The theorem is proved.

Let us recall that a ring R is called a semidirect sum of an ideal J and a subring S if $S+J=R$, $S\cap J=0$. In connection with our theorem we pose a natural

Question. What are the ring varieties in which the Jacobson radical of 1) every, 2) every finitely generated member is a semidirect summand?

Note that these classes of varieties are sufficiently large. Thus, all locally finite varieties of prime characteristic belong to the second of them by Wedderburn's classical theorem about separation of the radical.

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Disjoint sublattices of lattices

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1. Introduction

M. Sekanina asked whether there exist lattices A and B such that A contains an arbitrarily large finite number of pairwise disjoint sublattices isomorphic to B but does not contain infinitely many pairwise disjoint sublattices isomorphic to B. Independently, I. KOREC [2] and V. KOUBEK [3] have shown that such lattices do indeed exist. In fact, Koubek has shown that both A and B may be chosen to be distributive.

The aim of the present paper is to strengthen Koubek's result by showing that the distributive lattices A and B may be chosen to be totally ordered sets. Actually more will be shown. The principal result will be the following:

Theorem. *There exist totally ordered sets A and B_α, for $\alpha<2^{2^{\aleph_0}}$, such that* (i) $|A|=2^{\aleph_0}$, (ii) $B_\alpha\cong B_\beta$ *if and only if* $\alpha=\beta$, *and* (iii) *if* $\alpha<2^{2^{\aleph_0}}$ *then, for* $n<\omega$, *A contains n disjoint copies of B_α, but it does not contain infinitely many such copies.*

That A is uncountable is no coincidence. A routine proof, using Hausdorff's classification of the countable order types, shows that if A is a countable totally ordered set that contains an arbitrarily large number of finite disjoint copies of a totally ordered set B then A contains infinitely many disjoint copies of B. (We shall not include the details.)

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2. The construction

The construction of the totally ordered set A involves a new variation of a technique first introduced by B. DUSHNIK and E. W. MILLER [1].

Let λ denote the real line $[0, 1)$ and η its rational members. The Dedekind completion of a totally ordered set C will be given by C^+. Observe that for two totally ordered sets C and D any order preserving injection of C into D can be extended to an order preserving injection of C^+ into D^+. Since a monotone function on λ has at most countably many discontinuities, it is readily seen that there are $2^{\aleph_0}$ order preserving injections of λ into itself. With the exception of the identity function, let $G=\{g_\beta \mid 1 \leqq \beta < 2^{\aleph_0}\}$ be a list of all the order preserving injections of λ into itself.

We now define a distinguished countable subset of G. For $1 \leqq i < \omega$ and $1 \leqq k \leqq i!$ define

$$I_{ik} = [(k-1)/(i!),\ k/(i!));$$

that is, for each i, $\{I_{ik}\colon 1 \leqq k \leqq i!\}$ is a system of pairwise disjoint intervals of length $1/(i!)$ covering λ. If $1 \leqq j \leqq i+1$, define an order preserving injection $f_{ij}\colon \lambda \to \lambda$ by

$$f_{ij}(x) = x/(i+1) + ((k-1)i + (j-1))/(i+1!)$$

for $x \in I_{ik}$ and $k=1, \ldots, i!$. Observe that $f_{ij}(I_{ik}) = [(k-1)/(i!)+(j-1)/(i+1!), (k-1)/(i!)+j/(i+1!)) = J_{ijk} \subseteq I_{ik}$ for every $j=1, \ldots, i+1$. The function f_{ij} is said to be of *type* i.

By way of example, it follows that there are exactly two functions of type one: f_{11} is an order preserving bijection of $[0, 1)$ to $[0, 1/2)$ given by $f_{11}(x)=\frac{1}{2}x$; $f_{12}(x)=\frac{1}{2}x+\frac{1}{2}$ is an order preserving bijection of $[0, 1)$ to $[1/2, 1)$. There are three functions of type two: f_{21} is the order preserving bijection of $[0, 1)$ to $\left[0, \frac{1}{6}\right) \cup \left[\frac{1}{2}, \frac{2}{3}\right)$ defined by $f_{21}(x)=\frac{1}{3}x$, for $0 \leqq x < \frac{1}{2}$, and $f_{21}(x)=\frac{1}{3}x+\frac{1}{3}$ for $\frac{1}{2} \leqq x < 1$; f_{22} is the order preserving bijection from $[0, 1)$ to $\left[\frac{1}{6}, \frac{1}{3}\right) \cup \left[\frac{2}{3}, \frac{5}{6}\right)$ given by, for $0 \leqq x < \frac{1}{2}$, $f_{22}(x)=\frac{1}{3}x+\frac{1}{6}$ and, for $\frac{1}{2} \leqq x < 1$, $f_{22}(x)=\frac{1}{3}x+\frac{1}{2}$; finally, f_{23} is the order preserving bijection from $[0, 1)$ to $\left[\frac{1}{3}, \frac{1}{2}\right) \cup \left[\frac{5}{6}, 1\right)$ such that $f_{23}(x)=\frac{1}{3}x+\frac{1}{3}$, for $0 \leqq x < \frac{1}{2}$, and $f_{23}(x)=\frac{1}{3}x+\frac{2}{3}$ for $\frac{1}{2} \leqq x < 1$.

Let $F=\{f_{ij} \mid 1 \leqq i < \omega \text{ and } 1 \leqq j \leqq i+1\}$; for $x \in \lambda$, denote $F(x)=\{f(x) \mid f \in F\}$;

and, for $X\subseteqq\lambda$, let $F(X)=\bigcup(F(x)\colon x\in X)$. Note that, for every $f\in F$, x is rational if and only if $f(x)$ is rational. Since F is countable, we may conclude the following:

Lemma 1. $|\{x\in\lambda|x\in F(x)\}|=\aleph_0$.

We shall also need the following lemma.

Lemma 2. *For* $X, Y\subseteqq\lambda$, *if* $|X|=2^{\aleph_0}$ *and* $|Y|<2^{\aleph_0}$ *then there exists* $x\in X$ *such that* $F(x)\cap Y=\emptyset$.

Proof. Suppose that for every $x\in X$ there exists an $f\in F$ with $f(x)\in Y$. For $y\in Y$, let $X_y=\{x\in X\mid y\in F(x)\}$. Thus, $X\subseteqq\bigcup(X_y\colon y\in Y)$. Since $|X|=2^{\aleph_0}$ and $|Y|<2^{\aleph_0}$, it follows that X_y is uncountable for some $y\in Y$. However, F is countable. Hence, there are two distinct elements x of X_y such that $f(x)=y$ for the same $f\in F$. Since each $f\in F$ is one-to-one, this is a contradiction. The proof is complete.

Some further notation is necessary. For $g\in G$, define $g_F=\{x\in\lambda\mid g(x)\notin F(x)\}$. Then set $G_F=\{g\in G\,\|g_F|<2^{\aleph_0}\}$. Clearly, $F\subseteqq G_F$ follows from $f_F=\emptyset$ for every $f\in F$; it is also easy to see that the inclusion is proper.

We are now ready to define the totally ordered sets A and B_α for $\alpha<2^{\aleph_0}$. As will transpire, the totally ordered set A will be a subset of λ that contains η; the definition will be given by transfinite induction. For $\beta<2^{\aleph_0}$, sets A_β, C_β, $D_\beta\subseteqq\lambda$ will be defined; subsequently, A will be the set $\lambda\setminus\bigcup(A_\beta\colon\beta<2^{\aleph_0})$ and, for $\alpha<2^{2^{\aleph_0}}$, $\bigcup(C_\beta\colon\beta<2^{\aleph_0})\subseteqq B_\alpha\subseteqq\bigcup(C_\beta\cup D_\beta\colon\beta<2^{\aleph_0})$. Intuitively, the mappings from F will be used to exhibit arbitrarily many finite disjoint copies of B_α in A and the construction will ensure that no $g\notin G_F$ can be used to provide an order preserving injection of B_α into A.

Let $A_0=\emptyset$, $A'_0=\eta$, $C_0=\eta$, $C'_0=\emptyset$, and $D_0=\emptyset$. By transfinite induction we will define, for $\beta<2^{\aleph_0}$, $A_\beta, A'_\beta, C_\beta, C'_\beta, D_\beta\subseteqq\lambda$ such that (i) $|A_\beta|, |A'_\beta|, |C_\beta|, |C'_\beta|, |D_\beta|<2^{\aleph_0}$, (ii) for $\gamma<\beta$, $A_\gamma\subseteqq A_\beta$, $A'_\gamma\subseteqq A'_\beta$, $C_\gamma\subseteqq C_\beta$, $C'_\gamma\subseteqq C'_\beta$, and $D_\gamma\subset D_\beta$, (iii) $A_\beta\cap A'_\beta=\emptyset$, $C_\beta\cap C'_\beta=\emptyset$, and $(C_\beta\cup C'_\beta)\cap D_\beta=\emptyset$ and (iv) $F(C_\beta)\subseteqq A'_\beta$ and $F(D_\beta)\subseteqq A'_\beta$. (Note that these conditions are satisfied for $\beta=0$.) Suppose that, for $\gamma<\beta<2^{\aleph_0}$, A_γ, A'_γ, C_γ, C'_γ, D_γ are defined and satisfy (i), (ii), (iii), and (iv).

Since $g_\beta\colon\lambda\to\lambda$ is not the identity and is order preserving, there are $2^{\aleph_0}$ elements $x\in\lambda$ such that $x\neq g_\beta(x)$. Thus, because g_β is injective, the set of all elements $x\in\lambda$ such that $x\neq g_\beta(x)$, $x\notin\bigcup(C'_\gamma\colon\gamma<\beta)\cup\bigcup(D_\gamma\colon\gamma<\beta)$, and $g_\beta(x)\notin\bigcup(C_\gamma\colon\gamma<\beta)\cup\bigcup\bigcup(D_\gamma\colon\gamma<\beta)$ has cardinality $2^{\aleph_0}$. By Lemma 2, choose such an $x\in\lambda$ for which $F(x)\cap\bigcup(A_\gamma\colon\gamma<\beta)=\emptyset$. Let $C'_\beta=\{g_\beta(x)\}\cup\bigcup(C'_\gamma\colon\gamma<\beta)$.

By Lemma 2, there exists $y\in\lambda\setminus(\{x\}\cup\bigcup(C_\lambda\colon\lambda<\beta)\cup C'_\beta\cup\bigcup(D_\gamma\colon\gamma<\beta)$ such that $F(y)\cap\bigcup(A_\gamma\colon\gamma<\beta)=\emptyset$. Choose such a $y\in\lambda$. Let $D_\beta=\{y\}\cup\bigcup(D_\gamma\colon\gamma<\beta)$.

There are now two cases to consider.

First, suppose $g_\beta \in G_F$. Let $A_\beta = \bigcup(A_\gamma\colon \gamma<\beta)$, $A'_\beta = F(x)\cup F(y)\cup\bigcup(A'_\gamma\colon \gamma<\beta)$, and $C_\beta=\{x\}\cup\bigcup(C_\gamma\colon\gamma<\beta)$. Clearly (i) and (ii) are satisfied. By the choice of $x\in\lambda$, $C_\beta\cap C'_\beta=\emptyset$ and, by the choice of $x, y\in\lambda$, $A_\beta\cap A'_\beta=\emptyset$ and $(C_\beta\cup C'_\beta)\cap D_\beta=\emptyset$; thus, (iii) holds. Obviously, by definition, (iv) is also valid.

Second, suppose $g_\beta\notin G_F$. Thus, $|(g_\beta)_F|=2^{\aleph_0}$. Thus there are $2^{\aleph_0}$ elements $z\in(g_\beta)_F$ such that $z\notin C'_\beta\cup D_\beta$ and, since g_β is an injection, $g_\beta(z)\notin F(x)\cup F(y)\cup \cup\bigcup(A'_\gamma\colon\gamma<\beta)$. By Lemma 2, we may choose the element z such that, in addition, $F(z)\cap\bigcup(A_\gamma\colon\gamma<\beta)=\emptyset$. Let $A_\beta=\{g_\beta(z)\}\cup\bigcup(A_\gamma\colon\gamma<\beta)$, $A'_\beta=F(x)\cup F(y)\cup F(z)\cup \cup\bigcup(A'_\gamma\colon\gamma<\beta)$, and $C_\beta=\{x\}\cup\{z\}\cup\bigcup(C_\gamma\colon\gamma<\beta)$. Clearly, (i) and (ii) are valid. The choice of $z\in\lambda$ is such that $g_\beta(z)\notin F(z)$; thus, since $\big(F(x)\cup F(y)\cup F(z)\big)\cap \cap\bigcup(A_\gamma\colon\gamma<\beta)=\emptyset$, it follows that $A_\beta\cap A'_\beta=\emptyset$. By choice, $C_\beta\cap C'_\beta=\emptyset$. As in the first case $C'_\beta\cap D_\beta=\emptyset$ and, by inspection, $C_\beta\cap D_\beta=\emptyset$; thus (iii) also holds. Once more it is clear that (iv) is valid.

As indicated earlier, we set $A=\lambda\setminus\bigcup(A_\beta\colon\beta<2^{\aleph_0})$, $A'=\bigcup(A'_\beta\colon\beta<2^{\aleph_0})$, $C=\bigcup(C_\beta\colon\beta<2^{\aleph_0})$, $D=\bigcup(D_\beta\colon\beta<2^{\aleph_0})$, and $B=C\cup D$. It follows, by (iii), that $A'\subseteqq A$. However, by (iv), $F(B)\subseteqq A'\subseteqq A$. Thus $f\restriction B$ is an order preserving injection from B into A for each $f\in F$. By (ii), $|D|=2^{\aleph_0}$. Let $(S_\alpha\colon\alpha<2^{2^{\aleph_0}})$ be an indexing of the power set of D, let $B_\alpha=C\cup S_\alpha$ for $\alpha<2^{2^{\aleph_0}}$. Since $B_\alpha\subseteqq B$, the mapping $f\restriction B_\alpha$ is an order preserving injection of B_α into A for $\alpha<2^{2^{\aleph_0}}$ and $f\in F$.

3. Proof of the theorem

We first show that, for distinct $\alpha,\beta<2^{2^{\aleph_0}}$, $B_\alpha\not\cong B_\beta$. If $\alpha\neq\beta$, then $S_\alpha\neq S_\beta$. Suppose, with no loss of generality, that there exists $s\in S_\alpha\setminus S_\beta$. If $S_\alpha\cong S_\beta$ then there is an order preserving injection $g\colon B_\alpha\to B_\beta$. In which case, g extends to an order preserving injection $g^+\colon B_\alpha^+\to B_\beta^+$. Since $\eta\subseteqq B_\alpha, B_\beta\subseteqq\lambda$, it follows that $g^+\colon\lambda\to\lambda$. By (iii), $C\cap D=\emptyset$; thus, $s\notin B_\beta$. Consequently, g^+ is not the identity function and, hence, $g^+\in G$. Whence, for some $\gamma<2^{\aleph_0}$, $g^+=g_\gamma$. However, for g_γ, there is $x\in\lambda$ for which $x\in C$ and $g_\gamma(x)\in C'$. By (iii), $C\cap C'=\emptyset$ and $D\cap C'=\emptyset$. Since $C\subseteqq B_\alpha, B_\beta\subseteqq C\cup D$, we conclude that $x\in B_\alpha$ and $g_\gamma(x)\notin B_\beta$. However, g_γ is an extension of $g\colon B_\alpha\to B_\beta$; that is to say, $g_\gamma(x)=g(x)\in B_\beta$. By contradiction, we conclude that there is no order preserving injection $g\colon B_\alpha\to B_\beta$. We have shown the following:

Lemma 3. *For $\alpha,\beta<2^{2^{\aleph_0}}$, $B_\alpha\cong B_\beta$ if and only if $\alpha=\beta$.*

For the interested reader, we remark that, in the construction, a more judicious choice of subsets of D yields the following stronger result: for distinct $\alpha,\beta<2^{2^{\aleph_0}}$, B_α is not a sublattice of B_β and B_β is not a sublattice of B_α.

For $\alpha<2^{2^{\aleph_0}}$, we have already observed that, for $1\leqq i<\omega$ and $1\leqq j\leqq i+1$, $f_{ij}\upharpoonright B_\alpha$ is an order preserving injection from B_α into A. We now show that, for $n<\omega$, A contains n disjoint copies of B_α. As stated previously, for $1\leqq i<\omega$ and $1\leqq j\leqq i+1$, $f_{ij}\colon I_{ik}\to J_{ijk}$ is an order preserving bijection for every $1\leqq k\leqq i!$. Since, for distinct $1\leqq j, l\leqq i+1, J_{ijk}\cap J_{ilk}=\emptyset$, it follows that $f_{ij}(\lambda)\cap f_{il}(\lambda)=\emptyset$. Consequently, the restrictions of the functions of type i to B_α yield $i+1$ order preserving injections of B_α into A such that, for distinct $1\leqq j, l\leqq i+1, f_{ij}\upharpoonright(B_\alpha)\cap \cap f_{il}\upharpoonright(B_\alpha)=\emptyset$. Thus, we have shown:

Lemma 4. *Let $\alpha<2^{2^{\aleph_0}}$. For $n<\omega$, the totally ordered set A contains n disjoint copies of B_α.*

It remains to show that, for $\alpha<2^{2^{\aleph_0}}$, A does not contain infinitely many disjoint copies of B_α. Since, for every $\alpha<2^{2^{\aleph_0}}, C\subseteqq B_\alpha$, it is sufficient to show that A does not contain infinitely many disjoint copies of C.

Suppose that $g\colon C\to A$ is an order preserving injection. Then g extends to an order preserving injection $g^+\colon C^+\to A^+$. Again, since $\eta\subseteqq A, C\subseteqq\lambda$, it follows that $g^+\colon\lambda\to\lambda$; that is to say, if g^+ is not the identity function then $g^+\in G$.

Lemma 5. *Let $g\colon C\to A$ be an order preserving injection. If g is not the identity function, then $g^+\in G_F$.*

Proof. Suppose $g^+\notin G_F$. By the above comments, there exists $1\leqq\beta<2^{\aleph_0}$ such that $g^+=g_\beta$; thus, $g_\beta\notin G_F$. Hence, by the definition of A_β and C_β, there is $z\in(g_\beta)_F$ such that $z\in C_\beta$ and $g_\beta(z)\in A_\beta$. Consequently, $z\in C$ and $g_\beta(z)\notin A$. However, g_β is an extension of g; whence, $g_\beta(z)\in A$. By contradiction, we conclude $g^+\in G_F$.

Before considering infinitely many order preserving injections from C into A we must derive Lemma 8.

Let $g\in G_F$ and I be a nonempty open interval of λ. Since $g\in G_F$, $|\{x\in I\mid g(x)\notin \notin F(x)\}|\leqq|g_F|<2^{\aleph_0}$. Hence, $|\{x\in I\mid g(x)\in F(x)\}|=2^{\aleph_0}$ and, by Lemma 1, $|\{x\in I\mid x\neq g(x)\}|=2^{\aleph_0}$. Consequently, there exists $x\in I$ such that $x\neq g(x)$ and $x\neq(k-1)/(i!)$ for any $1\leqq i<\omega$ and $1\leqq k\leqq i!$. Select such an x. Since I is open there exists $d>0$ such that $(x-d, x+d)\subseteqq I$. For $d'=|g(x)-x|$, choose $1\leqq p<\omega$ such that $1/(p!)<\min\{d, d'\}$. Hence, there exists $1\leqq r\leqq p!$ such that $x\in I_{pr}\subseteqq I$ but $g(x)\notin I_{pr}$.

Lemma 6. *There is a nonempty open interval $I'\subseteqq I_{pr}$ such that, for $y\in I'$, either $y\in g_F$ or $g(y)=f_{ij}(y)$ for some $1\leqq i<p$ and $1\leqq j\leqq i+1$.*

Proof. For $1\leqq q\leqq p+1$, $f_{pq}(I_{pr})=J_{pqr}\subseteqq I_{pr}$. Furthermore, by definition, for $p\leqq i<\omega$ and $1\leqq j\leqq i+1$, $f_{ij}(I_{pr})\subseteqq I_{pr}$. Since, by hypothesis, $x\neq(r-1)/(p!)$

and $x \neq g(x)$, there is a nonempty open interval $I' \subseteqq I_{pr}$ such that $g(I') \cap I_{pr} = \emptyset$. Thus, for $y \in I'$, either $g(y) \notin F(y)$ (in which case, $y \in g_F$), or there exists $1 \leqq i < \omega$ and $1 \leqq j \leqq i+1$ such that $f_{ij}(y) = g(y) \notin I_{pr}$. Since $y \in I_{pr}$, it follows that $i < p$. The proof is complete.

Since $g \in G_F$ is assumed, it follows that the set of all $y \in I'$ with $g(y) = f_{ij}(y)$ for some $1 \leqq i < p$ and $i \leqq j \leqq i+1$ has cardinality $2^{\aleph_0}$. Furthermore, any nonempty open interval contained in I' has the same property.

Lemma 7. *There is a nonempty open interval* $I'' \subseteqq I'$, $1 \leqq i < p$, *and* $1 \leqq j \leqq i+1$ *such that, for* $y \in I''$, $g(y) = f_{ij}(y)$.

Proof. Since I' is nonempty and open, $I' = (u_0, v_0)$ for some distinct $u_0, v_0 \in \lambda$. Let $I_0 = I'$. For $n < \omega$, we inductively define a nonempty open interval $I_n = (u_n, v_n)$ such that, for $n \leqq m < \omega$, $I_n \supseteqq I_m$. Assume that I_n has been defined and choose, if possible, distinct $u_{n+1}, v_{n+1} \in I_n$ such that, for some $y \in I_n$, either there exist $1 \leqq i < p$ and $1 \leqq j \leqq i+1$ such that $g(y) = f_{ij}(y)$ but, for all $z \in (u_{n+1}, v_{n+1})$, $g(z) \neq f_{ij}(z)$, or $y \in g_F$ but, for all $z \in (u_{n+1}, v_{n+1})$, $z \notin g_F$. If u_{n+1} and v_{n+1} exist then set $I_{n+1} = (u_{n+1}, v_{n+1})$; otherwise, let $I_{n+1} = I_n$. Since there are only finitely many possibilities for i and j, there exists some $n < \omega$ such that $I_n = I_m$ for all $n \leqq m < \omega$. Let $I'' = I_n$. We must show that I'' satisfies the requirements of the lemma. By the remark preceding Lemma 7, there exists $y \in I''$ such that, for some $1 \leqq i < p$ and $1 \leqq j \leqq i+1$, $\langle y, g(y) \rangle \in f_{ij}$. Hence, by construction, for any distinct $u, v \in I''$, there exists $u < z < v$ such that $\langle z, g(z) \rangle \in f_{ij}$ for the same i and j; that is to say, the set of all elements $z \in I''$ such that $g(z) = f_{ij}(z)$ is dense in I''. Since g is order preserving and f_{ij} is continuous on I'' (recall that $I'' \subseteqq I' \subseteqq I_{pr}$ and f_{ij} is continuous on I_{pr}), it follows that $g(z) = f_{ij}(z)$ for all $z \in I''$. The lemma is verified.

The statement of the next lemma is immediate from the discussion following Lemma 5 together with Lemma 6 and Lemma 7.

Lemma 8. *Let* $g \in G_F$ *and let* I *be a nonempty open interval of* λ. *Then there exist a nonempty open interval* $J \subseteqq I$ *and* $f \in F$ *such that* $g(x) = f(x)$ *for all* $x \in J$.

Suppose that, for $n < \omega$, $h_n \colon C \to A$ is an order preserving injection.

Lemma 9. *There exists a nonempty open interval* $I \subseteqq \lambda$ *such that if* $y \in I$ *is rational then* $y = h_0(x)$ *for some rational* x.

Proof. If h_0 is the identity function then, since $\eta \subseteqq C$, any open interval $I \subseteqq \lambda$ will satisfy the lemma. If h_0 is not the identity then, by Lemma 5, $h_0^+ \in G_F$. Thus, by Lemma 8, there is a nonempty open interval $J \subseteqq \lambda$, $1 \leqq i < \omega$, and $1 \leqq j \leqq$ $\leqq i+1$ such that, for $x \in J$, $h_0^+(x) = f_{ij}(x)$. Since $\lambda = \bigcup (I_{ik} \colon 1 \leqq k \leqq i!)$, there is some $1 \leqq k \leqq i!$ such that $I_{ik} \cap J \neq \emptyset$. Choose a nonempty interval $I' \subseteqq I_{ik} \cap J$.

By definition, f_{ij} is continuous on I_{ik} and, hence, it is a continuous order preserving injection on I'. Thus, $f_{ij}(I')$ is a nonempty open interval of λ. Let $I=f_{ij}(I')$. If $y\in I$ then $y=f_{ij}(x)=h_0^+(x)$ for some $x\in I'$. By the definition of f_{ij}, if y is rational it follows that x is rational. Again, since $\eta\subseteqq C$, $h_0^+(x)=h_0(x)$ and the proof is complete.

Lemma 10. *There exist $x, y\in\eta$ and distinct $n, m<\omega$ such that $h_n(x)=h_m(y)$.*

Proof. Let I be given as in Lemma 9. Suppose that, for some $1\leqq n<\omega$, h_n is the identity function. In particular, for $y\in I$, $y=h_n^+(y)$. If y is rational $h_n^+(y)=h_n(y)$ and, by Lemma 9, the proof is complete. Thus, we assume that, for $1\leqq n<\omega$, h_n is not the identity function.

Choose $1\leqq p<\omega$ such that for some $1\leqq r\leqq p!$, $I_{pr}\subseteqq I$. Recall that, for all $f_{ij}=f\in F$ of type $i\geqq p$, $f_{ij}(I_{pr})\subseteqq I_{pr}\subseteqq I$.

By Lemma 5, all h_n^+ belong to G_F. Lemma 8 yields the existence of an open nonempty interval $I_1\subseteqq I_{pr}$ such that h_1^+ agrees with some $f_{(1)}\in F$ on I_1. Define inductively $I_{n+1}\subseteqq I_n$ as a nonempty open subinterval on which h_{n+1}^+ agrees with some $f_{(n+1)}\in F$. If some $f_{(n)}$ is of type $i\geqq p$, choose a rational $x\in I_n$. Then $h_n(x)=h_n^+(x)=f_{(n)}(x)\in I$ is rational, and, by Lemma 9, $h_n(x)=h_0(x')$ for some rational x'. Therefore, each $f_{(n)}$ for $1\leqq n<\omega$ is of type $i_n<p$. Since there are only finitely many of these functions, there exist $1\leqq m<n<\omega$ with $h_n^+\restriction I_n=f_{(n)}\restriction I_n=f_{(m)}\restriction I_n=h_m^+\restriction I_n$. For any rational $x\in I_n$ it follows that $h_n(x)=h_n^+(x)=h_m^+(x)=h_m(x)$. The proof is complete.

Since $\eta\subseteqq C$, Lemma 10 implies that there are distinct $n, m<\omega$ such that $h_n(C)\cap h_m(C)\neq\emptyset$.

Lemma 11. *If, for $n<\omega$, $h_n\colon C\to A$ is an order preserving injection then there exist distinct $n, m<\omega$ such that $h_n(C)\cap h_m(C)\neq\emptyset$; that is to say, A does not contain infinitely many disjoint copies of C.*

Lemmas 3, 4, and 11 yield the Theorem.

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Schwach distributive Verbände. II

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In [4] und [5] wurde der Begriff des n-distributiven Verbandes eingeführt. In denselben Arbeiten wurden Charakterisierungen der n-distributivität und, ohne diese weiter auszuführen, Beispiele für n-distributive Verbände angegeben. Ziel vorliegender Arbeit ist, diese früheren Arbeiten durch Angabe der noch nicht veröffentlichten Beweisen zu vervollständigen.

In der Einführung von [5] haben wir unserer Meinung Ausdruck gegeben, dass die wichtigsten Gebiete in dieser Theorie die folgenden sind:

a) Verallgemeinerung der „reinen Theorie" der distributiven Verbände, vor allem bei Anwesenheit der Modularität, die für $n \geqq 2$ keine Folgerung der n-Distributivität ist (vgl. die nachfolgende Definition).

b) Untersuchung der Beziehungen zwischen der n-Distributivität und der Dimension von projektiven Geometrien.

c) Anwendungen auf die Theorie der Varietäten von Verbänden.

d) Untersuchung der n-Distributivität in Kongruenzverbänden universeller Algebren, hauptsächlich in Normalteilerverbänden von Gruppen.

Untersuchungen zu a) haben wir in [5] begonnen und in [11] fortgesetzt. Die Gebiete b) und c) wurden in [9] bzw. [8] und [10] behandelt. Hier werden wir uns mit dem Gebiet d) beschäftigen. Da die Definitionen seit dem Erscheinen von [5] in neueren Arbeiten verändert worden sind, ist es nötig zuerst die Begriffe festzulegen.

Ein Verband heisst n-distributiv, wenn er der Identität

$$x \wedge \bigvee_{i=0}^{n} y_i = \bigvee_{j=0}^{n} \Big[x \wedge \bigvee_{\substack{i=0 \\ i \neq j}}^{n} y_i \Big]$$

genügt. Diese Definition ist dual zu der Definition in [5], und die Modularität wird

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nicht mehr wie in [5] gefordert. Es sei aber bemerkt, dass in dieser Arbeit nicht-modulare n-distributive Verbände fast keine Rolle spielen. Diese werden in zwei anderen Arbeiten betrachtet [12], wo wir uns mit Kontraktionen-Verbänden von Graphen und mit Verbänden von konvexen Mengen beschäftigen.

1. Der Chinesische Restsatz in universellen Algebren

Genau so, wie im Bereich der ganzen Zahlen, können Kongruenzsysteme in beliebigen universellen Algebren definiert werden. Es seien A eine universelle Algebra, $\Theta(A)$ der Kongruenzverband von A und $a_1, a_2, \ldots, a_k \in A$, $\theta_1, \theta_2, \ldots, \theta_k \in \Theta(A)$. Dann heisst das System

$$x \equiv a_i\,(\theta_i), \quad i = 1, 2, \ldots, k \tag{1}$$

ein Kongruenzsystem über A mit der Unbekannten x. Es ist klar, wie die Lösbarkeit und die Lösungen eines solchen Systems zu definieren sind.

Definition. Eine Algebra A genügt dem Chinesischen Restsatz der Ordnung n (oder in Zeichen: dem C_n-Satz), wenn für beliebige

$$a_1, a_2, \ldots, a_k \in A \quad \text{und} \quad \theta_1, \quad \theta_2, \ldots, \theta_k \in \Theta(A), \quad k > n+1,$$

die Lösbarkeit aller $(n+1)$-elementigen Teilsysteme von (1) auch die Lösbarkeit des ganzen Systems (1) nach sich zieht. (Ein n-elementiges „Teilsystem" braucht nicht aus n verschiedenen Kongruenzen zu bestehen, da identische Kongruenzen in (1) unter verschiededen Indizes aufgezählt werden können.)

Wie leicht zu sehen ist, besagt der klassische Chinesische Restsatz, dass der Ring der ganzen Zahlen dem C_1-Satz genügt. Eine Verbingung des C_n-Satzes mit der n-Distributivität ist in dem nächsten Satz enthalten.

1.1. Satz. *Damit eine universelle Algebra A dem C_n-Satz genügt, ist es notwendig und hinreichend, dass für beliebige Kongruenzen $\varphi, \theta_0, \theta_1, \ldots, \theta_n \in \Theta(A)$ die Identität*

$$\varphi \cdot \bigwedge_{i=0}^{n} \theta_i = \bigwedge_{j=0}^{n} \left[\varphi \cdot \bigwedge_{\substack{i=0 \\ i \neq j}}^{n} \theta_i\right] \tag{2}$$

gilt, wobei $\cdot$ und $\bigwedge$ das Produkt bzw. den Durschschnitt von Relationen bezeichnet. Wenn die Kongruenzen von A vertauschbar sind, d. h., wenn für beliebige $\theta, \varphi \in \Theta(A)$ $\theta\varphi = \varphi\theta$ gilt, so genügt A genau dann dem C_n-Satz, wenn $\Theta(A)$ n-distributiv ist.

Beweis. Die zweite Aussage des Satzes folgt aus der ersten. In der Tat stimmt unter den Bedingungen der zweiten Aussage das Produkt der Kongruenzen von

A mit dem Supremum überein. Hieraus folgt die duale n-distributivität von $\Theta(A)$. Im Falle der Vertauschbarkeit der Kongruenzen ist aber $\Theta(A)$ auch modular und in modularen Verbänden ist die n-Distributivität selbstdual ([5]). Also reicht es, nur die erste Aussage zu beweisen. Wir schicken die folgenden zwei einfachen Bemerkungen voraus:

1. Ist x_0 eine Lösung des Systems (1), so ist die allgemeine Lösung von (1)

$$x \equiv x_0 \left(\bigwedge_{i=1}^{k} \theta_i \right).$$

2. Für $k=2$ ist (1) genau dann lösbar, wenn $a_1\, \theta_1 \theta_2\, a_2$ gilt.

Nun zeigen wir dass die Bedingung des Satzes hinreichend ist. Es sei $1 \leqq i_1 < < i_2 < \ldots < i_r \leqq k$, und es bezeichne $\mathfrak{K}\,(i_1, i_2, \ldots, i_r)$ das folgende Teilsystem von (1):

$$x \equiv a_{i_j}\ (\theta_{i_j}), \quad j = 1, 2, \ldots, r.$$

$\mathfrak{K}\,(1, 2, \ldots, n+1)$ ist lösbar. Es sei $n+1 \leqq r < k$. Wir zeigen, dass die Lösbarkeit von $\mathfrak{K}\,(1, 2, \ldots, r+1)$ aus der Lösbarkeit von $\mathfrak{K}\,(1, 2, \ldots, r)$ folgt.

Es sei x_0 eine Lösung von $\mathfrak{K}\,(1, 2, \ldots, r)$. Dann ist die allgemeine Lösung von $\mathfrak{K}\,(1, 2, \ldots, r)$

$$x \equiv x_0 \left(\bigwedge_{i=1}^{r} \theta_i \right).$$

Es genügt zu zeigen, dass diese Kongruenz zusammen mit $\mathfrak{K}\,(r+1)$ ein lösbares System bildet, d. h., dass die folgende Relation gilt:

$$a_{r+1}\ \theta_{r+1} \cdot \bigwedge_{i=1}^{r} \theta_i\ x_0. \tag{3}$$

Es sei $\{i_1, i_2, \ldots, i_n\} \subseteqq \{1, 2, \ldots, r\}$. Dann ist x_0 eine Lösung von $\mathfrak{K}\,(i_1, \ldots, i_n)$. Die allgemeine Lösung von $\mathfrak{K}\,(i_1, \ldots, i_n)$ ist

$$x \equiv x_0 \left(\bigwedge_{j=1}^{n} \theta_{i_j} \right).$$

$\mathfrak{K}\,(i_1, \ldots, i_n, r+1)$ ist aber lösbar, also gilt

$$a_{r+1}\ \theta_{r+1} \cdot \bigwedge_{j=1}^{n} \theta_{i_j}\ x_0.$$

So erhalten wir

$$a_{r+1} \bigwedge_{\substack{K \subseteqq \{1, 2, \ldots, r\} \\ |K| = n}} \left[\theta_{r+1} \cdot \bigwedge_{i \in K} \theta_i\right] x_0. \tag{4}$$

Wir werden nun zeigen, dass folgende Gleichung gilt:

$$\theta_{r+1}\cdot\bigwedge_{i=1}^{r}\theta_i=\bigwedge_{\substack{K\subseteq\{1,2,\dots,r\}\\|K|=n}}\Big[\theta_{r+1}\cdot\bigwedge_{i\in K}\theta_i\Big]. \tag{5}$$

Durch Einsetzen von (5) in (4) ergibt sich dann (3).

Um (5) aus (2) herzuleiten, zeigen wir durch Induktion, dass für beliebige $s\geqq n$ und Kongruenzen $\varphi,\psi_0,\psi_1,\dots,\psi_s\in\Theta(A)$ die Identität

$$\varphi\cdot\bigwedge_{i=0}^{s}\psi_i=\bigwedge_{\substack{K\subseteq\{0,1,\dots,s\}\\|K|=n}}\Big[\varphi\cdot\bigwedge_{i\in K}\psi_i\Big] \tag{2_s}$$

gilt. (Man erhält dann (5) aus (2_{r-1}), indem man φ durch θ_{r+1} und ψ_i durch θ_{i+1} ersetzt.)

Für $s=n$ ist (2_s) mit (2) identisch. Es sei $s>n$ und nehmen wir an, dass (2_{s-1}) bewiesen ist. Es seien $\varphi,\psi_0,\psi_1,\dots,\psi_s\in\Theta(A)$. Es sei ferner

$$\chi_i=\begin{cases}\psi_i & \text{für}\quad i=0,1,\dots,n-1,\\ \bigwedge\limits_{j=n}^{s}\psi_j & \text{für}\quad i=n.\end{cases}$$

Dann können wir (2) anwenden:

$$\varphi\cdot\bigwedge_{i=0}^{s}\psi_i=\varphi\cdot\bigwedge_{i=0}^{n}\chi_i=\bigwedge_{j=0}^{n}\Big[\varphi\cdot\bigwedge_{\substack{i=0\\i\neq j}}^{n}\chi_i\Big]=\Big[\varphi\cdot\bigwedge_{i=0}^{n-1}\psi_i\Big]\wedge\bigwedge_{j=0}^{n-1}\Big[\varphi\cdot\bigwedge_{\substack{i=0\\i\neq j}}^{s}\psi_i\Big].$$

Auf der rechten Seite kann die Induktionsvoraussetzung angewendet werden und der somit erhaltene Ausdruck ist die rechte Seite von (2_s). Damit ist die Hinlänglichkeit der Bedingung bewiesen.

Um die Notwendigkeit zu zeigen, nehmen wir an, dass für gewisse Kongruenzen $\varphi,\theta_0,\theta_1,\dots,\theta_n\in\Theta(A)$ gilt:

$$\varphi\cdot\bigwedge_{i=0}^{n}\theta_i\neq\bigwedge_{j=0}^{n}\Big[\varphi\cdot\bigwedge_{\substack{i=0\\i\neq j}}^{n}\theta_i\Big].$$

Dann ist die rechte Seite kleiner als die linke Seite, d. h., es gibt Elemente $a,b\in A$, so dass $a\ \varphi\cdot\bigwedge_{i=0}^{n}\theta_i\ b$ ungültig, aber $a\ \bigwedge_{j=0}^{n}\Big[\varphi\cdot\bigwedge_{\substack{i=0\\i\neq j}}^{n}\theta_i\Big]\ b$ gültig ist. Daraus folgt, dass die $(n+1)$-elementigen Teilsysteme des Systems

$$\begin{aligned}x&\equiv a\ (\varphi)\\x&\equiv b\ (\theta_0)\\&\ \ \vdots\\x&\equiv b\ (\theta_n)\end{aligned}$$

lösbar sind, das ganze System aber unlösbar ist. Damit ist der Satz bewiesen.

Der Fall $n=1$ dieses Satzes ist bekannt (siehe Grätzer [2]). Grätzer hat das folgendes bewiesen: Es sei A eine universelle Algebra. Damit für alle $k>2$ und Kongruenzen $x \equiv a_i\,(\theta_i)$, $i=1,2,\dots,k$, über A die Bedingungen $a_i \equiv a_j\,(\theta_i \vee \theta_j)$, $i \neq j$, $i, j=1,2,\dots,k$, die Lösbarkeit von (1) nach sich ziehen, ist es notwendig und hinreichend, dass $\Theta(A)$ distributiv ist und seine Elemente vertauschbar sind. Es ist noch eine offene Frage, wie dieser Satz sich für beliebige n verallgemeinern älsst.

2. Mal'cev-Polynome

Im folgenden beschäftigen wir uns mit dem C_n-Satz für Varietäten. Man findet den folgenden Satz in [6]. Weitere, äquivalente Bedingungen wurden von BAKER und PIXLEY [1] und von PIXLEY [16] gefunden.

2.1. Satz. *Für eine beliebige Varietät* $\mathbf{V}$ *und natürliche Zahl* n *sind die folgenden Bedingungen äquivalent.*

(A) *Jede Algebra* $A \in \mathbf{V}$ *genügt dem* C_n*-Satz.*

(B) *Für beliebige* $A \in \mathbf{V}$ *und Kongruenzen* $\varphi, \theta_0, \theta_1, \dots, \theta_n \in \Theta(A)$ *gilt*

$$\varphi \cdot \bigwedge_{i=0}^{n} \theta_i = \bigwedge_{j=0}^{n} \Big[\varphi \cdot \bigwedge_{\substack{i=0\\ i\neq j}}^{n} \theta_i\Big].$$

(C) *Es gibt ein Term* μ *in* $n+2$ *Variablen über* $\mathbf{V}$, *so dass*

$$\mu(x, \dots, x, y) = \mu(x, \dots, x, y, x) = \dots = \mu(y, x, \dots, x) = x$$

eine Identität von $\mathbf{V}$ *ist.*

Bemerkung. Der Fall $n=1$ ist schon von WILLE in [19] behandelt worden.

Beweis. (A)$\Leftrightarrow$(B) folgt aus Satz 1.1.

(B)$\Leftrightarrow$(C). Nehmen wir an daß (B) gilt. Es bezeichne $F(n+2)$ die freie Algebra in $\mathbf{V}$ mit den freien Erzeugenden $a_0, a_1, \dots, a_{n+1}$. Es sei θ_i die kleinste Kongruenz, so dass $a_0, \dots, a_{i-1}, a_{i+1}, \dots, a_{n+1}$ modulo θ_i untereinander kongruent sind. Dann gelten

$$a_0 \;\theta_{n+1}\; a_i \quad (i=0,1,\dots,n)$$

$$a_i \;\theta_j\; a_{n+1} \quad (i, j=0,1,\dots,n, \quad i\neq j)$$

Daraus folgt

$$a_0 \Big[\theta_{n+1} \cdot \bigwedge_{\substack{j=0\\ j\neq i}}^{n} \theta_j\Big]\, a_{n+1}.$$

Somit gibt es ein Element $\mu(a_0, a_1, \ldots, a_{n+1}) \in F(n+2)$, mit

$$a_0 \ \theta_{n+1} \ \mu(a_0, a_1, \ldots, a_{n+1}) \ \bigwedge_{i=0}^{n} \theta_i \ a_{n+1}.$$

Es folgt also

$$a_0 \ \theta_{n+1} \ \mu(a_0, a_1, \ldots, a_{n+1}) \ \ \theta_{n+1} \ \ \mu(a_0, a_0, \ldots, a_0, a_{n+1}).$$

θ_{n+1} ist aber trivial auf der durch $\{a_0, a_{n+1}\}$ erzeugten Teilalgebra, daher ergibt sich

$$a_0 = \mu(a_0, \ldots, a_0, a_{n+1}).$$

Genau so folgt aus $\mu(a_0, a_1, \ldots, a_{n+1}) \, \theta_i \, a_{n+1} \ (i=0, 1, \ldots, n)$, dass

$$\mu(a_{n+1}, \ldots, a_{n+1}, \ a_i, \ a_{n+1}, \ldots, a_{n+1}) = a_{n+1}$$

für $i=0, 1, \ldots, n$ gilt. Damit ist (C) bewiesen.

Nehmen wir umgekehrt an, dass (C) gilt, d. h., dass ein μ mit der obigen Eigenschaft existiert. Wir werden zeigen, dass für beliebige Kongruenzen $\varphi, \theta_0, \theta_1, \ldots, \theta_n$ irgendeiner Algebra A in $\mathbf{V}$

$$\bigwedge_{j=0}^{n} \Big[\varphi \cdot \bigwedge_{\substack{i=0\\ i\neq j}}^{n} \theta_i\Big] \leqq \varphi \cdot \bigwedge_{i=0}^{n} \theta_i$$

gilt. (Die umgekehrte Ungleichung ist klar.)

In der Tat, es seien $x, y \in A$ mit

$$x \ \bigwedge_{j=0}^{n} \Big[\varphi \cdot \bigwedge_{\substack{i=0\\ i\neq j}}^{n} \theta_i\Big] \ y.$$

Dann existieren Elemente $t_0, t_1, \ldots, t_n \in A$, so dass gilt:

$$x \ \varphi \ t_j \ \bigwedge_{\substack{i=0\\ i\neq j}}^{n} \theta_i \ y \quad (j = 0, 1, \ldots, n).$$

Es sein ferner $t=\mu(t_0, t_1, \ldots, t_n, y)$. Aufgrund der Identitäten für μ in (C) erhält man die folgenden Relationen:

$$x \ \varphi \ t \ \bigwedge_{i=0}^{n} \theta_i \ y.$$

Zum Beispiel erhält man $x \, \varphi \, t$ wie folgt:

$$x = \mu(x, x, \ldots, x, y) \ \varphi \ \mu(t_0, t_1, \ldots, t_n, y) = t.$$

Damit ist der Satz bewiesen.

3. n-Distributivität in Untergruppenverbänden abelscher Gruppen

Zweck dieses Abschnitt ist es, weitere Beispiele für n-distributive Verbände zu erwähnen und eine notwendige und hinreichende Bedingung dafür anzugeben, dass der Untergruppenverband einer abelschen Gruppe n-distributiv ist. $S(G)$ (bzw. $N(G)$) wird den Untergruppenverband (bzw. Normalteilerverband) der Gruppe G bezeichnen. Für die Gruppenoperationen werden wir eine multiplikative Schreibweise verwenden. Dementsprechend bezeichnen wir das neutrale Element mit e. $[a_1, a_2, \ldots]$ bezeichnet das Erzeugnis der Elemente in den eckigen Klammern.

3.1. Satz. *Für eine beliebige natürliche Zahl n ist der Untergruppenverband der durch n Elemente erzeugten freien abelschen Gruppe U_n ein n-distributiver, aber kein $(n-1)$-distributiver Verband.*

Beweis. Es sein $n \geqq 2$, und es seien die Elemente $u_1, u_2, \ldots, u_n$ die freien Erzeugenden von U_n. Ist $v = u_1 u_2 \ldots u_n$, so haben wir offensichtlich:

$$[v] \wedge \bigvee_{i=0}^{n} [u_i] = [v] > [e] = \bigvee_{j=1}^{n} \Big[[v] \wedge \bigvee_{\substack{i=1 \\ i \neq j}}^{n} [u_i] \Big],$$

d. h., $S(U_n)$ ist nicht $(n-1)$-distributiv. Für $n=1$ ist dieser Teil der Behauptung trivial. (In Harmonie mit der Definition für $n \geqq 1$ sollen genau die ein-elementigen Verbände als 0-distributiv definiert werden.)

Umgekehrt ist wohlbekannt (vgl. Ore [15]), dass $S(U_1)$ distributiv ist. Sei nun $n > 1$ und nehmen wir an, dass für $k = 1, 2, \ldots, n-1$ $S(U_k)$ k-distributiv ist. Es ist die folgende Beziehung zu beweisen:

$$X = A \wedge \bigvee_{i=0}^{n} B_i \leqq \bigvee_{j=0}^{n} \Big[A \wedge \bigvee_{\substack{i=0 \\ i \neq j}}^{n} B_i \Big] = Y,$$

wobei $A, B_0, B_1, \ldots, B_n$ beliebige Elemente von $S(U_n)$ sind.

Es sei $a \in X$, d. h. $a = b_0 b_1 \ldots b_n \in A$ mit $b_j \in B_j$ $(j=0, 1, \ldots, n)$. Es seien $b_j = u_1^{\beta_{1j}} \ldots u_n^{\beta_{nj}}$ $(j=0, 1, \ldots, n)$, wobei $u_1, u_2, \ldots, u_n$ die freien Erzeugenden von U_n sind. Wir zeigen: $a \in Y$. Wenn der Rang der Matrix $B = (\beta_{ij})_{\substack{i=1,\ldots,n \\ j=0,\ldots,n}}$ kleiner als n ist, dann ist auch der Rang der Untergruppe $[b_0, b_1, \ldots, b_n]$ kleiner als n (siehe Kuroš [13]), und, da diese Untergruppe auch frei ist, folgt

$$[b_0, b_1, \ldots, b_n] \cong S(U_k)$$

für ein $k < n$. Nach der Induktionsvoraussetzung ist aber der Verband $S(U_k)$

k-distributiv, also ist er auch n-distributiv. Deshalb erhalten wir

$$[a] = [a] \wedge \bigvee_{i=0}^{n} [b_i] = \bigvee_{j=0}^{n} \left[[a] \wedge \bigvee_{\substack{i=0 \\ i \neq j}}^{n} [b_i] \right] \leqq Y,$$

d. h., $a \in Y$.

Also können wir annehmen, dass der Rang von B gleich n ist. Man betrachte nun für ein beliebiges aber festes $k \in \{0, 1, \ldots, n\}$ das Diophantische Gleichungssystem

$$(E_k) \qquad \sum_{\substack{j=0 \\ j \neq k}}^{n} \beta_{ij} x_{jk} = \left[\sum_{l=0}^{n} \beta_{il} \right] t_k \quad (i = 1, 2, \ldots, n)$$

in den Unbekannten x_{jk} $(j \neq k)$ und t_k. Es bezeichne D_k die Determinante von (E_k), d. h., es sei

$$D_k = |\beta_{ij}|_{\substack{i=1,\ldots,n \\ j=0,1,\ldots,n;\ j \neq k}}$$

Es sei ferner D_{jk} die Determinante, die Durch Ersetzen der Spalte $(\beta_{ij})_{i=1,2,\ldots,n}$ in D_k durch die Spalte $(\beta_{i0} + \beta_{i1} + \ldots + \beta_{in})_{i=1,2,\ldots,n}$ entsteht. Es ist leicht zu sehen, dass

$$(6) \qquad \begin{aligned} x_{jk} &= \frac{D_{jk}}{(D_{0k}, \ldots, D_{k-1,k}, D_k, D_{k+1,k}, \ldots, D_{nk})} \quad (j = 0, 1, \ldots, n;\ j \neq k), \\ t_k &= \frac{D_k}{(D_{0k}, \ldots, D_{k-1,k}, D_k, D_{k+1,k}, \ldots, D_{nk})} \end{aligned}$$

eine Lösung von (E_k) ist, wobei in den Nennern der grösste gemeinsame Teiler von $D_{0k}, \ldots, D_{k-1,k}, D_k, D_{k+1,k}, \ldots, D_{nk}$ steht. Dieser ist nicht 0, da $\operatorname{rang} B = n$ ist. Die Determinanten D_{jk} sind aber Summen oder Differenzen von D_j und D_k, somit gilt

$$(D_{0k}, \ldots, D_{k-1,k}, D_k, D_{k+1,k}, \ldots, D_{nk}) = (D_0, D_1, \ldots, D_k, \ldots, D_n).$$

Deshalb können wir die Lösungen (6) von (E_k) auch in der folgenden Form schreiben.

$$(7) \qquad \begin{aligned} x_{jk} &= \frac{D_{jk}}{(D_0, D_1, \ldots, D_n)} \quad (j = 0, 1, \ldots, n;\ j \neq k), \\ t_k &= \frac{D_k}{(D_0, D_1, \ldots, D_n)}. \end{aligned}$$

Nun können wir das Gleichungssystem betrachten, das sich aus den Systemen (E_k) $(k = 0, 1, \ldots, n)$ und der Gleichung $t_0 + t_1 + \ldots + t_n = 1$ zusammensetzt, d. h., das System

$$(8)\qquad \begin{aligned} &\sum_{\substack{j=0\\ j\neq k}}^{n} \beta_{ij} x_{jk} = \left[\sum_{l=0}^{n} \beta_{il}\right] t_k \quad (i = 1, \ldots, n;\ k = 0, 1, \ldots, n) \\ &\sum_{k=0}^{n} t_k = 1. \end{aligned}$$

Da der grösste gemeinsame Teiler der Lösungen $t_0, t_1, \ldots, t_n$ in (7) gleich 1 ist, können wir ganze Zahlen $y_0, y_1, \ldots, y_n$ finden, so dass $t_0 y_0 + t_1 y_1 + \ldots + t_n y_n = 1$ gilt. Es seien

$$t'_k = t_k \cdot y_k \quad (k = 0, 1, \ldots, n)$$

$$x'_{jk} = x_{jk} \cdot y_k \quad (k = 0, 1, \ldots, n;\ j = 0, 1, \ldots, n;\quad j \neq k).$$

t_k und x_{jk} genügen dem Diophantischen Gleichungssystem (8). Es sei $a_k = a^{t'_k}$. Es gilt offenbar $a_0 a_1 \ldots a_n = a$ und $a_k \in A$. Wir zeigen $a_k \in B_0 \vee \ldots \vee B_{k-1} \vee B_{k+1} \vee \ldots \vee B_n$. In der Tat gilt:

$$a_k = a^{t'_k} = (b_0 b_1 \ldots b_n)^{t'_k} =$$

$$= \{(u_1^{\beta_{10}} \ldots u_n^{\beta_{n0}})\,(u_1^{\beta_{11}} \ldots u_n^{\beta_{n1}}) \ldots (u_1^{\beta_{1n}} \ldots u_n^{\beta_{nn}})\}^{t'_k} =$$

$$= u_1^{(\beta_{10}+\ldots+\beta_{1n})\,t'_k} \ldots u_n^{(\beta_{n0}+\ldots+\beta_{nn})\,t'_k} = u_1^{\Sigma(\beta_{1j} x'_{jk} \mid j \neq k)} \ldots u_n^{\Sigma(\beta_{nj} x'_{jk} \mid j \neq k)} =$$

$$= (u_1^{\beta_{10}} \ldots u_n^{\beta_{n0}})^{x'_{0k}} \ldots (u_1^{\beta_{1,k-1}} \ldots u_n^{\beta_{n,k-1}})^{x'_{k-1,k}} \cdot (u_1^{\beta_{1,k+1}} \ldots u_n^{\beta_{n,k+1}})^{x'_{k+1,k}} \ldots (u_1^{\beta_{1n}} \ldots u_n^{\beta_{nn}})^{x'_{nk}} =$$

$$= b_0^{x'_{0k}} \ldots b_{k-1}^{x'_{k-1,k}} b_{k+1}^{x'_{k+1,k}} \ldots b_n^{x'_{nk}} \in B_0 \vee \ldots \vee B_{k-1} \vee B_{k+1} \vee \ldots \vee B_n.$$

Es ist also $a_k \in A(B_0 \vee \ldots \vee B_{k-1} \vee B_{k+1} \vee \ldots \vee B_n)$. Es folgt

$$a = a_0 a_1 \ldots a_n \in \bigvee_{j=0}^{n} \left[A \wedge \bigvee_{\substack{i=0\\ i\neq j}}^{n} B_i\right] = Y.$$

Q. E. D.

Wir bemerken, dass die Sätze 3.1 und 1.1. auch das Ergebnis von Rado [17] enthalten, dass für Kongruenzen von U_n der C_n-Satz gilt. Rado hat in [17] auch eine gemeinsame Verallgemeingrung des eben zitierten Satzes und des geometrischen Satzes von Helly bewiesen, diese Verallgemeinerung scheint aber von der Theorie der n-distributiven Verbände unabhängig zu sein.

Im nächsten Satz werden die abelsche Gruppen charakterisiert die einen n-distributiven Untergruppenverband haben. Der Rang $\operatorname{rang}(G)$ einer abelschen Gruppe G ist die kleinste natürliche Zahl n, so dass jede endlich erzeugte Untergruppe von G durch n Elemente erzeugt wird. Der Rang existiert natürlich nicht für jede abelsche Gruppe.

3.2. Satz. *Damit der Untergruppenverband einer abelschen Gruppe G n-distributiv ist, ist es notwendig und hinreichend, dass der Rang von G kleiner oder gleich n ist.*

Beweis. Es sei $\operatorname{rang}(G) \leqq n$. Wir zeigen, dass für beliebige Untergruppen $A_0, B_0, B_1, \ldots, B_n$ von G

$$X = A \wedge \bigvee_{i=0}^{n} B_i \leqq \bigvee_{j=0}^{n} \Big[A \wedge \bigvee_{\substack{i=0 \\ i \neq j}}^{n} B_i \Big] = Y$$

gilt. Es sei also $a \in X$, d. h., $a = b_0 b_1 \ldots b_n (\in A)$, mit $b_i \in B_i$ $(i = 0, 1, \ldots, n)$. Da die Untergruppe $[b_0, b_1, \ldots, b_n]$ durch n Elemente erzeugt werden kann, ist sie ein homomorphes Bild von U_n. Da die Untergruppenverbände abelscher Gruppen isomorph zu ihren Kongruenzverbänden sind, ist $S([b_0, b_1, \ldots, b_n])$ ein Teilverband von $S(U_n)$, und als solcher ist er auch n-distributiv. Folglich gilt

$$[a] = [a] \wedge \bigvee_{i=0}^{n} [b_i] = \bigvee_{j=0}^{n} \Big[[a] \wedge \bigvee_{\substack{i=0 \\ i \neq j}}^{n} [b_i] \Big] \subseteqq Y,$$

d. h., $a \in Y$.

Umgekehrt, nehmen wir an, dass $\operatorname{rang}(G) = r > n$. Dann gibt es eine Untergruppe H von G, die durch r Elemente erzeugbar ist, nicht aber durch $r-1$ Elemente. Es genügt zu zeigen, dass $S(H)$ nicht n-distributiv ist. Nach dem Fundamentalsatz abelscher Gruppen kann H als ein direktes Produkt von zyklischer Gruppen dargestellt werden

$$H = (C_{11} \times \ldots \times C_{1k_1}) \times \ldots \times (C_{s1} \times \ldots \times C_{sk_s}) \times \underbrace{C_\infty \times \ldots \times C_\infty}_{m \text{ Komponente}},$$

wobei C_∞ die unendliche zyklische Gruppe ist, und die anderen Komponenten so bezeichnet sind, dass für gewisse Primzahlen $p_1, p_2, \ldots, p_s$ $(p_i \neq p_j$ für $i \neq j)$, die Mächtigkeiten von C_{ij} $(j = 1, 2, \ldots, k_i)$ Potenzen von p_i sind.

Wäre $\max\limits_{1 \leqq i \leqq s} k_i + m < r$, so könnte H als ein direktes Produkt von weniger als r zyklischen Gruppen dargestellt werden (diese sind: $C_{11} \times \ldots \times C_{s1}, C_{12} \times \ldots \times C_{s2}, \ldots, C_\infty$ (m Exemplare)), und so könnte H durch weniger als r Elemente erzeugt werden. Folglich gibt es ein k_i, $k_i + m \geqq r$, so dass $C_{p_i}^r$ ein homomorphes Bild von H ist. Der Verband $S(C_{p_i}^r)$ ist nach [5] nicht $(r-1)$-distributiv, also ist er auch nicht n-distributiv, und dasselbe gilt für $S(H)$. Q. E. D.

Bemerkung. Dieser Satz enthält als Spezialfall das folgende Resultat von Ore [15]: Für eine Gruppe G ist $S(G)$ genau dann distributiv, wenn G lokal zyklisch ist (d. h., wenn $\operatorname{rang} G \leqq 1$ ist).

4. n-Distributivität in Normalteilerverbänden

In diesem Teil möchten wir eine Charakterisierung der n-Distributivität des Normalteilerverbandes einer Gruppe beweisen, die unseren Hauptsatz für abelsche Gruppen auch enthält. Eine solche Charakterisierung folgt durch Anwendung des Hauptsatzes des ersten Teiles [5] dieser Arbeit, d. h. des Satzes, der die n-distributiven Verbände in der Klasse aller modularen Verbände durch den Ausschluss des n-Diamanten (einer speziellen modularen Konfiguration) beschreibt (vgl. auch [8]). Es ist in [3] bewiesen worden, dass in dieser Beschreibung der n-Diamant auch durch den von Neumannschen $(n+1)$-Rahmen ersetzt werden kann. So erhält man:

4.1. Lemma. *Es sei G eine Gruppe. Dann ist $N(G)$ genau dann nicht n-distributiv, wenn Normalteiler A_i $(i=0, 1, \ldots, n)$ und C_{ij} $(i, j=0, 1, \ldots, n,\ i\neq j)$ existieren, so dass $A_0, A_1, \ldots, A_n$ ein unabhängiges System in $N(G)$ bilden und für alle i, j $(i\neq j)$ C_{ij} ein relatives Komplement von A_i und A_j in dem Interval $[A_i \wedge A_j, A_i \vee A_j]$ des Verbandes $N(G)$ ist.*

Um die versprochene Charakterisierung zu formulieren, ist es nötig einige weiteren Begriffe einzuführen. Sind A und B Normalteiler der Gruppe G so dass $A \leqq B$ in $N(G)$ gilt, dann heisst die Faktorgruppe B/A ein Faktor von G. Der Faktor B/A heisst transponiert zu dem Faktor D/C (in Zeichen $B/A \rightarrow D/C$), wenn entweder $A\vee D=B$ und $A\wedge D=C$ oder $B\vee C=D$ und $B\wedge C=A$ gelten. B/A heisst projektiv zu D/C, wenn es Faktoren Y_i/X_i $(i=0, 1, \ldots, m)$ gibt, so dass

$$B/A = Y_0/X_0 \rightarrow Y_1/X_1 \rightarrow \ldots \rightarrow Y_m/X_m = D/C$$

gilt. Die primitive Breite von $N(G)$ ist die grösste natürliche Zahl n, so dass $N(G)$ ein unabhängiges System $A_0, A_1, \ldots, A_{n-1}$ enthält, für das die Faktoren A_i/U mit $U=\bigwedge_{j=0}^{n-1} A_j$ paarweise projektiv sind. Bezüglich der allgemeinen Definition primitiver Begriffe siehe Wille [20].

Wir brauchen einen weiteren Begriff aus der Gruppentheorie. Es seien A, B, C, D Normalteiler der Gruppe G und es sei $\varphi\colon B/A \rightarrow D/C$ ein Isomorphismus. φ heisst zentral, wenn gegenüber allen inneren Automorphismen von G invariant ist, mit anderen Worten, wenn für jede $g\in G$ und $x\in B/A$

$$((g^{-1}A)x(gA))\varphi = (g^{-1}C)(x\varphi)(gC)$$

gilt.

4.2. Satz. *Es sei G eine Gruppe. Für eine beliebige natürliche Zahl n sind die folgenden drei Aussagen äquivalent*

(A) *$N(G)$ ist nicht n-distributiv.*

(B) *Die primitive Breite von $N(G)$ ist grösser als n.*

(C) *Es gibt ein unabhängiges System* $A_0, A_1, \ldots, A_n$ *von Elementen von* $N(G)$, *so dass die Faktoren* A_i/U $\left(mit\ U = \bigwedge_{j=0}^{n} A_j\right)$ *aufeinander durch zentralen Isomorphismen von* G *abgebildet werden können.*

Beweis. (A)$\Rightarrow$(B) folgt unmittelbar aus Lemma 4.1.

(B)$\Rightarrow$(C). Sind zwei Faktoren projektiv, so gibt es einen zentralen Isomorphismus zwischen den beiden Faktoren. (In der Tat ist der kanonische Isomorphismus transponierter Faktoren zentral.) Somit ist dieser Teil der Behauptung klar.

(C)$\Rightarrow$(A). Nehmen wir an, dass (C) gilt. Wir definieren die C_{ij} von Lemma 4.1. Es sei φ_{ij} $(i \neq j)$ ein zentraler Isomorphismus von A_i/U auf A_j/U. Es sei $\bar{C}_{ij} = \{x(x\varphi_{ij}) \mid x \in A_i/U\}$. Dann ist $\bar{C}_{ij} \subseteq G/U$. Es sei C_{ij} die Vereinigung aller U-Nebenklassen in $\bar{C}_{ij}$, d. h. $C_{ij} = \bigcup \bar{C}_{ij}$. So erhalten wir eine Teilmenge von G. Wir haben zu beweisen, dass C_{ij} die in Lemma 4.1 formulierten Eigenschaften besitzt. Allgemein wird für einen Normalteiler X mit $U \subseteq X \subseteq G$ der Faktor X/U mit $\bar{X}$ bezeichnet. Wir zeigen, dass die folgenden Aussagen gelten:

(i) $\bar{C}_{ij}$ ist ein Normalteiler von $\bar{G}$.

(ii) $\bar{A}_i \vee \bar{C}_{ij} = \bar{A}_j \vee \bar{C}_{ij} = \bar{A}_i \vee \bar{A}_j$,

(iii) $\bar{A}_i \wedge \bar{C}_{ij} = \bar{A}_j \wedge \bar{C}_{ij} = \bar{A}_i \wedge \bar{A}_j$.

Dann folgen die analogen Eigenschaften für C_{ij}, G, A_i, A_j unmittelbar.

Um (i) zu zeigen, bemerken wir, dass $\bar{C}_{ij}$ eine Untergruppe von $\bar{G}$ ist. In der Tat, ist $\bar{A}_i \vee \bar{A}_j$ das direkte Produkt von $\bar{A}_i$ und $\bar{A}_j$. Deshalb sind die Elemente von $\bar{A}_i$ mit den Elementen von $\bar{A}_j$ vertauschbar. Mit $\varphi = \varphi_{ij}$ sind $x(x\varphi)$ und $y(y\varphi)$ Elemente von $\bar{C}_{ij}$. Dann gelten

$$x(x\varphi)y(y\varphi) = xy(x\varphi)(y\varphi) = (xy)\big((xy)\varphi\big) \in \bar{C}_{ij}$$

und

$$x(x\varphi)x^{-1}(x^{-1}\varphi) = xx^{-1}(x\varphi)(x^{-1}\varphi) = e(xx^{-1})\varphi = e(e\varphi) = e,$$

d. h. $\big(x(x\varphi)\big)^{-1} = x^{-1}(x^{-1}\varphi) \in \bar{C}_{ij}$. Somit ist C_{ij} eine Untergruppe. Nun zeigen wir die Normalität. Es sei $a \in \bar{G}$ und $x(x\varphi) \in \bar{C}_{ij}$. Dann gilt

$$a^{-1}\big(x(x\varphi)\big)a = (a^{-1}xa)\big(a^{-1}(x\varphi)a\big) = (a^{-1}xa)\big((a^{-1}xa)\varphi\big) \in \bar{C}_{ij}.$$

Damit ist (i) bewiesen.

Es sei z ein Element von $\bar{A}_i \vee \bar{A}_j$. Dann ist z von der Form $z = x(y\varphi)$, $x, y \in \bar{A}_i$. Wir erhalten

$$z = x(y\varphi) = (xy^{-1})\big(y(y\varphi)\big) \in \bar{A}_i \vee \bar{C}_{ij},$$
$$z = x(y\varphi) = \big(x(x\varphi)\big)\big((x^{-1}y)\varphi\big) \in \bar{C}_{ij} \vee \bar{A}_j,$$

d. h. es gilt (ii).

Schliesslich zeigen wir (iii). Es sei $x \in \bar{A}_i \wedge \bar{C}_{ij}$, d. h. $x = y(y\varphi)$ für irgendein Element $y \in \bar{A}_i$. Da jedes Element von $\bar{A}_i \vee \bar{A}_j$ eindeutig als ein Produkt $a_i a_j$

mit $a_i \in \bar{A}_i$, $a_j \in \bar{A}_j$ ausgedrückt werden kann, erhält man aus der Beziehung $xe = y(y\varphi)$ die Relationen $x=y$ und $y\varphi=e$. Somit gilt $x=y=e$, d. h. $\bar{A}_i \wedge \bar{C}_{ij} = \{e\}$, wobei e das Einselement von $\bar{G}$ (d. h. die Untergruppe U) bezeichnet. Ähnlich erhält man $\bar{C}_{ij} \wedge \bar{A}_j = \{U\}$. Damit ist der Satz bewiesen.

Als Anwendung geben wir einen neuen Beweis von Satz 3.2. Der Beweis der Notwendigkeit war leicht. Wir brauchen also nur zu beweisen, dass die angegebene Bedingung hinreichend für die n-Distributivität des Untergruppenverbandes ist. Es sei A eine abelsche Gruppe mit $\operatorname{rang}(A) \leqq n$. Es ist leicht zu sehen, dass $\operatorname{rang}(A') \leqq n$ für jedes homomorphe Bild A' einer Untergruppe von G gilt. Deshalb kann A'' nicht die $(n+1)$-ste direkte Potenz einer Gruppe sein. Also ist kein Faktor von A die $(n+1)$-ste Potenz einer Gruppe, d. h. (C) ist unmöglich: $S(A)(=N(A))$ ist n-distributiv. Q. E. D.

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n-Distributivgesetze

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0. Einführung und Überblick

András Huhn prägte 1971 den Begriff der n-Distributivität, der eine Verallgemeinerung des gewöhnlichen Distributivgesetzes in Verbänden darstellt [10]. Wir nennen hier einen Verband X *n-distributiv*, wenn für jedes $x \in X$ und jede n-elementige Teilmenge Y von X die Gleichung $x \wedge \bigvee Y = \bigvee \{x \wedge \bigvee M \mid M \overset{n}{\subsetneqq} Y\}$ gilt. Dabei bedeutet $M \overset{n}{\subsetneqq} Y$, daß M eine Teilmenge von Y mit weniger als n Elementen ist. Für $n=2$ ist dies das gewöhnliche Distributivgesetz.

Es ist klar, daß für einen distributiven Verband das Distributivgesetz nicht nur für zweielementige Mengen Y, sondern sogar für alle endlichen Mengen gilt. Wir werden zeigen, daß in n-distributiven Verbänden die n-Distributivitätsgleichung auch für alle endlichen Mengen Y gilt. So wie man die gewöhnliche Distributivität zur $\bigvee$-Distributivität verschärft, indem man die Distributivitätseigenschaft für alle Teilmengen Y von X verlangt, liegt es nun nahe, auf dieselbe Weise eine Verschärfung der n-Distributivität zu definieren, die sogenannte *unendliche n-Distributivität*. Die unendliche n-Distributivität ist gleichbedeutend zu den beiden Eigenschaften $\wedge$-Stetigkeit und n-Distributivität (in Analogie zu dem bekannten Sachverhalt für $n=2$).

So wie man aber auch die gewöhnliche ($\bigvee$-)Distributivität zur vollständigen Distributivität verschärft, läßt sich analog die (unendliche) n-Distributivität zur *vollständigen n-Distributivität* verschärfen.

Als Werte für n lassen wir alle natürlichen Zahlen größer oder gleich 2 und $\aleph_0$ zu. Für $n=2$ erhält man die gewöhnlichen Distributivgesetze, für $n=\aleph_0$ die $\wedge$-Stetigkeit und die Stetigkeit (im Sinne von D. SCOTT [12]), so daß die übrigen n-Distributivgesetze als „interpolierende" Eigenschaften zwischen $\bigvee$-Distributivität und $\wedge$-Stetigkeit bzw. vollständiger Distributivität und Stetigkeit angesehen werden

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können. Die Abschnitte 2, 3 und 4 dieser Arbeit sind in Anlehnung an Arbeiten von MARCEL ERNÉ [3], [6] entstanden. Es werden die n-Distributivgesetze in einem sehr allgemeinen Rahmen behandelt, nämlich für Mengen, auf denen lediglich ein Hüllenoperator definiert ist. Dieser Idee liegt die Erkenntnis zugrunde, daß die ($\bigvee$-, vollständige) Distributivität eines Verbandes eigentlich eine Homomorphieeigenschaft des Schnittoperators ist ([6], Seite 20). Nur die Anwendung der Distributivgesetze für die mengentheoretische Durchschnitts- und Vereinigungsbildung (die ja immer gelten, also keine besondere Eigenschaft des Verbandes darstellen) führt auf das bekannte Aussehen der Distributivgesetze für Verbände. Der Vorteil dieser so allgemeinen Behandlung der n-Distributivität besteht in folgendem: Man erhält zum einen Charakterisierungen für diejenigen Verbände (sogar allgemeiner: quasigeordenete Mengen), die eine (vollständig, unendlich) n-distributive Schnittvervollständigung oder Idealvervollständigung besitzen. (Unter anderem wurden die Fälle $n=2$ und $n=\aleph_0$ in [4], [6] behandelt.) Zum anderen ergeben sich durch die Wahl des Hüllenoperators als diejenige Abbildung, die jeder Teilmenge einer gegebenen Algebra die kleinste sie enthaltende Subalgebra zuordnet, Charakterisierungen für die (vollständige, unendliche) n-Distributivität des Verbandes der Subalgebren oder Kongruenzrelationen. Insbesondere ergibt sich hier ein Satz über die n-Distributivität, der zwei derartige Sätze von András Huhn betreffend abelsche Gruppen und idempotente Algebren umfaßt. Weiter stellt sich heraus, daß für den Verband der Subalgebren einer idempotenten Algebra die (unendliche) n-Distributivität und die vollständige n-Distributivität gleichwertige Eigenschaften sind. Die abelschen Gruppen mit vollständig n-distributivem Untergruppenverband sind genau diejenigen, die keine Elemente unendlicher Ordnung besitzen, und deren endlich erzeugte Untergruppen immer schon von weniger als n Elementen erzeugt werden (letzteres bedeutet, daß der Verband der Untergruppen n-distributiv ist). Dies gilt für alle $n<\aleph_0$; für $n=\aleph_0$ ist jeder Untergruppenverband vollständig n-distributiv (d. h. stetig). Insbesondere gilt: Der Untergruppenverband einer abelschen Gruppe ist genau dann vollständig distributiv, wenn jede nicht triviale, endlich erzeugte Untergruppe von Primzahlordnung ist.

Weitere Anwendungen der n-distributivgesetze für Hüllenoperatoren erhält man durch die Wahl des Hüllenoperators als Abschlußoperator in topologischen Räumen. So ergibt sich zum Beispiel, daß für den Verband der abgeschlossenen Teilmengen eines T_1-Raumes aus der unendlichen n-Distributivität die vollständige n-Distributivität volgt.

Für alternative Verallgemeinerungen der klassischen Distributivgesetze wird der Leser verwiesen auf die Arbeiten [1], [5].

In den Notationen lehnen wir uns an die in [6] benutzte an. Zum Beispiel wird bei vorgegebenem Hüllenoperator Γ auf der Menge X der *Abschnittoperator* mit $\downarrow$ bezeichnet, d. h. $\downarrow Y=\bigcup\{\Gamma y \mid y\in Y\}$ $(Y\subseteq X)$. Ist X eine quasigeordnete

Menge, so ist Δ der *Schnittoperator* und I der *Idealoperator* auf X, also $\Delta Y = \bigcap\{\downarrow y \mid Y \subseteqq \downarrow y\}$, $IY = \bigcup\{\Delta M \mid M \subseteqq_e Y\}$ für $Y \subseteqq X$, wobei $M \subseteqq_e Y$ bedeutet, daß M eine endliche Teilmenge der Menge Y ist. $\mathfrak{P}Y$ bezeichnet die Menge aller Teilmengen von Y, $\mathfrak{P}_n Y$ die Menge aller Teilmengen von Y mit weniger als n Elementen und $\mathfrak{P}_e Y$ die Menge aller endlichen Teilmengen von Y.

1. n-distributive Verbände

Sei n eine natürliche Zahl, $n \geqq 2$. Ein Verband X heißt *n-distributiv*, wenn für jedes $x \in X$ und jede n-elementige Menge $Y \subseteqq X$ die Gleichung (d_n) gilt.

$$(d_n) \qquad x \wedge \bigvee Y = \bigvee \{x \wedge \bigvee M \mid M \overset{n}{\subseteqq} Y\}.$$

Die 2-Distributivität ist die gewöhnliche Distributivität.

Satz 1.1. *Der Verband X ist genau dann n-distributiv, wenn für jede endliche Menge $Y \subseteqq_e X$ die Gleichung (d_n) gilt.*

Beweis. Ist X n-distributiv, so gilt (d_n) für jede höchstens n-elementige Teilmenge von X. Angenommen, (d_n) gilt für jede höchstens m-elementige Teilmenge, $m \geqq n$. Wir zeigen, daß dann (d_n) auch für alle $(m+1)$-elementigen Teilmengen Y von X gilt. Sei also $Y = Z \cup \{a, b\}$, $|Z| = m-1$. Setze $z = a \vee b$. Da die Menge $Y_1 = Z \cup \{z\}$ höchstens m Elemente hat, gilt $x \wedge \bigvee Y = x \wedge \bigvee Y_1 = \bigvee\{x \wedge \bigvee M \mid M \overset{n}{\subseteqq} Z\} \vee \bigvee\{x \wedge \bigvee(N \cup \{z\}) \mid N \overset{n-1}{\subseteqq} Z\}$. Sei $N \overset{n-1}{\subseteqq} Z$ fest gewählt. Setze $Y_2 = N \cup \{a, b\}$. Wegen $|Y_2| \leqq n \leqq m$ gilt $x \wedge \bigvee(N \cup \{z\}) = x \wedge \bigvee Y_2 = \bigvee\{x \wedge \bigvee M \mid M \overset{n}{\subseteqq} Y_2\}$. Also gilt $x \wedge \bigvee Y = \bigvee\{x \wedge \bigvee M \mid M \overset{n}{\subseteqq} Z\} \vee \bigvee\{\bigvee\{x \wedge \bigvee M \mid M \overset{n}{\subseteqq} N \cup \{a, b\}\} \mid N \overset{n-1}{\subseteqq} Z\} = \bigvee\{x \wedge \bigvee M \mid M \overset{n}{\subseteqq} Y\}$.

Aufgrund von 1.1 liegt es nahe, den Begriff der $\bigvee$-Distributivität zu verallgemeinern: Ein vollständiger Verband X heiße *unendlich n-distributiv*, wenn für jede Menge $Y \subseteqq X$ die Beziehung (d_n) gilt. Die unendliche 2-Distributivität ist die $\bigvee$-Distributivität. Hier ist es sinnvoll, auch $\aleph_0$ als Wert für n zuzulassen: Die unendliche $\aleph_0$-Distributivität ergibt den bekannten Begriff der $\wedge$-Stetigkeit (vgl. [2], Seite 15).

Offensichtlich gilt: Erfüllen $x \in X$ und $Y \subseteqq X$ die Gleichung (d_n), so erfüllen sie auch (d_m) für jedes $m \geqq n$. Insbesondere ist ein unendlich n-distributiver Verband auch $\wedge$-stetig. Wir erhalten sogar (als Verallgemeinerung des Satzes, daß ein vollständiger Verband genau dann $\bigvee$-distributiv ist, wenn er $\wedge$-stetig und distributiv ist):

Satz 1.2. *Ein vollständiger Verband X ist genau dann unendlich n-distributiv, wenn er n-distributiv und $\wedge$-stetig ist.*

Beweis. Ist X $\wedge$-stetig, so gilt für $x \in X$ und $Y \subseteq X$: $x \wedge \bigvee Y = \bigvee \{x \wedge \bigvee N \mid N \subseteq_e Y\}$. Wenn X n-distributiv ist, gilt für jede endliche Teilmenge N von Y: $x \wedge \bigvee N = \bigvee \{x \wedge \bigvee M \mid M \overset{n}{\subseteq} N\}$. Es folgt somit: $x \wedge \bigvee Y = \bigvee \{\bigvee \{x \wedge \bigvee M \mid M \overset{n}{\subseteq} N\} \mid N \subseteq_e Y\} = \bigvee \{x \wedge \bigvee M \mid M \overset{n}{\subseteq} Y\}$.

Ist also der Verband der Subalgebren oder der Verband der Kongruenzrelationen einer Algebra A n-distributiv, dann ist er sogar unendlich n-distributiv.

Satz 1.3. *Ein (vollständiger) Verband X ist genau dann (unendlich) n-distributiv, wenn für jedes endliche System $\mathscr{Y}$ endlicher (beliebiger) Teilmengen von X gilt:*

$$(\mathrm{D}_n) \qquad \bigwedge \{\bigvee Y \mid Y \in \mathscr{Y}\} = \bigvee \{\bigwedge \{\bigvee f(Y) \mid Y \in \mathscr{Y}\} \mid f \in \prod_{Y \in \mathscr{Y}} \mathfrak{P}_n Y\}.$$

Beweis. Es gelte $x \wedge \bigvee Y = \bigvee \{x \wedge \bigvee M \mid M \overset{n}{\subseteq} Y\}$ für alle $x \in X$ und alle $Y \subseteq X$ ($\subseteq_e X$). Sei $\mathscr{Y} = \{Y_1, \ldots, Y_k\}$ eine Menge von (endlichen) Teilmengen von X. Es wird mittels vollständiger Induktion gezeigt, daß für alle $r = 1, \ldots, k$ gilt: $y := \bigvee Y_1 \wedge \ldots \wedge \bigvee Y_k = \bigvee \{\bigvee M_1 \wedge \ldots \wedge \bigvee M_r \wedge \bigvee Y_{r+1} \wedge \ldots \wedge \bigvee Y_k \mid M_i \overset{n}{\subseteq} Y_i$ für $i = 1, \ldots, r\}$. (Für $r = k$ erhält man die Behauptung.) Für $r = 1$ setze $x := \bigvee Y_2 \wedge \ldots \wedge \bigvee Y_k$. Dann ist $y = x \wedge \bigvee Y_1 = \bigvee \{x \wedge \bigvee M_1 \mid M_1 \overset{n}{\subseteq} Y_1\} = \bigvee \{\bigvee M_1 \wedge \bigvee Y_2 \wedge \ldots \wedge \bigvee Y_k \mid M_1 \overset{n}{\subseteq} Y_1\}$. Nehmen wir nun an, die Behauptung gilt für r, $1 \leq r < k$. Seien $M_1 \overset{n}{\subseteq} Y_1, \ldots, \ldots, M_r \overset{n}{\subseteq} Y_r$ gewählt. Mit $z := \bigvee M_1 \wedge \ldots \wedge \bigvee M_r \wedge \bigvee Y_{r+2} \wedge \ldots \wedge \bigvee Y_k$ folgt $\bigvee M_1 \wedge \ldots \wedge \bigvee M_r \wedge \bigvee Y_{r+1} \wedge \ldots \wedge \bigvee Y_k = z \wedge \bigvee Y_{r+1} = \bigvee \{z \wedge \bigvee M_{r+1} \mid M_{r+1} \overset{n}{\subseteq} Y_{r+1}\}$. Hieraus ergibt sich: $y = \bigvee \{\bigvee M_1 \wedge \ldots \wedge \bigvee M_r \wedge \bigvee M_{r+1} \wedge \bigvee Y_{r+2} \wedge \ldots \wedge \bigvee Y_k \mid M_i \overset{n}{\subseteq} Y_i$ für $i = 1, \ldots, r+1\}$. Damit ist der Induktionsbeweis beendet.

Es gelte umgekehrt (D_n) für jedes endliche System von (endlichen) Teilmengen von X. Seien $x \in X$ und $Y \subseteq X$ ($\subseteq_e X$) gewählt. Für $\mathscr{Y} = \{\{x\}, Y\}$ folgt $x \wedge \bigvee Y = \bigvee \{x\} \wedge \bigvee Y = \bigvee \{\bigvee \{x\} \wedge \bigvee M \mid M \overset{n}{\subseteq} Y\} = \bigvee \{x \wedge \bigvee M \mid M \overset{n}{\subseteq} Y\}$.

Wir nennen einen vollständigen Verband X *vollständig n-distributiv*, wenn die Gleichung (D_n) für jedes Mengensystem $\mathscr{Y} \subseteq \mathfrak{P} X$ erfüllt ist. Die vollständige 2-Distributivität ist der bekannte Begriff der vollständigen Distributivität. Vollständige $\aleph_0$-Distributivität ist dasselbe wie Stetigkeit (vgl. [8], Seite 58).

Es bietet sich noch die folgende Variante für einen n-Distributivitätsbegriff an: Ein vollständiger Verband X heiße *endlich n-distributiv*, wenn die Gleichung (D_n) für jedes Mengensystem $\mathscr{Y}$ bestehend aus endlichen Teilmengen von X gilt.

Erfüllt das Mengensystem $\mathscr{Y}$ die Gleichung (D_n), so erfüllt $\mathscr{Y}$ offensichtlich auch (D_m) für jedes $m \geqq n$. Insbesondere ist ein vollständig n-distributiver Verband auch stetig. In Analogie zu 1.2 gilt sogar:

Satz 1.4. *Ein vollständiger Verband ist genau dann vollständig n-distributiv, wenn er endlich n-distributiv und stetig ist.*

Beweis. Es ist noch zu zeigen, daß ein endlich n-distributiver und stetiger Verband X vollständig n-distributiv ist. Wenn X stetig ist, so gilt für ein beliebiges Mengensystem $\mathscr{Y} \subseteqq \mathfrak{P}X$: $\bigwedge\{\bigvee Y \mid Y \in \mathscr{Y}\} = \bigvee\{\bigwedge\{\bigvee f(Y) \mid Y \in \mathscr{Y}\} \mid f \in \prod_{Y \in \mathscr{Y}} \mathfrak{P}_e Y\}$. Wegen der endlichen n-Distributivität von X gilt für jedes $f \in \prod_{Y \in \mathscr{Y}} \mathfrak{P}_e Y$: $\bigwedge\{\bigvee f(Y) \mid Y \in \mathscr{Y}\} = \bigvee\{\bigwedge\{\bigvee g(f(Y)) \mid Y \in \mathscr{Y}\} \mid g \in \prod_{Y \in \mathscr{Y}} \mathfrak{P}_n f(Y)\}$. Es folgt somit: $\bigwedge\{\bigvee Y \mid Y \in \mathscr{Y}\} = \bigvee\{\bigvee\{\bigwedge\{\bigvee g(f(Y)) \mid Y \in \mathscr{Y}\} \mid g \in \prod_{Y \in \mathscr{Y}} \mathfrak{P}_n f(Y)\} \mid f \in \prod_{Y \in \mathscr{Y}} \mathfrak{P}_e Y\} = \bigvee\{\bigwedge\{\bigvee h(Y) \mid Y \in \mathscr{Y}\} \mid h \in \prod_{Y \in \mathscr{Y}} \mathfrak{P}_n Y\}$.

2. Charakterisierungen der n-Distributivgesetze durch Eigenschaften des Schnittoperators

Ist Γ ein Hüllenoperator auf der Menge X, so heißt der durch $\overset{n}{\downarrow} Y := \bigcup\{\Gamma M \mid M \overset{n}{\subseteqq} Y\}$ definierte Operator $\overset{n}{\downarrow} : \mathfrak{P}X \to \mathfrak{P}X$ *n-Abschnittoperator*. Die Bilder von $\overset{n}{\downarrow}$ heißen *n-Abschnitte*. Ist Γ der Schnittoperator einer quasigeordneten Menge X, so ist der Operator $\overset{n}{\downarrow}$ für $n=2$ der gewöhnliche Abschnittoperator und für $n=\aleph_0$ der Idealoperator. Für den (hier nicht vorkommenden) Fall $n>|X|$ erhält man den Schnittoperator. Ist X sogar ein Verband, so gilt $\overset{n}{\downarrow} Y = \downarrow\{\bigvee M \mid M \overset{n}{\subseteqq} X\}$.

Ist Γ ein Hüllenoperator auf der Menge X und ist $\mathscr{X}$ eine Menge von n-Abschnitten bzgl. Γ, so sagen wir Γ *erhält den Durchschnitt* $\bigcap\mathscr{X}$, wenn gilt: $\bigcap\Gamma[\mathscr{X}] = \Gamma(\bigcap\mathscr{X})$. (Hierbei kann „$=$“ durch „$\subseteqq$“ ersetzt werden.)

Satz 2.1. *Sei Δ der Schnittoperator auf dem (vollständigen) Verband X, und sei $\mathscr{Y} \subseteqq \mathfrak{P}X$. Δ erhält genau dann den Durchschnitt $\bigcap\{\overset{n}{\downarrow} Y \mid Y \in \mathscr{Y}\}$, wenn für $\mathscr{Y}$ die Gleichung* (D_n) *gilt.*

Beweis. $\bigcap\{\Delta Y \mid Y \in \mathscr{Y}\} = \bigcap\{\downarrow\bigvee Y \mid Y \in \mathscr{Y}\} = \downarrow\bigwedge\{\bigvee Y \mid Y \in \mathscr{Y}\}$. Andererseits gilt $\Delta(\bigcap\{\overset{n}{\downarrow} Y \mid Y \in \mathscr{Y}\}) = \downarrow\bigvee(\bigcap\{\overset{n}{\downarrow} Y \mid Y \in \mathscr{Y}\})$. Nun ist $\bigcap\{\overset{n}{\downarrow} Y \mid Y \in \mathscr{Y}\} = \bigcap\{\bigcup\{\downarrow\bigvee M \mid M \overset{n}{\subseteqq} Y\} \mid Y \in \mathscr{Y}\} = \bigcup\{\bigcap\{\downarrow\bigvee f(Y) \mid Y \in \mathscr{Y}\} \mid f \in \prod_{Y \in \mathscr{Y}} \mathfrak{P}_n Y\} = \bigcup\{\downarrow\bigwedge\{\bigvee f(Y) \mid Y \in \mathscr{Y}\} \mid f \in \prod_{Y \in \mathscr{Y}} \mathfrak{P}_n Y\} = \downarrow\{\bigwedge\{\bigvee f(Y) \mid Y \in \mathscr{Y}\} \mid f \in \prod_{Y \in \mathscr{Y}} \mathfrak{P}_n Y\}$. Hieraus ergibt sich die Behauptung.

Korollar 2.2. *Ein vollständiger Verband ist genau dann vollständig n-distributiv, wenn der Schnittoperator beliebige Durchschnitte von n-Abschnitten erhält.*

Ein vollständiger Verband ist genau dann unendlich n-distributiv, wenn der Schnittoperator endliche Durchschnitte von n-Abschnitten erhält.

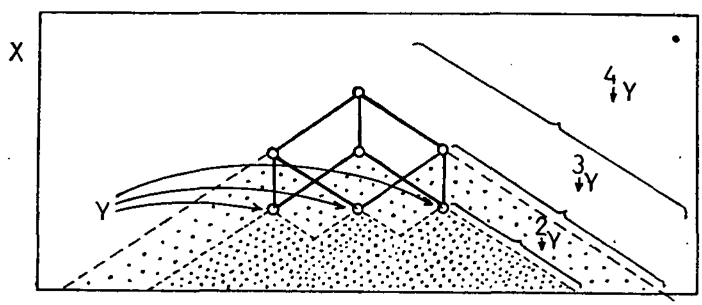
Ein vollständiger Verband ist genau dann endlich n-distributiv, wenn der Schnittoperator beliebige Durchschnitte von endlich erzeugten n-Abschnitten erhält.

Ein Verband ist genau dann n-distributiv, wenn der Schnittoperator endliche Durchschnitte von endlich erzeugten n-Abschnitten erhält.

Dabei heißt bei gegebener Abbildung Φ von X in X eine Menge $Z \subseteq X$ *endlich erzeugt*, wenn Z das Bild einer endlichen Teilmenge von X unter Φ ist.

Wir stellen noch einige Eigenschaften des n-Abschnittoperators zusammen. Sei Γ ein Hüllenoperator auf der Menge X und sei $\overset{n}{\downarrow}$ der zu Γ gehörige n-Abschnittoperator. Wie leicht zu sehen ist, gelten für jede Teilmenge Y von X die Beziehungen $\Gamma Y = \overset{n}{\downarrow}\Gamma Y = \Gamma \overset{n}{\downarrow} Y$ und $\overset{m}{\downarrow} Y \subseteq \overset{n}{\downarrow} Y$ für $m \leqq n$. Der Operator $\overset{n}{\downarrow}$ ist extensiv und monoton, aber für $2 < n < \aleph_0$ im allgemeinen nicht idempotent (also kein Hüllenoperator). Wenn jedoch für jede Menge $N \subseteq X$ ein $x_N \in X$ mit $\Gamma x_N = \Gamma N$ existiert (was zum Beispiel für den Schnittoperator eines vollständigen Verbandes der Fall ist), so gilt $\overset{m}{\downarrow} \circ \overset{n}{\downarrow} = \overset{r}{\downarrow}$ mit $r = (n-1)(m-1)+1$ bzw. $r = \aleph_0$, wenn n oder m den Wert $\aleph_0$ hat.

Beweis. $\overset{m}{\downarrow}(\overset{n}{\downarrow} Y) = \bigcup\{\Gamma M \mid M \overset{m}{\subseteq} \bigcup\{\Gamma N \mid N \overset{n}{\subseteq} Y\}\} = \bigcup\{\Gamma M \mid M \overset{m}{\subseteq} \bigcup\{\Gamma x_N \mid \cdot$ $\cdot \mid N \overset{n}{\subseteq} Y\}\} \overset{(*)}{=} \bigcup\{\Gamma(\{x_N \mid N \in \mathcal{N}\}) \mid \mathcal{N} \overset{m}{\subseteq} \mathfrak{P}_n Y\} \overset{(*)}{=} \bigcup\{\Gamma(\bigcup \mathcal{N}) \mid \mathcal{N} \overset{m}{\subseteq} \mathfrak{P}_n Y\} = \bigcup\{\Gamma S \mid \cdot$ $\cdot \mid S \overset{r}{\subseteq} Y\} = \overset{r}{\downarrow} Y$. Dabei geht an den mit $(*)$ gekennzeichneten Stellen die folgende Beziehung ein: Aus $K_i \subseteq \Gamma L_i$ für jedes $i \in I$ folgt $\Gamma(\bigcup_{i \in I} K_i) \subseteq \Gamma(\bigcup_{i \in I} L_i)$.



3. Die (unendliche) n-Distributivität

Die Ergebnisse des nächsten Satzes stammen größtenteils von Marcel Erné und sind grundlegend für die Charakterisierungen der n-Distributivität und der unendlichen n-Distributivität. Wir geben hier einen etwas anderen Beweis dafür, daß aus der Aussage (b) die Aussage (c) folgt. Der Vollständigkeit wegen werden hier die Beweise aus [6] für „aus (a) folgt (b)“ und „aus (c) folgt (a)“ mit aufgeführt.

Lemma 3.1. *Seien Γ ein Hüllenoperator und $\mathcal{M}$ ein Mengensystem auf der Menge X, so daß für alle $x\in X$ die Menge $\{x\}$ oder der Punktabschluß Γx ein Element von $\mathcal{M}$ ist. Die folgenden Aussagen* (a)—(c) *sind äquivalent.*

(a) *Für jedes $x\in X$ und alle $M\in\mathcal{M}$ mit $x\in\Gamma M$ gibt es eine Menge $N\subseteq{\downarrow}M$ mit $\Gamma N={\downarrow}x$.*

(b) ${\downarrow}x\cap\Gamma M=\Gamma({\downarrow}x\cap{\downarrow}M)$ *für alle* $x\in X$, $M\in\mathcal{M}$.

(c) *Γ erhält endliche Durchschnitte von Mengen ${\downarrow}M$, $M\in\mathcal{M}$.*

Ist Γ ein algebraischer Hüllenoperator und ist $\mathcal{M}$ eine Menge von n-Abschnitten, die alle endlich erzeugten n-Abschnitte enthält, so ist (d) *zu* (a)—(c) *äquivalent.*

(d) *Für jedes $x\in X$ und alle $E\subseteq_e X$ mit $x\in\Gamma E$ gibt es eine Menge $N\subseteq_e\overset{n}{\downarrow}E$ mit $\Gamma N={\downarrow}x$.*

Gilt $\Gamma x=\{x\}$ für alle $x\in X$, dann ist (e) *zu* (a)—(c) *äquivalent.*

(e) *Für alle $M\in\mathcal{M}$ ist $\Gamma M={\downarrow}M$.*

Beweis. (a)→(b): Seien $x\in X$, $M\in\mathcal{M}$ und $y\in{\downarrow}x\cap\Gamma M$. Dann gibt es eine Menge $N\subseteq{\downarrow}M$ mit $\Gamma N={\downarrow}y$. Es folgt $N\subseteq\Gamma N={\downarrow}y\subseteq{\downarrow}x$, also $N\subseteq{\downarrow}x\cap{\downarrow}M$, und damit $y\in\Gamma N\subseteq\Gamma({\downarrow}x\cap{\downarrow}M)$.

(b)→(c): Sei $\mathcal{Y}\subseteq\{{\downarrow}M\mid M\in\mathcal{M}\}$. Ist $\mathcal{Y}=\emptyset$, so ist (c) erfüllt. Es sei nun $\mathcal{Y}=\{Y_1,\dots,Y_k\}$, $k\geqq 1$. Es wird mittels vollständiger Induktion gezeigt, daß für $r=1,\dots,k$ gilt:

$$(*)\qquad \bigcap\Gamma[\mathcal{Y}]=\Gamma(Y_1\cap\dots\cap Y_r\cap\Gamma Y_{r+1}\cap\dots\cap\Gamma Y_k).$$

Sei $x\in\bigcap\Gamma[\mathcal{Y}]$. Wegen ${\downarrow}x\subseteq\Gamma Y_2\cap\dots\cap\Gamma Y_k$ ist $x\in{\downarrow}x\cap\Gamma Y_1=\Gamma({\downarrow}x\cap Y_1)\subseteq\Gamma(Y_1\cap\cap\Gamma Y_2\cap\dots\cap\Gamma Y_k)$. Es gelte nun $(*)$ für ein r, $1\leqq r<k$. Sei $x\in Y_1\cap\dots\cap Y_r\cap\cap\Gamma Y_{r+1}\cap\dots\cap\Gamma Y_k$. Dann ist $x\in{\downarrow}x\cap\Gamma Y_{r+1}=\Gamma({\downarrow}x\cap Y_{r+1})\subseteq\Gamma(Y_1\cap\dots\cap Y_{r+1}\cap\cap\Gamma Y_{r+2}\cap\dots\cap\Gamma Y_k)$. Also gilt $\Gamma(Y_1\cap\dots\cap Y_r\cap\Gamma Y_{r+1}\cap\dots\cap\Gamma Y_k)\subseteq\Gamma(Y_1\cap\dots\dots\cap Y_{r+1}\cap\Gamma Y_{r+2}\cap\dots\cap\Gamma Y_k)$. Damit ist der Induktionsbeweis beendet.

(c)→(a): Seien $M\in\mathcal{M}$ und $x\in\Gamma M$. Es gilt ${\downarrow}x={\downarrow}x\cap\Gamma M=\Gamma({\downarrow}x\cap{\downarrow}M)=\Gamma N$, wobei $N:={\downarrow}x\cap{\downarrow}M$ eine in ${\downarrow}M$ enthaltene Menge ist.

Zu (d): Ist Γ ein algebraischer Hüllenoperator und gilt $\{\overset{n}{\downarrow}E\mid E\subseteq_e X\}\subseteq\mathcal{M}$, so ist (a) zu der folgenden Bedingung äquivalent: $(*)$ Ist $x\in\Gamma E$, $E\subseteq_e X$, so ist $\Gamma K={\downarrow}x$ für eine Menge $K\subseteq\overset{n}{\downarrow}E$. Denn einerseits muß $(*)$ gelten, da $\mathcal{M}$ alle endlich erzeugten n-Abschnitte enthält. Ist andererseits $M\in\mathcal{M}$, so ist $M=\overset{n}{\downarrow}S$ für eine Menge $S\subseteqq X$. Ist nun $x\in\Gamma M=\Gamma S$, so ist $x\in\Gamma E$ für eine Menge $E\subseteqq_e S$, da Γ algebraisch ist; aus $K\subseteqq\overset{n}{\downarrow}E$ folgt $K\subseteqq\overset{n}{\downarrow}S=M$. Wegen $\Gamma K=\bigcup\{\Gamma F\mid F\subseteqq_e K\}$ und $x\in\Gamma K$ gibt es eine Menge $N\subseteqq_e K$ mit $x\in\Gamma N$ und somit $\Gamma N=\downarrow x$.

Zu (e): Gilt (a) und ist $x\in\Gamma M$ für eine Menge $M\in\mathcal{M}$, so ist $\Gamma N=\Gamma x$ für eine Menge $N\subseteqq\downarrow M$. Aus $\Gamma x=\{x\}$ folgt $N=\{x\}$. Also ist $x\in\downarrow M$. Damit ist gezeigt: $\Gamma M=\downarrow M$. Gilt umgekehrt $\Gamma M=\downarrow M$ für alle $M\in\mathcal{M}$, so ist (b) offensichtlich erfüllt.

Satz 3.2. *Sei Γ ein Hüllenoperator auf der Menge X und sei $\mathfrak{X}$ das zu Γ gehörige Hüllensystem. Die folgenden Aussagen* (a)—(d) *sind äquivalent.*

(a) *Für jedes $x\in X$ und jede endliche Teilmenge Y von X mit $x\in\Gamma Y$ gibt es eine Menge $N\subseteqq\overset{n}{\downarrow}Y$ mit $\Gamma N=\downarrow x$.*

(b) $\overline{\downarrow} x\cap\Gamma Y=\Gamma(\downarrow x\cap\overset{n}{\downarrow}Y)$ *für alle $x\in X$, $Y\subseteqq_e X$.*

(c) *Γ erhält endliche Durchschnitte von endlich erzeugten n-Abschnitten.*

(d) *Das n-Distributivgesetz* (d_n) *wird von allen Elementen von $\mathfrak{X}$ erfüllt, die endlich erzeugt sind (bzgl. Γ).*

Ist Γ ein algebraischer Hüllenoperator, so ist (e) *zu* (a)—(d) *äquivalent.*

(e) *Für jedes $x\in X$ und jede endliche Teilmenge Y von X mit $x\in\Gamma Y$ gibt es eine Menge $N\subseteqq_e\overset{n}{\downarrow}Y$ mit $\Gamma N=\downarrow x$.*

Enthält $\mathfrak{X}$ alle einelementigen Mengen, so ist (f) *zu* (a)—(d) *äquivalent.*

(f) *Für alle $Y\subseteqq_e X$ ist $\Gamma Y=\overset{n}{\downarrow}Y$.*

Beweis. Mit der Menge aller endlich erzeugten n-Abschnitte als Mengensystem $\mathcal{M}$ ergibt 3.1 die Äquivalenz von (a), (b) und (c) und, unter den angegebenen Voraussetzungen, auch die Äquivalenz von (e) bzw. (f) zu (a)—(c).

(c)→(d): Seien $Z\in\mathfrak{X}$ und $\mathcal{Y}=\{Y_1,\dots,Y_n\}\subseteqq\mathfrak{X}$, $Z=\Gamma E$ und $Y_i=\Gamma M_i$ für Mengen $E\subseteqq_e X$, $M_i\subseteqq_e X$ $(i=1,\dots,n)$. Es ist $\bigvee\mathcal{Y}=\Gamma(\bigcup\mathcal{Y})=\Gamma M$ mit $M=$ $=M_1\cup\dots\cup M_n$. Aufgrund von (c) gilt $Z\wedge\bigvee\mathcal{Y}=Z\cap\Gamma M=\Gamma(Z\cap\overset{n}{\downarrow}M)$. Nun gilt $Z\cap\overset{n}{\downarrow}M=Z\cap\bigcup\{\Gamma N\mid N\overset{n}{\subseteqq}M\}\subseteqq\bigcup\{Z\cap\Gamma N\mid N\overset{n}{\subseteqq}\bigcup\mathcal{Y}\}\subseteqq\bigcup\{Z\cap\Gamma(\bigcup\mathcal{Z})\mid\mathcal{Z}\overset{n}{\subseteqq}\mathcal{Y}\}$. Es folgt $Z\wedge\bigvee\mathcal{Y}\subseteqq\bigvee\{Z\wedge\bigvee\mathcal{Z}\mid\mathcal{Z}\overset{n}{\subseteqq}\mathcal{Y}\}$.

(d)→(b): Seien $x\in X$ und $Y\subseteqq_e X$. Es gilt $\downarrow x\cap\Gamma Y=\Gamma x\cap\Gamma(\bigcup\{\Gamma y\mid y\in Y\})=$ $=\Gamma x\wedge\bigvee\{\Gamma y\mid y\in Y\}=\bigvee\{\Gamma x\wedge\bigvee\{\Gamma y\mid y\in Z\}\mid Z\overset{n}{\subseteqq}Y\}=\Gamma(\bigcup\{\downarrow x\cap\Gamma Z\mid Z\overset{n}{\subseteqq}Y\})=$ $=\Gamma(\downarrow x\cap\bigcup\{\Gamma Z\mid Z\overset{n}{\subseteqq}Y\})=\Gamma(\downarrow x\cap\overset{n}{\downarrow}Y)$.

Satz 3.3. *Sei Γ ein Hüllenoperator auf der Menge X und sei $\mathfrak{X}$ das zu Γ gehörige Hüllensystem. Die folgenden Aussagen* (a)—(d) *sind äquivalent.*

(a) *Für jedes* $x \in X$ *und jede Teilmenge* Y *von* X *mit* $x \in \Gamma Y$ *gibt es eine Menge* $N \subseteq \overset{n}{\downarrow} Y$ *mit* $\Gamma N = \downarrow x$.

(b) $\downarrow x \cap \Gamma Y = \Gamma(\downarrow x \cap \overset{n}{\downarrow} Y)$ *für alle* $x \in X$, $Y \subseteq X$.

(c) Γ *erhält endliche Durchschnitte von n-Abschnitten.*

(d) $\mathfrak{X}$ *ist ein unendlich n-distributiver Verband.*

Ist Γ *ein algebraischer Hüllenoperator, so ist* (e) *zu* (a)—(d) *äquivalent.*

(e) *Für jedes* $x \in X$ *und jede endliche Teilmenge* Y *von* X *mit* $x \in \Gamma Y$ *gibt es eine Menge* $N \subseteq_e \overset{n}{\downarrow} Y$ *mit* $\Gamma N = \downarrow x$.

Enthält $\mathfrak{X}$ *alle einelementigen Mengen, so ist* (f) *zu* (a)—(d) *äquivalent.*

(f) $\Gamma = \overset{n}{\downarrow}$.

Beweis. Aus 3.1 erhält man die Äquivalenz von (a), (b) und (c) und, unter den angegebenen Zusatzvoraussetzungen, auch die Äquivalenz von (e) bzw. (f) zu (a)—(c), wenn man als Mengensystem $\mathcal{M}$ die Menge aller n-Abschnitte bzgl. Γ nimmt.

(c)$\to$(d): Analog zum Beweis „(c)$\to$(d)" von 3.2.

(d)$\to$(b): Analog zum Beweis „(d)$\to$(b)" von 3.2.

Wir wollen die Aussagen von 3.2 und 3.3 kurz für den Fall betrachten, daß Γ der Schnittoperator Δ eines (vollständigen) Verbandes X ist. Die Aussagen von 1.3 erhält man als Spezialfälle der Äquivalenz von (b) und (c) in 3.2 und 3.3 (wenn man die Charakterisierung der n-Distributivität von 1.1 voraussetzt). Die Äquivalenz von (a) und (b) in 3.2 ergibt für $n=2$ die lokale Charakterisierung der Distributivität von Grätzer ([9], Seite 99). Daß die Aussagen (b) und (d) in 3.3 äquivalent sind, bedeutet in diesem Fall, daß die unendliche n-Distributivität von X gleichbedeutend ist mit der unendlichen n-Distributivität der Schnittvervollständigung von X. Dies ist aber klar, da vollständige Verbände isomorph zu ihrer Schnittvervollständigung sind.

Die Äquivalenz von 3.3 (d) zu den anderen Bedingungen von 3.3 ist aber keineswegs für andere Hüllenoperatoren wertlos. Dies soll im folgenden verdeutlicht werden.

Aus einem Satz über Polynomidentitäten (siehe [2], Seite 68) folgt, daß ein Verband genau dann n-distributiv ist, wenn dies für seinen Idealverband zutrifft. Aus 3.3 erhalten wir ein allgemeineres Resultat, wenn wir für Γ den Idealoperator I wählen und beachten, daß der Operator I, auf endliche Mengen angewandt, mit dem Schnittoperator Δ übereinstimmt:

Korollar 3.4. *Für eine quasigeordnete Menge* X *sind äquivalent:*

(a) *Ist* $Y \subseteq X$ *und ist* $x \in \Delta Y$, *so existiert eine endliche Teilmenge von* $\overset{n}{\downarrow} Y$, *für die* x *kleinste obere Schranke ist.*

(b) *Für alle* $x \in X$ *und* $Y \subseteq X$ *gilt* $\downarrow x \cap IY = I(\downarrow x \cap \overset{n}{\downarrow} Y)$.
(c) I *erhält endliche Durchschnitte von n-Abschnitten.*
(d) *Der Idealverband von* X *ist (unendlich) n-distributiv.*

Bezeichnet Γ den Schnittoperator Δ eines Verbandes X, so charakterisieren die äquivalenten Bedingungen (a)—(d) von 3.2 die n-Distributivität von X. In diesem Fall müssen also 3.2 (a) und 3.4 (a) übereinstimmen. Es soll nun untersucht werden, für welche quasigeordneten Mengen X die Bedingungen 3.2 (a) (für $\Gamma = \Delta$) und 3.4 (a) sonst noch identisch sind: Im Gegensatz zu 3.2 (a) wird in 3.4 (a) zu vorgegebener Menge $Y \subseteq_e X$ und $x \in \Delta Y$ eine *endliche* Teilmenge N von $\overset{n}{\downarrow} Y$ mit $\Delta N = \downarrow x$ gefordert. Wie aus dem Beweis von 3.1 hervorgeht, kann in 3.2 (a) die Menge N als Durchschnitt eines Hauptabschnitts mit einem endlich erzeugten n-Abschnitt gewählt werden. Falls also X die Eigenschaft hat, daß im Durchschnitt $\downarrow x \cap \overset{n}{\downarrow} F$ eines Hauptabschnitts $\downarrow x$ mit einem endlich erzeugten n-Abschnitt $\overset{n}{\downarrow} F$ eine endliche Menge E enthalten ist mit $\Delta E = \Delta(\downarrow x \cap \overset{n}{\downarrow} F)$, so sind 3.2 (a) und 3.4 (a) äquivalent für X. Hier reicht es, diese Bedingung nur für alle (nicht leeren) Mengen F mit weniger als n Elementen zu fordern, denn für eine beliebige Teilmenge Z von X gilt $\downarrow x \cap \overset{n}{\downarrow} Z = \bigcup \{\downarrow x \cap \Delta F \mid F \overset{n}{\subseteq} Z\}$.

In einem Verband ist aber $\downarrow x \cap \overset{n}{\downarrow} F = \downarrow x \cap \Delta F = \downarrow x \cap \downarrow \bigvee F = \downarrow (x \wedge \bigvee F)$ für $\emptyset \neq F \subseteq X$. Als die geforderte Menge E kann man hier also $\{x \wedge \bigvee F\}$ nehmen. Im Fall $n=2$ kann man offensichtlich genauso schließen, wenn X lediglich ein $\wedge$-Halbverband ist ([6]). Daneben gilt die Äquivalenz von 3.2 (a) und 3.4 (a) natürlich auch für alle endlichen quasigeordneten Mengen.

Es sei noch bemerkt, daß die Bedingung 3.4 (a) für $n=2$ mit der von Katriňák [11] gegebenen Definition der Distributivität eines $\vee$-Halbverbandes übereinstimmt. Die von Katriňák bemerkte Tatsache, daß ein $\vee$-Halbverband genau dann distributiv ist, wenn dies für seinen Idealverband zutrifft, ist also ein Spezialfall von 3.4.

Die Äquivalenz der Bedingungen (d) und (e) von 3.3 ergibt insbesondere auch eine Charakterisierung für die (unendliche) n-Distributivität des Verbandes Su (A) der Subalgebren einer Algebra A. Wir stellen dieses Ergebnis noch einmal besonders heraus:

Korollar 3.5. *Sei* A *eine Algebra.* Su (A) *ist genau dann (unendlich) n-distributiv, wenn für jedes* $x \in X$ *und jede endliche Teilmenge* Y *von* X *mit* $x \in [Y]$ *eine Menge* $N \subseteq_e \overset{n}{\downarrow} Y$ *existiert mit* $[N] = [x]$.

Die Bedingung in 3.5 läßt sich auch so formulieren: Ist $x \in [Y]$, $Y \subseteq_e X$, so ist $x \in [N]$, wobei jedes der endlich vielen Elemente von N in $[x]$ und in einer Menge $[M]$, $M \overset{n}{\subseteq} Y$, liegt.

Ist A eine idempotente Algebra, so ist die n-Distributivität von $\mathrm{Su}(A)$ auch gleichwertig mit der Bedingung $\overset{n}{\downarrow}=\Gamma$ (siehe 3.3 (f)). Diese Charakterisierung der idempotenten Algebren mit n-distributivem Subalgebrenverband stammt von András Huhn.

Die abelschen Gruppen mit n-distributivem Untergruppenverband wurden von András Huhn wie folgt charakterisiert:

Sei G eine abelsche Gruppe. $\mathrm{Su}(G)$ *ist genau dann n-distributiv, wenn jede endlich erzeugte Untergruppe schon von weniger als n Elementen erzeugt wird.*

Beweis (mit Hilfe von 3.5). Angenommen, U ist eine endlich erzeugte Untergruppe von G, die nicht von weniger als n Elementen erzeugt wird. Wir können o.B.d.A. annehmen, daß U isomorph ist zu $\mathbf{Z}_{r_1}\times\ldots\times\mathbf{Z}_{r_n}$ mit Zahlen $r_i\in\mathbf{N}_0:=$ $:=\mathbf{N}\cup\{0\}$. Der größte gemeinsame Teiler von $r_1, \ldots, r_n$ ist ungleich 1, denn sonst ließe sich $\mathbf{Z}_{r_1}\times\ldots\times\mathbf{Z}_{r_n}$ in ein direktes Produkt mit weniger als n Faktoren verwandeln: Ist etwa $r_1\neq 0$ und $r_1=q_1\ldots q_k$ die Zerlegung von r_1 in Primpotenzen, so ist $\mathbf{Z}_{r_1}\cong\mathbf{Z}_{q_1}\times\ldots\times\mathbf{Z}_{q_k}$; wäre $\mathrm{ggT}(r_1, \ldots, r_n)=1$, so könnte jeder Faktor $\mathbf{Z}_{q_j}$ mit einem Faktor $\mathbf{Z}_{r_i}$, $i\in\{2, \ldots, n\}$, vermöge $\mathbf{Z}_{r_i}\times\mathbf{Z}_{q_j}\cong\mathbf{Z}_{r_i\cdot q_j}$ verschmolzen werden. Nimmt man nun als Elemente von Y die den Vektoren $(1, 0, \ldots, 0), \ldots$, $\ldots, (0, \ldots, 0, 1)$ entsprechenden Elemente von U und für x das dem Vektor $(1, \ldots, 1)$ entsprechende Element, so ist zwar $x\in[Y]$, aber $x\notin[\overset{n}{\downarrow}Y\cap[x]]$, denn in $\overset{n}{\downarrow}Y\cap[x]$ liegen nur Elemente von G, die Vektoren der Form $(k_1 r_1, \ldots, k_1 r_1), \ldots$, $\ldots, (k_n r_n, \ldots, k_n r_n)$ entsprechen (die j-te Komponente jeweils modulo r_j); wäre x die Summe solcher Elemente, so müßte eine Gleichung der Form $1=k_1 r_1+\ldots+k_n r_n$ mit ganzen Zahlen $k_1, \ldots, k_n$ gelten, im Widerspruch dazu, daß $r_1, \ldots, r_n$ teilerfremd sind.

Andererseits gilt die Bedingung aus 3.5 für die n-Distributivität offensichtlich für alle Teilmengen Y von G mit weniger als n Elementen. Nehmen wir also an, daß die Bedingung aus 3.5 für alle $(m-1)$-elementigen Teilmengen von G erfüllt ist für ein $m\geqq n$. Es sei nun $Y=\{y_1, \ldots, y_m\}\subseteq G$, $x\in[Y]$, o.B.d.A. $x=y_1+\ldots+y_m$. Vorausgesetzt, $[Y]$ wird schon von weniger als n Elementen erzeugt, dann gilt $[Y]\cong Z_{r_2}\times\ldots\times Z_{r_n}$ für gewisse Zahlen $r_2, \ldots, r_n\in\mathbf{N}_0$. Die Elemente von Y können also als $(n-1)$-komponentige Vektoren angesehen werden. Somit gibt es teilerfremde Zahlen $k_1, \ldots, k_m\in\mathbf{Z}$ mit $k_1 y_1+\ldots+k_m y_m=0$. Also gilt $x_j:=k_j x=$ $=k_j y_1+\ldots+k_j y_m=(k_j-k_1)\,y_1+\ldots+(k_j-k_m)y_m\in[Y_j]$ für $Y_j:=Y\setminus y_j$. Nach Induktionsvoraussetzung ist $x_j\in[\overset{n}{\downarrow}Y_j\cap[x_j]]\subseteq[\overset{n}{\downarrow}Y\cap[x]]$. Da $k_1, \ldots, k_m$ teilerfremd sind, gibt es ganze Zahlen $t_1, \ldots, t_m$ mit $k_1 t_1+\ldots+k_m t_m=1$. Es folgt: $x=(k_1 t_1+\ldots$ $\ldots+k_m t_m)x=t_1 x_1+\ldots+t_m x_m\in[\overset{n}{\downarrow}Y\cap[x]]$.

4. Die vollständige n-Distributivität

Satz 4.1. *Sei Γ ein Hüllenoperator auf der Menge X und sei $\mathfrak{X}$ das zu Γ gehörige Hüllensystem. Die folgenden Aussagen* (a)—(c) *sind äquivalent:*

(a) *Für jedes $x\in X$ gibt es eine Menge $N\subseteq X$, so daß $\Gamma N=\downarrow x$ und $N\subseteq\overset{n}{\downarrow}Y$ für alle $Y\subseteq X$ mit $x\in\Gamma Y$ gilt.*

(b) *Γ erhält beliebige Durchschnitte von n-Abschnitten.*

(c) *$\mathfrak{X}$ ist ein vollständig n-distributiver Verband.*

Ist Γ ein algebraischer Hüllenoperator, so ist (d) *zu* (a)—(c) *äquivalent.*

(d) *Für jedes $x\in X$ gibt es eine Menge $N\subseteq_e X$, so daß $\Gamma N=\downarrow x$ und $N\subseteq_e\overset{n}{\downarrow}Y$ für alle $Y\subseteq_e X$ mit $x\in\Gamma Y$ gilt.*

Enthält $\mathfrak{X}$ alle einelementigen Teilmengen, so ist (e) *zu* (a)—(c) *äquivalent.*

(e) $\Gamma=\overset{n}{\downarrow}$.

Beweis. (a)$\to$(b): Sei $\mathscr{Y}\subseteq\mathfrak{P}X$. Ist $x\in\bigcap\Gamma[\mathscr{Y}]$, so gilt $x\in\Gamma Y$ für jedes $Y\in\mathscr{Y}$. Nach Voraussetzung existiert eine Menge N mit $x\in\Gamma N$ und $N\subseteq\overset{n}{\downarrow}Y$ für alle $Y\in\mathscr{Y}$. Also ist $N\subseteq\bigcap\{\overset{n}{\downarrow}Y\mid Y\in\mathscr{Y}\}$, und somit gilt $x\in\Gamma N\subseteq\Gamma(\bigcap\{\overset{n}{\downarrow}Y\mid Y\in\mathscr{Y}\})$.

(b)$\to$(a): Sei $x\in X$. Setze $N=\bigcap\{\overset{n}{\downarrow}Y\mid x\in\Gamma Y\}$. Nach Voraussetzung ist $\Gamma N=\bigcap\{\Gamma Y\mid x\in\Gamma Y\}$. Also gilt $\Gamma N=\downarrow x$ und $N\subseteq\overset{n}{\downarrow}Y$ für alle $Y\subseteq X$ mit $x\in\Gamma Y$.

(b)$\to$(c): Sei $\mathscr{S}\subseteq\mathfrak{P}\mathfrak{X}$. Wegen (b) gilt $\bigwedge\{\bigvee\mathscr{X}\mid\mathscr{X}\in\mathscr{S}\}=\bigcap\{\Gamma(\bigcup\mathscr{X})\mid\mathscr{X}\in\mathscr{S}\}=$ $=\Gamma(\bigcap\{\overset{n}{\downarrow}(\bigcup\mathscr{X})\mid\mathscr{X}\in\mathscr{S}\})$. Es ist aber $\bigcap\{\overset{n}{\downarrow}(\bigcup\mathscr{X})\mid\mathscr{X}\in\mathscr{S}\}=\bigcap\{\bigcup\{\Gamma M\mid M\overset{n}{\subseteq}\bigcup\mathscr{X}\}\mid$ $\mid\mathscr{X}\in\mathscr{S}\}=\bigcup\{\bigcap\{\Gamma\psi(\mathscr{X})\mid\mathscr{X}\in\mathscr{S}\}\mid\psi\in\prod_{\mathscr{X}\in\mathscr{S}}\mathfrak{P}_n(\bigcup\mathscr{X})\}\subseteq\bigcup\{\bigcap\{\Gamma(\bigcup f(\mathscr{X}))\mid\mathscr{X}\in\mathscr{S}\}\mid f\in$ $\in\prod_{\mathscr{X}\in\mathscr{S}}\mathfrak{P}_n\mathscr{X}\}$. Also folgt: $\bigwedge\{\bigvee\mathscr{X}\mid\mathscr{X}\in\mathscr{S}\}\subseteq\bigvee\{\bigwedge\{\bigvee f(\mathscr{X})\mid\mathscr{X}\in\mathscr{S}\}\mid f\in\prod_{\mathscr{X}\in\mathscr{S}}\mathfrak{P}_n\mathscr{X}\}$.

(c)$\to$(b): Sei $\mathscr{Y}\subseteq\mathfrak{P}X$. Es gilt $\bigcap\{\Gamma Y\mid Y\in\mathscr{Y}\}=\bigwedge\{\bigvee\{\Gamma y\mid y\in Y\}\mid Y\in\mathscr{Y}\}=$ $=\bigvee\{\bigwedge\{\bigvee\{\Gamma y\mid y\in f(Y)\}\mid Y\in\mathscr{Y}\}\mid f\in\prod_{Y\in\mathscr{Y}}\mathfrak{P}_nY\}=\bigvee\{\bigwedge\{\Gamma f(Y)\mid Y\in\mathscr{Y}\}\mid f\in\prod_{Y\in\mathscr{Y}}\mathfrak{P}_nY\}=$ $=\Gamma(\bigcup\{\bigcap\{\Gamma f(Y)\mid Y\in\mathscr{Y}\}\mid f\in\prod_{Y\in\mathscr{Y}}\mathfrak{P}_nY\})=\Gamma(\bigcap\{\bigcup\{\Gamma M\mid M\overset{n}{\subseteq}Y\}\mid Y\in\mathscr{Y}\})=\Gamma(\bigcap\{\overset{n}{\downarrow}Y\mid$ $\mid Y\in\mathscr{Y}\})$.

Zu (d): Ist Γ ein algebraischer Hüllenoperator, so kann man sich in der Bedingung (a) offensichtlich auf endliche Mengen beschränken. Die nach (a) existierende Menge N kann endlich gewählt werden, denn ist M eine beliebige Menge mit $\Gamma M=\downarrow x$, so gibt es eine endliche Teilmenge N von M mit $\Gamma N=\downarrow x$.

Zu (e): Gilt $\Gamma=\overset{n}{\downarrow}$, so ist (b) offensichtlich erfüllt. Umgekehrt folgt aber schon aus der unendlichen n-Distributivität, wenn $\mathfrak{X}$ alle einelementigen Teilmengen enthält: $\Gamma=\overset{n}{\downarrow}$.

Bezeichnet Γ den Schnittoperator oder den Idealoperator, so erhält man aus 4.1 eine Charakterisierung derjenigen quasigeordneten Mengen, deren Schnittvervollständigung bzw. Idealvervollständigung vollständig n-distributiv ist (insbesondere also stetig oder vollständig distributiv). Zum Beispiel ergibt sich für die Idealvervollständigung quasigeordneter Mengen:

Korollar 4.2. *Für eine quasigeordnete Menge* X *sind äquivalent:*

(a) *Für jedes* $x \in X$ *gibt es eine Menge* $N \subseteq_e X$, *für die* x *kleinste obere Schranke ist und die in jeder Menge* $\overset{n}{\downarrow} Y$, $Y \subseteq_e X$, *mit* $x \in \Delta Y$ *enthalten ist.*

(b) I *erhält beliebige Durchschnitte von n-Abschnitten.*

(c) *Der Idealverband von* X *ist vollständig n-distributiv.*

Nimmt man für Γ den Operator [ ], der jeder Teilmenge einer gegebenen Algebra A die kleinste sie enthaltende Subalgebra zuordnet, so ergibt sich aus 4.1:

Korollar 4.3. *Sei* A *eine Algebra.* Su (A) *ist genau dann vollständig n-distributiv, wenn für alle* $x \in A$ *eine Menge* $N \subseteq_e A$ *existiert, so daß* $[N]=[x]$ *und* $N \subseteq_e \overset{n}{\downarrow} Y$ *für alle* $Y \subseteq_e X$ *mit* $x \in [Y]$ *gilt.*

Im Detail bedeutet die Bedingung in 4.3: Für alle $x \in A$ gibt es eine Menge $N \subseteq_e A$ mit $[N]=[x]$ und jedes Element von N liegt in einer Menge $[M]$, $M \overset{n}{\subseteq} Y$, wenn $x \in [Y]$ gilt.

Für idempotente Algebren gilt darüber hinaus (wegen (4) siehe [7]):

Satz 4.4. *Für eine idempotente Algebra* A *sind die folgenden vier Aussagen äquivalent:*

(1) Su (A) *ist villständig n-distributiv.*

(2) Su (A) *ist n-distributiv.*

(3) $[\ ] = \overset{n}{\downarrow}$.

(4) *Für alle* $Y \overset{n+s-1}{\subseteq} A$ *gilt* $[Y] = \overset{n}{\downarrow} Y$, *wobei die Stelligkeit von jeder Operation von* A *nicht größer als* s *ist.*

Bei abelschen Gruppen sind jedoch die n-Distributivität und die vollständige n-Distributivität des Subalgebrenverbandes keine äquivalenten Eigenschaften:

Satz 4.5. *Sei* G *eine abelsche Gruppe.* Su (G) *ist genau dann vollständig n-distributiv* $(n < \aleph_0)$, *wenn* G *keine Elemente unendlicher Ordnung enthält und jede endlich erzeugte Untergruppe schon von weniger als* n *Elementen erzeugt wird (d. h. jede endlich erzeugte Untergruppe ist isomorph zu einem Produkt* $\mathbf{Z}_{k_1} \times \ldots \times \mathbf{Z}_{k_n}$ *für natürliche Zahlen* $k_1, \ldots, k_n \neq 0$*).*

Beweis. Nehmen wir an, G enthält ein Element x unendlicher Ordnung.

Für jede Menge $P=\{p_1, \ldots, p_n\}$ von n verschiedenen Primzahlen sei $Y_P = \{(\prod_{i \neq j} p_i)x \mid j=1, \ldots, n\}$. Offensichtlich gilt $\overset{n}{\downarrow} Y_P = [p_1 x] \cup \ldots \cup [p_n x]$.

Angenommen, die Menge $\bigcap\{\overset{n}{\downarrow} Y_P \mid P\ n\text{-elementige Menge von Primzahlen}\}$ enthält ein Element $y \neq 0$. Wähle eine n-elementige Menge Q von Primzahlen. Wegen $y \in \overset{n}{\downarrow} Y_Q$ gilt $y=kqx$ für ein $q \in Q$, $k \in \mathbf{Z}$. Sei R eine Menge von n Primzahlen, die alle größer als $|kq|$ sind. Dann ist $y \notin \overset{n}{\downarrow} Y_R$, da x unendliche Ordnung hat, Widerspruch.

Hieraus folgt, daß die in 4.3 geforderte Menge N nicht existiert. $\mathrm{Su}(G)$ ist also nicht vollständig n-distributiv. Nehmen wir nun umgekehrt an, daß jede endlich erzeugte Untergruppe schon von weniger als n Elementen erzeugt wird und G keine Elemente unendlicher Ordnung enthält.

Sei $x \in G$. Sei P die Menge aller maximalen Primpotenzen, die $\operatorname{ord} x$ teilen. Wir setzen $N_x = \{(\operatorname{ord} x/p)x \mid p \in P\}$. Offensichtlich gilt $[N_x]=[x]$. Es wird nun gezeigt, daß $N_x \subseteq_e \overset{n}{\downarrow} Y$ für alle $Y \subseteq_e X$ mit $x \in [Y]$ gilt.

Hat Y weniger als n Elemente, so gilt $[Y] = \overset{n}{\downarrow} Y$. Wenn also in diesem Fall x ein Element von $[Y]$ ist, so ist $N_x \subseteq_e \overset{n}{\downarrow} Y$. Angenommen, für jedes $x \in G$ und für alle $(m-1)$-elementigen Teilmengen Y von G mit $x \in [Y]$ gilt: $N_x \subseteq_e \overset{n}{\downarrow} Y$ $(m \geqq n)$. Sei $Y=\{y_1, \ldots, y_m\} \subseteq G$, $x \in [Y]$, o.B.d.A. $x=y_1+\ldots+y_m$. Die Elemente von Y können als $(n-1)$-komponentige Vektoren angesehen werden. Somit gibt es Zahlen $k_1, \ldots, k_m \in \mathbf{Z}$ mit größtem gemeinsamen Teiler 1 und $k_1 y_1 + \ldots + k_m y_m = 0$. Sei p ein Primpotenzteiler von $r := \operatorname{ord} x$. Da $k_1, \ldots, k_m$ teilerfremd sind, gibt es ein $j \in \{1, \ldots, m\}$ mit $\operatorname{ggT}(k_j, p)=1$. Es ist $x_j := k_j x = k_j y_1 + \ldots + k_j y_m = (k_j - k_1) y_1 + \ldots + (k_j - k_m) y_m \in [Y_j]$ für $Y_j := Y \setminus y_j$. Wegen $\operatorname{ord} x_j = r/\operatorname{ggT}(k_j, r)$ ist $(\operatorname{ord} x_j/p)x_j = (rs_j/p)x$ mit $s_j := k_j/\operatorname{ggT}(k_j, r)$. Nach Induktionsvoraussetzung ist $(\operatorname{ord} x_j/p)x_j \in \overset{n}{\downarrow} Y_j \subseteq \overset{n}{\downarrow} Y$. Wegen $\operatorname{ggT}(s_j, r)=1$ gibt es Zahlen $a, b \in \mathbf{Z}$ mit $ars_j/p = r/p + br$. Es folgt $(ars_j/p)x = (r/p)x \in \overset{n}{\downarrow} Y$.

Aus dieser Charakterisierung der abelschen Gruppen mit vollständig n-distributivem Untergruppenverband ergibt sich insbesondere, daß der Untergruppenverband von $\mathbf{Z}$ nicht vollständig distributiv (d. h. kein A-Verband) ist. Für den Fall $n=\aleph_0$ wird der vorangegangene Satz falsch: Für jede (endlich-stellige) Algebra A ist $\mathrm{Su}(A)$ algebraisch, also insbesondere stetig.

Daß die Bedingungen 3.3 (f) und 4.1 (e) gleich lauten, hat unter anderem noch die folgende Konsequenz: Ist der Verband der abgeschlossenen Mengen eines T_1-Raumes unendlich n-distributiv, so ist er schon vollständig n-distributiv. Insbesondere sind also für eine T_1-Topologie die $\bigwedge$-Distributivität und die vollständige Distributivität gleichwertige Eigenschaften.

Schlußbemerkung

Wir haben uns in dieser Arbeit zwar auf Werte von n beschränkt, die zwischen 2 und $\aleph_0$ liegen, es soll jedoch nicht unerwähnt bleiben, daß man auch für andere Kardinalzahlen sinnvolle Sätze erhalten kann. Zum Beispiel gilt: Ist X ein topologischer Raum, der das 1. Abzählbarkeitsaxiom erfüllt, so ist der Verband der abgeschlossenen Mengen von X vollständig $\aleph_1$-distributiv. Dies gilt, weil in solch einem topologischen Raum jedes Element aus der topologischen Hülle einer Teilmenge schon Limes einer Folge von Elementen dieser Teilmenge ist, d. h. der topologische Hüllenoperator stimmt mit dem zugehörigen $\aleph_1$-Abschnittoperator überein (woraus sich die Gültigkeit von 4.1 (b) ergibt).

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On lightly compact spaces

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1. Introduction. A topological space X is called *lightly compact* if every locally finite family of open sets of X is finite. Several characterizations of light compactness are given in [1] and [2]. Two well-known characterizations of these spaces are: a space X is lightly compact iff every countable open cover of X contains a finite subfamily whose union is dense in X; and, every countable open filter base has an adherent point. The aim of this note is an investigation of lightly compact spaces. We give some characterizations of light compactness in term of regular-open, regular-closed sets. We also prove some structural properties of such spaces.

Recall that a set U is regular-open if $U=\overset{\circ}{\overline{U}}$ and a set F is regular-closed if $F=\overline{\overset{\circ}{F}}$ where $^{-}$ denotes the closure of a set and $^{\circ}$ denotes the interior of a set.

2. Results. We first prove a lemma.

Lemma 1. *The family of closures of members of a locally finite, infinite family is not finite.*

Proof. Let $\Psi=\{W_\alpha \mid \alpha\in\Delta\}$ be a locally finite, infinite family of subsets of a topological space X. Suppose $\overline{\Psi}=\{\overline{W}_\alpha \mid \alpha\in\Delta\}$ is finite, say only the sets $\overline{W}_{\alpha_1}, \overline{W}_{\alpha_2}, \dots, \overline{W}_{\alpha_n}$ are distinct. Since $\Psi=\{W_\alpha \mid \alpha\in\Delta\}$ is an infinite family, then at least one of the sets $\overline{W}_{\alpha_1}, \overline{W}_{\alpha_2}, \dots, \overline{W}_{\alpha_n}$ is the closure of infinitely many W_α. Suppose $\overline{W}_{\alpha_1}$ is the closure of infinitely many W_α. Take any $x\in\overline{W}_{\alpha_1}$. Then this implies that every neighbourhood of x meets infinitely many W_α. This is a contradiction with Ψ being a locally finite family.

The following theorem shows that the open sets in the definition of lightly compactness may be replaced with regular-closed sets.

Theorem 1. *A space X is lightly compact iff every locally finite family of regular-closed sets is finite.*

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Proof. Let X be a lightly compact space and $\mathscr{E}=\{F_\alpha \mid \alpha\in\Delta\}$ be a locally finite family of regular-closed sets. Since $F_\alpha=\overline{\mathring{F}_\alpha}$ for each $\alpha\in\Delta$, $\{\mathring{F}_\alpha \mid \alpha\in\Delta\}$ is a locally finite family of open sets of the lightly compact space X. Hence $\{\mathring{F}_\alpha \mid \alpha\in\Delta\}$ is finite. Thus the family $\mathscr{E}$ is finite.

Conversely, suppose $\{G_\alpha \mid \alpha\in\Delta\}$ is a locally finite family of open sets. Then $\{\overline{G}_\alpha \mid \alpha\in\Delta\}$ is a locally finite family of regular-closed sets. By hypothesis, $\{\overline{G}_\alpha \mid \alpha\in\Delta\}$ is finite. By Lemma 1, the family $\{G_\alpha \mid \alpha\in\Delta\}$ is finite. Hence X is lightly compact.

We next give another characterization theorem for light compactness.

Theorem 2. *In a topological space X the following are equivalent:*

(i) *X is lightly compact.*

(ii) *Every countable regular-open cover of X contains a finite subfamily whose union is dense in X.*

(iii) *For any countable family of regular-open sets $\{G_n \mid =1,2,\ldots\}$ with the finite intersection property, $\bigcap_{n=1}^{\infty} \overline{G}_n\neq\emptyset$.*

(iv) *For any countable family of regular-closed sets $\{F_n \mid n=1,2,\ldots\}$ such that $\bigcap_{n=1}^{\infty} F_n=\emptyset$, there exists a finite subfamily $\{F_1, F_2, \ldots, F_m\}$ such that $\bigcap_{i=1}^{m} \mathring{F}_i=\emptyset$.*

Proof. It is straightforward.

We next give a sufficient condition for a space X to be lightly compact.

Theorem 3. *Let X be any topological space. If every point of X is contained in only finitely many open sets, then X is lightly compact.*

Proof. Suppose X is not lightly compact. Then there exists a locally finite family Ψ of open sets which is not finite. Let $x\in X$ and let N_x be an open neighbourhood of x meeting only finitely many $W\in\Psi$, say $N_x\cap W_{\alpha_i}\neq\emptyset$ $(i=1,2,\ldots,n)$ and $N_x\cap W_\alpha=\emptyset$ for all $\alpha\neq\alpha_1,\alpha_2,\ldots,\alpha_n$. This implies that $x\notin\overline{W}_\alpha$ if $\alpha\neq\alpha_1,\alpha_2,\ldots,\alpha_n$. By Lemma 1, there are infinitely many $\overline{W}_\alpha$ and $x\in X-\overline{W}_\alpha$, $\alpha\neq\alpha_1,\alpha_2,\ldots,\alpha_n$. That is, x is contained in infinitely many open sets. This is a contradiction which completes the proof.

Theorem 4. *A space X is lightly compact whenever a dense subset of it is lightly compact.*

Proof. Let A be a lightly compact dense subset of X. If $\{G_n \mid n=1,2,\ldots\}$ is a countable open filter base in X, then $\{G_n\cap A \mid n=1,2,\ldots\}$ is a countable open filter base in A. Since $\bigcap_{n=1}^{\infty} \overline{(G_n\cap A)}^A\neq\emptyset$, then $\bigcap_{n=1}^{\infty} \overline{G}_n^X\neq\emptyset$. Hence X is lightly compact.

We know that in a first countable Hausdorff space every countably compact subset is closed. The following theorem shows that a similar result can also be obtained for lightly compact spaces.

Theorem 5. *Every lightly compact subset of a first countable Hausdorff space is closed.*

Proof. Let Y be a lightly compact subset of a first countable Hausdorff space X. Suppose Y is not closed in X. Take $y \in \bar{Y}-Y$. Let $\{G_n \mid n=1, 2, \ldots\}$ be a countable open neighbourhood base at y. Then $\{G_n \cap Y \mid n=1, 2, \ldots\}$ is a countable open filter base in Y which has no adherent point because

$$\bigcap_{n=1}^{\infty} \overline{(G_n \cap Y)}^Y \subseteqq \bigcap_{n=1}^{\infty} (\bar{G}_n^X \cap Y) = \{y\} \cap Y = \emptyset.$$

This is a contradiction.

It is known that a continuous image of a lightly compact space is lightly compact. For a weakly continuous function we have the following theorem. First recall that a function $f: X \to Y$ is *weakly continuous* [3] if for each $x \in X$ and each open set V containing $f(x)$, there exists an open set U containing x such that $f(U) \subseteqq \bar{V}$. Equivalently, $f: X \to Y$ is weakly continuous iff for each open set V in Y, we have $f^{-1}(V) \subseteqq [f^{-1}(\bar{V})]^\circ$ ([3], Theorem 1).

Theorem 6. *A weakly continuous image of a countably compact space is lightly compact.*

Proof. Let X be a countably compact space and $f: X \to Y$ be a weakly continuous onto function. If $\{G_n \mid n=1, 2, \ldots\}$ is a countable open cover of Y, then $\bigcup_{n=1}^{\infty} f^{-1}(G_n) = X$. Since f is weakly continuous, $f^{-1}(G_n) \subseteqq [f^{-1}(\bar{G}_n)]^\circ$ for $n=1, 2, \ldots$. Hence $\{[f^{-1}(\bar{G}_n)]^\circ \mid n=1, 2, \ldots\}$ is a countable open cover of X. Since X is countably compact, there exists a finite subfamily $\{G_1, G_2, \ldots, G_n\}$ such that $\bigcup_{i=1}^{n} [f^{-1}(\bar{G}_i)]^\circ = X$. Take any $y \in Y$. Since f is onto, there exists an $x \in X$ such that $f(x)=y$. Suppose $x \in [f^{-1}(\bar{G}_j)]^\circ$, $1 \leqq j \leqq n$. So $x \in f^{-1}(\bar{G}_j)$, that is $f(x)=y \in \bar{G}_j$, $1 \leqq j \leqq n$. Hence $\bigcup_{i=1}^{n} \bar{G}_i = Y$. Thus Y is lightly compact.

N. Levine [4] has introduced the concept of strongly continuous function. A function $f: X \to Y$ is said to be *strongly continuous* iff $f(\bar{A}) \subseteqq f(A)$ for every subset A of X. For a strongly continuous function we have:

Theorem 7. *A strongly continuous image of a lightly compact space is countably compact.*

Proof. Let X be a lightly compact space and $f\colon X\to Y$ be a strongly continuous and onto function. If $\{G_n \mid n=1,2,\ldots\}$ is a countable open cover of Y then $\bigcup_{n=1}^{\infty} f^{-1}(G_n)=X$. Since f is strongly continuous, hence continuous, $\{f^{-1}(G_n) \mid n=1,2,\ldots\}$ is a countable open cover of X. Since X is lightly compact there exists a finite subfamily $\{f^{-1}(G_1), f^{-1}(G_2), \ldots, f^{-1}(G_m)\}$ such that $\bigcup_{i=1}^{m} \overline{f^{-1}(G_i)}=X$. This implies that

$$f(X) = Y = f\left(\bigcup_{i=1}^{m} \overline{f^{-1}(G_i)}\right) = \bigcup_{i=1}^{m} f(\overline{f^{-1}(G_i)}) \subseteq \bigcup_{i=1}^{m} f(f^{-1}(G_i)) = \bigcup_{i=1}^{m} G_i.$$

That is, Y is countably compact.

Theorem 8. *A one-to-one continuous map from a regular lightly compact space X onto a first countable Hausdorff space Y is a homeomorphism.*

Proof. Let $f\colon X\to Y$ be a continuous one-to-one and onto map. Let F be a closed subset of X. It can be shown that F can be written as an intersection of regular-closed subsets of the regular space X. Say $F=\bigcap_{\alpha\in\Delta} C_\alpha$, where all C_α are regular-closed subsets. Since X is lightly compact, for all $\alpha\in\Delta$, C_α is a lightly compact subset of X [1]. Hence for all $\alpha\in\Delta$, $f(C_\alpha)$ is a lightly compact subset of Y. By Theorem 5, for all $\alpha\in\Delta$, $f(C_\alpha)$ is a closed subset of Y. Since f is one-to-one, therefore

$$f(F) = \bigcap_{\alpha\in\Delta} f(C_\alpha).$$

That is, $f(F)$ is closed in Y. Thus f is a closed map, and hence it is a homeomorphism.

Recall that a space (X,τ) is called *first countable and Hausdorff minimal* if τ is first countable and Hausdorff, and if no first countable Hausdorff topology on X is strictly weaker than τ.

Corollary. [6. 2. 6. Theorem (vii)] *A first countable, regular, lightly compact Hausdorff space is first countable and Hausdorff minimal.*

Singal [5] has introduced the concept of nearly compact space. A space X is called *nearly compact* if every open cover of X has a finite subfamily such that the interiors of closures of sets in this family covers X. It can be shown that a space is nearly compact iff the intersection of a family of regular-closed sets with finite intersection property is not empty.

It is known that the product of a lightly compact space and a compact space is lightly compact. The next theorem gives a generalization of this result.

Theorem 9. *The product of a lightly compact space and a nearly compact space is lightly compact.*

Proof. Let X be a nearly compact space and Y be a lightly compact space. To show that the product space $X\times Y$ is lightly compact, it is enough to prove that every countable open filter base has an adherent point in $X\times Y$. Let $\mathscr{E}=\{G_n \mid n=1,2,3,\ldots\}$ be a countable open filter base in $X\times Y$. Then $\Pi_2(\mathscr{E})=\{\Pi_2(G_n) \mid n=1,2,3,\ldots\}$ is a countable open filter base in Y, where Π_2 is the second projection. Since Y is lightly compact, $\Pi_2(\mathscr{E})$ has an adherent point, that is $\bigcap_{n=1}^{\infty}\overline{\Pi_2(G_n)}\neq\emptyset$. Take $y\in\bigcap_{n=1}^{\infty}\overline{\Pi_2(G_n)}$. If V is an open set containing y, then for all n, $V\cap\Pi_2(G_n)\neq\emptyset$. Hence for all n, $\Pi_2^{-1}(V)\cap G_n\neq\emptyset$. Let $\Pi_1(\Pi_2^{-1}(V)\cap G_n)=U_{V,n}$. All $U_{V,n}$ are open sets in X. Now the family

$$\{U_{V,n} \mid V \text{ is open in } Y \text{ and } y\in V,\ n=1,2,3,\ldots\}$$

has the finite intersection property in X. In fact,

$$U_{V_1,n_1}\cap U_{V_2,n_2}=\Pi_1(\Pi_2^{-1}(V_1)\cap G_{n_1})\cap\Pi_1(\Pi_2^{-1}(V_2)\cap G_{n_2})\supseteq$$
$$\supseteq\Pi_1\{[\Pi_2^{-1}(V_1)\cap G_{n_1}]\cap[\Pi_2^{-1}(V_2)\cap G_{n_2}]\}=\Pi_1[\Pi_2^{-1}(V_1\cap V_2)\cap(G_{n_1}\cap G_{n_2})]\neq\emptyset.$$

Hence the family $\{\overline{U}_{V,n} \mid V$ is open in Y and $y\in V$, $n=1,2,3,\ldots\}$ is a collection of regular-closed sets with the finite intersection property. Since X is nearly-compact, $\bigcap_{V,n}\overline{U}_{V,n}\neq\emptyset$. Let $x\in\bigcap_{V,n}\overline{U}_{V,n}$. If we show that (x,y) is an adherent point of the filter base $\mathscr{E}$ in $X\times Y$, then the proof will be completed. Suppose $M\times N$ is a basic open set containing (x,y) in $X\times Y$. It is clear that $M\cap U_{N,n}\neq\emptyset$ for all n. Thus $M\cap\Pi_1(\Pi_2^{-1}(N)\cap G_n)\neq\emptyset$ for $n=1,2,\ldots$. Consequently $(M\times N)\cap\cap G_n\neq\emptyset$ for all n, that is $(x,y)\in\bigcap_{n=1}^{\infty}\overline{G}_n$. So $X\times Y$ is lightly compact.

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A note on multifunctions

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1. Introduction

A function $F: X \to p(Y) - \{\emptyset\}$ is called a multifunction from X to Y and is usually denoted by $F: X \to Y$, where $p(Y)$ is the power set of Y. The *graph of* F is the subset $\{(x, y) \mid x \in X$ and $y \in F(x)\}$ of $X \times Y$. We will denote the graph of F by $G(F)$. If X and Y are topological spaces and $F: X \to Y$ is a multifunction we will say that F has a *closed graph* if $G(F)$ is a closed subset of $X \times Y$. The graph $G(F)$ is closed iff for each point $(x, y) \notin G(F)$, there exist open sets $U \subset X$ and $V \subset Y$ containing x and y, respectively, such that $F(U) \cap V = \emptyset$. The graph $G(F)$ is said to be *strongly closed* [4] if for each point $(x, y) \notin G(F)$, there exist open sets $U \subset X$ and $V \subset X$ containing x and y respectively, such that $F(U) \cap \bar{V} = \emptyset$, where $\bar{V}$ denotes the closure of V. A multifunction $F: X \to Y$ is called *upper semicontinuous (weakly upper semicontinuous)* if for each $x \in X$ and each open set $V \subset Y$ containing $F(x)$, there exists an open set $U \subset X$ containing x such that $F(U) \subset V$ $(F(U) \subset \bar{V})$. It is not difficult to see that F is upper semicontinuous iff $F^{-1}(K) = \{x \in X \mid F(x) \cap K \neq \emptyset\}$ is closed in X whenever K is closed in Y. We will say that a multifunction $F: X \to Y$ is *point closed (point compact)* if $F(x)$ is closed (compact) in Y for each $x \in X$. The definition of an open or closed multifunction is analogous to the definition of an open or closed single valued mapping.

A multifunction $F: X \to Y$ is said to be *almost upper semicontinuous* if for each point $x \in X$ and each open set $V \subset Y$ containing $F(x)$, there exists an open set $U \subset X$ containing x such that $F(U) \subset \mathring{\bar{V}}$, where $\mathring{\bar{V}}$ denotes the interior of the closure of V.

A subset K of a topological space X is called *quasi H-closed relative to* X if for each open cover $\{G_\alpha \mid \alpha \in \Delta\}$ of K, there exists a finite subfamily $\{G_{\alpha_i} \mid i = 1, 2, \dots, n\}$ such that $K \subset \bigcup_{i=1}^{n} \bar{G}_{\alpha_i}$. If X is quasi H-closed relative to X, then it is

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called quasi H-closed. When X is Hausdorff, the word "quasi" is omitted in these two definitions.

A Hausdorff space X is said to be locally *H-closed* [4] if every point of X has a neighbourhood which is H-closed. A space X is called *c-compact* [3] if every closed set of X is quasi H-closed relative to X.

Let X be a topological space and $A\subset X$. If D is a directed set and $\Phi: D\to A$ is a net, then we say it *r-accumulates* [3] to $x\in A$ if for each open set $V\subset X$ containing x and every $b\in D$, $\Phi(T_b)\cap\bar{V}\neq\emptyset$, where $T_b=\{c\in D \mid c\geqq b\}$. A space X is c-compact iff for each closed set $A\subset X$ and each net $\{x_\alpha\}$ in A, there exists a point $x\in A$ such that $\{x_\alpha\}$ r-accumulates to x [3, Th. 3].

2. c-compact, H-closed spaces and multifunctions with strongly closed graph

Theorem 2.1. *Let $F: X\to Y$ be a multifunction and Y be a c-compact space. If F has strongly closed graph, then F is upper semicontinuous.*

Proof. Suppose there exists a closed subset K in Y such that $F^{-1}(K)$ is not closed in X. Take $x_0\in\overline{F^{-1}(K)}-F^{-1}(K)$. Hence there exists a net $\{x_\alpha\}_{\alpha\in A}$ in $F^{-1}(K)$ such that $x_\alpha\to x_0$. Now let $\{y_\alpha\}_{\alpha\in A}$ be a net in K such that $y_\alpha\in F(x_\alpha)\cap K$ for each α. Since K is closed and Y is c-compact, there exists a point $y_0\in K$ such that the net $\{y_\alpha\}_{\alpha\in A}$ r-accumulates to y_0. Since $y_0\notin F(x_0)$, then $(x_0, y_0)\notin G(F)$ and since $G(F)$ is strongly closed, there are open sets $U\subset X$ and $V\subset Y$ containing x_0 and y_0, respectively, such that $(U\times\bar{V})\cap G(F)=\emptyset$. But $x_\alpha\to x_0$ implies there exists an $\alpha_0\in A$ such that for every $\alpha\in A$ and $\alpha\geqq\alpha_0$, $x_\alpha\in U$, and $\{y_\alpha\}_{\alpha\in A}$ r-accumulates to y_0 implies there exists some $\alpha_1\in A$ and $\alpha_1\geqq\alpha_0$ such that $y_{\alpha_1}\in\bar{V}$. From this it follows that $(x_{\alpha_1}, y_{\alpha_1})\in(U\times\bar{V})\cap G(F)$ which is a contradiction. Hence F is upper semicontinuous.

Theorem 2.2. *Let $F: X\to Y$ be a point compact multifunction and Y a locally H-closed (H-closed) space. If for each subset K, H-closed in Y, $F^{-1}(K)$ is closed in X then F has strongly closed graph.*

Proof. Suppose Y is locally H-closed. Take any point $(x, y)\notin G(F)$. Then $y\notin F(x)$. Since Y is Hausdorff, $F(x)$ is compact and $y\notin F(x)$, there are disjoint open sets V_1 and W in Y such that $y\in V_1$ and $F(x)\subset W$ [1, p. 225]. $V_1\cap W=\emptyset$ implies $\bar{V}_1\cap W=\emptyset$. On the other hand, there exists a neighbourhood V_2 of y which is H-closed. Put $V=V_1\cap\mathring{V}_2$. Then V is an open set containing y and $W\cap\bar{V}=\emptyset$. Since Y is Hausdorff and V_2 is H-closed in Y, then V_2 is closed in Y. Thus $\bar{V}\subset V_2$. $\bar{V}$ is a regularly closed subset in the H-closed set V_2. Therefore $\bar{V}$ is H-closed in V_2, so $\bar{V}$ is H-closed in Y. According to our assumption, $F^{-1}(\bar{V})$

is closed in X. Put $U=X-F^{-1}(\overline{V})$. Then U is an open set in X containing x and $F(U)\cap\overline{V}=\emptyset$. This shows that $G(F)$ is strongly closed.

Theorem 2.3. *Let $F:X\to Y$ be an almost upper semicontinuous point compact multifunction and Y Hausdorff. Then F has a strongly closed graph.*

Proof. Let $(x, y)\notin G(F)$. Since $F(x)$ is compact, $y\notin F(x)$ and Y is Hausdorff, there are disjoint open sets V and W containing y and $F(x)$, respectively. We can write $\overline{V}\cap\overset{\circ}{\overline{W}}=\emptyset$. Since F is almost upper semicontinuous there is an open set U in X containing x such that $F(U)\subset\overset{\circ}{\overline{W}}$. Now we have $F(U)\cap\overline{V}=\emptyset$. That is, $G(F)$ is strongly closed.

Corollary. *Let $F:X\to Y$ be a point compact multifunction and Y an H-closed space. The following are equivalent:*

(i) *F is almost upper semicontinuous,*

(ii) *F has strongly closed graph,*

(iii) *For each subset K, H-closed relative to Y, $F^{-1}(K)$ is closed in X,*

(iv) *For each H-closed subset K of Y, $F^{-1}(K)$ is closed in X.*

Proof. According to Theorem 2.3, (i) implies (ii). (ii) implies (iii), by Theorem 4.15 [4]. Since an H-closed subset of Y is H-closed relative to Y (the converse need not be true), the implication (iii)$\Rightarrow$(iv) is obvious.

Let us prove that (iv) implies (i). For any $x\in X$, let W be an open set containing $F(x)$. $\overset{\circ}{\overline{W}}$ is a regularly open set containing $F(x)$. $Y-\overset{\circ}{\overline{W}}$ is a regularly closed set. Since Y is H-closed then $Y-\overset{\circ}{\overline{W}}$ is H-closed. Hence by (iv), $F^{-1}(Y-\overset{\circ}{\overline{W}})$ is closed in X and $x\notin F^{-1}(Y-\overset{\circ}{\overline{W}})$. Thus there exists an open set U containing x such that $U\cap F^{-1}(Y-\overset{\circ}{\overline{W}})=\emptyset$. This implies that $F(U)\subset\overset{\circ}{\overline{W}}$, that is, F is almost upper semicontinuous.

Our next result is a generalization of Theorem 11 in [3], which was proved for a single valued mapping.

Theorem 2.4. *If $F:X\to Y$ is an open and closed multifunction from a regular space X into a c-compact space Y, and if $F^{-1}(y)$ is closed for each $y\in Y$, then F is upper semicontinuous.*

Proof. According to Theorem 3.4, Corollary 3.5 [5] F has closed graph. For an open multifunction the condition closed graph and strongly closed graph are identical. Hence $F:X\to Y$ has a strongly closed graph and Y is c-compact, so by Theorem 2.1, F is upper semicontinuous.

Theorem 2.5. *If $F: X \to Y$ is an upper semicontinuous point compact multifunction, then F is compact preserving.*

Proof. Let K be a compact subset of X and suppose $\{W_\alpha \mid \alpha \in \Delta\}$ is an open cover of $F(K)$. Take any $x \in K$, then $F(x)$ is a compact subset of Y and $F(x) \subset \subset F(K)$. Thus $\{W_\alpha \mid \alpha \in \Delta\}$ is an open cover of $F(x)$. Hence there is a finite subcover, say $\{W_{\alpha_1}(x), \dots, W_{\alpha_n}(x)\}$. Now put $V(x) = \bigcup_{i=1}^{n} W_{\alpha_i}(x)$. $V(x)$ is an open set containing $F(x)$. Since F is upper semicontinuous, there exists an open set $U(x) \subset X$ containing x such that $F(U(x)) \subset V(x)$. Now $\{U(x) \mid x \in K\}$ is an open cover of K and K is a compact subset of X. Take $x_1, x_2, \dots, x_m \in K$ such that $\{U(x_i) \mid i = 1, \dots, m\}$ is a subcover. Let $V(x_1), V(x_2), \dots, V(x_m)$ be the open sets corresponding to $U(x_1), U(x_2), \dots, U(x_m)$, respectively. Thus

$$F(K) \subset F\left(\bigcup_{i=1}^{m} U(x_i)\right) = \bigcup_{i=1}^{m} F(U(x_i)) \subset \bigcup_{i=1}^{m} V(x_i) =$$

$$= \bigcup \{W_{\alpha_1}(x_1), \dots, W_{\alpha_n}(x_1), \dots, W_{\beta_1}(x_m), \dots, W_{\beta_s}(x_m)\}$$

That is, we have a finite subcover of $\{W_\alpha \mid \alpha \in \Delta\}$. Hence $F(K)$ is compact in Y.

Corollary. *Let $F: X \to Y$ be an onto closed multifunction. If F has compact point inverses, then for each compact subset K of Y $F^{-1}(K)$ is compact in X.*

Proof. Since $(F^{-1})^{-1} = F$, then $F^{-1}: Y \to X$ is an upper semicontinuous point compact multifunction, hence F^{-1} is compact preserving.

Theorem 2.6. *Let $F: X \to Y$ be a weakly upper semicontinuous point compact multifunction. Then F maps a compact subset K of X onto subset $F(K)$ quasi H-closed relative to Y.*

Proof. The proof is the same as in Theorem 2.5.

Let $F: X \to Y$ be a multifunction. We can define a new multifunction $\bar{F}: X \to Y$ by setting $\bar{F}(x) = \overline{F(x)}$ for all $x \in X$. If Y is normal and $F: X \to Y$ is upper semicontinuous then $\bar{F}: X \to Y$ is upper semicontinuous [2]. We have the following new result.

Theorem 2.7. *If $F: X \to Y$ is weakly upper semicontinuous, then $\bar{F}: X \to Y$ is weakly upper semicontinuous.*

Proof. Let $x \in X$ and W an open set in Y containing $\bar{F}(x)$. Since $F(x) \subset \subset \overline{F((x} = \bar{F}(x) \subset W$ and F is weakly upper semicontinuous there is an open set U in X containing x such that $F(U) \subset \bar{W}$. This implies that $\overline{F(U)} \subset \bar{W}$. On the other hand

$$\bar{F}(U)=\bigcup_{x\in U}\bar{F}(x)=\bigcup_{x\in U}\overline{F(x)}\subset\overline{F(U)}.$$

Hence $\bar{F}(U)\subset\bar{W}$, that is, $\bar{F}$ is weakly upper semicontinuous.

Theorem 2.8. *If $F: X\to Y$ is weakly upper semicontinuous and Y is regular, then the graph of $\bar{F}$ is closed in $X\times Y$.*

Proof. $\bar{F}: X\to Y$ is weakly upper semicontinuous, by Theorem 2.7. Now suppose $(x, y)\notin G(\bar{F})$. $y\notin\bar{F}(x)=\overline{F(x)}$. Since Y is regular, there are open sets V and W containing y and $\bar{F}(x)$, respectively, such that $V\cap W=\emptyset$. Hence $V\cap\bar{W}=\emptyset$. From the weakly upper semicontinuity of $\bar{F}$, we have an open set U in X containing x such that $\bar{F}(U)\subset\bar{W}$. Hence $\bar{F}(U)\cap V=\emptyset$. That is, $G(\bar{F})$ is closed in $X\times Y$.

Corollary. [5, Theorem 3.3] *If $F: X\to Y$ is a point closed upper semicontinuous multifunction into a regular space, then F has a closed graph.*

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On variation spaces of harmonic maps into spheres

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1. Introduction

Given a harmonic map $f: M \to S^n$ [3] of a compact oriented Riemannian manifold M into the Euclidean n-sphere S^n, $n \geqq 2$, a vector field v along f, i.e. a section of the pull-back bundle $\mathcal{F} = f^*(T(S^n))$, gives rise to a (one-parameter, geodesic) variation $f_t = \exp \circ (tv): M \to S^n$, $t \in \mathbf{R}$, where $\exp: T(S^n) \to S^n$ is the exponential map. The element $v \in C^\infty(\mathcal{F})$ is said to be a *harmonic variation* if f_t is harmonic for all $t \in \mathbf{R}$ and the set of all harmonic variations v (or the variation space) of f is denoted by $V(f) \subset C^\infty(\mathcal{F})$. Then [11] $v \in V(f)$ if and only if $\|v\| = \text{const.}$ and

(i) $\nabla^2 v = \operatorname{trace} R(f_*, v) f_*$ (i.e. v is a Jacobi field along f [3]),

(ii) $\operatorname{trace} \langle f_*, \nabla v \rangle = 0$,

where $\langle\, ,\rangle$ and ∇ are the induced metric and connection of the Riemannian-connected bundle $\mathcal{F} \otimes \Lambda^*(T^*(M))$, $\nabla^2 = \operatorname{trace} \nabla \circ \nabla$ [9], R is the curvature tensor of S^n and the differential f_* of f is considered as a section of $\mathcal{F} \otimes T^*(M)$. Denote by $K(f)$ the linear space of all vector fields v along f satisfying (i) and (ii). The equation (i) being (strongly) elliptic [9] $\dim K(f) < \infty$ and $V(f) = \{v \in K(f) \mid \|v\| = \text{const.}\} \subset K(f)$ is a subset with the obvious property $\mathbf{R} V_0(f) = V(f)$, where $V_0(f) = \{v \in K(f) \mid \|v\| = 1\}$.

The purpose of this paper is to give a geometric description of the variation space $V(i) \subset K(i)$ $(\cong \mathbf{R}^N)$ of the canonical inclusion $i: S^m \to S^n$, where $N = m(m+1)/2 + (n-m)(m+1)$. In Section 2 we collect the necessary tools from matrix theory used in the sequel, especially we describe the singular value decomposition of rectangular matrices (see e.g. [7]). In Section 3 the problem of determining $V_0(i)$ is reduced to the geometric characterization of an (algebraic) set of matrices. Then the singular value decomposition of these matrices are exploited to get a description of $V_0(i) \subset K(i)$ as a set of orbits (under a linear Lie group action) which contains a (twisted) simplex as a global section (Theorem 1). In particular, we prove that

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$V(\mathrm{id}_{S^{2r-1}})$, $r\in\mathbf{N}$, is the double cone over the irreducible Hermitian symmetric space $SO(2r)/U(r)$ $(=V_0(\mathrm{id}_{S^{2r-1}}))$. (Note that $V(\mathrm{id}_{S^{2r}})=0$ because $\chi(S^{2r})=2$ [11].) In Section 4 we first give an alternative description of the linear space $K(f)$. In particular, we obtain that there is a one-to-one correspondence between the elements of $V_0(f)$ and the orthogonal pairs $f, f^1\colon M\to S^n$ of harmonic maps with the same energy density $e(f)=e(f^1)$ [3]. Second, as an example, we determine $K(f)$ for the Veronese surface $f\colon S^2\to S^4$ and prove that $K(f)\cong K(\mathrm{id}_{S^4})$ and $V(f)=V(\mathrm{id}_{S^4})=0$ hold.

Throughout this paper all manifolds, maps, bundles, etc. will be smooth, i.e. of class C^∞. The report [3] is our general reference for harmonic maps though we adopt the sign conventions of [6].

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2. Preliminaries from matrix theory

First we fix some notations used in the sequel. Denote by $M(p, q)$ the linear space of $(p\times q)$ matrices and, as usual, let I_p and 0_p the unit and zero elements of $M(p, p)$. A matrix $A\in M(p, q)$ with entries a_{ij}, $i=1, \dots, p$, $j=1, \dots, q$, is said to be (rectangular) diagonal if

$$a_{ij} = \begin{cases} 0, & \text{if } i\neq j,\ i=1, \dots, p,\ j=1, \dots, q, \\ \sigma_i, & \text{if } i=j,\ i=1, \dots, \min(p, q) \end{cases}$$

holds. We write $A=\mathrm{diag}(\sigma_1, \dots, \sigma_d)^p_q$ with $d=\min(p, q)$ and, in case $p=q$, we omit the indices p and q.

The singular value decomposition of rectangular matrices is given in the following theorem. (For the proof, see [7].)

Theorem A. *For any matrix $B\in M(p, q)$ there exist orthogonal matrices $V\in O(p)$ and $U\in O(q)$ such that*

$$V^{\mathrm{T}}BU = \mathrm{diag}(\sigma_1, \dots, \sigma_d)^p_q$$

with $\sigma_i\geqq 0$, $i=1, \dots, d=\min(p, q)$. The matrices V, U and the values σ_i are determined by the relations:

(A$_1$) $V^{\mathrm{T}}BB^{\mathrm{T}}V = \mathrm{diag}(\sigma_1^2, \dots, \sigma_d^2, \dots, \sigma_p^2)$,

(A$_2$) $U^{\mathrm{T}}B^{\mathrm{T}}BU = \mathrm{diag}(\sigma_1^2, \dots, \sigma_d^2, \dots, \sigma_q^2)$,

(A$_3$) $BU = V\,\mathrm{diag}(\sigma_1, \dots, \sigma_d)^p_d$,

where $\sigma_i=0$ for $d<i\leqq\max(p, q)$.

Remark. The numbers $\sigma_i \geqq 0$, $i=1, \dots, d$, are called the singular values of B. Clearly, V and U can always be chosen such that $\sigma_1 \geqq \sigma_2 \geqq \dots \geqq \sigma_d$ holds.

Denote by $\Lambda_r \in so(2r)$ the skew-symmetric matrix

$$\Lambda_r = \operatorname{diag}\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right)$$

and put $\Lambda_1 = \Lambda$. In the next theorem we collect some properties of skew-symmetric matrices (cf. [8] pp. 151, 231).

Theorem B. *For any matrix* $\mathscr{J} \in so(p)$ *we have*

(B_1) rank $\mathscr{J} = 2r \leqq p$;

(B_2) *The* $2r$ *nonzero eigenvalues of* $\mathscr{J}$ *appear in pairs* $\lambda_{2i-1} = \lambda_{2i} = \pm\sqrt{-1}\sigma_i$ *with* $\sigma_i > 0$, $i = 1, \dots, r$, *while zero is an eigenvalue with multiplicity* $p-2r$;

(B_3) *There exists* $U \in O(p)$ *such that*

$$U^{\mathrm{T}} \mathscr{J} U = \operatorname{diag}(0_{p-2r}, \sigma_1 \Lambda, \dots, \sigma_r \Lambda) \tag{1}$$

or equivalently

$$U^{\mathrm{T}} \mathscr{J} U = \begin{cases} \operatorname{diag}(\hat{\sigma}_1 \Lambda, \dots, \hat{\sigma}_{p/2} \Lambda), & \text{if } p \text{ is even,} \\ \operatorname{diag}(0, \hat{\sigma}_1 \Lambda, \dots, \hat{\sigma}_{[p/2]} \Lambda), & \text{if } p \text{ is odd,} \end{cases} \tag{1'}$$

where $\hat{\sigma}_1 = \dots = \hat{\sigma}_{[(p-2r)/2]} = 0$ *and* $\hat{\sigma}_{[(p-2r)/2]+i} = \sigma_i$, $i = 1, \dots, r$;

(B_4) *With the same matrix* $U \in O(p)$ *we have*

$$U^{\mathrm{T}}(-\mathscr{J}^2) U = \begin{cases} \operatorname{diag}(\hat{\sigma}_1^2 I_2, \dots, \hat{\sigma}_{p/2}^2 I_2) & \text{if } p \text{ is even,} \\ \operatorname{diag}(0, \hat{\sigma}_1^2 I_2, \dots, \hat{\sigma}_{[p/2]}^2 I_2), & \text{if } p \text{ is odd,} \end{cases} \tag{2}$$

in particular, the nonzero singular values of $\mathscr{J}$ *have even multiplicities.*

3. Variation space of the canonical inclusion $i: S^m \to S^n$

Let $i: S^m \to S^n$ be the canonical inclusion and let $W^1, \dots, W^k$, $k = n-m$, denote the system of orthonormal parallel sections of the normal bundle of i defined by the standard base vectors $e_{m+2}, \dots, e_{n+1} \in \mathbf{R}^{n+1}$.

According to a result of [11] $v \in K(i)$ if and only if the tangential part $\mathscr{J}$ of v is a Killing vector field on S^m and there exist vectors $b_1, \dots, b_k \in \mathbf{R}^{m+1}$ such that the orthogonal decomposition

$$v_x = \mathscr{J}_x + \sum_{j=1}^{k} \langle b_j, x \rangle W_x^j, \quad x \in S^m,$$

is valid. Hence the linear map $\Psi: K(i) \to so(m+1) \times M(k, m+1)$ defined by $\Psi(v) = (\mathscr{J}, B)$, $v \in K(i)$, where $\mathscr{J}$ is the tangential part of v and $B \in M(k, m+1)$

consists of the row vectors $b_1, \ldots, b_k \in \mathbf{R}^{m+1}$ occurring in the decomposition of v above, is a linear isomorphism. In what follows, we identify $K(i)$ and $so(m+1)\times M(k, m+1)$ via Ψ. Further, $V(i)=\mathbf{R}V_0(i)\subset K(i)$, where $V_0(i)= =\{v\in K(i) \mid \|v\|=1\}$. Thus, for $v=(\mathcal{J}, B)\in V_0(i)$, we have

$$1 = \|v_x\|^2 = \|\mathcal{J}_x\|^2 + \sum_{j=1}^{k} \langle b_j, x\rangle^2 = \langle -\mathcal{J}^2 x, x\rangle + \langle B^{\mathrm{T}} Bx, x\rangle, \quad x\in S^m,$$

i.e.

$$V_0(i) = \{(\mathcal{J}, B)\in so(m+1)\times M(k, m+1) \mid -\mathcal{J}^2 + B^{\mathrm{T}}B = I_{m+1}\}.$$

The objective of this section is to give a geometric description of the set $V_0(i)\subset \subset K(i)$. Before stating our main theorem we introduce some notations. For the given positive integers m and n, $m\leqq n$, set

$$t = \begin{cases} \min((m+1)/2, [k/2]), & \text{if } m+1 \text{ is even,} \\ \min(m/2, [(k-1)/2]), & \text{if } m+1 \text{ is odd,} \end{cases}$$

where $k=n-m$, and define

$$\Delta_t = \{(\sigma_1, \ldots, \sigma_t)\in \mathbf{R}^t \mid 1 \geqq \sigma_1 \geqq \ldots \geqq \sigma_t \geqq 0\}.$$

So $\Delta_t\subset \mathbf{R}^t$ is a (linear) simplex which reduces to a point if $t=0$. (Note that $t\geqq -1$ and equality holds if and only if $m=n$ is even, in which case $V_0(i)=\emptyset$ [11] and we put $\Delta_{-1}=\emptyset$.)

A linear representation of the Lie group $O(m+1)\times O(k)$ on the vector space $K(i)=so(m+1)\times M(k, m+1)$ is given by

$$(U, V)\cdot(\mathcal{J}, B) = (U\mathcal{J}U^{\mathrm{T}}, VBU^{\mathrm{T}}),$$

$(U, V)\in O(m+1)\times O(k)$, $(\mathcal{J}, B)\in so(m+1)\times M(k, m+1)$. Clearly, the subset $V_0(i)\subset \subset K(i)$ is invariant, i.e. $V_0(i)$ is the union of orbits crossing $V_0(i)$. Finally we introduce certain subgroups of $O(m+1)\times O(k)$ which will be the isotropy subgroups at points of $V_0(i)$. For given nonnegative integers $a_0, b_0, c_1, c_2, \ldots, c_{s+1}$ with $m+1=a_0+2c_1+\ldots+2c_{s+1}$ and $k=a_0+2c_1+\ldots+2c_s+b_0$ define the subgroups

$$\mathcal{G}(c_1, \ldots, c_{s+1}) = \{(A_0, C_1, \ldots, C_{s+1}; A_0, C_1, \ldots, C_s, B_0)\in O(m+1)\times O(k) \mid$$
$$A_0\in O(a_0), \quad B_0\in O(b_0), C_i\in U(c_i), \quad i = 1, \ldots, s+1\},$$

where $U(c_i)$ is considered as a subgroup of $SO(2c_i)$ via the canonical embedding $U(c_i)\to SO(2c_i)$, $i=1, \ldots, s+1$. The isotropy type i.e. the set of all conjugacy classes of a subgroup $\mathcal{G}\subset O(m+1)\times O(k)$ is denoted by $(\mathcal{G})$. The main result of this section is the following:

Theorem 1. *There exists an embedding $\Phi: \Delta_t\to K(i)$ such that $\Phi(\Delta_t)$ is a global section of the invariant subset $V_0(i)$ (i.e. $\Phi(\Delta_t)\subset V_0(i)$ and any orbit on*

$V_0(i)$ cuts $\Phi(\Delta_t)$ at exactly one point). Moreover, for $\sigma=(\sigma_0, \ldots, \sigma_0, \sigma_1, \ldots, \sigma_1, \ldots, \sigma_{s+1}, \ldots, \sigma_{s+1})\in\Delta_t$, where $1=\sigma_0>\sigma_1>\ldots>\sigma_s>\sigma_{s+1}=0$ and σ_i occurs c_i times in σ, $i=0, \ldots, s+1$, the isotropy type of the orbit through $\Phi(\sigma)$ is $(\mathscr{G}(c_1, \ldots, c_s, c_{s+1}+ +([(m+1)/2]-t)^+)$ ($^+=$ positive part) or equivalently this orbit has the form

$$(O(m+1)\times O(k))/\mathscr{G}(c_1, \ldots, c_s, c_{s+1}+([m+1)/2]]-t)^+).$$

In particular, for each open face Δ of the simplex Δ_t the orbits through $\Phi(\Delta)$ have the same type.

Remarks 1. Each orbit consists of 1, 2 or 4 components. More precisely, the subgroups $\mathscr{G}(c_1, \ldots, c_{s+1})\subset SO(m+1)\times SO(k)$ being connected, the orbit $(O(m+1)\times O(k))/\mathscr{G}(c_1, \ldots, c_s, c_{s+1}+([(m+1)/2]-t)^+)$ has N components, where

$$N=\begin{cases}1, & \text{if } k>0 \text{ and } a_0b_0>0,\\ 2, & \text{if } k>0,\ a_0b_0=0 \text{ and } a_0+b_0>0 \text{ or if } k=0,\\ 4, & \text{if } k>0 \text{ and } a_0=b_0=0.\end{cases}$$

2. By a result of [13] for any locally rigid harmonic embedding $f: M\to S^n$ we have $V(f)=V(i)$, where $i: S^m\to S^n$ is the inclusion and m is the dimension of the least totally geodesic submanifold of S^n containing the image of f. Thus Theorem 1 gives a description of the variation space of all locally rigid harmonic embeddings.

The proof of Theorem 1 is broken up into a few lemmas. Let $(\mathscr{J}, B)\in V_0(i)$ be fixed. Then, by Theorem B, there exists $U\in O(m+1)$ such that $U^T\mathscr{J}U$ and $U^T(-\mathscr{J}^2)U$ have the form (1′) and (2), resp., with

$$0\leqq\hat{\sigma}_1\leqq\ldots\leqq\hat{\sigma}_{[(m+1)/2]}.$$

Thus, by $B^TB=I_{m+1}+\mathscr{J}^2$, we obtain

$$U^TB^TBU=\begin{cases}\operatorname{diag}(\sigma_1^2I_2, \ldots, \sigma_{[(m+1)/2]}^2I_2), & \text{if } m+1 \text{ is even},\\ \operatorname{diag}(1, \sigma_1^2I_2, \ldots, \sigma_{[(m+1)/2]}^2I_2), & \text{if } m+1 \text{ is odd},\end{cases}$$

where $\sigma_i^2=1-\hat{\sigma}_i^2$, $i=1, \ldots, [(m+1)/2]$. Clearly, $1\geqq\sigma_1^2\geqq\ldots\geqq\sigma_{[(m+1)/2]}^2\geqq0$ is satisfied. Then the values σ_i^2, $i=1, \ldots, [(m+1)/2]$, occurring twice in B^TB, are the eigenvalues of the positive semidefinite matrix B^TB. The nonzero eigenvalues of B^TB and BB^T being the same, the system of eigenvalues of $BB^T\in M(k,k)$ can be obtained from that of $B^TB\in M(m+1, m+1)$ by supplementing or omitting $|k-(m+1)|$ zeros according as $k\geqq m+1$ or $k<m+1$. In the latter case, for some index $t_0\leqq[k/2]$, $\sigma_i=0$, $i>t_0$, must be valid. The determination of the minimal value of t_0 can be done by making distinction according to the parity of k. Hence we have

$$U^{\mathrm{T}}B^{\mathrm{T}}BU=$$

$$=\begin{cases}\operatorname{diag}(\sigma_1^2 I_2, \ldots, \sigma_{(m+1)/2}^2 I_2) & \text{for } k \geqq m+1,\ m+1 \text{ even},\\ \operatorname{diag}(\sigma_1^2 I_2, \ldots, \sigma_{k/2}^2 I_2, 0_{m+1-k}) & \text{for } k \text{ even},\ k < m+1,\ m+1 \text{ even},\\ \operatorname{diag}(\sigma_1^2 I_2, \ldots, \sigma_{[k/2]}^2 I_2, 0_{m+1-2[k/2]}) & \text{for } k \text{ odd},\ k < m+1,\ m+1 \text{ even},\\ \operatorname{diag}(1, \sigma_1^2 I_2, \ldots, \sigma_{[(m+1)/2]}^2 I_2) & \text{for } k \geqq m+1,\ m+1 \text{ odd},\\ \operatorname{diag}(1, \sigma_1^2 I_2, \ldots, \sigma_{(k-1)/2}^2 I_2, 0_{m+1-k}) & \text{for } k \text{ odd},\ k < m+1,\ m+1 \text{ odd},\\ \operatorname{diag}(1, \sigma_1^2 I_2, \ldots, \sigma_{[(k-1)/2]}^2 I_2, 0_{m-2[(k-1)/2]}) & \text{for } k \text{ even},\ k < m+1,\ m+1 \text{ odd}.\end{cases}$$

A case-by-case verification shows that the minimal value of t_0 is the number t defined before Theorem 1. Thus we obtain

$$U^{\mathrm{T}}B^{\mathrm{T}}BU=\begin{cases}\operatorname{diag}(\sigma_1^2 I_2, \ldots, \sigma_t^2 I_2, 0_{m+1-2t}), & \text{if } m+1 \text{ is even},\\ \operatorname{diag}(1, \sigma_1^2 I_2, \ldots, \sigma_t^2 I_2, 0_{m-2t}), & \text{if } m+1 \text{ is odd},\end{cases}$$

and consequently (1′) has the form

$$U^{\mathrm{T}}\mathscr{J}U=\begin{cases}\operatorname{diag}(\hat\sigma_1 \Lambda, \ldots, \hat\sigma_t \Lambda, \Lambda_{(m+1-2t)/2}), & \text{if } m+1 \text{ is even},\\ \operatorname{diag}(0, \hat\sigma_1 \Lambda, \ldots, \hat\sigma_t \Lambda, \Lambda_{(m-2t)/2}), & \text{if } m+1 \text{ is odd}.\end{cases}$$

Lemma 1. *Let* $(\mathscr{J}, B)\in K(i)$. *Then* $(\mathscr{J}, B)\in V_0(i)$ *if and only if there exists* $(U, V)\in O(m+1)\times O(k)$ *such that* $(\mathscr{J}, B)=(U\mathscr{J}(\hat\sigma)U^{\mathrm{T}}, VB(\sigma)U^{\mathrm{T}}$, *where*

$$\mathscr{J}(\hat\sigma)=\begin{cases}\operatorname{diag}(\hat\sigma_1 \Lambda, \ldots, \hat\sigma_t \Lambda, \Lambda_{(m+1-2t)/2}) & \text{if } m+1, \text{ is even}\\ \operatorname{diag}(0, \hat\sigma_1 \Lambda, \ldots, \hat\sigma_t \Lambda, \Lambda_{(m-2t)/2}), & \text{if } m+1, \text{ is odd},\end{cases}$$

$$B(\sigma)=\begin{cases}\operatorname{diag}(\sigma_1 I_2, \ldots, \sigma_t I_2, 0_{d-2t})_k^{m+1}, & \text{if } m+1, \text{ is even},\\ \operatorname{diag}(1, \sigma_1 I_2, \ldots, \sigma_t I_2, 0_{d-1-2t})_k^{m+1}, & \text{if } m+1, \text{ is odd},\end{cases}$$

with $\sigma\in\Delta_t$, $\hat\sigma_i=\sqrt{1+\sigma_i^2}$, $i=1, \ldots, t$, *and* $d=\min(m+1, k)$.

Proof. If $(\mathscr{J}, B)\in V_0(i)$ then there exists $U\in O(m+1)$ such that $U^{\mathrm{T}}B^{\mathrm{T}}BU=$ $=B(\sigma)^{\mathrm{T}}B(\sigma)$ and $U^{\mathrm{T}}\mathscr{J}U=\mathscr{J}(\hat\sigma)$ with $0\leqq\hat\sigma_1\leqq\ldots\leqq\hat\sigma_{[(m+1)/2]}$. The diagonal entries of $U^{\mathrm{T}}B^{\mathrm{T}}BU$ are the eigenvalues of $B^{\mathrm{T}}B$ and hence, by Theorem A, there exists $V\in O(k)$ such that the pair (U, V) perform the singular value decomposition of B, i.e. we have $V^{\mathrm{T}}BU=B(\sigma)$. Thus, $(U^{\mathrm{T}}\mathscr{J}U, V^{\mathrm{T}}BU)=(\mathscr{J}(\hat\sigma), B(\sigma)), \sigma\in\Delta_t$. The converse being obvious the proof is finished.

By the lemma above the map $\Phi\colon \Delta_t\to K(i)$, $\Phi(\sigma)=(\mathscr{J}(\hat\sigma), B(\sigma)), \sigma\in\Delta_t$, is an embedding with $(O(m+1)\times O(k))\cdot\Phi(\Delta_t)=V_0(i)$. Moreover, the eigenvalues of $\mathscr{J}$ and the singular values of B are invariants characterizing the orbit through $(\mathscr{J}, B)$ uniquely. Thus $\Phi(\Delta_t)$ is a global section on $V_0(i)$ which accomplishes the proof of the first statement of Theorem 1.

Let $\sigma=(\sigma_0, \ldots, \sigma_0, \sigma_1, \ldots, \sigma_1, \ldots, \sigma_{s+1}, \ldots, \sigma_{s+1})\in\Delta_t$ be fixed with $1=\sigma_0>\sigma_1>\ldots>\sigma_s>\sigma_{s+1}=0$ and σ_i occurs c_i times in σ, $i=0, \ldots, s+1$. It remains to compute the isotropy type of the orbit through $\Phi(\sigma)$. The isotropy subgroup at $\Phi(\sigma)$ consists of pairs (U, V) such that $U\mathscr{J}(\hat{\sigma})=\mathscr{J}(\hat{\sigma})U$ and $VB(\sigma)=B(\sigma)U$. First we study the second relation. Consider $B(\sigma)\in M(k, m+1)$ as a matrix

$$B(\sigma)=\begin{bmatrix}\Sigma & 0\\ 0 & 0\end{bmatrix},$$

where $\Sigma=\operatorname{diag}(\sigma_0 I_{a_0}\, \sigma_1 I_{2c_1}, \ldots, \sigma_s I_{2c_s})\in M(r, r)$, $r=a_0+2\sum_{i=1}^{s} c_i$,

$$a_0=\begin{cases}2c_0, & \text{if } m+1 \text{ is even},\\ 2c_0+1, & \text{if } m+1 \text{ is odd},\end{cases}$$

and 0 on the right lower corner is of size $(k-r)\times(m+1-r)$.

Lemma 2. *Let* $(U, V)\in O(m+1)\times O(k)$ *such that* $VB(\sigma)=B(\sigma)U$ *holds. Then we have* $V=\operatorname{diag}(A_0, C_1, \ldots, C_s, B_0)$ *and* $U=\operatorname{diag}(A_0, C_1, \ldots, C_s, C_{s+1})$, *where* $A_0\in O(a_0)$, $B_0\in O(k-r)$, $C_i\in O(2c_i)$, $i=1, \ldots, s$, $C_{s+1}\in O(m+1-r)$.

Proof. Let $V\in O(k)$ and $U\in O(m+1)$ have the partitioned forms (conformal to that of $B(\sigma)$ above):

$$V=\begin{bmatrix}V_0 & R\\ S & B_0\end{bmatrix} \quad\text{and}\quad U=\begin{bmatrix}U_0 & P\\ Q & C_{s+1}\end{bmatrix},$$

where $V_0, U_0\in M(r, r)$, $B_0\in M(k-r, k-r)$, $C_{s+1}\in M(m+1-r, m+1-r)$. (The size of C_{s+1} can be expressed as $m+1-r=2c_{s+1}+2([(m+1)/2]-t)^+$). Substituting these into the equations $VB(\sigma)=B(\sigma)U$, $VV^{\mathrm{T}}=I_k$, $UU^{\mathrm{T}}=I_{m+1}$ we obtain $R=0$, $S=0$, $V_0\in O(r)$, $B_0\in O(k-r)$ and $P=0$, $Q=0$, $U_0\in O(r)$, $C_{s+1}\in O(m+1-r)$. Thus the first equation reduces to $V_0\Sigma=\Sigma U_0$, i.e. by $\det\Sigma=\sigma_1^{2c_1}\ldots\sigma_s^{2c_s}>0$, $V_0=\Sigma U_0\Sigma^{-1}$. Substituting this into the orthogonality relation $V_0^T V_0=I_r$ we get $U_0\Sigma^2=\Sigma^2 U_0$ which gives for $U_0=(C_{ij})$, $C_{00}\in M(a_0, a_0)$, $C_{i0}\in M(2c_i, a_0)$, $C_{0j}\in M(a_0, 2c_j)$, $C_{ij}\in M(2c_i, 2c_j)$, $i, j=1, \ldots, s$, the relations $C_{ij}=0$, if $i\neq j$. Hence, using the notations $C_{00}=A_0$ and $C_{ii}=C_i$, $i=1, \ldots, s$, we obtain $U_0=\operatorname{diag}(A_0, C_1, \ldots, C_s)$ with $A_0\in O(a_0)$, $C_i\in O(2c_i)$, $i=1, \ldots, s$. As U_0 and Σ commute we have $V_0=U_0$ which accomplishes the proof.

Consider now the second equation $U\mathscr{J}(\hat{\sigma})=\mathscr{J}(\hat{\sigma})U$, where U has the form given in Lemma 2. Clearly, this equation is satisfied if and only if $C_i\in Z(\Lambda_{c_i})$, $i=1, \ldots, s$, $C_{s+1}\in Z(\Lambda_{(m+1-r)/2})$, where $Z(\Lambda_p)$ denotes the centralizer of Λ_p in $O(2p)$.

Lemma 3. *The centralizer* $Z(\Lambda_p)\subset O(2p)$ *is connected and there exists* $U_0\in O(2p)$ *such that* $\mathrm{Ad}\,(U_0)Z(\Lambda_p)=U(p)\subset SO(2p)$, *where* Ad *denotes the adjoint representation of* $O(2p)$.

Proof. It is well-known that $Z(\Lambda_p)\subset SO(2p)$ (cf. [8], Ch. IV. § 29, p. 248). First we prove that $Z(\Lambda_p)\subset SO(2p)$ is connected. Clearly, $\exp\big((\pi/2)\Lambda_p\big)=\Lambda_p$, where $\exp\colon so(2p)\to SO(2p)$ is the exponential map. Hence $T=\overline{\exp(\mathbf{R}\Lambda_p)}\subset SO(2p)$ is a toroidal subgroup which contains Λ_p, i.e. its centralizer $Z(T)$ is contained in $Z(\Lambda_p)$. On the other hand, if $U\in Z(\Lambda_p)$ then the geodesics $s\mapsto\exp\big((\pi/2)s\Lambda_p\big)\cdot U$, $s\mapsto U\cdot\exp\big((\pi/2)s\Lambda_p\big)$, $s\in\mathbf{R}$, (with respect to a biinvariant metric on $SO(2p)$) have common tangent vector at $s=0$, i.e. $\exp\big((\pi/2)s\Lambda_p\big)U=U\exp\big((\pi/2)s\Lambda_p\big)$ which implies that $U\in Z(T)$. Thus $Z(\Lambda_p)=Z(T)$ and hence connected (cf. [4], Cor. 2.8. p. 287). Finally, let

$$\mathscr{I}_p=\begin{bmatrix}0_p & I_p\\ -I_p & 0_p\end{bmatrix}$$

and choose $U_0\in O(2p)$ with $\mathrm{Ad}\,(U_0)\Lambda_p=\mathscr{I}_p$. Then $\mathrm{Ad}\,(U_0)Z(\Lambda_p)=Z\big(\mathrm{Ad}\,(U_0)\Lambda_p\big)=Z(\mathscr{I}_p)$ and the fixed point set of the automorphism $\mathrm{Ad}\,(\mathscr{I}_p)$ of $SO(2p)$ is $Z(\mathscr{I}_p)$. It is known that $Z(\mathscr{I}_p)=U(p)\subset SO(2p)$ ([4], p. 453—454) which accomplishes the proof.

By Lemmas 1—3, (U, V) belongs to the isotropy subgroup at $\Phi(\sigma)$ if and only if $(U, V)\in O(m+1)\times O(k)$ is conjugate to an element of $\mathscr{G}\big(c_1, \dots, c_s, c_{s+1}+([(m+1)/2]-t)^+\big)$ (under a conjugation which does not depend on (U, V)) which completes the proof of Theorem 1.

Example (Variation space of the identity of odd spheres). Consider the special case when $m=n=2r-1$ odd. Then $t=0$ and $V_0(\mathrm{id}_{S^{2r-1}})$ reduces to a single orbit through $\Lambda_r\in so(2r)$ under the adjoint representation of $O(2r)$ on $so(2r)$. We claim that this orbit is a disjoint union

$$\mathrm{Ad}\,\big(SO(2r)\big)\Lambda_r\cup\mathrm{Ad}\,\big(SO(2r)\big)\Lambda_r^-,$$

where $\Lambda_r^-=\mathrm{diag}\,(\Lambda, \dots, \Lambda, -\Lambda)\in so(2r)$. Indeed, denoting $R=\mathrm{diag}\,(1, \dots, 1, -1)\in O(2r)$, we have $R\Lambda_rR=\Lambda_r^-$ and hence if $U\in O(2r)$ such that $\mathrm{Ad}\,(U)\Lambda_r=\Lambda_r^-$ then $\mathrm{Ad}\,(RU)\Lambda_r=\Lambda_r$ which implies $RU\in SO(2r)$, i.e. $\det U=-1$.

The Killing form of $so(2r)$ is a negative definite Ad-invariant scalar product on $so(2r)$ and so it follows easily that any ray in $so(2r)$ starting at the origin cuts the orbit $\mathrm{Ad}\,\big(SO(2r)\big)\Lambda_r$ (or $\mathrm{Ad}\,\big(SO(2r)\big)\Lambda_r^-$) at most once.

Case I: r is even. Then $\mathrm{Ad}\,(U_0)\Lambda_r=-\Lambda_r$ with $U_0=\mathrm{diag}\,(1, -1, 1, -1, \dots, 1, -1)\in SO(2r)$, i.e. the orbit $\mathrm{Ad}\big(SO(2r)\big)\Lambda_r$ (and $\mathrm{Ad}\big(SO(2r)\big)\Lambda_r^-$) is central symmetric to the origin. Thus $V(\mathrm{id}_{S^{2r-1}})=\mathbf{R}\cdot V_0(\mathrm{id}_{S^{2r-1}})$ is a double cone over $\mathrm{Ad}\,\big(SO(2r)\big)\Lambda_r=SO(2r)/U(r)$.

Case II: r is odd. It follows easily that any line through the origin cuts $V_0(\mathrm{id}_{S^{2r-1}})$ twice and that the components $\mathrm{Ad}(SO(2r))\Lambda_r$ and $\mathrm{Ad}(SO(2r))\Lambda_r^-$ are central symmetric to each other, i.e. $V(\mathrm{id}_{S^{2r-1}})$ is again a double cone over $SO(2r)/U(r)$.

Remark. In the special case $r=2$ the space $V_0(\mathrm{id}_{S^3})$ is the disjoint union of two samples of $S^2(=SO(4)/U(2))$ which was already noticed in [13].

4. The Veronese surface

Let M be a compact oriented Riemannian manifold and consider a harmonic map $f: M \to S^n$. By the inclusion $j: S^n \to \mathbf{R}^{n+1}$ the map f becomes a vector-valued function $f: M \to \mathbf{R}^{n+1}$. Moreover, translating vectors tangent to $S^n \subset \mathbf{R}^{n+1}$ to the origin, a vector field v along $f: M \to S^n$ gives rise to a map $\hat{v}: M \to \mathbf{R}^{n+1}$ with the property $\langle f, \hat{v} \rangle = 0$. The following lemma characterizes the elements of $K(f)$ in terms of the induced functions $\hat{v}$.

Lemma 4. *Let v be a vector field along $f: M \to S^n$. Then $v \in K(f)$ if and only if $\Delta^M \hat{v} = 2e(f)\hat{v}$ holds, where $e(f) = \|f_*\|^2/2$ denotes the energy density of f.*

Proof. The covariant differentiation on S^n can be obtained from that of $\mathbf{R}^{n+1}$ by performing the orthogonal projection to the corresponding tangent space of S^n and thus, for $X \in \mathfrak{X}(M)$, we have

$$(\nabla_X v)^\wedge = X(\hat{v}) - \langle X(\hat{v}), f \rangle f,$$

where X acts on $\hat{v}$ componentwise. An easy computation shows that

$$(\nabla_Y \nabla_X v)^\wedge = YX(\hat{v}) - \langle YX(\hat{v}), f \rangle f - \langle X(\hat{v}), f \rangle Y(f), \quad X, Y \in \mathfrak{X}(M),$$

i.e.

$$(\nabla^2 v)^\wedge = -\Delta^M \hat{v} + \langle \Delta^M \hat{v}, f \rangle f - \mathrm{trace}\, \langle d\hat{v}, f \rangle\, df$$

holds. On the other hand, we have

$$\big(\mathrm{trace}\, R(f_*, v) f_*\big)^\wedge = (\mathrm{trace}\, \langle f_*, v \rangle f_*)^\wedge - 2e(f)\hat{v} =$$

$$= \mathrm{trace}\, \langle df, \hat{v} \rangle df - 2e(f)\hat{v} = -\mathrm{trace}\, \langle f, d\hat{v} \rangle\, df - 2e(f)\hat{v}.$$

The identities yield that v is a Jacobi vector field along f if and only if

$$\Delta^M \hat{v} - \langle \Delta^M \hat{v}, f \rangle f = 2e(f)\hat{v} \tag{1}$$

is satisfied. Moreover, we have

$$\mathrm{trace}\, \langle f_*, \nabla v \rangle = \mathrm{trace}\, \langle df, d\hat{v} \rangle - \mathrm{trace}\, \langle d\hat{v}, f \rangle \langle df, f \rangle.$$

By $\|f\|^2=1$ the second term vanishes and so equation (ii) of Section 1 is equivalent to the following

(2) $$\operatorname{trace}\langle df, d\hat{v}\rangle = 0.$$

Further, harmonicity of f means that $\Delta^M f = 2e(f)f$ is valid and hence we get

$$\langle \Delta^M \hat{v}, f\rangle = -\langle \nabla^2 \hat{v}, f\rangle = -\operatorname{trace}\nabla\langle d\hat{v}, f\rangle + \operatorname{trace}\langle d\hat{v}, df\rangle =$$
$$= \operatorname{trace}\nabla\langle \hat{v}, df\rangle + \operatorname{trace}\langle d\hat{v}, df\rangle = 2\operatorname{trace}\langle d\hat{v}, df\rangle + \langle \hat{v}, \Delta^M f\rangle = 2\operatorname{trace}\langle d\hat{v}, df\rangle.$$

Assuming $v \in K(i)$ we obtain that $\langle \Delta^M \hat{v}, f\rangle = 0$ and hence (1) reduces to the equation given in the lemma. Conversely, multiplying this equation with f we get $\langle \Delta^M \hat{v}, f\rangle = 0$ and hence (1) and (2) are satisfied which accomplishes the proof.

Corollary. *Let $f, f' : M \to S^n$ be orthogonal harmonic maps with $e(f) = e(f')$. Then the (unique) vector field v along f with $\|v\| = 1$ and* $\exp \circ ((\pi/2)v) = f'$ *is a harmonic variation.*

Proof. By hypothesis $\hat{v} = f_{\pi/2} = f'$ and harmonicity of f' yields $\Delta^M \hat{v} = 2e(f')\hat{v} = 2e(f)\hat{v}$. Applying the lemma above we obtain that $v \in K(f)$ which accomplishes the proof.

Remark. According to a result of [11] a vector field v along f is a harmonic variation if and only if v is a Jacobi field along f and $e(f_t) = e(f)$ holds for all $t \in \mathbf{R}$. Hence there is a one-to-one correspondence between the harmonic variations of $V_0(f)$ and the orthogonal pairs of harmonic maps $f, f' : M \to S^n$ with $e(f) = e(f')$.

Now we turn to the variation space of the Veronese surface. Consider the eigenspace $\mathscr{H}_2$ of the Laplacian $\Delta = \Delta^{S^2}$ of the Euclidean sphere S^2 corresponding to the (second) eigenvalue $\lambda_2 = 6$ [1]. An element of $\mathscr{H}_2$ is the restriction (to S^2) of a homogeneous polynomial $p: \mathbf{R}^3 \to \mathbf{R}$ of degree 2 which has the form

$$p = \sum_{k=1}^{3} a_k \varphi_k + 2 \sum_{i<j} b_{ij} \varphi_{ij},$$

where $a_k, b_{ij} \in \mathbf{R}$ with $\sum_{k=1}^{3} a_k = 0$ and φ_k, φ_{ij}, $k = 1, 2, 3$, $1 \leq i < j \leq 3$, are scalars on S^2 defined by $\varphi_k(x) = x_k^2$, $\varphi_{ij}(x) = x_i x_j$, $x = (x_1, x_2, x_3) \in S^2$. (cf. [1] p. 176), in particular $\dim \mathscr{H}_2 = 5$.

Integration over S^2 defines a Euclidean scalar product on $\mathscr{H}_2$. Denoting $I = \|\varphi_k\|^2$ and $J = \|\varphi_{ij}\|^2$, the Veronese surface $f: S^2 \to S^4$ is defined by

$$f(x_1, x_2, x_3) = \frac{N}{I-J} \sum_{k=1}^{3} \left(x_k^2 - \frac{1}{3}\right) \varphi_k + \frac{2N}{J} \sum_{i<j} x_i x_j \varphi_{ij}, \quad (x_1, x_2, x_3) \in S^2,$$

where $N > 0$ is a normalizing factor given by the condition $\|f\| = 1$. Then f is full and homothetic [1]. It is well-known [1] that f factors through the canonical projection $\pi: S^2 \to \mathbf{R}P^2$ yielding an embedding of $\mathbf{R}P^2$ into S^4.

Lemma 5. *For the Veronese surface* $f: S^2 \to S^4$, *if* $v \in K(f)$ *then* $\hat{v}: S^2 \to \mathcal{H}_2$ *has the decomposition*

$$\hat{v} = \sum_{k=1}^{3} a_k \varphi_k + 2 \sum_{i<j} b_{ij} \varphi_{ij},$$

where a_k, b_{ij}, $k=1,3$, $1 \leqq i < j \leqq 3$, *are scalars on* S^2 *determined by the formulas*

$$a_1(x) = -\varepsilon x_2^2 + \varepsilon x_3^2 + 2\alpha_1 x_1 x_2 + 2\beta_1 x_1 x_3 - 2(\alpha_2 + \beta_3) x_2 x_3,$$

$$a_2(x) = \varepsilon x_1^2 - \varepsilon x_3^2 + 2\beta_2 x_1 x_2 - 2(\beta_1 + \alpha_3) x_1 x_3 + 2\alpha_2 x_2 x_3,$$

$$a_3(x) = -\varepsilon x_1^2 + \varepsilon x_2^2 - 2(\alpha_1 + \beta_2) x_1 x_2 + 2\alpha_3 x_1 x_3 + 2\beta_3 x_2 x_3,$$

$$b_{12}(x) = -\frac{\alpha_1}{2} x_1^2 - \frac{\beta_2}{2} x_2^2 + \frac{\alpha_1 + \beta_2}{2} x_3^2 - 2\gamma_1 x_1 x_3 + 2\gamma_2 x_2 x_3,$$

$$b_{23}(x) = \frac{\alpha_2 + \beta_3}{2} x_1^2 - \frac{\alpha_2}{2} x_2^2 - \frac{\beta_3}{2} x_3^2 - 2\gamma_2 x_1 x_2 + 2\gamma_3 x_1 x_3,$$

$$b_{13}(x) = -\frac{\beta_1}{2} x_1^2 + \frac{\alpha_3 + \beta_1}{2} x_2^2 - \frac{\alpha_3}{2} x_3^2 + 2\gamma_1 x_1 x_2 - 2\gamma_3 x_2 x_3,$$

$x = (x_1, x_2, x_3) \in S^2$, $\varepsilon, \alpha_k, \beta_k, \gamma_k \in \mathbf{R}$, $k = 1, 2, 3$. *In particular,* $\dim K(f) = 10$.

Proof. As $\hat{v}$ maps into $\mathcal{H}_2$ we have the decomposition of $\hat{v}$ as above with $\sum_{k=1}^{3} a_k = 0$. On the other hand, Lemma 4 implies that

$$0 = \Delta\hat{v} - 6\hat{v} = \sum_{k=1}^{3} (\Delta a_k - 6a_k) \varphi_k + 2 \sum_{i<j} (\Delta b_{ij} - 6b_{ij}) \varphi_{ij}$$

and hence orthogonality of the polynomials φ_{ij}, $i<j$, and the relations $\langle \varphi_k, \varphi_{ij} \rangle = 0$, $\langle \varphi_k, \varphi_r \rangle = J + \delta_{kr}(I - J)$, $k, r = 1, 2, 3$, $i<j$, yield that the scalars a_k, b_{ij}, $k = 1, 2, 3$, $i<j$, belong to $\mathcal{H}_2$. Thus

$$a_r = \sum_{k=1}^{3} a_k^r \varphi_k + 2 \sum_{i<j} b_{ij}^r \varphi_{ij}, \quad r = 1, 2, 3,$$

and

$$b_{pq} = \sum_{k=1}^{3} a_k^{pq} \varphi_k + 2 \sum_{i<j} b_{ij}^{pq} \varphi_{ij}, \quad 1 \leqq p < q \leqq 3,$$

where $a_k^r, b_{ij}^r, a_k^{pq}, b_{ij}^{pq} \in \mathbf{R}$ such that

$$(\mathrm{C}_1) \quad \sum_{k=1}^{3} a_k^r = 0 \quad \text{and} \quad \sum_{k=1}^{3} a_k^{pq} = 0, \quad r = 1, 2, 3, \ 1 \leqq p < q \leqq 3,$$

hold. Moreover, from the equation $\sum_{k=1}^{3} a_k = 0$ we obtain

$$(\mathrm{C}_2) \quad \sum_{r=1}^{3} a_k^r = 0 \quad \text{and} \quad \sum_{r=1}^{3} b_{ij}^r = 0.$$

Finally, the orthogonality relations for φ_k and φ_{ij} above imply that the condition $\langle f, \hat{v}\rangle = 0$ is equivalent to the equation

$$\sum_{k=1}^{3} a_k x_k^2 + 4 \sum_{i<j} b_{ij} x_i x_j = 0, \quad (x_1, x_2, x_3) \in S^2.$$

Substituting the explicit expressions of a_k and b_{ij} we get

$$\sum_{k=1}^{3} \sum_{r=1}^{3} a_k^r \varphi_k \varphi_r + 2 \sum_{i<j} \sum_{r=1}^{3} (b_{ij}^r + 2a_r^{ij}) \varphi_r \varphi_{ij} + 8 \sum_{i<j} \sum_{p<q} b_{ij}^{pq} \varphi_{ij} \varphi_{pq} = 0.$$

A straightforward computation, determining the coefficients of the fourth order homogeneous polynomial on the left hand side, shows that this equation is satisfied if and only if the following relations hold:

(C$_3$) $a_k^k = 0$ for $k = 1, 2, 3$,

(C$_4$) $b_{12}^1 + 2a_1^{12} = b_{12}^2 + 2a_2^{12} = b_{13}^1 + 2a_1^{13} = b_{13}^3 + 2a_3^{13} = b_{23}^2 + 2a_2^{23} = b_{23}^3 + 2a_3^{23} = 0$,

(C$_5$) $a_i^j + a_j^i + 8b_{ij}^{ij} = 0$ for $1 \leqq i < j \leqq 3$,

(C$_6$) $b_{23}^1 + 2a_1^{23} + 4b_{12}^{13} + 4b_{13}^{12} = b_{13}^2 + 2a_2^{13} + 4b_{12}^{23} + 4b_{23}^{12} = b_{12}^3 + 2a_3^{12} + 4b_{13}^{23} + 4b_{23}^{13} = 0$.

Putting $\varepsilon = a_1^2$, the relations (C$_1$)—(C$_2$)—(C$_3$) imply that the matrix $A = (a_k^r) \in M(3, 3)$ has the form

$$A = \begin{bmatrix} 0 & \varepsilon & -\varepsilon \\ -\varepsilon & 0 & \varepsilon \\ \varepsilon & -\varepsilon & 0 \end{bmatrix}$$

and consequently, by (C$_5$), $b_{ij}^{ij} = 0$ for $i < j$. Introducing the new (independent) variables

$$\alpha_1 = b_{12}^1, \; \alpha_2 = b_{23}^2, \; \alpha_3 = b_{13}^3,$$
$$\beta_1 = b_{13}^1, \; \beta_2 = b_{12}^2, \; \beta_3 = b_{23}^3,$$
$$\gamma_1 = b_{12}^{13}, \; \gamma_2 = b_{23}^{12}, \; \gamma_3 = b_{13}^{23},$$

we see that all the remaining coefficients are expressible in terms of the variables $\{\varepsilon, \alpha_k, \beta_k, \gamma_k \mid k = 1, 2, 3\}$ and a straightforward computation leads to the coefficients given in Lemma 5.

Our last result asserts that the Veronese surface is rigid. More precisely, we have the following

Theorem 2. *For the Veronese surface* $f: S^2 \to S^4$ *the variation space* $V(f)$ *is zero.*

Proof. Using the notations of Lemma 5 we parametrize $K(f)$ with the variables $\{\varepsilon, \alpha_k, \beta_k, \gamma_k \mid k = 1, 2, 3\}$. Putting $v \in K(f)$ we have

$$\hat{v} = \sum_{k=1}^{3} a_k \varphi_k + 2 \sum_{i<j} b_{ij} \varphi_{ij},$$

where the coefficients a_k, b_{ij}, $k = 1, 2, 3$, $1 \leqq i < j \leqq 3$, are given in Lemma 5.

Note that the parametrization of $K(f)$ is chosen in such a way as the cyclic permutation $\pi=(123)$ of the indices on the right hand sides will permute the scalars a_1, a_2, a_3 and b_{12}, b_{23}, b_{13} cyclically. Now suppose, on the contrary, that $V(f)\neq\{0\}$, i.e. we may choose $v\in V(f)$ with $\|v\|^2=4\mathcal{J}$. Then we have

$$4J=\|v\|^2=\sum_{k=1}^{3}\sum_{r=1}^{3}a_k a_r\langle\varphi_k,\varphi_r\rangle+4J\sum_{i<j}b_{ij}^2=(I-J)\sum_{k=1}^{3}a_k^2+4J\sum_{i<j}b_{ij}^2,$$

or equivalently

$$1=\frac{1}{2}\sum_{k=1}^{3}a_k^2+\sum_{i<j}b_{ij}^2 \tag{3}$$

on S^2, where we used the equality $\dfrac{I-J}{4J}=\dfrac{1}{2}$ which can be obtained by integrating the polynomials φ_3^2 and φ_{23}^2 on S^2. Thus

$$(x_1^2+x_2^2+x_3^2)^2=\frac{1}{2}\sum_{k=1}^{3}a_k(x_1,x_2,x_3)^2+\sum_{i<j}b_{ij}(x_1,x_2,x_3)^2$$

is satisfied for all $(x_1,x_2,x_3)\in\mathbf{R}^3$. By computing the coefficients of the fourth order homogeneous polynomial on the right hand side we obtain a system of 15 quadratic equations in which the first 5 are given as follows

(i) $4\varepsilon^2+\alpha_1^2+\beta_1^2+(\alpha_2+\beta_3)^2=4,$

(ii) $\varepsilon(\alpha_1+2\beta_2)-\beta_1\gamma_1-(\alpha_2+\beta_3)\gamma_2=0,$

(iii) $-\varepsilon(\beta_1+2\alpha_3)+\alpha_1\gamma_1+(\alpha_2+\beta_3)\gamma_3=0,$

(iv) $\varepsilon(\alpha_2-\beta_3)+2\big(\alpha_1\beta_1-\beta_2(\beta_1+\alpha_3)-\alpha_3(\alpha_1+\beta_2)\big)-\alpha_1\gamma_2+\beta_1\gamma_3-4\gamma_2\gamma_3=0,$

(v) $-2\varepsilon^2+4\big(\alpha_1^2+\beta_2^2+(\alpha_1+\beta_2)^2\big)+\alpha_1\beta_2-\beta_1(\beta_1+\alpha_3)-\alpha_2(\alpha_2+\beta_3)+8(\gamma_1^2+\gamma_2^2)=4,$

and, the equation (3) being invariant under the cyclic permutation $\pi=(123)$ of the indices, the last 10 equations are obtained from (i)—(v) by performing the index permutations π and π^2. Denote the equations of the permuted systems by $(\mathrm{i})_\pi$—$(\mathrm{v})_\pi$ and $(\mathrm{i})_{\pi^2}$—$(\mathrm{v})_{\pi^2}$, respectively. Our purpose is to show that these equations have no solution. To do this, first denote by s the symmetric polynomial given by $s(x,y)=x^2+xy+y^2$, $x,y\in\mathbf{R}$. Then (v) can be written as

$$-2\varepsilon^2+8s(\alpha_1,\beta_2)+(\alpha_1\beta_2-\beta_1^2-\beta_1\alpha_3-\alpha_2^2-\alpha_2\beta_3)+8(\gamma_1^2+\gamma_2^2)=4.$$

Performing the index permutations π and π^2 and adding these three equations we get

$$-6\varepsilon^2+7\big(s(\alpha_1,\beta_2)+s(\alpha_2,\beta_3)+s(\alpha_3,\beta_1)\big)+16(\gamma_1^2+\gamma_2^2+\gamma_3^2)=12.$$

In a similar way, from (i)—$(\mathrm{i})_\pi$—$(\mathrm{i})_{\pi^2}$ it follows that

$$12\varepsilon^2+2\big(s(\alpha_1,\beta_2)+s(\alpha_2,\beta_3)+s(\alpha_3,\beta_1)\big)=12,$$

i.e. eliminating the terms containing the polynomial s we have

$$24(1-\varepsilon^2)+8(\gamma_1^2+\gamma_2^2+\gamma_3^2) = 9. \tag{4}$$

On the other hand, fixing γ_i, $i=1, 2, 3$, the equations (ii)—(ii)$_\pi$—(ii)$_{\pi^2}$ and (iii)—(iii)$_\pi$—(iii)$_{\pi^2}$ form a linear system for the variables α_i, β_i, $i=1, 2, 3$. Denoting by $M(\gamma_1, \gamma_2, \gamma_3)\in M(6, 6)$ its matrix, we compute $\det M(\gamma_1, \gamma_2, \gamma_3)$. For $\xi, \eta, \zeta\in\mathbf{R}$ define

$$S(\xi, \eta, \zeta) = \begin{bmatrix} \varepsilon & 2\varepsilon & 0 & -\xi & -\eta & -\eta \\ -2\varepsilon & -\varepsilon & \xi & \xi & \eta & 0 \\ -\xi & -\xi & \varepsilon & 2\varepsilon & 0 & -\zeta \\ \xi & 0 & -2\varepsilon & -\varepsilon & \zeta & \zeta \\ 0 & -\eta & -\zeta & -\zeta & \varepsilon & 2\varepsilon \\ \eta & \eta & \zeta & 0 & -2\varepsilon & -\varepsilon \end{bmatrix}.$$

Permuting the rows and the coloumns of $M(\gamma_1, \gamma_2, \gamma_3)$ by the permutation (25) we obtain $S(\gamma_1, \gamma_2, \gamma_3)$ and consequently $\det M(\gamma_1, \gamma_2, \gamma_3)=\det S(\gamma_1, \gamma_2, \gamma_3)$. Similarly, by performing (135462) and (132465) on the rows and coloumns of $M(\gamma_1, \gamma_2, \gamma_3)$ we get $S(\gamma_2, \gamma_3, \gamma_1)$ and $S(\gamma_3, \gamma_1, \gamma_2)$ i.e. $\det S(\gamma_1, \gamma_2, \gamma_3) = \det S(\gamma_2, \gamma_3, \gamma_1)=\det S(\gamma_3, \gamma_1, \gamma_2)$. Thus, it is enough to compute $\det S(\xi, \eta, \zeta)$. To do this, let $S(\xi, \eta, \xi)$ have the decomposition

$$S(\xi, \eta, \zeta) = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where $A\in M(4, 4)$. The matrix A is centroskew and so by using a result of [2] a direct computation shows that $\det A=(3\varepsilon^2-\xi^2)^2$. Assuming $3\varepsilon^2\neq\xi^2$ we have [2]

$$\det S(\xi, \eta, \zeta) = \det A \det (D-CA^{-1}B) = 3\varepsilon^2\big(3\varepsilon^2-(\xi^2+\eta^2+\zeta^2)\big)^2.$$

Suppose now that $\gamma_1^2=\gamma_2^2=\gamma_3^2=3\varepsilon^2$. Then equation (4) implies that $15+8\varepsilon^2=0$ which is impossible. Hence there exists $i\in\{1, 2, 3\}$ such that $\gamma_i\neq 3\varepsilon^2$. Then, by the above, $\det M(\gamma_1, \gamma_2, \gamma_3)=3\varepsilon^2(3\varepsilon^2-(\gamma_1^2+\gamma_2^2+\gamma_3^2))^2$. Further, $\det M(\gamma_1, \gamma_2, \gamma_3)\neq 0$ since otherwise $\gamma_1^2+\gamma_2^2+\gamma_3^2=3\varepsilon^2$ which contradicts to (4). Thus the linear system in question has only trivial solution $\alpha_1=\alpha_2=\alpha_3=\beta_1=\beta_2=\beta_3=0$. Then equations (iv)—(iv)$_\pi$—(iv)$_{\pi^2}$ imply that two of the numbers $\gamma_1, \gamma_2, \gamma_3$ vanish. By equations (v)—(v)$_\pi$—(v)$_{\pi^2}$ we obtain $\varepsilon=0$ which again contradicts to (4).

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On partial asymptotic stability and instability. II (The method of limiting equation)

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1. Introduction

In [1] we established criteria on the partial asymptotic stability and instability based on Ljapunov functions with semidefinite derivatives not requiring boundedness of solutions. We proved an alternative for every solution of an autonomous system saying that either all the controlled coordinates tend to zero or the vector of the uncontrolled coordinates tends to infinity as $t\to\infty$ (see [1], Theorem 3.1). Combining this result with additional hypotheses *on the Ljapunov function* we found sufficient conditions for the partial asymptotic stability and instability of the zero solution. By the aid of these theorems we could study stability properties of equilibrium positions of certain mechanical systems in the presence of dissipative forces. However, as it was mentioned in [1], to apply the alternative to certain mechanical systems one needs additional conditions of other types. For example, consider a material point moving on a surface in a constant field of gravity in the inertial frame of reference $0xyz$ ($0z$ directed vertically upward) and subject to viscous friction [1]. Let the point be constrained to move on the surface of the equation $z=(1/2)y^2\times\times[1+1/(1+x^2)]$. Theorems in [1] cannot be applied to prove asymptotic y-stability for the equilibrium position $x=y=0$. Nevertheless, it is reasonable to conjecture that the equilibrium position possesses this property. For, if a motion $(x(t), y(t))$ is bounded, then $|y(t)|\to 0$ as $t\to\infty$ (see [1], Theorem A). On the other hand, if $|x(t)|\to\infty$ as $t\to\infty$, then the motion $(x(t), y(t))$ is "asymptotically near" to a motion of the point on the surface of the equation $z=(1/2)y^2$, for which the equilibrium position $x=y=0$ is asymptotically y-stable.

The purpose of this paper is to establish partial asymptotic stability and instability of the zero solution of such system whose *right-hand side* allows a limiting process as the vector of the uncontrolled coordinates tends to infinity in norm.

The paper is organized as follows. In Section 3 we treat such autonomous

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system whose right-hand side has a uniform limit as the vector of uncontrolled coordinates tends to infinity in norm. In Section 4 results of the previous section will be applied to study partial stability properties of the equilibrium position with respect to all generalized velocities and some of generalized coordinates in the scleronomic holonomic mechanical systems being under the action of viscous friction. The method to be presented works also for the nonautonomous differential systems. Section 5 is devoted to this generalization. Whilst Section 3 is based upon the standard sphere of concepts of stability theory and is selfcontained, Section 5 is strongly connected with a recent topic of the theory of limiting equations developed by Z. Artstein [2]—[4], some of whose results are necessary preliminaires for applying our main theorem.

2. A nonautonomous invariance principle

All the necessary notations and definitions have been introduced in [1] (see Section 2) excepting the following one. Consider the system of differential equation

$$\dot{x} = X(x, t) \quad (t \in R_+, x \in R^k), \tag{2.1}$$

where the function X is continuous in x, is measurable in t, and satisfies the Carathéodory condition locally on the set Γ_y. Let us given a Ljapunov function $V: \Gamma'_y \to R$ (for $\Gamma'_y \subset \Gamma_y \subset R^m \times R^n \times R_+$ see [1], Sec. 2). For $c \in R$ denote by $V_m^{-1}[c, \infty]_0$ the set of the points $y \in R^m$ for which there exists a sequence $\{(y_i, z_i, t_i)\}$ such that $y_i \to y$, $|z_i| \to \infty$, $t_i \to \infty$, $V(y_i, z_i, t_i) \to c$ and $\dot{V}(y_i, z_i, t_i) \to 0$ as $i \to \infty$. Obviously, $V_m^{-1}[c, \infty]_0$ is closed relative to Γ'_y.

We shall need the following nonautonomous invariance principle even in Section 3 where the basic differential system is assumed to be autonomous.

Theorem A. [5]—[7] *Assume that for every compact set $K \subset R^k$ there is a $\mu_K \in \mathcal{K}$ such that if $u: [\alpha, \beta] \to K$ is continuous then*

$$\left| \int_\alpha^\beta X(u(t), t)\, dt \right| \leqq \mu_K(\beta - \alpha). \tag{2.2}$$

If $V: \Gamma'_x \to R$ is a Ljapunov function bounded below, and $\varphi: [t_0, \infty) \to R^k$ is a solution of (2.1) for which $|\varphi(t)| \leqq H'' < H'$ holds for all $t \geqq t_0$, then $\Omega_x(\varphi)$ is contained in a component of $V_k^{-1}[c, \infty]_0$ for some constant c.

In order to make Section 3 selfcontained we sketch the proof. Since V is locally Lipschitzian, the function $v(t) = V(\varphi(t), t)$ is locally absolutely continuous and

$$\frac{d}{dt} v(t) = \dot{V}(\varphi(t), t) \leqq 0 \tag{2.3}$$

for almost all $t \geqq t_0$. Thus $v(t)$ is nonincreasing and $v(t) \to c$ as $t \to \infty$ for some constant c. Suppose that the statement is false. Then there exist $p \in \Omega_x(\varphi)$ and $\varepsilon > 0$ such that $\bar{B}_k(p, 2\varepsilon) \cap V_k^{-1}[c, \infty]_0 = \emptyset$, where $\bar{B}_k(p, 2\varepsilon)$ denotes the closed ball in R^k with center p and radius 2ε. Obviously,

$$\limsup_{T \to \infty} \{\dot{V}(\varphi(t), t) \colon t \geqq T, \ \varphi(t) \in \bar{B}_k(p, 2\varepsilon)\} < 0, \tag{2.4}$$

thus, however large the time T^* may be, the point $\varphi(t)$ cannot be contained in the set $\bar{B}_k(p, 2\varepsilon)$ for all $t \geqq T^*$ since v is bounded below. Therefore, $\varphi(t)$ enters $\bar{B}_k(p, \varepsilon)$ and leaves $B_k(p, 2\varepsilon)$ infinite number of times. In view of (2.2)—(2.4) this means that v is not of bounded variation, which is a contradiction.

3. Autonomous equations

Consider the differential system

$$\dot{x} = X(x) \quad (x \in R^k; \ X(0) = 0), \tag{3.1}$$

where $X \colon G_y \to R^k$ is continuous. By the partition $x = (y, z)$ ($y \in R^m$, $z \in R^n$; $1 \leqq m \leqq k$, $n = k - m$) the system (3.1) can be written in the form

$$\dot{y} = Y(y, z), \quad \dot{z} = Z(y, z). \tag{3.2}$$

Throughout this section we assume that $Y(y, z) \to Y_*(y)$ uniformly in $y \in \bar{B}_m(H')$ as $|z| \to \infty$.

Theorem 3.1. *Suppose that there is a Ljapunov function $V \colon G'_y \to R$ of (3.2) satisfying the following conditions:*

(i) *V is positive y-definite;*

(ii) *for every $c > 0$ the set $(\dot{V}_{(3.2)})^{-1}(0) \cap V^{-1}(c)$ contains no complete trajectory of (3.2), and*

(iii) *the set $V_m^{-1}[c, \infty]_0$ contains no complete trajectory of the system*

$$\dot{y} = Y_*(y) \tag{3.3}$$

except the origin of R^m.

Then the zero solution of (3.2) is asymptotically y-stable.

Proof. Since V is positive y-definite and $\dot{V}_{(3.2)}(y, z) \leqq 0$ on G'_y, the zero solution of (3.2) is y-stable (see [8], p. 15), i.e. for every $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that $|x_0| < \delta(\varepsilon)$ implies $|y(t; x_0)| < \varepsilon$ for all $t \geqq 0$. Let $0 < \varepsilon_0 < H'$ and define $\sigma = \delta(\varepsilon_0) > 0$. We shall prove that for every $x_0 \in B_k(\sigma)$ we have $|y(t; x_0)| \to 0$ as $t \to \infty$.

Let $x=\varphi(t)=(\psi(t), \chi(t))$ be a solution of (3.2) such that $\varphi(0)\in B_k(\sigma)$. The function $v(t)=V(\varphi(t))$ is nonincreasing and nonnegative, hence $v(t)\to v_0\geqq 0$ as $t\to\infty$. If $v_0=0$ then $|\psi(t)|\to 0$ as $t\to\infty$ since V is positive y-definite. Assume that $v_0>0$. By Theorem 3.1 in [1], this assumption together with (ii) imply

$$\lim_{t\to\infty} |\chi(t)| = \infty. \tag{3.4}$$

Consider the system

$$\dot{y} = Y(y, \chi(t)) \quad (y\in B_m(H'),\ t\in R_+) \tag{3.5}$$

and the function $U: B_m(H')\times R_+\to R$ defined by $U(y, t)=V(y, \chi(t))$. Obviously,

$$\dot{U}_{(3.5)}(y, t) = \dot{V}_{(3.2)}(y, \chi(t)) \leqq 0, \tag{3.6}$$

therefore U is a Ljapunov function of (3.5) and $u(t)=U(\psi(t), t)\to v_0$ as $t\to\infty$. The function $y=\psi(t)$ is a solution of equation (3.5), whose right-hand side is bounded for $(y, t)\in\bar{B}_m(H')\times R_+$, and $|\psi(t)|\leqq\varepsilon_0<H'$ for all $t\geqq 0$. By Theorem A in Section 2 we have the inclusion $\Omega_y(\psi)\subset U_m^{-1}[v_0, \infty]_0$. Furthermore, in view of (3.4) and (3.6), $U_m^{-1}[v_0, \infty]_0\subset V_m^{-1}[v_0, \infty]_0$. Taking into account the obvious fact that the positive y-limit set $\Omega_y(\varphi)$ of the solution $x=\varphi(t)$ of (3.2) coincides with the positive limit set $\Omega_y(\psi)$ of ψ, being a solution of (3.5), we obtain

$$\Omega_y(\varphi) = \Omega_y(\psi) \subset V_m^{-1}[v_0, \infty]_0. \tag{3.7}$$

On the other hand, property (3.4) implies that $Y(y, \chi(t))\to Y_*(y)$ uniformly in $y\in\bar{B}_m(H')$ as $t\to\infty$. Thus (3.3) is the limit equation of (3.5) and $\Omega_y(\psi)$ is semiinvariant with respect to (3.3) (see [8], p. 304). Now we can conclude the proof by showing that $\Omega_y(\varphi)=\{0\}$, i.e. $|\psi(t)|\to 0$ as $t\to\infty$. Indeed, if the nonempty set $\Omega_y(\varphi)$ contains any point besides the origin of R^m, then it contains also a complete trajectory of (3.3) different from the origin because it is semiinvariant with respect to (3.3). But, in consequence of (3.7), this contradicts condition (iii) of the theorem. The proof is complete.

In certain applications condition (ii) in Theorem 3.1 proves to be rather restrictive. For example, it may happen that the potential energy $P(\hat{q}, \tilde{q})$ of a mechanical system is $\hat{q}$-definite, in every neighbourhood of the origin $\hat{q}=\tilde{q}=0$ there exists an equilibrium position belonging to the set $P(\hat{q}, \tilde{q})>0$, nevertheless the origin is asymptotically $\hat{q}$-stable (see [1], Examples). Now we relax this condition of the theorem (compare with Theorem 3.3 in [1]).

Theorem 3.2. *Suppose that the function Z in* (3.2) *is bounded on the set G_y', and there is a Ljapunov function $V: G_y'\to R$ of* (3.2) *satisfying conditions* (i), (iii) *in Theorem* 3.1. *Assume, in addition, that* (ii′) *for every* $c>0$, *if the set* $(\dot V_{(3.2)})^{-1}(0)\cap V^{-1}(c)$ *contains a complete trajectory of* (3.2) *then this trajectory is contained in the set* $\{(y,z)\colon y=0\}$.

Then the zero solution of (3.2) *is asymptotically y-stable.*

Proof. We have to modify the proof of Theorem 3.1 only from that point where we assumed $v_0>0$. It is enough to prove that in this case $\Omega_y(\varphi)=\{0\}$.

Let $0\neq q\in\Omega_y(\varphi)$. Then, by Lemmas 3.1—3.2 in [1], either there exists an $r\in R^n$ such that $(q,r)\in\Omega_x(\varphi)\subset M(v_0)=(\dot V_{(3.2)})^{-1}(0)\cap V^{-1}(v_0)$ or $|\chi(t_i)|\to\infty$ whenever $t_i\to\infty$ and $\psi(t_i)\to q$ as $i\to\infty$. In the first case, by the semiinvariance property of $\Omega_x(\varphi)$ with respect to (3.2), the set $M(v_0)$ contains a trajectory of (3.2) not contained in the set $\{(y,z)\colon y=0\}$, which contradicts (ii′). Therefore, if $t_i\to\infty$ and $\psi(t_i)$ converges to a point different from the origin of R^m, then $|\chi(t_i)|\to\infty$ as $i\to\infty$.

We shall prove that in the case $\Omega_y(\varphi)\neq\{0\}$ the inclusion $\Omega_y(\varphi)\subset N(v_0)=$ $=V_m^{-1}[v_0,\infty]_0$ holds. But $\Omega_y(\varphi)$ is compact and connected, and $N(v_0)$ is closed, so it is enough to show that $\Omega_y(\varphi)\setminus\{0\}\subset N(v_0)$. Suppose the contrary. Then there exist $q\in\Omega_y(\varphi)$ $(q\neq 0)$ and $\varepsilon>0$ such that $\bar B_m(q,2\varepsilon)\cap[N(v_0)\cup\{0\}]=\emptyset$. We state that

$$\alpha=\limsup_{T\to\infty}\{\dot V(\psi(t),\chi(t))\colon t\geqq T,\ \psi(t)\in\bar B_m(q,2\varepsilon)\}<0. \tag{3.8}$$

Indeed, otherwise there is a sequence $\{t_i\}$ for which $t_i\to\infty$, $\dot V(\varphi(t_i))\to 0$, $\psi(t_i)\to$ $\to q'\in\bar B_m(q,2\varepsilon)$ and, consequently, $|\chi(t_i)|\to\infty$ as $i\to\infty$, i.e. $q'\in N(v_0)$, which contradicts the definition of ε. Since V is bounded below, (3.8) implies that $\psi(t)\in$ $\in\bar B_m(q,2\varepsilon)$ cannot be satisfied on any whole interval $[T,\infty)$. From this fact it follows that there exist sequences $\{t_i'\}$, $\{t_i''\}$ with the properties

$$t_i'<t_i''<t_{i+1}',\quad t_i'\to\infty;\quad |\psi(t_i')-q|=\varepsilon,\ |\psi(t_i'')-q|=2\varepsilon,$$
$$\varepsilon\leqq|\psi(t)-q|\leqq 2\varepsilon\ \ (t_i'\leqq t\leqq t_i'';\ i=1,2,\ldots).$$

Since $Y(\psi(t),\chi(t))$ is bounded, $t_i''-t_i'\geqq\beta>0$ for all i with some constant β and

$$v(t_i'')-v(t_1')\leqq\sum_{j=1}^{i}\int_{t_j'}^{t_j''}\dot V(\varphi(t))\,dt\leqq i\alpha\beta\to-\infty,$$

which is a contradiction.

It remains to prove that for every $q\in\Omega_y(\varphi)$ $(q\neq 0)$ the system (3.3) has a complete trajectory through q lying in $\Omega_y(\varphi)$. Consider the sequence of the functions $\{\psi^i(t)=\psi(t_i+t)\}$ whose i-th member is a solution of the initial value problem

$$\dot y=Y(y,\chi(t_i+t)),\quad y(0)=\psi(t_i)\quad (i=1,2,\ldots).$$

Since Z is bounded, $|\chi(t_i+t)|\to\infty$ uniformly with respect to t on each compact interval $[a,b]$ as $i\to\infty$. Thus, $Y(y,\chi(t_i+t))\to Y_*(y)$ uniformly in $(y,t)\in\bar B_m(H')\times$ $\times[a,b]$, and $\psi(t_i)\to q$. Consequently, there exists a subsequence of $\{\psi^i(t)\}$ which converges uniformly on $[a, b]$ to a solution γ of the initial value problem $\dot{y}=Y_*(y)$, $y(0)=q$ (see [8], p. 297). For each $t\geqq 0$ the point $\gamma(t)$ is the limit of a subsequence of $\psi(t+t_i)$. But also $t_i+t\to\infty$, so $\gamma(t)\in\Omega_y(\varphi)$, which means that $\Omega_y(\varphi)$ contains a complete trajectory of (3.3) different from the origin.

We have proved that if there exists a $q\in\Omega_y(\varphi)$ $(q\neq 0)$ then there exists also a complete trajectory of (3.3) different from the origin that is contained by $\Omega_y(\varphi)$ and, because of $\Omega_y(\varphi)\subset N(v_0)$, by $N(v_0)$ as well, in contradiction to assumption (iii). The proof is complete.

Our method can be used for deriving instability theorems, too.

Theorem 3.3. *Suppose that there is a Ljapunov function* $V: G'_y\to R$ *of* (3.2) *satisfying the following conditions:*

(i) V *is bounded below;*

(ii) *for every* $\delta>0$ *there exists an* $x_0\in B_k(\delta)$ *such that* $V(x_0)<0$;

(iii) *for every* $c<0$ *the set* $(\dot{V}_{(3.2)})^{-1}(0)\cap V^{-1}(c)$ *contains no complete trajectory of* (3.2), *and*

(iv) *the set* $V_m^{-1}[c, \infty]_0$ *contains no complete trajectory of* (3.3).

Then the zero solution of (3.2) *is* y*-unstable.*

Proof. We have to prove that there is an $\varepsilon_0>0$ such that from every neighbourhood of the origin in R^k there starts a solution of (3.2) which leaves the set $\bar{B}_m(\varepsilon_0)\times R^n$.

Let $0<\varepsilon_0<H'$. For an arbitrary δ $(0<\delta<\varepsilon_0)$ take an $x_0\in B_k(\delta)$ such that $V(x_0)<0$, and consider a solution $x=\varphi(t)=(\psi(t), \chi(t))$ of (3.2) with $\varphi(0)=x_0$. We shall prove that $\psi(T)>\varepsilon_0$ for some $T>0$. Suppose the contrary, i.e. $|\psi(t)|\leqq\varepsilon_0$ for all $t\geqq 0$. Then $v(t)\to v_0<V(x_0)<0$ as $t\to\infty$. By Lemma 3.1 in [1] and invariance property of $\Omega_x(\varphi)$, assumption (iii) implies (3.4). As it was shown in the proof of Theorem 3.1, from these facts it follows that the nonempty set $\Omega_y(\varphi)$ is a subset of $V_m^{-1}[v_0, \infty]_0$ (see (3.7)) and it is semiinvariant with respect to (3.3). Consequently, the set $V_m^{-1}[v_0, \infty]_0$ contains at least one complete trajectory of (3.3) in contradiction to assumption (iv) of the theorem. The proof is complete.

Remark 3.1. Let $y=(y_1, y_2)$ be a partition of $y\in R^m$ $(y_1\in R^{m_1}, y_2\in R^{m_2}$, $1\leqq m_1<m$, $m_1+m_2=m)$ and suppose that for some $\varepsilon_0>0$ the inequalities $|y|_1\leqq\varepsilon_0$, $V(y_1, y_2, z)<0$ imply $|y_2|\leqq H'$. Analysing the proof of Theorem 3.3 one can easily see that, in fact, in this case the zero solution of (3.2) is y_1-unstable.

As we shall see in the applications, we often have an estimate of the type $\dot{V}_{(3.2)}(y, z)\leqq U(y)$, which allows us to simplify the last condition in Theorems 3.1—3.3. In the following simple proposition even a slightly more general case is considered.

Proposition 3.1. *Suppose that for a Ljapunov function* $V: G'_y \to R$ *of* (3.2) *there exists a continuous function* $W: G'_y \to R$ *such that*

(i) $\dot{V}_{(3.2)}(y, z) \leqq W(y, z) \leqq 0 \quad ((y, z) \in G'_y)$;

(ii) $W(y, z) \to U(y)$ *uniformly in* $y \in \bar{B}_m(H')$ *as* $|z| \to \infty$.

Then for every $c \in R$,

$$E(c) = U^{-1}(0) \cap V_m^{-1}[c, \infty] \supset V_m^{-1}[c, \infty]_0.$$

4. An application

Consider again the holonomic mechanical system of r degrees of freedom described by the Lagrangian equation

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}} - \frac{\partial T}{\partial q} = -\frac{\partial P}{\partial q} + Q \quad (q, \dot{q} \in R^r), \tag{4.1}$$

where the following notations are used (see [1]): $P(q)$ is the potential energy $(P(0)=0)$, $T(q, \dot{q}) = (1/2)\dot{q}^T A(q)\dot{q}$ is the kinetic energy, and $Q(q, \dot{q})$ is the resultant of non-energic and dissipative forces with complete dissipation.

Let $q = \mathrm{col}\,(q_1, q_2)$ be a partition of the vector of generalized coordinates ($q_1 \in R^{r_1}$, $q_2 \in R^{r_2}$, $1 \leqq r_1 \leqq r$, $r_1 + r_2 = r$). Applying our results we give sufficient conditions for asymptotic stability and instability of the equilibrium $q = \dot{q} = 0$ (possibly non-isolated) with respect to the velocities $\dot{q}$ and coordinates q_1 in the case when the system is "asymptotically q_2-independent". It is worth emphasizing that the coordinates of q_2 are not supposed to be bounded along the motions.

The system (4.1) is defined to be *asymptotically* q_2*-independent* if for some constant $H' > 0$ and for every compact set $K \subset R^r$

(a) there are $\lambda > 0$ and $c \in \mathcal{K}$ such that

$$\lambda |\dot{q}|^2 \leqq \frac{1}{2}\dot{q}^T A(q_1, q_2)\dot{q}, \quad Q^T(q_1, q_2, \dot{q})\dot{q} \leqq -c(|\dot{q}|)$$

for all $q_1 \in \bar{B}_{r_1}(H')$, $q_2 \in R^{r_2}$, $\dot{q} \in R^r$;

(b) $A(q_1, q_2) \to A_*(q_1)$, $P(q_1, q_2) \to P_*(q_1)$ as $|q_2| \to \infty$; in addition, $Q(q_1, q_2, \dot{q}) \to Q_*(q_1, \dot{q})$ uniformly in $q_1 \in \bar{B}_{r_1}(H')$, $\dot{q} \in K$ as $|q_2| \to \infty$, as well as $\partial A/\partial q$, $\partial P/\partial q$ converge uniformly in $q_1 \in \bar{B}_{r_1}(H')$ as $|q_2| \to \infty$.

We are going to apply Theorems 3.2 and 3.3 while $z = q_2$ and V is the total mechanical energy. For this purpose we introduce the Hamiltonian variables $q, p = A(q)\dot{q}$, by the aid of which the system (4.1) can be rewritten in the form

$$\begin{aligned} \dot{p} &= -\frac{1}{2}p^T\left(\frac{\partial A^{-1}(q)}{\partial q}\right)p - \frac{\partial P}{\partial q} + Q(q, A^{-1}(q)p), \\ \dot{q} &= A^{-1}(q)p, \end{aligned} \tag{4.2}$$

In view of asymptotic q_2-independence, the equilibrium $q=\dot{q}=0$ of (4.1) and the zero solution $p=q=0$ of (4.2) have the same stability properties.

Consider the total mechanical energy H defined by $H=H(p, q_1, q_2)=T+P$. As is known (see [8], p. 358),

$$\dot{H}_{(4.2)}(p, q_1, q_2) = Q^T(q, A^{-1}(q)p)A^{-1}(q)p \leqq -d(|p|) \tag{4.3}$$

for all $(p, q_1)\in \bar{B}_{r_1+r}(H')$, $q_2\in R^{r_2}$ with a suitable $d\in\mathcal{K}$. Consequently, H is a Ljapunov function of (4.2), and

$$(\dot{H}_{(4.2)})^{-1}(0)\cap H^{-1}(c) = \{\mathrm{col}\,(p, q)\colon P(q) = c,\ p = 0\} \quad (c\in R), \tag{4.4}$$

so the trajectories of (4.2) contained in this set are the equilibria $p=0, q=q_0$ for which $P(q_0)=c$.

Now let us determine the set

$$E(c) = H_{r_1+r}^{-1}[c, \infty]\cap d^{-1}(0) = \{\mathrm{col}\,(p, q_1)\colon p = 0,\ q_1 = P_{r_1}^{-1}[c, \infty]\},$$

figuring in Proposition 3.1. Since $\partial P/\partial q_1$ is continuous and converges uniformly as $|q_2|\to\infty$, the function $P(\cdot, q_2)\colon \bar{B}_{r_1}(H')\to R$ is continuous uniformly in $q_2\in R^{r_2}$. From this fact it follows that

$$E(c) = \{\mathrm{col}\,(p, q_1)\colon p = 0,\ P_*(q_1) = c\}. \tag{4.5}$$

The system (4.2) is asymptotically q_2-independent, hence its limit system as $|q_2|\to\infty$ reads as follows:

$$\begin{aligned} \dot{p}^i &= -\frac{1}{2}p^T\left(A_*^{-1}\frac{\partial A_*}{\partial q^i}A_*^{-1}\right)p - \frac{\partial P_*}{\partial q^i} + Q_*^i(q_1, A_*^{-1}p) \\ \dot{p}^j &= Q_*^j(q_1, A_*^{-1}, p) \\ \dot{q}^i &= \sum_{k=1}^{r}[A_*^{-1}(q_1)]_{ik}p^k \end{aligned} \tag{4.6}$$

for $i=1, 2, \ldots, r_1$; $j=r_1+1, \ldots, r$. In view of (4.5), if $E(c)$ contains a trajectory of (4.6) then it is of the form $p=0$, $q_1=(q_1)_0=$const., furthermore

$$P_*((q_1)_0) = c, \quad \left.\frac{dP_*}{dq_1}\right|_{q_1=(q_1)_0} = 0. \tag{4.7}$$

Theorem 4.1. *Suppose that the mechanical system* (4.1) *is asymptotically q_2-independent.*

I. *If* (i) *the potential energy P is positive q_1-definite,* (ii) *system* (4.1) *has no equilibrium position in the region $\{(q_1, q_2)\colon P(q_1, q_2)>0,\ q_1\neq 0\}$, and* (iii) *the equality $dP_*(q_1)/dq_1=0$ implies either $q_1=0$ or $P_*(q_1)=0$, then the equilibrium $q=\dot{q}=0$ of* (4.1) *is asymptotically $(q_1, \dot{q})$-stable.*

II. *If* (i) *the potential energy* P *has no local minimum at* $q=0$, (ii) *the system* (4.1) *has no equilibrium position in the region* $\{q: P(q)<0\}$, *and* (iii) *the equality* $dP_*(q_1)/dq_1=0$ *implies* $P_*(q_1)\geqq 0$, *then the equilibrium* $q=\dot{q}=0$ *of* (4.1) *is* q_1*-unstable.*

Proof. I. We show that (4.2) and the total mechanical energy H as a Ljapunov function satisfy the conditions of Theorem 3.2. Condition (a) in the definition of the asymptotic q_2-independence and (i) assure H to be positive (q_1, p)-definite. In consequence of (4.4), for the system (4.2) condition (ii) precludes the possibility of having such a complete trajectory in the set $(\dot{H}_{(4.2)})^{-1}(0)\cap H^{-1}(c)$ $(c>0)$ that is not in $\{(q_1, q_2, p): q_1=0, p=0\}$. Finally, using (4.7), condition (ii), and Proposition 3.1 we obtain that the limit system (4.6) cannot have any trajectory in the set $H^{-1}_{r_1+r}[c, \infty]_0$ $(c>0)$ except the origin.

II. One can similarly check the conditions of Theorem 3.3, from which (q_1, p)-instability follows. According to Remark 3.1, for the purpose of proving q_1-instability it is enough to show that $|q_1|\leqq\varepsilon_0$, $H(q_1, p, q_2)<0$ imply $|p|\leqq M$ for some constants $\varepsilon_0>0$, M. Observe, that P is bounded below on the set $\bar{B}_{r_1}(\varepsilon_0)\times R^{r_2}$ because of q_2-independence. Therefore, T is bounded above, which together with (a) imply that p belongs to a bounded set. The proof is complete.

Concluding this section we note that in possession of Theorem 4.1 one can easily prove the conjecture made in the Introduction in connection with the motion of a material point on the surface $z=(1/2)y^2[1+1/(1+x^2)]$.

5. A generalization to nonautonomous systems

The LaSalle principle and the invariance property of limit sets with respect to the limiting equation, which served as the two main tools in the proofs of Section 3 have been extended to quite general nonautonomous systems. These extensions enabel us to generalize our results to the equation

$$\dot{x} = X(x, t) \quad (X(0, t) \equiv 0). \tag{5.1}$$

Namely, we give a theorem on the partial asymptotic stability of the zero solution of (5.1) without any assumptions on the boundedness of solutions. To formulate and prove it we need some concepts and results from topological dynamics given in [2]—[4]. The theorem will be followed by a corollary, containing only analytical conditions and, consequently, more suitable for applications.

As is known, (5.1) is equivalent to the integral equation $x(t)=x(a)+\int_a^t X(s,x(s))ds$, i.e. to the functional equation $x=x(a)+I_a x$, where the operator I_a is defined by

$I_a x(t)=\int_a^t X(s, x(s))ds$. In the method of limiting equation there occur such functional equations in which the operator I_a is more general than the integral with a kernel. An *ordinary integral-like operator* I is a mapping which associates with each continuous function $\varphi: [\alpha, \beta)\to R^k$ and $a\in[\alpha, \beta)$ a continuous function $I_a\varphi$ so that (1) if $\varphi_i: [\alpha, \beta)\to R^k$ are continuous and $\varphi_i(t)\to\varphi(t)$ uniformly, then $I_a\varphi_i(t)\to I_a\varphi(t)$ uniformly in $t\in[a, b]$, as $i\to\infty$ for all $[a, b]\subset[\alpha, \beta)$; (2) $I_a\varphi(t)=I_a\varphi(s)+I_s\varphi(t)$ for all $a, s, t\in[\alpha, \beta)$. We shall denote by $u=Iu$ the functional equation $u=u(a)+I_a u$ associated with the ordinary integral-like operator I.

For $t\in R_+$ we define the *translate* X^t of X by $X^t(x, s)=X(x, t+s)$ $(s\in R_+)$. We denote by $\operatorname{tran}(X)$ the collection of all translates X^t of X $(t\in R_+)$. An ordinary integral-like operator equation $u=Iu$ is a *limiting equation* of (5.1) if there exists a sequence $\{t_i\}$ converging to infinity so that X^{t_i} integrally converges to I as $i\to\infty$, i.e. whenever $\varphi_i: [a, b]\to R^k$ converges uniformly to φ then

$$\int_a^b X(\varphi_i(s), t_i+s)\,ds \to I_a\varphi(b) \quad (i\to\infty).$$

The set $\operatorname{tran}(X)$ is said to be *precompact* if every sequence in it has an integrally converging subsequence.

Theorem B. [4] *Suppose that* $\operatorname{tran}(X)$ *is precompact and* $\varphi: [t_0, \infty)\to R^n$ *is a solution of* (5.1). *Then* $\Omega_x(\varphi)$ *is semiinvariant with respect to the family of the limiting equations of* (5.1), *i.e. for each* $p\in\Omega_x(\varphi)$ *there is a limiting equation* $u=Iu$ *of* (5.1) *and a solution* γ *of the equation* $u=p+I_0u$ *so that* $\gamma(t)\in\Omega_x(\varphi)$ *for all* t *in the domain of* γ.

By our standard partition $x=(y, z)$ the system (5.1) can be written in the form

$$\dot y = Y(y, z, t), \quad \dot z = Z(y, z, t) \quad ((y, z, t)\in\Gamma_y). \tag{5.2}$$

Let $0<H'<H$.

Theorem 5.1. *Suppose that the right-hand sides of* (5.2) *satisfy the following conditions:*

(i) *for each compact set* $K\subset R^n$ *and continuous function* $\chi: R_+\to R^n$ *with* $|\chi(t)|\to\infty$ *as* $t\to\infty$, *there are functions* $p, q\in\mathscr{K}$ *so that for arbitrary continuous functions* $v: [a, b]\to\bar B_m(H')$, $w: [a, b]\to K$

$$\left|\int_a^b Y(v(t), \chi(t), t)\,dt\right| \leq p(b-a), \quad \left|\int_a^b X(v(t), w(t), t)\,dt\right| \leq q(b-a);$$

(ii) $\operatorname{tran}(X(x, t))$ *is precompact;*

(iii) $\operatorname{tran}(Y(y, \chi(t), t))$ *is precompact for every continuous function* $\chi: R_+\to R$ *with* $|\chi(t)|\to\infty$ *as* $t\to\infty$.

Suppose, in addition, that there is a positive y-definite Ljapunov function $V\colon \Gamma'_y \to R$ *of* (5.2) *having the following properties:*

(iv) *for each* $c>0$ *neither limiting equation of* (5.2) *has a positive semitrajectory in the set* $V_k^{-1}[c, \infty]_0$;

(v) *for each* $c>0$ *and continuous function* $\chi\colon R_+ \to R^n$ *such that* $|\chi(t)| \to \infty$ *as* $t\to\infty$, *neither limiting equation of* $\dot{y}=Y(y, \chi(t), t)$ *has a positive semitrajectory in the set* $V_m^{-1}[c, \infty]_0$ *different from* 0.

Then the zero solution of (5.2) *is asymptotically y-stable.*

Proof. The zero solution of (5.2) is y-stable (see [8], p. 15); therefore, it is sufficient to prove that if $x=\varphi(t)=(\psi(t), \chi(t))$ is a solution of (5.2) and $|\psi(t)| \leqq H''<H'$ for all $t\geqq t_0$, then $\psi(t)\to 0$ as $t\to\infty$.

Let us introduce the notations $v(t)=V(\varphi(t), t)$ and $v_0=\lim\limits_{t\to\infty} v(t)$. We distinguish two cases:

a) Assume that $|\chi(t)| \not\to \infty$ as $t\to\infty$. We show that in this case $v_0=0$, which implies $\psi(t)\to 0$ because V is positive y-definite.

The limit set $\Omega_x(\varphi)$ is not empty and, by Theorem A, $\Omega_x(\varphi)\subset V_k^{-1}[v_0, \infty]_0$. On the other hand, $\Omega_x(\varphi)$ is semiinvariant with respect to the family of the limiting equations of (5.2) (see Theorem B). Consequently, one of them has at least one positive semitrajectory in $V_k^{-1}[v_0, \infty]_0$. Thus, in view of (iv), $v_0=0$.

b) Let $|\chi(t)|\to\infty$ as $t\to\infty$. We show that either $v_0=0$ or $\Omega_y(\varphi)=\{0\}$.

Consider the equation

$$\dot{y} = Y(y, \chi(t), t) \quad (y\in \bar{B}_m(H'),\ t\in R_+) \tag{5.3}$$

and its Ljapunov function $U(y, t)=V(y, \chi(t), t)$. Using again Theorem A we obtain

$$\Omega_y(\varphi) = \Omega_y(\psi) \subset U_m^{-1}[v_0, \infty]_0 \subset V_m^{-1}[v_0, \infty]_0. \tag{5.4}$$

On the other hand, $\Omega_y(\varphi)$ as the limit set of the solution $y=\psi(t)$ of (5.3) is semiinvariant with respect to the family of the limiting equations of (5.3). If there is a $q\in\Omega_y(\varphi)$, $q\neq 0$, this means that one of the limiting equations of (5.3) has a positive semitrajectory different from $\{0\}$ which is a subset of $\Omega_y(\varphi)$. Then, according to (5.4) and hypothesis (v), $v_0=0$. The proof is complete.

Corollary 5.1. *Suppose that*

(i) *for each compact set* $K\subset R^n$ *there are locally integrable functions* $\mu_j, \nu_j\colon R_+ \to R_+$, $j=1, 2$ *so that the functions* $\int_0^t \mu_j(s)\,ds$ *are uniformly continuous on* R_+, *the functions* $\int_t^{t+1} \nu_j(s)ds$ *are bounded on* R_+, *and*

$$|Y(y, z, t)| \leqq \mu_1(t), \quad |Z(w, t)| \leqq \mu_2(t),$$
$$|Y(y, z, t)-Y(y', z, t)| \leqq \nu_1(t)|y-y'|, \quad |X(w, t)-X(w', t)| \leqq \nu_2(t)|w-w'|$$

for all $y, y' \in \bar{B}_m(H')$, $z \in R^n$, $w, w' \in \bar{B}_m(H) \times K$, $t \in R_+$. *Suppose, furthermore, that there is a positive y-definite Ljapunov function* $V: \Gamma'_y \to R$ *of* (5.2) *having the following properties:*

(ii) *if for a function* $X_*: \Gamma'_y \to R$ *there is a sequence* $\{t_i\}$ *so that* $t_i \to \infty$ *and*

$$\int_0^t X(x, s+t_i)\, ds \to \int_0^t X_*(x, s)\, ds \quad (i \to \infty)$$

for every fixed $(x, t) \in \Gamma'_y$, *moreover, if* $c>0$, *then the set* $V_k^{-1}[c, \infty]_0$ *contains no positive semitrajectory of the equation* $\dot{x} = X_*(x, t)$;

(iii) *if for a function* $Y_*: \bar{B}_m(H') \times R_+ \to R^m$ *there exist a sequence* $\{t_i\}$ *and a continuous function* $\chi: R_+ \to R^n$ *so that* $t_i \to \infty$ $(i \to \infty)$, $|\chi(t)| \to \infty$ $(t \to \infty)$ *and*

$$\int_0^t Y(y, \chi(s+t_i), s+t_i)\, ds \to \int_0^t Y_*(y, s)\, ds \quad (i \to \infty)$$

for every fixed $(y, t) \in \bar{B}_m(H') \times R_+$, *moreover, if* $c>0$, *then the set* $V_m^{-1}[c, \infty]_0$ *contains no positive semitrajectory of the equation* $\dot{y} = Y_*(y, t)$ *except the origin* $y=0$.

Then the zero solution of (5.2) *is asymptotically y-stable.*

Proof. As it follows from [2] (Theorem 4.1), under assumption (i) both $\operatorname{tran}(X(x, t))$ and $\operatorname{tran}(Y(y, \chi(t), t))$ are precompact, and all the limiting equations are ordinary differential equations whose right-hand sides are the almost-everywhere derivatives of

$$\lim_{i \to \infty} \int_0^t X(x, s+t_i)\, ds, \quad \lim_{i \to \infty} \int_0^t Y(y, \chi(s+t_i), s+t_i)\, ds,$$

respectively. This means that all assumptions of Theorem 5.1 are satisfied.

Theorem 5.1 can be used for the case when $X(x, t)$ is periodic in t. For example, if we assume that $Y(y, z, t) \to Y_*(y, t)$ uniformly in $(y, t) \in \bar{B}_m(H') \times R_+$ as $|z| \to \infty$, then both $\operatorname{tran}(X(x, t))$ and $\operatorname{tran}(Y(y, \chi(t), t))$ are precompact, and the limiting equations read

$$\dot{x} = X(x, t+t_0), \quad \dot{y} = Y_*(y, t+t_0),$$

respectively.

Remark 5.1. Suppose assumptions (i), (ii), (iv) in Theorem 5.1 to be satisfied. Suppose, in addition, that

(v′) for every continuous function $\chi: R_+ \to R^n$, for which $|\chi(t)| \to \infty$ as $t \to \infty$, there is a limiting equation $u = Ju$ of $Y(y, \chi(t), t)$ so that for every $c>0$ the set $V_m^{-1}[c, \infty]_0$ contains no positive trajectory of $u = u(0) + J_0 u$.

Then the zero solution of (5.2) is equiasymptotically y-stable, i.e. it is y-stable and for every $t_0 \in R_+$ there is a $\sigma(t_0)>0$ such that $|y(t; x_0, t_0)| \to 0$ uniformly in $x_0 \in \bar{B}_k(\sigma(t_0))$ as $t \to \infty$.

To show this we have to modify only part b) of the proof of Theorem 5.1. Namely, we prove that also in this case $v_0=0$. After proving (5.4) consider the limiting equation

$$u = u(0) + J_0 u. \tag{5.5}$$

For a sequence $\{t_i\}$ the sequence of translates $Y^{t_i}(y, \chi(t), t)$ tends to J integrally as $i \to \infty$. From assumption (i) it follows that the functions $\{\psi_i(t)=\psi(t+t_i)\}$ being solutions of the equations $\dot{y}=Y^{t_i}(y, \chi(t), t)$ are uniformly bounded and equicontinuous on every fixed interval $[0, T]$. By Arzela—Ascoli theorem, we can assume that $\psi_i \to \psi_*$ uniformly on $[0, T]$, thus ψ_* is a solution of (5.5). Obviously, $\psi_*(t) \in \Omega_y(\varphi)$ for all $t \geqq 0$. According to (5.4) and assumption (v′), $v_0=0$.

So we have proved that $V(x(t; x_0, t_0)) \to 0$ as $t \to \infty$ for every fixed $t_0 \in R_+$ and for all x_0 with sufficiently small $|x_0|$. By the classic covering theorem of Heine—Borel—Lebesgue, this convergence is uniform with respect to x_0 [9], which implies equiasymptotic y-stability since V is positive y-definite.

Remark 5.2. The statement in Remark 5.1 remains valid if assumption (v′) is weakened so that $V_m^{-1}[c, \infty]_0$ contains no positive semitrajectory of the limiting equation $u=u(0)+J_0 u$ except the origin $y=0$, but it is supposed, in addition, that $V(y, z, t) \to 0$ uniformly in $(z, t) \in R^n \times R_+$ as $y \to 0$.

To see this one has to observe only that the additional condition on V obviously precludes the possibility of $\psi_*(t) \equiv 0$ for the function $\psi_*(t)$ occurring in the argument in Remark 5.1.

These two remarks make it easier to see that our result generalizes and improves the main theorem of [10].

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On the stability and convergence of solutions of differential equations by Liapunov's direct method

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1. Introduction

By means of a modification of Liapunov's direct method we give sufficient conditions for the stability of solutions of ordinary differential equations and for the existence of finite limits of certain functions (specially, of a part of coordinates) along solutions as $t\to\infty$. For the study of this problem, T. A. BURTON [2], J. R. HADDOCK [5, 6] and L. HATVANI [8, 13] used modifications in which the derivative of the Liapunov function was estimated by the norm of a linear combination of components of the right-hand side of the system. T. A. BURTON [3] has extended this method for the estimate in which a power of a linear combination of the right-hand sides occurs. In this paper we investigate the case when the estimate contains a monotone function of a linear combination of the right-hand sides. We apply our results to studying the asymptotic behaviour of solutions of certain second order non-linear differential equations and the stability properties of motions of mechanical systems under the action of potential and dissipative forces depending also on the time.

2. The main results

Consider the differential system

$$\dot{x}(t) = X(t, x), \tag{2.1}$$

where $t\in R_+=[0, \infty)$, x belongs to the n-dimensional Euclidean space R^n, $X\in C(R_+\times\Gamma, R^n)$; $\Gamma\subset R^n$ is an open set.

Let us introduce some notations. Denote by (x, y) the scalar product of vectors $x, y\in R^n$. $\|x\|=(x, x)^{1/2}$ is the norm of the vector $x\in R^n$. Let B_H denote the set of elements $x\in R^n$ such that $\|x\|<H$ $(H>0)$. The distance $\varrho(H_1, H_2)$

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between the sets $H_1, H_2 \subset R^n$ is defined by

$$\varrho(H_1, H_2) = \inf\{\|x-y\|:\ x\in H_1,\ y\in H_2\}.$$

$\bar{H}$ denotes the closure of the set H. Let K denote the class of increasing functions $a\in C(R_+, R_+)$ for which $a(0)=0$ and $a(s)>0$ for all $s>0$. Denote by L^+ the class of Lebesgue measurable functions $f: R_+ \to R_+ \cup \{\infty\}$, by L_p^+ $(0<p<\infty)$ and L_∞^+ the classes of the functions $f\in L^+$ with

$$\int_0^\infty f^p(s)\,ds < \infty, \quad \sup_{s\in R_+} \operatorname{ess} f(s) < \infty,$$

respectively. Let $u(t;\, t_0, u_0)$ be the maximal noncontinuable solution of the equation

(2.2) $$\dot{u} = r(t, u)$$

through (t_0, u_0), where $r\in C(R_+\times R_+, R_+)$.

Let us given a function $\omega\in C(R_+\times R_+, R_+)$ with $\omega(t, \cdot)\in K$. In the sequel we shall often have to solve an inequality of type $\omega(t, f(t)) \leqq g(t)$ for the function f. This motivates the following notations:

$$\omega(t, \infty) = \lim_{u\to\infty} \omega(t, u) \quad (\leqq \infty),$$

$$\omega^{-1}(t, v) = \max\{u:\ \omega(t, u) \leqq v\},$$

$$\omega^{-1}\big(t, w(t, \infty)\big) = \infty.$$

The function $\omega^{-1}(t, v)$ is defined for $t\in R_+$, $0\leqq v\leqq \omega(t, \infty)$, it is increasing in u, continuous on the right and satisfies the inequality

$$\omega^{-1}\big(t, \omega(t, u)\big) \geqq u \quad (t\in R_+,\ u\in R_+).$$

For every δ $(0<\delta\leqq\infty)$ denote by D_δ the set of functions $f\in L^+$ for which $f(t)\leqq \leqq\omega(t, \delta)$ $(t\in R_+)$, and define the map $\Omega_\delta: D_\delta \to L^+$ by

$$(\Omega_\delta f)(t) = \omega^{-1}\big(t, f(t)\big) \quad (t\in R_+, f\in D_\delta).$$

For a function $V\in C^1(R_+\times\Gamma', R^k)$ $(\Gamma'\subset\Gamma)$ we define the derivative $\dot{V}\in \in C(R_+\times\Gamma', R^k)$ of the function V with respect to (2.1) as follows

$$\dot{V}(t, x) = \frac{\partial V(t, x)}{\partial t} + \frac{\partial V(t, x)}{\partial x} X(t, x) \quad (t\in R_+,\ x\in\Gamma').$$

Obviously, if $x(t)$ is a solution of equation (2.1), then

$$\frac{d}{dt} V\big(t, x(t)\big) = \dot{V}\big(t, x(t)\big).$$

Let us given a function $W \in C^1(R_+ \times \Gamma, R^k)$. In the sequel we examine the asymptotic behavior of W along solutions of (2.1), i.e. the asymptotic behavior of the function $W(t, x(t))$. In the following theorem we use the set $\bigcap_{t \geqq 0} \overline{W([t, \infty), \Gamma)}$, which consists of all $w \in R^k$ for which there exist sequences $\{t_i\}$, $\{x_i\}$ with $x_i \in \Gamma$, $t_i \to \infty$, $W(t_i, x_i) \to w$ as $i \to \infty$.

Theorem 2.1. *Suppose that for each* $w_1, w_2 \in \bigcap_{t \geqq 0} \overline{W([t, \infty), \Gamma)}$ *there exist functions* $V \in C^1(R_+ \times \Gamma, R_+)$, $r, r_1, \omega \in C(R_+ \times R_+, R_+)$, *open sets* $H_1, H_2 \subset R^k$ *and a constant* $T > 0$ *satisfying the following conditions:*

(A) $w_1 \in H_1, w_2 \in H_2, \varrho(H_1, H_2) > 0$;

(B) $r(t, u)$ *is increasing in* u *and the solutions of equation* (2.2) *are bounded;*

(C) $r_1(t, u)$ *is increasing in* u *and* $r_1(\cdot, u) \in L_1^+$ $(u \in R_+)$;

(D) $\omega(t, \cdot) \in K$ $(t \in R_+)$ *and* Ω_∞ *maps* $D_\infty \cap L_1^+$ *into* L_1^+;

(E) $\dot{V}(t, x) \leqq r(t, V(t, x))$ $(t \in R_+, x \in \Gamma)$;

(F) $\dot{V}(t, x) \leqq -\omega(t, \|\dot{W}(t, x)\|) + r_1(t, V(t, x))$

for all (t, x) *such that* $t \geqq T$, $x \in \Gamma$, $W(t, x) \notin \bar{H}_1 \cup \bar{H}_2$.

Then for every solution $x(t)$ *of* (2.1) *defined on* $[t_0, \infty)$ *either* $\|W(t, x(t))\| \to \infty$ *or* $W(t, x(t)) \to$const. *as* $t \to \infty$.

Proof. First of all, observe that

$$r(\cdot, u_0) \in L_1^+ \quad (u_0 \in R_+). \tag{2.3}$$

Indeed, let $u_0 \in R_+$. By virtue of the monotonicity of $r(t, u)$ in u we have

$$\dot{u}(t; t_0, u_0) = r(t, u(t; t_0, u_0)) \geqq r(t, u_0);$$

therefore, assertion (2.3) holds.

Now, consider a solution $x: [t_0, \infty) \to R^n$ of (2.1) and put $w(t) = W(t, x(t))$. Suppose that the assertion of the theorem is not true, i.e., there exist two distinct elements w_1, w_2 of the set $\bigcap_{t \geqq t_0} \overline{w([t, \infty))}$. Consider some sets H_1, H_2, functions V, r, r_1, ω and some constant T corresponding to w_1, w_2 in the sense of the assumptions of the theorem.

By the basic theorem on differential inequalities, from assumptions (B) and (E) we obtain the estimate

$$V(t, x(t)) \leqq u(t; t_0, V(t_0, x_0)) \leqq C = \text{const} \quad (t \in [t_0, \infty)).$$

So,

$$\frac{d}{dt}\Big(V(t, x(t)) + \int_t^\infty r(s, C)\, ds\Big) = \dot{V}(t, x(t)) - r(t, C) \leqq 0,$$

consequently,

$$f(t) = r(t, C) - \dot{V}(t, x(t)) \in L_1^+.$$

Since $w_1, w_2 \in \bigcap_{t \geqq t_0} \overline{w([t, \infty))}$, there exist two sequences $\{t_i\}$, $\{t_i^*\}$ such that

$$T \leqq t_i < t_i^* < t_{i+1} \quad (i = 1, 2, \ldots), \quad \lim_{i\to\infty} t_i = \infty; \tag{2.3}$$

$$w(t_i) \in \overline{H}_1, \quad w(t_i^*) \in \overline{H}_2 \quad (i = 1, 2, \ldots),$$

$$w(t) \notin \overline{H}_1 \cup \overline{H}_2 \quad \left(t \in \bigcup_{i=1}^{\infty} (t_i, t_i^*)\right).$$

Introduce the notation

$$g(t) = \max\left(0, \min\left(\omega(t, \infty), r_1(t, C) - \dot{V}(t, x(t))\right)\right).$$

Then by condition (F) we have

$$g(t) \geqq \omega\left(t, \|\dot{w}(t)\|\right) \quad \left(t \in \bigcup_{i=1}^{\infty} (t_i, t_i^*)\right).$$

So,

$$\|\dot{w}(t)\| \leqq \omega^{-1}\left(t, g(t)\right) \quad \left(t \in \bigcup_{i=1}^{\infty} (t_i, t_i^*)\right).$$

Therefore,

$$N\varrho(H_1, H_2) \leqq \sum_{i=1}^{N} \|w(t_i) - w(t_i^*)\| =$$

$$= \sum_{i=1}^{N} \left\| \int_{t_i}^{t_i^*} \dot{w}(t)\, dt \right\| \leqq \sum_{i=1}^{N} \int_{t_i}^{t_i^*} \omega^{-1}\left(t, g(t)\right) dt.$$

This means that $\omega^{-1}(\cdot, g(\cdot)) \notin L_1$. Consequently, by condition (D), $g \notin L_1^+$.

On the other hand, we have

$$g(t) \leqq r_1(t, C) - \dot{V}\left(t, x(t)\right) \leqq f(t) + r_1(t, C)$$

for all t such that $r_1(t, C) - \dot{V}\left(t, x(t)\right) \geqq 0$. By virtue of $f(t) \geqq 0$, $r_1(t, C) \geqq 0$ we have

$$g(t) \leqq f(t) + r_1(t, C) \quad (t \in R_+),$$

which contradicts $f, r_1(\cdot, C) \in L_1^+$. The theorem is proved.

Theorem 2.2. *Suppose that there exist functions* $V \in C^1(R_+ \times \Gamma, R_+)$, $r, \omega \in C(R_+ \times R_+, R_+)$ *such that assumptions* (B), (D) *and*

$$\dot{V}(t, x) \leqq -\omega\left(t, \|\dot{W}(t, x)\|\right) + r\left(t, V(t, x)\right) \quad (t \in R_+,\ x \in \Gamma) \tag{F$_1$}$$

are fulfilled. Then $W\left(t, x(t)\right) \to \text{const.}$ *as* $t \to \infty$ *for every solution* $x(t)$ *of* (2.1) *defined on* $[t_0, \infty)$.

Proof. By Theorem 2.1, it is sufficient to show that $w(t) = W\left(t, x(t)\right)$ is bounded for every solution of (2.1) defined on $[t_0, \infty)$.

Suppose the contrary. Then there exist two sequences $\{t_i\}$, $\{t_i^*\}$ and a natural number $M>0$ such that

$$T \leqq t_i < t_i^* \leqq t_{i+1} \quad (i = 1, 2, \dots), \quad \lim_{i\to\infty} t_i = \infty,$$
$$\|w(t_i)\| = i, \ \|w(t_i^*)\| = i+1 \quad (i = M, M+1, \dots),$$
$$i < \|w(t)\| < i+1, \ t\in(t_i, t_i^*) \quad (i = M, M+1, \dots),$$

are fulfilled. So

$$N \leqq \sum_{i=M}^{N+M} (\|w(t_i^*)\| - \|w(t_i)\|) =$$
$$= \sum_{i=M}^{M+N} \int_{t_i}^{t_i^*} \frac{d}{dt}\|w(t)\|\,dt \leqq \sum_{i=M}^{M+N} \int_{t_i}^{t_i^*} \|\dot w(t)\|\,dt.$$

Hence, by virtue of (F_1) we have

$$N \leqq \sum_{i=M}^{M+N} \int_{t_i}^{t_i^*} \omega^{-1}(t, g_1(t))\,dt \leqq \int_{t_M}^{t_{M+N}^*} \omega^{-1}(t, g_1(t))\,dt,$$

where

$$g_1(t) = \min\big(\omega(t, \infty),\ r\big(t, \sup_{t\geqq T} V(t, x(t))\big) - \dot V(t, x(t))\big).$$

This inequality contradicts $g_1\in L_1^+$, which concludes the proof.

Theorem 2.3. *Let $0\in\Gamma$ and $X(t,0)\equiv 0$ for all $t\in R_+$. Suppose there exist functions $a, b\in K$, $V\in C^1(R_+\times B_H, R_+)$ $(B_H\subset\Gamma)$, $\omega, r\in C(R_+\times R_+, R_+)$ such that*

(B_1) *$r(t,0)=0$ for all $t\in R_+$, $r(\cdot, u)\in L_1^+$ for all $u>0$, $r(t,u)$ is increasing in u and the zero solution of equation (2.2) is unique;*

(D_1) *$\omega(t,\cdot)\in K$ $(t\in R_+)$ and the map $\Omega_\infty\colon D_\infty\cap L_1^+ \to L_1^+$ is continuous at $u(t)\equiv 0$ in L_1-norm;*

(F_2) *$\dot V(t,x) \leqq -a(\|W(t,x)\|)\omega(t, \|\dot W(t,x)\|) + r\big(t, V(t,x)\big)$ for all $t\in R_+, x\in B_H$;*

(G) *$V(t,0)=0$, $W(t,0)=0$ for all $t\in R_+$ and $b(\|x\|)\leqq V(t,x)+\|W(t,x)\|$ $(t\in R_+, x\in B_H)$.*

Then the zero solution of equation (2.1) is stable, and for every solution $x(t)$ of (2.1) with sufficiently small $\|x(t_0)\|$ the function $W\big(t, x(t)\big)$ has a finite limit as $t\to\infty$.

Proof. We first prove that the zero solution of equation (2.2) is stable. Suppose the contrary. Then there exist a number $\varepsilon_0>0$, sequences $\{u_i\}$, $\{t_i\}$ of solutions of (2.2) and positive numbers, respectively, such that

$$u_i(0) \to 0 \quad \text{as} \quad i\to\infty,$$
$$u_i(t_i) = \varepsilon_0, \quad u_i(t) < \varepsilon_0 \quad (0 \leqq t < t_i,\ i = 1, 2, \dots).$$

Define

$$r_i(t) = \begin{cases} r(t, u_i(t)) & 0 \leqq t \leqq t_i, \\ 0 & t_i \leqq t. \end{cases}$$

By virtue of (B_1) we have

$$0 \leqq r_i(t) \leqq r(t, \varepsilon_0) \quad (i = 1, 2, \dots), \quad r_i(t) \to 0 \quad \text{as} \quad i \to \infty \quad (t \in R_+).$$

Applying Lebesgue's dominated convergence theorem we obtain

$$\int_0^{t_i} r(t, u_i(t))\, dt = \int_0^{\infty} r_i(t)\, dt \to 0 \quad \text{as} \quad i \to \infty.$$

By integration of (2.2) it follows that

$$\varepsilon_0 - u_i(0) = \int_0^{t_i} r(t, u_i(t))\, dt.$$

Hence, if $i \to \infty$, we get $\varepsilon_0 = 0$, which is a contradiction. Consequently, the zero solution of (2.2) is stable.

Let us denote by $\delta_1(\varepsilon)$, $\delta_2(\varepsilon)$ the numbers corresponding to ε in the definition of stability of the zero solution of (2.2) and in the definition of continuity of Ω_∞, respectively. Let $0 < \varepsilon < H$, $t_0 \in R_+$ be fixed arbitrarily. Choose ε_1 so that

$$\varepsilon_1 < b(\varepsilon), \quad \int_0^{\infty} r(t, \varepsilon_1)\, dt + \varepsilon_1 < \delta_2\left(\frac{b(\varepsilon) - \varepsilon_1}{2}\right) a\left(\frac{b(\varepsilon) - \varepsilon_1}{2}\right) \tag{2.4}$$

and define $\delta = \delta(\varepsilon, t_0)$ such that $0 < \delta < \frac{\varepsilon}{2}$ and $\|x_0\| < \delta$ imply

$$V(t_0, x_0) < \delta_1(\varepsilon_1), \quad \|W(t_0, x_0)\| < (b(\varepsilon) - \varepsilon_1)/2. \tag{2.5}$$

Consider a solution $x(t)$ of (2.1) with $\|x(t_0)\| < \delta$. Denote by $[t_0, A)$ the maximal interval to the right in which $\|x(t)\| < H$ is true. By assumption (F_2) we have

$$\dot V(t, x(t)) \leqq r(t, V(t, x(t))) \quad (t \in [t_0, A)),$$

hence and from (2.5) it follows

$$V(t, x(t)) \leqq u(t, V(t_0, x(t_0))) \leqq \varepsilon_1 \quad (t \in [t_0, A)).$$

We show that the inequality $\|x(t)\| < \varepsilon$ also is satisfied for $t \in [t_0, A)$. Otherwise there exists a $T \in (t_0, A)$ such that $\|x(T)\| = \varepsilon$. Consequently,

$$\|W(T, x(T))\| \geqq b(\|x(T)\|) - V(T, x(T)) \geqq b(\varepsilon) - \varepsilon_1.$$

So, by (2.5) there are $t_1, t_2 \in (t_0, A)$ such that the function $w(t) = W(t, x(t))$ satisfies

$$\|w(t_1)\| = (b(\varepsilon) - \varepsilon_1)/2, \quad \|w(t_2)\| = b(\varepsilon) - \varepsilon_1,$$

$$(b(\varepsilon) - \varepsilon_1)/2 < \|w(t)\| < b(\varepsilon) - \varepsilon_1 \quad (t \in (t_1, t_2)).$$

Using assumption (F_2), we obtain

$$\|\dot{w}(t)\| \leqq \omega^{-1}(t, u(t)) \quad (t\in(t_1, t_2))$$

where

$$u(t) = \min\left(\omega(t, \infty), \frac{r(t, \varepsilon_1) - \dot{V}(t, x(t))}{a((b(\varepsilon)-\varepsilon_1)/2)}\right).$$

By integration over (t_1, t_2) this implies that

$$\int_{t_1}^{t_2} \omega^{-1}(t, u(t))\,dt \geqq (b(\varepsilon)-\varepsilon_1)/2. \tag{2.6}$$

On the other hand, from (2.4) it follows that

$$\int_{t_1}^{t_2} u(t)\,dt \leqq \left(\int_{t_1}^{t_2} r(t, \varepsilon_1)\,dt + V(t_1, x(t_1)) - V(t_2, x(t_2))\right)\frac{1}{a((b(\varepsilon)-\varepsilon_1)/2)} \leqq$$

$$\leqq \frac{1}{a((b(\varepsilon)-\varepsilon_1)/2)}\left(\int_{t_1}^{\infty} r(t, \varepsilon_1)\,dt + V(t_1, x(t_1))\right) < \delta_2((b(\varepsilon)-\varepsilon_1)/2),$$

which contradicts (2.6). This means that $\|x(t)\| < \varepsilon$ is satisfied for all $t\in[t_0, A)$. Therefore, $A=\infty$ and the zero solution is stable.

The other statements of the theorem follows from Theorem 2.1.

Remark 2.1. If we put $W(t, x)=(x_1, \dots, x_k)$ $(1\leqq k\leqq n)$, where $x_1, \dots, x_k, \dots, x_n$ are the components of the vector x, then our theorems with

$$\|\dot{W}(t, x)\| = \left(\sum_{i=1}^{k} X_i^2(t, x)\right)^{1/2}$$

yield conditions on the convergence of the components $x_1, \dots, x_k$ along solutions.

Remark 2.2. If

$$V(t, x) + \|W(t, x)\| \to \infty \quad \text{as,} \quad x \to R^n\setminus\Gamma \quad \text{or} \quad \|x\| \to \infty$$

for every $t\in R_+$, then under the assumptions of Theorem 2.2 every solution of equation (2.1) can be continued to $[t_0, \infty)$.

Remark 2.3. If there exists $d\in K$ such that

$$\|\dot{W}(t, x)\| \leqq d(\|x\|) \quad t\in R_+, \ x\in B_H)$$

then in Theorem 2.3 assumption (D_1) may be replaced by the following:

(D_2) $\omega(t, \cdot)\in K$ $(t\in R_+)$ and the map $\Omega_\delta\colon D_\delta\cap L_1^+ \to L_1^+$ is continuous at $u(t)\equiv 0$ in L_1-norm for some $\delta>0$.

In the following we give realization of assumptions (D), (D_1), (D_2) in some important special cases. Let $N(u)$ be a continuous convex function which satisfies the following conditions:

$$N \in K, \quad \lim_{u\to\infty} \frac{N(u)}{u} = 0, \quad \lim_{u\to\infty} \frac{N(u)}{u} = \infty.$$

Put

$$M(u) = \int_0^u \sup\left\{t: \frac{d}{dt} N(t) \leqq s\right\} ds.$$

If $s(t), r(t)$ are measurable on $[0, T]$ and

$$\int_0^T N(s(t))\,dt < \infty, \quad \int_0^T M(r(t))\,dt < \infty$$

then, by the generalized Hölder inequality (see [10], p. 222—233) the function $s(t)\cdot r(t)$ is integrable and

$$\int_0^T s(t)r(t)\,dt \leqq \Big(1+\int_0^T N(s(t))\,dt\Big)\Big(1+\int_0^T M(r(t))\,dt\Big). \tag{2.7}$$

Lemma 2.1. *Let a continuous function $\lambda(t) \geqq 0$ satisfy the inequality*

$$\int_0^\infty M\left(\frac{1}{\lambda(t)}\right) dt < \infty.$$

If $\omega(t, u)$ is defined by $\omega(t, u) = N(\lambda(t)u)$ $(t \in R_+, u \in R_+)$ then (D) *is satisfied.*

Proof. It is easy to see that $\lambda(t) > 0$ almost everywhere, and

$$\omega(t, \infty) = \begin{cases} \infty, & \lambda(t) > 0, \\ 0, & \lambda(t) = 0, \end{cases}$$

$$\omega^{-1}(t, u) = \frac{N^{-1}(u)}{\lambda(t)} \quad (\lambda(t) > 0,\ u \in R_+)$$

Let $u \in L_1^+ \cap D_\infty$. Applying inequality (2.7) we have

$$\int_0^T \omega^{-1}(t, u(t))\,dt = \int_{\substack{\lambda(t)>0\\ t<T}} \frac{N^{-1}(u(t))}{\lambda(t)}\,dt \leqq$$

$$\leqq \Big(1+\int_0^T N(N^{-1}(u(t)))\,dt\Big)\left(1+\int_0^T M\left(\frac{1}{\lambda(t)}\right) dt\right) \leqq$$

$$\leqq \Big(1+\int_0^\infty u(t)\,dt\Big)\left(1+\int_0^\infty M\left(\frac{1}{\lambda(t)}\right) dt\right) < \infty$$

for all $T > 0$. So, $\int_0^\infty \omega^{-1}(t, u(t))\,dt < \infty$ which was to be proved.

Remark 2.4. If $\omega(t,u)=\mu(t)u^\alpha$ $(t\in R_+, u\in R_+)$, where $1<\alpha=\text{const.}$, $\mu\in C(R_+, R_+)$, $1/\mu\in L^+_{1/(\alpha-1)}$ then assumptions (D), (D_1) are satisfied.

This assertion follows from the ordinary Hölder inequality. T. A. BURTON [3] considered this case studying the boundedness and the existence of the limit of solutions.

Obviously, if $\omega(t,u)=\mu(t)u$ where $\mu\in C(R_+;[c,\infty))$ and $0<c=\text{const.}$, then (D), (D_1) are satisfied. This case was studied in [2, 5, 6, 7, 13].

Lemma 2.2. *Let g be a continuous strictly increasing function such that*

$$\lim_{u\to\infty} g(u)=\infty,\quad g(u)\geqq cu^\nu \quad (0\leqq u\leqq u_0),$$

where $c>0$, $\nu\geqq 1$ are some constants. Let us choose a continuous function $\lambda(t)$ such that $1/\lambda\in L^+_{1/(\nu-1)}\cap L^+_\infty$ and put $\omega(t,u)=\lambda(t)g(u)$. Then ($D_2$) is satisfied. Moreover, if

$$0<\liminf_{u\to\infty}\frac{g(u)}{u} \tag{2.8}$$

then (D_1) is also true.

Proof. The assumptions imply

$$\lambda(t)\geqq c_1=\text{const.}>0\quad (t\in R_+),\quad \omega(t,\infty)=\infty\quad (t\in R_+),$$

$$\omega^{-1}(t,v)=g^{-1}(v/\lambda(t))\ (v\in R_+,\ t\in R_+),\quad g^{-1}(v)\leqq (v/c)^{1/\nu}\ (0\leqq v\leqq g(u_0)).$$

Let $u\in L^+_1\cap D_\delta$. Then, for $\nu>1$ by means of Hölder inequality we obtain

$$\int_{u(t)\leqq c_1 g(u_0)}\omega^{-1}(t,u(t))\,dt\leqq\frac{1}{c^{1/\nu}}\int_{u(t)\leqq c_1 g(u_0)}\left(\frac{u(t)}{\lambda(t)}\right)^{1/\nu}dt\leqq$$

$$\leqq\frac{1}{c^{1/\nu}}\Big(\int_0^\infty u(t)\,dt\Big)^{1/\nu}\Big(\int_0^\infty(\lambda(t))^{1/(1-\nu)}\,dt\Big)^{\nu/(\nu-1)}$$

and

$$\int_{c_1 g(u_0)\leqq u(t)}\omega^{-1}(t,u(t))\,dt\leqq\int_{c_1 g(u_0)\leqq u(t)}g^{-1}(g(\delta))\,dt\leqq\frac{\delta}{c_1 g(u_0)}\int_0^\infty u(t)\,dt.$$

Consequently,

$$\int_0^\infty\omega^{-1}(t,u(t))\,dt\leqq c_2\Big(\int_0^\infty u(t)\,dt\Big)^{1/\nu}+c_3\int_0^\infty u(t)\,dt$$

for some $c_2, c_3>0$. This inequality is obvious for $\nu=1$, therefore (D_2) is satisfied, indeed.

By (2.8) there exist positive constants K and u_1 such that $g^{-1}(u) \leqq Ku$ $(u_1 \leqq u)$. If $u \in L_1^+$, then

$$\int_{u_1 \leqq u(t)} \omega^{-1}(t, u(t))\, dt \leqq \frac{K}{c_1} \int_0^\infty u(t)\, dt,$$

$$\int_{u_0 \leqq u(t) \leqq u_1} \omega^{-1}(t, u(t))\, dt \leqq \frac{g^{-1}(u_1/c)}{u_0} \int_0^\infty u(t)\, dt,$$

so, using the preceding argument, it is easy to verify assumption (D_1).

Example 2.1. Let us define

$$\omega(t, u) = \begin{cases} \lambda(t) \exp[\log^3 u], & u > 0 \\ 0, & u = 0 \end{cases} \quad (t, u \in R_+),$$

where $\lambda(t)$ is continuous, $\lambda(t) \geqq c = \text{const.} > 0$ and

$$\int_0^\infty \exp\left[\log^{1/3} \frac{ce}{\lambda(t)}\right] dt < \infty$$

(e.g., $\lambda(t) = \exp[t^3]$ or $\exp[\delta \log^3(1+t)]$, where $\delta > 1$). Then (D) is satisfied. Indeed,

$$\omega^{-1}(t, u) = \exp\left[\log^{1/3} \frac{u}{\lambda(t)}\right] \quad (t \in R_+,\ u \in R_+),$$

and if $u \in L_1^+$, then

$$\int_0^\infty \omega^{-1}(t, u(t))\, dt \leqq \int_{u(t) \leqq ce} \exp\left[\log^{1/3} \frac{ce}{\lambda(t)}\right] dt + \int_{u(t) \geqq ce} \exp\left[\log^{1/3} \frac{u(t)}{\lambda(t)}\right] dt$$

$$\leqq \int_0^\infty \exp\left[\log^{1/3} \frac{ce}{\lambda(t)}\right] dt + \frac{1}{c} \int_0^\infty u(t)\, dt < \infty.$$

3. Applications to second order differential equations and mechanical system

I. Consider the differential equation

$$(p(t)\dot{x})^{\cdot} + q(t) g(x) = 0, \tag{3.1}$$

where $p, q \in C^1(R_+, R_+)$, $g \in C(R, R)$, $p(t) > 0$, $q(t) > 0$ $(t \in R_+)$, $\int_0^x g(u)\, du \geqq 0$ $(x \in R)$. Attractivity and asymptotic stability of the trivial solution $x = \dot{x} = 0$ have been studied by many authors under the assumption that $x = 0$ is an isolated solution of the equation $g(x) = 0$ [8, 9, 12]. Now we are going to apply Theorem 2.2, 2.3

to get sufficient conditions for the existence of $\lim_{x\to\infty} x(t)$ in the case when $x=0$ is, possibly, a non-isolated solution of $g(x)=0$.

By introducing the variable $y=p(t)\dot{x}$, equation (3.1) can be written in the form

(3.2) $$\dot{x} = y/p(t), \quad \dot{y} = -q(t)g(x).$$

For this equation let us choose the Liapunov function

$$V(t, x, y) = \frac{\varrho(t)}{2p(t)} y^2 + \varrho(t)\, q(t) \int_0^x g(u)\, du,$$

where $\varrho \in C^1(R_+, R_+ \setminus \{0\})$. The derivative of V with respect to (3.2) reads as follows:

$$\dot{V}(t, x, y) = \left(\frac{\varrho(t)}{p(t)}\right)^{\cdot} \frac{y^2}{2} + (\varrho(t)q(t))^{\cdot} \int_0^x g(u)\, du.$$

Let the functions W, r, ω be defined by

$$W(t, x, y) = x, \quad r(t, u) = \frac{[(\varrho(t)\, q(t))^{\cdot}]_+}{\varrho(t)\, q(t)}\, u, \quad \omega(t, u) = -\frac{p^2(t)}{2}\left(\frac{\varrho(t)}{p(t)}\right)^{\cdot} u^2.$$

Then we have

$$\dot{V} \leqq -\omega(t, |\dot{W}|) + r(t, V), \quad \dot{W} = y/p(t).$$

We note that in this case the solutions of equation (2.2) are bounded provided that the inequality

(3.3) $$\int_0^\infty \frac{[(\varrho(t)\, q(t))^{\cdot}]_+}{\varrho(t)\, q(t)}\, dt < \infty$$

is fulfilled. By virtue of Remark 2.4

(3.4) $$\int_0^\infty \frac{ds}{p^2(s)(\varrho(s)/p(s))^{\cdot}} > -\infty, \quad (\varrho(t)/p(t))^{\cdot} < 0 \quad (t \in R_+)$$

imply assumption (D). Consequently, from Theorem 2.2 it follows the following:

Corollary 3.1. *If there exists a function* $\varrho \in C^1(R_+, R_+)$ *such that* (3.3) *and* (3.4) *are true, then the limit of every solution* $x(t)$ *of* (3.1) *defined on* $[t_0, \infty)$ *exists as* $t \to \infty$.

Suppose that

(3.5) $$\frac{\varrho(t)}{p(t)} \geqq c = \text{const.} > 0 \quad (t \in R_+).$$

Then $V(t, x, y) + |W(t, x, y)| \geqq (y^2/2)c + |x|$. Using Remark 2.2 and Theorem 2.3, and taking into consideration the fact that the function $V(t, x, y)$ is non-increasing along the solutions of (3.2), provided that $\lim_{t\to\infty} \varrho(t)q(t)$ exists, we get

Corollary 3.2. *Suppose, that* (3.3)—(3.5) *are fulfilled. Then the zero solution of system* (3.2) *is stable. For every solution* $x(t)$ *of equation* (3.1), $\lim_{t\to\infty} x(t)$ *exists. Moreover, if* $\lim_{t\to\infty} q(t)\varrho(t)$ *exists, then* $\lim_{t\to\infty} \varrho(t)\dot{x}(t)$ *exists, too.*

It is worth noticing that these corollaries work in case $\int_0^\infty \frac{ds}{p(s)} < \infty$, whose interest consists in the fact that it cannot be reduced to an equation of type $\ddot{x}+a(t)g(x)=0$.

On the other hand, one can easily see that if

$$p(t)q(t) \geqq c = \text{const.} > 0, \quad (p(t)q(t))^{\cdot}\left[\varepsilon + \int_t^\infty \frac{ds}{p(s)}\right] \leqq q(t)$$

for t sufficiently large with some $\varepsilon > 0$, and $\int_0^\infty \frac{ds}{p(s)} < \infty$, then the fuctinon

$$\varrho(t) = p(t)\left[\varepsilon + \int_t^\infty \frac{ds}{p(s)}\right]$$

satisfies the conditions of Corollary 3.2.

II. Consider the differential equation

$$\ddot{x} + f(t, x, \dot{x})|\dot{x}|^\alpha \dot{x} + g(x) = 0, \tag{3.6}$$

where $f \in C(R_+ \times R \times R, R_+)$, $0 \leqq \alpha = \text{const.}$, $g \in C(R, R)$. A great number of papers have been devoted to the study of the conditions of the asymptotic stability and attractivity of the zero solution $x = \dot{x} = 0$. In these papers it is assumed that f is either bounded above or tends to infinity sufficiently slowly as $t \to \infty$ [1, 7, 8]. R. J. Ballieu and K. Peiffer [1] analyzed whether this condition is necessary. They proved for the case $\alpha = 0$ $f(t, x, \dot{x}) = \vartheta(t)$, $\limsup_{x\to 0} g(x)/x < \infty$ the following assertions: a) If $\vartheta(t)$ is increasing and $\int_0^\infty \frac{dt}{\vartheta(t)} = \infty$, then the zero solution of (3.3) is asymptotically stable. b) If $\vartheta(t)$ is increasing and $\int_0^\infty \frac{dt}{\vartheta(t)} < \infty$, then the zero solution of (3.3) is not attractive. Applying Theorem 2.3 we obtain that in the latter case the zero solution of (3.3) is stable, and every solution has a finite limit as $t \to \infty$.

Corollary 3.3. *Suppose that*

$$\int_0^x g(u)\,du \geqq 0 \quad (|x| \leqq c = \text{const.}),$$

$$f(t, x, \dot{x}) \geqq \vartheta(t)\, b(|x|) \quad (t \in R_+,\ |x|,\ |\dot{x}| \leqq c),$$

where $b\in K$ *and* $1/\vartheta\in L^1_{1/(1+\alpha)}$, $\vartheta(t)$ *is continuous. Then the zero solution of* (3.6) *is stable and for every solution* $x(t)$ *of* (3.6), $x(t)\to\text{const.}$, $\dot{x}(t)\to 0$ *as* $t\to\infty$ *provided that* $x^2(t_0)+\dot{x}^2(t_0)$ *is sufficiently small.*

Proof. Equation (3.6) may be written in the form

$$\dot{x} = y, \quad \dot{y} = -f(t, x, y)|y|^\alpha y - g(x). \tag{3.7}$$

Let the Liapunov function V be defined by

$$V(x, y) = y^2/2 + \int_0^y g(u)\,du.$$

Since

$$\dot{V}(t, x, y) = -f(t, x, y)|y|^{\alpha+2} \quad (x, y\in R,\ t\in R_+),$$

we have the estimate

$$\dot{V}(t, x, y) \leqq -\vartheta(t)b(|x|)|y|^{\alpha+2} \quad (t\in R_+,\ |x|, |y| \leqq c).$$

Therefore, by $\omega(t, u)=\vartheta(t)|u|^{\alpha+2}$, $W(t, x, y)=x$ we obtain

$$\dot{V}(t, x, y) \leqq -b(|x|)\,\omega(t, |\dot{W}(t, x, y)|) \quad (t\in R_+,\ |x|, |y| \leqq c).$$

Consequently, by Remark 2.4 the assumptions of Theorem 2.3 are fulfilled. So, $x=y=0$ is stable and $\lim_{t\to\infty} x(t)$ exists if $x^2(t_0)+y^2(t_0)$ is small. On the other hand, $V(t, x, y)$ is nonincreasing along solutions. This implies the existence of the limit $\lim_{t\to\infty} y(t)$, which, obviously, cannot differ from zero.

III. Corollary 3 can be generalized to mechanical systems with friction if the damping is increasing sufficiently fast in the time.

Consider a holonomic, rheonomic mechanical system being under the action of conservative, gyroscopic and dissipative forces, which may depend also on the time. The equation of motions in Lagrange's form reads as follows:

$$\frac{d}{dt}\frac{\partial T(q, \dot{q})}{\partial \dot{q}} - \frac{\partial T(q, \dot{q})}{\partial q} = -\frac{\partial \Pi(t, q)}{\partial q} + Q(t, q, \dot{q}), \tag{3.8}$$

where $q\in\Gamma\subset R^n$, $\dot{q}\in R^n$ denote the vectors of the generalized coordinates and velocities, respectively; $T\in C^2(\Gamma\times R^n, R_+)$ is the kinetic energy, $\Pi\in C^1(R_+\times\Gamma, R)$ is the potential energy of the system, $Q\in C(R_+\times\Gamma\times R^n, R^n)$ denotes the resultant of the gyroscopic and dissipative forces. We assume that

$$T(q, \dot{q}) = \dot{q}^{\mathrm{T}} A(q)\dot{q}/2,$$

where $A(q)$ is a symmetric positive definite matrix for each $q\in\Gamma$. Suppose that $0\in\Gamma$, $\partial\Pi(t, 0)/\partial q=0$, $Q(t, q, 0)=0$ $(t\in R_+, q\in\Gamma)$. Under these conditions the state $q=\dot{q}=0$ is an equilibrium of (3.8).

Corollary 3.4. *Suppose*

$$\Pi(t, q) \geqq 0, \quad \partial\Pi(t, q)/\partial t \leqq r(t, \Pi(t, q)) \quad (t\in R_+, \; q\in B_H\subset\Gamma),$$
$$(Q(t, q, \dot q), \dot q) \leqq -\vartheta(t)a(\|q\|)g(\|\dot q\|) \quad (t\in R_+, \; q, \dot q\in B_H),$$

where $a\in K$, $r\in C^1(R_+\times R_+, R_+)$, $r(t, \cdot)\in K$, $\int_0^\infty r(\tau, u)\,d\tau<\infty$ $(t, u\in R_+)$; *furthermore, suppose there exists a natural number* μ *such that* $g\in K$, $g'(0)=\ldots=g^{(\mu-1)}(0)=0$, $g^{(\mu)}(0)\neq 0$, $1/\vartheta\in L^+_{1/\mu}$, ϑ *is continuous.*

Then the equilibrium $q=\dot q=0$ *is stable and* $q(t)\to\text{const.}\in R^n$ *as* $t\to\infty$ *provided that* $q^2(t_0)+\dot q^2(t_0)$ *is sufficiently small.*

Proof. $A(q)$ is positive definite, so, introducing the new variables $x=q, y=\dot q$ equation (3.8) can be written in the form

$$\dot x = y, \quad \dot y = F(t, x, y). \tag{3.9}$$

In the capacity of Liapunov function choose the total mechanical energy

$$V(t, x, y) = T(x, y)+\Pi(t, x).$$

As is known [4],

$$\dot V(t, x, y) = (Q(t, x, y), y)+\frac{\partial\Pi(t, x)}{\partial t} \quad (t\in R_+, \; x\in\Gamma, \dot x\in R).$$

Consequently, if we define $W(t, x, y)=x$, $\omega(t, u)=\vartheta(t)g(u)$ we obtain

$$\dot V(t, x, y) \leqq -a(\|x\|)\vartheta(t)g(\|y\|)+r(t, \Pi(t, x)) \leqq$$
$$\leqq -a(\|x\|)\omega(t, \|\dot W(t, x, y)\|)+r(t, V(t, x, y))$$

for every $t\in R_+$, $x, y\in B_H$. Therefore, the assertion follows from Theorem 2.3., Lemma 2.2 and Remark 2.3.

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Meromorphic functions of operators

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Let T be a bounded operator on a separable Hilbert space. Combining previous results of HALMOS [4] and FILLMORE [3] concerning operator identities of the forms $0=f(T^*)$ and $T=f(T^*)$ with f entire, MOORE [6] proved the following general theorem:

Theorem A. [6] *Suppose that p is a polynomial, f is an entire function, and $p(T)=f(T^*)$. Then there is a polynomial q (of the same degree as p when T is not algebraic) such that $p(T)=q(T^*)$.*

The proof of this theorem required a key replacement of the operator identity by a complex variable identity, followed by a version of the Jacobi polynomial expansion theorem, resultant arguments, and a theorem of Picard. In this paper we begin with the complex variable identity and generalize Theorem A utilizing a more geometric argument, motivated by FILLMORE [3] and based on the monodromy theorem and the Weierstrass preparation theorem. A good reference for the classical complex variable theorems is HILLE [5]. We prove:

Theorem 1. *Let r be a rational function, M a meromorphic function in the complex plane, and assume that $r(T)=M(T^*)$. (Thus the poles of r and M lie outside of $\sigma(T)$ and $\sigma(T^*)$, respectively.) Then there is a rational function q such that $r(T)=q(T^*)$. Moreover, when T is not algebraic, M itself must be rational and of the same order as r.*

Before beginning the proof we state the replacement theorem of MOORE [6] for convenience.

Theorem B. [6] *Let f and g be analytic in neighborhoods of $\sigma(T)$ and $\sigma(T^*)$, respectively, and suppose that $g(T)=f(T^*)$. Then for $z\in\sigma(T)$, $g(z)=f(\bar{z})$.*

Proof of Theorem 1. If $\sigma(T)$ is finite then $\sigma(r(T))$ is finite and $r(T)$ is normal, hence algebraic. Thus T and T^* are algebraic, and M may be replaced by a rational function.

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Otherwise $\sigma(T)$ is infinite and contains a limit point α. First note that if r has order less than one, $F(T^*)=0$ for some entire function F, so T^* and thus T is algebraic and the theorem holds. Hence we may assume that N, the order of r, satisfies $N \geqq 1$.

By Theorem B we know that $r(z)=M(\bar{z})$ for z in $\sigma(T)$. Clearly we may take $\alpha=0$ and $r(0)=0$, modifying T, M, and r, if necessary. Then $z^n r_1(z)=\bar{z}^m M_1(\bar{z})$ for z in $\sigma(T)$, where $r_1(0)\neq 0$, $M_1(0)\neq 0$ and n and m are positive integers. Taking the modulus and letting $z\in\sigma(T)\to 0$, we see that $n=m$. Thus we may write $r(z)=h(z)^m$ and $M(\bar{z})=k(\bar{z})^m$, where h and k are analytic and invertible in a neighborhood W of 0. Since $(h(z))^m=(k(\bar{z}))^m$ for z's in $\sigma(T)$ (which is an infinite set), we may assume that h and k are chosen so that $h(z_n)=k(\bar{z}_n)$ for some sequence $\{z_n\}\subset\sigma(T)$ with $z_n\to 0$.

Computing, $z_n=h^{-1}\circ k(\bar{z}_n)$, so $\bar{z}_n=\overline{h^{-1}}\circ\bar{k}(z_n)=\bar{h}^{-1}\circ\bar{k}(z_n)$, where $\bar{h}(z)$ is defined to be $\overline{h(\bar{z})}$.

Let $S=\{z\in W: \bar{z}=\bar{h}^{-1}\circ\bar{k}(z)\}$. Then $\sigma(T)\subset S$. Using a consequence of the Weierstrass preparation theorem [7], we conclude that S is the intersection of (real analytic) arcs with only a finite number in any compact set. Using the fact that S contains a limit point we conclude that S contains a real analytic arc γ. Choose a point z_0 in γ so that $r'(z_0)$ is not zero or infinity.

Thus $\bar{z}=\bar{h}^{-1}\circ\bar{k}(z)$ for z in γ, so $h(z)=k\circ\bar{h}^{-1}\circ\bar{k}(z)$ for z in γ. Because all functions are analytic in W we conclude that $h(z)=k\circ\bar{h}^{-1}\circ\bar{k}(z)$ for all z in W.

By choice of z_0, r is invertible in a connected neighborhood of z_0 contained in W, Ω. Again, let $R(w)=r(w+z_0)-r(z_0)$, $\mathcal{M}(\bar{w})=M(\bar{w}+\bar{z}_0)-M(\bar{z}_0)$, $\Omega_0=\Omega-z_0$, and $\gamma_0=\gamma-z_0$. Hence $R(z)=\mathcal{M}(\bar{z})$ for $z\in\gamma_0$ and R is invertible in Ω_0. Then arguing as before $z=R^{-1}\circ\mathcal{M}(\bar{z})$ so $R(z)=M\circ\bar{R}^{-1}\circ\bar{\mathcal{M}}(z)$ for all z in Ω_0.

Denote the complex plane by $\mathbf{C}$. Define $Z(f,a)=\{z\in\mathbf{C}: f(z)=a\}$. Let $Z_1=\{z\in\mathbf{C}: \bar{\mathcal{M}}(z)\in Z(\bar{R}',0)\cup Z(\bar{R}',\infty)\}$. Each of the sets $Z(R',a)$ contains at most $2N$ elements since R has order N. But $\bar{\mathcal{M}}$ is meromorphic in the complex plane, so the set of points with $\bar{\mathcal{M}}(z)=c$ for any fixed c has no finite limit points. Thus we may join each of the points of Z_1 by a simple curve γ_1 accumulating only at ∞, chosen so that if $\Omega_1=\mathbf{C}-\gamma_1$, then Ω_1 is connected and simply connected. $\bar{R}^{-1}$ is one branch of the inverse of $\bar{R}$ in $\bar{\Omega}_0$. By construction branches of the inverse of $\bar{R}$ exist at every point of $\mathbf{C}-\gamma_1$. By the monodromy theorem (see [1, p. 134]) we see that $\bar{R}^{-1}$ can be continued into Ω_1, defining a single-valued analytic function (again denoted by $\bar{R}^{-1}$) in Ω_1.

Recall that $R(z)=\mathcal{M}\circ\bar{R}^{-1}\circ\bar{\mathcal{M}}(z)$ for $z\in\Omega_0$. Thus by permanence of functional relations $R(z)=\mathcal{M}\circ\bar{R}^{-1}\circ\bar{\mathcal{M}}(z)$ for $z\in\Omega_1$.

Suppose that for some $c\in\Omega_1$, $|Z(\bar{\mathcal{M}},c)|>N$. Let $d=\mathcal{M}\circ R^{-1}(c)$. Then

$$N \geqq |Z(R,d)| \geqq |Z(\mathcal{M}\circ\bar{\mathcal{R}}^{-1}\circ\bar{\mathcal{M}},d)| \geqq |Z(\bar{\mathcal{M}},c)| > N.$$

This contradiction shows that $\bar{\mathcal{M}}$ is at most N-valent in Ω_1. Since Ω_1 is open and dense in $\mathbf{C}$, the open mapping principle shows that $\bar{\mathcal{M}}$ is at most N-valent in $\mathbf{C}$. Applying the Casorati—Weierstrass theorem and the open mapping principle (or using the great Picard theorem), we see that ∞ is not an essential singularity of $\bar{\mathcal{M}}$. Thus M is a rational function of order less than or equal to N. A symmetric argument shows that the order of M equals the order of r.

Note that in the case when both r and M are entire, the conclusion that M has order N means that M is a polynomial of degree N.

Remarks. (a) Letting T be a unitary operator shows that taking r to be a polynomial with M meromorphic does not allow us to conclude that M is itself a polynomial.

(b) Theorem 1 covers the case that $g(T^*)p(T)=f(T^*)q(T)$, where f and g are entire, p and q are polynomials, and $q(T)$ and $g(T^*)$ are invertible. We do not know how to handle more general identities with T and T^* appearing on both sides.

(c) There should be some "Riemann surface" version of Theorem 1 valid for r an algebraic function with appropriate hypotheses concerning M.

We briefly wish to consider what compact sets K can be the spectrum of an operator T satisfying

$$f(T) = F(T)^*, \tag{1}$$

where f and F are analytic in a neighborhood of K. Notice that if

$$f(z) = \overline{F(z)} \tag{2}$$

for z in K, then (1) can be solved for a normal operator T and in many cases nonnormal operator solutions can be constructed as well.

Denote the real and imaginary parts of f and F by u, v and U, V, respectively. We see that (2) is equivalent to

$$u-U=0 \quad \text{and} \quad v+V=0 \quad \text{for } z \text{ in } K. \tag{3}$$

On the otherhand, let P and Q be any real-valued harmonic functions in a neighborhood of K with single-valued conjugates (denoted by $\tilde{P}$ and $\tilde{Q}$, respectively) in a neighborhood of K. Then if

$$P=0 \quad \text{and} \quad Q=0 \quad \text{for } z \text{ in } K \tag{4}$$

we may write $P=u-U$ and $Q=v+V$ where $u=(P-\tilde{Q})/2$, $U=(-\tilde{Q}-P)/2$, $v=\tilde{u}$, and $V=\tilde{U}$. Thus letting $f=u+iv$ and $F=U+iV$ we have established

Theorem 2. *There exist analytic functions f and F in a neighborhood of K with $f(z)=\overline{F(z)}$ for z in K if and only if there exist real harmonic functions P, Q in a neighborhood of K with single-valued conjugates in a neighborhood of K and with $P(z)=0$ and $Q(z)=0$ for z in K.*

Corollary. *Suppose that* Q *is a real harmonic function with single-valued conjugate in a neighborhood of* K *and* $Q(z)=0$ *for* z *in* K. *Then* $-\tilde{Q}+iQ= =-\overline{\tilde{Q}+iQ}$ *for* z *in* K.

Proof. Take $P\equiv 0$ in Theorem 2.

Theorem 2 and the corollary are useful for constructing various examples.

By the corollary, to understand K we must look at the zero set of a harmonic function. We mention a few well-known facts. Simply because a harmonic function h is locally a real analytic function in x and y, the Weierstrass preparation theorem [6] shows that locally $Z(h, 0)$ is a finite union of analytic arcs. Moreover, if the gradient of h vanishes at some point s, then the derivative of $h+i\tilde{h}$ vanishes at s. Thus locally the number of arcs and the types of singularities of $Z(h, 0)$ are restricted. In the case when f and F are analytic in a simply connected set, the maximum principle says that $Z(h, 0)$ contains no closed curves.

It may be of interest to see how the paks revious remmarnd geoetric considerations lead to a proof of a special case of Theorem A. Let K be an infinite compact set. Suppose that $p(z)=\overline{q(z)}$ for z in K, where p and q are polynomials with $\max(\deg p, \deg q)=m$. Let $u_1=\operatorname{Re} p-\operatorname{Re} q$ and $u_2=\operatorname{Im} p+\operatorname{Im} q$. Then $u_1=0$ and $u_2=0$ for z in K, where u_1 and u_2 are real harmonic polynomials of degree m. Since u_1 and u_2 vanish at so many common points (see [2, Chapter 1]), it follows that u_1 and u_2 have a common polynomial factor, h, of degree greater than 0. Let $f=u_1+i\tilde{u}_1$. Then f is a polynomial in z of degree m. So, at ∞, $Z(u_1, 0)$ has $2m$ branches. However the degree of u_1/h is less than m, so $Z(h, 0)$ must contain some branch which extends to ∞. But then $p(z)=\overline{q(z)}$ holds for some sequence of z's approaching ∞. Since p and q are polynomials, the degrees of p and q are equal.

I do not know whether Theorem 1 or even Theorem A can be proved analogously to the above special case with a more thorough understanding of the zero sets involved.

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An exact description of Lorentz spaces

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1. Introduction

We assume that f is a measurable complex-valued function on a measure space (Ω, μ), where μ is a σ-finite positive measure. The function f can be rearranged to a non-increasing function, denoted f^*, on $[0, \infty[$. The function f^* is continuous from the right and equidistributed with f (see e.g. [13, p. 131]).

We suppose that p and q are real numbers satisfying $0<p<\infty$, $0<q<\infty$. The Lorentz space $L(p, q)$ consists of all functions f satisfying

$$\|f\|_{p,q}^* = \Big(\int_0^\infty (f^*)^q t^{q/p-1}\,dt\Big)^{1/q} < \infty.$$

See [7], [9] or [13, p. 132]. The $L(p, q)$-spaces are of great interest in pure and applied mathematics. In particular, they appear as intermediate spaces in the theory of interpolation (see e.g. [6, p. 264] or [13, p. 134]).

Obviously $L(p, p)=L^p$. It is well known that if $q_2 \leqq q_1$, then $\|f\|_{p,q_1}^* \leqq \|f\|_{p,q_2}^*$ (see [6, p. 253]). In particular, $L(p, q) \supset L^p$ when $p<q$ and $L(p, q) \subset L^p$ when $p>q$. Moreover, in a sense, every $L(p, q)$-space is "close to" the corresponding L^p-space. In particular, by generalizing the definition of the $L(p, q)$-norm in the natural way we obtain the usual weak L^p-space when $q=\infty$. However, it is not possible to identify an $L(p, q)$-space by some Orlicz space of the type $L^p(\log L)^a$. One aim of this paper is to give an exact description of the $L(p, q)$-spaces at least in similar terms.

Throughout this paper we let the letter h stand for a strictly positive and continuous function on $[0, \infty[$ which is constant on $[0, 1]$.

The following theorem by the present author can be found in [12, p. 270].

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Theorem A. *Let* $p>q$. *Then*

$$\int_0^1 (f^*)^q t^{q/p-1}\,dt<\infty,$$

if and only if

$$\int_0^1 (f^* h(\log^+ f^*))^p\,dt \quad (1.1)$$

for some function h *such that, for some* $a>0$,

(1.2) $h(x)a^x$ *is a decreasing or an increasing function of* x

and

$$\int_1^\infty (h(x))^{pq/(q-p)}\,dx<\infty. \quad (1.3)$$

We may assume, without loss of generality, that $\log=\log_2$.

In Section 2 of this paper we shall state a theorem (Theorem 2.1) which generalizes Theorem A in two directions. On the one hand, by also studying conditions of the type $\int_1^\infty (f^*)^q t^{q/p-1}dt<\infty$ and, on the other hand, by also considering the case $p<q$. In this way we obtain an exact characterization of the $L(p,q)$-spaces not only for the special case when $\mu(\Omega)<\infty$ and $p>q$. Some applications to the theory of Fourier series (and transforms) are also given in Section 2. In particular, we shall see that the conclusion we usually extract from Hausdorff—Young's inequality (see e.g. [14, vol II, p. 101]) is, in a sense, far from being the sharpest possible. Some useful lemmas can be found in Section 3. The proof of the main theorem in Section 2 is carried out in Sections 4 (the case $p>q$) and 5 (the case $p<q$).

We say that the function f belongs to the Lorentz—Zygmund space $L^{p,q}(\log L)^\alpha$, $0<p<\infty$, $0<q<\infty$, $-\infty<\alpha<\infty$ if the quasi-norm

$$\|f\|^*_{p,q;\alpha}=\Big(\int_0^\infty (f^*(t)t^{1/p}(|\log t|+1)^\alpha)^q\,dt/t\Big)^{1/q}$$

is finite (see [2, p. 7]). In particular, we have $L^{p,q}(\log L)^0=L(p,q)$ and $L^{p,p}(\log L)^\alpha$ can be identified with the Zygmund space $L^p(\log L)^\alpha$ (see [2, p. 35]).

In Section 6 we shall generalize our main theorem so that we obtain an exact characterization of the spaces $L^{p,q}(\log L)^\alpha$. We shall also point out the fact that a recent embedding result by Bennett and Rudnick [2, p. 31] is a consequence of this characterization.

In Section 7 we shall give some concluding remarks. In particular, we shall compare the functional spaces introduced in this paper with the similarly defined Beurling—Herz spaces (see [1, p. 2] and [5, pp. 298—300]).

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2. A description of the $L(p, q)$-spaces

We make the following definition.

Definition. Let $p>q$. Then

a) $f\in E_0(p, q)$ if

$$\int_0^1 (f^* h(\log^+ f^*))^p \, dt < \infty \tag{2.1}$$

for some function h such that, for some $a>0$,

$$h(x)a^x \text{ is a decreasing or an increasing function of } x \tag{2.2}$$

and

$$\int_1^\infty (h(x))^{pq/(q-p)} \, dx < \infty. \tag{2.3}$$

b) $f\in E_\infty(p, q)$ *if* $f^*(t)>0$ *and*

$$\int_1^\infty \left(f^* h\left(\log^+ \frac{1}{f^*}\right)\right)^p dt < \infty \tag{2.4}$$

for some function h satisfying (2.2) and (2.3).

Let $p<q$. Then

c) $f\in E_0(p, q)$ if (2.1) holds for every function h satisfying (2.2) and (2.3).

d) $f\in E_\infty(p, q)$ if (2.4) holds for every function h satisfying (2.2) and (2.3).

Let $p\neq q$. Then

e) $f\in E(p, q)$ if $f\in E_0(p, q)$ and $f\in E_\infty(p, q)$.

The main theorem in this section can now be formulated in the following way.

Theorem 2.1. *Let* $0<p<\infty$ *and* $0<q<\infty$. *Then*

a) $\int_0^1 (f^*)^q t^{q/p-1} dt < \infty$ *if and only if* $f\in E_0(p, q)$

and

b) $\int_1^\infty (f^*)^q t^{q/p-1} dt < \infty$ *if and only if* $f\in E_\infty(p, q)$.

We see that part a) of this theorem gives an exact description of the desired type for the case when the μ-measure of Ω is finite. By combining the equivalences in Theorem 2.1 we obtain a characterization of the $L(p, q)$-spaces in the general case, namely that

$$f \in L(p, q) \quad \text{if and only if} \quad f \in E(p, q). \tag{2.5}$$

It can be somewhat difficult to see what this equivalence really means so we shall formulate it in another way. Therefore we let D be a subset of Ω such that $|f| \geqq 1$ on D and $|f| \leqq 1$ on $\Omega \setminus D$. Then we can make some elementary calculations to find that $f \in E(p, q)$ if and only if

$$\int_D (|f|\, h(\log |f|))^p \, d\mu + \int_{\Omega \setminus D} \left(|f|\, h\left(\log \frac{1}{|f|}\right)\right)^p d\mu < \infty \tag{2.6}$$

for *some* (the case $p>q$) or *every* (the case $p<q$) *function* h *satisfying* (2.2) and (2.3). In the sequel we say that $f \in L^p h(\log L)$ when (2.6) holds. For the special case $h(x)=x^\alpha$ we get the Zygmund space $L^p(\log L)^\alpha$. We can now formulate the equivalence (2.5) in the following way.

Theorem 2.2. *Let* $0<p<\infty$ *and* $0<q<\infty$.

a) *Let* $p>q$. *Then* $f \in L(p, q)$ *if and only if* $f \in L^p h(\log L)$ *for some function* h *satisfying* (2.2) *and* (2.3).

b) *Let* $p<q$. *Then* $f \in L(p, q)$ *if and only if* $f \in L^p h(\log L)$ *for every function* h *satisfying* (2.2) *and* (2.3).

We apply Theorem 2.2 with $h(x)=x^{(1+\delta)(1/q-1/p)}$, $\delta>0$, and find that if $p>q$, then, for every $\varepsilon>0$,

$$L(p, q) \supset L^p(\log L)^{1/q-1/p+\varepsilon} \tag{2.7}$$

and if $p<q$, then, for every $\varepsilon>0$,

$$L(p,q) \subset L^p(\log L)^{1/q-1/p-\varepsilon}. \tag{2.8}$$

The inclusions (2.7) and (2.8) are the sharpest possible in the sense that they are in general false if we permit $\varepsilon=0$. In order to verify this fact we set $(\Omega, \mu)= =([0, 1], dx)$ and study the function

$$f(x) = \frac{1}{x^{1/p} (\log 1/x)^{1/q} (\log(\log 1/x+2))^\alpha}.$$

Then, as $t \to 0$,

$$(f^*)^q t^{q/p-1} \simeq \frac{1}{t \log 1/t \,(\log(\log 1/t+2))^{\alpha q}},$$

and

$$(f^*)^p (\log^+ f^* + 1)^{p/q-1} \simeq \frac{1}{t \log 1/t \,(\log(\log 1/t+2))^{\alpha p}}.$$

We obtain suitable counterexamples by choosing α satisfying $1/p<\alpha<1/q$ for the case $p>q$ and $1/q<\alpha\leqq 1/p$ for the case $p<q$.

We shall now consider a function f on $[0, 1]$. Let c_n, $n\in Z$, be the complex Fourier coefficients of f (with respect to a uniformly bounded system of orthonormal functions). The sequence $(c_n^*)_0^\infty$ is the sequence $(|c_n|)_{-\infty}^\infty$ rearranged in non-increasing order. Hausdorff—Young's inequality (see e.g. [14, vol. II, p. 101]) can be used to obtain the following implication:

$$\text{(2.9)} \qquad \text{if } f\in L^p,\ 1<p<2,\ p'=p/(p-1), \text{ then } \sum_{-\infty}^{\infty}|c_n|^{p'}<\infty.$$

By an estimate of Paley it is also well known that if $f\in L^p$, $1<p<2$, then $\sum_1^\infty (c_n^*)^p n^{p-2}<$ $<\infty$ (see e.g. [14, vol II, p. 123]).

Therefore we can use Theorem 2.1 b) and make some straightforward calculations to obtain the following more precise implication than that in (2.9).

Corollary 2.3. *If $f\in L^p$, $1<p<2$, $p'=p/(p-1)$, then*

$$\sum_{-\infty}^{\infty}|c_n|^{p'}\left(h\left(\log^+\frac{1}{|c_n|}\right)\right)^{(2-p)/(p-1)}<\infty$$

for some function h, $h\geqq 1$, satisfying (2.2) *and*

$$\text{(2.10)} \qquad \int_1^\infty \frac{1}{h(x)}\,dx<\infty.$$

Remark. The result in Corollary 2.3 cannot be improved. In fact, by using the results obtained in [12, p. 268] we find that the implication in Corollary 2.3 can be replaced by an equivalence in a relatively large class of functions. This class consists at least of all non-negative functions f satisfying the condition that

$$\int_0^t f^*(u)\,du\leqq K\int_0^t f(x)\,dx$$

for some constant K. Of course it is impossible to replace the implication in (2.9) by an equivalence in some similar relatively large class of functions.

In Corollary 2.3 we have seen that the condition $f\in L^p$ is an unnecessarily restricted condition to ensure the convergence of the series $\sum_{-\infty}^{\infty}|c_n|^{p'}$. However, it is well known that also the condition $\int_0^1 (f^*)^{p'} t^{p'-2}\,dt$ (that is $f\in L(p, p')$) implies that $\sum_{-\infty}^{\infty}|c_n|^{p'}<\infty$ (see [14, vol II, p. 124]). Therefore we can use Theorem 2.1 a) and obtain the following more precise criterion.

Corollary 2.4. *Let* $1<p<2$ *and* $p'=p/(p-1)$. *If*

$$\int_0^1 |f|^p (h(\log^+ |f|))^{p-2}\, dx < \infty$$

for every function $h, h \geqq 1$, *satisfying* (2.2) *and* (2.10), *then*

$$\sum_{-\infty}^{\infty} |c_n|^{p'} < \infty.$$

Remark. We can use the estimates obtained in [12, p. 268] to see that the implication in Corollary 2.4 can be replaced by an equivalence in the same class of functions as that in the remark after Corollary 2.3.

Finally we note that we can use Theorem 2.1 and similar arguments as before to obtain the corresponding results for a function $f \in R^n$ and its Fourier transform $\hat{f} \in R^n$. For example the corollary corresponding to Corollary 2.3 can be formulated in the following way.

Corollary 2.5. *If* $f \in L^p(R^n)$, $1<p<2$, $p'=p/(p-1)$, *then*

$$\int_{R^n} |\hat{f}|^{p'} (h(|\log |\hat{f}||))^{(2-p)/(p-1)}\, d\xi < \infty$$

for some function h *satisfying* (2.2) *and* (2.10).

Remark. It may be tempting to try to find some function h_0, *not* depending on f, such that

$$\|f\|_p \leqq 1 \Rightarrow \int_{R^n} |\hat{f}|^{p'} h_0(|\log |\hat{f}||)\, d\xi \leqq K_0 < \infty. \tag{2.11}$$

However, this is not possible for any positive function h_0 such that $h_0(x) \to \infty$ as $x \to \infty$. This fact follows when using the following homogeneity argument: Let f be a function on R^n such that $\hat{f}(\xi) \geqq a_0 > 0$ on a set E of positive measure. If $f_a(x) = a^{1/p} f(ax_1, x_2, \ldots, x_n)$, then

$$\|f_a(x)\|_p = \|f\|_p \leqq 1, \quad \hat{f}_a(\xi) = a^{1/p-1} \quad \hat{f}\left(\frac{\xi_1}{a}, \xi_2, \ldots, \xi_n\right)$$

and

$$I_a = \int_{R^n} |\hat{f}_a(\xi)|^{p'} h_0(|\log \hat{f}_a(\xi)|)\, d\xi = \int_{R^n} |\hat{f}(\eta)|^{p'} h_0(|\log (a^{-1/p'} \hat{f}(\eta))|)\, d\eta.$$

Since $h_0(x) \to \infty$ as $x \to \infty$ we can choose a small enough to obtain that

$$h_0(|\log (a^{-1/p'} \hat{f}(\eta))|) \geqq 2K_0/(m(E) a_0^{p'}) \quad \text{on} \quad E.$$

Therefore $I_a \geqq m(E) a_0^{p'} 2K_0/(m(E) a_0^{p'}) = 2K_0$. We conclude that (2.11) does not hold for any of the functions h_0 considered.

3. Some lemmas

Lemma 3.1. *Let* $\sum_1^\infty c_k$ *be a non-negative and divergent series. If* $S_k=\sum_1^k c_n$, *then the series* $\sum_1^\infty c_k/S_k$ *is divergent and, for every* $a>0$, *the series* $\sum_1^\infty c_k/S_k^{1+a}$ *is convergent.*

A proof of this lemma by Abel can be found for example in [4, p. 121]. We shall now state two useful regularization lemmas.

Lemma 3.2. *Let* $\sum_{k=0}^\infty a_k$ *be a positive and convergent series and let* $c>1$. *Then there exists a sequence* $(b_k)_{k=0}^\infty$ *such that, for* $k=0, 1, 2, \ldots$, *we have* $a_k \leqq b_k$, $c^{-1} \leqq \leqq b_{k+1}/b_k \leqq c$ *and*

$$\sum_{k=0}^\infty b_k \leqq \frac{c+1}{c-1} \sum_{k=0}^\infty a_k.$$

Lemma 3.3. *Let* δ *be a positive number and let* g *be a positive, integrable function on* $[1, \infty[$ *such that, for some* $b>0$, $g(x)x^b$ *is a decreasing or an increasing function of* x. *Then there exists a constant* K *(depending only on* b *and* δ*) and a function* $g_1(x)$, *such that* $g_1(x) \geqq g(x)$,

$$g_1(x)x^{1+\delta} \quad \textit{is increasing,} \tag{3.1}$$

$$g_1(x)x^{1-\delta} \quad \textit{is decreasing,} \tag{3.2}$$

and

$$\int_1^\infty g_1(x)\,dx \leqq K \int_1^\infty g(x)\,dx.$$

Somewhat less precise versions of Lemmas 3.2 and 3.3 have been proved in [11, pp. 292—294]. The proofs we shall give here are elementary and based on convolutions.

Proof of Lemma 3.2. We choose $b_k=\sum_{n=0}^\infty a_n c^{-|k-n|}$. Then

$$\sum_{k=0}^\infty b_k = \sum_{k=0}^\infty \sum_{n=0}^k a_n c^{-(k-n)} + \sum_{k=0}^\infty \sum_{n=k+1}^\infty a_n c^{(k-n)} =$$

$$= \sum_{n=0}^\infty a_n c^n \sum_{k=n}^\infty c^{-k} + \sum_{n=1}^\infty a_n c^{-n} \sum_{k=0}^{n-1} c^k \leqq \frac{c+1}{c-1} \sum_{n=0}^\infty a_n.$$

Moreover,

$$b_{k+1}=\sum_{n=0}^{\infty} a_n c^{-|k+1-n|}=\sum_{n=0}^{k} a_n c^{-(k+1-n)}+\sum_{n=k+1}^{\infty} a_n c^{k+1-n}=$$

$$=c^{-1}\sum_{n=0}^{k} a_n c^{-(k-n)}+c\sum_{n=k+1}^{\infty} a_n c^{(k-n)}.$$

Therefore, we find that $b_{k+1}\leqq cb_k$ and $b_{k+1}\geqq c^{-1}b_k$. Trivially $a_k\leqq b_k$. The proof is complete.

Proof of Lemma 3.3. Let $g(x)x^b$ be an increasing function of x. Then, for $2^k\leqq x\leqq 2^{k+1}$, $k=0, 1, 2, \ldots$,

$$2^{-b}g(2^k)\leqq g(x)\leqq 2^b g(2^{k+1}). \tag{3.3}$$

Therefore

$$\sum_0^{\infty} g(2^k)2^k\leqq 2^b\sum_0^{\infty}\int_{2^k}^{2^{k+1}} g(x)\,dx\leqq 2^b\int_1^{\infty} g(x)\,dx<\infty. \tag{3.4}$$

Now we can use Lemma 3.2 with $c=2^\delta$ to obtain real numbers d_k, $k=0, 1, 2, \ldots$, such that $d_k\geqq g(2^k)$, $\sum_0^{\infty} d_k 2^k<\infty$,

$$2^{-(1+\delta)}\leqq d_{k+1}/d_k\leqq 2^{-1+\delta}, \tag{3.5}$$

and

$$\sum_0^{\infty} d_k 2^k\leqq\frac{2^\delta+1}{2^\delta-1}\sum_0^{\infty} g(2^k)2^k. \tag{3.6}$$

We define the function g_1 in the following way:

$$g_1(x)=g_1(2^{k+u})=2^b(d_k)^{1-u}(d_{k+1})^u,\quad k=0, 1, 2, \ldots,\ 0\leqq u\leqq 1.$$

Observe that, for $0\leqq u_1\leqq u_2\leqq 1$,

$$2^{-(\delta+1)(u_2-u_1)}\leqq\frac{g_1(2^{k+u_2})}{g_1(2^{k+u_1})}=\left(\frac{d_{k+1}}{d_k}\right)^{u_2-u_1}\leqq 2^{(\delta-1)(u_2-u_1)} \tag{3.7}$$

and, for $k_2>k_1$,

$$2^{-(\delta+1)(k_2-k_1)}\leqq\frac{g_1(2^{k_2})}{g_1(2^{k_1})}=\frac{d_{k_2}}{d_{k_1}}\leqq 2^{(\delta-1)(k_2-k_1)}. \tag{3.8}$$

According to the estimates (3.7)—(3.8) we find that our function g_1 satisfies the growth conditions (3.1) and (3.2).

We may, without loss of generality, assume that $\delta<1$. Then, by (3.3), (3.5), and the fact that $d_{k+1}\geqq g(2^{k+1})$, we get

$$g_1(x)=g_1(2^{k+u})=2^b(d_k)^{1-u}(d_{k+1})^u\geqq 2^b 2^{(1-\delta)(1-u)}d_{k+1}\geqq 2^b d_{k+1}\geqq 2^b g(2^{k+1})\geqq g(x).$$

Finally, by (3.4), (3.6), and (3.7), we have

$$\int_1^\infty g_1(x)\,dx=\sum_0^\infty \int_{2^k}^{2^{k+1}} g_1(x)\,dx \leqq \sum_0^\infty g_1(2^k)2^k=$$

$$=2^b\sum_0^\infty d_k 2^k \leqq 2^b\frac{2^\delta+1}{2^\delta-1}\sum_0^\infty g(2^k)2^k \leqq 2^{2b}\frac{2^\delta+1}{2^\delta-1}\int_1^\infty g(x)\,dx.$$

The case when $g(x)x^b$ is a decreasing function of x can be carried out analogously. The proof is complete.

4. Proof of Theorem 2.1; the case $p>q$

In this case part a) of Theorem 2.1 is identical with Theorem A so it is sufficient to prove part b) of the theorem.

First we assume that

$$\int_1^\infty (f^*)^q t^{q/p-1}\,dt<\infty,$$

and choose ε satisfying $0<\varepsilon<q/p$. We can now use Lemma 3.3 to find a function $g(t)$, such that $g(t)\geqq f^*(t)$,

$$(g(t))^q t^{q/p+\varepsilon} \text{ is increasing,} \tag{4.1}$$

$$(g(t))^q t^{q/p-\varepsilon} \text{ is decreasing,} \tag{4.2}$$

and

$$\int_1^\infty (g(t))^q t^{q/p-1}\,dt<\infty. \tag{4.3}$$

For $k=0, 1, 2, \dots$ we set $b_k=(g(2^k)2^{k/p})^q$ and observe that, by (4.1)—(4.3), the series $\sum_0^\infty b_k$ converges. We also note that we may, without loss of generality, assume that $g(t)\leqq 1$.

We define the function h at the points $x_k=\log(1/g(2^k))$ by $h(x_k)=b_k^{(q-p)/pq}$, $k=0, 1, 2, \dots$. According to (4.1)—(4.2) we find, for $0\leqq u\leqq 1$ and $k=0, 1, 2, \dots$,

$$g^q(2^k)2^{-u(q/p+\varepsilon)} \leqq g^q(2^{k+u}) \leqq g^q(2^k)2^{u(\varepsilon-q/p)}. \tag{4.4}$$

We can now use (4.4) and make some elementary calculations to obtain the following useful estimates:

$$2^{-\varepsilon} \leqq \frac{b_{k+1}}{b_k} = \left(\frac{g(2^{k+1})2^{(k+1)/p}}{g(2^k)2^{k/p}}\right)^q \leqq 2^{\varepsilon}, \tag{4.5}$$

$$2^{-\varepsilon(p-q)/pq} \leqq \frac{h(x_{k+1})}{h(x_k)} \leqq 2^{\varepsilon(p-q)/pq}, \tag{4.6}$$

and

$$0 < \frac{1}{q}\left(\frac{q}{p}-\varepsilon\right) \leqq x_{k+1}-x_k \leqq \frac{1}{q}\left(\frac{q}{p}+\varepsilon\right). \tag{4.7}$$

We extend the definition of the function h by setting

$$h(x) = \left((h(x_k))^{x-x_k}(h(x_{k+1}))^{x_{k+1}-x}\right)^{1/(x_{k+1}-x_k)}$$

for $x_k \leqq x \leqq x_{k+1}$, $k=0, 1, 2, \ldots$. We can make some elementary (but rather laborious) calculations and find, for some $\delta > 0$, that

$$h(x)2^{\delta x} \text{ is increasing} \tag{4.8}$$

and

$$h(x)2^{-\delta x} \text{ is decreasing.} \tag{4.9}$$

(We can for example choose $\delta = \varepsilon(p-q)/(q-p\varepsilon)$.)

According to (4.5)—(4.9) we obtain, for $x_k \leqq x \leqq x_{k+1}$, $k=0, 1, 2, \ldots$, and for some $\delta_0 > 0$,

$$2^{-\delta_0} h(x_k) \leqq h(x) \leqq 2^{\delta_0} h(x_k).$$

(If we choose $\delta = \varepsilon(p-q)/(q-p\varepsilon)$, then we can have $\delta_0 = \varepsilon(p-q)/pq$.) Therefore, by (4.7), we have

$$\int_{x_0}^{\infty} (h(x))^{pq/(q-p)}\,dx \leqq \sum_0^{\infty} \int_{x_k}^{x_{k+1}} (h(x))^{pq/(q-p)}\,dx \leqq$$

$$\leqq 2^{\delta_0 pq/(q-p)} \sum_0^{\infty} (h(x_k))^{pq/(q-p)}(x_{k+1}-x_k) \leqq 2^{\delta_0 pq/(q-p)} \sum_0^{\infty} b_k(x_{k+1}-x_k) \leqq \tag{4.10}$$

$$\leqq 2^{\delta_0 pq/(q-p)} \frac{1}{q}\left(\frac{q}{p}+\varepsilon\right) \sum_0^{\infty} b_k < \infty.$$

We use (4.4) once more and obtain, for $2^k \leqq t \leqq 2^{k+1}$, $k=0, 1, 2, \ldots$,

$$g(2^k)2^{-(q+p\varepsilon)/pq} \leqq g(t) \leqq g(2^k). \tag{4.11}$$

Hence we can use (4.8)—(4.9) to obtain that, for $2^k \leqq t \leqq 2^{k+1}$,

$$h\left(\log\frac{1}{g(t)}\right) \leqq h\left(\log\left(\frac{1}{g(2^k)}+\frac{q+p\varepsilon}{pq}\right)\right) 2^{\delta(\log(g(t)/g(2^k))+(q+p\varepsilon)/pq)} \leqq$$

$$\leqq h\left(\log\frac{1}{g(2^k)}\right) 2^{2\delta(q+p\varepsilon)/pq}. \tag{4.12}$$

Furthermore, according to (4.11)—(4.12),

$$\int_1^\infty \left(g(t)\,h\left(\log\frac{1}{g(t)}\right)\right)^p dt = \sum_0^\infty \int_{2^k}^{2^{k+1}} \left(g(t)\,h\left(\log\frac{1}{g(t)}\right)\right)^p dt \leqq$$
$$\leqq K_0 \sum_0^\infty (g(2^k)h(x_k))^p 2^k = K_0 \sum_0^\infty b_k^{p/q} 2^{-k} b_k^{1-p/q} 2^k = K_0 \sum_0^\infty b_k < \infty. \tag{4.13}$$

(We can for example choose $K_0 = 2^{2\delta(q+p\varepsilon)/q}$.)

By choosing ε small enough and using the growth condition (4.8) we see that $yh(\log(1/y))$ is an increasing function of y, $0<y<1$. Therefore, by (4.13) and the fact that $f^*(t) \leqq g(t)$, we have

$$\int_1^\infty \left(f^* h\left(\log^+\frac{1}{f^*}\right)\right)^p dt < \infty.$$

Since the function h satisfies (4.8)—(4.10) we conclude that $f \in E_\infty(p, q)$.

In order to prove the converse implication we assume that $f \in E_\infty(p, q)$. Let $(\alpha_k)_0^\infty$ be the nondecreasing sequence of the least real numbers α_k such that $2^{-k-1} \leqq f^*(t) \leqq 2^{-k}$, when $\alpha_{k-1} \leqq t < \alpha_k$, $k = 0, 1, 2, \ldots$. Let $h(x)$ be the function associated with the definition of $E_\infty(p, q)$. We assume that $h(x)2^{\delta x}$, for some $\delta > 0$, is an increasing function of x. Therefore, if $\alpha_{k-1} \leqq t \leqq \alpha_k$, then

$$h(k)2^{-\delta} \leqq h\left(\log\frac{1}{f^*(t)}\right) \leqq h(k+1)\,2^{\delta}.$$

Thus the assumption

$$\int_1^\infty \left(f^* h\left(\log^+\frac{1}{f^*}\right)\right)^p dt < \infty$$

implies that

$$\sum_{k=0}^\infty 2^{-pk}(h(k))^p(\alpha_k - \alpha_{k-1}) < \infty. \tag{4.14}$$

Moreover,

$$\int_{\alpha_0}^\infty (f^*)^q t^{q/p-1}\,dt = \sum_1^\infty \int_{\alpha_{k-1}}^{\alpha_k} (f^*)^q t^{q/p-1}\,dt \leqq \frac{p}{q} \sum_1^\infty 2^{-qk}(\alpha_k^{q/p} - \alpha_{k-1}^{q/p}). \tag{4.15}$$

We use Hölder's inequality and an elementary estimate and obtain

$$\sum_1^\infty 2^{-qk}(\alpha_k^{q/p} - \alpha_{k-1}^{q/p}) \leqq \sum_1^\infty 2^{-qk}(\alpha_k - \alpha_{k-1})^{q/p} \leqq$$
$$\leqq \left(\sum_1^\infty 2^{-pk}(h(k))^p(\alpha_k - \alpha_{k-1})\right)^{q/p} \left(\sum_1^\infty (h(k))^{pq/(q-p)}\right)^{1-q/p}. \tag{4.16}$$

From the growth and integrability properties of h we deduce that the series $\sum_1^\infty (h(k))^{pq/(q-p)}$ converges. Hence, by (4.14)—(4.16), we obtain

$$\int_1^\infty (f^*)^q t^{q/p-1}\, dt < \infty.$$

The case when $h(x)2^{-\delta x}$ is a decreasing function of x can be handled analogously. The proof is complete.

5. Proof of Theorem 2.1; the case $p<q$

We assume

$$\int_0^1 (f^*)^q t^{q/p-1}\, dt < \infty.$$

Let h be any function on $[0, \infty[$ such that for some $\delta, 0<\delta<p$,

$$h(x)2^{\delta x} \text{ is increasing,} \tag{5.1}$$

$$h(x)2^{-\delta x} \text{ is decreasing} \tag{5.2}$$

and

$$\int_1^\infty (h(x))^{pq/(q-p)}\, dx < \infty. \tag{5.3}$$

Let $(\beta_k)_0^\infty$ be the nonincreasing sequence of the least real numbers β_k, such that $2^{k-1} \leqq f^*(t) \leqq 2^k$, when $\beta_k \leqq t < \beta_{k-1}$, $k=0, 1, 2, \dots$. Then

$$\int_0^{\beta_0} (f^*)^q t^{q/p-1}\, dt = \sum_1^\infty \int_{\beta_k}^{\beta_{k-1}} (f^*)^q t^{q/p-1}\, dt \geqq \frac{p}{q} 2^{-q} \sum_1^\infty 2^{qk}(\beta_{k-1}^{q/p} - \beta_k^{q/p}). \tag{5.4}$$

Moreover, by (5.1),

$$\begin{aligned} \int_0^{\beta_0} (f^* h(\log^+ f^*))^p\, dt &= \sum_1^\infty \int_{\beta_k}^{\beta_{k-1}} (f^* h(\log^+ f^*))^p\, dt \leqq \\ &\leqq 2^{\delta p} \sum_1^\infty 2^{pk} (h(k))^p (\beta_{k-1} - \beta_k). \end{aligned} \tag{5.5}$$

We use Hölder's inequality and find

$$\sum_1^\infty 2^{pk} (h(k))^p (\beta_{k-1} - \beta_k) \leqq \left(\sum_1^\infty 2^{qk} (\beta_{k-1} - \beta_k)^{q/p} \right)^{p/q} \left(\sum_1^\infty (h(k))^{pq/(q-p)} \right)^{1-p/q}. \tag{5.6}$$

Since $(\beta_{k-1}-\beta_k)^{q/p} \leqq \beta_{k-1}^{q/p}-\beta_k^{q/p}$ we can use (5.4) and the integrability assumption on f^* to obtain

$$\sum_1^\infty 2^{qk}(\beta_{k-1}-\beta_k)^{q/p} < \infty. \tag{5.7}$$

The conditions (5.1)—(5.3) imply that the series $\sum_1^\infty (h(k))^{pq/(q-p)}$ converges. Therefore, according to (5.6)—(5.7), $\sum_1^\infty 2^{pk}(h(k))^p(\beta_{k-1}-\beta_k) < \infty$. In view of (5.5) we conclude that

$$\int_0^1 (f^* h(\log^+ f^*))^p \, dt < \infty$$

for every function h satisfying (5.1)—(5.3).

Finally we suppose that the conditions (5.1) and (5.2) on the function h, are replaced by the general condition that, for some $a>0$, $h(x)a^x$ is increasing or decreasing. Then we can use Lemma 3.3 to obtain a function $h_1 \geqq h$ satisfying (5.1)—(5.3). We have just proved that

$$\int_0^1 (f^* h_1(\log^+ f^*))^p \, dt < \infty$$

and, thus, since $h_1 \geqq h$,

$$\int_0^1 (f^* h(\log^+ f^*))^p \, dt < \infty$$

so that $f \in E_0(p, q)$.

In order to prove the converse implication we assume that $f \in E_0(p, q)$. Let h be an arbitrary function satisfying (5.1)—(5.3). Then

$$\begin{aligned} \int_0^{\beta_0} (f^* h(\log^+ f^*))^p \, dt &= \sum_1^\infty \int_{\beta_k}^{\beta_{k-1}} (f^* h(\log^+ f^*))^p \, dt \geqq \\ &\geqq 2^{-p(1+\delta)} \sum_1^\infty 2^{pk}(h(k))^p (\beta_{k-1}-\beta_k). \end{aligned} \tag{5.8}$$

Hence, by assumption and (5.8), the series $\sum_1^\infty 2^{pk}(h(k))^p(\beta_{k-1}-\beta_k)$ converges. We make an Abelian transformation on this series and find

$$\sum_1^\infty 2^{pk}(h(k))^p \beta_k < \infty. \tag{5.9}$$

Since

$$\int_0^{\beta_0} (f^*)^q t^{q/p-1} \, dt = \sum_1^\infty \int_{\beta_k}^{\beta_{k-1}} (f^*)^q t^{q/p-1} \, dt \leqq \frac{p}{q} \sum_1^\infty 2^{qk} \beta_{k-1}^{q/p},$$

it is sufficient if we can prove that $\sum_1^\infty 2^{qk}\beta_k^{q/p}<\infty$. We assume the contrary, viz. $\sum_1^\infty 2^{qk}\beta_k^{q/p}=\infty$. For $k=1,2,3,\ldots$ we set $c_k=2^{qk}\beta_k^{q/p}$ and $d_k=2^{k(q-p)}\beta_k^{q/p-1}$. By assumption the series $\sum_1^\infty c_k$ diverges so we can use Lemma 3.1 and obtain

$$\sum_1^\infty 2^{pk}\beta_k\frac{d_k}{S_k}=\sum_1^\infty\frac{c_k}{S_k}=\infty \tag{5.10}$$

and, for $a=p/(q-p)$,

$$\sum_1^\infty\left(\frac{d_k}{S_k}\right)^{q/(q-p)}=\sum_1^\infty\frac{c_k}{S_k^{1+a}}<\infty.$$

We choose $\delta,\ 0<\delta<p$, and set $a_k=d_k/S_k$. We apply Lemma 3.2 to the series $\sum_1^\infty a_k^{q/(q-p)}$ to obtain a sequence $(b_k)_1^\infty$ such that $b_k\geqq a_k$,

$$(b_k 2^{\delta pk})_1^\infty \text{ is an increasing sequence,} \tag{5.11}$$

$$(b_k 2^{-\delta pk})_1^\infty \text{ is a decreasing sequence,} \tag{5.12}$$

$$\sum_1^\infty b_k^{q/(q-p)}<\infty, \tag{5.13}$$

and, by (5.10),

$$\sum_1^\infty 2^{pk}\beta_k b_k=\infty. \tag{5.14}$$

For $k=1,2,3,\ldots$ and $0\leqq u\leqq 1$ we define $h(x)=h(k+u)=(b_k^{1-u}b_{k+1}^u)^{1/p}$. Then, by (5.11)—(5.14), we can see that there exists a function h satisfying (5.1)—(5.3) but

$$\sum_1^\infty 2^{pk}\beta_k(h(k))^p=\infty.$$

This fact contradicts the condition (5.9). We conclude that our assumption is false so that

$$\int_0^1 (f^*)^q t^{q/p-1}\,dt<\infty.$$

The proof of part a) of the theorem is complete.

In order to prove part b) we study the nondecreasing sequence $(\alpha_k)_0^\infty$ of the least real numbers α_k such that $2^{-k-1}\leqq f^*(t)\leqq 2^{-k}$, when $\alpha_{k-1}\leqq t<\alpha_k$, $k=0,1,2,\ldots$. The proof of part b) can now be carried out by arguing exactly as in the proof of part a). Therefore we leave out the details.

6. A description of the spaces $L^{p,q}(\log L)^\alpha$

Theorem 2.1 can be generalized in the following way.

Theorem 6.1. *Let* $0<p<\infty$, $0<q<\infty$ *and* $-\infty<\alpha<\infty$.

a) *Let* $p>q$. *Then*

$$\int_0^1 (f^* t^{1/p} (|\log t|+1)^\alpha)^q \, dt/t < \infty \tag{6.1}$$

if and only if

$$\int_0^1 (f^* (\log^+ f^* + 1)^\alpha h(\log^+ f^*))^p \, dt < \infty \tag{6.2}$$

for some function h, *such that, for some real number* a,

(6.3) $\quad h(x)a^x$ *is a decreasing or an increasing function of* x

and

$$\int_1^\infty (h(x))^{pq/(q-p)} \, dx < \infty. \tag{6.4}$$

b) *Let* $p<q$. *Then* (6.1) *holds if and only if* (6.2) *holds for every function* h *satisfying* (6.3) *and* (6.4).

c) *Let* $p>q$. *Then*

$$\int_1^\infty (f^* t^{1/p} (|\log t|+1)^\alpha)^q \, dt/t < \infty \tag{6.5}$$

if and only if

$$\int_1^\infty \left[f^* \left(\log^+ \frac{1}{f^*} + 1 \right)^\alpha h\left(\log^+ \frac{1}{f^*} \right) \right]^p dt < \infty \tag{6.6}$$

for some function h *satisfying* (6.3) *and* (6.4).

d) *Let* $p<q$. *Then* (6.5) *holds if and only if* (6.6) *holds for every function* h *satisfying* (6.3) *and* (6.4).

The proof of Theorem 6.1 can be carried out in a similar way as the proof of Theorem 2.1 so we omit the details. Moreover, we can use Theorem 6.1 and argue in a similar way as before to obtain the following exact characterization of the Lorentz—Zygmund spaces.

Theorem 6.2. *Let* $0<p<\infty$, $0<q<\infty$ *and* $-\infty<\alpha<\infty$.

a) *Let* $p>q$. *Then* $f\in L^{p,q}(\log L)^\alpha$ *if and only if* $f\in L^p h(\log L)$ *for some function* h *satisfying* (6.3) *and*

$$\int_1^\infty (h(x)x^{-\alpha})^{pq/(q-p)} \, dx < \infty. \tag{6.7}$$

b) *Let* $p<q$. *Then* $f\in L^{p,q}(\log L)^{\alpha}$ *if and only if* $f\in L^p h(\log L)$ *for every function* h *satisfying* (6.3) *and* (6.7).

The following recent embedding result by BENNETT and RUDNICK [2, p. 31] can be deduced from Theorem 6.2.

Corollary 6.3. *Let* $0<p<\infty$, $0<q<\infty$, $0<q_1<\infty$, $-\infty<\alpha<\infty$ *and* $-\infty<\alpha_1<\infty$. *Then*

$$L^{p,q}(\log L)^{\alpha} \subseteqq L^{p,q_1}(\log L)^{\alpha_1} \tag{6.8}$$

whenever either

$$q > q_1 \quad \text{and} \quad \alpha+1/q > \alpha_1+1/q_1 \tag{6.9}$$

or

$$q \leqq q_1 \quad \text{and} \quad \alpha \geqq \alpha_1. \tag{6.10}$$

Remark. It is easy to find elementary examples showing that the inclusion (6.8) does not hold in general if we permit some α satisfying $\alpha\leqq\alpha_1+1/q_1-1/q$ when $q>q_1$ or some α satisfying $\alpha<\alpha_1$ when $q\leqq q_1$ (see [2, p. 33]).

In our introduction we have noted that $L(p,q)\subset L^p$ when $p>q$ and $L(p,q)\supset L^p$ when $p<q$. Therefore, by applying Corollary 6.3 with $q=p, \alpha=0$ and $q_1=p$, $\alpha_1=0$ and by using the inclusions (2.7) and (2.8), we obtain the following chains of inclusions: If $0<q<p<\infty$, then, for every $\varepsilon>0$,

$$L^p(\log L)^{1/q-1/p+\varepsilon} \subset L(p,q) \subset L^p \subset L^{p,q}(\log L)^{1/p-1/q-\varepsilon}$$

and if $0<p<q<\infty$, then, for every $\varepsilon>0$,

$$L^{p,q}(\log L)^{1/p-1/q+\varepsilon} \subset L^p \subset L(p,q) \subset L^p(\log L)^{1/q-1/p-\varepsilon}.$$

All inclusions are the sharpest possible in the sense that we can nowhere permit that $\varepsilon=0$.

Proof of the corollary. We assume that $f\in L^{p,q}(\log L)^{\alpha}$ and $q>q_1$. First we consider the case $p>q$. Then, by Theorem 6.2 a), $f\in L^p h(\log L)$ for some function h satisfying (6.3) and

$$\int_1^{\infty} \left(h(x)x^{-\alpha}\right)^{pq/(q-p)}\,dx = \int_1^{\infty}\left(\frac{x^{\alpha}}{h(x)}\right)^{pq/(p-q)} dx < \infty. \tag{6.11}$$

We put $a=q(p-q_1)/q_1(p-q)$ and use Hölder's inequality to obtain

$$\int_1^{\infty}\left(\frac{x^{\alpha_1}}{h(x)}\right)^{pq_1/(p-q_1)} dx \leqq \left(\int_1^{\infty}\left(\frac{x^{\alpha}}{h(x)}\right)^{pq/(p-q)} dx\right)^{1/a} \left(\int_1^{\infty} x^{(\alpha_1-\alpha)qq_1/(q-q_1)}\,dx\right)^{1-1/a}.$$

The assumption $\alpha+1/q>\alpha_1+1/q_1$ implies that $(\alpha_1-\alpha)qq_1/(q-q_1)<-1$. Therefore, according to (6.11),

$$(6.12)\qquad \int_1^\infty (h(x)x^{-\alpha_1})^{pq_1/(q_1-p)}\,dx = \int_1^\infty \left(\frac{x^{\alpha_1}}{h(x)}\right)^{pq_1/(p-q_1)} dx < \infty.$$

We have just proved that $f\in L^p h(\log L)$ for some function h satisfying (6.3) and (6.12). Thus, by Theorem 6.2 a), $f\in L^{p,q_1}(\log L)^{\alpha_1}$.

For the case $p<q_1$ we assume that h is an arbitrary function satisfying (6.3) and (6.12). We put $a=q_1(p-q)/q(p-q_1)$ and use Hölder's inequality and the assumption that $(\alpha_1-\alpha)qq_1/(q-q_1)<-1$ to see that h also satisfies the condition (6.11). Therefore, according to Theorem 6.2 b), $f\in L^p h(\log L)$. By using Theorem 6.2 b) once more we conclude that $f\in L^{p,q_1}(\log L)^{\alpha_1}$.

For the case $p=q$ our assumption means that $f\in L^p h(\log L)$ for $h(x)=x^\alpha$. We note that the function h satisfies (6.3) and (6.12). We use Theorem 6.2 a) and conclude that $f\in L^{p,q_1}(\log L)^{\alpha_1}$.

When $p=q_1$ we can use Theorem 6.2 b) to see that $f\in L^p h(\log L)$ for every function h satisfying (6.3) and (6.11). We note that the function $h(x)=x^{\alpha_1}$ satisfies these conditions. Thus, $f\in L^p(\log L)^{\alpha_1}$ which in this case is equivalent to that $f\in L^{p,q_1}(\log L)^{\alpha_1}$.

Finally we suppose that $q_1<p<q$. Then we can use Theorem 6.2 b) to see that $f\in L^p h(\log L)$ for every function h satisfying the conditions (6.3) and (6.11). In particular, the assumption $(\alpha_1-\alpha)qq_1/(q-q_1)<-1$ implies that the function

$$h(x) = x^{((\alpha_1-\alpha)qq_1/p+(\alpha q-\alpha_1 q_1))/(q-q_1)}$$

satisfies these conditions. But this function $h(x)$ satisfies also the condition (6.12) so we can use Theorem 6.2 a) to conclude that $f\in L^{p,q_1}(\log L)^{\alpha_1}$. Thus the proof of the case $q_1<q$ is complete.

If $q_1\geqq q$ we may, without loss of generality, assume that $\alpha_1=\alpha$. The proof of this case is analogous and even simpler so we leave out the details.

7. Some concluding remarks

Professor Jaak Peetre has made me aware of the fact that our description of the $L(p,q)$-spaces is similar to the definition of the spaces $B^p_{\theta,q}(\omega)$, defined by Peetre [10] and Gilbert [3, pp. 242—243] in the following way: Let ω be a nonnegative weight function, $0<\theta<1$, $1\leqq p<\infty$, $1\leqq q\leqq\infty$ and $\gamma=1/p-1/q$. Let Φ_G be the set of nonnegative functions φ on $[0,\infty[$, such that

$$(7.1)\qquad \|\varphi\|_{L^*} = \int_0^\infty \varphi(t)\frac{dt}{t} = 1,$$

and

(7.2) $t^{\theta}\varphi^{\gamma}(t)$ is nondecreasing.

Then

$$B^p_{\theta,q}(\omega) = \begin{cases} \bigcup_{\varphi\in\Phi_G} \{L^p_\sigma | \sigma = \omega^\theta \varphi^\gamma(\omega)\}, & \text{when} \quad \gamma \leqq 0, \\ \bigcap_{\varphi\in\Phi_G} \{L^p_\sigma | \sigma = \omega^\theta \varphi^\gamma(\omega)\}, & \text{when} \quad \gamma \geqq 0. \end{cases}$$

In particular, when the underlying measure space is (R^n, dx) we obtain the usual Beurling—Herz spaces

$${}^pL^q = \begin{cases} B^p_{-\gamma,q}(|x|^n), & \text{when} \quad q < p, \\ B^p_{\gamma,q}\left(\dfrac{1}{|x|^n}\right), & \text{when} \quad q > p. \end{cases}$$

The Beurling spaces A^p and B^p are the special cases ${}^pL^1$ and ${}^pL^\infty$, respectively (see [3, p. 247] and [5, pp. 298—300]).

We can use our Theorem 2.2 and make some elementary calculations to see that the $L(p, q)$-spaces can be characterized in similar terms. More exactly, we can in fact *define* the $L(p, q)$-spaces in the following way: Let $0<p<\infty$, $0<q<\infty$ and $\gamma=1/p-1/q$. Let Φ_P be the set of nonnegative functions φ on $[0, \infty[$, satisfying (7.1) and, for some real number a,

(7.2)′ $t^a\varphi(t)$ is nondecreasing (or nonincreasing).

Then

$$L(p, q) = \begin{cases} \bigcup_{\varphi\in\Phi_P} \{L^p(\varphi(L))^\gamma\}, & \text{when} \quad \gamma \leqq 0, \\ \bigcap_{\varphi\in\Phi_P} \{L^p(\varphi(L))^\gamma\}, & \text{when} \quad \gamma \geqq 0. \end{cases}$$

It is also interesting to compare how the spaces $L(p, q)$ (or, equivalently, $E(p, q)$) and $B^p_{\theta,q}(\omega)$ (and, thus, the Beurling—Herz spaces ${}^pL^q$) occur as intermediate spaces in analogous situations in the theory of interpolation. For example we have

$$(L^{p_0}, L^{p_1})_{\theta,q;K} = L(p, q) \ (=E(p, q))$$

when $1/p=(1-\theta)/p_0+\theta/p_1$ (see e.g. [13, p. 134]) and

$$(L^p, L^p_\omega)_{\theta,q;K} = B^p_{\theta,q}(\omega)$$

(see [3, p. 243] and [10, pp. 64—66]).

Lorentz has in [7] defined that a function f belongs to the space $\Lambda(\varphi, q)$ if

$$\int_0^\infty (f^*)^q \varphi \, dt < \infty.$$

Here φ is a nonnegative and integrable function on $[0, \infty[$. Lorentz has also given an exact characterization of the spaces $\Lambda(\varphi, 1)$ which are also Orlicz spaces (see [8, pp. 130—132]. Roughly speaking, the result of Lorentz shows that this can happen if and only if we impose integrability conditions on φ such that the space $\Lambda(\varphi, 1)$ is fairly close to L^1.

In this context we also note that it is feasible to generalize Theorem 6.1 for example by replacing the factor $(\log —)^a$ in the conditions (6.1)—(6.2) and (6.5)—(6.6) by any "logarithmic varying" function φ. (We say that a function φ is logarithmic varying if there exist x_0 and a such that, for $x \geqq x_0$, $\varphi(x)(\log x)^a$ is a decreasing or an increasing function of x.) We can still use essentially the same techniques as in the proofs in Sections 4 and 5.

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Compact and Hilbert—Schmidt composition operators on Hardy spaces of the upper half-plane

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Introduction. Let $H^p(\pi^+)$ denote the Banach space of functions f holomorphic in π^+ (the upper half-plane) for which

$$\|f\|_p = \operatorname*{Sup}_{y>0}\left\{\left|\int_{-\infty}^{\infty} |f(x+iy)|^p\,dx\right|^{1/p}\right\} < \infty.$$

Let $T:\pi^+\to\pi^+$ be analytic. Then the composition mapping C_T, defined by

$$C_T f = f\circ T,$$

maps $H^p(\pi^+)$ into the vector space of all analytic functions on π^+. This mapping C_T is a linear transformation. If the range of C_T is a subspace of $H^p(\pi^+)$ and C_T happens to be bounded, we call it the composition operator induced by T. We are interested in the case when $p=2$. In this case $H^2(\pi^+)$ becomes a Hilbert space. For the sake of simplicity we will denote $\|\ \|_2$ simply by $\|\ \|$. Composition operators on $H^2(\pi^+)$ have been studied by SINGH [6] and SINGH and SHARMA [7]. In [7], we have proved that if T is an analytic function from π^+ into itself and the only singularity that T can have is a pole at infinity, then C_T is a bounded operator on $H^2(\pi^+)$ if and only if the point at infinity is a pole of T. In Section 2, we give a characterization of compact composition operators on $H^2(\pi^+)$. A sufficient condition for a composition operator to be compact is also provided. In Section 3, Hilbert—Schmidt composition operators are characterized.

2. Compact composition operators on $H^2(\pi^+)$. A linear operator A on a Hilbert space H is called compact if A takes bounded sets into sets with compact closures. This definition is equivalent to the statement that the image of every bounded sequence under A has a convergent subsequence [2]. This is further equivalent to saying that if $f_n\to f$ weakly in H, then $Af_n\to Af$ strongly in H. In this section we give a characterization of compact composition operators on $H^2(\pi^+)$.

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Theorem 2.1. *Let C_T be a composition operator on $H^2(\pi^+)$. Then C_T is compact if and only if for every sequence $f_n \to f$ uniformly on compact subsets of π^+ and bounded in $H^2(\pi^+)$ norm, the image sequence $C_T f_n \to C_T f$ strongly.*

We need the following lemmas to prove the theorem.

Lemma 2.1. *Let $f \in H^2(\pi^+)$. Then $|f(x+iy)|^2 \leqq \|f\|^2/2\pi y$ for $x+iy \in \pi^+$.*

Proof. First suppose $f \in H^1(\pi^+)$. Then by Cauchy-formula [1, p. 195],

$$f(w) = (2\pi i)^{-1} \int_{-\infty}^{\infty} \frac{f(r)}{r-w}\, dr.$$

Writing $w = x+iy$ and taking absolute values we get

$$|f(x+iy)| \leqq (2\pi y)^{-1} \int_{-\infty}^{\infty} |f(r)|\, dr = \|f\|_1/2\pi y.$$

(Here $\|f\|_1$ is the $H^1(\pi^+)$-norm.)

Let $f \in H^2(\pi^+)$. Then we can write $f = B \cdot g$, where B is a Blaschke product and g is an analytic function in π^+ and does not have any zero in π^+ [3, pp. 132—133]. It is obvious that $\|f\| = \|g\|$. Let $h_1 = g^2$. Then $h_1 \in H^1(\pi^+)$. Hence $f = B \cdot h_1^{1/2}$ and

$$|f(x+iy)| = |B(x+iy)|\, |h_1(x+iy)|^{1/2} \leqq \|h_1\|_1^{1/2}/\sqrt{2\pi y},$$

which implies that $|f(x+iy)|^2 \leqq \|f\|^2/2\pi y$ for every $x+iy \in \pi^+$.

Lemma 2.2. *Let $\{f_n\}$ be a sequence in $H^2(\pi^+)$. Then $f_n \to f$ in norm implies that $f_n \to f$ uniformly on compact subsets of π^+.*

Proof. Suppose $f_n \to f$ strongly. Let K be a compact subset of π^+. Then by Lemma 2.1

$$|f_n(x+iy) - f(x+iy)| \leqq (2\pi)^{-1/2} M_K \|f_n - f\|,$$

where $M_K = \sup_{x+iy \in K} \{y^{-1/2}\}$. The right hand side tends to zero as $n \to \infty$ for every point $x+iy \in K$. Since K is an arbitrary compact subset of π^+ and $f_n \to f$ uniformly on the compact subset K of π^+, the proof follows.

Lemma 2.3. *If $\{f_n\}$ is a bounded sequence in $H^2(\pi^+)$, then there exists a subsequence $\{f_{n_k}\}$ which converges uniformly on compact subsets.*

Proof. In the light of Theorem 14.6 of [4] it is enough to show that the sequence $\{f_n\}$ is uniformly bounded on each compact subset of π^+. If K is a compact subset of π^+, then again by Lemma 2.1 we have for $x+iy \in K$ that

$$|f_n(x+iy)| \leqq (2\pi)^{-1/2} M_K M,$$

where M_K is as in Lemma 2.2 and $M \geqq 0$ is such that $\|f_n\| \leqq M$ for all n. This finishes the proof.

Proof of Theorem 2.1. Suppose C_T is compact. Let $\{f_n\}$ be a sequence in $H^2(\pi^+)$ and $f \in H^2(\pi^+)$ such that $f_n \to f$ uniformly on compact subsets of π^+ and let $M \geqq 0$ be such that $\|f_n\| \leqq M$ for all n. Then we want to show that

$$\|C_T f_n - C_T f\| \to 0 \quad \text{as} \quad n \to \infty.$$

Suppose this is not true. Then there exists a subsequence $C_T f_{n_k}$ and an $\varepsilon > 0$ such that $\|C_T f_{n_k} - C_T f\| \geqq \varepsilon > 0$. Since $\{f_{n_k}\}$ is norm bounded and C_T is compact, there exists a subsequence $\{f_{n_{k_l}}\}$ such that $C_T f_{n_{k_l}} \to g$ strongly for some $g \in H^2(\pi^+)$. Hence, by Lemma 2.2, $C_T f_{n_{k_l}} \to g$ uniformly on compact subsets. Also, the subsequence $\{f_{n_{k_l}}\}$ converges to f uniformly on compact subsets, implying that $\{C_T f_{n_{k_l}}\}$ converges to $C_T f$ uniformly on compact subsets. This shows that $C_T f = g$, which is a contradiction. This proves that $\|C_T f_n - C_T f\| \to 0$ as $n \to \infty$.

In order to prove the converse, let F be a bounded set in $H^2(\pi^+)$. We want to show that the closure of $\{C_T f : f \in F\}$ is compact. Let $\{C_T f_n\}$ be a sequence in this closure. Then, since $\{f_n\}$ is norm bounded, by Lemma 2.3 there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ converging uniformly on compact subsets to some function f. Hence, by our hypothesis, $C_T f_{n_k} \to C_T f$ strongly, which shows that $\{C_T f_n\}$ has an accumulation point. Thus the closure of $\{C_T f : f \in F\}$ is countably compact and hence compact. This completes the proof.

In the next theorem the above result is used to give a sufficient condition for a composition operator to be compact.

Theorem 2.2. *Let $T : \pi^+ \to \pi^+$ be an analytic function such that C_T is a bounded operator on $H^2(\pi^+)$. Suppose $T_*(x) = \lim_{y \to 0} T(x+iy)$ exists a.e. and $T_*(x) \in \pi_+$ for almost all $x \in R$ (the set of reals). If $\int_{-\infty}^{\infty} [\operatorname{im} T_*(x)]^{-1} dx < \infty$, then C_T is a compact composition operator on $H^2(\pi^+)$.*

The following lemma is required to prove the theorem.

Lemma 2.4. *If $T_*(x) = \lim_{y \to 0} T(x+iy)$ exists a.e. and $T_*(x) \in \pi^+$ for almost all $x \in R$, then for every $f \in H^2(\pi^+)$*

$$(f \circ T_*)(x) = (f \circ T)_*(x) \quad a.e. \text{ on } R.$$

Proof. Let $E_1 = \{x \in R : T_*(x)$ does not exist$\}$, $E_2 = \{x \in R : T_*(x) \notin \pi^+\}$ and $E = E_1 \cup E_2$. Then for $x \in R \setminus E$, $T_*(x) = \lim_{y \to 0} T(x+iy)$ belongs to π^+. Since f is analytic at $T_*(x)$, it follows by the continuity of f that

$$(f\circ T_*)(x) = f(\lim_{y\to 0} T(x+iy)) = \lim_{y\to 0}(f\circ T)(x+iy) = (f\circ T)_*(x)$$

for every $x\in R\setminus E$. Since the set E has Lebesgue measure zero, the result follows.

Proof of Theorem 2.2. Let $\{f_n\}$ be a bounded sequence in $H^2(\pi^+)$ such that $f_n\to f$ uniformly on compact subsets. If we show that $C_Tf_n\to C_Tf$ strongly, then we are done. Using Lemmas 2.4 and 2.1 we have

$$|(C_Tf_n-C_Tf)_*(x)|^2 = |(f_n\circ T)_*-(f\circ T)_*(x)|^2 =$$
$$= |(f_n\circ T_*)(x)-(f\circ T_*)(x)|^2 = |(f_n-f)(T_*(x))|^2 \leqq M/im\, T_*(x),$$

where $M\geqq 0$ is such that $\|f_n-f\|/2\pi\leqq M$ for all n. Since $T_*(x)\in\pi^+$ for $x\in R\setminus E$ and the convergence is uniform on compact subsets, we have

$$(C_Tf_n-C_Tf)_*(x) = f_n(T_*(x))-f(T_*(x)) \to 0 \quad \text{as} \quad n\to\infty \quad \text{for all} \quad x\in R\setminus E;$$

where E is the set as described in Lemma 2.4. This shows that $|(C_Tf_n-C_Tf)_*|^2\to 0$ as $n\to\infty$ pointwise on $R\setminus E$ and the functions $|C_Tf_n-C_Tf)|^2$ are bounded by an integrable function g defined by $g(x)=1/\text{im}\, T_*(x)$ for $x\in R$. Hence, by Lebesgue's dominated convergence theorem and by the equality

$$\|f\|^2 = \int_{-\infty}^{\infty} |f_*(x)|^2\,dx \quad \text{for every} \quad f\in H^2(\pi^+)$$

(see [1, p. 190]), it follows that $\|C_Tf_n-C_Tf\|^2\to 0$ as $n\to\infty$. This completes the proof.

3. Hilbert—Schmidt composition operators on $H^2(\pi^+)$. A linear operator A on an infinite dimensional separable Hilbert space is said to be Hilbert—Schmidt if there exists an orthonormal basis $\{e_n: n\in\mathrm{N}\}$ in H such that

$$\sum_{n\in\mathrm{N}} \|Ae_n\|^2 < \infty. \tag{3.1}$$

It is easy to see that the sum on the right side of (3.1) does not depend upon the particular choice of the orthonormal basis $\{e_n: n\in\mathrm{N}\}$ [5].

In Theorem 2.2 it has been analysed that if an analytic function T maps the upper half-plane into the upper half-plane and C_T is a composition operator on $H^2(\pi^+)$, then the following condition

$$\int_{-\infty}^{\infty} 1/im\, T_*(x)\,dx < \infty \tag{3.2}$$

is sufficient for C_T to be a compact composition operator on $H^2(\pi^+)$. In fact, the condition (3.2) turns out to be a necessary and sufficient condition for C_T to be a Hilbert—Schmidt composition operator on $H^2(\pi^+)$. This we demonstrate in the following theorem.

Theorem 3.1. *Let* $T:\pi^+\to\pi^+$ *be an analytic function such that* C_T *is a composition operator on* $H^2(\pi^+)$. *Suppose* $T_*(x)=\lim_{y\to 0} T(x+iy)$ *exists a.e. and* $T_*(x)\in\pi^+$ *for almost all* $x\in R$. *Then the condition* (3.2) *is necessary and sufficient for* C_T *to be Hilbert—Schmidt.*

Proof. We know that the family of functions S_n defined by

$$S_n(w)=\frac{(w-i)^n}{\sqrt{\pi}(w+i)^{n+1}} \quad (n=0,1,\ldots)$$

forms an orthonormal basis for $H^2(\pi^+)$. Therefore, C_T is Hilbert—Schmidt if and only if

$$\infty>\sum_{n=0}^{\infty}\|C_T S_n\|^2=\sum_{n=0}^{\infty}\int_{-\infty}^{\infty}|(S_n\circ T)_*(x)|^2\,dx=\sum_{n=0}^{\infty}\int_{-\infty}^{\infty}\left|S_n(T_*(x))\right|^2 dx$$

(the equalities above follow from [1, p. 190] and Lemma 2.4, respectively). A simple computation yields that C_T is Hilbert—Schmidt if and only if

$$\infty>\pi^{-1}\int_{-\infty}^{\infty}[4\,im\,T_*(x)]^{-1}\,dx.$$

Hence the theorem.

Remark. It is worthwhile to remark here that Theorem 2.2 follows as an easy consequence of Theorem 3.1. In spite of this we have presented an independent proof to Theorem 2.2 because of the following reason: With a little modification Theorem 2.1 and consequently Theorem 2.2 can easily be developed for the Banach spaces $H^p(\pi^+)\,(1\le p<\infty)$. Hence if we consider a composition operator on $H^p(\pi^+)$, the condition (3.2) turns out to be sufficient for a composition operator C_T to be compact on $H^p(\pi^+)$. Whereas, in case of $H^2(\pi^+)$, the condition (3.2) is necessary as well as sufficient for a composition operator C_T to be a Hilbert—Schmidt operator.

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On the absolute Riesz summability of orthogonal series

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1. Let Σa_n be a given infinite series and s_n denote its nth partial sum. If $\{p_n\}$ is a sequence of positive numbers, and

$$P_n = \sum_{k=0}^{n} p_k \to \infty \quad \text{as} \quad n \to \infty,$$

then the nth Riesz mean R_n of Σa_n is defined by

$$R_n = \frac{1}{P_n} \sum_{k=0}^{n} p_k s_k. \tag{1.1}$$

If the series

$$\sum_{n=1}^{\infty} |R_n - R_{n-1}| \tag{1.2}$$

converges, then the series Σa_n is said to be summable $|R, P_n, 1|$. It is clear that if $p_k = 1$ then (1.1) reduces to the classical $(C, 1)$-mean, and $|R, n+1, 1|$ means that the series Σa_n is absolute $(C, 1)$-summable.

Let $\{\varphi_n(x)\}$ be an orthonormal system defined on the finite interval (a, b). We consider the orthogonal series

$$\sum_{k=0}^{\infty} c_k \varphi_k(x) \quad \text{with} \quad \sum_{k=0}^{\infty} c_k^2 < \infty. \tag{1.3}$$

Furthermore let $P(x)$ be a strictly increasing function such that $P(n) = P_n$ and linear between n and $n+1$. We denote the inverse function of $P(x)$ by $\Lambda(x)$ and put $\nu_m = [\Lambda(2^m)]$, where $[x]$ denotes the integral part of x.

K. TANDORI [5] proved that *the condition*

$$\sum_{m=0}^{\infty} \left\{ \sum_{n=2^m+1}^{2^{m+1}} c_n^2 \right\}^{1/2} < \infty \tag{1.4}$$

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is necessary and sufficient that series (1.3) *for every orthonormal system* $\{\varphi_n(x)\}$ *should be absolute* $(C, 1)$*-summable, or summable* $|R, n+1, 1|$ *almost everywhere in* (a, b).

We ([1]) showed that *condition* (1.4) *is also necessary and sufficient that series* (1.3) *for every orthonormal system* $\{\varphi_n(x)\}$ *be absolute* (C, α)*-summable with* $\alpha > 1/2$ *almost everywhere.* In [1] we also gave conditions implying the absolute $(C, 1/2)$- and (C, α)-summability with $-1<\alpha<1/2$.

The result of Tandori was generalized by F. Móricz [3] to the absolute Riesz summability as follows.

Theorem A. *Orthogonal series* (1.3) *for every orthonormal system* $\{\varphi_n(x)\}$ *is summable* $|R, P_n, 1|$ *almost everywhere if and only if*

$$\sum_{m=0}^{\infty}\left\{\sum_{n=\nu_m+1}^{\nu_{m+1}} c_n^2\right\}^{1/2} < \infty, \tag{1.5}$$

where $C_m=\left\{\sum_{n=\nu_m+1}^{\nu_{m+1}} c_n^2\right\}^{1/2}=0$ *if* $\nu_{m+1}=\nu_m$.

Recently Y. Okuyama and T. Tsuchikura [4] gave a condition which is equivalent to (1.5) and it does not use the concept of $\Lambda(x)$.

More precisely they proved

Theorem B. *Condition* (1.5) *is equivalent to*

$$\sum_{n=1}^{\infty}\frac{p_n}{P_nP_{n-1}}\left\{\sum_{k=1}^{n} P_{k-1}^2 c_k^2\right\}^{1/2} < \infty. \tag{1.6}$$

Using these theorems and some lemmas the authors of [4] also proved the following

Theorem C. *If the series*

$$\sum_{n=1}^{\infty}\frac{p_n}{P_nP_{n-1}}\left\{\sum_{k=1}^{n} P_{k-1}^2 (a_k^2+b_k^2)\right\}^{1/2} \tag{1.7}$$

converges, then almost all series of

$$\sum_{n=1}^{\infty} \pm(a_n\cos nx+b_n\sin nx) \tag{1.8}$$

are summable $|R, P_n, 1|$ *almost everywhere, and if series* (1.7) *diverges, then almost all series of* (1.8) *are non-summable* $|R, P_n, 1|$ *almost everywhere.*

2. In the present note we prove certain symmetrical analogues of Theorems B and C.

Theorem 1. *Condition* (1.5) *is equivalent to*

$$\sum_{n=0}^{\infty} p_n \left\{ \sum_{k=n}^{\infty} P_k^{-2} c_k^2 \right\}^{1/2} < \infty. \tag{2.1}$$

By Theorem A and Theorem 1 we immediately obtain

Corollary 1. *Condition* (2.1) *is necessary and sufficient that series* (1.3) *for any orthonormal system* $\{\varphi_n(x)\}$ *should be summable* $|R, P_n, 1|$ *almost everywhere.*

Hence we get

Corollary 2. *If*

$$\sum_{n=0}^{\infty} \frac{p_n}{P_n} \left\{ \sum_{k=n}^{\infty} c_k^2 \right\}^{1/2} < \infty \tag{2.2}$$

then series (1.3) *for every orthonormal system* $\{\varphi_n(x)\}$ *is summable* $|R, P_n, 1|$ *almost everywhere.*

It is well known, by the Riesz—Fischer theorem, that series (1.3) converges in L^2 to a square-integrable function f; and if $E_n^{(2)}(f)$ denotes the best approximation to f in the metric of L^2 by means of polynomials of $\varphi_0, \ldots, \varphi_{n-1}$, then

$$E_n^{(2)}(f) = \left\{ \sum_{k=n}^{\infty} c_k^2 \right\}^{1/2}.$$

Thus, by Corollary 2, *condition*

$$\sum_{n=0}^{\infty} \frac{p_n}{P_n} E_n^{(2)}(f) < \infty \tag{2.3}$$

also implies the $|R, P_n, 1|$ *summability of* (1.3) *for every orthonormal system* $\{\varphi_n\}$ *almost everywhere.*

If $\{\varphi_n\}$ is the trigonometric system, i.e., if we consider the following orthogonal series

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \equiv \sum_{n=0}^{\infty} A_n(x), \tag{2.4}$$

then using Corollary 2 and the following estimation (see [2], Hilfssatz II)

$$E_n^{(2)}(f) \leqq w_n^{(2)}\left(\frac{1}{n}, f\right),$$

$$w_2^{(2)}(\delta, f) := \left\{ \frac{1}{\delta} \int_0^{\delta} \left(\int_0^{2\pi} [f(x+2t) + f(x-2t) - 2f(x)]^2 \, dx \right) dt \right\}^{1/2},$$

we also have a further

Corollary 3. *If*

$$\sum_{n=0}^{\infty} \frac{p_n}{P_n} w_2^{(2)}\left(\frac{1}{n}, f\right) < \infty \tag{2.5}$$

then series (2.4) *is summable* $|R, P_n, 1|$ *almost everywhere.*

The next theorem is the analogue of Theorem C.

Theorem 2. *If the series*

$$\sum_{n=1}^{\infty} p_n \left\{ \sum_{k=n}^{\infty} (a_k^2 + b_k^2) P_k^{-2} \right\}^{1/2} \tag{2.6}$$

converges, then all series of (1.8) *are summable* $|R, P_n, 1|$ *almost everywhere, and if series* (2.6) *diverges, then almost all series of* (1.8) *are non-summable* $|R, P_n, 1|$ *almost everywhere.*

3. In order to prove our theorems we require the following lemmas.

Lemma 1 ([3]). *Suppose that the set of points for which the Rademacher series* $\sum_{n=0}^{\infty} c_n r_n(x)$ *is summable* $|R, P_n, 1|$ *is of positive measure, then condition* (1.5) *holds.*

Lemma 2. *Let*

$$A_n(x) = \varrho_n \cos(nx + Q_n) \quad \textit{with} \quad \varrho_n = (a_n^2 + b_n^2)^{1/2}.$$

If the series

$$\sum_{n=1}^{\infty} p_n \left\{ \sum_{k=n}^{\infty} A_k^2(x) P_k^{-2} \right\}^{1/2} \tag{3.1}$$

converges on a set E_0 *of positive measure, then the series*

$$\sum_{n=1}^{\infty} p_n \left\{ \sum_{k=n}^{\infty} \varrho_k^2 P_k^{-2} \right\}^{1/2} \tag{3.2}$$

converges. Conversely, the convergence of (3.2) *implies that of* (3.1) *for every* x.

The proof of Lemma 2 follows the same line as that of an analogous lemma of Y. Okuyama and T. Tsuchikura [4].

Proof. First we prove the implication (3.1)⇒(3.2). By the assumption there exists a set $E \subset E_0$ of positive measure such that

$$I \equiv \sum_{n=1}^{\infty} p_n \int_E \left\{ \sum_{k=n}^{\infty} P_k^{-2} \varrho_k^2 \cos^2(kx + Q_k) \right\}^{1/2} dx \leqq K\mu(E), \tag{3.3}$$

where K denotes a positive constant and $\mu(E)$ denotes the Lebesgue measure of

E. Using the Minkowski inequality with $p=1/2$, we obtain that

$$I \geqq \sum_{n=1}^{\infty} p_n \left\{ \sum_{k=n}^{\infty} \left(\int_E P_k^{-1} \varrho_k |\cos(kx+Q_k)|\, dx \right)^2 \right\}^{1/2} =$$

(3.4)

$$= \sum_{n=1}^{\infty} p_n \left\{ \sum_{k=n}^{\infty} P_k^{-2} \varrho_k^2 \left(\int_E |\cos(kx+Q_k)|\, dx \right)^2 \right\}^{1/2}.$$

Using the Riemann—Lebesgue theorem and the following estimation

$$\int_E |\cos(kx+Q_k)|\, dx \geqq \int_E \cos^2(kx+Q_k)\, dx = \frac{1}{2} \int_E (1+\cos 2(kx+Q_k))\, dx =$$

$$= \frac{1}{2}\mu(E) + \frac{1}{2} \int_E \cos 2(kx+Q_k)\, dx$$

we obtain that for sufficiently large $k \geqq k_0$

$$\int_E |\cos(kx+Q_k)|\, dx \geqq \frac{1}{4}\mu(E) \equiv A. \tag{3.5}$$

Thus, by (3.4) and (3.5), we have that

$$I \geqq A \sum_{n=k_0}^{\infty} p_n \left\{ \sum_{k=n}^{\infty} P_k^{-2} \varrho_k^2 \right\}^{1/2}, \tag{3.6}$$

whence

$$\sum_{k=1}^{\infty} P_k^{-2} \varrho_k^2 < \infty$$

follows obviously, and this implies that

$$\sum_{n=0}^{k_0-1} p_n \left\{ \sum_{k=n}^{\infty} P_k^{-2} \varrho_k^2 \right\}^{1/2} < \infty. \tag{3.7}$$

Summing up, by (3.3), (3.6) and (3.7), the implication (3.1)⇒(3.2) is proved.

Since $A_n^2(x) \leqq \varrho_n^2$, the implication (3.2)⇒(3.1) is trivial. Thus the proof is completed.

4. Now we can start the proofs of the theorems.

Proof of Theorem 1. First we prove that condition (1.5) implies (2.1). An elementary calculation shows that

$$\sum_{k=\nu_0+1}^{\infty} p_k \left\{ \sum_{n=k}^{\infty} P_n^{-2} c_n^2 \right\}^{1/2} \leqq \sum_{m=0}^{\infty} \sum_{k=\nu_m+1}^{\nu_{m+1}} p_k \left\{ \sum_{n=k}^{\infty} P_n^{-2} c_n^2 \right\}^{1/2} \leqq$$

(4.1)

$$\leqq \sum_{m=0}^{\infty} \sum_{k=\nu_m+1}^{\nu_{m+1}} p_k \left\{ \sum_{n=\nu_m+1}^{\infty} P_n^{-2} c_n^2 \right\}^{1/2} \leqq \sum_{m=0}^{\infty} \sum_{k=\nu_m+1}^{\nu_{m+1}} p_k \sum_{i=m}^{\infty} P_{\nu_i+1}^{-1} \left\{ \sum_{n=\nu_i+1}^{\nu_{i+1}} c_n^2 \right\}^{1/2} \equiv \sum\nolimits_1 .$$

Since

$$P_{\nu_i+1} = P(\nu_i+1) \geqq P(\Lambda(2^i)) = 2^i, \tag{4.2}$$

thus

$$\sum_1 \leqq \sum_{m=0}^{\infty} \sum_{k=\nu_m+1}^{\nu_{m+1}} p_k \sum_{i=m}^{\infty} 2^{-i} C_i = \sum_{i=0}^{\infty} 2^{-i} C_i \sum_{m=0}^{i} \sum_{k=\nu_m+1}^{\nu_{m+1}} p_k \leqq$$

$$\leqq \sum_{i=0}^{\infty} 2^{-i} C_i \sum_{k=0}^{\nu_{i+1}} p_k \leqq \sum_{i=0}^{\infty} 2^{-i} C_i P(\Lambda(2^{i+1})) \leqq 2 \sum_{i=0}^{\infty} C_i. \tag{4.3}$$

By (4.1) and (4.3) the implication (1.5)⇒(2.1) is proved.

Next we prove the converse implication. It is clear that

$$P_{\nu_m} \leqq P(\Lambda(2^m)) = 2^m, \tag{4.4}$$

thus, by (4.2) and (4.4), we have that

$$\left(\sum_{k=\nu_{m-1}+1}^{\nu_m+1} p_k\right) P_{\nu_{m+1}}^{-1} = (P_{\nu_m+1} - P_{\nu_{m-1}}) P_{\nu_{m+1}}^{-1} \geqq (2^m - 2^{m-1}) 2^{-m-1} = \frac{1}{4}.$$

Using this inequality we obtain that

$$\sum_{m=1}^{\infty} C_m \leqq 4 \sum_{m=1}^{\infty} \left(\sum_{k=\nu_{m-1}+1}^{\nu_m+1} p_k\right) P_{\nu_{m+1}}^{-1} C_m \leqq$$

$$\leqq 4 \sum_{m=1}^{\infty} \sum_{n=\nu_{m-1}+1}^{\nu_{m+1}} p_k \left\{\sum_{n=\nu_m+1}^{\nu_{m+1}} P_n^{-2} c_n^2\right\}^{1/2} \equiv \sum_2, \tag{4.5}$$

where

$$C_m(p) := \left\{\sum_{n=\nu_m+1}^{\nu_{m+1}} P_n^{-2} c_n^2\right\}^{1/2}$$

means zero if $\nu_m = \nu_{m+1}$. Therefore

$$\sum_2 = 4 \sum_m{}' \sum_{n=\nu_{m-1}+1}^{\nu_m+1} p_k C_m(p), \tag{4.6}$$

where $\sum_m{}'$ denotes that the summation runs just through such indices m which have the property $\nu_{m+1} \geqq \nu_m + 1$. Then

$$\sum_m{}' \sum_{k=\nu_{m-1}+1}^{\nu_m+1} p_k C_m(p) \leqq \sum_m{}' \sum_{k=\nu_{m-1}+1}^{\nu_m+1} p_k \left\{\sum_{n=k}^{\infty} P_n^{-2} c_n^2\right\}^{1/2} \leqq$$

$$\leqq \sum_m{}' \sum_{k=\nu_{m-1}+1}^{\nu_{m+1}} p_k \left\{\sum_{n=k}^{\infty} P_n^{-2} c_n^2\right\}^{1/2} \leqq \tag{4.7}$$

$$\leqq 2 \sum_{m=1}^{\infty} \sum_{k=\nu_{m-1}+1}^{\nu_m} p_k \left\{\sum_{n=k}^{\infty} P_n^{-2} c_n^2\right\}^{1/2} \leqq 2 \sum_{k=0}^{\infty} p_k \left\{\sum_{n=k}^{\infty} P_n^{-2} c_n^2\right\}^{1/2}.$$

By (4.5), (4.6) and (4.7) we have

$$\sum_{m=1}^{\infty} C_m \leqq 8 \sum_{k=0}^{\infty} p_k \left\{ \sum_{n=k}^{\infty} P_n^{-2} c_n^2 \right\}^{1/2},$$

which proves the implication (2.1)⇒(1.5), and this completes the proof of Theorem 1.

Proof of Theorem 2. The proof is the same as that of Theorem C, the only difference is that we use Theorem 1 and Lemma 2 instead of Theorem B and Lemma 2 of [4].

The sketch of the proof is the following: By Lemmas 1 and 2 and Theorem 1 we have to follow the Paley and Zygmund argument (cf. [6, p. 214]).

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Approximation in L^1 by Kantorovich polynomials

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1.

This paper is a continuation of two earlier ones [11, 12]. Let

$$K_n(f;x)=\sum_{k=0}^{n}\Big((n+1)\int\limits_{k/(n+1)}^{(k+1)/(n+1)} f(u)\,du\Big)\,b_{n,k}(x),\quad b_{n,k}(x)=\binom{n}{k}x^k(1-x)^{n-k}$$

be the Kantorovich-variant of the Bernstein operator. A series of papers contains results for the approximation properties of $K_n(f)$ in integral metrics (for references see the survey article [3]). However, the analogue of the well-known equivalence theorem of BERENS and LORENTZ [5] or that of LORENTZ and SCHUMAKER [7] and DITZIAN [6] is not known for them. The problem is the characterization of $\|K_n(f)-f\|_{L^1(0,1)}=O(n^{-\alpha})$ $(0<\alpha<1)$ in terms of a certain modulus of smoothness, and the aim of this paper is to give this characterization.

For $f\in L^p(0,1)$, $p>1$ we proved in [12]

Theorem A. *If* $1<p<\infty$, $0<\alpha<1$ *and* $f\in L^p(0,1)$ *then*

(i) $\|K_n(f)-f\|_{L^p}=O(n^{-\alpha})$

and

(ii) (α) $\|\Delta^*_{h\sqrt{x(1-x)}}(f;x)\|_{L^p(h^2,1-h^2)}=O(h^{2\alpha})$,

(β) $\|f(\cdot+h)-f(\cdot)\|_{L^p(0,1-h)}=O(h^\alpha)$

are equivalent.

Here

$$\Delta^*_h(f;x)=f(x-h)-2f(x)+f(x+h)$$

(we deviate from the custom and write $\|f(x)\|_{L^p}$ instead of $\|f(\cdot)\|_{L^p}$ if the former is more suggestive).

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For the saturation case $\alpha=1$ we have (see [9, 10, 4, 12])

Theorem B. *If* $1<p<\infty$ *and* $f\in L^p(0,1)$ *then the following are equivalent:*

(i) $\|K_n(f)-f\|_{L^p}=O(n^{-1})$,
(ii) *f has an absolutely continuous derivative with* $x(1-x)f''(x)\in L^p(0,1)$
(iii) $\|(x(1-x)\Delta_h^*(F;x))'\|_{L^p(h,1-h)}=O(h^2)$,
(iv) $\|x(1-x)\Delta_h^*(f;x)\|_{L^p(h,1-h)}=O(h^2)$,
(v) $\|\Delta^*_{h\sqrt{x(1-x)}}(f;x)\|_{L^p(h^2,1-h^2)}=O(h^2)$.

Here $F(x)=\int_0^x f(u)\,du$ and naturally (ii) means that "f coincides a.e. with a function which has absolutely continuous derivative".

Turning to L^1 let us mention the saturation result (see [8, 2]):

Theorem C. *For* $f\in L^1(0,1)$ *the following conditions are equivalent:*
(i) $\|K_n(f)-f\|_{L^1}=O(n^{-1})$,
(ii) *f is absolutely continuous and* $x(1-x)f'(x)$ *is of bounded variation,*
(iii) $\|x(1-x)\Delta_h^*(F,x)\|_{BV+L^\infty(h,1-h)}=O(h^2)$

Here $BV+L^\infty$ denotes the sum of the two norms: total variation and ess. supremum. Examples show that Theorem B does not hold for L^1, i.e., the BV-norm in Theorem C seems to be the appropriate one and we cannot hope in replacing it by an L^1-norm. The difference between Theorems B and C suggests also that we should exchange the L^p-norm in Theorem A for a BV-norm or something like that to obtain a correct result in L^1 (see also the conjecture in [3]). Thus, it is rather surprising that Theorem A holds word for word when $p=1$:

Theorem 1. *If* $0<\alpha<1$ *and* $f\in L^1(0,1)$ *then*

(i) $\|K_n(f)-f\|_{L^1}=O(n^{-\alpha})$

and

(ii) (α) $\|\Delta^*_{h\sqrt{x(1-x)}}(f;x)\|_{L^1(h^2,1-h^2)}=O(h^2)$,
(β) $\|f(\cdot+h)-f(\cdot)\|_{L^1(0,1-h)}=O(h^\alpha)$

are equivalent.

Let us mention that although (ii)$\Rightarrow$(i) holds also for $\alpha=1$, neither (ii) (α) nor (ii) (β) is necessary for (i) in the case $\alpha=1$. This is shown by the function $f(x)=\log x$ $(x\in(0,1))$.

The first result with the modulus of smootheness $\sup_{0<h\le\delta}\|\Delta^*_{h\sqrt{x(1-x)}}(f,x)\|_{L^p(h^2,1-h^2)}$ (more precisely with its analogue) was proved in [11] for the Szász—Kantorovich operators:

$$M_n(f;x)=\sum_{k=0}^{\infty}\Big(n\int_{k/n}^{(k+1)/n}f(u)\,du\Big)p_{n,k}(x),\quad p_{n,k}(x)=e^{-nx}\frac{(nx)^k}{k!},\quad x\ge 0.$$

Theorem D. *For* $1<p<\infty$, $0<\alpha<1$ *and* $f\in L^p(0,\infty)$ *the following conditions are equivalent:*

(i) $\|M_n(f)-f\|_{L^p(0,\infty)} = O(n^{-\alpha})$,
(ii) (α) $\|\Delta^*_{h\sqrt{x}}(f;\, x)\|_{L^p(h^2,\infty)} = O(h^{2\alpha})$
(β) $\|f(\cdot+h)-f(\cdot)\|_{L^p(0,\infty)} = O(h^\alpha)$.

This is true just as well for $p=1$:

Theorem 2. *Theorem* D *holds also when* $p=1$.

We shall prove only Theorem 2, but our method works also for K_n (the technical details are somewhat easier for M_n); we refer to [12] for the necessary changes in the proof (observe that [12] relates to [11] about as Theorem 1 relates to Theorem 2). The only point in our proof which might not be obvious for K_n is the delicate formula (2.5) but the analogue of this was given in [12, (4.5)].

Although Theorems A and 1 (D and 2) have the same form, here we have to use a different method since in the case $p>1$ the proof rested heavily on the maximal inequality. Nevertheless, the roots of the proofs of the inverse parts are the same: the so called elementary method of inverse results developed by Berens and Lorentz [5], and Becker and Nessel [1].

2. Proof of Theorem 2

I. Proof of (ii)$\Rightarrow$(i). First we derive from (ii) three further inequalities.
Inequality 1.

$$\int_0^h\!\!\int |f(x)-f(y)|\,dx\,dy = 2\int_0^h d\varepsilon \int_0^{h-\varepsilon} |f(x+\varepsilon)-f(x)|\,dx \leqq K\int_0^h \varepsilon^\alpha\,d\varepsilon \leqq Kh^{\alpha+1}.$$

Inequality 2.

$$A(f,h) \overset{\text{def}}{=} \left\|\frac{1}{x}\int_0^{h\sqrt{x}} |f(x\pm\tau)-f(x)|\,d\tau\right\|_{L^1(h^2,\infty)} \leqq Kh^{2\alpha} \quad (h\geqq 0).$$

Proof. For any $f\in L^1(0,\infty)$,

$$\int_{h^2}^{\infty} \frac{1}{x}\int_0^{h\sqrt{x}} |f(x\pm\tau)|\,d\tau\,dx \leqq Kh^{-2}\int_0^{2h^2}\!\!\int |f(x+u)|\,du\,dx +$$

$$+K\int_{2h^2}^{\infty} |f(u)|\frac{h\sqrt{u}}{u}\,du \leqq K\|f\|_{L^1}$$

and if f is absolutely continuous with $f'\in L_1$ then

$$A(f,h)\leqq\int_{h^2}^{\infty}\frac{1}{x}\int_0^{h\sqrt{x}}d\tau\left|\int_0^{\pm\tau}|f'(x+u)|\,du\right|dx\leqq$$

$$\leqq\int_{h^2}^{\infty}\frac{1}{x}\int_0^{h\sqrt{x}}(h\sqrt{x}-|u|)\,|f'(x\pm u)|\,du\,dx\leqq Kh^2\|f'\|_{L^1}$$

Let now $f\in L^1$ be arbitrary for which (ii) (β) holds, and let

$$g_h(x)=\frac{1}{h^2}\int_0^{h^2}f(x+\tau)\,d\tau.$$

For this

$$\|f-g_h\|_{L^1}\leqq h^{-2}\int_0^{h^2}\|f(\cdot+\tau)-f(\cdot)\|_{L^1}\,d\tau\leqq Kh^{-2}\int_0^{h^2}\tau^{\alpha}\,d\tau\leqq Kh^{2\alpha}$$

and

$$\|g_h'\|_{L^1}=h^{-2}\|f(\cdot+h^2)-f(\cdot)\|_{L^1}\leqq Kh^{2\alpha-2}$$

by which

$$A(f,h)\leqq A(f-g_h,h)+A(g_h,h)\leqq K(\|f-g_h\|_{L^1}+h^2\|g_h'\|_{L^1})\leqq Kh^{2\alpha}.$$

Inequality 3.

$$\left\|\frac{1}{h\sqrt{x}}\int_0^{h\sqrt{x}}|\Delta_\tau^*(f;x)|\,d\tau\right\|_{L^1(h^2,\infty)}=\left\|\frac{1}{h}\int_0^{h}|\Delta_{u\sqrt{x}}^*(f;x)|\,du\right\|_{L^1(h^2,\infty)}\leqq$$

$$\leqq\frac{1}{h}\int_0^h\|\Delta_{u\sqrt{x}}^*(f;x)\|_{L^1(h^2,\infty)}\,du\leqq K\frac{1}{h}\int_0^h u^{2\alpha}\,du\leqq Kh^{2\alpha}.$$

Now the analogous inequalities for L^p were the only tools used at the proof of (ii)$\Rightarrow$(i) in [11, Theorem 1], and this proof equally holds, using Inequalities 1—3, for $p=1$. For the details see [11].

II. Proof of (i)$\Rightarrow$(ii) (β). Let

$$v(f;\delta)=v(f)=\sup_{0\leqq h\leqq\delta}\|f(\cdot+h)-f(\cdot)\|_{L^1(0,\infty)}.$$

It is sufficient to prove that for $0<h\leqq 1$, $n\geqq 1$,

$$v(h)\leqq K\left(n^{-\alpha}+nhv\left(\frac{1}{n}\right)\right);$$

see [1, Lemma 2.1].

But

$$v(f;h)\leqq v(f-M_n(f);\ h)+v(M_n(f);h)$$

and here, by (i),

$$v(f-M_n(f);\, h) \leqq 2\|f-M_n(f)\|_{L^1} \leqq Kn^{-\alpha}.$$

By

$$M_n'(f;x) = n\sum_{k=0}^{\infty}\left(n\int_0^{1/n}\left(f\left(\frac{k}{n}+u\right)-f\left(\frac{k+1}{n}+u\right)\right)du\right)p_{n,k}(x)$$

we have

$$v(M_n(f);\, h) \leqq \int_0^{\infty} dx \int_0^{h} |M_n'(f;\, x+u)|\,du \leqq \int_0^{h} \|M_n'(f)\|_{L^1}\,du \leqq$$

$$\leqq hn\sum_{k=0}^{\infty}\int_0^{1/n}\left|f\left(\frac{k}{n}+u\right)-f\left(\frac{k+1}{n}+u\right)\right|du\int_0^{\infty} np_{n,k}(x)\,dx =$$

$$= hn\sum_{k=n}^{\infty}\int_0^{1/n}\left|f\left(\frac{k}{n}+u\right)-f\left(\frac{k+1}{n}+u\right)\right|du = hn\left\|f\left(\cdot+\frac{1}{n}\right)-f(\cdot)\right\|_{L^1} \leqq hnv\left(\frac{1}{n}\right),$$

and the proof is complete.

For later application let us prove also the inequality

$$I(f;\,\delta) = \left\|\frac{\delta}{\sqrt{x}}\left(f(x+\delta\sqrt{x})-f(x-\delta\sqrt{x})\right)\right\|_{L^1(\delta^2,\infty)} \leqq K\delta^{2\alpha}. \tag{2.1}$$

In fact, for the function

$$g_\delta(x) = \frac{1}{\delta^2}\int_0^{\delta^2} f(x+u)\,du$$

we have proved above

$$\|f-g_\delta\|_{L^1} \leqq \delta^{-2}\int_0^{\delta^2} \|f(\cdot+u)-f(\cdot)\|_{L^1}\,du \leqq K\delta^{2\alpha}$$

and

$$\|g_\delta'\|_{L^1} \leqq \delta^{-2}\|f(\cdot+\delta^2)-f(\cdot)\|_{L^1} \leqq K\delta^{2\alpha-2},$$

by which

$$I(f;\,\delta) \leqq I(f-g_\delta;\,\delta)+I(g_\delta;\,\delta) \leqq$$

$$\leqq \|(f-g_\delta)(x+\delta\sqrt{x})\|_{L^1(\delta^2,\infty)}+\|(f-g_\delta)(x-\delta\sqrt{x})\|_{L^1(\delta^2,\infty)}+$$

$$+\int_{\delta^2}^{\infty}\frac{\delta}{\sqrt{x}}\left(\int_{-\delta\sqrt{x}}^{\delta\sqrt{x}} |g_\delta'(x+u)|\,du\right)dx \leqq K\delta^{2\alpha}+\int_{\delta^2}^{\infty}\delta\int_{-\delta}^{\delta}|g_\delta'(x+u\sqrt{x})|\,du\,dx \leqq$$

$$\leqq K\delta^{2\alpha}+K\delta\int_{-\delta}^{\delta}\|g_\delta'\|_{L^1} \leqq K(\delta^{2\alpha}+\delta^2\|g_\delta'\|_{L^1}) \leqq K\delta^{2\alpha}.$$

III. Proof of (i)⇒(ii) (α). First let us prove the following

Lemma. *Let* $0<h\leqq 1$, $h^2\leqq n^{-1}\leqq h$, $k=0,1,2,\ldots$. *Then there is an absolute constant* K *for which*

$$(1)\quad \int_{h^2}^{\infty} dx \iint_{-h\sqrt{x}/2}^{h\sqrt{x}/2} p_{n,k}(x+u+v)\,du\,dv \leqq K\frac{h^2(k+1)}{n^2},$$

$$(2)\quad \int_{h^2}^{\infty} dx \iint_{-h\sqrt{x}/2}^{h\sqrt{x}/2} \frac{k}{(x+u+v)^2}\,p_{n,k}(x+u+v)\,du\,dv \leqq Kh^2,$$

$$(3)\quad \int_{h^2}^{\infty} dx \iint_{-h\sqrt{x}/2}^{h\sqrt{2}/x} \frac{\left(\frac{k}{n}-(x+u+v)\right)^2}{(x+u+v)^2}\,p_{n,k}(x+u+v)\,du\,dv \leqq K\frac{h^2}{n^2}.$$

Proof. $p_{n,k}(x)$ increases on $(0, k/n)$ and decreases on $(k/n, \infty)$, hence

$$\iint_{-h\sqrt{x}/2}^{h\sqrt{x}/2} p_{n,k}(x+u+v)\,du\,dv \leqq$$

$$\leqq \begin{cases} h^2xp_{n,k}(x+h\sqrt{x}) & \text{for} \quad x\in(0, k/n-h\sqrt{k/n}),\\ h^2x\max\limits_{y} p_{n,k}(y) & \text{for} \quad x\in(k/n-h\sqrt{k/n},\ k/n+2h\sqrt{k/n}),\\ h^2xp_{n,k}(x-h\sqrt{x}) & \text{for} \quad x\in(k/n+2h\sqrt{k/n},\ \infty).\end{cases}$$

Since

$$\int_{h^2}^{\infty} |g(x\pm h\sqrt{x})|\,dx \leqq 2\int_0^{\infty} g(x)\,dx, \quad xp_{n,k}(x)=\frac{k+1}{n}p_{n,k+1}(x),$$

$$\int_0^{\infty} p_{n,k}(x)\,dx=\frac{1}{n}$$

and $\max\limits_{y} p_{n,k}(y)=p_{n,k}(k/n)\leqq K/\sqrt{k+1}$ (use Stirling's formula), we obtain easily

$$\int_{h^2}^{\infty} dx \iint_{-h\sqrt{x}/2}^{h\sqrt{x}/2} p_{n,k}(x+u+v)\,du\,dv \leqq$$

$$\leqq K\left(\frac{(k+1)h^2}{n}\|p_{n,k+1}\|_{L^1}+h^2\frac{k}{n}\frac{1}{\sqrt{k+1}}h\sqrt{\frac{k}{n}}\right) \leqq K\left(\frac{h^2(k+1)}{n^2}\right) \quad (k=0,1,2,\ldots).$$

For $k\geqq 2$ inequality (2) follows from (1), since $kx^{-2}p_{n,k}(x)=(n^2/(k-1))p_{n,k-2}(x)$. For $k=1$ we have

$$\int_{h^2}^{\infty} dx \iint_{-h\sqrt{x}/2}^{h\sqrt{x}/2} n(x+u+v)^{-1}e^{-n(x+u+v)}\,du\,dv = n\int_{h^2}^{\infty} dx \int_{-h\sqrt{x}}^{h\sqrt{x}} \frac{h\sqrt{x}-|\tau|}{x+\tau}e^{-n(x+\tau)}\,d\tau \leqq$$

$$\leqq 2n\int_{h^2}^{\infty}\left(\frac{h}{\sqrt{x}}\int_{-h\sqrt{x}}^{h\sqrt{x}} e^{-n(x-\tau)}\,d\tau\right)dx \leqq Kh^2.$$

Finally, (3) follows from (1) for $k=0$, and for $k\geqq 1$ we have

$$\int_{h^2}^{\infty} dx \iint_{-h\sqrt{x}/2}^{h\sqrt{x}/2} \frac{(k/n-(x+u+v))^2}{(x+u+v)^2}\, p_{n,k}(x+u+v)\, du\, dv \leqq$$

$$\leqq K\int_{h^2}^{2h^2} dx \iint_{-h\sqrt{x}/2}^{h\sqrt{x}/2} \left(\frac{k+1}{n(x+u+v)}\right)^2 p_{n,k}(x+u+v)\, du\, dv +$$

$$+K\int_{2h^2}^{\infty} \frac{dx}{x^2} \iint_{-h\sqrt{x}/2}^{h\sqrt{x}/2} \left(\frac{k}{n}-(x+u+v)\right)^2 p_{n,k}(x+u+v)\, du\, dv.$$

Here the first term is at most Kh^2/n^2 for $k=1$ (see (2)) and

$$K\int_{h^2}^{2h^2} dx \iint_{-h\sqrt{x}/2}^{h\sqrt{x}/2} p_{n,k-2}(x+u+v)\, du\, dv \leqq Kh^6 \leqq Kh^2/n^2$$

for $k\geqq 2$.

The second term can be estimated as we have done in inequality (1) (use that $(k/n-x)^2 p_{n,k}(x)$ increases on $(0, (k+1)/n-\sqrt{2k+1}/n)$ and decreases on $((k+1)/n+ +\sqrt{2k+1}/n, \infty)$ together with the facts

$$\int_0^{\infty} \frac{n}{x}\left(\frac{k}{n}-x\right)^2 p_{n,k}(x)\, dx = \frac{1}{n},$$

$$\int_{\max\left(\frac{k+1}{n}-\frac{\sqrt{2k+1}}{n}-h\sqrt{\frac{k+1}{n}};\, h^2\right)}^{\frac{k+1}{n}+\frac{\sqrt{2k+1}}{n}+4h\sqrt{\frac{k+1}{n}}} \frac{dx}{x^2} \iint_{-h\sqrt{x}/2}^{h\sqrt{x}/2} \left(\frac{k}{n}-(x+u+v)\right)^2 p_{n,k}(x+u+v)\, du\, dv \leqq$$

$$\leqq K\frac{\sqrt{k}}{n}\left(\frac{k}{n}\right)^{-2} h^2\frac{k}{n}\left(\frac{\sqrt{k}}{n}\right)^2 \frac{1}{\sqrt{k}} \leqq K\frac{h^2}{n^2}.$$

Let us turn back to (ii) (α), and let

$$\omega(f;\delta) = \omega(\delta) = \sup_{0<h\leqq\delta} \int_{h^2}^{\infty} |\Delta^*_{h\sqrt{x}}(f;x)|\, dx.$$

It is sufficient to prove that for $0<h^2\leqq 1/n\leqq h\leqq 1$ we have

$$\omega(h) \leqq K\left(n^{-\alpha}+h^2 n\left(n^{-\alpha}+\omega\left(\frac{1}{n}\right)\right)\right),$$

see [1, Lemma 2.1]. Since (i) yields

$$\left\|\Delta^*_{h\sqrt{x}}(f-M_n(f);x)\right\|_{L^1(h^2,\infty)} \leqq K\,\|f-M_n(f)\|_{L^1} \leqq Kn^{-\alpha},$$

an easy consideration shows that it is enough to prove

$$\text{(2.2)}\qquad \left\|\Delta^*_{h\sqrt{x}}(M_n(f);x)\right\|_{L^1(h^2,\infty)} \leqq Kh^2n\left(n^{-\alpha}+\omega\left(\frac{1}{\sqrt{n}}\right)\right)\quad \left(h^2\leqq\frac{1}{n}\leqq h\right).$$

Let

$$\text{(2.3)}\quad \mathscr{M}_n(f;x)=\sum_{k=1}^{\infty}\left(n\int_{k/n}^{(k+1)/n}f(u)\,du\right)p_{n,k}(x)=M_n(f;x)-ne^{-nx}\int_0^{1/n}f(u)\,du.$$

a) (ii) (β) (which we have proved above) gives

$$\left\|\left(n\int_0^{1/n}f(u)\,du\right)\Delta^*_{h\sqrt{x}}(e^{-nt};x)\right\|_{L^1(h^2,\infty)} \leqq Knh^2\left|\int_0^{1/n}f(u)\,du\right| \leqq$$

$$\leqq Knh^2\left\|f\left(\cdot+\frac{1}{n}\right)-f(\cdot)\right\|_{L^1} \leqq Knh^2n^{-\alpha}.$$

b) Let $F_1(x)=\int_0^x f(t)\,dt$, $F_2(x)=\int_0^x F_1(t)\,dt$ and

$$f_\delta(x)=\frac{1}{\delta^2}\int_0^\delta du\int_{-u}^{u}f(x+v\sqrt{x})\,dv=\frac{1}{\delta^2}\int_0^\delta du\int_0^u\left(f(x+v\sqrt{x})+f(x-v\sqrt{x})\right)dv=$$

$$=\frac{1}{\delta^2x}\Delta^*_{\delta\sqrt{x}}(F_2;x).$$

We have

$$\text{(2.4)}\qquad \|f-f_\delta\|_{L^1(\delta^2,\infty)} \leqq \frac{1}{\delta^2}\int_0^\delta du\int_0^u \|\Delta^*_{v\sqrt{x}}(f;x)\|_{L^1(\delta^2,\infty)}\,dv \leqq \omega(\delta)$$

and

$$f''_\delta(x)=\frac{2}{\delta^2x^3}\Delta^*_{\delta\sqrt{x}}(F_2;x)-\frac{2}{\delta^2x^2}\Delta^*_{h\sqrt{x}}(F_1;x)-$$

$$-\frac{5}{4\delta x^{5/2}}\left(F_1(x+\delta\sqrt{x})-F_1(x-\delta\sqrt{x})\right)+\frac{1}{\delta^2x}\Delta^*_{\delta\sqrt{x}}(f;x)+$$

$$+\frac{1}{\delta x^{3/2}}\left(f(x+\delta\sqrt{x})-f(x-\delta\sqrt{x})\right)+\frac{1}{4x^2}\left(f(x+\delta\sqrt{x})+f(x-\delta\sqrt{x})\right),$$

and the key point in our theorem is that the latter is equal to

$$f_\delta''(x) = \frac{2}{x^2}\big(f_\delta(x)-f(x)\big)-\frac{5}{4x^2}\frac{1}{\delta}\int_0^\delta \Delta^*_{h\sqrt{x}}(f;\, x)\, dt+$$

$$\text{(2.5)}\quad +\frac{1}{4x^2}\Delta^*_{\delta\sqrt{x}}(f;\, x)+\frac{1}{\delta^2 x}\Delta^*_{\delta\sqrt{x}}(f;\, x)+\frac{1}{\delta x^{3/2}}\big(f(x+\delta\sqrt{x})-f(x-\delta\sqrt{x})\big)-$$

$$-\frac{2}{\delta x^{3/2}}\frac{1}{\delta}\int_0^\delta \big(f(x+t\sqrt{x})-f(x-t\sqrt{x})\big)\, dt.$$

Now

$$\|\Delta^*_{h\sqrt{x}}(\mathcal{M}_n(f);\, x)\|_{L^1(h^2,\infty)} \leqq$$

$$\leqq \|\Delta^*_{h\sqrt{x}}(\mathcal{M}_n(f-f_{\frac{1}{\sqrt{n}}});\, x)\|_{L^1(h^2,\infty)}+\|\Delta^*_{h\sqrt{x}}(\mathcal{M}_n(f_{\frac{1}{\sqrt{n}}});\, x)\|_{L^1(h^2,\infty)}$$

and below we estimate the two terms on the right side separately.

c) Since

$$(p_{n,k}(x))'' = \frac{n^2}{x^2}\left[\left(\frac{k}{n}-x\right)^2-\frac{k}{n^2}\right]p_{n,k}(x),$$

we obtain by (2) and (3) from the Lemma, and by (2.4) that

$$\int_{h^2}^\infty |\Delta^*_{h\sqrt{x}}(\mathcal{M}_n(f-f_{\frac{1}{\sqrt{n}}});\, x)|\, dx = \int_{h^2}^\infty dx\left|\iint_{-h\sqrt{x}/2}^{h\sqrt{x}/2} \mathcal{M}_n''(f-f_{\frac{1}{\sqrt{n}}};\, x+u+v)\, du\, dv\right| \leqq$$

$$\leqq \sum_{k=1}^\infty\left(n\int_{k/n}^{(k+1)/n} |f-f_{\frac{1}{\sqrt{n}}}|(u)\, du\right)\left\{\int_{h^2}^\infty dx\iint_{-h\sqrt{x}/2}^{h\sqrt{x}/2}\frac{n^2}{(x+u+v)^2}\left(\frac{k}{n}-(x+u+v)\right)^2\times\right.$$

$$\left.\times p_{n,k}(x+u+v)\, du\, dv+\int_{h^2}^\infty dx\iint_{-h\sqrt{x}/2}^{h\sqrt{x}/2}\frac{k}{(x+u+v)^2}p_{n,k}(x+u+v)\, du\right\} \leqq$$

$$\leqq Kh^2 n\sum_{k=1}^\infty\int_{k/n}^{(k+1)/n}|f-f_{\frac{1}{\sqrt{n}}}|(u)\, du = Kh^2 n\|f-f_{\frac{1}{\sqrt{n}}}\|_{L^1\left(\frac{1}{n},\infty\right)} \leqq Kh^2 n\omega\left(\frac{1}{\sqrt{n}}\right).$$

d) We have also

$$(p_{n,k}(x))'' = n^2\big(p_{n,k-2}(x)-2p_{n,k-1}(x)+p_{n,k}(x)\big)\quad (k = 1, 2, \ldots, p_{n-1}(x) \equiv 0),$$

thus

$$\int_{h^2}^{\infty} |\Delta^*_{h\sqrt{x}}(\mathcal{M}_n(f_{\frac{1}{\sqrt{n}}});\, x)|\, dx =$$

$$= n^2 \int_{h^2}^{\infty} dx \left| \sum_{k=1}^{\infty} \left(n \iiint_0^{1/n} f''_{\frac{1}{\sqrt{n}}}\left(\frac{k}{n}+u+v+w\right) du\, dv\, dw \right) \iint_{-h\sqrt{x}/2}^{h\sqrt{x}/2} p_{n,k}(x+s+t)\, ds\, dt + \right.$$

$$\left. + \left(-2n \int_{1/n}^{3/n} f_{\frac{1}{\sqrt{n}}}(u)\, du + n \int_{2/n}^{3/n} f_{\frac{1}{\sqrt{n}}}(u)\, du \right) \iint_{-h\sqrt{x}/2}^{h\sqrt{x}/2} p_{n,0}(x+s+t)\, ds\, dt \right| \leqq$$

$$\leqq K \sum_{k=1}^{\infty} \left(n \iiint_0^{1/n} \left| f''_{\frac{1}{\sqrt{n}}}\left(\frac{k}{n}+u+v+w\right) \right| du\, dv\, dw \right) kh^2 +$$

$$+ Knh^2 \left(\left| \int_{1/n}^{2/n} f_{\frac{1}{\sqrt{n}}}(u)\, du \right| + \left| \int_{2/n}^{3/n} f_{\frac{1}{\sqrt{n}}}(u)\, du \right| \right) \leqq$$

$$\leqq Knh^2 \sum_{k=1}^{\infty} k \iiint_0^{1/n} \left| f''_{\frac{1}{\sqrt{n}}}\left(\frac{k}{n}+u+v+w\right) \right| du\, dv\, dw +$$

$$+ Knh^2 \left(n^{-\alpha} + \omega\left(\frac{1}{\sqrt{n}}\right) \right) = Knh^2 A + Knh^2 \left(n^{-\alpha} + \omega\left(\frac{1}{\sqrt{n}}\right) \right),$$

where

$$A = \sum_{k=1}^{\infty} k \iiint_0^{1/n} \left| f''_{\frac{1}{\sqrt{n}}}\left(\frac{k}{n}+u+v+w\right) \right| du\, dv\, dw,$$

and where we used that

$$\left| \int_{1/n}^{2/n} f_{\frac{1}{\sqrt{n}}}(u)\, du \right| + \left| \int_{2/n}^{3/n} f_{\frac{1}{\sqrt{n}}}(u)\, du \right| \leqq \|f - f_{\frac{1}{\sqrt{n}}}\|_{L^1\left(\frac{1}{n},\infty\right)} +$$

$$+ \left| \int_{1/n}^{\infty} f(u)\, du - \int_{2/n}^{\infty} f(u)\, du \right| + \left| \int_{2/n}^{\infty} f(u)\, du - \int_{3/n}^{\infty} f(u)\, du \right| \leqq K\left(\omega\left(\frac{1}{\sqrt{n}}\right) + n^{-\alpha} \right).$$

To estimate A we apply (2.5). Taking absolute value in (2.5) term by term we increase $|f''_{\frac{1}{\sqrt{n}}}(x)|$. Now the first term on the right of (2.5) contributes to A at most by

$$\sum_{k=1}^{\infty} k \iiint_0^{1/n} \frac{2}{\left(\frac{k}{n}+u+v+w\right)^2} \left| n \int_0^{1/\sqrt{n}} ds \int_0^s \Delta^*_{t\sqrt{\frac{k}{n}+u+v+w}} \left(f; \frac{k}{n}+u+v+w\right) dt \right| du\, dv\, dw \leqq$$

$$\leqq 2n \int_0^{1/\sqrt{n}} ds \int_0^s dt \sum_{k=1}^{\infty} \frac{n^2}{k} \iiint_0^{1/n} \left| \Delta^*_{t\sqrt{\frac{k}{n}+u+v+w}} \left(f; \frac{k}{n}+u+v+w\right) \right| du\, dv\, dw \leqq$$

$$\leqq Kn \int_0^{1/\sqrt{n}} ds \int_0^s \|\Delta^*_{t\sqrt{x}}(f; x)\|_{L^1\left(\frac{1}{n}, \infty\right)} dt \leqq K\omega\left(\frac{1}{\sqrt{n}}\right).$$

Quite similarly the contribution of the second, third and fourth terms to A is at most $K\omega\left(\frac{1}{\sqrt{n}}\right)$.

Using inequality (2.1), the fifth term contributes to A at most by

$$\sum_{k=1}^{\infty} k \iiint_0^{1/n} \frac{\sqrt{n}}{\left(\frac{k}{n}+u+v+w\right)^{3/2}} \left| f\left(\frac{k}{n}+u+v+w+\frac{1}{\sqrt{n}}\sqrt{\frac{k}{n}+u+v+w}\right) - \right.$$

$$\left. -f\left(\frac{k}{n}+u+v+w-\frac{1}{\sqrt{n}}\sqrt{\frac{k}{n}+u+v+w}\right) \right| du\, dv\, dw \leqq$$

$$\leqq K \int_{1/n}^{\infty} \frac{1}{\sqrt{nx}} \left| f\left(x+\frac{1}{\sqrt{n}}\sqrt{x}\right) - f\left(x-\frac{1}{\sqrt{n}}\sqrt{x}\right) \right| dx \leqq Kn^{-\alpha},$$

and a similar estimate can be given for the contribution of the sixth term:

$$\sum_{k=1}^{\infty} k \iiint_0^{1/n} \frac{2\sqrt{n}}{\left(\frac{k}{n}+u+v+w\right)^{3/2}} \left(\sqrt{n} \int_0^{1/\sqrt{n}} \left| f\left(\frac{k}{n}+u+v+w+t\sqrt{\frac{k}{n}+u+v+w}\right) - \right.\right.$$

$$\left.\left. -f\left(\frac{k}{n}+u+v+w-t\sqrt{\frac{k}{n}+u+v+w}\right) \right| dt \right) du\, dv\, dw \leqq$$

$$\leqq \int_0^{1/\sqrt{n}} \frac{1}{t} \left(\int_{1/n}^{\infty} \frac{t}{\sqrt{x}} |f(x+t\sqrt{x})-f(x-t\sqrt{x})|\, dx \right) dt \leqq K \int_0^{1/\sqrt{n}} t^{2\alpha-1}\, dt \leqq Kn^{-\alpha}.$$

Collecting our estimates from a) to d) we obtain (2.2) by which the proof is complete.

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On the equiconvergence of different kinds of partial sums of orthogonal series

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Let N^d $(d \geqq 1)$ be the set of d-tuples $\mathrm{i}=(i_1, \dots, i_d)$ with non-negative integral coordinates. Let $\varphi=\{\varphi_\mathrm{i} \mid \mathrm{i} \in N^d\}$ be an orthonormal system (ONS) on $[0, 1]$. Consider the d-multiple orthogonal series

(1) $$\sum_{\mathrm{i} \in N^d} a_\mathrm{i} \varphi_\mathrm{i}(x), \quad \sum_{\mathrm{i} \in N^d} a_\mathrm{i}^2 < \infty.$$

Fixing a sequence $Q=\{Q_k \mid k=0, 1, \dots\}$ of finite sets in N^d with properties

(2) $$\emptyset = Q_0 \subset Q_1 \subset Q_2 \subset \dots, \quad \bigcup_{k=0}^{\infty} Q_k = N^d \stackrel{\text{def}}{=} Q_\infty$$

we can define the Q-partial sums of (1) (see e.g. [1]):

$$s_k^Q(x) = \sum_{\mathrm{i} \in Q_k} a_\mathrm{i} \varphi_\mathrm{i}(x) \quad (k = 1, 2, \dots).$$

If $P=\{P_k\}$ is another sequence satisfying similar conditions to (2) we write $Q \Rightarrow P$ when the a.e. convergence of $\{s_k^Q(x)\}_{k=1}^\infty$ always implies that of $\{s_k^P(x)\}_{k=1}^\infty$. If not $Q \Rightarrow P$ then we write shortly $Q \not\Rightarrow P$.

F. MÓRICZ [1] proved among others that if

$$Q_k' = \{\mathrm{i} \in N^d \mid \max_{1 \leqq j \leqq d} i_j \leqq k\}$$

and

$$P_k' = \left\{\mathrm{i} \in N^d \,\middle|\, \left(\sum_{j=1}^d i_j^2\right)^{1/2} \leqq k\right\}$$

then $Q' \not\Rightarrow P'$ and $P' \not\Rightarrow Q'$.

The aim of this note is to give necessary and sufficient conditions for $Q \Rightarrow P$. Our result has several corollaries which are interesting in themselves.

With the notation $\bar{P}_k = N^d \setminus P_k$ we prove

Theorem 1. *We have $Q \Rightarrow P$ if and only if there is a number K such that*

(i) *each $Q_{k+1} \setminus Q_k$ is the union of at most K sets $(Q_{k+1} \setminus Q_k) \cap (P_{m+1} \setminus P_m)$,*

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(ii) *for every* k, P_k *and* $\bar{P}_k$ *are the (not necessarily disjoint) union of at most* K *sets of the form* $Q_{r+s}\setminus Q_r$ $(s=1, 2, \ldots, \infty)$, $P_{m+1}\setminus P_m$, $(Q_{r'+1}\setminus Q_{r'})\cap(P_{m'+1}\setminus P_{m'})$.

Corollary 1. *The systems* Q *and* P *are equivalent (i.e.* $P\Rightarrow Q$ *and* $Q\Rightarrow P$*) if and only if there is a* K *such that*

(i) *each* $(Q_{k+1}\setminus Q_k)\cup(P_{m+1}\setminus P_m)$ *is the union of at most* K *sets* $(Q_{s+1}\setminus Q_s)\cap$ $\cap(P_{r+1}\setminus P_r)$,

(ii) *each* Q_k *and* P_k *is the union of at most* K *sets* $(Q_{s+1}\setminus Q_s)\cap(P_{r+1}\setminus P_r)$ *and* K *sets of the form* $P_{\varrho+\tau}\setminus P_\varrho$ *and* $Q_{\varrho'+\tau'}\setminus Q_{\varrho'}$, *respectively.*

With the notation

$$(k, l) = \{k, k+1, \ldots, l\} \quad (k \leqq l,\ k, l \in N^1) \tag{3}$$
$$(k, \infty) = \{k, k+1, \ldots\}$$

we have

Corollary 2. *Let* $\{p_k\}$ *and* $\{q_k\}$ *be two subsequences of the natural numbers. Then the a.e. convergence of* $\{s_{p_k}(x)\}_{k=1}^{\infty}$ *implies that of* $\{s_{q_k}(x)\}_{k=1}^{\infty}$ *for every orthogonal series*

$$\sum_{k=0}^{\infty} a_k \varphi_k(x), \quad \sum_{k=0}^{\infty} a_k^2 < \infty \tag{4}$$

if and only if the number of the q_k's *in the intervals* (p_m, p_{m+1}) *is bounded* (here s_k is the ordinary k-th partial sum of (4)).

Corollary 3. *With the above notations the a.e. equiconvergence of* $\{s_{p_k}(x)\}_{k=1}^{\infty}$ *and* $\{s_{q_k}(x)\}_{k=1}^{\infty}$ *for every orthogonal series* (4) *is equivalent to the existence of a* K *for which* $p_k<q_l$ *implies* $p_{k+1}<q_{l+K}$ *and* $q_k<p_l$ *implies* $q_{k+1}<p_{l+K}$.

Corollary 1 follows easily from the proof of Theorem 1. Corollaries 2 and 3 were also proved by H. Schwinn [3].

To formulate another consequence of Theorem 1 let $d=1$, $N=N^1$ and $\pi: N\to N$ be a mapping of N onto N for which the inverse image $\pi^{-1}(k)$ of every number k is finite (one can see easily that the following problem becomes trivial if some of the $\pi^{-1}(k)$ are infinite). Our problem is the following: determine which π has the property: if the orthogonal series (4) converges a.e. then the same is true for the rearranged and bracketed series

$$\sum_{k=0}^{\infty}\Big(\sum_{i\in\pi^{-1}(k)} a_i \varphi_i(x)\Big). \tag{5}$$

The answer is given by

Theorem 2. *The a.e. convergence of* (4) *implies that of* (5) *for every orthogonal series* (4) *if and only if there is a* K *such that for every* k, $\pi^{-1}(0, k)$ *and* $\pi^{-1}(k, \infty)$

are the (not necessarily disjoint) union of at most K *sets of the form* (l, m) $(m=1, 2, \dots, \infty)$ *or* $\pi^{-1}(s)$.

For the definition of (l, m) see (3).

Corollary 4. *If* $\pi: N \to N$ *is a permutation of* N *then the a.e. convergence of* (4) *implies the a.e. convergence of*

$$\sum_{k=0}^{\infty} a_{\pi(k)} \varphi_{\pi(k)}(x)$$

for every orthogonal series (4) *if and only if there is a* K *such that for every* k, $\pi(0, k)$ *consists of at most* K *chains of consecutive integers.*

Remarks. 1. Although we formulated Theorem 1 in d dimensions, the problem and the solution is essentially one-dimensional, namely Theorems 1 and 2 are equivalent (see the proof of Theorem 1 below).

2. If $Q \Rightarrow P$ then our proof yields an orthogonal series (1) for which $\{s_k^Q(x)\}_{k=1}^{\infty}$ converges a.e. but $\{s_k^P(x)\}_{k=1}^{\infty}$ diverges on a set of positive measure. By a standard modification of the proof one could achieve also the a.e. divergence of $\{s_k^P(x)\}_{k=1}^{\infty}$.

3. The ONS $\{\varphi_i\}$ above could be defined on any non-atomic measure space instead of $[0, 1]$ (compare to [2]).

4. Our proof shows that if $Q \Rightarrow P$ and $\{s_k^Q(x)\}_{k=1}^{\infty}$ converges on a set E then $\lim_{k\to\infty} s_k^P(x) = \lim_{k\to\infty} s_k^Q(x)$ a.e. on E, i.e. the P-sums and Q-sums are equal a.e. automatically.

5. Finally, let us remark that to the proof of Corollaries 2 and 3 needs only the consideration used in the proof of the necessity of Theorem 1 (i), by which we obtain a very short proof of Schwinn's results (see [3]). The same is true for a part of Móricz's theorem mentioned earlier (see [1, Theorem 3]).

After these we turn to the proofs our theorems. First we prove Theorem 2.

Proof of Theorem 2. *I. Necessity.* Let us suppose on the contrary that e.g. for each n there is a k such that $\pi^{-1}\{0, \dots, k\} = \pi^{-1}(0, k)$ (see (3)) cannot be represented as the union of at most n sets (l, m) and at most n sets $\pi^{-1}(l)$.

We define sequences $\{N_n\}, \{M_n\}, \{m_n\}, \{m_n^*\}, k_1^{(n)} < k_2^{(n)} < \dots < k_n^{(n)}$ and $\{i_1^{(n)}, \dots, i_n^{(n)}\}, \{j_1^{(n)}, \dots, j_n^{(n)}\}$ in the following way: put $N_0 = M_0 = m_0 = m_0^* = 0$ and if all of the above numbers are already defined up to $n-1$, let N_n and m_n^* be so large that

$$N_n > M_{n-1}, \pi^{-1}(0, N_n) \supseteqq (0, m_{n-1}), \quad m_n^* > m_{n-1}, (0, m_n^*) \supseteqq \pi^{-1}(0, N_n)$$

be satisfied. By our assumption there is an $M_n > N_n$ such that $\pi^{-1}(N_n+1, M_n) \setminus (0, m_n^*)$ cannot be represented as the union of at most n sets (l, m) and at most

n sets $\pi^{-1}(l)$. Let $\pi^{-1}(N_n+1, M_n)\setminus(0, m_n^*)=(r_1, s_1)\cup(r_2, s_2)\cup\ldots\cup(r_t, s_t)$ where $s_i\geqq r_i$ and $r_{i+1}>s_i+1$. We claim that there are n numbers $k_1^{(n)}<\ldots<k_n^{(n)}$ belonging to (N_n+1, M_n) and numbers $i_\varrho\in\pi^{-1}(k_\varrho^{(n)})\setminus(0, m_n^*)$ $(\varrho=1, \ldots, n)$ such that neither two of the i_ϱ belong to the same (r_τ, s_τ). In fact, let $i_1^*\in(r_1, s_1)$, $\pi(i_1^*)=k_1^*$ and if i_ϱ^*, k_ϱ^* $(\varrho<n)$ are already defined and $i_u^*\in(r_{\tau_u}, s_{\tau_u})$ $(1\leqq u\leqq\varrho)$, then, since the ϱ intervals (r_{τ_u}, s_{τ_u}) and the ϱ sets $\pi^{-1}(k_u^*)$ $(1\leqq u\leqq\varrho)$ do not cover $\pi^{-1}(N_n+1, M_n)\setminus(0, m_n^*)$, there is an

$$i_{\varrho+1}^*\in(\pi^{-1}(N_n+1, M_n)\setminus(0, m_n^*))\setminus\left(\left(\bigcup_{u=1}^{\varrho}(r_{\tau_u}, s_{\tau_u})\right)\cup\left(\bigcup_{u=1}^{\varrho}\pi^{-1}(k_u^*)\right)\right).$$

Let $k_{\varrho+1}^*=\pi(i_{\varrho+1}^*)$. We can continue this up to $\varrho=n$, and all what we have to do is to rearrange the set $\{k_1^*, \ldots, k_n^*\}$ into an increasing order $k_1^{(n)}<k_2^{(n)}<\ldots<k_n^{(n)}$ and to carry over this rearrangement to $\{i_1^*, \ldots, i_n^*\}$, by which we obtain $\{i_1^{(n)}, \ldots, i_n^{(n)}\}$. Let $i_\varrho^{(n)}$ belong to $(r_{\tau'_\varrho}, s_{\tau'_\varrho})$ and let us put $j_\varrho^{(n)}=s_{\tau'_\varrho}+1$ $(\varrho=1, \ldots, n)$. Finally, let $m_n>m_n^*$ be so large that $(0, m_n)$ contains $\pi^{-1}(0, M_n)$ as well as the numbers $j_1^{(n)}, \ldots, j_n^{(n)}$.

Our definition is complete and let us observe the following:

$$\pi^{-1}(0, M_{n-1})\subseteqq(0, m_{n-1})\subseteqq\pi^{-1}(0, N_n)\subseteqq(0, m_n^*), \tag{6}$$

$$m_n^*<i_\varrho^{(n)}<j_\varrho^{(n)}\leqq m_n\quad(\varrho=1, \ldots, n), \tag{7}$$

$$M_{n-1}<N_n<k_1^{(n)}<\ldots<k_n^{(n)}\leqq M_n, \tag{8}$$

$$i_\varrho^{(n)}\in\pi^{-1}(k_\varrho^{(n)}),\quad j_\varrho^{(n)}\notin\pi^{-1}(0, M_n)\quad(\varrho=1, \ldots, n), \tag{9}$$

$$\max_{1\leqq\varrho\leqq n-1} j_\varrho^{(n-1)}<\min_{1\leqq\varrho\leqq n} i_\varrho^{(n)}, \tag{10}$$

(11) every two $i_{\varrho_1}^{(n)}<i_{\varrho_2}^{(n)}$ is separated by $j_{\varrho_1}^{(n)}$: $i_{\varrho_1}^{(n)}<j_{\varrho_1}^{(n)}<i_{\varrho_2}^{(n)}$.

Now we shall use that there is an orthogonal series (4) with partial sums $S_k(x)$ which diverges unboundedly a.e. on $[0, 1]$. This gives that there is a sequence $p_1<p_2<\ldots$ such that with $q_k=\sum_{l=1}^{k-1}p_l$ we have

$$\sup_n\max_{0<l\leqq p_n}|S_{q_n+l}(x)-S_{q_n}(x)|=\infty\quad\text{(a.e.)}. \tag{12}$$

Let now

$$\psi_{i_\varrho^{(p_n)}}(x)=\psi_{j_\varrho^{(p_n)}}(x)=\frac{1}{2}\varphi_{q_n+\varrho}(x)\quad(x\in[0, 1]), \tag{13}$$

$$b_{i_\varrho^{(p_n)}}=-b_{j_\varrho^{(p_n)}}=a_{q_n+\varrho} \tag{14}$$

for $n=1, 2, \ldots$ and $\varrho=1, \ldots, p_n$ and let $\psi_k(x)=0$ $(x\in[0, 1])$, $b_k=0$ otherwise. Since each ψ_k is orthogonal to all but at most one ψ_l, $l\neq k$ and since $\int_0^1|\psi_k\psi_l|\leqq 1/4$ $(k, l=0, 1, \ldots)$, a standard argument yields that the system $\{\psi_k\}_{k=0}^\infty$ can be extended onto $[-1, 1]$ in such a way that it constitutes an ONS on $[-1, 1]$, and for every $x\in[-1, 0)$ all but at most two of the numbers $\{\psi_k(x)\}_{k=0}^{\infty}$ are zero.

By (10), (11), (13) and (14) the k-th partial sum $s_k(x)$ of

$$\sum_{l=0}^{\infty} b_l\psi_l(x)$$

is equal either to 0 or to some $a_l\varphi_l(x)/2$ if $x\in[0, 1]$. Here l tends to infinity together with k (take into account that if $k>m_{p_n}$ then necessarily $l>q_n$), and by

$$\sum_{l=0}^{\infty}\int_0^1 (a_l\varphi_l(x))^2\,dx = \sum_{l=0}^{\infty} a_l^2 < \infty,$$

$a_l\varphi_l(x)$ tends to 0 a.e. as $l\to\infty$. Hence, $s_k(x)$ tends to zero a.e. on $[0, 1]$ as $k\to\infty$ and so $\{s_k(x)\}_{k=1}^{\infty}$ is convergent a.e. on $[-1, 1]$ (for $x\in[-1, 0)$, $\{s_k(x)\}_{k=1}^{\infty}$ is constant from a certain point on).

However, by (6), (7), (13) and (14)

$$\sum_{k=0}^{N_n}\sum_{l\in\pi^{-1}(k)} b_l\psi_l(x) = 0 \quad (x\in[0, 1]),$$

hence by (8) and (9)

$$\sum_{k=0}^{k_\varrho^{(p_n)}}\sum_{l\in\pi^{-1}(k)} b_l\psi_l(x) = \sum_{k=N_n+1}^{k_\varrho^{(p_n)}}\sum_{l\in\pi^{-1}(k)} b_l\psi_l(x) = \sum_{s=1}^{\varrho} b_{i_\varrho^{(p_n)}}\psi_{i_\varrho^{(p_n)}}(x) =$$

$$= \sum_{s=1}^{\varrho}\frac{1}{2}a_{q_n+s}\varphi_{q_n+s}(x) = \frac{1}{2}\left(S_{q_n+\varrho}(x)-S_{q_n}(x)\right) \quad (1\leqq\varrho\leqq p_n)$$

and thus, using (12), we obtain that

$$\sum_{k=0}^{\infty}\sum_{l\in\pi^{-1}(k)} b_l\psi_l(x)$$

diverges a.e. on $[0, 1]$.

The necessity of the assumption concerning $\pi^{-1}(k, \infty)$ can be proved similarly, we omit the details.

The proof of the necessity is thus complete (clearly, it is indifferent that the constructed system $\{\psi_k\}_{k=0}^{\infty}$ is orthonormal on $[-1, 1]$ and not on $[0, 1]$).

II. Sufficiency. 1. First we prove that there are no integers

$$x_1 < y_1 < x_2 < y_2 < \ldots < y_{4K+2} < x_{4K+3}$$

with $\pi(x_j)=\pi(x_l)$ $(0\leqq j, l\leqq 4K+3)$ but $\pi(y_j)\neq\pi(y_l)$ $(1\leqq j, l\leqq 4K+2,\ j\neq l)$. Let us suppose the contrary and let $\pi(x_j)=k$ $(1\leqq j\leqq 4K+3)$. We distinguish two cases.

(a) At least $2K+1$ of the distinct numbers $\pi(y_j)$ $(1\leqq j\leqq 4K+2)$ are less than k. We may suppose without loss of generality that

$$x_1 < y_1 < x_2 < \ldots < y_{2K+1} < x_{2K+2}, \quad \pi(y_j) < k \ (1 \leqq j \leqq 2K+1).$$

For any n, $\pi^{-1}(0, n)$ is the disjoint union of sets of consecutive integers, i.e., for some τ_n,

$$\pi^{-1}(0, n) = (a_1^{(n)}, b_1^{(n)}) \cup \ldots \cup (a_{\tau_n}^{(n)}, b_{\tau_n}^{(n)}) \tag{15}$$

where $a_{j+1}^{(n)} > b_j^{(n)}$ $(1\leqq j<\tau_n)$. Let us put $n=k-1$ into (15) and let us determine the numbers i_j $(1\leqq j\leqq 2K+1)$ by $y_j \in (a_{i_j}^{(k-1)}, b_{i_j}^{(k-1)})$. Since x_j $(1\leqq j\leqq 2K+2)$ does not belong to $\pi^{-1}(0, k-1)$, we have

$$x_j < a_{i_j}^{(k-1)} \leqq y_j \leqq b_{i_j}^{(k-1)} < x_{j+1} < a_{i_{j+1}}^{(k-1)} \quad (1 \leqq j < 2K+1),$$

hence the numbers $i_1, i_2, \ldots, i_{2K+1}$ are all different from each other.

By the assumption of our theorem there are numbers $1\leqq l_1<\ldots<l_K\leqq\tau_{k-1}$ and $0\leqq n_1<\ldots<n_k\leqq k-1$ so that

$$\pi^{-1}(0, k-1) = (a_{l_1}^{(k-1)}, b_{l_1}^{(k-1)}) \cup \ldots \cup (a_{l_K}^{(k-1)}, b_{l_K}^{(k-1)}) \cup \pi^{-1}(n_1) \cup \ldots \cup \pi^{-1}(n_K). \tag{16}$$

Now at least $K+1$, say $i_1, i_2, \ldots, i_{K+1}$, of the numbers $i_1, i_2, \ldots, i_{2K+1}$ are different from every l_j $(1\leqq j\leqq K)$ (i.e., we may suppose without loss of generality that $i_j\neq l_{j'}$ for $1\leqq j\leqq K+1$, $1\leqq j'\leqq K$) and at least one, say $\pi(y_1)$, of the $K+1$ distinct numbers $\pi(y_1), \pi(y_2), \ldots, \pi(y_{K+1})$ is different from every n_j $(1\leqq j\leqq K)$. Thus, y_1 does not belong to

$$(a_{l_1}^{(k-1)}, b_{l_1}^{(k-1)}) \cup \ldots \cup (a_{l_K}^{(k-1)}, b_{l_K}^{(k-1)})$$

since $y_1 \in (a_{i_1}^{(k-1)}, b_{i_1}^{(k-1)})$ and $i_1\neq l_j$ for $1\leqq j\leqq K$ and also y_1 does not belong to

$$\pi^{-1}(n_1) \cup \ldots \cup \pi^{-1}(n_k)$$

since $\pi(y_1)$ is different from every n_j $(1\leqq j\leqq K)$. By (16) this means that $y_1 \notin \pi^{-1}(0, k-1)$ which contradicts the assumed inequality $\pi(y_1)<k$. This contradiction proves our assertion in the case (a).

(b) If at most $2K$ of the numbers $y_1, \ldots, y_{4K+2}$ are less than k then at least $2K+1$ of them are greater than k. Now using $\pi^{-1}(k+1, \infty)$ instead of $\pi^{-1}(0, k-1)$ we arrive at a contradiction exactly as above.

2. Let for $k=0, 1, 2, \ldots$

$$\Pi_k = \{\pi^{-1}(k) \cap (a_j^{(n)}, b_j^{(n)}) \mid n = 0, 1, 2, \ldots, 1 \leqq j \leqq \tau_n\}$$

(for the definition of $a_j^{(n)}$ and $b_j^{(n)}$ see (15)). Our next claim is that for each k and $x\in\pi^{-1}(k)$ there are at most $8K+3$ distinct sets $A\in\Pi_k$ with $x\in A$. In fact, if there were numbers $n_1<n_2<\ldots<n_{8K+4}$ and for each $1\leqq j\leqq 8K+4$ an $1\leqq i_j\leqq\tau_{n_j}$

such that the sets $(x\in)(a_{i_j}^{(n_j)}, b_{i_j}^{(n_j)})\cap\pi^{-1}(k)$ are all different then either for at least $4K+2$ of the j's we would have

$$(a_{i_{j+1}}^{(n_{j+1})}, a_{i_j}^{(n_j)}-1)\cap\pi^{-1}(k)\neq\emptyset \tag{17}$$

or for at least $4K+2$ of the j's

$$(b_{i_j}^{(n_j)}+1, b_{i_{j+1}}^{(n_{j+1})})\cap\pi^{-1}(k)\neq\emptyset.$$

We might suppose the first case and also that (17) holds for $j=1, 2, ..., 4K+2$, i.e., for $j=1, ..., 4K+2$ there would be numbers

$$x_{j+1}\in(a_{i_{j+1}}^{(n_{j+1})}, a_{i_j}^{(n_j)}-1)\cap\pi^{-1}(k).$$

Putting $x_1=x\in(a_{i_1}^{(n_1)}, b_{i_1}^{(n_1)})\cap\pi^{-1}(k)$ and $y_j=a_{i_j}^{(n_j)}-1$ $(1\leqq j\leqq 4K+2)$ we would have $y_j\in\pi^{-1}(0, n_{j+1})$ but $y_j\notin\pi^{-1}(0, n_j)$, i.e., $\pi(y_j)\leqq n_{j+1}<\pi(y_{j+1})$ $(1\leqq j\leqq 4K+1)$, and also $y_j\notin\pi^{-1}(k)$. Thus, we would get a system of numbers

$$x_{4K+3}<y_{4K+1}<x_{4K+2}<\ldots<y_1<x_1$$

with $\pi(x_j)\in k$ $(1\leqq j\leqq 4K+3)$ but $\pi(y_j)\neq\pi(y_{j'})$ $(1\leqq j, j'\leqq 4K+2, j\neq j')$ and this would contradict the fact proved in point 1 above.

3. After these preliminary considerations we turn to the proof of the sufficiency part of our theorem. First of all, by point 2 above

$$\sum_{k=0}^{\infty}\sum_{A\in\pi_k}\int_0^1\Big(\sum_{i\in A}a_i\varphi_i(x)\Big)^2dx\leqq(8K+3)\sum_{i=0}^{\infty}a_i^2<\infty$$

and hence

$$\lim_{k\to\infty}\sum_{i\in A_k}a_i\varphi_i(x)=0 \quad \text{(a.e.)}$$

independently of the choice of the sets $A_k\in\Pi_k$.

Let us suppose that the series (4) converges a.e. and let x be any point in $[0, 1]$ for which

$$\lim_{k\to\infty}\sum_{i\in A_k}a_i\varphi_i(x)=0 \quad (A_k\in\Pi_k) \tag{18}$$

$$\lim_{k\to\infty}s_k(x)=s(x) \quad \Big(s_k(x)=\sum_{i=0}^{k}a_i\varphi_i(x)\Big) \tag{19}$$

exist. It is enough to show that (5) converges at this point x.

From (19) we have also

$$\lim_{k\to\infty}\big(s_{k+l_k}(x)-s_k(x)\big)=0 \tag{20}$$

whatever $l_k\geqq 1$ be.

For a given p let $p<p_1<p_2<p_3$ be chosen so that $\pi(0, p)\subseteqq(0, p_1)$, $\pi^{-1}(0, p_1)\subseteqq$ $\subseteqq(0, p_2)$, $\pi(0, p_2)\subseteqq(0, p_3)$ be satisfied. For $n\geqq p_3$ we have $\pi^{-1}(0, n)\supseteqq(0, p_3)\supseteqq$

$\supseteqq (0, p_2)$ and by the assumption of the theorem

$$\pi^{-1}(0, n) = (a_{i_1}^{(n)}, b_{i_1}^{(n)}) \cup \ldots \cup (a_{i_\varrho}^{(n)}, b_{i_\varrho}^{(n)}) \cup \pi^{-1}(k_1) \cup \ldots \cup \pi^{-1}(k_\tau) \tag{21}$$

for some $i_1 < \ldots < i_\varrho$ and $k_1 < \ldots < k_\tau$, where $\tau + \varrho \leqq K$ (if $\tau = 0$ or $\varrho = 0$ then the corresponding terms are missing). Since $(0, p_2) \subseteq \pi^{-1}(0, n)$ we may assume (by increasing K by 1 if necessary) $i_1 = 1, (0, p_2) \subseteqq (a_1^{(n)}, b_1^{(n)})$ and then, since $\pi^{-1}(0, p_1) \subseteqq \subseteqq (0, p_2)$, we can drop those of the k_j's for which $k_j \leqq p_1$. Thus, we may assume that in (21) each $k_j > p_1$ and so, since $\pi^{-1}(0, p_1) \supseteqq (0, p)$,

$$\pi^{-1}(k_j) \cap (p+1, b_1^{(n)}) = \pi^{-1}(k_j) \cap (a_1^{(n)}, b_1^{(n)}) \overset{\text{def}}{=} A_j^{(1)} \quad (1 \leqq j \leqq \tau).$$

For $1 \leqq j \leqq \tau$ and $2 \leqq l \leqq \varrho$ let $A_j^{(l)} = \pi^{-1}(k_j) \cap (a_{i_l}^{(n)}, b_{i_l}^{(n)})$. Then $A_j^{(l)} \in \Pi_{k_j}$ $(1 \leqq j \leqq \tau$, $1 \leqq l \leqq \varrho)$ and for $n \geqq p_3$ we have the representation

$$\pi^{-1}(0, n) = (0, p) \cup (p+1, b_1^{(n)}) \cup (a_{i_2}^{(n)}, b_{i_2}^{(n)}) \cup \ldots \cup (a_{i_\varrho}^{(n)}, b_{i_\varrho}^{(n)}) \cup$$
$$\cup \bigcup_{j=1}^{\tau} \left(\pi^{-1}(k_j) \setminus \bigcup_{l=1}^{\varrho} A_j^{(l)} \right)$$

and here the terms on the right are already disjoint. According to this

$$\left| \sum_{k=0}^{n} \sum_{i \in \pi^{-1}(k)} a_i \varphi_i(x) - \sum_{i=0}^{p} a_i \varphi_i(x) \right| =$$

$$= \left| \left(\sum_{i=0}^{p} + \sum_{i=p+1}^{b_1^{(n)}} + \sum_{j=2}^{\varrho} \sum_{i=a_{i_j}^{(n)}}^{b_{i_j}^{(n)}} + \sum_{j=1}^{\tau} \sum_{i \in \pi^{-1}(k_j)} - \sum_{j=1}^{\tau} \sum_{l=1}^{\varrho} \sum_{i \in A_j^{(l)}} \right) a_i \varphi_i(x) - \sum_{i=0}^{p} a_i \varphi_i(x) \right| \leqq$$

$$\leqq |s_{b_1^{(n)}}(x) - s_p(x)| + \sum_{j=2}^{\varrho} |s_{b_{i_j}^{(n)}}(x) - s_{a_{i_j-1}^{(n)}}(x)| + \sum_{j=1}^{\tau} \Big| \sum_{i \in \pi^{-1}(k_j)} a_i \varphi_i(x) \Big| +$$
$$+ \sum_{j=1}^{\tau} \sum_{l=1}^{\varrho} \Big| \sum_{i \in A_j^{(l)}} a_i \varphi_i(x) \Big|$$

and (18) and (20) give that here the right hand side tends to zero as $p \to \infty$ by $b_{i_j}^{(n)} \geqq a_{i_j}^{(n)} > b_1^{(n)} > p$ $(2 \leqq j \leqq \varrho)$ and $k_j > p$ (notice that $\pi^{-1}(k_j) \in \Pi_{k_j}$ for $1 \leqq j \leqq \tau$ and take into account that $\varrho + \tau \leqq K$). Since $s_p(x) \to s(x)$ as $p \to \infty$ and $n > p_3 = p_3(p)$ was arbitrary, we get the convergence of the series (5) at x and the proof is complete.

Proof of Theorem 1. Let us arrange the non-void sets $(Q_{k+1} \setminus Q_k) \cap \cap (P_{m+1} \setminus P_m)$ into a sequence $A_0, A_1, \ldots, A_n, \ldots$ in such a way that $Q_k = \bigcup_{l=0}^{n_k} A_l$ $(k \geqq 1)$ be satisfied for some sequence $n_1 < n_2 < \ldots$.

I. Sufficiency. Let us suppose (i), (ii) and the a.e. convergence of $\{s_k^Q(x)\}$

where s_k^Q are the Q-partial sums of the series (1). Let for $k=0, 1, 2, \dots$

$$\Phi_k(x) = \frac{1}{\sqrt{\sum_{\mathbf{i}\in A_k} a_{\mathbf{i}}^2}} \sum_{\mathbf{i}\in A_k} a_{\mathbf{i}}\varphi_{\mathbf{i}}(x), \quad b_k = \sqrt{\sum_{\mathbf{i}\in A_k} a_{\mathbf{i}}^2} \tag{22}$$

if $b_k \neq 0$ and

$$\Phi_k(x) = \frac{1}{\sqrt{\sum_{\mathbf{i}\in A_k} 1}} \sum_{\mathbf{i}\in A_k} \varphi_{\mathbf{i}}(x), \quad b_k = 0 \tag{23}$$

in the opposite case. Then $\{\Phi_k\}_{k=0}^{\infty}$ is an ONS on $[0, 1]$ and if S_k denotes the k-th partial sum of the ordinary orthogonal series $\sum_{l=0}^{\infty} b_l \Phi_l(x)$ then

$$s_k^Q(x) = S_{n_k}(x) \quad (k = 1, 2, \dots). \tag{24}$$

(i) gives $n_{k+1}-n_k \leqq K$ by which

$$\sum_{k=1}^{\infty} \sum_{n_k \leqq l < n_{k+1}} \int_0^1 (S_l(x)-S_{n_k}(x))^2\,dx \leqq K \sum_{k=0}^{\infty} b_k^2 = K \sum_{\mathbf{i}\in N^d} a_{\mathbf{i}}^2 < \infty,$$

and so

$$\lim_{k\to\infty} S_l(x)-S_{n_k}(x) = 0 \quad (n_k \leqq l < n_{k+1})$$

almost everywhere. This, (24) and the assumed a.e. convergence of $\{s_k^Q(x)\}_{k=1}^{\infty}$ imply the a.e. convergence of $\sum_{l=0}^{\infty} b_l \Phi_l(x)$.

Let now $\pi: N \to N$ be defined by $\pi(l)=k$ iff $A_l \subseteqq P_{k+1} \setminus P_k$ $(l, k=0, 1, \dots)$. Clearly, π is "onto", $\pi^{-1}(k)$ is a finite set for each k and $P_{k+1} = \bigcup_{l\in\pi^{-1}(0,k)} A_l$, i.e.

$$s_{k+1}^P(x) = \sum_{l=0}^{k} \sum_{i\in\pi^{-1}(l)} b_i \Phi_i(x).$$

By (ii) $\pi^{-1}(0, k)$ and $\pi^{-1}(k, \infty)$ are the union of at most K sets of the form (l, m), $\pi^{-1}(l)$ or $\{l\}=(l, l)$, hence this π satisfies the assumptions of Theorem 2.

Applying Theorem 2 to π and $\sum_{l=0}^{\infty} b_l \Phi_l(x)$ and taking into account the above proved fact that the a.e. convergence of $\{s_k^Q(x)\}_{k=1}^{\infty}$ implies that of $\sum_{l=0}^{\infty} b_l \Phi_l(x)$, we obtain the sufficiency of conditions (i) and (ii).

II. Necessity. First let us prove the necessity of (i). Let us write shortly $Q_k^* = Q_{k+1} \setminus Q_k$, $P_k^* = P_{k+1} \setminus P_k$. If (i) does not hold then for each n there are a k_n and numbers $k_1^{(n)} < \dots < k_n^{(n)} < l_1^{(n)} < \dots < l_n^{(n)}$ such that

$$\emptyset \neq Q_{k_n}^* \cap P_{k_1^{(n)}} \subset Q_{k_n}^* \cap P_{k_2^{(k)}} \subset \dots \subset Q_{k_n}^* \cap P_{l_n^{(n)}}.$$

We may suppose $l_{n-1}^{(n-1)} < k_1^{(n)}$ $(n=1, 2, \dots)$. Let $\mathbf{i}_\varrho^{(n)} \in Q_{k_n}^* \cap P_{k_{\varrho-1}^{(n)}}^*$, $\mathbf{j}_\varrho^{(n)} \in Q_{k_n}^* \cap P_{l_{\varrho-1}^{(n)}}^*$ $(1 \leqq \varrho \leqq n)$. Using the orthogonal series $\sum_{k=0}^{\infty} a_k \varphi_k(x)$ and the sequences p_n, q_n from

(12), putting

$$b_{\mathbf{i}_\varrho^{(p_n)}} = -b_{\mathbf{j}_\varrho^{(p_n)}} = a_{q_n+\varrho}, \quad \psi_{\mathbf{i}_\varrho^{(p_n)}}(x) = \psi_{\mathbf{j}_\varrho^{(p_n)}}(x) = \frac{1}{2}\varphi_{q_n+\varrho}(x) \quad (x\in[0,1])$$

for $n=1,2,\ldots,\varrho=1,\ldots,n$ and $b_{\mathbf{i}}=0$, $\psi_{\mathbf{i}}(x)=0$ $(x\in[0,1])$ otherwise, and extending these ψ_i to an ONS on $[-1,1]$ exactly as above in the necessity proof of Theorem 2 we get a series $\sum_{\mathbf{i}\in N^d} b_{\mathbf{i}}\psi_{\mathbf{i}}(x)$ for which $\sum_{\mathbf{i}\in Q_k} b_{\mathbf{i}}\psi_{\mathbf{i}}(x)=0$ and

$$\sum_{\mathbf{i}\in P_{k_\varrho^{(p_n)}}} b_{\mathbf{i}}\psi_{\mathbf{i}}(x) = \Big(\sum_{\mathbf{i}\in P_{l_{p_n-1}^{(p_n-1)}}} + \sum_{\mathbf{i}\in P_{k_\varrho^{(p_n)}}\setminus P_{l_{p_n-1}^{(p_n-1)}}}\Big) =$$

$$= 0+\sum_{s=1}^{\varrho} b_{\mathbf{i}_s^{(p_n)}}\psi_{\mathbf{i}_s^{(p_n)}}(x) = \frac{1}{2}\big(S_{q_n+\varrho}(x)-S_{q_n}(x)\big) \quad (x\in[0,1],\ 1\leqq\varrho\leqq p_n,\ n=1,2,\ldots).$$

Hence $\{s_k^Q(x)\}_{k=1}^\infty$ converges everywhere on $[-1,1]$ but $\{s_k^P(x)\}_{k=1}^\infty$ diverges a.e. on $[0,1]$ (see (12)).

Thus, the necessity of (i) is proved and from now on we assume its validity.

Let us now consider the sequence of the sets A_n introduced at the beginning of the proof and the mapping π used in the sufficiency proof. Using (i), (ii) can be expressed as: there is a K_1 such that for every k, $\pi^{-1}(0,k)$ and $\pi^{-1}(k,\infty)$ are the union of at most K_1 sets (l,m) and $\pi^{-1}(l)$. By (i) the a.e. convergence of $\{s_k^Q(x)\}_{k=1}^\infty$ is equivalent to that of

$$\sum_{l=0}^{\infty}\sum_{\mathbf{i}\in A_l} a_{\mathbf{i}}\varphi_{\mathbf{i}}(x) = \sum_{l=0}^{\infty} b_l\Phi_l(x)$$

(see point I above) where we used the notations of (19) and (20). Since the a.e. convergence of $\{s_k^P(x)\}_{k=1}^\infty$ is the same as the a.e. convergence of

$$\sum_{k=0}^{\infty}\sum_{\mathbf{i}\in P_{k+1}\setminus P_k} a_{\mathbf{i}}\varphi_{\mathbf{i}}(x) = \sum_{k=0}^{\infty}\sum_{l\in\pi^{-1}(k)} b_l\Phi_l(x),$$

the necessity of (ii) easily follows from Theorem 2.

We have completed our proof.

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L^r inequalities for Walsh series, $0<r<1$

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1. Introduction. Let $w_0, w_1, \dots$ denote the *Walsh—Paley functions* (see [5]). Thus, for each integer $k\geqq 0$ and each point x belonging to the unit interval [0, 1], the identity

$$w_k(x) = \prod_{j=0}^{\infty} \exp(i\pi x_{j+1} k_j) \tag{1}$$

holds, where the numbers x_j and k_j are either 0 or 1 and come from the binary expansions of x and k:

$$x = \sum_{j=1}^{\infty} x_j 2^{-j}, \quad k = \sum_{j=0}^{\infty} k_j 2^j.$$

(When $x\in[0, 1)$ is a dyadic rational the finite binary expansion is used.)

Given any *Walsh series* $W = \sum_{k=1}^{\infty} a_k w_k$, denote its *n-th partial sums* by

$$W_n = \sum_{k=1}^{n-1} a_k w_k,$$

its *n-th partial Cesaro sums* by

$$\sigma_n = \sum_{k=1}^{n-1} \left(1-\frac{k}{n}\right) a_k w_k,$$

and its *n-th layer* by

$$\Delta_n = \sum_{k=2^{n-1}}^{2^n-1} a_k w_k,$$

for $n=1, 2, \dots$. Notice that the Walsh series W has no constant term, and thus that $W = \sum_{n=1}^{\infty} \Delta_n$. This has been done for convenience to avoid writing a separate constant term in each of the inequalities derived below. It does not affect the generality of our results.

In Section 2 a basic inequality is derived which is a Walsh series analogue for

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L^r norms, $0<r<1$, of a trigonometric result for L^p norms, $1<p<\infty$, due to MARCINKIEWICZ [7]. In Section 3 we shall apply this basic inequality to estimate the L^r norms of the following three series:

$$S_1 = \left(\sum_{k=1}^{\infty} \frac{(W_k-\sigma_k)^2}{k}\right)^{1/2}, \tag{2}$$

$$S_2 = \left(\sum_{k=1}^{\infty} (W_{2^k}-\sigma_{2^k})^2\right)^{1/2}, \tag{3}$$

and

$$S_3 = \left(\sum_{n=1}^{\infty} \Delta_n^2\right)^{1/2}. \tag{4}$$

The results of Section 3 are summarized as follows.

Theorem. *Let $0<r<1$. There is an absolute constant α_r depending only on r such that given any Walsh series W the following three inequalities hold:*

$$\|S_1\|_{L^r} \leqq \alpha_r \|S_2\|_{L^1}, \tag{5}$$

$$\|S_2\|_{L^r} \leqq \alpha_r \|S_3\|_{L^1}, \tag{6}$$

and

$$\|S_3\|_{L^r} \leqq \alpha_r \|S_1\|_{L^1}. \tag{7}$$

In the case that W is a trigonometric series and $1<r<\infty$, the theorem above was obtained by ZYGMUND [13]. SUNOUCHI [11] used Zygmund's techniques to show that for Walsh series, the L^p norms of the series S_1, S_2, and S_3 are equivalent, for each $1<p<\infty$.

In Section 4 we apply the theorem above to obtain some inequalities relating a Walsh series to its term by term dyadic derivative. The surprising thing is that under suitable hypotheses, there is a direct relationship between the H^r norm of a function f and the growth of the partial sums of the formal dyadic derivative of the Walsh series representing f.

It should be pointed out that if the series S_1, S_2, and S_3 are replaced by appropriate maximal functions, then equivalence in L^r norms, $0<r<1$, can be restored. In connection with this remark see BURKHOLDER and GUNDY [2], especially Section 5. We do not proceed in this manner because the maximal function form of the theorem above proves intractable for studying the term by term dyadic derivative.

2. The basic inequality. Given a Walsh series W, denote its *maximal function* by

$$W^* = \sup_{n>0} |W_{2^n}|.$$

BURKHOLDER and GUNDY [2] have shown that given $0<r<\infty$ there exist constants

a_r and A_r depending only on r such that

$$a_r \|W^*\|_{L^r} \leqq \left\| \left(\sum_{n=1}^{\infty} \Delta_n^2 \right)^{1/2} \right\|_{L^r} \leqq A_r \|W^*\|_{L^r} \tag{8}$$

holds for all Walsh series W.

Given any function f, integrable over the interval $[0, 1]$, denote its *Walsh—Fourier series* by $W[f]$. Denote the partial Cesaro sums of $W[f]$ by $\sigma_n[f]$ and the n-th layer of $W[f]$ by $\Delta_n[f]$ $n = 1, 2, \ldots$. It is well-known (see [5]) that if $\dot{+}$ represents dyadic addition then

$$\Delta_n[f, x] = \int_0^1 f(t) \left(\sum_{k=2^{n-1}}^{2^n-1} w_k(x \dot{+} t) \right) dt \tag{9}$$

for $x \in [0, 1]$ and $n = 1, 2, \ldots$.

Our main goal in this section is to sketch a proof of the following inequality.

Lemma. *Let $0<r<1$ and suppose that $p_1, p_2, \ldots$ is a sequence of integers which diverges to $+\infty$. There exists a constant β_r, depending only on r, such that*

$$\left\{ \int_0^1 \left(\sum_{n=1}^{\infty} W_{p_n}^2 [f_n, x] \right)^{r/2} dx \right\}^{1/r} \leqq \beta_r \int_0^1 \left(\sum_{n=1}^{\infty} f_n^2(x) \right)^{1/2} dx$$

holds for any sequence $f_1, f_2, \ldots$ of functions which belong to $L^1[0, 1]$.

To prove this lemma set $\varphi(x) = \left(\sum_{n=1}^{\infty} f_n^2(x) \right)^{1/2}$, for $x \in [0, 1]$, and assume without loss of generality that $\varphi \in L^1[0, 1]$. Let $r_1, r_2, \ldots$, denote the *Rademacher functions,* i.e., $r_n = w_{2^{n-1}}$ for $n = 1, 2, \ldots$, and consider the series

$$F(x, y) = \sum_{n=1}^{\infty} r_n(y) f_n(x), \quad x, y \in [0, 1].$$

We claim that the assumption $\varphi \in L^1(0, 1]$ guarantees that for a.e. $y \in [0, 1]$ some subsequence of the series $F(x, y)$ converges in the $L^1(dx)$ norm. In fact, according to Khinchin's inequality there exist constants b_r and B_r, for $0<r<\infty$, such that

$$b_r \|\varphi\|_{L^r}^r \leqq \int_0^1 \int_0^1 |F(x, y)|^r \, dx \, dy \leqq B_r \|\varphi\|_{L^r}^r \tag{10}$$

In particular, for $r=1$ we have that

$$\int_0^1 \int_0^1 \left| \sum_{k=n}^{m} r_k(y) f_k(x) \right| dx \, dy \leqq B_1 \int_0^1 \left(\sum_{k=n}^{m} f_k^2(x) \right)^{1/2} dx. \tag{11}$$

By the Lebesgue dominated convergence theorem the left-hand-side of (11) converges to zero, as $n, m \to \infty$. Therefore $F(x, y)$ converges in the L^1 norm on the unit square $[0, 1]\times[0, 1]$. In particular, for a.e. $y\in[0, 1]$ there exists a subsequence $n_1<n_2<\dots$ such that $\sum_{n=1}^{n_j} r_n(y)f_n\to F(\cdot, y)$ in $L^1[0, 1]$ norm, as $j\to\infty$, and the claim is established. It follows from (9) that

$$\Delta_k(F(\cdot, y)) = \sum_{n=1}^{\infty} r_n(y)\Delta_k(f_n) \tag{12}$$

holds for a.e. $y\in[0, 1]$ and for all $k\geqq 1$.

Next, we show that there exist constants c_r and C_r, depending only on r, such that

$$c_r\|\varphi\|_{L^r}\leqq\left\{\int_0^1\left(\sum_{n,k=1}^{\infty}\Delta_k^2[f_n, x]\right)^{r/2}dx\right\}^{1/r}\leqq C_r\|\varphi\|_{L^1} \tag{13}$$

holds for $0<r<1$. Toward this let I denote the middle term of (13) and apply the two-dimensional version of Khinchin's inequality (see p. 84 of [7]) to I^r. Follow up by applying Khinchin's inequality to the inner-most integral of the resulting triple integral. What eventuates is that there exist constants d_r and D_r, depending only on r, such that

$$d_r\int_0^1\int_0^1\left(\sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}r_n(y)\Delta_k[f_n, x]\right)^2\right)^{r/2}dy\,dx\leqq$$

$$\leqq I^r\leqq D_r\int_0^1\int_0^1\left(\sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}r_n(y)\Delta_k[f_n, x]\right)^2\right)^{r/2}dy\,dx.$$

Continuing, we apply (12) and the Burkholder—Gundy inequality (8) to conclude that

$$a_rd_r\int_0^1\int_0^1|F^*(x, y)|^r\,dx\,dy\leqq I^r\leqq A_rD_r\int_0^1\int_0^1|F^*(x, y)|^r\,dx\,dy, \tag{14}$$

where $F^*(x, y)$ represents the maximal function $\sup_{n>0}|W_{2^n}[F(\cdot, y), x]|$ for each $x, y\in[0, 1]$. However, since for a.e. y the function $F(\cdot, y)$ is integrable, it is easy to see that

$$\int_0^1\int_0^1|F(x, y)|^r\,dx\,dy\leqq\int_0^1\int_0^1|F^*(x, y)|^r\,dx\,dy\leqq\gamma_r\int_0^1\int_0^1|F(x, y)|\,dx\,dy$$

for $0<r<1$. (The constant γ_r either follows from known martingale inequalities or from a weak type (1, 1) estimate of Yano [12]. In connection with this see the comment on p. 734 in [1].) Consequently, inequality (13) follows from (14) and (10) with $c_r=a_rb_rd_r$ and $C_r=\gamma_rA_rB_rD_r$.

To complete the proof of the lemma, observe by SUNOUCHI [11] (pp. 7—8) that corresponding to each p_n there are numbers $\varepsilon_j^{(n)}\in\{0, 1\}$ $(j, n=1, 2, \dots)$ such that

$$(15)\qquad w_{p_n}W_{p_n}[f_n] = \sum_{j=1}^{\infty} \varepsilon_j^n \Delta_j [w_{p_n} f_n].$$

It follows from (13), then, that

$$\int_0^1 \left(\sum_{n=1}^{\infty} W_{p_n}^2[f_n, x]\right)^{r/2} dx \equiv \int_0^1 \left(\sum_{n=1}^{\infty}\left(\sum_{j=1}^{\infty} \varepsilon_j^n \Delta_j [w_{p_n} f_n, x]\right)^2\right)^{r/2} dx \leqq$$

$$\leqq c_r^{-r} \int_0^1 \left(\sum_{n=1}^{\infty}\sum_{j=1}^{\infty} \varepsilon_j^n \Delta_j^2 [w_{p_n} f_n, x]\right)^{r/2} dx.$$

In particular, another application of (13) results in the following inequality:

$$\int_0^1 \left(\sum_{n=1}^{\infty} W_{p_n}^2[f_n, x]\right)^{r/2} dx \leqq c_r^{-r} C_r^{+r} \left(\int_0^1 \left(\sum_{n=1}^{\infty} |w_{p_n}(x) f_n(x)|^2\right)^{1/2} dx\right)^r.$$

The proof of the lemma is now complete with $\beta_r = C_r/c_r$ since $|w_{p_n}|\equiv 1$ for all integers n.

3. A proof of the theorem. To prove (5), set $p_k=k$ and

$$f_k(x) = k^{-3/2} \sum_{j=0}^{2^n-1} j a_j w_j(x),$$

for $2^{n-1}\leqq k<2^n$, $x\in[0, 1]$, and observe that $(W_k-\sigma_k)=k^{-1}\sum_{j=0}^{k-1} j a_j w_j$. It follows from the lemma proved in Section 2 that

$$\|S_1\|_{L^r} \equiv \left\{\int_0^1 \left(\sum_{n=1}^{\infty} \sum_{k=2^{n-1}}^{2^n-1} k^{-3} \left(\sum_{j=0}^{k-1} j a_j w_j(x)\right)^2\right)^{r/2} dx\right\}^{1/r} \leqq \beta_r \int_0^1 \left(\sum_{k=1}^{\infty} f_k^2(x)\right)^{1/2} dx.$$

Since f_k^2 is dominated by

$$(16)\qquad 8\cdot 2^{-3n}\left(\sum_{j=0}^{2^n-1} j a_j w_j\right)^2 \equiv 8\cdot 2^{-n}(W_{2^n}-\sigma_{2^n})^2,$$

for $2^{n-1}\leqq k<2^n$, it follows that (5) holds with $\alpha_r=\sqrt{8}\,\beta_r$.

To verify (6) begin by observing that $W_n-\sigma_n=n^{-1}\sum_{j=1}^{n-1}(W_n-W_j)$ holds for any integer $n\geqq 1$. It follows from the Schwarz inequality that

$$\sum_{n=1}^{\infty}(W_{2^n}-\sigma_{2^n})^2 \leqq \sum_{n=0}^{\infty} 2^{-n} \sum_{k=1}^{n} \sum_{j=2^{k-1}}^{2^k-1} (W_{2^n}-W_j)^2.$$

If we set

$$G=\sum_{n=0}^{\infty}2^{-n}\sum_{k=1}^{n}2^{k}(W_{2^n}-W_{2^{k-1}})^2$$

we have by (3) and the lemma that

$$\|S_2\|_{L^r}\leqq\beta_r\int_0^1[G(x)]^{1/2}\,dx. \tag{17}$$

Here we have used the lemma on a connected block of terms of a Walsh series instead of partial sums of Walsh series. This application is justified since such blocks are differences of partial sums of Walsh series.

Continuing, observe that

$$|W_{2^n}-W_{2^{k-1}}|\leqq|\Delta_k|+|\Delta_{k+1}|+\ldots+|\Delta_n|$$

holds and write $\Delta_j=2^{j/4}\cdot 2^{-j/4}\Delta_j$ for each $j\in[k,n]$. Hence another application of the Schwarz inequality followed by a routine calculation results in the inequalities:

$$G\leqq\sum_{n=0}^{\infty}2^{-n}\sum_{k=1}^{n}2^{k}\left(\sum_{j=k}^{n}2^{-j/2}\right)\left(\sum_{j=k}^{n}2^{j/2}\Delta_j^2\right)\leqq 3\sqrt{2}\sum_{n=0}^{\infty}2^{-n}\sum_{k=1}^{n}2^{k/2}\left(\sum_{j=k}^{n}2^{j/2}\Delta_j^2\right).$$

Reverse the two inner-most sums, and sum $2^{k/2}$ from $k=1$ to $k=j$ to verify that

$$G\leqq 6\sum_{n=0}^{\infty}2^{-n}\sum_{j=1}^{n}2^{j}\Delta_j^2.$$

Now, interchange the order of summation again, and sum 2^{-n} from $n=j$ to $n=\infty$ to conclude that

$$G\leqq 6\sum_{j=0}^{\infty}\Delta_j^2.$$

Finally, combine this inequality with (17) to verify that (6) holds with $\alpha_r=6\beta_r$.

To establish (7) begin with the trivial identity

$$W_{2^n}-W_{2^{n-1}}=(W_{2^n}-\sigma_{2^n})+(\sigma_{2^n}-\sigma_{2^{n-1}})+(\sigma_{2^{n-1}}-W_{2^{n-1}})$$

which holds for $n=1,2,\ldots,$ and apply the Schwarz inequality to conclude that

$$S_3\leqq 2S_2+\left(\sum_{n=1}^{\infty}|\sigma_{2^n}-\sigma_{2^{n-1}}|^2\right)^{1/2}. \tag{18}$$

Let S_4 represent the second term on the right hand side of (18). Correcting a misprint which appears on p. 9 of [11], it is known that

$$|\sigma_{2^n}-\sigma_{2^{n-1}}|^2\leqq 2\sum_{k=2^{n-1}}^{2^n-1}\frac{|W_k-\sigma_k|^2}{k}.$$

Indeed,

$$|\sigma_{2^n}-\sigma_{2^{n-1}}| \leqq \sum_{k=2^{n-1}}^{2^n-1} |\sigma_{k+1}-\sigma_k| \leqq \sqrt{\sum_{k=2^{n-1}}^{2^n-1} k(\sigma_{k+1}-\sigma_k)^2} \cdot \sqrt{\sum_{k=2^{n-1}}^{2^n-1} k^{-1}} \leqq$$

$$\leqq \sqrt{2}\sqrt{\sum_{k=2^{n-1}}^{2^n-1} k(\sigma_{k+1}-\sigma_k)^2} \leqq \sqrt{2}\sqrt{\sum_{k=2^{n-1}}^{2^n-1} \frac{|W_k-\sigma_k|^2}{k}}.$$

It follows that $S_4 \leqq \sqrt{2}S_1$. Moreover, by Jensen's inequality it is known that $\|S_4\|_{L^r}^r \leqq \|S_4\|_{L^1}^r$. In particular,

(19) $$\|S_4\|_{L^r}^r \leqq 2^{r/2}\|S_1\|_{L^1}^r.$$

To estimate S_2, observe that $2^{-2n} \leqq \sum_{j=2^{n-1}}^{2^n-1} j^{-3}$ and therefore by (16) that

$$|W_{2^n}-\sigma_{2^n}|^2 \leqq \left(\sum_{k=2^{n-1}}^{2^n-1} j^{-3}\right)\left(\sum_{i=1}^{2^n-1} ia_i w_i\right).$$

A final application of the lemma proved in Section 2 yields the following inequality:

$$\|S_2\|_{L^r} \leqq \beta_r \int_0^1 \left(\sum_{n=1}^{\infty} \sum_{j=2^{n-1}}^{2^n-1} j^{-3}\left|\sum_{i=1}^{j} ia_i w_i(x)\right|^2\right)^{1/2} dx \equiv \beta_r \|S_1\|_{L^1}.$$

Hence by (18) and (19), we conclude that

$$\|S_3\|_{L^r}^r \leqq (2\beta_r^r + 2^{r/2})\|S_1\|_{L^1}^r.$$

Inequality (7) therefore holds with $\alpha_r = (2\beta_r^r + 2^{r/2})^{1/r}$.

4. An application. Butzer and Wagner [3] introduced the following definition. A function f defined at points $x, x\dotplus 2^{-k}$ $(k=1, 2, \dots)$ on the unit interval is said to have a *dyadic derivative* $\dot{d}f$ at x if the following limit exists:

$$\dot{d}f(x) = \lim_{n\to\infty} \sum_{k=0}^{n} 2^{k-1}[f(x)-f(x\dotplus 2^{-k})].$$

It is not difficult to prove that $\dot{d}w_k(x) = kw_k(x)$ for every $x \in [0, 1]$ and every integer $k \geqq 0$. Thus given a Walsh series $W = \sum_{k=1}^{\infty} a_k w_k$, its term by term dyadic derivative is given by

$$\dot{W}(x) = \sum_{k=1}^{\infty} ka_k w_k(x).$$

Notice that

(20) $$\dot{W}_N \equiv \sum_{k=1}^{N} ka_k w_k = N(W_N - \sigma_N)$$

holds for all integers $N \geqq 1$.

Let $0<r\leqq 1$ and let W be a Walsh series. We shall use the following measurements of how rapidly $k^{-3/2}W_k$ and $2^{-2}W_{2^k}$ decay:

$$\|W\|_{\mathscr{U}_r}=\left\|\left(\sum_{k=1}^{\infty}|W_k^2/k^3|\right)^{1/2}\right\|_{L^r},\quad \|W\|_{\mathscr{W}_r}=\left\|\left(\sum_{k=1}^{\infty}|2^{-k}W_{2^k}|^2\right)^{1/2}\right\|_{L^r}.$$

In spite of the suggestive notation, neither of these measurements are norms; the triangle inequality fails to hold. Observe by Jensen's inequality that

$$\|W\|_{\mathscr{U}_r}^2\leqq\sum_{k=1}^{\infty}\|W_k^2/k^3\|_{L^1}\quad\text{and}\quad \|W\|_{\mathscr{W}_r}^2\leqq\sum_{k=1}^{\infty}\|2^{-2k}W_{2^k}^2\|_{L^1}.$$

Thus $\|W\|_{\mathscr{U}_r}$ and $\|W\|_{\mathscr{W}_r}$ are both finite when W is a Walsh—Fourier series.

Recall that given a Walsh series W, the partial sums $\{W_{2^n},\ n\geqq 0\}$ form a dyadic martingale. Hence if S_3 is given by (4), then the dyadic H_r norm of W is given by $\|W\|_{H_r}\equiv\|S_3\|_{L^r}$ (see [6], especially the remarks on p. 193). Moreover, by the Burkholder—Gundy inequality (8), it follows that W belongs to dyadic H_r if and only if $W^*\in L^r$. In particular, since

$$\|W\|_{\mathscr{W}_r}\leqq\|W^*\|_{L^r}\left(\sum_{k=1}^{\infty}2^{-2k}\right)^{1/2},$$

we have that $\|W\|_{\mathscr{W}_r}$ is finite when W belongs to dyadic H^r.

It is now easy to see that for $0<r<1$ there exists an absolute constant α_r (depending only on r) such that

$$\|\dot W\|_{\mathscr{U}_r}\leqq\alpha_r\|\dot W\|_{\mathscr{W}_1},\tag{21}$$

$$\|\dot W\|_{\mathscr{W}_r}\leqq\alpha_r\|W\|_{H^1},\tag{22}$$

and

$$\|W\|_{H^r}\leqq\alpha_r\|\dot W\|_{\mathscr{U}_1}.\tag{23}$$

Indeed, by (20) $\|\dot W\|_{\mathscr{U}_r}=\|S_1\|_{L^r}$ and $\|\dot W\|_{\mathscr{W}_r}=\|S_2\|_{L^r}$ so inequalities (21), (22), and (23) are restatements of inequalities (5), (6), and (7).

Inequalities (22) and (23) are most useful. According to inequality (22), if W is the Walsh—Fourier series of some function f belonging to dyadic H_1, then $\|\dot W\|_{\mathscr{W}_r}<\infty$ for all $0<r<1$. In the case when $\dot df=\dot W$ (see [8], [9], or [10]) we have that $\dot df$ can be represented by a convergent Walsh series whose partial sums are reasonably well-behaved. According to inequality (23), if the partial sums of $\dot W$ are suitably well-behaved, then the original Walsh series must belong to dyadic H_r. In particular, if $\sum_{k=1}^{\infty}\|\dot W_k^2/k^3\|_{L^1}<\infty$, then W belongs to dyadic H_r, $0<r<1$.

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Ergodic sequences of integers

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1. Introduction. Let $S=\{k_1, k_2, \ldots\}$ be an increasing sequence of positive integers. We shall call S an ergodic sequence provided

$$\lim_{n\to\infty} \frac{1}{n} e^{ik_j\alpha} = 0 \quad \text{for} \quad 0<\alpha<2\pi. \tag{1.1}$$

The reason for the terminology is as follows. If U is any unitary operator on a Hilbert space H, then if (1.1) holds we have

$$\operatorname*{strong\,limit}_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} U^{k_j} = P, \tag{1.2}$$

where P is the orthogonal projection of U on $\{f\in H \mid Uf=f\}$. Moreover if (1.2) is to hold for every such U, then (1.1) is both necessary and sufficient. For details see e.g., [1].

Ergodic sequences of integers have been constructed in [1] and [2]. In [4] NIDERREITER gives a method of constructing ergodic sequences which have density zero. In this paper we use a result due to WIENER and WINTNER [5], to construct large classes of such sequences, both random sequences and nonrandom sequences. Here is what we mean by a random sequence. Let $(\Omega, \mathscr{F}, P)$ be a probability space and let τ be a measure preserving, ergodic transformation defined on Ω. Let $A\in\mathscr{F}$ such that $0<P(A)$. Then there exists a measurable set $\Omega_0\subseteqq\Omega$ with $P(\Omega_0)=1$ with the following property. Let $\omega\in\Omega$ and define the sequence $S(\omega, A)$ by

$$S(\omega, A)=\{k \mid \chi_A(\tau^k\omega)=1\} \tag{1.3}$$

where χ_A is the indicator function of A. Then for each $\omega\in\Omega_0$ we shall see that $S(\omega, A)$ is an ergodic sequence. In Section 2 we present the necessary background material. In Section 3 we consider nonrandom sequences and in Section 4 we construct

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the random sequences mentioned above. Finally in Section 5 we mention some possible generalizations and related matters.

2. Background material. In this section we state two results which we shall use subsequently. Let μ be a Borel measure on the circle group T with Fourier coefficients $\hat{\mu}(n)$, $n=0, \pm 1, \ldots$. Then we have

Theorem 1. (i) *Let* $\tau \in T$. *Then* $\mu(\{\tau\}) = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{-N}^{N} \hat{\mu}(n) e^{in\tau}$, *and*

(ii) $\sum_{\tau} |\mu(\{\tau\})|^2 = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} |\hat{\mu}(n)|^2$.

The proof may be found in Katznelson [3, p. 42].

We shall primarily consider measures for which the Fourier coefficients are real so that in (i) we will have $\mu(\{\tau\}) = \lim_{N\to\infty} \frac{1}{N} \sum_{1}^{N} \hat{\mu}(n) e^{in\tau}$.

Now let $\{a_n, n=0, \pm 1, \pm 2, \ldots\}$ be a bounded sequence of numbers. Suppose for each $k=0, \pm 1, \ldots$ the limit

$$\mu_k = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{-N}^{N} a_n \overline{a_{n-k}} \tag{2.1}$$

exists. The following result is due to Wiener and Wintner [5].

Theorem 2. (i) *There exists a positive Borel measure* μ *on* T *such that* $\hat{\mu}(k) = \mu_k$, $k=0, \pm 1, \pm 2, \ldots$ *and*

(ii) $\lim_{N\to\infty} \frac{1}{2N+1} \sum_{-N}^{N} a_n e^{in\lambda} = 0$ *for every* λ *with* $0 \leqq \lambda \leqq 2\pi$ *which is a continuity point of* μ.

Since we shall only consider real sequences $\{a_n\}$, we shall restrict ourselves to one-sided sequences $\{a_n, n=1, 2, \ldots\}$ and (ii) becomes

$$\lim_{N\to\infty} \frac{1}{N} \sum_{1}^{N} a_n e^{in\lambda} = 0 \tag{2.2}$$

for λ a continuity point of μ.

3. Nonrandom sequences. Let $S = \{k_1, k_2, \ldots\}$ be a sequence of positive integers. For each $n=1, 2, \ldots$ let $S_n = \{k_1, \ldots, k_n\}$, and for each $r=1, 2, \ldots$ let $S_n^{(r)} = \{k_1+r, \ldots, k_n+r\}$. Assume that for each $r=1, \ldots$

$$\nu_r = \lim_{n\to\infty} \frac{1}{n} |S_n \cap S_n^{(r)}| \tag{3.1}$$

exists, where $|A|$ is the cardinality of A. Let $\chi_S(\cdot)$ be the indicator function of S. Then we have

Theorem 3. *Suppose S has positive density, i.e.,*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\chi_S(j)=d>0. \tag{3.2}$$

Let $\nu_0=1$. Then μ_k exists for all $k=0,\pm 1,\ldots$ for the sequence $\{\chi_S(j)\}$ and $\mu_k=d\nu_k$, $k=0,\pm 1,\ldots$. Let μ be the measure with Fourier coefficients $\hat{\mu}(k)=\mu_k$. Then we have

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}e^{ik_j\lambda}=0 \tag{3.3}$$

for every λ which is a continuity point of μ.

The proof follows easily from Theorem 2. We see that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\chi_S(j)\chi_S(j+r)=\lim_{n\to\infty}\frac{\alpha(n)}{n}\frac{1}{\alpha(n)}|S_{\alpha(n)}\cap S_{\alpha(n)}^{(r)}|=d\nu_r,$$

$r=1,2,\ldots$, where $\alpha(n)$ is the number of ones among $\chi_S(j)$, $j=1,\ldots,n$. Let $\mu_r=\mu_{-r}=d\nu_r$, $r=0,1,2,\ldots$, and let μ be the measure guaranteed by Theorem 2. Then we have

$$\lim\frac{1}{\alpha(n)}\sum_{j=1}^{\alpha(n)}e^{ik_j\lambda}=\lim\frac{n}{\alpha(n)}\frac{1}{n}\sum_{j=1}^{n}\chi_S(j)e^{ij\lambda}=\frac{1}{d}\mu(\{\lambda\})=0$$

for $0<\lambda<2\pi$ and λ a continuity point of μ. But $\alpha(n)\to\infty$ and $\dfrac{\alpha(n)}{n}\to d$.

This result allows us to give many simple examples of ergodic sequences of integers. If $\nu_1=1$, and hence $\nu_k=1$ for all k, then μ is the measure which puts mass d at $e^{2\pi i}$ and every Borel set of T which does not include the point $e^{2\pi i}$ has μ-measure zero, and we have an ergodic sequence.

We can apply the theorem in two ways. One way is to look at simple measures on T, calculate their Fourier coefficients and then construct ergodic sequences which give rise to these coefficients. The other is to look at certain sequences and verify the appropriate conditions.

Here is a simple example of the first technique. Consider the measure μ which puts mass 1/2 each on $e^{\pi i}$ and $e^{2\pi i}$. Then $\hat{\mu}(n)=0$ for n odd and $\hat{\mu}(n)=1$ for n even. If S is the sequence of even integers then the numbers $\chi_S(j)$ satisfy the conditions of Theorem 2. However the sequence $k_n=2n$ is not ergodic since $\lim\limits_{n}\frac{1}{n}\sum_{j=1}^{n}e^{\pi i k_j}=1$. Now let $\{r_k, k=1,\ldots\}$ be an increasing sequence of positive integers such that $\lim\limits_{n}(r_{k+1}-r_k)=\infty$, $\lim\limits_{n}r_{k+1}/r_k=1$. Modify the sequence S in the following way. When k is even leave the elements of S between r_k and r_{k+1} as they are. When k is odd, add one to each k_n between r_k and r_{k+1}. The resulting sequence $S'=\{k_1', k_2', \ldots\}$ will be ergodic from Theorem 3, and the fact that now $\lim_n \frac{1}{n} \sum_{j=1}^{n} e^{\pi i k_j'}=0$. Clearly we can play the same game for any measure μ which puts mass $1/k$ on each of $e^{2\pi i j/k}$, $j=0, \ldots, k-1$.

Now let x be a normal number to the base two in the unit interval and let $\{x_n, n=1, 2, \ldots\}$ be its coordinates. The measure μ corresponding to this sequence then has Fourier coefficients $\hat{\mu}(0)=1/2$ and $\hat{\mu}(k)=1/4$, $k\neq 0$. It then follows from Theorem 1 that $\mu\{e^{i\lambda}\}=0$ for $0<\lambda<2\pi$ and therefore the sequence $\{k_n, n=1, \ldots\}$ consisting of those integers for which $x_n=1$ is ergodic by Theorem 2.

4. Random sequences. Let $(\Omega, \mathscr{F}, P)$ be a probability space and let τ be a measure preserving transformation mapping Ω onto Ω. Now let $A\in\mathscr{F}$ with $0<P(A)$. It follows from the individual ergodic theorem that there exists $\Omega_0\in\mathscr{F}$ with $P(\Omega_0)=1$ such that for $\omega\in\Omega_0$ the following limit relations hold

$$(4.1)\qquad \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \chi_A(\tau^j\omega)\chi_A(\tau^{j-k}\omega) = P\{A\cap\tau^k A\}, \quad k = 0, 1, 2, \ldots.$$

Now let $\omega\in\Omega_0$ and consider the sequence $\{\chi_A(\tau^j\omega), j=1, 2, \ldots\}$. By Theorem 2 there is a measure μ on T with $\hat{\mu}(k)=P\{A\cap\tau^k A\}$, $k=0, \pm 1, \pm 2, \ldots$. Now suppose τ is mixing. Then we have $\lim_{k\to\infty} P\{A\cap\tau^k A\}=P^2\{A\}$. Moreover from Theorem 1 we see that μ is continuous except at $e^{2\pi i}$. We summarize in

Theorem 4. *Let τ measure-preserving and mixing, and let $A\in\mathscr{F}$ with $0<P(A)$. Then for almost all ω the sequence $\{k_n(A, \omega), n=1, 2, \ldots\}$ consisting of those integers for which $\chi_A(\tau^j\omega)=1$ is an ergodic sequence.*

Theorem 3 enables us to give a simple proof of a theorem of NIEDERREITER [4]. Let r be a positive integer and suppose we are given r and α with $0<\alpha<1$. Then Niederreiter exhibited an ergodic sequence with $\nu_r=\alpha$.

We shall show that the existence of such a sequence follows from Theorem 3. For let $\{X_n(\omega), n=1, 2, \ldots\}$ be a sequence of independent Bernoulli random variables with $P\{X_n(\omega)=1\}=\alpha=1-P\{X_n(\omega)=0\}$. It follows from the law of large numbers that there exists a set of probability one such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} X_j(\omega) X_{j+r}(\omega) = \begin{cases} \alpha, & r=0 \\ \alpha^2, & r>0 \end{cases}$$

for ω in this set. If ω is in this set and $\{k_n(\omega)\}$ is the sequence of integers for which $X_j(\omega)=1$ then $\{k_n(\omega)\}$ is ergodic and $\nu_r=\alpha$ for all $r>0$.

This method can easily be generalized to yield for certain values of $r_1, r_2, \alpha_1, \alpha_2$ ergodic sequences for which $\nu_{r_1}=\alpha_1$, $\nu_{r_2}=\alpha_2$. Whether this can be done in full generality is not clear.

5. Concluding remarks. The method used in this paper does not apply when a sequence has density zero. For example it is easy to show that the sequence $\{n^k, n=1, \dots\}$, $k>1$ and an integer, is not ergodic. On the other hand NIEDERREITER [4] has shown that the sequence of integer parts of n^k when $k>1$ is *not* an integer is ergodic. In both cases we have $v_r=0$, $r=1, 2, \dots$.

Moreover even when a sequence has positive density and each v_k exists and is positive, the situation is not entirely clear. For example, it is possible to construct for each ε such that $0<\varepsilon<1$ a nonergodic sequence with density $d>1-\varepsilon$ and also each $v_k>1-\varepsilon$. Thus we are a long way having convenient necessary and sufficient conditions for ergodicity of a sequence.

Another unresolved question concerns the individual ergodic theorem. As mentioned in Section 1, if $S=\{k_1, k_2, \dots\}$ is an ergodic sequence then the mean ergodic theorem holds for every unitary operator U. Now suppose τ is a measure-preserving transformation on a probability space $(\Omega, \mathcal{F}, P)$. We can then ask if $\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} f(\tau^{k_j}\omega)$ exists a.e. for every $f \in L_1(\Omega, \mathcal{F}, P)$. When S consists of all positive integers, this is of course the individual ergodic theorem. However, when S is significantly different nothing is known.

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Weighted translation semi-groups with operator weights

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1. Introduction. If φ is a continuous nonzero complex-valued function on $\mathscr{R}^+$ and $(S_t f)(x)=[\varphi(x)/\varphi(x-t)]f(x-t)$ for $x\geqq t$ and 0 otherwise, then S is a semi-group of linear transformations on $\mathscr{L}^2(\mathscr{R}^+,\mathscr{C})$. S is a strongly continuous semi-group of bounded operators if φ satisfies certain boundedness conditions. These semi-groups, called *weighted translation semi-groups (w.t.s.) with symbol* φ, were introduced in [4] and the subnormal w.t.s. characterized in [5].

In [4] it was shown that S is quasinormal if and only if $\varphi(x)=Ma^x$ for some constants M and a. In this case $S_t=a^t L_t$, where L is the forward translation semi-group on $\mathscr{L}^2(\mathscr{R}^+,\mathscr{C})$. In [6] we proved that any strongly continuous quasinormal semi-group S on a separable Hilbert space $\mathscr{H}$ is unitarily equivalent to the direct sum of a normal semigroup and a pure quasinormal semi-group Q on $\mathscr{L}^2(\mathscr{R}^+,\mathscr{K})$ for some Hilbert space $\mathscr{K}$. Furthermore, $Q_t=\overline{h(t)}L_t$ where L is the forward translation semi-group on $\mathscr{L}^2(\mathscr{R}^+,\mathscr{K})$, h is a strongly continuous self-adjoint semi-group on $\mathscr{K}$, and $(\overline{h(t)}f)(x)=h(t)f(x)$ a.e. for each f in $\mathscr{L}^2(\mathscr{R}^+,\mathscr{K})$. Thus, the pure quasinormal semi-groups behave like quasinormal w.t.s.

In this paper, we shall introduce w.t.s. on $\mathscr{L}^2(\mathscr{R}^+,\mathscr{K})$ for which the symbol φ is $\mathscr{K}$-operator-valued and study a few of their properties.

In Section 2, we specify which operator-valued functions φ will be allowed. This class of semi-groups gives a rich supply of easily constructed examples. In particular, every pure quasinormal semi-group is (unitarily equivalent to) a weighted translation semi-group. Section 3 is devoted to characterizing subnormal w.t.s. on $\mathscr{L}^2(\mathscr{R}^+,\mathscr{K})$. In Theorem 3 we show that S with symbol φ is subnormal if and only if φ^2 is the compression of a strongly continuous self-adjoint semi-group; equivalently, there exists an operator measure on an interval $[0, a]$ such that $\varphi(x)^2=$

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$=\int_0^a r^x d\varrho(r)$. This last condition is precisely the characterization of subnormal w.t.s. in [5] in the numerical case $\mathcal{K}=\mathcal{C}$.

Throughout the paper, we shall assume all Hilbert spaces to be separable. $\mathcal{L}^2(\mathcal{R}^+, \mathcal{K})$ is the Hilbert space of (equivalence classes of) square integrable weakly measurable functions from the nonnegative reals $\mathcal{R}^+$ to the separable Hilbert space $\mathcal{K}$. $\mathcal{B}(\mathcal{K})$ or $\mathcal{B}(\mathcal{L}^2(\mathcal{R}^+, \mathcal{K}))$ stands for the Banach algebra of continuous linear operators on $\mathcal{K}$ or $\mathcal{L}^2(\mathcal{R}^+, \mathcal{K})$, respectively. A function $S\colon \mathcal{R}^+\to\mathcal{B}(\mathcal{K})$ is a *semi-group* if $S_0=I$, the identity operator, and $S_tS_r=S_{t+r}$ for all r and t in $\mathcal{R}^+$. A function $\varphi\colon \mathcal{R}^+\to\mathcal{B}(\mathcal{K})$ is *strongly continuous* if $\lim_{t\to r}\|\varphi(t)f-\varphi(r)f\|=0$ for each f in $\mathcal{K}$ and r in $\mathcal{R}^+$. In this case, we write $\text{s-}\lim_{t\to r}\varphi(t)=\varphi(r)$. The *forward translation semi-group* L $((L_tf)(x)=f(x-t)$ if $x\geqq t$ and 0 otherwise) on $\mathcal{L}^2(\mathcal{R}^+, \mathcal{K})$ plays a special role in ideas developed in this paper.

A semi-group S of operators is *normal* if $S_t^*S_t=S_tS_t^*$ for all t, *quasinormal* if $S_t(S_t^*S_t)=(S_t^*S_t)S_t$ for all t and *subnormal* if S is the restriction of a normal semi-group to an invariant subspace. An *operator measure* ϱ on $[a, b]$ is a function defined on the Borel sets of $[a, b]$ with values in $\mathcal{B}(\mathcal{K})$ such that $\varrho(\emptyset)=0$, $\varrho(E)$ is a positive Hermitian operator for each Borel set E, $\varrho(E)\ll\varrho(F)$ whenever $E\subseteqq F$ and $\varrho(E)=\text{s-}\lim_{n\to\infty}\sum_{i=1}^n\varrho(E_i)$ whenever E is the union of a collection of pairwise disjoint sets E_i. If the values of ϱ are projections and $\varrho[a, b]=I$, then ϱ is a *spectral measure* on $[a, b]$. Two integral representations which reoccur frequently in this paper are as follows:

1) [8, Theorem 22.3.1, p. 588]. If H is a strongly continuous self-adjoint semi-group of operators, there exists a spectral measure ϱ on an interval $[0, a]$ such that

$$H_t=\int_0^a r^t\,d\varrho(r),$$

2) [5, Theorem 2.1]. S is a strongly continuous subnormal semi-group if and only if there exists an operator measure ϱ on an interval $[0, a]$ such that $\varrho([0,a])=I$ and

$$S_t^*S_t=\int_0^a r^t\,d\varrho(r).$$

We shall also say that a semi-group S on $\mathcal{H}$ is the *compression* of a semi-group T on $\mathcal{K}$ if $\mathcal{H}\subseteqq\mathcal{K}$ and $S_t=PT_tP$ for each t where P is the orthogonal projection of $\mathcal{K}$ onto $\mathcal{H}$.

2. Weighted translation semi-groups. Let $\varphi\colon \mathcal{R}^+\to\mathcal{B}(\mathcal{K})$ have properties:

i) for each x in $\mathcal{R}^+$, $\varphi(x)$ is a one-to-one positive Hermitian operator,

ii) $\{\varphi(x)\colon x\in\mathcal{R}^+\}$ is abelian,

iii) φ is strongly continuous,

iv) there exist numbers M and a such that for all x and t in $\mathcal{R}^+$,

$$\varphi(x+t)^2 \ll M^2 a^{2t} \varphi(x)^2.$$

Such a φ will be called a *symbol.* We are requiring $\varphi(x)$ to be positive Hermitian for simplicity. We use the other requirements to prove that the mapping $t \to \varphi(x-t)^{-1}\varphi(x)(L_t f)(x)$ defines a semi-group which is strongly continuous.

Conditions i) and iv) imply that if $t \leqq x$, there exists a unique element C of $\mathcal{B}(\mathcal{H})$ such that $\varphi(x)=\varphi(t)C$. In this case, we write $C=\varphi(t)^{-1}\varphi(x)$. Even if $\varphi(x)$ is not one-to-one, this factorization of $\varphi(x)$ can be obtained [2]; however, $\varphi(x)^{-1}\varphi(x)$ would be the projection onto the closure of the range of $\varphi(x)$ and the semi-group that we are interested in constructing would not have $S_0=I$.

Lemma 1. *Let φ be a symbol on $\mathcal{H}$. Then*

i) $\varphi(x)^{-1}\varphi(x)=I$ for all x,

ii) $\varphi(r)$ *commutes with* $\varphi(t)^{-1}\varphi(x)$ *for all* r *whenever* $t \leqq x$,

iii) $\varphi(t)^{-1}\varphi(x)$ *commutes with* $\varphi(a)^{-1}\varphi(b)$ *whenever* $t \leqq x$ *and* $a \leqq b$,

iv) $[\varphi(r)^{-1}\varphi(t)][\varphi(t)^{-1}\varphi(s)]=\varphi(r)^{-1}\varphi(s)$ *whenever* $r \leqq t \leqq s$,

v) $\varphi(t)^{-1}\varphi(x)$ *is one-to-one and positive Hermitian whenever* $t \leqq x$ *and satisfies* $\|\varphi(t)^{-1}\varphi(x)\| \leqq Ma^{x-t}$,

vi) $\operatorname{s-lim}_{t\to 0} \varphi(x)^{-1}\varphi(x+t)=I$,

vii) $[\varphi(x)^{-1}\varphi(x+t)]^2=\big(\varphi(x)^2\big)^{-1}\varphi(x+t)^2$ *for all* x *and* t.

Proof. i) follows immediately from definition of $\varphi(x)^{-1}\varphi(x)$. ii) by definition $\varphi(x)=\varphi(t)[\varphi(t)^{-1}\varphi(x)]$. Since $\{\varphi(s)\}$ is abelian and $\varphi(t)$ is one-to-one, $\varphi(r)$ commutes with $\varphi(t)^{-1}\varphi(x)$. Therefore, $\varphi(x)=\varphi(t)^{1/2}[\varphi(t)^{-1}\varphi(x)]\varphi(t)^{1/2}$ and v) now follows from the fact that $\varphi(x)$ and $\varphi(t)^{1/2}$ are one-to-one positive Hermitian operators. The inequality in v) follows from condition iv) of the definition of φ. iii) follows from ii) and the facts that each $\varphi(x)$ is one-to-one and $\{\varphi(x)\}$ is abelian. iv) $\varphi(r)^{-1}\varphi(s)$ is the unique operator satisfying $\varphi(s)=\varphi(r)[\varphi(r)^{-1}\varphi(s)]$. But $[\varphi(r)^{-1}\varphi(t)][\varphi(t)^{-1}\varphi(s)]$ also satisfies this equation. vi) Note that for each k in $\mathcal{H}$,

$$\|[\varphi(x)^{-1}\varphi(x+t)-I]\varphi(x)k\| = \|\varphi(x+t)k-\varphi(x)k\|.$$

Since φ is strongly continuous, then $\operatorname{s-lim}_{t\to 0}\varphi(x)^{-1}\varphi(x+t)=I$ on the range of $\varphi(x)$ which is dense in $\mathcal{H}$. Since $\|\varphi(x)^{-1}\varphi(x+t)-I\| \leqq Ma^t+1 \leqq M_0$ for t in $[0, 1]$, we see that $\operatorname{s-lim}_{t\to 0}\varphi(x)^{-1}\varphi(x+t)=I$ on all of $\mathcal{H}$. vii) Since $\varphi(x+t)^2 \ll M^2a^{2t}\varphi(x)^2$ and $\{\varphi(x)\}$ is abelian, then $\varphi(x+t)^4 \ll M^4a^{4t}\varphi(x)^4$ and $\big(\varphi(x)^2\big)^{-1}\varphi(x+t)^2$ can be defined in a fashion similar to $\varphi(x)^{-1}\varphi(x+t)$: that is, $\big(\varphi(x)^2\big)^{-1}\varphi(x+t)^2$ is the unique operator C satisfying $\varphi(x+t)^2=\varphi(x)^2C$. Since $[\varphi(x)^{-1}\varphi(x+t)]^2$ also satisfies this equation (using the definition of $\varphi(x)^{-1}\varphi(x+t)$ and the fact that it commutes with $\varphi(x+t)$), the proof of vii) is complete.

Now let φ be a symbol on $\mathscr{K}$. For each t in $\mathscr{R}^+$ define the operator S_t on $\mathscr{L}^2(\mathscr{R}^+, \mathscr{K})$ by

$$(S_t f)(x) = \begin{cases} \varphi(x-t)^{-1}\varphi(x)f(x-t) & \text{if } x \geqq t \\ 0 & \text{if } x < t. \end{cases} \tag{1}$$

An argument directly paralleling one in [7, p. 211] can be given to show that

$$\|S_t\| = \operatorname*{ess\,sup}_{x \in \mathscr{R}^+} \|\varphi(x)^{-1}\varphi(x+t)\|. \tag{2}$$

Theorem 2. *If φ is a symbol on $\mathscr{K}$, then S is a strongly continuous semi-group of operators on $\mathscr{L}^2(\mathscr{R}^+, \mathscr{K})$.*

Proof. Note that $(S_0 f)(x) = \varphi(x)^{-1}\varphi(x)f(x) = f(x)$ by Lemma 1 i) so that $S_0 = I$ on $\mathscr{L}^2(\mathscr{R}^+, \mathscr{K})$. A straightforward computation, making use of Lemma 1 iii) and iv), shows that $S_{t+r} = S_t S_r$ for all $t, r \geqq 0$. It remains to be shown that S is strongly continuous. By equation (2) and Lemma 1 v) we have

$$\|S_t\| \leqq Ma^t. \tag{3}$$

We argue as in [4, p. 211]. Assume first that $a=1$. Let f be a continuous function of compact support in $\mathscr{L}^2(\mathscr{R}^+, \mathscr{K})$. Then

$$\|S_t f - f\|^2 = \int \|\varphi(x)^{-1}\varphi(x+t)f(x) - f(x+t)\|^2\,dx.$$

Let $b = \operatorname{ess\,sup}|f|$, $\operatorname{supp} f \subseteqq [0, k]$ and $g(x) = b$ if $x \in [0, k+1]$ and $g(x) = 0$ otherwise. Then $g \in \mathscr{L}^2(\mathscr{R}^+, \mathscr{K})$ and for $t \leqq 1$,

$$\|\varphi(x)^{-1}\varphi(x+t)f(x) - f(x+t)\| \leqq (M+1)g(x).$$

By Lemma 1 vi) and the continuity of f,

$$\lim_{t\to 0} \|\varphi(x)^{-1}\varphi(x+t)f(x) - f(x+t)\| = 0.$$

Thus, by the Lebesgue dominated convergence theorem, $\lim_{t\to 0}\|S_t f - f\|^2 = 0$. S is strongly continuous on a dense subset of $\mathscr{L}^2(\mathscr{R}^+, \mathscr{K})$ and consequently on all of $\mathscr{L}^2(\mathscr{R}^+, \mathscr{K})$ since S is uniformly bounded by M.

Now assume that a is arbitrary in (3) and let $T_t = a^{-t}S_t$ and $\varrho(t) = a^{-t}\varphi(t)$. Then ϱ is a symbol on $\mathscr{K}$ and defines T by (1). Hence, the preceding result implies, that T is strongly continuous on $\mathscr{L}^2(\mathscr{R}^+, \mathscr{K})$; the same must be true for S.

Hereafter, if φ is a symbol on $\mathscr{K}$ and S is the semi-group defined by (1) we shall say that (S, φ) *is a weighted translation semi-group (w.t.s.) on* $\mathscr{L}^2(\mathscr{R}^+, \mathscr{K})$. Note that $(S_t^* f)(x) = \varphi(x)^{-1}\varphi(x+t)f(x+t)$ and, consequently, by Lemma 1 for f in $\mathscr{L}^2(\mathscr{R}^+, \mathscr{K})$,

$$(S_t^* S_t f)(x) = \varphi(x)^{-2}\varphi(x+t)^2 f(x) \quad \text{a.e.} \tag{4}$$

Thus, if P_t is the positive square root of $S_t^* S_t$, then $(P_t f)(x)=\varphi(x)^{-1}\varphi(x+t)f(x)$ by Lemma 1 vii) and v). A straightforward argument shows that $S_t=L_t P_t$ and $\ker L_t=\ker S_t$ where L is the forward translation semi-group on $\mathcal{L}^2(\mathcal{R}^+,\mathcal{K})$. That is, P_t is the positive factor and L_t the isometric factor in the polar decomposition of S_t.

The following examples give two ways in which to construct symbols and the associated w.t.s.

Example 1. Let $\varphi:\mathcal{R}^+\to\mathcal{B}(\mathcal{K})$ and assume that φ is one-to-one positive Hermitian-valued, nonincreasing and strongly continuous. If $\{\varphi(x)\}$ is abelian, then it follows that φ satisfies the properties of a symbol. Consequently, (S,φ) is a strongly continuous semi-group.

Example 2. Let φ be a strongly continuous self-adjoint semi-group of operators on $\mathcal{K}$. It follows easily that φ satisfies properties i)—iii) of a symbol. Moreover, there exists a spectral measure ϱ such that $\varphi(x)=\int\limits_0^a r^x d\varrho(r)$ [8, p. 588]. The inequality $\varphi(x+t)^2\ll a^{2t}\varphi(x)^2$ readily follows. In this case, (S,φ) has a simpler form than the general w.t.s.:

$$(S_t f)(x)=\varphi(x-t)^{-1}\varphi(x)f(x-t)=\varphi(t)f(x-t) \quad \text{if} \quad x\geqq t.$$

We shall see in the following section that these are the only quasinormal w.t.s. Indeed, every pure quasinormal semi-group is unitarily equivalent to (S,φ) where φ is a strongly continuous self-adjoint semi-group (Corollary 6).

In the next section, it will be convenient to consider symbols φ for which $\varphi(0)=I$. There is no loss of generality in making this assumption for if φ is a symbol, define $\varphi_1(x)=\varphi(0)^{-1}\varphi(x)$. Then $\varphi_1(0)=I$ by Lemma 5. Furthermore, by Lemma 5 $\varphi_1(x)$ is a one-to-one positive Hermitian operator, $\{\varphi_1(x)\}$ is abelian and φ_1 is strongly continuous. To see that $\varphi_1(x+t)^2\ll M^2a^{2t}\varphi_1(x)^2$ we argue as follows. By definition of φ, $\|\varphi(x+t)k\|\leqq Ma^t\|\varphi(x)k\|$ for all k in $\mathcal{K}$. Therefore, $\|[\varphi(0)^{-1}\varphi(x+t)]\varphi(0)k\|\leqq Ma^t\|[\varphi(0)^{-1}\varphi(x)]\varphi(0)k\|$. Consequently, $\|\varphi_1(x)k\|\leqq$ $\leqq Ma^t\|\varphi_1(x)k\|$ for all k in the range of $\varphi(0)$, a dense subset of $\mathcal{K}$. Thus, the inequality holds for all k so that φ_1 satisfies condition iv) of the definition of a symbol.

3. Subnormal weighted translation semi-groups. Throughout this section, we assume $\varphi(0)=I$ when φ is a symbol.

Example 3. Let ϱ be an abelian operator measure on $[0,a]$ with $\varrho[0,a]=I$. Define $\varphi(x)^2=\int\limits_0^a r^x d\varrho(r)$ where $\varphi(x)\gg 0$ for each x. It will follow from Lemma 4 that φ is a symbol. Indeed, we see in the following theorem that these are exactly the symbols which define the subnormal w.t.s.

Theorem 3. *Let* (S, φ) *be a w.s.t. on* $\mathscr{L}^2(\mathscr{R}^+, \mathscr{K})$. *The following statements are equivalent:*

i) (S, φ) *is subnormal,*

ii) φ^2 *is the compression of a strongly continuous self-adjoint semi-group,*

iii) *there exists an operator measure* ϱ *on* $[0, a]$ *with* $\varrho[0, a]=I$ *such that*

$$\varphi(x)^2 = \int_0^a r^x \, d\varrho(r).$$

Before proving Theorem 3, we shall prove a lemma which includes the equivalence of ii) and iii).

Lemma 4. *Let* $\mathscr{K}$ *be a Hilbert space and* $h\colon \mathscr{R}^+ \to \mathscr{B}(\mathscr{K})$. *The following are equivalent:*

1. h *is a compression of a strongly continuous self-adjoint semi-group,*
2. *there exists an operator measure on a finite interval* $[0, a]$ *such that* $\varrho[0, a]=I$ *and*

$$h(x) = \int_0^a r^x d\varrho(r),$$

3. h *satisfies the following four conditions:*

i) $h(0)=I$,

ii) h *is strongly continuous,*

iii) *there exists a number* a *such that* $h(x+t) \ll a^t h(x)$ *for all* x *and* t *in* $\mathscr{R}^+$,

iv) $\sum_{i,j=0}^{n} \langle h(x_i+x_j)k_i, k_j\rangle \geqq 0$ *for all finite collections* $\{x_0, \ldots, x_n\}$ *in* $\mathscr{R}^+$ *and* $\{k_0, \ldots, k_n\}$ *in* $\mathscr{K}$.

Proof. We shall show that $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 1$.

$1 \Rightarrow 2$. Assume that $h(t)=PH(t)P$ where P is the projection of a larger Hilbert space onto $\mathscr{K}$ and H is a strongly continuous self-adjoint semi-group on the larger space. There exists a spectral measure [8, p. 588] on an interval $[0, a]$ such that $H(t)=\int_0^a r^t d\mu(r)$. Consequently, $h(t)=\int_0^a r^t dP\mu(r)P$ and $P\mu P$ is an operator measure on $\mathscr{K}$ with $(P\mu P)[0, a]=I$ on $\mathscr{K}$.

$2 \Rightarrow 3$. Assume 2 holds. 3 i) and iii) are immediate. 3 ii) follows from an application of the monotone convergence theorem. To see that 3 iv) holds, observe that if E is any measurable subset of $[0, a]$, then

$$\sum_{i,j=0}^{n} r^{x_i+x_j} \langle \varrho(E)k_i, k_j\rangle = \left\langle \varrho(E) \sum_{i=0}^{n} r^{x_i} k_i, \sum_{j=0}^{n} r^{x_j} k_j \right\rangle \geqq 0$$

and consequently,

$$\sum_{i,j=0}^{n} \int_0^a r^{x_i+x_j} \langle d\varrho(r) k_i, k_j \rangle \geqq 0.$$

$3 \Rightarrow 1$. The techniques used in this part of the proof are standard and will only be outlined. They are patterned after proofs in [1] and [3]. Assume that h satisfies the properties given in 3. Let M be the set of all functions $f: \mathscr{R} \to \mathscr{K}$ such that $f(x)=0$ except possibly for a finite number of real x. If f and g are in M, define

$$(f, g) = \sum_{a,b} \langle h(a+b) f(a), g(b) \rangle.$$

(See [3, p. 1254] for details.) Since $(f, f) \geqq 0$ by hypothesis, it is easily checked that $(\ ,\ )$ is a semi-inner product on M. Let $M_0 = \{f: (f, f) = 0\}$ and $H_0 = M/M_0$. Let $(\ ,\ )$ also be the inner product on H_0 induced by $(\ ,\ )$ on M and let $\mathscr{H}$ be the completion of H_0.

For each t in $\mathscr{R}^+$ define $H(t): M \to M$ by $(H(t)f)(x) = f(x-t)$. Then H is a semi-group and for f and g in M

$$(H(t)f, g) = \sum_{a,b} \langle h(a+b) f(a-t), g(b) \rangle =$$
$$= \sum_{a,b} \langle h(a+b+t) f(a), g(b) \rangle = (f, H(t) g).$$

It follows from the Cauchy—Schwarz inequality that M_0 is invariant under $H(t)$; consequently $H(t)$ induces a self-adjoint semi-group of linear transformations on H_0. If we can show that $H(t)$ is a bounded transformation, then $H(t)$ can be extended continuously to $\mathscr{H}$.

To prove that $H(t)$ is bounded, we need to show that there exists K such that $(H(t)f, H(t)f) \leqq K(f, f)$ for all f in M. Equivalently,

$$\sum_{a,b} \langle h(a+b+2t) f(a), f(b) \rangle \leqq K \sum_{a,b} \langle h(a+b) f(a), f(b) \rangle.$$

The argument given by Bram [1, p. 76] can be duplicated in this situation to show that this inequality holds with $K = a^{2t}$ (we use condition iii) here).

Thus, H is a semi-group of self-adjoint operators on $\mathscr{H}$. We next show that H is strongly continuous. Let $f \in M$ and compute

$$(H(t)f - f, H(t)f - f) = \operatorname{Re} \sum_{a,b} \langle [h(a+b+2t) - 2h(a+b+t) + h(a+b)] f(a), f(b) \rangle.$$

Since h is strongly continuous on $\mathscr{K}$, the right-hand side converges to 0 as $t \to 0$. We conclude that H is strongly continuous on $\mathscr{H}$.

We complete the proof by identifying $\mathscr{K}$ with a subspace of $\mathscr{H}$ and h with the compression of H to that subspace. For each k in $\mathscr{K}$ define $(Uk)(x) = k$ if $x = 0$ and $(Uk)(x) = 0$ otherwise. Then $Uk \in M$, U is linear, and $(Uk, Uk) =$

$=\langle h(0)k, k\rangle=\|k\|^2$ by condition i). Therefore, we may consider Uk to be an element of M/M_0 and consequently of $\mathscr{H}$. $\|Uk\|_{\mathscr{H}}=\|k\|_{\mathscr{K}}$ so that U is an isometry from $\mathscr{K}$ onto a subspace of $\mathscr{H}$. UU^* is the projection P of $\mathscr{H}$ onto that subspace. We complete the proof by showing that $U^*PH(t)PU=h(t)$, so that h is unitarily equivalent to this compression of the strongly continuous selfadjoint semi-group H. For k and j in $\mathscr{K}$,

$$\langle U^*PH(t)PUk, j\rangle = \langle U^*H(t)Uk, j\rangle = \big(H(t)Uk, Uj\big) =$$
$$= \sum_{a,b} \langle h(a+b)(Uk)(a-t), (Uj)(b)\rangle = \langle h(t)k, j\rangle$$

and $h(t)=U^*PH(t)PU$, as desired.

Remark. If h satisfies Lemma 4.2, then $h(x)=PH(x)P$ where P is a projection and H a self-adjoint semi-group. Therefore, if $h(x)k=0$, then $H(x/2)Pk=0$ and $h(x/2)k=0$. Consequently, we can construct a sequence $x_n\to 0$ for which $h(x_n)k=0$. Since h is strongly continuous and $h(0)=I$, then $k=0$ and we see that $h(x)$ is one-to-one. Indeed, we see that h satisfies all of the properties of a symbol except possibly $\{h(x)\}$ being abelian.

Proof of Theorem 3. ii)$\Leftrightarrow$iii) by Lemma 4.

Assume that i) holds and (S, φ) is subnormal. By [5, Theorem 2.1] there exists an operator measure ϱ in $\mathscr{B}(\mathscr{L}^2(\mathscr{R}^+, \mathscr{K}))$ such that $\varrho[0, a]=I$ and

$$S_t^* S_t = \int_0^a r^t \, d\varrho(r).$$

By equation (4) then for each f in $\mathscr{L}^2(\mathscr{R}^+, \mathscr{K})$,

$$\varphi(x)^{-2}\varphi(x+t)^2 f(x) = \int_0^a r^t\big(d\varrho(r)f\big)(x)$$

except on a set of measure zero. We conclude then that for a given finite collection $f_0, \ldots, f_n$ of elements of $\mathscr{L}^2(\mathscr{R}^+, \mathscr{K})$ and all positive rational numbers t this equation holds except on a set E of measure zero. In particular, if $k_0, \ldots, k_n$ are elements of $\mathscr{K}$ and for $i=0, \ldots, n, f_i(x)=k_i$ for x in $[0, 1]$ and zero otherwise, then

$$\varphi(x)^{-2}\varphi(x+t)^2 k_i = \int_0^a r^t\big(d\varrho(r)f_i\big)(x)$$

for t rational and x in $[0, 1]\cap E$. Consequently, if $t_0, \ldots, t_n$ are rational and $x\in[0, 1]\cap E$, then

$$\sum_{i,j=0}^n \langle \varphi(x)^{-2}\varphi(x+t_i+t_j)^2 k_i, k_j\rangle = \sum_{i,j=0}^n \int_0^a r^{t_i+t_j}\big\langle d\big(\varrho(r)f_i\big)(x), f_j(x)\big\rangle.$$

We argue as in the proof of Lemma 4 to see that the right-hand side of the last equation is nonnegative. Therefore,

$$\sum_{i,j=0}^{n} \langle \varphi(x)^{-2}\varphi(x+t_i+t_j)^2 k_i, k_j\rangle \geqq 0$$

for all x in E. Using arguments similar to those in Lemma 1, we can show that $\operatorname{s-lim}_{x\to 0} \varphi(x)^{-2}\varphi(x+t)^2=\varphi(t)^2$ for all real t. Consequently,

$$\sum_{i,j=0}^{n} \langle \varphi(t_i+t_j)^2 k_i, k_j\rangle \geqq 0$$

for all finite collections $k_0, \ldots, k_n$ in $\mathscr{K}$ and $t_0, \ldots, t_n$ in the rationals (and hence, in the reals since φ^2 is strongly continuous). We now apply Lemma 4 to φ^2 and see that iii) holds.

Conversely, assume that iii) holds: ϱ_0 is an operator measure on $[0, a]$ such that $\varrho_0[0, a]=I$ and

$$\varphi^2(x) = \int_0^a r^x \, d\varrho_0(r). \tag{5}$$

Note that for each Borel set E, $\Big(\int_E r^x d\varrho_0(r)\Big)^2 \ll \varphi(x)^4$, so by the factorization theorem [2] $\varphi(x)^{-2}\int_E r^x d\varrho_0(r)$ is the unique operator C on $\mathscr{K}$ satisfying $\varphi(x)^2 C = \int_E r^x d\varrho_0(r)$, $\ker C = \ker \int_E r^x d\varrho_0(r)$, $C(\mathscr{K}) \subseteqq \varphi(x)^2(\mathscr{K})^-$ and $\|C\| \leqq 1$. Thus, we can define

$$\varrho(E, x) = \varphi(x)^{-2} \int_E r^x d\varrho_0(r) \tag{6}$$

for all x in $\mathscr{R}^+$. Since $\{\varphi(x)\}$ is abelian, so is ϱ_0 and it follows that $\varrho(E, x)$ is positive Hermitian and further, that if $F \subseteqq E$,

$$\varphi(x)[\varrho(E, x)-\varrho(F, x)]\varphi(x) = \int_{E-F} r^x \, d\varrho_0(r).$$

Since $\varphi(x)$ is one-to-one, $\varrho(E, x)$ is a monotone $\mathscr{B}(\mathscr{K})$-valued function on the Borel sets of $[0, a]$. Further, if

$$E = \bigcup_n E_n, \quad E_j \cap E_i = \emptyset \quad \text{for } (i \neq j), \text{ then}$$

$$\varphi(x)^2\varrho(E, x) = \int_E r^x \, d\varrho_0(r) = \operatorname{s-lim}_{n\to\infty} \sum_{i=1}^{n} \int_{E_i} r^x \, d\varrho_0(r) = \operatorname{s-lim}_{n\to\infty} \sum_{i=1}^{n} \varphi(x)^2 \varrho(E_i, x).$$

Thus, $\sum_{i=1}^{n} \varrho(E_i, x)$ converges strongly to $\varrho(E, x)$ on the dense set $\varphi(x)^2$ and consequently, on $\mathscr{K}$ since $\Big\|\sum_{i=1}^{n} \varrho(E_i, x)\Big\| \leqq 1$. Finally, observe that $\varrho(\emptyset, x)=0$ and $\varrho([0, a], x)=I$ for all x.

Define the set function ϱ with values in $\mathscr{B}(\mathscr{L}^2(\mathscr{R}^+, \mathscr{H}))$ by $(\varrho(E)f)(x) = \varrho(E, x)f(x)$. From our previous remarks concerning $\varrho(E, x)$, it follows that ϱ is monotone, $\varrho(\emptyset)=0$ and $\varrho([0, a])=I$. Using an argument similar to that used in Theorem 2 $\varrho(E)=\text{s-}\lim_{n\to\infty} \sum_{i=1}^{n} \varrho(E_i)$ when $E=\bigcup_n E_i$, $E_i \cap E_j = \emptyset$ (for $i \neq j$). Therefore, ϱ defines an operator measure. Finally, from (6) and (5), we have

$$\varphi(x)^2 \int_0^a s^t \, d\varrho(s, x) = \int_0^a s^{x+t} \, d\varrho_0(s) = \varphi(x+t)^2.$$

Therefore,

$$\varphi(x)^{-2} \varphi(x+t)^2 = \int_0^a s^t \, d\varrho(s, x).$$

We combine the last equation with equation (4) to conclude that

$$S_t^* S_t = \int_0^a s^t \, d\varrho(s).$$

Once again, we invoke [5, Theorem 2.1] and conclude that S is subnormal.

In [8, Theorem 22.3.1], it is shown that if T is a strongly continuous semi-group of self-adjoint operators, then T has a holomorphic extension whose maximal domain of analyticity is either the whole plane or the right half-plane. It follows immediately from Theorem 3 that the symbol φ of a subnormal w.t.s. has a holomorphic extension. Therefore, if two such symbols φ_1 and φ_2 agree on an infinite set with cluster point in their common domain of analyticity, they must agree everywhere.

Prior to characterizing quasinormal w.t.s., we restate a general characterization of quasinormal semi-groups [6, Theorem 6] in the w.t.s. terminology.

Theorem 5. *Let Q be a strongly continuous semi-group on a separable Hilbert space $\mathscr{H}$. Q is quasinormal if and only if Q is unitarily equivalent to the direct sum of a strongly continuous normal semi-group N and a w.t.s. (S, φ) where φ is a strongly continuous self-adjoint semi-group.*

A quasinormal semi-group is *pure* if there exists no nontrivial invariant subspace on which it is normal.

Corollary 6. *Every strongly continuous pure quasinormal semi-group is unitarily equivalent to a w.t.s. (S, φ) where φ is a strongly continuous self-adjoint semi-group.*

Corollary 7. *Let (S, φ) be a w.t.s. The following are equivalent:*

i) *(S, φ) is quasinormal,*

ii) *φ is a strongly continuous self-adjoint semi-group,*

iii) *there exists a spectral measure* ϱ *on* $[0, a]$ *such that*

$$\varphi(x) = \int_0^a r^x \, d\varrho(r).$$

Proof. Observe first that if (S, φ) is any w.t.s., then (S, φ) has no normal part. If $S_t^* S_t f = S_t S_t^* f$ for all t, then $\varphi(x)^{-2}\varphi(x+t)^2 f(x) = 0$ for $0 \leqq x \leqq t$ and for all t. But $\varphi(x)^{-2}\varphi(x+t)^2$ is one-to-one so that $f=0$. The equivalence of i) and ii) now follows immediately from Theorem 5. The equivalence of ii) and iii) can either be derived from Theorem 5 or from [8, p. 588].

Example 4. Let s be a strongly continuous subnormal semi-group on a separable Hilbert space $\mathscr{K}$. Let $\varphi(x) = (s_x^* s_x)^{1/2}$. By [5, Theorem 2.1], there exists an operator measure ϱ on $[0, a]$ with $\varrho[0, a] = I$ and $\varphi(x)^2 = \int_0^a r^x d\varrho(r)$. We noted in the remark after Lemma 4 that φ satisfies all properties of a symbol except $\{\varphi(x)\}$ being abelian. If we assume $\{\varphi(x)\}$ abelian, then φ is a symbol and it follows from Theorem 3 that (S, φ) is a subnormal w.t.s.

During the development of the material in this paper, several questions arose which remain unanswered.

1. If φ is the symbol of a subnormal w.t.s. (S, φ), does there exist a strongly continuous semi-group s such that $\varphi(x) = (s_x^* s_x)^{1/2}$? In the last example, we saw that if φ is of this type, it does generate a subnormal w.t.s. However, if we start with a subnormal (S, φ), then by Theorem 3 $\varphi(x)^2 = \int_0^a r^t d\varrho(r)$. Thus, by [5, Theorem 2.1] φ acts like the positive part of some subnormal semi-group. The trick is to construct a function $u: \mathscr{R}^+ \to \mathscr{B}(\mathscr{K})$ such that each $u(x)$ is an isometry and $u\varphi$ is a strongly continuous semi-group.

2. More generally, we can ask whether each of the functions $h(x) = \int_0^a r^x d\varrho(r)$ in Lemma 4 is the square of the positive part of some strongly continuous subnormal semi-group. (Here, we do not require $\{h(x)\}$ to be abelian as we do for symbols.)

3. When are two w.t.s. (S, φ) and (T, ψ) unitarily equivalent or similar? In [5] it was shown in the numerical case, $\mathscr{K} = \mathscr{C}$, that similarity occurs if and only if there exist constants m and M such that $0 < m \leqq |\varphi(x)/\psi(x)| \leqq M < \infty$ for all x in $\mathscr{R}^+$ and in [4], it was shown that (T, ψ) is unitarily equivalent to (S, φ) if and only if $|\varphi(x)/\psi(x)|$ is constant on $\mathscr{R}^+$. Other questions were answered in [4] and [5] for the numerical case which may have interesting analogues in the operator case.

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On the polar decomposition of an operator

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1. Introduction

An operator means a bounded linear operator on a Hilbert space. An operator T can be decomposed into $T=UP$ where U is a partial isometry and $P=|T|=(T^*T)^{1/2}$ with $N(U)=N(P)$, where $N(X)$ denotes the kernel of an operator X. The kernel condition $N(U)=N(P)$ uniquely determines U and P of this polar decomposition $T=UP$ [2]. In this paper, $T=UP$ denotes the right-handed polar decomposition which satisfies the kernel condition $N(U)=N(P)$. In order to prove our results, this kernel condition $N(U)=N(P)$ is essential. When $T=UP$ where U is partial isometry and $P=|T|$, but the kernel condition $N(U)=$ $=N(P)$ is not necessarily satisfied, we say that $T=UP$ is merely "a decomposition" (not the polar decomposition) of T. When T commutes with S and S^*, we say that T doubly commutes with S.

Our two main results are as follows. When $T_1=U_1P_1$ and $T_2=U_2P_2$ are polar decompositions of T_1 and T_2 with $N(U_1)=N(P_1)$ and $N(U_2)=N(P_2)$, respectively, then T_1 doubly commutes with T_2 if and only if U_1^*, U_1 and P_1 commute with U_2^*, U_2 and P_2. As an application of this result we show that for every normal operator T, there exists a unitary U such that $T=UP=PU$ and U and P commute with V^*, V and $|A|$ of the polar decomposition $A=V|A|$ of any operator A which commutes with T and T^*. This second result yields a familiar and well-known result, see RIESZ and SZ.-NAGY [4].

An operator T is called quasinormal if T commutes with T^*T and hyponormal if $T^*T\geqq TT^*$. The inclusion relation of these classes of nonnormal operators is as follows:

$$\text{Normal} \subset \text{Quasinormal} \subset \text{Hyponormal}$$

and the inclusions above are all proper [2].

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2. A necessary and sufficient condition for $T_1T_2=T_2T_1$ and $T_1T_2^*=T_2^*T_1$

Theorem 1. *If $T=UP$ is the polar decomposition of an operator T, then U and P commute with A and A^*, where A denotes any operator which commutes with T and T^*.*

Proof. Let A be an operator such that $AT=TA$ and $AT^*=T^*A$. Then $(T^*T)A=A(T^*T)$, that is, $P^2A=AP^2$ where $P=|T|$, whence $PA=AP$, or equivalently $PA^*=A^*P$. The conditions $AT-TA=0$ and $PA=AP$ yield $AUP-UPA=(AU-UA)P=0$, so that $AU-UA$ annihilates $\overline{R(P)}$. If $x\in N(P)=N(U)$, then $Px=0$ and $Ux=0$, so that $PAx=APx=0$, that is, $Ax\in N(P)=N(U)$, hence $UAx=0$, therefore $AU-UA$ annihilates $N(P)$ too, and it follows that $AU-UA=0$ on $H=\overline{R(P)}\oplus N(P)$. Similarly, the conditions $AT^*-T^*A=0$ and $PA=AP$ imply $APU^*-PU^*A=P(AU^*-U^*A)=0$. By taking adjoint of this equation, $(UA^*-A^*U)P=0$, so that UA^*-A^*U annihilates $\overline{R(P)}$. If $x\in N(P)=N(U)$, then $Px=0$ and $Ux=0$, so that $PA^*x=A^*Px=0$ (since $PA^*=A^*P$ holds), therefore $A^*x\in N(P)=N(U)$, $UA^*x=0$, whence UA^*-A^*U annihilates $N(P)$, too, and it follows that $UA^*-A^*U=0$ and so the proof is complete.

Our main result is the following extension of Theorem 1 which gives a necessary and sufficient condition under which an operator doubly commutes with another.

Theorem 2. *Let $T_1=U_1P_1$ and $T_2=U_2P_2$ be the polar decompositions of T_1 and T_2, respectively. Then the following conditions are equivalent:*

(A) *T_1 doubly commutes with T_2.*

(B) *U_1^*, U_1 and P_1 commute with U_2^*, U_2 and P_2.*

(C) *The following five equations are satisfied:* (1) $P_1P_2=P_2P_1$, (2) $U_1P_2=P_2U_1$, (3) $P_1U_2=U_2P_1$, (4) $U_1U_2=U_2U_1$ *and* (5) $U_1^*U_2=U_2U_1^*$.

Proof. (B)$\leftrightarrow$(C). (B)$\rightarrow$(C) is trivial and (B) follows from (C) by taking adjoints of (2), (3), (4) and (5).

(A)$\rightarrow$(C). Assume (A), then by Theorem 1, we have

$$(*) \qquad T_1P_2=P_2T_1 \quad \text{and} \quad T_1^*P_2=P_2T_1^*,$$

$$(**) \qquad T_1U_2=U_2T_1 \quad \text{and} \quad T_1^*U_2=U_2T_1^*.$$

By $(*)$ and by Theorem 1, we have (1), (2), and also by $(**)$ and by Theorem 1, we have (3), (4), and $U_1U_2^*=U_2^*U_1$, or equivalently (5).

(B)$\rightarrow$(A). (A) easily follows from (B). Hence the proof is complete.

Theorem 2 yields the following well-known fact. In Theorem 2, $U_1^*U_1$ and $U_1U_1^*$ commute with U_2, P_2 and T_2, that is, both the initial space and the final space of U_1 reduce U_2, P_2 and T_2. Similarly, both the initial space and the final space of U_2 reduce U_1, P_1 and T_1. In Section 4, Theorem 2 will be extended to Theorem 5 in the intertwining case.

Corollary 1. *Let $T_1=U_1P_1$ and $T_2=U_2P_2$ be the polar decompositions of T_1 and T_2, respectively. If T_1 doubly commutes with T_2, then $T_1T_2=(U_1U_2)(P_1P_2)$ is the polar decomposition of T_1T_2.*

Proof. By (4) and (5) in (C) of Theorem 2, we have

$$U_1U_2(U_1U_2)^*U_1U_2 = U_1U_2U_2^*U_1^*U_1U_2 = U_1U_1^*U_1U_2U_2^*U_2 = U_1U_2$$

since U_1 and U_2 are both partial isometries, whence U_1U_2 is a partial isometry. By (1) in (C) of Theorem 2, we have

$$|T_1T_2|^2 = (T_1T_2)^*(T_1T_2) = (T_1^*T_1)(T_2^*T_2) = P_1^2P_2^2 = (P_1P_2)^2.$$

$N(U_2U_1)=N(U_1U_2)=N(P_1P_2)$ is obtained by (2) and (4) in (C) of Theorem 2 as follows: $x\in N(U_2U_1)\leftrightarrow U_2U_1x = 0\leftrightarrow U_1x\in N(U_2) = N(P_2)\leftrightarrow P_2U_1x = 0\leftrightarrow U_1P_2x = 0\leftrightarrow P_2x\in N(U_1)=N(P_1)\leftrightarrow P_1P_2x=0\leftrightarrow x\in N(P_1P_2)$, so the proof is complete.

Theorem 2 easily implies the following result which is a more precise statement than Theorem 1 on the polar decomposition.

Corollary 2 (The polar decomposition). *Every operator T can be expressed in the form $U|T|$ where U is a partial isometry with $N(U)=N(|T|)$. This kernel condition uniquely determines U; U and $|T|$ commute with V^*, V and $|A|$ of the polar decomposition $A=V|A|$ of any operator A commuting with T and T^*.*

Proof. The first half of the result follows by [2] and the second follows by Theorem 2 since we put $T=T_2$ and $A=T_1$ in Theorem 2.

Theorem 2 also yields the following result which is a characterization of normal operators.

Corollary 3. *Let $T=UP$ be the polar decomposition of an operator T. Then T is normal if and only if U commutes with P and U is unitary on $N(T)^\perp$.*

Proof. Put $T=T_1=T_2$ in Theorem 2, then the condition (A) in Theorem 2 is equivalent to the normality of T and the condition (C) is equivalent to that U commutes with P and $U^*U=UU^*$. So U is unitary on the initial space of U which equals $N(T)^\perp$.

Theorem 3. *Let T be normal. Then there exists a unitary operator U such that $T=UP=PU$ and both U and P commute with V^*, V and $|A|$ of the polar decomposition $A=V|A|$ of any operator commuting with T and T^*.*

Proof. Let $T=U_1P=PU_1$ be the polar decomposition of a normal operator T and let $A=V|A|$ be the polar decomposition of A. By Corollary 3, $U_1^*U_1=U_1U_1^*$, that is, the initial space M of U_1 coincides with the final space N, so that M reduces U_1; consequently $U_1P_M=P_MU_1$ where $P_M=U_1^*U_1$ denotes the projection of H onto M. Put $U=U_1P_M+1-P_M$ by the standard technique [4], then $U_1^*U_1==U_1U_1^*$ and $U_1P_M=P_MU_1$ yield the following:

$$U^*U=(P_MU_1^*+1-P_M)(U_1P_M+1-P_M)=1,$$

$$UU^*=(U_1P_M+1-P_M)(P_MU_1^*+1-P_M)=1.$$

Hence U is unitary and we show that U is the desired unitary as follows. As $P_MP=P$, that is, $PP_M=P$, so we have

$$UP=(U_1P_M+1-P_M)P=U_1P_MP+P-P_MP=U_1P=T$$

and similarly we have $T=PU=PU_1$, therefore $T=UP=PU$. By Corollary 2' U_1 and P commute with V^*, V and $|A|$, so $P_M=U_1^*U_1$ commutes with V^*, V and $|A|$, that is, $P_M|A|=|A|P_M$, $P_MV=VP_M$ and $P_MV^*=V^*P_M$. By Corollary 2, P commutes with V^*, V and $|A|$. Hence we have only to show that U commutes with V^*, V and $|A|$.

Clearly,

$$VU=V(U_1P_M+1-P_M)=VU_1P_M+V(1-P_M)=$$

$$=U_1VP_M+V(1-P_M)=(U_1P_M+1-P_M)V=UV.$$

Similarly we have $V^*U=UV^*$ and $|A|U=U|A|$, so the proof is complete.

We remark that U and P commute with $A=V|A|$ in Theorem 3, so that Theorem 3 yields the following well-known result.

Theorem A. [4] *Every normal operator T can be written in the form UP where P is positive and U may be taken to be unitary and such that U and P commute with each other and with all operators commuting with T and T^*.*

Corollary 4. *Let $T_1=U_1P_1$ be the polar decomposition of an operator T_1, and let $T_2=U_2P_2$ be the decomposition described in Theorem 3 of a normal operator T_2. Then the following conditions are equivalent.*

(A) T_1 *commutes with* T_2.

(B) U_1^*, U_1 *and* P_1 *commute with* U_2^*, U_2 *and* P_2.

(C) U_1 *and* P_1 *commute with* U_2 *and* P_2.

Proof. As T_2 is normal, (A) implies $T_1T_2^*=T_2^*T_1$ by the Fuglede—Putnam Theorem [2], so by Theorem 3, U_2 and P_2 commute with U_1^*, U_1 and P_1, whence (B) is shown. (C) trivially follows from (B) and also (A) easily follows from (C), so the proof is complete.

3. Nonnormal operators

Theorem 4. *Suppose that $N(T)\subset N(T^*)$ and let $T=UP$ be the polar decomposition of T. Then there exists an isometry U_1 such that $T=U_1P$ and both U_1 and P commute with V^*, V and $|A|$ of the polar decomposition $A=V|A|$ of any operator A commuting with T and T^*. In case $N(T)=N(T^*)$, U_1 can be chosen to be unitary.*

Proof. The condition $N(T)\subset N(T^*)$ implies $N(T)^\perp\supset N(T^*)^\perp=\overline{R(T)}$, so that U is a partial isometry from the initial space $M=N(T)^\perp$ into M, whence M reduces U; consequently $UP_M=P_MU$ where P_M denotes the projection of H onto M and $P_M=U^*U$. Put $U_1=UP_M+1-P_M$. In the same way as in the proof of Theorem 3, $U_1^*U_1=1$, $U_1P=UP=T$, and the commutativity stated in Theorem 4 can be shown. If $N(T)=N(T^*)$, then U is unitary on M, so that U_1 defined above turns out to be unitary since $U_1U_1^*=1$ can also be shown.

Remark 1. If T is invertible or hyponormal, then $N(T)\subset N(T^*)$ holds, so that Theorem 4 holds for these operators.

Corollary 5. *Let T be quasinormal. Then there exists an isometry U such that $T=UP=PU$ and U and P commute with V^*, V and $|A|$ of the polar decomposition $A=V|A|$ of any operator A commuting with T and T^*.*

Proof. If T is quasinormal, then T is hyponormal, so that T satisfies $N(T)\subset N(T^*)$. T commutes with T^*T by the quasinormality of T, so that $P=(T^*T)^{1/2}$ commutes with T and T^*. Put $A=P$ in Theorem 4, so the isometry U chosen in Theorem 4 commutes with P and the rest follows from Theorem 4.

We remark that Theorem 3 can be alternatively derived from Theorem 4 and Corollary 5.

4. Intertwining case

Theorem 2 yields the following result which is closely related to the Fuglede—Putnam theorem.

Theorem 5. *Let $T_k=U_kP_k$ be the polar decompositions of T_k for $k=1,2$ and 3. Then the following conditions are equivalent.*

(A) $$T_1T_2=T_2T_3 \quad \text{and} \quad T_1^*T_2=T_2T_3^*.$$

(B) (1) $P_3P_2=P_2P_3$, (2) $P_1U_2=U_2P_3$, (3) $U_3P_2=P_2U_3$, (4) $U_1U_2=U_2U_3$ *and* (5) $U_1^*U_2=U_2U_3^*$.

Proof. We put $\hat{A}$ and $\hat{T}$ on $H\oplus H$ as follows:

$$\hat{A} = \begin{pmatrix} T_1 & 0 \\ 0 & T_3 \end{pmatrix} \quad \text{and} \quad \hat{T} = \begin{pmatrix} 0 & T_2 \\ 0 & 0 \end{pmatrix}.$$

Let $\hat{A}=\hat{U}_1\hat{P}_1$ and $\hat{T}=\hat{U}_2\hat{P}_2$ be the polar decompositions of $\hat{A}$ and $\hat{T}$, respectively, where $\hat{U}_1, \hat{P}_1, \hat{U}_2$ and $\hat{P}_2$ are as follows on $H\oplus H$:

$$\hat{U}_1 = \begin{pmatrix} U_1 & 0 \\ 0 & U_3 \end{pmatrix}, \quad \hat{P}_1 = \begin{pmatrix} P_1 & 0 \\ 0 & P_3 \end{pmatrix}, \quad \hat{U}_2 = \begin{pmatrix} 0 & U_2 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \hat{P}_2 = \begin{pmatrix} 0 & 0 \\ 0 & P_2 \end{pmatrix}.$$

The condition (A) assures that $\hat{A}\hat{T}=\hat{T}\hat{A}$ and $\hat{A}^*\hat{T}=\hat{T}\hat{A}^*$, so by Theorem 2, these relations are equivalent to that $\hat{U}_1^*$, $\hat{U}_1$ and $\hat{P}_1$ commute with $\hat{U}_2$ and $\hat{P}_2$. Then, by simple calculations, $\hat{P}_1\hat{P}_2=\hat{P}_2\hat{P}_1 \leftrightarrow$ (1), $\hat{U}_2\hat{P}_1=\hat{P}_1\hat{U}_2 \leftrightarrow$ (2), $\hat{U}_1\hat{P}_2=\hat{P}_2\hat{U}_1 \leftrightarrow$ (3), $\hat{U}_1\hat{U}_2=\hat{U}_2\hat{U}_1 \leftrightarrow$ (4), and $\hat{U}_1^*\hat{U}_2=\hat{U}_2\hat{U}_1^* \leftrightarrow$ (5), whence the proof is complete.

Combining the techniques in Corollary 4 and Theorem 5, we have

Corollary 6. *Let* $T_1=U_1P_1$, $T_3=U_3P_3$ *be the decompositions described in Theorem* 3 *of some normal operators* T_1, T_3, *and let* $T_2=U_2P_2$ *be the polar decomposition of an operator* T_2. *Then the following conditions are equivalent:*

(A) $T_1T_2=T_2T_3$.

(B) (1), (2), (3), (4), *and* (5) *in Theorem* 5 *hold.*

(C) (1), (2), (3), *and* (4) *in Theorem* 5 *hold.*

Let $\{p_\alpha\}$ be a family of polynomials of T and T^*. A property Σ of T is said to be algebraic definite (resp. semidefinite) with $\{p_\alpha\}$ provided that T has Σ if and only if $p_\alpha(T, T^*)=0$ $\big($resp. $p_\alpha(T, T^*)\geqq 0\big)$ for all α [1].

Next we show an application of Theorem 5.

Corollary 7. *Let* $T_k=U_kP_k$ *be the polar decompositions of* T_k *for* $k=1, 2$ *and* 3 *and let* $T_1T_2=T_2T_3$ *and* $T_1^*T_2=T_2T_3^*$. *Then*

(1) $\overline{R(T_2)}$ *reduces* U_1, P_1 *and* T_1; $N(T_2)$ *reduces* U_3, P_3 *and* T_3.

(2) $U_1 \mid \overline{R(T_2)}$ $\big($*resp.* $P_1 \mid \overline{R(T_2)}$, $T_1 \mid \overline{R(T_2)}\big)$ *is unitary equivalent to* $U_3 \mid N(T_2)^\perp$ $\big($resp. $P_3 \mid N(T_2)^\perp$, $T_3 \mid N(T_2)^\perp\big)$.

(3) *When* T_2 *has dense range, then if* U_3 *(resp.* P_3 *and* T_3*) has an algebraic definite property* Σ *with polynomials* $\{p_\alpha\}$, *then so has* U_1 *(resp.* P_1 *and* T_1*).*

(4) *When* T_2 *is injective, then if* U_1 *(resp.* P_1 *and* T_1*) has an algebraic definite property* Σ *with polynomials* $\{p_\alpha\}$, *then so has* U_3 *(resp.* P_3 *and* T_3*).*

Proof. (1) By (5), (4) and (2) in Theorem 5

$$(U_2U_2^*)U_1 = U_2U_3U_2^* = U_1(U_2U_2^*), \quad (U_2U_2^*)P_1 = U_2P_3U_2^* = P_1(U_2U_2^*),$$

whence $\overline{R(T_2)}$ reduces U_1, P_1 and also T_1. By (4), (5) and (2) in Theorem 5,

$$(U_2^* U_2)U_3 = U_2^* U_1 U_2 = U_3(U_2^* U_2), \quad (U_2^* U_2)P_3 = U_2^* P_1 U_2 = P_3(U_2^* U_2),$$

whence $N(T_2)$ reduces U_3, P_3 and also T_3.

(2) By (2) and (1) in Theorem 5,

$$\text{(i)} \qquad P_1 U_2 P_2 x = U_2 P_3 P_2 x = U_2 P_2 P_3 x \quad \text{for all } x.$$

Let $P_1' = P_1 \mid \overline{R(T_2)}$ and $P_3' = P_3 \mid N(T_2)^\perp$. Let V be defined by $Vy = U_2 y$ for all $y \in N(T_2)^\perp$. This V maps from $N(T_2)^\perp = N(P_2)^\perp = \overline{R(P_2)}$ onto $\overline{R(T_2)}$, so V is a surjective isometry, i.e., V is unitary. As $P_2 x$ and $P_2 P_3 x$ belong to $N(T_2)^\perp$ and $U_2 P_2 x$ belongs to $\overline{R(T_2)}$, (i) implies $P_1' V y = V P_3' y$ for all $y \in N(T_2)^\perp$, so P_1' is unitary equivalent to P_3'. Similarly (4) and (3) in Theorem 5 yield

$$\text{(ii)} \qquad U_1 U_2 P_2 x = U_2 U_3 P_2 x = U_2 P_2 U_3 x \quad \text{for all } x.$$

Let $U_1' = U_1 \mid \overline{R(T_2)}$ and $U_3' = U_3 \mid N(T_2)^\perp$. As $P_2 x$ and $P_2 U_3 x$ belong to $N(T_2)^\perp$ and $U_2 P_2 x$ belongs to $\overline{R(T_2)}$, (ii) implies $U_1' V y = V U_3' y$ for all $y \in N(T_2)^\perp$. The third unitary equivalence relation follows by the first and second relations obtained above.

(3) When T_2 has dense range, then by (2), $U_1 \mid \overline{R(T_2)} = U_1$ is unitary equivalent to $U_3' = U_3 \mid N(T_2)^\perp$. If U_3 has an algebraic definite property, then U_3' also has it, and consequently so has U_1. The rest can be shown similarly.

(4) When T_2 is injective, then by (2), $U_3 \mid N(T_2)^\perp = U_3$ is unitary equivalent to $U_1' = U_1 \mid \overline{R(T_2)}$ and the proof goes in a similar way as above.

We remark that the algebraic definite property can be replaced by semidefinite property in (3) and (4) of Corollary 7. Also we remark that in [3] only the equivalence relation between $T_1 \mid \overline{R(T_2)}$ and $T_3 \mid N(T_2)^\perp$ is shown, and in [1] the algebraic definite property relation between T_1 and T_3 is shown when T_2 has dense range, and in [5] also when T_2 has dense range and injective.

Added in proof. Theorem 2 is also found in M. Takesaki, Theory of operator algebras I, Springer, 1979, however, the proof we gave here is more elementary in that it merely uses kernel conditions and avoids operator algebraic considerations. We would express our thanks to Professor J. Tomiyama for his valuable comments after reading our preprint.

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Spectral properties of elementary operators

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1. Introduction. Let $\mathfrak{H}$ denote an infinite dimensional complex Hilbert space and let $\mathfrak{L}(\mathfrak{H})$ denote the algebra of all bounded linear operators on $\mathfrak{H}$. For an integer $N \geqq 1$, let $A=(A_1, \ldots, A_N)$ and $B=(B_1, \ldots, B_N)$ denote N-tuples of mutually commuting operators in $\mathfrak{L}(\mathfrak{H})$. The *elementary operator* $\mathfrak{R} \equiv \mathfrak{R}(A, B)$ acting on $\mathfrak{L}(\mathfrak{H})$ is defined by $\mathfrak{R}(X)=A_1XB_1+\ldots+A_NXB_N$ $(X \in \mathfrak{L}(\mathfrak{H}))$. Spectral, metric, and algebraic properties of elementary operators have been studied from a variety of viewpoints [1], [2], [5], [7], [14], [18], [20], [22]. In particular, the *generalized derivation* $\mathfrak{T}(A, B)$ defined by $\mathfrak{T}(X)=AX-XB$, has been analyzed in considerable detail, and various characterizations have been given for the cases when a generalized derivation has dense range [11], or is surjective, bounded below [6], [8], Fredholm [9], or semi- Fredholm [10]. Analogous results are also known for the restriction of a generalized derivation to a norm ideal in $\mathfrak{L}(\mathfrak{H})$ [8], [12].

In the present note we extend several results concerning generalized derivations to an arbitrary elementary operator $\mathfrak{R}$ and its restriction $\mathfrak{R}_{\mathfrak{J}}$ to a norm ideal $\mathfrak{J}$. Descriptions of the right and left spectra of $\mathfrak{R}$ were determined by R. HARTE [16] (cf. [5]) and in section 2 we obtain qualitative refinements of these results; we show that $\mathfrak{R}-\lambda$ is right invertible in $\mathfrak{L}(\mathfrak{L}(\mathfrak{H}))$ (and thus surjective) if its range contains each rank one operator, and is left invertible (hence bounded below) if its restriction to the set of rank one operators is bounded below. These results allow us to relate spectral properties of $\mathfrak{R}$ to those of $\mathfrak{R}_{\mathfrak{J}}$ (Theorem 2.3, Theorem 2.8). We also characterize the case when $\mathfrak{R}-\lambda$ has dense range, extending the characterization given for $\mathfrak{T}$ in [11].

In section 3 we specialize to study the *elementary multiplication operator* $\mathfrak{S} \equiv \mathfrak{S}(A, B)$ defined by $\mathfrak{S}(X)=AXB$. The essential spectrum and index function of $\mathfrak{S}$ was determined in [12] and here we describe the semi-Fredholm domain of

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$\mathfrak{S}$ and conditions for $\mathfrak{S}-\lambda$ to have index equal to $+\infty$ or $-\infty$. Analogous results are given for the semi-Fredholm domain of $\mathfrak{S}_{\mathfrak{J}}$. These results complement (but are independent of) the characterization of the semi-Fredholm domain of $\mathfrak{T}$ given in [9] and [10], and we believe they will prove helpful in studying the semi-Fredholm domain and index function of a general elementary operator.

We conclude this section with some preliminary results and notation. Let $\mathfrak{A}$ denote a complex Banach algebra with identity 1, and let $\mathfrak{A}^{(N)}$ denote an N-fold copy of $\mathfrak{A}$. For $a=(a_1, \dots, a_N)\in\mathfrak{A}^{(N)}$, the joint left spectrum of a in the sense of R. Harte [15] is defined by $\sigma_l(a)=\{\alpha\equiv(\alpha_1, \dots, \alpha_N)\in\mathbf{C}^{(N)}$: there exists no N-tuple $(b_1, \dots, b_N)\in\mathfrak{A}^{(N)}$ such that $b_1(a_1-\alpha_1)+\dots+b_N(a_N-\alpha_N)=1\}$; the joint right spectrum of a, $\sigma_r(a)$, is defined analogously, and the joint spectrum of a is defined by $\sigma(a)=\sigma_l(a)\cup\sigma_r(a)$ [15]. For $a\in\mathfrak{A}$, L_a and R_a denote, respectively, the left and right multiplication operators on $\mathfrak{A}$ induced by a, i.e. $L_a(x)=ax$ and $R_a(x)=xa\,(x\in\mathfrak{A})$. For $a=(a_1, \dots, a_N)\in\mathfrak{A}^{(N)}$, we set $L_a=(L_{a_1}, \dots, L_{a_N})$ and $R_a=(R_{a_1}, \dots, R_{a_N})$. When $\mathfrak{A}=\mathfrak{L}(\mathfrak{H})$, $A=(A_1, \dots, A_N)\in\mathfrak{A}^{(N)}$, and $\mathfrak{J}$ is a norm ideal in $\mathfrak{L}(\mathfrak{H})$, we define $L_A\mid\mathfrak{J}=(L_{A_1}\mid\mathfrak{J}, \dots, L_{A_N}\mid\mathfrak{J})$. In this case the left joint spectrum of A may be described in more detail as follows.

Lemma 1.1. [15, Theorem 2.5] *The following are equivalent.*

i) $\alpha\in\sigma_l(A)$;

ii) $\sum_{i=1}^{N}(A_i-\alpha_i)^*(A_i-\alpha_i)$ *is not invertible;*

iii) *There exists a sequence of unit vectors* $\{x_k\}\subset\mathfrak{H}$ *such that*

$$\lim_{k\to\infty}\sum_{i=1}^{N}\|(A_i-\alpha_i)x_k\|=0.$$

Let $\mathfrak{K}(\mathfrak{H})$ denote the ideal of all compact operators in $\mathfrak{L}(\mathfrak{H})$ and let $\mathfrak{A}(\mathfrak{H})=\mathfrak{L}(\mathfrak{H})/\mathfrak{K}(\mathfrak{H})$ denote the Calkin algebra; for $T\in\mathfrak{L}(\mathfrak{H})$, $\tilde{T}$ denotes the image of T in $\mathfrak{A}(\mathfrak{H})$ under the canonical projection. For an N-tuple of operators $T=(T_1, \dots, T_N)$, we set $\tilde{T}=(\tilde{T}_1, \dots, \tilde{T}_N)$ and denote the [left] [right] joint essential spectrum of T by $[\sigma_{le}(T)]$ $[\sigma_{re}(T)]$ $\sigma_e(T)$, i.e. $\sigma_{le}(T)=\sigma_l(\tilde{T})$, $\sigma_{re}(T)=\sigma_r(\tilde{T})$, and $\sigma_e(T)=\sigma(\tilde{T})$. The following result is contained in [24, Corollary 2.5, Theorem 2.6].

Lemma 1.2. *The following are equivalent.*

i) $\alpha\in\sigma_{le}(T)$;

ii) $\sum_{i=1}^{N}(T_i-\alpha_i)^*(T_i-\alpha_i)$ *is not Fredholm;*

iii) *There exists an orthonormal sequence* $\{e_n\}_{n=1}^{\infty}\subset\mathfrak{H}$ *such that*

$$\lim_{n\to\infty}\sum_{i=1}^{N}\|(T_i-\alpha_i)e_n\|=0.$$

For $T\in\mathfrak{L}(\mathfrak{H})^{(N)}$, let $\sigma_p(T)=\{\alpha\in\mathbf{C}^{(N)}$: there exists a unit vector $x\in\mathfrak{H}$ such that $(T_i-\alpha_i)x=0,\ 1\leqq i\leqq N\}$, the joint point spectrum of T. Lemmas 1.1 and 1.2 readily imply that $\sigma_l(T)=\sigma_{le}(T)\cup\sigma_p(T)$. For $T\in\mathfrak{L}(\mathfrak{H})^{(N)}$ and $\alpha\in\mathbf{C}^{(N)}$, let $T^*=(T_1^*,\ldots,T_N^*)$ and $\bar\alpha=(\bar\alpha_1,\ldots,\bar\alpha_N)$. Analogues of the preceding results for right spectra follow from the identity $\sigma_r(T)=[\sigma_l(T^*)]^*\equiv\{\bar\alpha\in\mathbf{C}^{(N)}: \alpha\in\sigma_l(T^*)\}$; in particular, $\sigma_r(T)=\sigma_{re}(T)\cup\sigma_p(T^*)^*$.

Let $(\mathfrak{J},|||\cdot|||)$ denote a norm ideal in $\mathfrak{L}(\mathfrak{H})$ in the sense of [21]. Clearly $\mathfrak{J}$ is $\mathfrak{R}$-invariant and $\mathfrak{R}_{\mathfrak{J}}$, the restriction of $\mathfrak{R}$ to $\mathfrak{J}$, is in $\mathfrak{L}(\mathfrak{J})$. If $\mathfrak{J}=C_p$ $(1\leqq p\leqq\infty)$ (the Schatten p-ideal [21]), we denote $\mathfrak{R}_{\mathfrak{J}}$ by $\mathfrak{R}_p$. For $x,y\in\mathfrak{H}$, $x\otimes y$ denotes the rank one operator defined by $(x\otimes y)h=(h,y)x$. $\mathfrak{J}_1$ denotes the set of all rank one operators in $\mathfrak{L}(\mathfrak{H})$; if $F\in\mathscr{F}_1$, then $|||F|||=\|F\|$ [21].

Let $\mathfrak{X}$ denote a complex Banach space and let $\mathfrak{L}(\mathfrak{X})$ denote the algebra of bounded linear operators on $\mathfrak{X}$. For $T\in\mathfrak{L}(\mathfrak{X})$, let $\ker(T)$ and $R(T)$ denote the kernel and range of T; we set $\operatorname{nul}(T)=\dim(\ker(T))$ and $\operatorname{def}(T)=\dim(\mathfrak{X}/R(T)^-)$ (where $R(T)^-$ denotes the norm closure of $R(T)$). T is *semi-Fredholm* if $R(T)$ is closed and either $\operatorname{nul}(T)<\infty$ or $\operatorname{def}(T)<\infty$; in this case, the index of T is defined by $\operatorname{ind}(T)=\operatorname{nul}(T)-\operatorname{def}(T)$ [17]. T is Fredholm if $R(T)$ is closed and both $\operatorname{nul}(T)$ and $\operatorname{def}(T)$ are finite; $\sigma_e(T)=\{\lambda\in\mathbf{C}: T-\lambda$ is not Fredholm$\}$ is the essential spectrum of T. The semi-Fredholm domain of T is defined by $\varrho_{SF}(T)=\{\lambda\in\mathbf{C}: T-\lambda$ is semi-Fredholm$\}$; we denote the complement $\mathbf{C}\setminus\varrho_{SF}(T)$ by $\sigma_{SF}(T)$. For $\alpha=(\alpha_1,\ldots,\alpha_N)$ and $\beta=(\beta_1,\ldots,\beta_N)$ in $\mathbf{C}^{(N)}$ we set $\alpha\circ\beta=\alpha_1\beta_1+\ldots+\alpha_N\beta_N$, and for $\sigma,\varrho\subset\mathbf{C}^{(N)}$, let $\sigma\circ\varrho=\{\alpha\circ\beta:\alpha\in\sigma,\ \beta\in\varrho\}$. If $N=1$, we abbreviate $\sigma\circ\varrho$ by $\sigma\varrho$. In [12] it is proved that $\sigma_e(\mathfrak{S}(A,B))=\sigma_e(\mathfrak{S}_{\mathfrak{J}})=\sigma(A,B)\equiv\sigma_e(A)\sigma(B)\cup\sigma(A)\sigma_e(B)$. In the sequel we will prove that $\sigma_{SF}(\mathfrak{S})=\sigma_{SF}(\mathfrak{S}_{\mathfrak{J}})=[\sigma_l(A)\sigma_{re}(B)\cup\sigma_{le}(A)\sigma_r(B)]\cap[\sigma_r(A)\sigma_{le}(B)\cup\sigma_{re}(A)\sigma_l(B)]$ (Corollary 3.12, Theorem 3.14).

2. Spectral properties of elementary operators. In this section we present several equivalent descriptions of the left and right spectra of elementary operators and we describe the elementary operators with dense range. The following result will be used to show that an elementary operator is surjective if its range contains each rank one operator: the proof is motivated by that of [8, Theorem 2.1].

Lemma 2.1. *If $\lambda\in\sigma_r(A)\circ\sigma_l(B)$, then the range of $\mathfrak{R}-\lambda$ does not contain every rank one operator.*

Proof. Let $\alpha\in\sigma_r(A)$ and $\beta\in\sigma_l(B)$ be such that $\lambda=\alpha\circ\beta$. We consider several cases for the location of α and β.

i) $\alpha\in\sigma_r(A)\setminus\sigma_{re}(A)$, $\beta\in\sigma_l(B)\setminus\sigma_{le}(B)$. In this case there exist unit vectors e and f in $\mathfrak{H}$ such that $(A_i-\alpha_i)^*f=(B_i-\beta_i)e=0$ $(1\leqq i\leqq n)$. Let $Yx=(x,e)f$ $(x\in\mathfrak{H})$. If $X\in\mathfrak{L}(\mathfrak{H})$ satisfies $(\mathfrak{R}-\lambda)(X)=Y$, then $1=(Ye,f)=$

$=\sum_{i=1}^{N}[((A_i-\alpha_i)XB_ie,f)+(\alpha_iX(B-\beta_i)e,f)]=\sum_{i=1}^{N}(XB_ie,\ (A_i-\alpha_i)^*f)=0$, a contradiction; thus the rank one operator Y is not in the range of $\mathfrak{R}-\lambda$.

ii) $\alpha\in\sigma_{re}(A)$, $\beta\in\sigma_{le}(B)$. (Clearly, we may assume that $\max\{\|B_i\|\}>0$.) The following argument is based on J. G. Stampfli's proof that the range of an inner derivation contains no nontrivial unitarily invariant subset of $\mathfrak{L}(\mathfrak{H})$ [23, Theorem 2]; we prove a similar result for $\mathfrak{R}-\lambda$. Let Y be an operator in $\mathfrak{L}(\mathfrak{H})$ that is not a scalar multiple of the identity. We will construct a unitary operator U such that U^*YU is not in the range of $\mathfrak{R}-\lambda$. Let $\{h_n\}_{n=1}^{\infty}$ denote an orthonormal seqence such that $(Yh_n, h_m)\neq 0$ for $n, m\geqq 1$ [19, Theorem 2]. Let $\delta_n=(Yh_{3n}, h_{3n+1})$ for $n\geqq 1$. Let $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ denote orthonormal sequences in $\mathfrak{H}$ such that the following properties are satisfied: i) $\sum_{i=1}^{N}\|(B_i-\beta_i)f_m\|\leqq|\delta_m|/m$ $(m\geqq 1)$; ii) $\sum_{i=1}^{N}\|(A_i-\alpha_i)^*g_m\|<|\delta_m|/m$; iii) $(f_n, g_m)=0$ for $1\leqq n, m$; iv) the subspace spanned by all of the vectors f_n and g_n $(n\geqq 1)$ has an infinite dimensional orthocomplement in $\mathfrak{H}$. Using iii) and iv) we may define a unitary operator U which satisfies $Uf_n=h_{3n}$ and $Ug_n=h_{3n+1}$ $(n\geqq 1)$. If $X\in\mathfrak{L}(\mathfrak{H})$ satisfies $(\mathfrak{R}-\lambda)(X)=U^*YU$, then

$$0<|\delta_n|=|(Yh_{3n}, h_{3n+1})|=|(YUf_n, Ug_n)|=|(U^*YUf_n, g_n)|=$$

$$=\left|\sum_{i=1}^{N}[((A_i-\alpha_i)XB_if_n, g_n)+(\alpha_iX(B_i-\beta_i)f_n, g_n)]\right|\leqq$$

$$\leqq\|X\|\,[\max\{\|B_i\|\}+\max\{|\alpha_i|\}]\,|\delta_n|/n.$$

Thus $\|X\|\geqq n/(\max\{\|B_i\|\}+\max\{|\alpha_i|\})$ for every $n\geqq 1$, so U^*YU is not in the range of $\mathfrak{R}-\lambda$. The proof is completed by taking Y to be a rank one operator.

iii) $\alpha\in\sigma_r(A)\setminus\sigma_{re}(A)$, $\beta\in\sigma_{le}(B)$. If $\beta\in\sigma_p(B)$ we may use the same proof as in part i). We may thus assume that $\beta\notin\sigma_p(B)$. Let $\{e_n\}_{n=1}^{\infty}$ denote an orthonormal sequence such that $0<\sum_{i=1}^{N}\|(B_i-\beta_i)e_n\|<1/n^2$ $(n\geqq 1)$, and set $\delta_n=n\sum_{i=1}^{N}\|(B_i-\beta_i)e_n\|$. Let f be a unit vector such that $(A_i-\alpha_i)^*f=0$ $(1\leqq i\leqq n)$. Since $0<\delta_n<1/n$, we may define a rank one operator Y by the relations $Ye_n=\delta_nf$ $(n\geqq 1)$ and $Yx=0$ if $(x, e_n)=0$ for each n. If $X\in\mathfrak{L}(\mathfrak{H})$ satisfies $(\mathfrak{R}-\lambda)(X)=Y$, then

$$\delta_n=(Ye_n,f)=\sum_{i=1}^{N}[((A_i-\alpha_i)XB_ie_n,f)+(\alpha_iX(B_i-\beta_i)e_n,f)]=$$

$$=\sum_{i=1}^{N}(\alpha_iX(B_i-\beta_i)e_n,f)\quad(n\geqq 1).$$

Thus $0<\delta_n\leqq(\max|\alpha_i|)\|X\|\left(\sum_{i=1}^{N}\|(B_i-\beta_i)e_n\|\right)$ and so $\|X\|\geqq n/\max\{|\alpha_i|\}$ $(n\geqq 1)$. This contradiction shows that Y is not in the range of $\mathfrak{R}-\lambda$.

iv) $\alpha\in\sigma_{re}(A)$, $\beta\in\sigma_l(B)\setminus\sigma_{le}(B)$. Since $\bar\alpha\in\sigma_{le}(A^*)$ and $\bar\beta\in\sigma_r(B^*)\setminus\sigma_{re}(B^*)$, then iii) implies that there exists a rank one operator Y such that $\left(\sum_{i=1}^{N} B_i^*X^*A_i^*\right)-$ $-\bar\alpha\circ\bar\beta X^*=Y^*$ has no solution. Then $(\mathfrak{R}-\lambda)(X)=Y$ has no solution and the proof is complete.

Lemma 2.2. i) $\sigma_r(\mathfrak{R}\,|\,\mathfrak{J})\subset\sigma_r(A)\circ\sigma_l(B)$;
ii) $\sigma_r(\mathfrak{R})\subset\sigma_r(A)\circ\sigma_l(B)$.

Proof. Part ii) is contained in [16, Theorem 3.4]. The proof of i) is similar. The argument is essentially that used in the proof of [5, Lemma 3]. We first note that $\sigma_r(L_A\,|\,\mathfrak{J}, R_B\,|\,\mathfrak{J})\subset\sigma_r(A)\times\sigma_1(B)$. Indeed, suppose $(\alpha,\beta)\in\mathbf{C}^N\times\mathbf{C}^N$ and $\alpha\notin\sigma_r(A)$. There exists an N-tuple of operators $(R_1, \ldots, R_N)$ such that $\sum_{i=1}^{N}(A_i-\alpha_i)R_i=1$, and thus $\sum_{i=1}^{N}(L_{A_i}\,|\,\mathfrak{J}-\alpha_i)(L_{R_i}\,|\,\mathfrak{J})=1\in\mathfrak{L}(\mathfrak{J})$, so that $(\alpha,\beta)\notin\sigma_r(L_A\,|\,\mathfrak{J}, R_B\,|\,\mathfrak{J})$. The proof for the case when $\beta\notin\sigma_l(B)$ is similar. For $z=(z_1, \ldots, z_N)$ and $w=$ $=(w_1, \ldots, w_N)$ we define the $2N$-variable polynomial p by $p(z,w)=p(z_1, \ldots, z_N,$ $w_1, \ldots, w_N)=\sum_{i=1}^{N} z_iw_i$. Since $(L_A\,|\,\mathfrak{J}, R_B\,|\,\mathfrak{J})$ is a commutative $2N$-tuple in $\mathfrak{L}(\mathfrak{J})$, the spectral mapping theorem for right spectra [15], [16, Theorem 1.2] implies that

$$\sigma_r(\mathfrak{R}|\mathfrak{J}) = \sigma_r(p(L_A|\mathfrak{J}, R_B|\mathfrak{J})) = p(\sigma_r(L_A|\mathfrak{J}, R_B|\mathfrak{J})) \subset p(\sigma_r(A)\times\sigma_l(B)) =$$
$$= \sigma_r(A)\circ\sigma_l(B).$$

Theorem 2.3. *For* $\lambda\in\mathbf{C}$ *and* $\mathfrak{R}=\mathfrak{R}(A,B)$, *the following are equivalent:*
i) $\mathfrak{R}-\lambda$ *is surjective;*
ii) *The range of* $\mathfrak{R}-\lambda$ *contains each rank one operator;*
iii) $\lambda\notin\sigma_r(A)\circ\sigma_l(B)$;
iv) $\mathfrak{R}-\lambda$ *is right invertible in* $\mathfrak{L}(\mathfrak{L}(\mathfrak{H}))$;
v) $\mathfrak{R}_{\mathfrak{J}}-\lambda$ *is right invertible for some norm ideal* $\mathfrak{J}$;
vi) $\mathfrak{R}_{\mathfrak{J}}-\lambda$ *is surjective for some norm ideal* $\mathfrak{J}$;
vii) $\mathfrak{R}_{\mathfrak{J}}-\lambda$ *is right invertible in* $\mathfrak{L}(\mathfrak{J})$ *for every norm ideal* $\mathfrak{J}$;
viii) $\mathfrak{R}_{\mathfrak{J}}-\lambda$ *is surjective for every norm ideal* $\mathfrak{J}$.

Proof. i)⇒ii) is clear, ii)⇒iii) follows from Lemma 2.1, iii)⇒iv) follows from Lemma 2.2, and iv)⇒i) is clear, so i)—iv) are equivalent. Lemma 2.1 implies that vi)⇒ii)⇒iii) and Lemma 2.2 implies that iii)⇒v)⇒vi), so iii), v) and vi) are equivalent. Similarly, we have vii)⇒viii)⇒ii)⇒iii)⇒vi).

We next begin our consideration of elementary operators with dense range.

Corresponding to $\mathfrak{R}(A, B)$ we define an operator $\tilde{\mathfrak{R}}(\tilde{A}, \tilde{B})$ on the Calkin algebra $\mathfrak{A}(\mathfrak{H})$ by $\tilde{\mathfrak{R}}(\tilde{X})=\widetilde{\mathfrak{R}(X)}=\sum_{i=1}^{N}\tilde{A}_i\tilde{X}\tilde{B}_i$ $(X\in\mathfrak{L}(\mathfrak{H}))$.

Lemma 2.4. $\sigma_r(\tilde{\mathfrak{R}})\subset\sigma_{re}(A)\circ\sigma_{le}(B)$.

Proof. The proof is similar to that of Lemma 2.2 ; as before, $\sigma_r(L_{\tilde{A}}, R_{\tilde{B}})\subset\subset\sigma_{re}(A)\times\sigma_{le}(B)$. Let $p(z, w)=p(z_1, \dots, z_N, w_1, \dots, w_N)=z_1w_1+\dots+z_Nw_N$. Since $\tilde{\mathfrak{R}}=p(L_{\tilde{A}}, R_{\tilde{B}})$ and $(L_{\tilde{A}}, R_{\tilde{B}})$ is a commutative $2N$-tuple of elements of $\mathfrak{L}(\mathfrak{A}(\mathfrak{H}))$, the spectral mapping theorem for right spectra [15] implies that

$$\sigma_r(\tilde{\mathfrak{R}}) = \sigma_r(p(L_{\tilde{A}}, R_{\tilde{B}})) = p(\sigma_r(L_{\tilde{A}}, R_{\tilde{B}}))\subset$$
$$\subset p(\sigma_{re}(A)\times\sigma_{le}(B)) = \sigma_{re}(A)\circ\sigma_{le}(B).$$

Recall that $C_\infty^*\approx C_1$; a trace class operator K corresponds to the functional $f_K\in C_\infty^*$ defined by $f_K(X)=\operatorname{tr}(KX)$ [21]. Under this identification $\mathfrak{S}_\infty(A, B)^*==\mathfrak{S}_1(B, A)$. Indeed, for $X\in C_\infty$ and $K\in C_1$ we have $\mathfrak{S}_\infty(A, B)^*(f_K)(X)==\operatorname{tr}(KAXB)=\operatorname{tr}(BKAX)=f_{BKA}(X)$. Recall also that $C_1^*\approx\mathfrak{L}(\mathfrak{H})$, where $T\in\mathfrak{L}(\mathfrak{H})$ corresponds to the functional $f_T\in C_1^*$ defined by $f_T(K)=\operatorname{tr}(TK)$. For $K\in C_1$ and $T\in\mathfrak{L}(\mathfrak{H})$, $\mathfrak{S}_1(B, A)^*(f_T)(K)=\operatorname{tr}(TBKA)=\operatorname{tr}(ATBK)=f_{ATB}(K)$, and therefore $\mathfrak{S}_1(B, A)^* = \mathfrak{S}(A, B)$. By linearity, we see that $\mathfrak{R}_\infty(A, B)^* = \mathfrak{R}_1(B, A)$ and $\mathfrak{R}_1(B, A)^*=\mathfrak{R}(A, B)$.

Theorem 2.5. *The following are equivalent for* $\lambda\in\mathbf{C}$.

i) $\mathfrak{R}(A, B)-\lambda$ *has norm dense range;*

ii) $\lambda\notin\sigma_{re}(A)\circ\sigma_{le}(B)$ *and* $\mathfrak{R}_1(B, A)$ *is injective;*

iii) *For* $\varepsilon>0$ *and* $Y\in\mathfrak{L}(\mathfrak{H})$, *there exists* $X\in\mathfrak{L}(\mathfrak{H})$ *such that* $(\mathfrak{R}-\lambda)(X)-Y$ *is a compact operator with norm less than* ε.

Proof. We first prove ii)$\Rightarrow$iii). Suppose ii) is satisfied, let $\varepsilon>0$, and let Y be in $\mathfrak{L}(\mathfrak{H})$. Lemma 2.4 shows that $\tilde{\mathfrak{R}}-\lambda$ is surjective; thus there exists $X\in\mathfrak{L}(\mathfrak{H})$ and $K\in\mathfrak{K}(\mathfrak{H})$ such that $(\mathfrak{R}-\lambda)(X)-Y=K$. Since $\mathfrak{R}_1(B, A)-\lambda$ is injective, $\mathfrak{R}_\infty(A, B)-\lambda$ has dense range. Thus there exists $\{K_n\}\subset\mathfrak{K}(\mathfrak{H})$ such that $\|(\mathfrak{R}-\lambda)(K_n)--K\|\to 0$. Now $(\mathfrak{R}-\lambda)(X-K_n)-Y=K-(\mathfrak{R}-\lambda)(K_n)\in\mathfrak{K}(\mathfrak{H})$, and for sufficiently large n, $\|K-(\mathfrak{R}-\lambda)(K_n)\|<\varepsilon$.

Clearly iii)$\Rightarrow$i), so it suffices to prove that i)$\Rightarrow$ii). If $\mathfrak{R}(A, B)-\lambda$ has dense range, then duality implies that $\mathfrak{R}_1(B, A)-\lambda$ is injective. Suppose $\lambda\in\sigma_{re}(A)\circ\circ\sigma_{le}(B)$; it suffices to prove that the range of $\mathfrak{R}(A, B)-\lambda$ is not dense. Let $\alpha\in\sigma_{re}(A)$ and $\beta\in\sigma_{le}(B)$ satisfy $\lambda=\alpha\circ\beta$. Let $\{e_n\}$ and $\{f_n\}$ denote orthonormal sequences such that $\sum_{i=1}^{N}\|(A_i-\alpha_i)^*e_n\|\to 0$ $(n\to\infty)$ and $\sum_{i=1}^{N}\|(B_i-\beta_i)f_n\|\to 0$ $(n\to\infty)$.

Let Y denote an operator in $\mathfrak{L}(\mathfrak{H})$ such that $Yf_n=e_n$ $(n\geqq 1)$. For $X\in\mathfrak{L}(\mathfrak{H})$,

$$\|(\mathfrak{R}-\lambda)(X)-Y\| \geqq \left|\sum_{i=1}^{N}(((A_i-\alpha_i)XB_i+\alpha_i X(B_i-\beta_i))f_n, e_n)-(Yf_n, e_n)\right| =$$

$$= \left|\left[\sum_{i=1}^{N}(XB_i f_n, (A_i-\alpha_i)^* e_n)+(\alpha_i X(B_i-\beta_i)f_n, e_n)\right]-1\right| \geqq$$

$$\geqq 1-\|X\| \max\{\|B_i\|\}\left(\sum_{i=1}^{N}\|(A_i-\alpha_i)^* e_n\|\right)-\max\{|\alpha_i|\}\|X\|\left(\sum_{i=1}^{N}\|(B_i-\beta_i) f_n\|\right),$$

and thus $\|(\mathfrak{R}-\lambda)(X)-Y\|\geqq 1$. The proof is complete.

We conclude this section with an analogue of Theorem 2.3 for left spectra of elementary operators.

Lemma 2.6. *If $\lambda\in\sigma_l(A)\circ\sigma_r(B)$, then $(\mathfrak{R}-\lambda)\mid\mathfrak{F}_1$ and $(\mathfrak{R}_3-\lambda)\mid\mathfrak{F}_1$ are not bounded below.*

Proof. Let $\alpha\in\sigma_l(A)$ and $\beta\in\sigma_r(B)$ be such that $\lambda=\alpha\circ\beta$. There exist sequences of unit vectors $\{x_k\}$, $\{y_k\}\subset\mathfrak{H}$ such that

$$\lim_{k\to\infty}\sum_{i=1}^{N}\|(A_i-\alpha_i)x_k\| = \lim_{k\to\infty}\sum_{i=1}^{N}\|(B_i-\beta_i)^* y_k\| = 0.$$

Now

$$|||(\mathfrak{R}_3-\lambda)(x_k\otimes y_k)||| \leqq$$

$$\leqq \sum_{i=1}^{N}[|||(A_i-\alpha_i)(x_k\otimes y_k)B_i|||+|\alpha_i|\;|||(x_k\otimes y_k)(B_i-\beta_i)|||] =$$

$$= \sum_{i=1}^{N}[\|(A_i-\alpha_i)(x_k\otimes y_k)B_i\|+|\alpha_i|\,\|\alpha_i(x_k\otimes y_k)(B_i-\beta_i)\|].$$

For $t\in\mathfrak{H}$, $\|t\|=1$, we have

$$\|(A_i-\alpha_i)(x_k\otimes y_k)B_i t\| =$$

$$= \|(A_i-\alpha_i)(B_i t, y_k)x_k\| \leqq \|B_i\|\,\|y_k\|\,\|(A_i-\alpha_i)x_k\|.$$

Thus

$$\sum_{i=1}^{N}\|(A_i-\alpha_i)(x_k\otimes y_k)B_i\| \to 0 \quad (k\to\infty);$$

similarly,

$$\sum_{i=1}^{N}\|(x_k\otimes y_k)(B_i-\beta_i)\| \to 0 \quad (k\to\infty).$$

Since $|||x_k\otimes y_k|||=\|x_k\otimes y_k\|=1$, it follows that neither $(\mathfrak{R}_3-\lambda)\mid\mathfrak{F}_1$ nor $(\mathfrak{R}-\lambda)\mid\mathfrak{F}_1$ is bounded below.

Lemma 2.7. i) $\sigma_l(\mathfrak{R})\subset\sigma_l(A)\circ\sigma_r(B)$;
ii) $\sigma_l(\mathfrak{R}_\mathfrak{J})\subset\sigma_l(A)\circ\sigma_r(B)$.

Proof. The proof is similar to the proof of Lemma 2.2, but using the spectral mapping theorem for left spectra [16].

Theorem 2.8. *For $\lambda\in\mathbf{C}$ the following are equivalent.*
i) $\mathfrak{R}-\lambda$ *is left invertible;*
ii) $\mathfrak{R}-\lambda$ *is bounded below;*
iii) $(\mathfrak{R}-\lambda)\mid\mathfrak{F}_1$ *is bounded below;*
iv) $\lambda\notin\sigma_l(A)\circ\sigma_r(B)$;
v) $\mathfrak{R}_\mathfrak{J}-\lambda$ *is left invertible in* $\mathfrak{L}(\mathfrak{J})$ *for some norm ideal* $\mathfrak{J}$;
vi) $\mathfrak{R}_\mathfrak{J}-\lambda$ *is bounded below for some norm ideal* $\mathfrak{J}$;
vii) $\mathfrak{R}_\mathfrak{J}-\lambda$ *is left invertible for every norm ideal* $\mathfrak{J}$;
viii) $\mathfrak{R}_\mathfrak{J}-\lambda$ *is bounded below for every norm ideal* $\mathfrak{J}$.

Proof. The implications i)⇒ii)⇒iii) are trivial; iii)⇒iv) follows from Lemma 2.6, and Lemma 2.7 implies that iv)⇒i); thus i)—iv) are equivalent. The implications v)⇒vi)⇒iv)⇒v) and vii)⇒viii)⇒iv)⇒vii) also follow by application of Lemmas 2.6 and 2.7.

Corollary 2.9. *For each norm ideal* $\mathfrak{J}$,

$$\sigma(R_\mathfrak{J}(A, B)) = \sigma(R(A, B)) = \sigma_r(A)\circ\sigma_l(B)\cup\sigma_l(A)\circ\sigma_r(B).$$

Proof. The result follows from Theorem 2.3 and Theorem 2.8.

Remark. The identity for $\sigma(R)$ given above is due to R. HARTE [16]; our contribution is the identity $\sigma(R_\mathfrak{J})=\sigma(R)$. A special case of the latter identity for the Hilbert—Schmidt ideal C_2 was obtained by R. CURTO [5, Lemma 3]. The main result of [5] presents a new description of $\sigma(R)$ in terms of Taylor joint spectra.

3. The semi-Fredholm domain of $\mathfrak{S}_\mathfrak{J}$. In the present section we describe the semi-Fredholm domain and index function of $\mathfrak{S}_\mathfrak{J}$ and $\mathfrak{S}$. To this end we define the following sets:

$$\sigma_{lr} \equiv \sigma_{lr}(A, B) = \sigma_l(A)\sigma_{re}(B)\cup\sigma_{le}(A)\sigma_r(B);$$
$$\sigma_{rl} \equiv \sigma_{rl}(A, B) = \sigma_r(A)\sigma_{le}(B)\cup\sigma_{re}(A)\sigma_l(B).$$

It follows from [12, Lemma 3.2] that if $\lambda\in\sigma_{lr}$ and $\mathfrak{S}_\mathfrak{J}-\lambda$ is semi-Fredholm, then $\operatorname{ind}(\mathfrak{S}_\mathfrak{J}-\lambda)=+\infty$; [12, Lemma 3.3] implies that if $\lambda\in\sigma_{rl}$ and $\mathfrak{S}_\mathfrak{J}-\lambda$ is semi-Fredholm, then $\operatorname{ind}(\mathfrak{S}_\mathfrak{J}-\lambda)=-\infty$. Thus $\sigma_{lr}\cap\sigma_{rl}\subset\sigma_{SF}(\mathfrak{S}_\mathfrak{J})$ and in the sequel we prove the reverse inclusion. We begin with the following special case.

Proposition 3.1. *If* $\lambda\in\mathbf{C}\setminus\sigma_{rl}$, *then* $\mathfrak{S}_\mathfrak{J}(A, B)-\lambda$ *is semi-Fredholm with* $\operatorname{ind}(\mathfrak{S}_\mathfrak{J}(A, B)-\lambda)>-\infty$.

Proof. If $\lambda\in\mathbf{C}\setminus\sigma_r(A)\sigma_l(B)$, then Theorem 2.3 implies that $\mathfrak{S}_3-\lambda$ is surjective, so the result is clear in this case. We may thus assume that $\lambda\in\sigma_r(A)\sigma_l(B)\setminus\sigma_{rl}$. We require the following preliminary lemmas.

Lemma 3.2. *If $\alpha\in\sigma_r(A)$, $\beta\in\sigma_l(B)$ and $\alpha\beta\notin\sigma_{rl}$ then α is isolated in $\sigma_r(A)$ or β is isolated in $\sigma_l(B)$.*

Proof. Since $\alpha\beta\notin\sigma_{rl}$, then $\alpha\in\sigma_r(A)\setminus\sigma_{re}(A)$ and $\beta\in\sigma_l(B)\setminus\sigma_{le}(B)$. Suppose that α is not isolated in $\sigma_r(A)$ and β is not isolated in $\sigma_l(B)$. Since $\alpha\notin\sigma_{re}(A)$, [10, Lemma 3.6 (i)] implies that $\alpha\in\mathfrak{U}\equiv\mathrm{int}\,(\sigma_r(A))$. Similarly, since $\beta\notin\sigma_{le}(B)$ and β is not isolated in $\sigma_l(B)$, then [10, Lemma 3.6 (ii)] implies that $\beta\in\mathfrak{B}\equiv\mathrm{int}\,(\sigma_l(B))$. $\mathfrak{U}$ and $\mathfrak{B}$ are nonempty, open, bounded subsets of the plane, so [12, Lemma 2.11] implies that there exists $t>0$ such that

i) $t\alpha\in\mathrm{bdry}\,(\mathfrak{U})$ and $\beta/t\in\mathfrak{B}^-$, or

ii) $t\alpha\in\mathfrak{U}$ and $\beta/t\in\mathrm{bdry}\,(\mathfrak{B})$.

It follows from [10, Lemma 3.6] that $\mathrm{bdry}\,(\mathfrak{U})\subset\sigma_{re}(A)$ and $\mathrm{bdry}\,(\mathfrak{B})\subset\sigma_{le}(B)$. In case i), $t\alpha\in\mathrm{bdry}\,(\mathfrak{U})\subset\sigma_{re}(A)$ and $\beta/t\in\mathfrak{B}^-\subset\sigma_l(B)$, so $\lambda=\alpha\beta=(t\alpha)(\beta/t)\in\sigma_{re}(A)\cdot\sigma_l(B)\subset\sigma_{rl}$, which is a contradiction. In case ii), $t\alpha\in\mathfrak{U}\subset\sigma_r(A)$ and $\beta/t\in\mathrm{bdry}(\mathfrak{B})\subset\sigma_{le}(B)$, so $\lambda=(t\alpha)(\beta/t)\in\sigma_r(A)\sigma_{le}(B)\subset\sigma_{rl}$, also a contradiction; the proof is now complete.

Lemma 3.3. *If $\lambda\in\sigma_r(A)\sigma_l(B)\setminus\sigma_{rl}$, then $\lambda\neq0$ and $X\equiv\{(\alpha,\beta)\in\sigma_r(A)\times\sigma_l(B)\colon\alpha\beta=\lambda\}$ is finite.*

Proof. If $0\in\sigma_r(A)\sigma_l(B)$, either $0\in\sigma_r(A)$ or $0\in\sigma_l(B)$, so $0\in\sigma_r(A)\sigma_{le}(B)$ or $0\in\sigma_{re}(A)\sigma_l(B)$, and so $0\in\sigma_{rl}$; thus $\lambda\neq0$.

Assume that X is infinite and let $\{(\alpha_n,\beta_n)\}_{n=1}^{\infty}$ be a sequence of distinct points of X. It follows readily that the α_n's are distinct and the β_n's are distinct. There exists a convergent subsequence $(\alpha_{n_k},\beta_{n_k})\to(\alpha,\beta)$, and clearly $\alpha\in\sigma_r(A)$, $\beta\in\sigma_l(B)$, and $\alpha\beta=\lambda$. Since α is not isolated in $\sigma_r(B)$ and β is not isolated in $\sigma_l(B)$, we have a contradiction to Lemma 3.2.

We return to the proof of Proposition 3.1 and consider $\lambda\in\sigma_r(A)\sigma_l(B)\setminus\sigma_{rl}$. Lemma 3.2 and Lemma 3.3 imply that $\lambda\neq0$ and that there exist integers p and n, $p\geqq n\geqq0$, $p>0$, distinct nonzero points $\alpha_1,\dots,\alpha_p\in\sigma_r(A)\setminus\sigma_{re}(A)$, and distinct nonzero points $\beta_1,\dots,\beta_p\in\sigma_l(B)\setminus\sigma_{le}(B)$ such that the following properties are satisfied:

1) $\{(\alpha,\beta)\in\sigma_r(A)\times\sigma_l(B)\colon\alpha\beta=\lambda\}=\{(\alpha_i,\beta_i)\}_{i=1}^{p}$;

2) if $n>0$, then α_i is isolated in $\sigma_r(A)$, $1\leqq i\leqq n$;

3) if $p>n$, then β_i is isolated in $\sigma_l(B)$, $n+1\leqq i\leqq p$.

If each β_i isolated in $\sigma_l(B)$ we may take $n=0$ and delete $\{(\alpha_1,\beta_1),\dots,(\alpha_n,\beta_n)\}$; likewise, if each α_i is isolated in $\sigma_r(A)$, we may take $p=n$ and delete $\{(\alpha_{n+1},\beta_{n+1}),$

$\ldots, (\alpha_p, \beta_p)\}$. We assume in the sequel that $1 \leqq n < p$, for the other cases require only obvious modifications of the argument for this case.

Let $\mathfrak{H}_1$ and $\mathfrak{H}_2$ denote copies of $\mathfrak{H}$ with $A \in \mathfrak{L}(\mathfrak{H}_1)$ and $B \in \mathfrak{L}(\mathfrak{H}_2)$. We identify $\mathfrak{L}(\mathfrak{H})$ with $\mathfrak{L}(\mathfrak{H}_2, \mathfrak{H}_1)$ and consider $\mathfrak{S}(A, B)$ as an operator on $\mathfrak{L}(\mathfrak{H}_2, \mathfrak{H}_1)$. [10, Corollary 2.4] implies that there exists an orthogonal decomposition $\mathfrak{H}_1 = \mathfrak{M}_0 \oplus \ldots \ldots \oplus \mathfrak{M}_n$ and operators $A_i \in \mathfrak{L}(\mathfrak{M}_i)$ $(0 \leqq i \leqq n)$ such that:

4) $\mathfrak{M}_i$ is finite dimensional $(1 \leqq i \leqq n)$;

5) $\sigma(A_i) = \{\alpha_i\}$ $(1 \leqq i \leqq n)$;

6) $\sigma_r(A_0) \cap \{\alpha_1, \ldots, \alpha_n\} = \emptyset$;

7) A is similar to $A' \equiv A_0 \oplus A_1 \oplus \ldots \oplus A_n$.

An application of [10, Corollary 2.3] implies that there is an orthogonal decomposition $\mathfrak{H}_2 = \mathfrak{K}_{n+1} \oplus \ldots \oplus \mathfrak{K}_{p+1}$ and operators $B_i \in \mathfrak{L}(\mathfrak{K}_i)$ $(n+1 \leqq i \leqq p+1)$ such that:

8) $\mathfrak{K}_i$ is finite dimensional, $n+1 \leqq i \leqq p$;

9) $\sigma(B_i) = \{\beta_i\}$, $n+1 \leqq i \leqq p$;

10) $\sigma_l(B_{p+1}) \cap \{\beta_{n+1}, \ldots, \beta_p\} = \emptyset$;

11) B is similar to $B' \equiv B_{n+1} \oplus \ldots \oplus B_{p+1}$.

[12, Proposition 2.5] implies that to complete the proof it suffices to prove that $\mathfrak{S}_3(A', B') - \lambda$ is semi-Fredholm with $\operatorname{ind}(\mathfrak{S}_3(A', B') - \lambda) > -\infty$. The argument is formally similar to that in the proof of [12, Theorem 3.1] so we give the outline and refer the reader to [12] for certain details.

Relative to the above decompositions of $\mathfrak{H}_1$ and $\mathfrak{H}_2$, let $(X_{ij})_{0 \leqq i \leqq n, n+1 \leqq j \leqq p+1}$ denote the operator matrix of an operator $X \in \mathfrak{L}(\mathfrak{H}) = \mathfrak{L}(\mathfrak{H}_2, \mathfrak{H}_1)$. A calculation (using 7) and 11)) shows that the row i, column j entry of the matrix of $S'(X) \equiv \equiv A'XB' - \lambda X$ is equal to $A_i X_{ij} B_j - \lambda X_{ij}$, $0 \leqq i \leqq n$, $n+1 \leqq j \leqq p+1$. For $X \in \mathfrak{L}(\mathfrak{H})$, let $R(X)$ be defined by the matrix which modifies the first row and last column of $S'(X)$ as follows:

$$\begin{bmatrix} (A_0 - \alpha_{n+1}) X_{0,n+1} \beta_{n+1} \ldots (A_0 - \alpha_p) X_{0,p} \beta_p & A_0 X_{0,p+1} B_{p+1} - \lambda X_{0,p+1} \\ & \alpha_1 X_{1,p+1} (B_{p+1} - \beta_1) \\ [A_i X_{ij} B_j - \alpha_j \beta_j X_{ij}] & \vdots \\ & \alpha_n X_{n,p+1} (B_{p+1} - \beta_n) \end{bmatrix}$$

We first prove that $R \mid \mathfrak{J}$ is semi-Fredholm with $\operatorname{ind}(R \mid \mathfrak{J}) > -\infty$. Let R_{ij} be the operator on $\mathfrak{L}(\mathfrak{K}_j, \mathfrak{H}_i)$ defined by the row i, column j entry of $R(X)$, $0 \leqq i \leqq n$, $n+1 \leqq j \leqq p+1$. It follows from 1), 6), 7), 10), and 11) above that $\lambda \notin \sigma_r(A_0) \sigma_l(B_{p+1})$, so Theorem 2.3 implies that $R_{0,p+1} = \mathfrak{S}(A_0, B_{p+1}) - \lambda$ is surjective; in particular, $\operatorname{ind}(R_{0,p+1}) \geqq 0$. Let $1 \leqq i \leqq n$ and $n+1 \leqq j \leqq p$. Since $\sigma(A_i) = \{\alpha_i\}$ $(1 \leqq i \leqq n)$ and $\sigma(B_j) = \{\beta_j\}$ $(n+1 \leqq j \leqq p)$, it follows that $\lambda = \alpha_j \beta_j \notin \notin \sigma(A_i) \sigma(B_j)$, and thus $R_{ij} = \mathfrak{S}(A_i, B_j) - \lambda$ is invertible for $1 \leqq i \leqq n$ and $n+1 \leqq \leqq j \leqq p$ [4].

We next consider the operators $R_{0,j}$ $(n+1 \leqq j \leqq p)$ defined by $R_{0,j}(X) = (A_0 - \alpha_j) X \beta_j$ $(X \in \mathfrak{L}(\mathfrak{K}_j, \mathfrak{H}_0))$. Since $\alpha_j \in \sigma_r(A) \setminus \sigma_{re}(A)$, 7) implies that $\alpha_j \notin \sigma_{re}(A_0)$, and thus $A_0 - \alpha_j$ is semi-Fredholm and $\operatorname{ind}(A_0 - \alpha_j) > -\infty$. Since $\dim(\mathfrak{K}_j) < \infty$, [10, Lemma 3.5] implies that $R_{0,j}$ is semi-Fredholm with $\operatorname{ind}(R_{0,j}) = \operatorname{ind}(A_0 - \alpha_j) \dim(\mathfrak{K}_j) > -\infty$. Similarly, since $\mathfrak{H}_i$ is finite dimensional and $\beta_i \notin \sigma_{le}(B_{p+1})$ $(1 \leqq i \leqq n)$, then [10, Lemma 3.5] and [12, Lemma 2.6] imply that $R_{i,p+1}$ is semi-Fredholm with $\operatorname{ind}(R_{i,p+1}) = \operatorname{ind}((B_{p+1} - \beta_i)^*) \dim(\mathfrak{H}_i) > -\infty$.

It now follows exactly as in the proof of [12, Theorem 3.1] that $R \mid \mathfrak{J}$ is semi-Fredholm with

$$\operatorname{ind}(R|\mathfrak{J}) = \sum_{j=n+1}^{p} \operatorname{ind}(R_{0,j}) + \sum_{i=1}^{n} \operatorname{ind}(R_{i,p+1}) + \operatorname{ind}(R_{0,p+1}) > -\infty.$$

Let $K_j \in \mathfrak{L}(\mathfrak{K}_j)$ be invertible $(n+1 \leqq j \leqq p)$ and let $M_i \in \mathfrak{L}(\mathfrak{M}_i)$ be invertible $(1 \leqq i \leqq n)$. For $X \in \mathfrak{J}$, $X = (X_{ij})$, define $T(X)$ by the matrix

$$\begin{bmatrix} A_0 X_{0,n+1} K_{n+1}^{-1} (B_{n+1} - \beta_{n+1}) K_{n+1} & \dots & A_0 X_{0,p} K_p^{-1} (B_p - \beta_p) K_p & 0 \\ 0 & \dots & 0 & M_1^{-1}(A_1 - \alpha_1) M_1 X_{1,p+1} B_{p+1} \\ \vdots & & \vdots & \vdots \\ 0 & \dots & 0 & M_n^{-1}(A_n - \alpha_n) M_n X_{n,p+1} B_{p+1} \end{bmatrix}.$$

Since $B_j - \beta_j$ $(n+1 \leqq j \leqq p)$ and $A_i - \alpha_i$ $(1 \leqq i \leqq n)$ are nilpotent, appropriate choices of the K_j's and M_i's insure that $Q \equiv R \mid \mathfrak{J} + T$ is semi-Fredholm with $\operatorname{ind}(Q) = \operatorname{ind}(R \mid \mathfrak{J}) > -\infty$. The matrix of $Q(X)$ $(X \in \mathfrak{J})$ is of the form

$$\begin{bmatrix} A_0 X_{0,n+1} K_{n+1}^{-1} B_{n+1} K_{n+1} - \lambda X_{0,n+1} \dots A_0 X_{0,p} K_p^{-1} B_p K_p - \lambda X_{0,p} & A_0 X_{0,p+1} B_{p+1} - \lambda X_{0,p+1} \\ & M_1^{-1} A_1 M_1 X_{1,p+1} B_{p+1} - \lambda X_{1,p+1} \\ [A_i X_{ij} B_j - \lambda X_{ij}] & \vdots \\ & M_n^{-1} A_n M_n X_{n,p+1} B_{p+1} - \lambda X_{n,p+1} \end{bmatrix}.$$

It now follows as in [12, Theorem 3.1] that $\mathfrak{S}_{\mathfrak{J}}(A', B') - \lambda$ is semi-Fredholm with $\operatorname{ind}(\mathfrak{S}_{\mathfrak{J}}(A', B') - \lambda) = \operatorname{ind}(Q) > -\infty$, so the proof is complete.

Corollary 3.4. *$\mathfrak{S}_{\mathfrak{J}}(A, B) - \lambda$ is semi-Fredholm with $\operatorname{ind}(\mathfrak{S}_{\mathfrak{J}} - \lambda) > -\infty$ if and only if $\lambda \in \mathbf{C} \setminus \sigma_{rl}(A, B)$.*

Proof. The result follows from [12, Lemma 3.3] and Proposition 3.1.

Corollary 3.5. *$\mathfrak{S}_{\mathfrak{J}}(A, B) - \lambda$ is semi-Fredholm with $\operatorname{ind}(\mathfrak{S}_{\mathfrak{J}} - \lambda) = +\infty$ if and only if $\lambda \in \sigma(A, B) \setminus \sigma_{rl}(A, B)$.*

Proof. Apply [12, Theorem 3.1] and Corollary 3.4.

We now consider the case when $\lambda \in \mathbf{C} \setminus \sigma_{lr}(A, B)$.

Proposition 3.6. *If $\lambda \in \mathbf{C} \setminus \sigma_{lr}$, then $\mathfrak{S}_{\mathfrak{J}} - \lambda$ is semi-Fredholm with* $\operatorname{ind}(\mathfrak{S}_{\mathfrak{J}} - \lambda) < +\infty$.

The proof is completely analogous to that of Proposition 3.1; for this reason we omit the details and merely mention the necessary preliminary results.

Lemma 3.7. *If* $\alpha \in \sigma_l(A)$, $\beta \in \sigma_r(B)$ *and* $\alpha\beta \notin \sigma_{lr}$, *then* α *is isolated in* $\sigma_l(A)$ *or* β *is isolated in* $\sigma_r(B)$.

Proof. The proof is similar to that of Lemma 3.2.

Using Lemma 3.7, the proof of the next result is based on that of Lemma 3.3.

Lemma 3.8. *If* $\lambda \in \sigma_l(A)\sigma_r(B) \setminus \sigma_{lr}(A, B)$, *then* $\lambda \neq 0$ *and* $\{(\alpha, \beta) \in \sigma_l(A) \times \times \sigma_r(B) : \alpha\beta = \lambda\}$ *is finite.*

Using the preceding two lemmas, the proof of Proposition 3.6 follows the argument in the proof of Proposition 3.1, except that instead of using Theorem 2.3, we now use Theorem 2.8 to show that $\mathfrak{S}(A_0, B_{p+1}) - \lambda$ is bounded below.

Corollary 3.9. $\mathfrak{S}_{\mathfrak{J}}(A, B) - \lambda$ *is semi-Fredholm with* $\operatorname{ind}(\mathfrak{S}_{\mathfrak{J}} - \lambda) < +\infty$ *if and only if* $\lambda \in \mathbf{C} \setminus \sigma_{lr}(A, B)$.

Proof. The result follows from [12, Lemma 3.2] and Proposition 3.6.

Corollary 3.10. $\mathfrak{S}_{\mathfrak{J}} - \lambda$ *is semi-Fredholm with* $\operatorname{ind}(\mathfrak{S}_{\mathfrak{J}} - \lambda) = -\infty$ *if and only if* $\lambda \in \sigma(A, B) \setminus \sigma_{lr}(A, B)$.

Proof. The result follows from Corollary 3.9 and [12, Theorem 3.1].

An immediate consequence of Corollary 3.4 and Corollary 3.9 is the following description of the semi-Fredholm domain of $\mathfrak{S}_{\mathfrak{J}}$.

Theorem 3.11. $\mathfrak{S}_{\mathfrak{J}} - \lambda$ *is semi-Fredholm if and only if* $\lambda \in \mathbf{C} \setminus (\sigma_{rl} \cap \sigma_{lr})$.

Corollary 3.12. $\sigma_{SF}(\mathfrak{S}_{\mathfrak{J}}) = \sigma_{rl} \cap \sigma_{lr}$.

For the case when $\mathfrak{S}_{\mathfrak{J}} - \lambda$ is Fredholm, a formula for $\operatorname{ind}(\mathfrak{S}_{\mathfrak{J}} - \lambda)$ is given in [12, Theorem 3.8]. The latter result, when combined with Corollary 3.5 and Corollary 3.10, thus gives a complete description of $\operatorname{ind}(\mathfrak{S}_{\mathfrak{J}} - \lambda)$ for $\lambda \in \varrho_{SF}(\mathfrak{S}_{\mathfrak{J}})$.

Example 3.13. Consider the case when $\mathfrak{J}$ is the ideal of all Hilbert—Schmidt operators endowed with its (separable) Hilbert space structure [4]. In this case $\mathfrak{S}_{\mathfrak{J}}(A, B)$ is again a Hilbert space operator; we will show that if A and B^* are quasitriangular, then so is $\mathfrak{S}_{\mathfrak{J}}$. By a theorem of C. Apostol, C. Foiaş, and D. Voiculescu [3], an operator T on a separable Hilbert space is quasitriangular if and only if $\operatorname{ind}(T - \lambda) \geqq 0$ for every $\lambda \in \varrho_{SF}(T)$.

Suppose A and B^* are quasitriangular; thus $\operatorname{ind}(A - \lambda) \geqq 0$ $(\lambda \in \varrho_{SF}(A))$ and $\operatorname{ind}(B - \lambda) \leqq 0$ $(\lambda \in \varrho_{SF}(B))$. It follows directly from the index formula of [12, Theorem 3.8] that $\operatorname{ind}(\mathfrak{S}_{\mathfrak{J}} - \lambda) \geqq 0$ for every $\lambda \in \mathbf{C} \setminus \sigma_e(\mathfrak{S}_{\mathfrak{J}})$. To complete the proof it thus suffices to verify that the case $\operatorname{ind}(\mathfrak{S}_{\mathfrak{J}} - \lambda) = -\infty$ cannot occur.

Suppose to the contrary that $\operatorname{ind}(\mathfrak{S}_3-\lambda)=-\infty$; from Corollary 3.10 we have

$$\lambda\in(\sigma_e(A)\sigma(B)\cup\sigma(A)\sigma_e(B))\setminus(\sigma_l(A)\sigma_{re}(B)\cup\sigma_{le}(A)\sigma_r(B)).$$

We consider the case $\lambda\in\sigma_e(A)\sigma(B)$ and let $\alpha\in\sigma_e(A)$ and $\beta\in\sigma(B)$ satisfy $\alpha\beta=\lambda$. If $\beta\in\sigma_r(B)$, then $\alpha\notin\sigma_{le}(A)$ and thus $\operatorname{ind}(A-\alpha)=-\infty$, a contradiction. Therefore $\beta\in\sigma(B)\setminus\sigma_r(B)$, so $\operatorname{ind}(B-\beta)>0$, which is also a contradiction. The case when $\lambda\in\sigma(A)\sigma_e(B)$ can be handled similarly, so we omit the details.

We note that the converse of this example is false. [12] contains an example of operators A and B such that A, A^*, B, and B^* are non-quasitriangular but $\mathfrak{S}_3(A, B)$ is biquasitriangular, i.e. $\mathfrak{S}_3$ and $\mathfrak{S}_3{}^*$ are both quasitriangular.

Systematic revision of the proofs of this section (replacing the norm ideal $\mathfrak{J}$ by $\mathfrak{L}(\mathfrak{H})$) yields a description of the semi-Fredholm domain of $\mathfrak{S}(A, B)$.

Theorem 3.14. i) $\sigma_{SF}(\mathfrak{S})=\sigma_{lr}\cap\sigma_{rl}$;
ii) $\mathfrak{S}-\lambda$ *is semi-Fredholm with* $\operatorname{ind}(\mathfrak{S}-\lambda)<+\infty$ *if and only if* $\lambda\in\mathbf{C}\setminus\sigma_{lr}$;
iii) $\mathfrak{S}-\lambda$ *is semi-Fredholm with* $\operatorname{ind}(\mathfrak{S}-\lambda)>-\infty$ *if and only if* $\lambda\in\mathbf{C}\setminus\sigma_{rl}$.

This result, together with [12, Theorem 3.9], completes the description of $\operatorname{ind}(\mathfrak{S}-\lambda)$ $(\lambda\in\varrho_{SF}(\mathfrak{S}))$. More generally, the present results, together with those of [9], [10] and [12], completely describe the semi-Fredholm domain and index function of the operators $\mathfrak{T}$, $\mathfrak{T}_3$, $\mathfrak{S}$, and $\mathfrak{S}_3$. Corresponding results for arbitrary elementary operators $\mathfrak{R}$, or the operators $\mathfrak{R}_3$, appear to be unknown at present. Some partial results are known for the general case. In [12, Theorem 3.14] it is proved that $\sigma_e(\mathfrak{R}_3)\subset\sum_{i=1}^{N}(\sigma(A_i)\sigma_e(B_i)\cup\sigma_e(A_i)\sigma(B_i))$ (and similarly for the operator $\mathfrak{R}$). By combining the techniques of [9], [10], [12] with the multi-variate techniques used in section 2, it is possible to prove the following result for the general case. The proof, and applications, will appear elsewhere. For n-tuples of operators A and B, let $\sigma_{lr}(A, B)=\sigma_{le}(A)\circ\sigma_r(B)\cup\sigma_l(A)\circ\sigma_{re}(B)$ and let $\sigma_{rl}(A, B)=\sigma_{re}(A)\circ\sigma_l(B)\cup\sigma_r(A)\circ\sigma_{le}(B)$. Let $\mathfrak{J}$ be an arbitrary norm ideal.

Theorem 3.15. i) $\sigma_e(A)\circ\sigma(B)\cup\sigma(A)\circ\sigma_e(B)\subset\sigma_e(\mathfrak{R}_3)$;
ii) *If* $\lambda\in\sigma_{lr}(A, B)$ *and* $\mathfrak{R}_3-\lambda$ *is semi-Fredholm, then* $\operatorname{ind}(\mathfrak{R}_3-\lambda)=+\infty$;
iii) *If* $\lambda\in\sigma_{rl}(A, B)$ *and* $\mathfrak{R}_3-\lambda$ *is semi-Fredholm, then* $\operatorname{ind}(\mathfrak{R}_3-\lambda)=-\infty$;
iv) $\sigma_{lr}(A, B)\cap\sigma_{rl}(A, B)\subset\sigma_{SF}(\mathfrak{R}_3)$.

We note that parts ii)—iv) are valid for elementary operators with arbitrary (non-commutative) coefficient sequences. A similar result holds for the operator $\mathfrak{R}$.

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Characterizations and invariant subspaces of composition operators

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1. Preliminaries. Let $(X, \mathcal{S}, \lambda)$ be a σ-finite measure space and let T be a non-singular measurable transformation from X into itself. Then the composition transformation C_T from $L^p(\lambda)$ into the space of all complex-valued functions on X is defined by

$$C_T f = f \circ T \quad \text{for every} \quad f \in L^p(\lambda).$$

If C_T happens to be a bounded operator on $L^p(\lambda)$, then we call it a composition operator induced by T.

Let $X=N$, the set of all non-zero positive integers and $\mathcal{S}=P(N)$, the power set of N. Then we can define the measure λ on $P(N)$ by

$$\lambda(E) = \sum_{n\in E}' w_n \quad \text{for every} \quad E \in P(N),$$

where $w=\{w_n\}$ is a sequence of strictly positive real numbers. If $p=2$, then $L^p(\lambda)$ is a Hilbert space with the inner product defined by

$$\langle f, g\rangle = \sum w_n f(n)\overline{g(n)}$$

for all $f, g \in L^p(\lambda)$. This Hilbert space is denoted by l_w^2, and is called a weighted sequence space. By $B(l_w^2)$ we mean the Banach algebra of all bounded linear operators on l_w^2. Let $\{e_n\}$ be the sequence defined by $e_n(p)=\delta_{np}$, the Kronecker delta. If C_T is a composition operator, then C_T^*, the adjoint of C_T, is given by

$$(C_T^* f)(n) = \frac{1}{w_n} \int_{T^{-1}(\{n\})} f\, d\lambda \quad \text{(cf. [4])}.$$

In the present note certain criteria for a bounded operator to be a composition operator are obtained. It is also shown that every composition operator on l_w^2 has an invariant subspace. This generalizes a result of SINGH and KOMAL [5] to the weighted sequence spaces.

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2. Criteria for a bounded operator to be a composition operator. In this section we obtain two different criteria for a bounded operator to be a composition operator.

Theorem 2.1. *Let $A \in B(l^2_w)$. Then A is a composition operator if and only if for every $n \in N$, there exists $m \in N$ such that $A^* e'_n = e'_m$, where $e'_n = e_n/w_n$.*

Proof. The proof follows from NORDGREN [2]. Here e'_n's play the role of kernel functions.

Theorem 2.2. *Let $A \in B(l^2_w)$. Then A is a composition operator if and only if there exists a partition $\{E_n\}$ of N such that $Ae_n = X_{E_n}$, where X_E denotes the characteristic function of a set E.*

Proof. Suppose A is a composition operator. Then $A = C_T$ for some mapping $T: N \to N$. The choice $T^{-1}(\{n\}) = E_n$ clearly suits our requirements.

Conversely, if A satisfies the condition of the theorem, then we may define a mapping $T: N \to N$ by $T(m) = n$ for $m \in E_n$. Now $Ae_n = C_T e_n$ and so $Ae_n/\sqrt{w_n} = C_T e_n/\sqrt{w_n}$ for every $n \in N$. Thus A and C_T agree on the basis vectors of l^2_w. It is easy to show that C_T is a bounded operator. Hence $Af = C_T f$ for every $f \in l^2_w$. This completes the proof.

Theorem 2.3. *Let $T: N \to N$ be a surjective mapping such that $C_T \in B(l^2_w)$ and let $A \in B(l^2_w)$. Then $C_T A$ is a composition operator if and only if A is a composition operator.*

Proof. The proof is an immediate consequence of Theorem 2.1. Indeed if $C_T A = C_S$ then $A^* C_T^* = C_S^*$, i.e., $A^* e'_{T(N)} = A^* C_T^* e'_k = C_S^* e'_k = e'_{S(K)}$ for every $k \in N$. Since $T(N) = N$, for every $m \in N$ there exists $n \in N$ such that $A^* e'_m = e'_n$.

Theorem 2.4. *Let $T: N \to N$ be an injection and let C_T, $A \in B(l^2_w)$. Then AC_T is a composition operator if and only if A is a composition operator.*

Proof. Suppose AC_T is a composition operator. Then there is a mapping $S: N \to N$ such that $AC_T = C_S$. Now $Ae_n = AC_T e_{T(n)} = C_S e_{T(n)} = X_{E_n}$, where $E_n = S^{-1}(\{T(n)\})$. By Theorem 2.2, $\{E_n\}$ is a partition of N. Hence A is a composition operator. The proof of the sufficient part of the theorem is straight forward.

Theorem 2.5. *Let $A \in B(l^2_w)$. Then A is a unitary composition operator if and only if*

$$\{Ae'_n \colon n \in N\} = \{e'_n \colon n \in N\} = \{A^* e'_n \colon n \in N\}.$$

Proof. Assume A is a unitary composition operator. Then by Theorem 2.1

$$\{A^* e'_n \colon n \in N\} \subseteq \{e'_n \colon n \in N\} = \{AA^* e'_n \colon n \in N\} \subseteq \{Ae'_n \colon n \in N\}.$$

From Theorem 3.1 of [6], A^* is a composition operator and hence also the converse inclusions hold.

If the conditions of the theorem are true, then by Theorem 2.1 both A and A^* are composition operators. Hence by Theorem 3.1 of [6] A is a unitary composition operator.

3. Invariant subspaces. Definition. Let $T: N \to N$ be a mapping. Then two integers m and n are said to be in the same orbit of T if each can be reached from the other by composing T and T^{-1} (T^{-1} means a multivalued function) sufficiently many times.

Definition. A closed subspace M of a Hilbert space is called an invariant subspace of A if $AM \subseteq M$.

One of the most outstanding unsolved problems of operator theory is the Invariant Subspace Problem. The problem is simple to state: Does every operator on an infinite dimensional separable Hilbert space have a non-trivial invariant subspace? The answer is not yet known. Recently SINGH and KOMÁL [5] obtained that every composition operator on l^2 has a non-trivial invariant subspace. In the following theorem we generalize this result to the weighted sequence spaces.

Theorem 3.1. *Let $C_T \in B(l^2_w)$. Then C_T has a non-trivial invariant subspace.*

Proof. Suppose C_T is a composition operator induced by a mapping $T: N \to N$. Then either T is invertible or T is not invertible. First assume that T is invertible. Then take $n \in N$. Now either the orbit of n is equal to N or it is not equal to N. Suppose $o(n) = N$, where $o(n)$ is the orbit of n. Then let

$$E_n = \{(T^m)^{-1}(\{n\}) \colon m \in N\}.$$

If $l^2_{E_n} = \operatorname{span}\{e'_m \colon m \in E_n\}$, then clearly $l^2_{E_n}$ is invariant under C_T. Next, if $o(n) \neq N$, then $l^2_{E_n} = \operatorname{span}\{e'_m \colon m \in o(n)\}$ is an invariant subspace of C_T.

Further, suppose T is not invertible. Then, either T is not an injection or T is not a surjection. If T is not an injection, then C_T has not dense range and hence $\overline{\operatorname{ran} C_T}$ is invariant under C_T. And, if T is not a surjection, then C_T has a non-trivial kernel and hence $\ker C_T$ is invariant under C_T. This completes the proof.

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Note on operators of class $C_0(1)$

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1. Introduction. Let H be a separable Hilbert space and $B(H)$ the algebra of all bounded linear operators on H. The ultraweak topology on $B(H)$ is the weak topology relative to the family of functionals φ of the form

$$\varphi(T) = \sum_{n=1}^{\infty} (Tx_n, y_n) \tag{1}$$

where $\{x_n\}$, $\{y_n\} \subset H$ and $\sum_{n=1}^{\infty} (\|x_n\|^2 + \|y_n\|^2) < \infty$.

The following theorem occurs in HADWIN and NORDGREN [1].

Theorem 1. *Let $\mathscr{L}$ be an ultraweakly closed subspace of $B(H)$ and φ an ultraweakly continuous linear functional on $\mathscr{L}$ with $\|\varphi\| \leqq 1$. Then for every $\varepsilon > 0$ there is an extension of φ to $B(H)$ which is a functional of the form* (1) *with*

$$\left(\sum_{n=1}^{\infty} \|x_n\|^2\right)^{1/2} \left(\sum_{n=1}^{\infty} \|y_n\|^2\right)^{1/2} < 1 + \varepsilon.$$

Let $\mathscr{A}$ be a unital ultraweakly closed subalgebra of $B(H)$. We say that $\mathscr{A}$ *has property* $D_\sigma(1)$ if every ultraweakly continuous linear functional φ on $\mathscr{A}$ can be represented in the form $\varphi(T) = (Tx, y)$ with some x, y in H. If in addition $r \geqq 1$ and, for every $s > r$, x and y can be chosen so that $\varphi(T) = (Tx, y)$ for all T in $\mathscr{A}$ and $\|x\|\,\|y\| \leqq s\|\varphi\|$, then we say that $\mathscr{A}$ *has property* $D_\sigma(r)$. An operator T is said to *have property* D_σ *or* $D_\sigma(r)$ if $\mathscr{A}(T)$ has the respective property, where $\mathscr{A}(T)$ denotes the unital ultraweakly closed algebra generated by T.

The main purpose of this note is to show that the operators of the class $C_0(1)$ have property $D_\sigma(1)$. From this we deduce that the commutant of the Volterra operator V defined by $(Vf)(x) = \int_0^x f(y)\,dy$ in $L^2(0, 1)$ is the minimal unital ultraweakly closed algebra with linearly ordered invariant subspace lattice containing V.

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In conclusion we prove that a contraction with sufficiently large spectra cannot be reductive.

2. Main result. For any subset $\mathcal{S}$ of $B(H)$, $\mathcal{S}'$ denotes its commutant, lat $\mathcal{S}$ the collection of closed subspaces in H invariant under every operator in $\mathcal{S}$, and alg lat $\mathcal{S}$ the algebra of all operators in $B(H)$ leaving each element of lat $\mathcal{S}$ invariant. $\mathcal{S}$ is called *reflexive* if $U(\mathcal{S})=\operatorname{alg}\operatorname{lat}\mathcal{S}$, where $U(\mathcal{S})$ is the weakly closed unital algebra generated by $\mathcal{S}$.

Let N be a positive integer. The class $C_0(N)$ of operators is defined as the set of completely non-unitary contractions $T\in B(H)$ for which $T^n\to 0$, $T^{*n}\to 0$ (strongly) and $\dim(I-T^*T)(H)=\dim(I-TT^*)(H)=N$. The operators of class $C_0(1)$ admit the following description [2]. Let U be the canonical unilateral shift, that is the operator of multiplication by the independent variable λ in H^2, and let $m(\lambda)$ be an inner function. Denote by $H(m)$ the subspace $H^2\ominus mH^2$ and define the operator $S(m)$ in $H(m)$ by

$$S(m)=P_{H(m)}U.$$

Then every operator of class $C_0(1)$ is unitarily equivalent to $S(m)$ for an appropriate inner function m. Alternatively, one can view the operators of class $C_0(1)$ as restrictions of the backward shift U^* to its invariant subspaces.

Theorem 2. *Every operator of the class $C_0(1)$ has property $D_\sigma(1)$.*

Proof. By virtue of the preceding remark, it is enough to show that if U is a (cyclic) unilateral shift in H and $L\in\operatorname{lat}U^*$, then $T=U^*\,|\,L$ has property $D_\sigma(1)$. Suppose φ is an ultraweakly continuous functional on $\mathcal{A}(T)$ with $\|\varphi\|\leqq 1$, and $\varepsilon>0$. By Theorem 1, we may assume that for every $S\in\mathcal{A}(T)$,

$$\varphi(S)=\sum_{n=1}^{\infty}(Sx_n,y_n).$$

where $\{x_n\},\{y_n\}\subset L$ and $\left(\sum_{n=1}^{\infty}\|x_n\|^2\right)^{1/2}\left(\sum_{n=1}^{\infty}\|y_n\|^2\right)^{1/2}<1+\varepsilon$. Let $\mathbf{H}$ denote the infinite Hilbert sum $H\oplus H\oplus\ldots\oplus H\oplus\ldots$. Then the vectors $\mathbf{x}=x_1\oplus x_2\oplus\ldots\oplus\ldots\oplus x_n\oplus\ldots$ and $\mathbf{y}=y_1\oplus y_2\oplus\ldots\oplus y_n\oplus\ldots$ are in $\mathbf{H}$ and the operator $\mathbf{U}=U\oplus U\oplus\ldots\oplus U\oplus\ldots$ is in $B(\mathbf{H})$. Let $M=\bigvee_{n=1}^{\infty}\mathbf{U}^n\mathbf{y}$. Since $\mathbf{U}\,|\,M$ is a cyclic completely non-unitary isometry, it is unitarily equivalent to U. Hence there is an isometry W from H into $\mathbf{H}$ such that $W(H)=M$ and

$$WU=\mathbf{U}W. \tag{2}$$

Let $T_n=P_nW$, where P_n is the projection in $\mathbf{H}$ onto the nth coordinate subspace. Clearly $T_n \in B(H)$ and for every $x \in H$,

$$Wx = T_1 x \oplus T_2 x \oplus \ldots \oplus T_n x \oplus \ldots. \tag{3}$$

From (2) and (3) it follows that $T_n U = U T_n$ for every n. Let $y_0 = W^* \mathbf{y}$. Then $T_n y_0 = y_n$ and $\|y_0\|^2 = \|\mathbf{y}\|^2 = \sum_{n=1}^{\infty} \|y_n\|^2$. Since

$$(W^* \mathbf{x}, z) = (\mathbf{x}, Wz) = \sum_{n=1}^{\infty} (x_n, T_n z) = \sum_{n=1}^{\infty} (T_n^* x_n, z),$$

for every $z \in H$, we can assert that the series $\sum_{n=1}^{\infty} T_n^* x_n$ converges weakly to some $x_0 \in H$ and

$$\|x_0\| = \|W^* \mathbf{x}\| \leqq \|\mathbf{x}\| = \left(\sum_{n=1}^{\infty} \|x_n\|^2 \right)^{1/2}.$$

Moreover, since L is a hyperinvariant subspace of U^*, x_0 is actually in L. Now for every $S \in \mathscr{A}(T)$,

$$\varphi(S) = \sum_{n=1}^{\infty} (Sx_n, y_n) = \sum_n (Sx_n, T_n y_0) = \sum_n (T_n^* S x_n, y_0) =$$

$$= \sum_n (S T_n^* x_n, y_0) = \bigl(S(\sum_n T_n^* x_n), y_0\bigr) = (Sx_0, y_0) = (Sx_0, Py_0),$$

where P denotes the projection in H onto L. Finally,

$$\|x_0\| \, \|Py_0\| \leqq \|x_0\| \, \|y_0\| \leqq \left(\sum_{n=1}^{\infty} \|x_n\|^2 \right)^{1/2} \left(\sum_{n=1}^{\infty} \|y_n\|^2 \right)^{1/2} < 1 + \varepsilon,$$

which completes the proof.

The proofs of the assertions of Corollary 3 below are either obvious or can be found in [1] (if we note that by [2, Corollary VI. 4.3.7], for every operator T of class $C_0(1)$ we have $\{T\}' = \mathscr{A}(T)$). For the definitions of an attainable family and direct integral see also [1].

Corollary 3. *Let (X, μ) be a measure space and $\{\mathscr{A}(T_x)\}_{x \in X}$ an attainable family of algebras, where T_x is an operator of class $C_0(1)$ for every $x \in X$. Denote by $\mathscr{A}$ the direct integral of the algebras $\mathscr{A}(T_x)$: $\mathscr{A} = \int_X^{\oplus} \mathscr{A}(T_x) \, d\mu(x)$. Let $\mathscr{A}_1$ be a unital ultraweakly closed subalgebra of $\mathscr{A}$. Then,*

(a) *the weak and ultraweak topologies coincide on $\mathscr{A}$,*

(b) *$\mathscr{A}_1$ has property $D_\sigma(1)$,*

(c) *if $\mathscr{A}_2$ is a unital ultraweakly closed subalgebra of $\mathscr{A}$ and* $\mathrm{lat}\, \mathscr{A}_1 \subseteqq \mathrm{lat}\, \mathscr{A}_2$, *then $\mathscr{A}_1 \supseteqq \mathscr{A}_2$,*

(d) *if* $\mathcal{A}_1$ *is reflexive, then every unital ultraweakly closed subalgebra of* $\mathcal{A}_1$ *is reflexive,*

(e) *if* T_x *reflexive for almost every* $x \in X$, *then* $\mathcal{A}_1$ *is reflexive,*

(f) $\mathcal{A}_1 = \mathcal{A}_1'' \cap \operatorname{alg\,lat} \mathcal{A}_1$,

(g) $\mathcal{A}_1^{(2)} = \{S \oplus S \mid S \in \mathcal{A}_1\}$ *is reflexive.*

We now give two examples to illustrate Theorem 2.

Corollary 4. *For every positive integer* n, *let* H_n *be a finite dimensional Hilbert space and* J_n *a Jordan cell in* $B(H_n)$ *(with respect to some orthonormal basis in* H_n*). If* $T_n J_n = J_n T_n$ *and* $T = T_1 \oplus T_2 \oplus \ldots \oplus T_n \oplus \ldots$ *is a bounded opera- then* $\mathcal{A}(T) = \{T\}' \cap \operatorname{alg\,lat} T$.

Proof. It suffices to note that, by a theorem of Brickman and Fillmore [3], for any operator S in a finite dimensional space, $\mathcal{A}(S) = \{S\}' \cap \operatorname{alg\,lat} S$, and then we apply corollary 3 (f).

Next we consider the Volterra operator V defined by $(Vf)(x) = \int_0^x f(y)\,dy$ in $L^2(0, 1)$. It is well known that V is quasinilpotent and unicellular. Foiaş and Williams [7] gave an example of a unicellular operator in $\{V\}'$ whose spectrum contains more than one point. Here we prove that every unicellular operator commuting with V is an ultraweak generator of the commutant of V.

Corollary 5. *Suppose* $\mathcal{B}$ *is an ultraweakly closed unital algebra strictly contained in* $\{V\}'$. *Then* $\operatorname{lat} \mathcal{B}$ *is not a chain. In particular, if* T *commutes with* V *and* T *is unicellular, then* $\mathcal{A}(T) = \{V\}'$.

Proof. Sarason [4] pointed out that V commutes with the operator $S(m)$ where $m(\lambda) = \exp\{(\lambda+1)/(\lambda-1)\}$. Since $\{S(m)\}' = \mathcal{A}(S(m))$, $V \in \mathcal{A}(S(m))$. On the other hand, by [5], the commutant of V is the weak closure of the polynomials of V, so that $\{V\}' = \mathcal{A}(V) \subseteq \mathcal{A}(S(m))$. Now suppose $\operatorname{lat} \mathcal{B}$ is a chain. Since $\operatorname{lat} V$ is a maximal chain, $\operatorname{lat} V = \operatorname{lat} \mathcal{B}$. By Corollary 3 (c), $\mathcal{B}$ coincides with $\{V\}'$. The obtained contradiction proves the assertion.

Now we note that Theorem 2 yields the following factorization theorem. By H_0^1 we denote the subspace of H^1 consisting of the functions vanishing at 0.

Theorem 6. *Let* m *be an inner function,* $f, g \in H(m) = H^2 \ominus mH^2$ and $\inf\{\|f\bar{g} - h\|_1 \mid h \in H_0^1\} \leq 1$. *Then for every* $\varepsilon > 0$, *there exist* $f_1, g_1 \in H(m)$ *such that* $\|f_1\|_2 \|g_1\|_2 < 1 + \varepsilon$ *and* $f\bar{g} - f_1 \bar{g}_1 \in H_0^1$.

Proof. Let $T = PU \mid H(m)$, where P is the projection onto $H(m)$, and denote by φ the functional on $\mathcal{A}(T)$ defined by $\varphi(X) = (Xf, g)$. Choose $X \in \mathcal{A}(T)$ with $\|X\| = 1$. Then, by [2, Corollary VI. 4.3.7], there exists an analytic Toeplitz operator Y, $\|Y\|=1$, such that $PY\,|\,H(m)=X$. Since L^1/H_0^1 is the pre-dual of H^∞,

$$|\varphi(X)| = |(Xf, g)| = |(Yf, g)| \leqq 1,$$

so that $\|\varphi\|\leqq 1$. By Theorem 2, φ can be represented in the form $\varphi(X)=(Xf_1, g_1)$ where $f_1, g_1 \in H(m)$ and $\|f_1\|_2\|g_1\|_2<1+\varepsilon$. But then, for every $Y\in\{U\}'$,

$$(Yf_1, g_1) = (PYf_1, g_1) = (PYf, g) = (Yf, g),$$

so that $f\bar{g}-f_1\bar{g}_1\in H_0^1$.

In Proposition 7 below we shall prove that for certain operator algebras the possibility of such "factorization" implies property $D_\sigma(r)$. $BC(H)$ denotes the ideal of compact operators in $B(H)$. $\omega_{x,y}$ is a functional on $B(H)$ defined by $\omega_{x,y}(T)=(Tx, y)$.

Proposition 7. *Let $\mathcal{A}$ be a unital ultraweakly closed operator algebra. Suppose that $\mathcal{A}'$ has a cyclic vector and that $\mathcal{A}\cap BC(H)$ is ultraweakly dense in $\mathcal{A}$. Then $\mathcal{A}$ has property $D_\sigma(r)$ if and only if for every $\varepsilon>0$ and every pair $x, y\in H$ such that $\|\omega_{x,y}\,|\,\mathcal{A}\|\leqq 1$ there are $\xi, \eta\in H$ such that $\|\xi\|\,\|\eta\|<r+\varepsilon$ and $\omega_{x,y}=\omega_{\xi,\eta}$ on $\mathcal{A}$.*

Proof. The "only if" part is obvious. Let us prove the "if" part. Choose an arbitrary $\varepsilon>0$ and let φ be an ultraweakly continous functional on $\mathcal{A}$ with $\|\varphi\|\leqq 1$. By Theorem 1, φ can be represented in the form $\varphi(T)=\sum_{n=1}^{\infty}(Tx_n, y_n)$ where $\left(\sum_{n=1}^{\infty}\|x_n\|^2\right)^{1/2}\left(\sum_{n=1}^{\infty}\|y_n\|^2\right)^{1/2}<1+\varepsilon$. Choose a number N such that $\sum_{n=N+1}^{\infty}(\|x_n\|^2+\|y_n\|^2)<\varepsilon$. If x_0 is a cyclic vector for $\mathcal{A}'$, there exist $\{T_i\}_{i=1}^N$ in $\mathcal{A}'$ satisfying the inequalities

$$\|T_i x_0 - x_i\| < \varepsilon/2\left(\sum_{i=1}^{N}\|y_i\|\right)^{-1} \quad (i = 1, 2, \ldots, N).$$

Then for every $T\in\mathcal{A}$ with $\|T\|\leqq 1$,

$$\left|\varphi(T)-\left(Tx_0, \sum_{i=1}^{N}T_i^* y_i\right)\right| \leqq \left|\sum_{i=1}^{N}(Tx_i, y_i)-\left(Tx_0, \sum_{i=1}^{N}T_i^* y_i\right)\right|+\left|\sum_{i=N+1}^{\infty}(Tx_i, y_i)\right| \leqq$$

$$\leqq \left|\sum_{i=1}^{N}(T(x_i-T_ix_0), y_i)\right|+\left|\sum_{i=N+1}^{\infty}(Tx_i, y_i)\right| \leqq$$

$$\leqq \sum_{i=1}^{N}\|x_i-T_ix_0\|\,\|y_i\|+\sum_{i=N+1}^{\infty}\|x_i\|\,\|y_i\| < \varepsilon/2+\varepsilon/2 = \varepsilon.$$

Hence it follows that the set of functionals $\omega_{x,y}$ is norm-dense in the family of all ultraweakly continuous functionals on $\mathcal{A}$. Let ω_{x_n,y_n} be a sequence converging to φ in the norm topology. We may assume that $\|\omega_{x_n, y_n} \mid \mathscr{A}\| < 1+\varepsilon$. Choose ξ_n, η_n in H such that $\omega_{x_n, y_n} = \omega_{\xi_n, \eta_n}$ on $\mathscr{A}$ and $\|\xi_n\| = \|\eta_n\| < r^{1/2}(1+\varepsilon)^{1/2}$. Passing to subsequences, we may suppose that $\xi_n \xrightarrow{w} \xi$, $\eta_n \xrightarrow{w} \eta$ for some ξ, η in H. If $K \in \mathscr{A} \cap BC(H)$, $\varphi(K) = \lim_n (K\xi_n, \eta_n)$. On the other hand,

$$|(K\xi_n, \eta_n) - (K\xi, \eta)| \leqq |(K(\xi_n - \xi), \eta_n)| + |(K\xi, \eta_n - \eta)| \leqq$$
$$\leqq r^{1/2}(1+\varepsilon)^{1/2} \|K(\xi_n - \xi)\| + |(K\xi, \eta_n - \eta)| \to 0,$$

so that $\varphi(K) = (K\xi, \eta)$. Now if $T \in \mathscr{A}$ and if $\{K_\alpha\}$ is a net of compact operators in $\mathscr{A}$ which converges ultraweakly to T, then $\varphi(T) = \lim_\alpha \varphi(K_\alpha) = \lim_\alpha (K_\alpha \xi, \eta) = (T\xi, \eta)$. Finally, since the norm on any Hilbert space is lower semicontinuous, we have $\|\xi\| \, \|\eta\| \leqq r(1+\varepsilon)$.

3. Reductive contractions with rich spectra. The properties D_σ or $D_\sigma(r)$ might be very useful in applications to various problems on invariant subspaces. Let us introduce the following definition. If G is an open non-empty subset of the complex plane, we say that $\sigma \subseteqq \mathbf{C}$ is *rich in* G if for every h in $H^\infty(G)$,

$$\sup_{z \in G} |h(z)| = \sup_{z \in \sigma \cap G} |h(z)|$$

where $H^\infty(G)$ denotes, as usual, the algebra of bounded functions analytic in G.

Recently, Brown, Chevreau and Pearcy [8] proved that if T is contraction in $B(H)$ whose spectrum $\sigma(T)$ is rich in the open unit disk D, then lat T is not trivial. Recall that an operator T is called *reductive* if lat T = lat T^*. Here we show that if $\sigma(T)$ is sufficiently large, then T cannot be reductive.

Theorem 8. *Let $T \in B(H)$ be a contraction. Suppose that $\sigma(T)$ has the following property: if $\sigma(T) = \sigma_1 \cup \sigma_2$ where σ_1 and σ_2 are closed subsets of* cl D, *then either there exists a non-empty open set G such that σ_1 is rich in G, or σ_2 is rich in D. Then T is not reductive.*

Proof. Clearly we may assume that T is completely non-unitary. Suppose T is reductive. Let H_1 denote the subspace of H spanned by all eigenvectors of T. Then $H_1 \in \operatorname{lat} T$, so that $H_2 = H_1^\perp \in \operatorname{lat} T$. Then $T = T_1 \oplus T_2$, where $T_i = T \mid H_i$. Clearly $\sigma(T) = \sigma(T_1) \cup \sigma(T_2)$. We claim that T_1 is normal. Indeed, since T is reductive, so is T_1, and every eigenvector of T_1 is also an eigenvector of T_1^*. Thus, if $x_1, x_2, \ldots, x_n$ is any finite set of eigenvectors of T_1 (and T_1^*), then $T_1 T_1^* \left(\sum_{k=1}^n \alpha_k x_k\right) = T_1^* T_1 \left(\sum_{k=1}^n \alpha_k x_k\right)$ for all scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$. Since H_1 is spanned by eigenvectors, we conclude that $T_1 T_1^* = T_1^* T_1$, that is T_1 is normal. Now T_1 is a normal reductive operator whose set of eigenvectors is total in H_1. By Sarason [6], there is no non-empty open set G such that $\sigma(T_1)$ is rich in G.

By assumption $\sigma(T_2)$ is rich in D. Now T_2 is a reductive completely non-unitary contraction with rich spectrum in D. Then $\sigma(T_2)$ coincides with the left essential spectrum of T_2, for otherwise, as pointed out in [8], T_2 or T_2^* (and therefore both of them) has an eigenvector, which contradicts the definition of H_1. Now by [8], T_2 has property D_σ and there exist a non-zero multiplicative ultraweakly continuous functional φ on $\mathcal{A}(T_2)$. Let

$$\varphi(S) = (Sx, y), \quad S \in \mathcal{A}(T_2).$$

Let $\mathcal{I}$ denote the null-space of φ. Then $\mathcal{I}$ is an ideal in $\mathcal{A}(T_2)$ such that the subspace $M = \operatorname{cl} \mathcal{I}x = \operatorname{cl}\{Sx, S \in \mathcal{I}\}$ is in $\operatorname{lat} T_2$. On the other hand, x is not in M, for $y \in M^\perp$, but $\varphi(I) = (x, y) \neq 0$. If we denote by N the subspace spanned by x and M, then $N \in \operatorname{lat} T_2$ and $\dim(N \ominus M) = 1$. Since T_2 is reductive, $N \ominus M \in \operatorname{lat} T_2$, which again contradicts the definition of H_1. This contradiction leads to the desired conclusion.

Corollary 9. *If T is a contraction and $\sigma(T)$ is an annulus $\{z \mid r \leqq |z| \leqq 1\}$, $0 \leqq r < 1$, then T is not reductive.*

Proof. Suppose $\sigma(T) = \sigma_1 \cup \sigma_2$ with $\sigma_1, \sigma_2 \subseteqq \operatorname{cl} D$ and σ_2 not rich in D. Then there are $\lambda \in \mathbf{C}$, $|\lambda| = 1$ and $\varepsilon, 0 < \varepsilon < 1 - r$ such that $\{z \,\big|\, |z - \lambda| < \varepsilon\} \cap \sigma_2 = \emptyset$. But then $G = \{z \,\big|\, |z - \lambda| < \varepsilon\} \cap D \subseteqq \sigma_1$, so that σ_1 is rich in G, which completes the proof.

Of course, an example of a rich subset of D not satisfying the conditions of Theorem 8 can be easily constructed.

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On ranges of adjoint operators in Hilbert space

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Introduction

Let A be a given densely defined operator in the (complex) Hilbert space H. Let further y and z be elements in H. The relation

$$(Ax, z) = (x, y) \quad (x \in \mathscr{D}(A)), \tag{1}$$

where $\mathscr{D}(A)$ stands as usual for the domain of A, is fundamental for the definition of A^*, the adjoint of A. Namely, z is in $\mathscr{D}(A^*)$ if

$$\sup \{|(Ax, z)|: x \in \mathscr{D}(A), \|x\| \leqq 1\} < \infty$$

holds, that is by the Riesz Representation Theorem if and only if there is an y in H satisfying (1). The reverse problem is the characterization of $\mathscr{R}(A^*)$, the range of $A^* : y$ is in $\mathscr{R}(A^*)$ if there is an element, z in $\mathscr{D}(A^*)$ for which (1) holds. We shall show that this is the case if and only if

$$\sup \{|(x, y)|: x \in \mathscr{D}(A), \|Ax\| \leqq 1\} < \infty$$

holds (Theorem 1).

As an application we obtain results concerning the factorization of a given densely defined operator C in H in the form $C \subset A^*B$ by which we mean that B is an operator in H defined at least on $\mathscr{D}(C)$, and for any x in $\mathscr{D}(C)$, $Bx \in \mathscr{D}(A^*)$ and $Cx = A^*(Bx)$. In general, as a Zorn's argument shows, $\mathscr{R}(A^*) \supset \mathscr{R}(C)$ is sufficient for such a factorization, but we produce a minimal B in the sense that

$$\|Bx\| \leqq \|u\| \quad \text{for} \quad x \in \mathscr{D}(C),\ u \in \mathscr{D}(A^*);\ Cx = A^*u. \tag{2}$$

The question of the boundedness of B is also analyzed in the hope that we shall be able to answer the question raised by R. G. DOUGLAS [1] concerning the factorization of unbounded operators, especially with a bounded cofactor.

Our constant reference is [2].

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Results

Theorem 1. *Let y and A be a unit vector and a densely defined operator, respectively, in a Hilbert space H. The following two assertions are equivalent:*

(i) *There exists a unique vector z in H such that*

$$y = A^*z \quad \text{and} \quad \|z\| \leqq \|u\| \quad \text{for} \quad u \in \mathscr{D}(A^*),\ y = A^*u. \tag{3}$$

(ii) $M_y := \sup \{|(x, y)|:\ x \in \mathscr{D}(A),\ \|Ax\| \leqq 1\} < \infty.$

If (i) *and* (ii) *are valid, then* $M_y = \|z\|$.

Proof. (ii) simply follows from (i) since for any x in $\mathscr{D}(A)$,

$$|(x, y)| = |(x, A^*z)| = |(Ax, z)| \leqq \|z\| \cdot \|Ax\|;$$

we see also that $M_y \leqq \|z\|$.

(ii) implies (i): Assuming (ii) we have $|(x, y)| \leqq M_y \|Ax\|$ for any x in $\mathscr{D}(A)$. So the map $Ax \mapsto (x, y)$ is a bounded linear functional on $\mathscr{R}(A)$. It has a unique bounded linear extension to $\overline{\mathscr{R}(A)}$, the norm closure of $\mathscr{R}(A)$. By the Riesz Representation Theorem there exists a unique vector, z, in $\overline{\mathscr{R}(A)}$ for which (1) holds. Then z is in $\mathscr{D}(A^*)$ and $y = A^*z$.

If $u \neq z$ is from $\mathscr{D}(A^*)$ and $y = A^*u$, then $(Ax, z) = (x, A^*z) = (x, A^*u) = (Ax, u)$ for every x in $\mathscr{D}(A)$. Since z is, while u is not in $\overline{\mathscr{R}(A)}$, it follows that

$$\begin{aligned} \|z\| &= \sup \{|(Ax, z)|:\ x \in \mathscr{D}(A),\ \|Ax\| \leqq 1\} = \\ &= \sup \{|(Ax, u)|:\ x \in \mathscr{D}(A),\ \|Ax\| \leqq 1\} < \|u\| \end{aligned}$$

Thus (3) holds and the z with this property is unique. The proof is complete.

Theorem 2. *Let A and C be densely defined operators in a Hilbert space H. The following three assertions are equivalent:*

(i) *There exists an operator B in H such that*

$$C \subset A^*B \quad \text{and } B \text{ fulfils (2).} \tag{4}$$

(ii) $\mathscr{R}(C) \subset \mathscr{R}(A^*)$.

(iii) $M_y(C) := \sup \{|(x, Cy)|:\ x \in \mathscr{D}(A),\ \|Ax\| \leqq 1\} < \infty \quad (y \in \mathscr{D}(C))$.

Proof. (i) clearly implies (ii). Further (ii) implies (iii) since for any y in $\mathscr{D}(C)$ there exists (by assumption) a u in $\mathscr{D}(A^*)$ such that $Cy = A^*u$ whence for any x in $\mathscr{D}(A)$,

$$|(x, Cy)| = |(x, A^*u)| = |(Ax, u)| \leqq \|u\| \cdot \|Ax\|,$$

and thus (iii) follows.

Lastly assume (iii) and prove (i). For a fixed x in $\mathscr{D}(C)$ there exists, by Theorem 1, a unique vector z in H such that (3) holds with $y = Cx$. Writing

$z=Bx$ we get just (2) as desired. We have to show only that Bx is a linear function of x if x varies on $\mathscr{D}(C)$.

Recall that, as the proof of Theorem 1 indicates, Bx is in $\overline{\mathscr{R}(A)}$ for any x in $\mathscr{D}(C)$. Thus if x, x' are arbitrary vectors from $\mathscr{D}(C)$, for any y belonging to $\mathscr{D}(A)$ we have

$$0 = \big(C(x+x'), y\big)-(Cx, y)-(Cx' y) = \big(A^*B(x+x'), y\big)-(A^*Bx, y)-(A^*Bx', y) =$$
$$= \big(B(x+x'), Ay\big)-(Bx, Ay)-(Bx', Ay) = \big(B(x+x')-Bx+Bx', Ay\big),$$

which shows that $B(x+x')=Bx+Bx'$. The proof of $B(\lambda x)=\lambda Bx$ for a scalar λ is similar. The proof is complete.

The following is analogous to [1, Theorem 2, (3)] due to Douglas.

Corollary 1. *If C of Theorem 2 is closed then*

$$\sup\{\|Bx\|:\ x\in\mathscr{D}(C),\ \|x\|+\|Cx\|\leqq 1\}<\infty.$$

In particular, B is bounded if C is.

Proof. By assumption, C has a closed graph. Hence we have to show that the linear operator given by $\{x, Cx)\mapsto Bx\ \big(x\in\mathscr{D}(C)\big)$ also has a closed graph. In other words, assuming that $x_n\to x$, $Cx_n\to Cx$ and $Bx_n\to u$, we must conclude $u=Bx$. Since $Cx_n\to Cx$ means that $A^*Bx_n\to Cx$, by the closedness of A^* we get $A^*u=Cx=A^*Bx$. But since Bx_n is in $\overline{\mathscr{R}(A)}$, u is in $\overline{\mathscr{R}(A)}$, too. As $(Ay, u)=$ $=(y, A^*u)=(y, A^*Bx)=(Ay, Bx)$ for every $y\in\mathscr{D}(A)$, it follows that

$$\|u\| = \sup\{|(Ay, u)|:\ y\in\mathscr{D}(A),\ \|Ay\|\leqq 1\} =$$
$$= \sup\{|(Ay, Bx)|:\ y\in\mathscr{D}(A),\ \|Ay\|\leqq 1\}\leqq\|Bx\|,$$

whence by the uniqueness of Bx we have $u=Bx$ indeed.

Remark 1. If in Theorem 2 the operator A is bounded and C is closed, further if we take $\mathscr{D}(B)=\mathscr{D}(C)$, then B is closed. Indeed, if $x_n\to x$ and $Bx_n\to u$, where $x_n\in\mathscr{D}(C)$ $(n=1, 2, \ldots)$, then $Cx_n=A^*Bx_n\to A^*u$ so that $A^*u=Cx=A^*Bx$, and an argument similar to that appearing in the proof of Corollary 1 shows $u=Bx$.

Theorem 3. *The following four assertions are equivalent:*

(i) *The operator B in Theorem* 2 (i) *is bounded.*

(ii) $\mathscr{R}(A^*)\supset\mathscr{R}(C)$ *and*

$$\sup\{\inf[\|z\|:\ z\in\mathscr{D}(A^*),\ Cy=A^*z]:\ y\in\mathscr{D}(C),\ \|y\|\leqq 1\}<\infty.$$

(iii) $\sup\{|(x, Cy)|:\ x\in\mathscr{D}(A),\ \|Ax\|\leqq 1,\ y\in\mathscr{D}(C),\ \|y\|\leqq 1\}<\infty.$

(iv) $\mathscr{D}(C^*)\supset\mathscr{D}(A)$ *and*

$$\sup\{\|C^*x\|:\ x\in\mathscr{D}(A), \|Ax\|\leqq 1\}<\infty.$$

Proof. Assume first (i). We know from Theorem 2 that for any y in $\mathcal{D}(C)$,

$$\inf [\|z\|:\ z \in \mathcal{D}(A^*),\ Cy = A^*z] = \|By\| \leqq \|B\|\, \|y\|$$

holds. This proves (ii). But (ii) implies (iii) since we know also from Theorem 2 that

$$\sup \{|(x, Cy)|:\ x \in \mathcal{D}(A),\ \|Ax\| \leqq 1\} = \inf [\|z\|:\ z \in \mathcal{D}(A)^*,\ Cy = A^*z]$$

for any y in $\mathcal{D}(C)$. For the same reason (iii) implies (i). But (iv) also follows from (iii) since by (iii) $\mathcal{D}(A) \subset \mathcal{D}(C^*)$ and since for any x in $\mathcal{D}(A)$,

$$\|C^*x\| = \sup \{|(C^*x, y)|:\ y \in \mathcal{D}(C),\ \|y\| \leqq 1\} =$$
$$= \sup \{|(x, Cy)|:\ y \in \mathcal{D}(C),\ \|y\| \leqq 1\}.$$

Finally (iv) implies (iii) since for any x in $\mathcal{D}(A)$, x is in $\mathcal{D}(C^*)$ and

$$|(x, Cy)| = |(C^*x, y)| \leqq \|C^*x\| \cdot \|y\|$$

holds for any y in $\mathcal{D}(C)$.

Remark 2. Assuming that A^* is densely defined or, what is the same, that A^{**} exists, assertions (i)—(iv) in Theorem 3 are equivalent to

(iv)′ $\sup \{\|C^*x\|:\ x \in \mathcal{D}(A^{**}),\ \|A^{**}x\| \leqq 1\} < \infty$.

Indeed, since $A^{**} \supset A$ in this case, (iv)′ implies (iv). On the other hand, (i) implies now that $C^* \supset (A^*B)^* \supset B^*A^{**}$ and that

$$\|C^*x\| = \|B^*A^{**}x\| \leqq \|B^*\| \cdot \|A^{**}x\|$$

holds for any x in $\mathcal{D}(A^{**})$, which proves (iv)′.

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Restrictions of positive operators

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The aim of this note is to give a necessary and sufficient condition for the existence of a positive linear operator on Hilbert space whose restriction to a subset of this space is given. As an application we have a result on division of Hilbert space operators. Krein's theorem on the extension of a bounded symmetric operator from a subspace to the whole space is also established.

Theorem. *Let H be a (complex) Hilbert space, H_0 its subset, and b a function on H_0 with values in H. There exists a positive operator B on H with restriction to H_0 identical to b if and only if*

$$\|\sum_h c_h b(h)\|^2 \leqq M(\sum_h c_h b(h), \sum_h c_h h) \tag{1}$$

holds with some constant $M \geqq 0$ for any finite sequence $\{c_h\}_{h \in H_0}$ of complex numbers indexed by elements of H_0. In this case, $\|B\| \leqq M$.

Proof. The necessity of condition (1) is a simple consequence of a property of positive operators:

$$\|\sum_h c_h b(h)\|^2 = \|B(\sum_h c_h h)\|^2 \leqq \|B\|(B(\sum_h c_h h), \sum_h c_h h) =$$
$$= \|B\|(\sum_h c_h b(h), \sum_h c_h h).$$

Hence (1) holds with $M = \|B\|$ for any finite sequence $\{c_h\}_{h \in H_0}$ of complex numbers.

Conversely, assume that (1) is valid for arbitrary finite sequences $\{c_h\}_{h \in H_0}$ of complex numbers. Since the linear span of H_0 in H, say X, consists of elements $\sum_h c_h h$ with such coefficients, we can introduce a semi-definite inner product on X by

$$\langle \sum_h c_h h, \sum_k d_k k \rangle := (\sum_h c_h b(h), \sum_k d_k k)$$

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for elements $\sum_h c_h h$ and $\sum_k d_k k$ of X. It is well defined because $\sum_h c_h h=0$ implies $\sum_h c_h b(h)=0$, in view of (1). As usual, we get a Hilbert space K from X by factorizing X with respect to the null space of $\langle\cdot,\cdot\rangle$ and by completing with respect to the norm arising on the factor space. For simplicity we denote the image in K of an element of X by the same symbol. With this convention there is a continuous linear operator V from K into H given by

$$V\bigl(\sum_h c_h h\bigr) = \sum_h c_h b(h)$$

for any element $\sum_h c_h h$ of X. Indeed, according to (1), V is well defined and has norm $\leqq\sqrt{M}$. We are going to prove that $B=VV^*$ is the desired positive operator on H. To see this it is enough to prove that

(2) $$V^*k = k \quad \text{for any } k \text{ in } H_0,$$

since then $Bk=VV^*k=Vk=b(k)$ $(k\in H_0)$.

To prove (2) we see

$$\bigl\langle\sum_h c_h h, V^*k\bigr\rangle = \bigl(V\bigl(\sum_h c_h h\bigr), k\bigr) = \bigl(\sum_h c_h b(h), k\bigr) = \bigl\langle\sum_h c_h h, k\bigr\rangle$$

for any element $\sum_h c_h h$ in K. Since these elements are dense in K, the statement follows. The proof of the Theorem is complete.

Corollary 1. *Let A and C be bounded linear operators on the Hilbert space H. There exists a positive operator B on H such that $A=BC$ if and only if there exists a constant $M\geqq 0$ such that*

(3) $$A^*A \leqq M\cdot C^*A$$

Proof. For $h=Ck$, k in H, let $b(h)=Ak$. Then (1) takes the form

$$\|Ak\|^2 \leqq M(Ak, Ck) = M(C^*Ak, k)$$

for any k in H, which is the same as (3).

Remark. If b is a linear map of a subspace $H_0\subset H$ into H such that for some constant $M\geqq 0$

(4) $$\|b(h)\|^2 \leqq M\bigl(b(h), h\bigr) \quad (h\in H_0),$$

then b has a positive extension B defined on H. This is a consequence of the Theorem. On the other hand, the usual condition for positivity

(5) $$0 \leqq \bigl(b(h), h\bigr) \quad (h\in H_0)$$

is not enough for the existence of such a positive extension: a simple example is the case when $\bigl(b(h), h\bigr)=0\neq b(h)$ for some element h in H_0.

Corollary 2. *Let b a function on a subset H_0 of the Hilbert space H with values in H. There exists a self-adjoint operator B on H such that $m \cdot I \leqq B \leqq M \cdot I$, where $m \leqq M$ are real constants, if and only if*

$$\Big\|\sum_h c_h\big(b(h)-m\cdot h\big)\Big\|^2 \leqq (M-m)\Big(\sum_h c_h\big(b(h)-m\cdot h\big),\ \sum_h c_h h\Big) \tag{6}$$

holds for any finite sequence $\{c_h\}$ of complex numbers indexed by elements of H_0.

Corollary 3 (M. G. Kreĭn, cf. [1]). *Let b a symmetric and bounded linear operator from a subspace H_0 of a Hilbert space H into H. Then there exists a self-adjoint extension of b to the whole space H with the same bound.*

Proof. Let M be the norm of the operator b, that is,

$$M = \sup \{\|b(h)\|:\ h \in H_0,\ \|h\| \leqq 1\}.$$

We have then for any h in H_0

$$\|b(h)+M\cdot h\|^2 = \|b(h)\|^2+2M\cdot\big(b(h),h\big)+M^2\cdot\|h\|^2 \leqq$$
$$\leqq 2M\cdot\big(b(h),h\big)+2M^2\cdot\|h\|^2 = 2M\big(b(h)+Mh,h\big).$$

But this is nothing else than (6) in case $-m=M$ and b is a linear function, an operator. As a consequence, $-M\cdot I\leqq B\leqq M\cdot I$ holds for a self-adjoint extension B of b to the space H. This was to be proved.

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A note on a paper of S. Watanabe

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In his paper WATANABE [1] asked (p. 38) if every closed positive linear map $\Phi: A_0 \to B$ (A_0 is a star subalgebra of a unital C^*-algebra containing the unit, B is a C^*-algebra and $\Phi(x^*x) \geqq 0$ for $x \in A_0$) is automatically continuous. He proved that when Φ is 2-positive. In the general case, too, the answer is "yes". The proof (similar to that of Theorem 1 in [1]) is based on the lemma of [1] on p. 37 and on a corollary from the following theorem of Palmer.

Theorem (T. PALMER [2]). *Let A be a complex unital Banach *-algebra with continuous involution and $H = \{x: x \in A, x = x^*\}$, $E = \{e^{ih}: h \in H\}$. If the set E is bounded, the algebra A is C^*-equivalent.*

Corollary. *In the above notations, if the set $K = \{u^2: u \in H, u^2 + v^2 = \mathbf{1}$ for some $v \in H\}$ is bounded, A is C^*-equivalent.*

Proof. We have $e^{ih} = \cos(h) + i \sin(h)$ $(h \in H)$ where $\cos(h) = (e^{ih} + e^{-ih})/2$, $\sin(h) = (e^{ih} - e^{-ih})/(2i) \in H$ and $\cos(h)^2 + \sin(h)^2 = \mathbf{1}$. If $\|\sin(h)^2\| \leqq N$ (a constant) for every $h \in H$, we obtain that $\|\cos(h)\| = \|\mathbf{1} - 2\sin(h/2)^2\| \leqq 2N+1$, $\|\sin(h)\| = \|\cos(\pi\mathbf{1}/2 - h)\| \leqq 2N+1$. Hence $\|e^{ih}\| \leqq 2(2N+1)$ and A is C^*-equivalent.

Now following the lines of the proof of Theorem 1 in [1] we obtain the modification

Theorem 1′. *Let Φ be a closed linear map of A_0 into a Banach space B. If Φ is norm bounded on the set K (defined for A_0, see the corollary above and the lemma in* [1]*), then A_0 is a C^*-algebra (the original C^*-norm in A_0 turns out to be equivalent to the graph norm in it) and Φ is bounded.*

When B is a C^*-algebra and Φ is positive (we need only $\Phi(x^2) \geqq 0$ when $x = x^* \in A_0$), this is fulfilled: if $u^2 + v^2 = \mathbf{1}$ (u, v are hermitian in A_0), it follows that $\Phi(u^2) + \Phi(v^2) = \Phi(\mathbf{1})$, hence $\|\Phi(u^2)\| \leqq \|\Phi(\mathbf{1})\|$, i.e., Φ is bounded on K.

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Conditions for hermiticity and for existence of an equivalent C*-norm

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The author has found a sufficient condition for a self-adjoint element in a Banach *-algebra to have purely real spectrum. This is contained in Theorem 1 below. Using this result it becomes possible to prove that a fairly weak condition provides for the existence of an equivalent C^*-norm (see Theorem 2).

The problem discussed here is a version of the Araki—Elliott problem. ARAKI and ELLIOTT [3] proved in 1973 that if the B^*-condition

$$\|a^*a\| = \|a^*\| \cdot \|a\|$$

holds for a linear norm and the * is continuous, then it is a C^*-norm. They conjectured that the continuity of the involution is also a consequence of the B^*-condition. Z. SEBESTYÉN and the author [4] verified this conjecture, and gave a condition for a norm to be a C^*-norm which can hardly be weakened.

We shall use [1] without further reference.

Theorem 1. *Let $\mathscr{A}$ be a Banach *-algebra, and let r be the spectral radius in it. Consider a self-adjoint element $h(\in\mathscr{A})$. Let $\langle h\rangle$ be the algebra generated by h. Assume there are a seminorm p on $\langle h\rangle$ and constants $0<M_1\leqq M_2$ such that*

(i) $M_1^2\cdot r(a^*a)\leqq p(a^*)\cdot p(a)\leqq M_2^2\cdot r(a^*a)$ *for all* $a\in\langle h\rangle$.

Then $\mathrm{Sp}\,(h)\subset\mathbf{R}$ *or* $\mathrm{Sp}\,(h)\subset\{0, w, \bar{w}\}$ *with a suitable* $w\in\mathbf{C}$. *Further, if p is a norm then* $\mathrm{Sp}\,(h)\subset\mathbf{R}$. ("Sp" denotes the spectrum in $\mathscr{A}$.)

The proof will consist of two parts. Part I contains independent propositions with independent notations. Then we shall prove Theorem 1 in Part II utilizing the results of the previous part.

Part I. We start with an easy lemma.

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Lemma 1.1. *Let* $\mathcal{A}$ *be a* **-algebra,* p, r *be seminorms on it such that* $r(a^2)=r(a)^2$, $r(a^*)=r(a)$ *and*

$$M_1^2\cdot r(a^*a) \leqq p(a^*)\cdot p(a) \leqq M_2^2\cdot r(a^*a) \quad \textit{for all} \quad a\in\mathcal{A}. \tag{1}$$

Then the following also hold:

$$M_1\cdot r(h) \leqq p(h) \leqq M_2\cdot r(h) \quad \textit{if} \quad h=h^*\in\mathcal{A}, \tag{2}$$

$$p(a) \leqq 2M_2\cdot r(a) \quad \textit{for all} \quad a\in\mathcal{A}. \tag{3}$$

Proof. Writing $a=h$, $a^*=h$, (2) is immediate from the properties of r. For an arbitrary element a consider the real and imaginary part of a, that is, $h=2^{-1}(a+a^*)$, $k=(2i)^{-1}(a-a^*)$. Then $r(a^*)=r(a)$ implies $r(k)\leqq r(a)$, $r(h)\leqq r(a)$, and so (3) follows from (2).

We call a set $K\subset\mathbf{C}$ symmetric if it is stable under conjugation, i.e. $\bar{z}\in K$ if $z\in K$. In the remainder of this part let K be a fixed symmetric non-void compact subset of the complex plain. Denote by $C(K)$ the algebra of continuous functions on K, and by r the customary sup-norm in $C(K)$. Define an involution in $C(K)$ setting $f^*(z)=\overline{f(\bar{z})}$. This definition is correct and this involution is norm-preserving, since K is symmetric.

Let $A\subset C(K)$ be the polynomials without constant terms. This is a *-subalgebra. Consider the following condition: there are a seminorm p on A and constants $0<M_1\leqq M_2$ such that

$$M_1^2\cdot r(f^*f) \leqq p(f^*)\cdot p(f) \leqq M_2^2\cdot r(f^*f) \quad \text{for all} \quad f\in A. \tag{P1}$$

Our goal is to prove that this condition implies that the shape of K is very special (see Propositions 1.2 and 1.5 below).

First we list some immediate consequences of (P1). We see from Lemma 1.1 that

$$M_1\cdot r(h) \leqq p(h) \leqq M_2\cdot r(h) \quad \text{if} \quad h=h^*\in A, \tag{P2}$$

$$p(f) \leqq 2M_2\cdot r(f) \quad \text{for all} \quad f\in A. \tag{P3}$$

Let B be the norm-closure of A in $C(K)$. Because of (P3) p has a unique continuous extension to B, which will also be denoted by p. Then this extended p will also be a seminorm and (P1), (P2), (P3) remain valid on B.

Notation. We say that a set $T\subset\mathbf{C}$ is a *cross* if there is a real number s such that $T\subset\mathbf{R}\cup\{s+it;\ t\in\mathbf{R}\}$.

Proposition 1.2. (P1) *implies that* K *is a cross.*

Proof. Suppose the contrary. Then we shall find f, g in B with $p(f)+p(g)<p(f+g)$, which is a contradiction. We need two lemmas for this.

Denote by C (resp. β) the maximum of $|z|$ (resp. $\operatorname{Im} z$) on K. Note that $C, \beta > 0$ because K is symmetric and not a cross. Let $\alpha \in \mathbf{R}$ be such that $\alpha + i\beta \in K$. Write $w_1 = \alpha + i\beta$, $w_2 = \bar{w}_1$, $m = |w_1|$.

Lemma 1.3. *For any $n \in \mathbf{R}$ there are a, b in B such that*
(4) $r(a^*a), r(b^*b) \leqq C^2$, (5) $r(a) = r(b) > n$, (6) $|b(w_1)| = |b(w_2)| = m$,
(7) $|a(w_1)| \geqq mC^{-1} \cdot r(a)$, (8) $|a(w_2)| < 2^{-1}m$.

Proof. Let $a_t(z) = z \cdot \exp(-it(z-\alpha))$, $b_t(z) = z \cdot \exp(-it(z-\alpha)^2)$ where t is real and $z \in K$. Then $a_t, b_t \in B$ for all t. Since K is not a cross, there is a $u = \gamma i\delta \in K$ such that $\gamma \neq \alpha$ and $\delta \neq 0$ ($\gamma, \delta \in \mathbf{R}$). Thus $|b_t(u)| = |u| \cdot \exp(2t(\gamma - \alpha)\delta)$ and hence there is a t for which $|b_t(u)| > n$. Let $b = b_t$ with such a t.

Since $|a_t(w_1)| = m \cdot \exp(t\beta)$, $|a_t(w_2)| = m \cdot \exp(-t\beta)$, there is a $t > 0$ with $|a_t(w_2)| < 2^{-1}m$, $r(a_t) > r(b)$. With such a t let $a = r(b) r(a_t)^{-1} a_t$. It is easy to check that (4)—(8) hold for this a, b (for (7) use that β is the maximum of $\operatorname{Im} z$ on K).

Lemma 1.4. *Assume that for an $a \in B$ the condition*

$$(9) \qquad r(a^*a)^{1/2} \leqq C \leqq 2^{-1} \cdot r(a)$$

holds. Then there is a constant L (e.g. $L = 4M_2^2 C^2 M_1^{-1}$ is appropriate) such that

$$(10) \qquad \min(p(a), p(a^*)) \leqq L \cdot r(a)^{-1}.$$

Proof. Choosing z in K with $r(a) = a(z)$ we have by (9)

$$|a^*(z)| \leqq C^2 \cdot r(a)^{-1} \leqq 2^{-1} C \leqq 4^{-1} \cdot r(a),$$

and thus

$$r(a+a^*) \geqq |(a+a^*)(z)| \geqq |a(z)| - |a^*(z)| \geqq r(a) - 4^{-1} \cdot r(a) \geqq 2^{-1} \cdot r(a).$$

Then we get from (P1), (P2), (9) and the subadditivity of p that

$$p(a) + p(a^*) \geqq 2^{-1} M_1 \cdot r(a) \quad \text{and} \quad p(a) \cdot p(a^*) \leqq M_2^2 C^2.$$

Writing $c = \min(p(a), p(a^*))$, $d = \max(p(a), p(a^*))$, we then have $2d \geqq c + d \geqq 2^{-1} M_1 \cdot r(a)$, $c \cdot d \leqq M_2^2 C^2$, and hence $c \leqq 4 M_2^2 C^2 M_1^{-1} r(a)^{-1}$.

We turn to the proof of Proposition 1.2. Let $a, b \in B$ be such that (4)—(8) hold with "large enough" n. Let further f (resp. g) be the one from a and a^* (resp. b and b^*) for which p is less. Since $r(g) = r(f) = r(a) > n$ and n is large ($> 2C$), we can apply Lemma 1.4 and have

$$(11) \qquad p(f) + p(h) < 2Ln^{-1}.$$

On the other hand, (P1) and (5)—(8) give us

$$M_1^{-2} \cdot p(f^* + g^*) \cdot p(f+g) \geqq r((f^*+g^*)(f+g)) \geqq |[(f^*+g^*)(f+g)](w_1)| \geqq$$
$$\geqq (mC^{-1} \cdot r(a) - m) \cdot (m - 2^{-1}m) \geqq (4C)^{-1} m^2 \cdot r(a)$$

if n is large (since $n>2C$ implies $m\leqq(2C)^{-1}m\cdot r(a)$). Further, by (P3)

$$p(f^*+g^*)\leqq 2M_2\cdot r(f^*+g^*)\leqq 4M_2\cdot r(a)$$

and thus

$$p(f+g)\geqq M_1^2m^2(16M_2C)^{-1}\geqq 2Ln^{-1}$$

if n is large. This and (11) show the desired contradiction. Proposition 1.2 is proved.

Proposition 1.5. *If* $\operatorname{card}(K-\mathbf{R})=2$ *and* (P1) *holds then* $K\cap\mathbf{R}\subset\{0\}$.

Proof. Suppose $K-\mathbf{R}=\{w,\bar{w}\}$. Since $\mathbf{C}-K$ is connected now, by Runge's theorem there are polynomials P_k converging to $w^{-1}\cdot 1_{\{w\}}$ in $C(K)$, where $1_{\{w\}}$ denotes the characteristic function of the one point set $\{w\}$. Hence $z\cdot P_k(z)$ converges to $1_{\{w\}}$ in $C(K)$, consequently $1_{\{w\}}\in B$.

Since $1^*_{\{w\}}\cdot 1_{\{w\}}=0$, thus by (P1) we infer that one of the functions $1_{\{w\}}$ and $1^*_{\{w\}}$, say f, is such that $p(f)=0$. This implies

(12) $$p(f+g)=p(g)\quad\text{for all}\quad g\in B.$$

Applying this to $g=f^*$ we get from (P2) that

(13) $$p(f^*)\geqq M_1.$$

Let $h(z)=z$ on K and let $h_0=h-w\cdot 1_{\{w\}}-\bar{w}\cdot 1^*_{\{w\}}$; thus $h_0\in B$. We will show that $h_0=0$, i.e. $K\cap\mathbf{R}\subset\{0\}$. Write $g=\alpha\cdot h_0$, where α is a real number, and let $k=f+g$. Since g is self-adjoint, further $g\cdot f=0=g\cdot f^*$, therefore $k^*k=g^2$ and so (P1) implies

(14) $$p(k^*)\cdot p(k)\leqq M_2^2\cdot r(g)^2.$$

On the other hand, we can see from (12), (13) and (P2) that $p(k)\geqq M_1\cdot r(g)$, $p(k^*)\geqq \geqq M_1-M_2\cdot r(g)$. This contradicts (14), if $r(g)$ is a small positive number. But if $h_0\neq 0$, then $r(g)$ runs over all of $\mathbf{R}_+$ when α does. Thus $h_0=0$ and the proof of Proposition 1.5 is complete.

Part II. If $P=\sum_{k=1}^{n}a_kX^k$ is a complex polynomial without constant term then we write $P^*=\sum_{k=1}^{n}\bar{a}_kX^k$. It is clear that $P^*(h)=P(h)^*$, where h is the self-adjoint element considered in Theorem 1.

Let $K=\operatorname{Sp}(h)$. Then K is symmetric, because in each *-algebra $\operatorname{Sp}(a^*)= =\overline{\operatorname{Sp}(a)}$ for any a. We will show that this K satisfies (P1). Consider the following relation between A and $\langle h\rangle$: $f\sim a$ if there is a polynomial P such that $P(h)=a$ and $P(z)=f(z)$ for all $z\in K$. Denote by r' the sup-norm in $C(K)$. Then $r'(f)=r(a)$ if $f\sim a$, because $P(\operatorname{Sp}(h))=\operatorname{Sp}(P(h))$. Further, $f\sim a$, $g\sim b$ ensure $f+\lambda g\sim a+\lambda b$, $f^*\sim a^*$, since $P^*(z)=\overline{P(\bar{z})}$. Finally we see from (i) and Lemma 1.1 that $p\leqq 2M_2\cdot r$.

Hence the following definition is correct: let $p'(f)=p(a)$ if $f\sim a$. Moreover, this p' shows that K satisfies (P1). Thus we know that

(15) $\mathrm{Sp}(h)$ is a cross,

(16) if $\mathrm{card}(\mathrm{Sp}(h)-\mathbf{R})=2$ then $\mathrm{Sp}(h)\cap\mathbf{R}\subset\{0\}$.

Suppose that $K=\mathrm{Sp}(h)\not\subset\mathbf{R}$ and $K\not\subset\{0, w, \bar{w}\}$ for any $w\in\mathbf{C}$. Then by (15) and (16) we can find w_1, w_2 in $K-\mathbf{R}$ such that $\mathrm{Re}\, w_1=\mathrm{Re}\, w_2$, $\mathrm{Im}\, w_1\neq\pm\mathrm{Im}\, w_2$. Thus $\mathrm{Re}(w_1+sw_1^2)\neq\mathrm{Re}(w_2+sw_2^2)$ for any $s\in\mathbf{R}-\{0\}$, and if $|s|$ is small then $w_1+sw_1^2$, $w_2+sw_2^2$ are not real. Therefore $\mathrm{Sp}(h+sh^2)$ is not a cross. But this is impossible, since $g=h+sh^2$ is self-adjoint and $\langle g\rangle\subset\langle h\rangle$.

It remains to prove the last statement of the theorem. Assume the contrary, that is, $K\not\subset\mathbf{R}$ and p is a norm. We know already that $K\cup\{0\}=\{0, w, \bar{w}\}$ where $w\in\mathbf{C}-\mathbf{R}$. Let $y=h^2-wh$. Then $y^*y=h^4-wh^3-\bar{w}h^3+w\bar{w}h^2$ and hence $\mathrm{Sp}(y)\neq\{0\}$, $\mathrm{Sp}(y^*y)=\{0\}$. Thus, on the one hand, $r(y^*y)=0$; on the other hand, $p(y^*)\cdot p(y)\neq 0$, since $y\in\langle h\rangle-\{0\}$ and p is a norm on $\langle h\rangle$. This contradicts (i). Theorem 1 is proved.

Theorem 2. *Let $\mathscr{A}$ be a *-algebra. Let p be a norm on it, and assume that the following hold with suitable positive constants C, D:*

(i) $p(a^*a)\leqq C\cdot p(a^*)\cdot p(a)$ *for all* $a\in\mathscr{A}$,

(ii) $p(b^*b)\geqq D\cdot p(b^*)\cdot p(b)$ *if* $b\in\langle h\rangle$, $h=h^*\in\mathscr{A}$.

Then $(\mathscr{A}, p)$ is an equivalent pre-C^*-algebra (that is, there is a norm on the completion of $(\mathscr{A}, p)$, equivalent to p and such that the completion with this norm is a C^*-algebra).

Proof. This identity holds in each *-algebra:

$$4xy=(x^*+y)^*(x^*+y)-(-x^*+y)^*(-x^*+y)+\\ +i(ix^*+y)^*(ix^*+y)-i(-ix^*+y)^*(-ix^*+y). \tag{1}$$

From this and (i) we get

$$4p(xy)\leqq 4C\cdot\big(p(x)+p(y^*)\big)\cdot\big(p(x^*)+p(y)\big). \tag{2}$$

Writing $x=\big(p(v^*)^{1/2}+\varepsilon\big)\big(p(v)^{1/2}+\varepsilon\big)u$, $y=\big(p(u^*)^{1/2}+\varepsilon\big)\big(p(u)^{1/2}+\varepsilon\big)v$ in (2) (where $\varepsilon>0$) and letting ε tend to 0, we infer

$$p(uv)\leqq C\cdot\big(p(u^*)^{1/2}p(v^*)^{1/2}+p(u)^{1/2}p(v)^{1/2}\big)^2. \tag{3}$$

Define a new norm on $\mathscr{A}$ by setting

$$\|a\|=4C\cdot\max\big(p(a^*), p(a)\big) \quad\text{for all}\quad a\in\mathscr{A}. \tag{4}$$

Then we have

$$\|ab\|\leqq\|a\|\cdot\|b\|,\ \|a^*\|=\|a\|,\ p(a)\leqq(4C)^{-1}\|a\| \quad\text{for all}\quad a, b\in\mathscr{A}. \tag{5}$$

Let $\mathscr{B}$ be the completion of $(\mathscr{A}, \|\cdot\|)$. Because of (5) the operations and p have unique continuous extensions to $\mathscr{B}$ and (i), (ii), (4), (5) remain valid in $\mathscr{B}$.

Let r be the spectral radius in $\mathscr{B}$. Since $\mathscr{B}$ is a Banach-algebra, thus

$$(6) \qquad r(a) = \lim \|a^n\|^{1/n} \quad \text{for all} \quad a \in \mathscr{B}.$$

If h is a self-adjoint element in $\mathscr{B}$, then $D \cdot p(h)^2 \leqq p(h^2)$, and hence $p(h) \leqq \leqq D^{-1/2} p(h^2)^{1/2} \leqq D^{-1/2} D^{-1/4} p(h^4)^{1/4} \leqq \dots$. Therefore $p(h) \leqq D^{-1} \cdot \limsup p(h^n)^{1/n}$. Thus we see from (5) and (6) that $p(h) \leqq D^{-1} \cdot r(h)$. On the other hand, $r(h) \leqq \leqq \|h\| = 4C \cdot p(h)$ and we have

$$(7) \qquad (4C)^{-1} \cdot r(h) \leqq p(h) \leqq D^{-1} \cdot r(h) \quad \text{if} \quad h^* = h \in \mathscr{B}.$$

From this and (i), (ii) we can see that

$$(8) \qquad (4C^2)^{-1} \cdot r(a^*a) \leqq p(a^*) \cdot p(a) \leqq D^{-2} \cdot r(a^*a) \quad \text{if} \quad a \in \langle h \rangle,\ h^* = h \in \mathscr{A};$$

furthermore, p is a norm on $\langle h \rangle$. Thus Theorem 1 shows that $\mathrm{Sp}(h) \subset \mathbf{R}$ if $h^* = h \in \mathscr{A}$. Then $r(\sin h) \leqq 1$, $r(\cos h - 1) \leqq 2$ via functional calculus. Since * is continuous in $\mathscr{B}$, hence $\sin h$, $\cos h - 1$ are self-adjoint. Therefore (7) and (4) imply $\|\sin h\| \leqq 4CD^{-1}$, $\|\cos h - 1\| \leqq 8CD^{-1}$, and so

$$(9) \qquad \|\exp(ih) - 1\| \leqq 12CD^{-1} \quad \text{if} \quad h^* = h \in \mathscr{A}.$$

The self-adjoint part of $\mathscr{A}$ is dense in that of $\mathscr{B}$, and hence (9) remains valid for $h = h^* \in \mathscr{B}$, too. But this ensures that $\|a\|_c = r(a^*a)^{1/2}$ is a C^*-norm on $\mathscr{B}$, which is equivalent to $\|\cdot\|$ (see [2]). Thus p is continuous with respect to $\|\cdot\|_c$; let $E > 0$ be such that

$$p(a) \leqq E \cdot \|a\|_c \quad \text{for all} \quad a \in \mathscr{B}.$$

Comparing this with (i) and (7) we see that for any $a \in \mathscr{B}$

$$E \cdot \|a\|_c \cdot p(a) = E \cdot \|a^*\|_c \cdot p(a) \geqq p(a^*) \cdot p(a) \geqq (4C^2)^{-1} r(a^*a) = (4C^2)^{-1} \|a\|_c^2,$$

that is, $p(a) \geqq (4EC^2)^{-1} \|a\|_c$. Therefore p is equivalent to $\|\cdot\|_c$. Theorem 2 is proved.

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On zeros of analytic multivalued functions

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It had been observed by F. V. ATKINSON [1] and B. SZ.-NAGY [13] that if $f(\lambda)=$ $=I+\lambda V_1+\ldots+\lambda^n V_n$, where $V_1, \ldots, V_n$ are compact operators on a Banach space, then the set of λ in $\mathbf{C}$ for which $0\in\operatorname{Sp} f(\lambda)$ is discrete and closed in the complex plane. For $n=1$ it is exactly the classical result of F. Riesz. For $n>1$ B. SZ.-NAGY [13] believed that this result is deeper than the classical one. The problem was also studied by JU. L. ŠMUL'JAN [12]. Here we show in Theorem 1, by a completely different method, that it comes from Riesz's theorem using only complex function theory. Moreover, we give a generalization of this result when $f(\lambda)$ is any analytic function from a domain Ω of $\mathbf{C}$ into a Banach algebra such that $\operatorname{Sp} f(\lambda)$ is countable for every λ in Ω.

It is known that $\lambda\to\operatorname{Sp} f(\lambda)$ is an analytic multivalued function [3] and that analytic multivalued functions have properties very similar to this special case. So it is better to formulate all the theorems of this paper in the more general situation (for more details see [3], [5], [8]). However, the reader not familiar with this theory can adapt immediately all the proofs to the spectral case.

Theorem 1. *Let $\lambda\to K(\lambda)$ be an analytic multivalued function defined on a domain Ω in $\mathbf{C}$. Suppose that $K(\lambda)$ has at most 0 as a limit point for every λ in Ω. Let $z\neq 0$ be a fixed complex number. Then the set of those λ in Ω for which $z\in K(\lambda)$ is either closed and discrete in Ω or it is all Ω.*

Proof. Suppose that $z\in K(\lambda_0)$ for some $\lambda_0\in\Omega$. We shall show that the point λ_0 is either isolated or interior in the set $E=\{\lambda\in\Omega\colon z\in K(\lambda)\}$. Because $z\neq 0$ there exists an open disk Δ centred at z and not containing 0 such that $\Delta^-\cap K(\lambda_0)=$ $=\{z\}$. By upper semi-continuity of the function K there exists $r>0$ such that $|\lambda-\lambda_0|<r$ implies $K(\lambda)\cap\operatorname{bdry}\Delta=\emptyset$. Moreover, by Newburgh's property we can also suppose that $K(\lambda)\cap\Delta\neq\emptyset$ for these λ, and in this situation $\lambda\to K(\lambda)\cap\Delta$ is an analytic multivalued function on the disk $B(\lambda_0, r)$, see [5], Theorem 3.14. Because

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Δ does not contain 0 the set $K(\lambda)\cap\Delta$ is finite for $|\lambda-\lambda_0|<r$. We apply the scarcity theorem for analytic multivalued functions [3], [5] (we can also use the subharmonicity of $\lambda\to\operatorname{Log}\delta_n(K(\lambda))$, where δ_n denotes the n-th diameter; in the case when $K(\lambda)=\operatorname{Sp}f(\lambda)$ we can use the scarcity theorem ([2], p. 67), or the subharmonicity of $\lambda\to\operatorname{Log}\delta_n(\operatorname{Sp}f(\lambda))$ [11]). So there exist an integer $n\geqq 1$, a closed discrete subset F of the disk $B(\lambda_0, r)$ and n functions $\alpha_1, \ldots, \alpha_n$ which are holomorphic on $B(\lambda_0, r)\setminus F$ such that

$$K(\lambda)\cap\Delta = \{\alpha_1(\lambda), \ldots, \alpha_n(\lambda)\} \quad \text{for} \quad \lambda\in B(\lambda_0, r)\setminus F.$$

There exists s such that $0<s\leqq r$ and $B(\lambda_0, s)\cap F\subset\{\lambda_0\}$. The functions $\alpha_1, \ldots, \alpha_n$ are holomorphic on $B(\lambda_0, s)$ except perhaps at λ_0.

Moreover, by the upper semi-continuity of the function $K(\lambda)$ we have $\lim\limits_{\lambda\to\lambda_0}\alpha_i(\lambda)=z$ for every $i=1, 2, \ldots, n$. Therefore the α_i's can be extended holomorphically to the whole disk $B(\lambda_0, s)$. It follows that either $\alpha_{i_0}(\lambda)\equiv z$ for some i_0, or there exists t with $0<t\leqq s$ such that $\alpha_i(\lambda)\neq z$ for all $\lambda\in B(\lambda_0, t)\setminus\{\lambda_0\}$ and $i=1, 2, \ldots, n$. In the first case λ_0 is an interior point of E, while in the second case λ_0 is isolated in E.

To finish the proof we consider the set E' of all limit points of E in Ω. Because of the upper semi-continuity of the function K the set E is closed in Ω, so $E'\subset E$. Let $\mu\in E'$. Since μ is not isolated in E it is an interior point of E, hence an interior point of E'. So E' is both closed and open in Ω. Consequently we have either $E'=\emptyset$ or $E'=\Omega$. This completes the proof.

Corollary 1. *Let $\lambda\to f(\lambda)$ be an analytic function from a domain Ω into the compact operators on a Banach space. Suppose that $z\notin\operatorname{Sp}f(0)$. Then the set of all λ for which $z\in\operatorname{Sp}f(\lambda)$ is closed and discrete in Ω.*

Remark 1. F. V. Atkinson [1] and B. Sz.-Nagy [13] consider the situation when $\Omega=\mathbf{C}$ and $f(\lambda)=\lambda V_1+\ldots+\lambda^p V_p$ with compact operators $V_1, \ldots, V_p$. Ju. L. Šmul'jan [12] studies the case when $f(\lambda)$ is an analytic family of compact operators, defined on a domain Ω.

We intend to generalize Theorem 1 to the situation when $K(\lambda)$ are general countable sets. Of course, in this situation it is impossible to conclude that the set $\{\lambda: z\in K(\lambda)\}$ is discrete. To see this take, for example, $K(\lambda)=\operatorname{Sp}(\lambda I+C)$ where C is a compact operator with infinite spectrum. In this case the preceding set has z as a limit point.

The situations studied in Theorem 1 and in the last example suggest to introduce the notion of *good isolated point*. Given an analytic multivalued function $\lambda\to K(\lambda)$ on a domain Ω, for $\lambda_0\in\Omega$ we say that $\mu\in K(\lambda_0)$ is a good isolated point of $K(\lambda_0)$ if there exist a disk Δ centred at μ such that $\Delta^-\cap K(\lambda_0)=\{\mu\}$ and an $r>0$ such that the set $K(\lambda)\cap\Delta$ is finite for $|\lambda-\lambda_0|<r$. By the scarcity theorem for analytic

multivalued functions (see [3], Theorem 7) there exists an integer $n \geqq 1$ such that $K(\lambda) \cap \Delta$ has exactly n points for all $|\lambda-\lambda_0|<r$ except perhaps on a closed discrete subset. By definition we put $DK(\lambda)$ to be the set of points of $K(\lambda)$ which are not good isolated points. By transfinite induction we can define $D^\alpha K(\lambda)$ for every ordinal α by

$$D^\alpha K(\lambda) = D(D^{\alpha-1}K(\lambda)) \quad \text{if } \alpha \text{ is not a limit ordinal,}$$

$$D^\alpha K(\lambda) = \bigcap_{\beta<\alpha} D^\beta K(\lambda) \quad \text{if } \alpha \text{ is a limit ordinal.}$$

It is a remarkable fact that if $D^\alpha K(\lambda)$ is not identically void then $\lambda \to D^\alpha K(\lambda)$ is an analytic multivalued function on Ω (see [8] and [5]).

In the situation of Theorem 1 we have $DK(\lambda)$ constant (either empty or equal to $\{0\}$) while in the previous example we have $DK(\lambda)=\{\lambda\}$.

Theorem 2. *Let $\lambda \to K(\lambda)$ be an analytic multivalued function defined on a domain Ω in $\mathbf{C}$. Let z be a fixed complex number. Then every point of the set $\{\lambda \in \Omega : z \in K(\lambda) \setminus DK(\lambda)\}$ is either isolated or interior.*

Proof. We omit the proof because it is similar to the proof of Theorem 1.

We shall need two lemmas the proofs of which are similar to some arguments given in [5].

Lemma 1. *Let $\lambda \to K(\lambda)$ be an analytic multivalued function defined on a domain Ω in $\mathbf{C}$, with $K(\lambda)$ countable for every λ in Ω. Then there exists a point μ in Ω such that $K(\mu) \neq DK(\mu)$.*

Proof. Suppose that $DK(\lambda)=K(\lambda)$ for every λ in Ω. From this we conclude that there exists some $\lambda_0 \in \Omega$ for which $K(\lambda_0)$ has an infinite number of points. Because $K(\lambda_0)$ is countable and compact we can assume that there exist two isolated points in $K(\lambda_0)$ (see [9], Theorem 2.43). We denote them by α_0 and α_1. We choose two open disks Δ_0 and Δ_1 centred respectively at α_0 and α_1, having disjoint closures and such that $\Delta_0^- \cap K(\lambda_0)=\{\alpha_0\}$ and $\Delta_1^- \cap K(\lambda_0)=\{\alpha_1\}$. Then we choose $r>0$ such that $B^-(\lambda_0, r) \subset \Omega$ and such that $|\lambda-\lambda_0|<r$ implies $K(\lambda) \cap \operatorname{bdry} \Delta_i=\emptyset$ for $i=0, 1$.

Because $K(\lambda_0)=DK(\lambda_0)$ the isolated point α_i is not a good isolated point of $K(\lambda_0)$, for $i=0, 1$. By applying the scarcity theorem for the two functions $\lambda \to K(\lambda) \cap \cap \Delta_i$ we conclude that the two sets $E_i=\{\lambda \in B(\lambda_0, r) : K(\lambda) \cap \Delta_i$ is finite$\}$ are of outer capacity zero. Consequently, $E_0 \cup E_1$ is of outer capacity zero, therefore there exists some λ_1 in $B(\lambda_0, r/2)$ such that the intersection of $K(\lambda_1)$ on both Δ_0 and Δ_1 is infinite.

As before we find four distinct isolated points in $K(\lambda_1)$, say α_{00}, α_{01} in Δ_0 and α_{10}, α_{11} in Δ_1. We take four open disks Δ_{ij} centred respectively at α_{ij}, having disjoint closures, such that $\Delta_{00} \cup \Delta_{01} \subset \Delta_0$, $\Delta_{10} \cup \Delta_{11} \subset \Delta_1$ and $\Delta_{ij}^- \cap K(\lambda_1)= =\{\alpha_{ij}\}$. By induction we can construct a sequence (λ_n) such that:

(i) $|\lambda_{n+1}-\lambda_n|\leqq r/2^{n+1}$ for $n=0, 1, 2, \ldots$,

(ii) $K(\lambda_n)$ contains at least 2^{n+1} distinct isolated points $\alpha_{i_1\ldots i_{n+1}}$ where i_k takes the values 0, 1,

(iii) each $\alpha_{i_1\ldots i_{n+1}}$ is the centre of an open disk $\Delta_{i_1\ldots i_{n+1}}$, all these 2^{n+1} disks have disjoint closures, and moreover we have $\Delta_{i_1\ldots i_n i_{n+1}}\subset\Delta_{i_1\ldots i_n}$.

Then (λ_n) is a Cauchy sequence converging to some $\mu\in B^-(\lambda_0, r)\subset\Omega$. To obtain a contradiction we shall show that $K(\mu)$ is uncountable.

Let $I=\{i_1, i_2, \ldots, i_n, \ldots\}$ be an arbitrary sequence of 0's and 1's. A subsequence of $\alpha_{i_1}, \alpha_{i_1i_2}, \alpha_{i_1i_2i_3}, \ldots$ converges to an α_I which is in $K(\mu)$ by upper semi-continuity. If $I\neq J$ then for some index k we have $i_k\neq j_k$ with $i_l=j_l$ for $1\leqq l<k$. We have $\alpha_I\in\bar{\Delta}_{i_1i_2\ldots i_k}$ while $\alpha_J\in\bar{\Delta}_{i_1i_2\ldots i_{k-1}j_k}$ and these two disks are disjoint by construction, so $\alpha_I\neq\alpha_J$. But the set of sequences I is uncountable so $K(\mu)$ is uncountable.

Remark 2. For any analytic multivalued function $K(\lambda)$ on Ω it is easy to see that the set of $\lambda\in\Omega$ for which $K(\lambda)\neq DK(\lambda)$ is open. If in addition the set $K(\lambda)$ are countable for $\lambda\in\Omega$, then this set is dense in Ω.

Lemma 2. *Let $\lambda\to K(\lambda)$ be an analytic multivalued function defined on a domain Ω, with $K(\lambda)$ countable for every λ in Ω. Then there exists a first or second class ordinal β such that $D^\beta K(\lambda)=\emptyset$ for every λ in Ω.*

Proof. Let $\mathcal{O}$ denote the set of ordinals in the first and second classes (see [10], p. 369). For every λ in Ω the family of $D^\alpha K(\lambda)$, for α in $\mathcal{O}$, is decreasing, consequently it stabilizes at some ordinal $\alpha(\lambda)$, i.e. we have $D^\gamma K(\lambda)=D^{\alpha(\lambda)}K(\lambda)$ for every $\gamma\geqq\alpha(\lambda)$, γ in $\mathcal{O}$ (see [7], p. 146). For every α in $\mathcal{O}$ we define

$$F_\alpha=\{\lambda\in\Omega\colon D^\gamma K(\lambda)=D^\alpha K(\lambda)\quad\text{for}\quad\gamma\geqq\alpha,\ \gamma\in\mathcal{O}\}.$$

Obviously this family is increasing and exhausts all Ω. Also the sets F_α are closed in Ω (even if the sets $K(\lambda)$ are not countable). Indeed, taking λ_0 in $\Omega\setminus F_\alpha$, we have $D^\gamma K(\lambda_0)\neq D^{\gamma+1}K(\lambda_0)$ for some $\gamma\geqq\alpha, \gamma\in\mathcal{O}$. Since $D^\gamma K(\lambda)\not\equiv\emptyset$, it follows by the Oka—Nishino theorem (see [5], Lemma 3.16) that $\lambda\to D^\gamma K(\lambda)$ is an analytic multivalued function. By the first part of Remark 2 we have $D^\gamma K(\lambda)\neq D^{\gamma+1}K(\lambda)$ in a neighbourhood of λ_0, so $\Omega\setminus F_\alpha$ is open. Using again the results in [7], p. 146, and [10], p. 370, we obtain that for some β in $\mathcal{O}$ we have $F_\beta=\Omega$.

Suppose that $D^\beta K(\lambda)\not\equiv\emptyset$ on Ω. By Oka—Nishino theorem $\lambda\to D^\beta K(\lambda)$ is analytic multivalued on Ω. By hypothesis $D^\beta K(\lambda)$ is countable for every λ in Ω hence by Lemma 1 we have $D^{\beta+1}K(\mu)\neq D^\beta K(\mu)$ for some $\mu\in\Omega$, that is $F_\beta\neq\Omega$, which is a contradiction.

Theorem 3. *Let $\lambda\to K(\lambda)$ be an analytic multivalued function on a domain Ω in $\mathbf{C}$. Suppose that $K(\lambda)$ is countable for every λ in Ω. Let z be a fixed complex number. Then the set of those λ in Ω for which $z\in K(\lambda)$ is either countable or it is all Ω.*

Proof. By Lemma 2 there exists a smallest ordinal β in the first or second class such that $D^{\beta}K(\lambda)\equiv\emptyset$ for λ in Ω. We have $E=\{\lambda\in\Omega\colon z\in K(\lambda)\}=\bigcup_{0\leqq\gamma<\beta} E_\gamma$ where $E_\gamma=\{\lambda\in\Omega\colon z\in D^{\gamma}K(\lambda)\setminus D^{\gamma+1}K(\lambda)\}$. By Theorem 2 applied to the analytic multivalued function $\lambda\to D^{\gamma}K(\lambda)$ we conclude that E_γ has only isolated or interior points. Therefore E_γ is the disjoint union of an open set and a countable set. Because the set of ordinals less than β is countable the set E is also the disjoint union of an open set and a countable set. If the interior of E is empty then E is countable and we have finished. If not, we shall show that $E=\Omega$. First we note that E is closed in Ω by upper semi-continuity and so the boundary of E in Ω is countable. Let F be the closure of the interior of E in Ω. It is enough to prove that $F=\Omega$. Because F is closed in Ω and Ω is a domain we have only to show that F is open. Let a be a point of F, and let $r>0$ be such that $B(a,r)\in\Omega$. There exists b in the interior of E such that $|a-b|<r$. The set of half-lines Γ with origin at b such that $\Gamma\cap B(a,r)$ contains a boundary point of E is at most countable. So the interior of E is dense in $B(a,r)$ and hence $F\supset B(a,r)$.

Now we give an application of Theorem 3 concerning the problem of spectral classification of projections. In [6] we obtained such result for finite-dimensional algebras. Here we extend it to algebras with countable spectrum.

We say that two idempotents e and f in a Banach algebra A are equivalent if they belong to the same connected component of the set of all idempotents in A. It is possible to prove that e and f are equivalent if and only if there exist elements $a_1, \dots, a_n$ in A such that $f=\exp(-a_n)\dots\exp(-a_1)\cdot e\cdot\exp(a_1)\dots\exp(a_n)$, see [4].

Corollary 2. *Let A be a (real or complex) Banach algebra. Suppose that every element in A has countable spectrum. Let e and f be given idempotents in A. Then e is not equivalent to f if and only if $1\in\operatorname{Sp}(e'+f')$ for all idempotents e', f' in neighbourhoods of e and f, respectively.*

Proof. As noted in [6] it is enough to prove that $1\in\operatorname{Sp}(e'+f')$ implies e not equivalent to f. Suppose on the contrary that e and f are equivalent. So there are elements $a_1, \dots, a_n$ in A such that

$$f=\exp(-a_n)\dots\exp(-a_1)\cdot e\cdot\exp(a_1)\dots\exp(a_n).$$

Consider the analytic function

$$g(\lambda)=\exp(-\lambda a_n)\dots\exp(-\lambda a_1)\cdot e\cdot\exp(\lambda a_1)\dots\exp(\lambda a_n)$$

defined for all complex λ and with values in the complexification of A. The values of this function are idempotents and for λ real they belong to A. Moreover we have $g(0)=e$, $g(1)=f$. We consider the analytic multivalued function defined on $\mathbf{C}$ by

$$\lambda\to K(\lambda)=\operatorname{Sp}\big(g(\lambda)+g(1-\lambda)\big)$$

which has countable values for λ real. (We recall that for real Banach algebras the spectrum is defined with respect to the complexification.) Hence by Oka—Nishino theorem on scarcity of elements with countable values (see [3], [5], [8]) we conclude that $K(\lambda)$ is countable for every λ in $\mathbf{C}$. But we know that $1 \in K(\lambda)$ if λ is in a small real segment containing zero. So by Theorem 3 we have $1 \in K(\lambda)$ for every λ. In particular, taking $\lambda = 1/2$ we get $1 \in \operatorname{Sp}(2g(1/2))$ which is impossible because $g(1/2)$ is an idempotent.

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Added in proof. Some related new results are given in [14], [15], [16].

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Infinite-dimensional Jordan models and Smith McMillan forms. II

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1. Introduction

This paper is a continuation of [3]. Throughout we follow the notation and terminology established there and in [11]. The k-dimensional space of complex k-tuples is denoted by $\mathscr{E}^k$ and $z=e^{it}$ for $t\in[0, 2\pi]$. The orthogonal projection onto a subspace $\mathscr{X}$ is denoted by $P_{\mathscr{X}}$. The greatest common inner divisor of the functions α, β in H^∞ is $\alpha\wedge\beta$. A *bounded analytic function* $\{\mathscr{E}^m, \mathscr{E}^n, \Omega\}$ is a Lebesgue measurable operator valued function such that $\Omega(z)$ maps $\mathscr{E}^m$ into $\mathscr{E}^n$ for all z, $\Omega(z)$ has analytic continuation into the open unit disc and $\|\Omega(z)\|\leqq M<\infty$ a.e. The Hardy H^2-space of analytic functions with values in $\mathscr{E}$ is denoted by $H^2(\mathscr{E})$. The forward shift U_+ on $H^2(\mathscr{E})$ is defined by $U_+f:=zf$ where f is in $H^2(\mathscr{E})$. Let $\{\mathscr{E}^k, \mathscr{E}^n, \Phi\}$ be an inner function. Then $\mathscr{H}(\Phi):=H^2(\mathscr{E}^n)\ominus\Phi H^2(\mathscr{E}^k)$ and $S(\Phi)$ is the compression of U_+ to $\mathscr{H}(\Phi)$. Recall [11] that $S(\Phi)$ is a C_0 contraction if and only if Φ is inner from both sides, i.e., $k=n$. Finally, let $\{\mathscr{E}^m, \mathscr{E}^n, \Omega\}$ be a bounded analytic function then $\{\mathscr{E}^k, \mathscr{E}^n, C(\Omega)\}$ is the inner function uniquely defined by

$$\mathscr{H}(C(\Omega)):=\bigvee_{j\geqq 0} U_+^{*j}\Omega\mathscr{E}^m \tag{1}$$

Note $C(\Omega)$ is well defined by the Beurling—Lax theorem [11].

Throughout $N(z)$ is a Lebesgue measurable function in $[0, 2\pi]$ whose values are a.e. nonnegative self adjoint operators mapping $\mathscr{E}^m$ into $\mathscr{E}^m$ and $\|N(z)\|\leqq\leqq M<\infty$ a.e. It is also assumed that N admits a factorization of the form $N(z)==\theta^*(z)\theta(z)$ a.e., where $\{\mathscr{E}^m, \mathscr{E}^n, \theta\}$ is a bounded analytic outer function; such a θ will be called an outer factor of N. In the previous paper [3] we gave a simple procedure to compute the Jordan model for $S(C(\theta))$ by means of θ. Here this is done without computing θ or the inner function $C(\theta)$ generated by θ. That is,

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our present procedure calculates this Jordan model directly from N, by using a generalized Smith—McMillan procedure. Our procedure, given in Theorem 1, plays an important role in infinite-dimensional stochastic realization theory [4]. The following is needed.

Lemma 1. [2] *Let* $\{\mathcal{E}^m, \mathcal{E}^n, \theta\}$ *be the outer factor for* N. *Then* $S(C(\theta))$ *is a* C_0 *contraction if and only if there exists an inner function* c *in* H^∞ *such that* cN *is a bounded analytic function.*

Remark 1. The above lemma allows us to determine if $S(C(\theta))$ is a C_0 contraction directly from N without obtaining θ or $C(\theta)$. Finally, if cN is a bounded analytic function for some c in H^∞ then N always admits an outer spectral factor [2]. (In this situation our factorization assumption on N is redundant.)

2. Main result

For convenience we recall some terminology in [9], [10]. Let $\{\mathcal{E}^n, \mathcal{E}^m, H\}$ and $\{\mathcal{E}^n, \mathcal{E}^m, H_1\}$ be two bounded analytic functions. H is *quasi-equivalent* to H_1 if for every scalar valued inner function c there exists two bounded analytic functions $\{\mathcal{E}^m, \mathcal{E}^m, A\}$, $\{\mathcal{E}^n, \mathcal{E}^n, B\}$ such that $\det(A)$ and $\det(B)$ are prime to c and $HB = AH_1$. Quasi-equivalence is an equivalence relation. It can be shown that $\{\mathcal{E}^n, \mathcal{E}^m, H\}$ is quasi-equivalent to $\{\mathcal{E}^n, \mathcal{E}^m, D\}$ where D is a diagonal analytic function of the form

$$D = \begin{bmatrix} D_1 & 0 \\ 0 & 0 \end{bmatrix} \tag{2}$$

and $D_1 = \operatorname{diag}[d_1, d_2, \ldots, d_k]$. The d_i's are scalar valued inner functions such that d_i divides d_{i+1} for $i=1, \ldots, k-1$. Furthermore, this representation is unique and called the *normal form of* H. The normal form D can be obtained from the invariant factors of H [9], [10]. Define $\mathcal{D}_r$ as the greatest common inner divisor of all minors in H of order r, with $\mathcal{D}_0 = 1$. The *invariant factors* for H are $\mathcal{E}_i(H) := \mathcal{D}_i/\mathcal{D}_{i-1}$ for $i=1, \ldots, \min(m, n)$. By convention $\mathcal{E}_j(H) = 0$ for all $j \geqq i-1$ if $\mathcal{D}_{i-1} = 0$. If $\mathcal{E}_i(H)$ is nonzero then $\mathcal{E}_{i-1}(H)$ divides $\mathcal{E}_i(H)$. It can be shown that the normal form for H is given by (2) where $D_1 = \operatorname{diag}[\mathcal{E}_1(H), \ldots, \mathcal{E}_k(H)]$ and k is the number of nonzero invariant factors for H.

A Jordan model is an operator of the form $S(m_1) \oplus S(m_2) \oplus \ldots \oplus S(m_k)$ where the m_i's are inner functions in H^∞, see [1], [12], [13], [14] for further details. Finally we need

Lemma 2. [6, Ch. 3] *Let* $\{\mathcal{E}^m, \mathcal{E}^n, \theta\}$ *be a bounded analytic function. Then* $S(C(\theta))$ *is a* C_0 *contraction if and only if* θ *admits a factorization of the form*

$\theta=\bar{z}G^*\psi$, where $\{\mathcal{E}^m, \mathcal{E}^m, \psi\}$ is inner from both sides, $\{\mathcal{E}^n, \mathcal{E}^m, G\}$ is a bounded analytic function, and the only common, inner from both sides, left factor to both ψ and G_i is a unitary constant. (The inner part of G is denoted by G_i.) Furthermore, when $S(C(\theta))$ is a C_0 contraction then $S(C(\theta))$ and $S(\psi)$ are quasi-similar. In particular, $S(C(\theta))$ and $S(\psi)$ admit the same Jordan model.

Theorem 1. *Let $\{\mathcal{E}^m, \mathcal{E}^n, \theta\}$ be the outer factor for N. Assume there exists a scalar inner function c such that $cN=zH$ is a bounded analytic function. Then*

(i) *$S(C(\theta))$ is a C_0 contraction.*

(ii) *The Jordan model for $S(C(\theta))$ is $S(m_1)\oplus S(m_2)\oplus\ldots\oplus S(m_k)$ where k is the number of nonzero invariant factors for $\{\mathcal{E}^m, \mathcal{E}^m, H\}$ and $m_i=c/(\mathcal{E}_i(H)\wedge c)$ for $i=1, \ldots, k$.*

Proof. Part (i) is an obvious consequence of Lemma 1. The proof of part (ii) is similar to Theorem 1 in [3]. Let

(3) $$D' = \operatorname{diag}[\mathcal{E}_1(H), \ldots, \mathcal{E}_k(H), 0, 0, \ldots, 0]$$

be the normal form for H, where $\mathcal{E}_k(H)\neq 0$. Choose any two bounded analytic functions $\{\mathcal{E}^m, \mathcal{E}^m, A\}$ and $\{\mathcal{E}^m, \mathcal{E}^m, B\}$ with $\det(A)\cdot\det(B)=a$ such that a is prime to $c\mathcal{E}_k(H)$ and $HB=AD'$. Lemma 2 and $N=\theta^*\theta$ gives $\psi^*G\theta=H\bar{c}$ where ψ and G satisfy the conclusion of Lemma 2. Applying B yields

(4) $$\psi^*G\theta B = AD'\bar{c}.$$

Let

(5) $$M = \operatorname{diag}[m_1, m_2, \ldots, m_k\ 1, 1, \ldots, 1],$$
$$D = \operatorname{diag}[d_1, d_2, \ldots, d_k, 0, 0, \ldots, 0]$$

where the m_i's are defined in statement (ii) above and $d_i:=\mathcal{E}_i(H)/(\mathcal{E}_i(H)\wedge c)$ for $i=1, \ldots, k$. By [12, Lemma 2b] we have d_i divides d_{i+1}. Using $D'\bar{c}=DM^*$ in (4):

(6) $$G\theta BM = \psi AD.$$

Equation (6) and [11, Theorem 3.6, p. 258] or [8], [14] implies $S(\psi)X=XS(M)$ where

(7) $$X = P_{\mathcal{H}(\psi)}G\theta B|\mathcal{H}(M).$$

To complete the proof it is sufficient to show that X is a quasiaffinity. By the results in [1], [12], [13], [14] this implies $S(M)$ is the Jordan model for $S(\psi)$. Then by Lemma 2, $S(M)$ is also the Jordan model for $S(C(\theta))$.

First it is shown that X is densely onto. By equation (6):

$$P_{\mathcal{H}(\psi)}G\theta BMH^2(\mathcal{E}^m) = \{0\}.$$

Using this in the following calculation with the fact that θ is outer gives:

$$\begin{aligned}(8)\quad &\overline{X\mathscr{H}(M)} = \overline{P_{\mathscr{H}(\psi)} G\theta B(\mathscr{H}(M) \vee MH^2(\mathscr{E}^m))} = \overline{P_{\mathscr{H}(\psi)} G\theta BH^2(\mathscr{E}^m)} \supseteqq \\ &\supseteqq \overline{P_{\mathscr{H}(\psi)} G\theta aH^2(\mathscr{E}^m)} = \overline{P_{\mathscr{H}(\psi)} GaH^2(\mathscr{E}^n)} = \overline{P_{\mathscr{H}(\psi)} GaH^2(\mathscr{E}^n) \vee \psi H^2(\mathscr{E}^m)} = \mathscr{H}(\psi).\end{aligned}$$

The last equality follows from Lemma 3 in [3] which shows that

$$(9) \qquad GaH^2(\mathscr{E}^n) \vee \psi H^2(\mathscr{E}^m) = H^2(\mathscr{E}^m).$$

Hence X is densely onto.

Finally we verify that X is one-to-one. Our technique is similar to some of the arguments in [14]. Assume $h \in \mathscr{H}(M)$ and $Xh=0$. Let $g \in L^2(\mathscr{E}^m)$ be such that $h=Mg$. To show that X is one-to-one we simply show that $g \in H^2(\mathscr{E}^m)$. Then $h \in MH^2(\mathscr{E}^m) \cap \mathscr{H}(M) = \{0\}$.

By using (6):

$$(10) \qquad 0 = P_{\mathscr{H}(\psi)} G\theta BMg = P_{\mathscr{H}(\psi)} \psi ADg.$$

Since Mg is analytic, ψADg is analytic. Equation (10) implies ψADg is in $\psi H^2(\mathscr{E}^m)$. Thus ADg is in $H^2(\mathscr{E}^m)$. Using $A'A = aI$ for the appropriate bounded analytic $\{\mathscr{E}^m, \mathscr{E}^m, A'\}$ yields $aDg \in H^2(\mathscr{E}^m)$. This with the definition of D places $ad_k g$ in $H^2(\mathscr{E}^m)$. (This follows because $m_j = 1$ if $j > k$ where k is defined in (3) or (5). Notice that $h = Mg$ is in $\mathscr{H}(M)$. Thus $g_j = 0$ for all $j > k$. Here g_j is the jth component of the m-vector g.) Clearly $h = Mg$ is in $H^2(\mathscr{E}^m)$. Therefore cg is in $H^2(\mathscr{E}^m)$. By [11, Proposition 1.5, p. 108] we have $(c \wedge (ad_k))\, g \in H^2(\mathscr{E}^m)$. By construction c and ad_k are prime. Hence g is in $H^2(\mathscr{E}^m)$, X is one-to-one and the proof is complete.

Lemma 3. ([5], [6]) *Let $\{\mathscr{E}^p, \mathscr{E}^m, \Omega\}$ be a bounded analytic function.*

(i) *$S(C(\Omega))$ is a C_0 contraction if and only if $S(C(\tilde{\Omega}))$ is a C_0 contraction.*

(ii) *If $S(C(\Omega))$ is a C_0 contraction then $S(C(\Omega))$ and $S^*(C(\tilde{\Omega}))$ are quasi-similar. In particular, they have the same Jordan model.*

Proof. This lemma follows from Theorem 2.1 in [5]. One can also obtain this result by using either Theorem 14.11, p. 206 and Theorem 3.5, p. 254 in [6] or Theorem 1 in [3].

Finally we are ready for

Corollary 1. *Assume there exists a scalar valued inner function c such that $cN = zH$ is a bounded analytic function. Then*

i) *N admits a *-outer factorization $N(z) = \Omega(z)\Omega^*(z)$ a.e. where $\{\mathscr{E}^p, \mathscr{E}^m, \Omega\}$ is *-outer.*

(ii) *$S(C(\Omega))$ is a C_0 contraction. Furthermore, $S(C(\Omega))$ and $S(C(\theta))$ have the same Jordan model. (θ is the outer factor for N.) In particular, the Jordan model for $S(C(\Omega))$ can be obtained directly from Theorem 1.*

Proof. (i) $\tilde{c}\tilde{N}=z\tilde{H}$ is a bounded analytic function. By Remark 1 or [2] N admits a *-outer factorization.

Now for part (ii). Clearly $\tilde{N}=\tilde{\Omega}^*\tilde{\Omega}$ is an outer factorization of $\tilde{N}$ and $\tilde{c}\tilde{N}=z\tilde{H}$. Lemmas 1 and 3 imply that $S(C(\Omega))$ and $S(C(\tilde{\Omega}))$ are C_0 contractions. By Theorem 1 the Jordan model for $S(C(\tilde{\Omega}))$ is $S(\tilde{m}_1)\oplus\ldots\oplus S(\tilde{m}_k)$ where k is the number of nonzero invariant factors for H and

$$\tilde{m}_j = [c/(c\wedge\mathscr{E}_j(H))]^{\sim} = [\tilde{c}/(\tilde{c}\wedge\mathscr{E}_j(\tilde{H}))]. \tag{11}$$

Recall [11] that $S(\tilde{m})$ is unitarily equivalent to $S^*(m)$ for an inner function m. Equation (11), Theorem 1 and Lemma 3 imply that $S(C(\Omega))$ and $S(C(\theta))$ have the same Jordan model.

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(0, A)-semigroups on $L_p(G)$ commuting with translations are (C_0)

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1. Introduction. Let X be a Banach space and let $B(X)$ denote the Banach algebra of all bounded linear operators on X with the operator norm. Suppose that $\{T(\xi);\ \xi \geqq 0\}$ is a family of operators in $B(X)$ satisfying the following conditions:

(i) $T(\xi_1+\xi_2) = T(\xi_1)T(\xi_2)$ for $\xi_1, \xi_2 \geqq 0$, $T(0) = I$;

(ii) $T(\xi)$ is strongly measurable on $\xi > 0$.

It is well known that (i) and (ii) imply that $T(\xi)$ is strongly continuous for $\xi > 0$ [2, p. 305] and we shall call the family $\{T(\xi)\}$ a *strongly continuous* semigroup of operators on X. In studying semigroups of operators, it is usual to assume that $T(\xi)$ converges to an operator J in one sense or another as $\xi \to 0^+$. In particular, semigroups have been classified in terms of the sense in which $T(\xi)$ converges to the identity operator. Thus a strongly continuous semigroup of operators satisfying

(iii) $\lim\limits_{\xi \to 0^+} T(\xi)x = x$ for all $x \in X$

is called a semigroup of class (C_0) [2, 10.6].

A semigroup $\{T(\xi)\}$ satisfying $\lim\limits_{\xi \to 0^+} T(\xi)x = Jx$ for all $x \in X$, where J is a bounded linear operator on X is said to *converge strongly in the sense of Cauchy* with J as its Cauchy limit. If $\lim\limits_{\xi \to 0^+} T(\xi) = J$ in the uniform operator topology then $\{T(\xi)\}$ is said to *converge uniformly in the sense of Cauchy* with J as its Cauchy limit.

To define the second class of semigroups that we shall be concerned with, we need the notion of the type of a semigroup. For any strongly continuous semigroup $\{T(\xi)\}$, the real number

$$\omega_0 = \inf_{\xi > 0} \frac{1}{\xi} \log \|T(\xi)\| = \lim_{\xi \to \infty} \frac{1}{\xi} \log \|T(\xi)\|$$

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is called the *type* of $\{T(\xi)\}$. (See [2, 10.2].) A strongly measurable semigroup of operators $\{T(\xi)\}$ on X of type ω_0 is said to be of *class* $(0, A)$ if it satisfies the following conditions:

(iv) $\int_0^1 \|T(\xi)x\|\, d\xi < \infty$ for each $x \in X$;

(v) for all λ with $\operatorname{re}(\lambda) > \omega_0$, the linear operator

$$R(\lambda)x \equiv \int_0^\infty e^{-\lambda\xi} T(\xi)x\, d\xi$$

is defined and bounded for all $x \in X$;

(vi) $\lim_{\lambda\to\infty} \lambda R(\lambda)x = x$ for each $x \in X$.

A semigroup of class (C_0) is of class $(0, A)$ [2, Theorem 10.6.1]. There are a number of classes between (C_0) and $(0, A)$ which we shall not define here. For a full discussion of the basic classes of semigroups, the reader is referred to [2, 10.6].

A semigroup $\{T(\xi)\}$ satisfying $\lim_{\lambda\to\infty} \lambda R(\lambda)x = Jx$ for all $x \in X$, where J is a bounded linear operator on X is said to be *strongly Abel-ergodic at zero* with the operator J as its *Abel limit*. The condition is then written

$$(\mathrm{A})\text{-}\lim_{\xi\to 0^+} T(\xi)x \equiv \lim_{\lambda\to\infty} \lambda R(\lambda)x = Jx \quad \text{for all} \quad x \in X.$$

If

$$(\mathrm{A})\text{-}\lim_{\xi\to 0^+} T(\xi) \equiv \lim_{\lambda\to\infty} \lambda R(\lambda) = J$$

in the uniform operator topology, then $\{T(\xi)\}$ is said to be *uniformly Abel-ergodic* at zero with J as its *Abel limit* [2, 18.4.3].

In this paper, we shall be concerned with semigroups $\{T(\xi)\}$ defined on $L_p(G)$ where G is an infinite compact group and $1 \leqq p < \infty$. Two of the results proved in [3] may be stated as follows:

1.1. Theorem. *Let $\{T(\xi)\}$ be a semigroup of operators on $L_p(G)$ each of which commutes with right translations and let $\{E_\xi\}$ be the associated semigroup of $L_p(G)$-multipliers. Then $\{E_\xi\}$ converges uniformly in the sense of Cauchy to the identity operator if and only if $\{T(\xi)\}$ converges strongly in the sense of Cauchy to the identity operator.*

Our first result in the present paper is in the same spirit: Let $\{T(\xi)\}$ be a semigroup of operators on $L_p(G)$ each of which commutes with right translations and let $\{E_\xi\}$ be the associated semigroup of $L_p(G)$-multipliers. Then $\{E_\xi\}$ is uniformly measurable if and only if $\{T(\xi)\}$ is strongly measurable.

In our next theorem we show that if $\{T(\xi)\}$ is strongly Abel-ergodic at zero with the identity operator as its Abel limit, then $\{E_\xi\}$ is uniformly Abel-ergodic at zero with the identity operator as its Abel limit. These results and the result quoted from [3] suggest that the strong version of a property of $\{T(\xi)\}$ implies the uniform version of the corresponding property of $\{E_\xi\}$.

Our main result is Theorem 2.5 in which the above results are used to prove that if $\{T(\xi)\}$ is strongly Abel-ergodic at zero with the identity operator as its Abel limit, then $T(\xi)$ actually converges strongly to the identity operator in the sense of Cauchy.

The work in this paper shows again the usefulness of studying semigroups of multipliers for a function space in order to obtain results about operators on the function space itself. In this connection, see [3], [4] and [5].

2. Semigroups of operators on $L_p(G)$. For G an infinite compact group with dual object Σ, we denote by $\mathfrak{G}(\Sigma)$ the set $\underset{\sigma\in\Sigma}{PB}(H_\sigma)$ where H_σ is the representation space of the representation U^σ [1, 28.24]. If $\mathfrak{A}$ and $\mathfrak{B}$ are subsets of $\mathfrak{G}(\Sigma)$, then an element $E\in\mathfrak{G}(\Sigma)$ is said to be an $(\mathfrak{A}, \mathfrak{B})$*-multiplier* if $EA\in\mathfrak{B}$ for all $A\in\mathfrak{A}$ [1, 35.1]. An $(\mathfrak{A}, \mathfrak{A})$-multiplier will be described simply as an $\mathfrak{A}$-multiplier and an $L_p(G)^\wedge$-multiplier will be called an $L_p(G)$-multiplier. Here $L_p(G)^\wedge$ denotes the set of Fourier transforms $\hat{f}$ of $f\in L_p(G)$.

A family $\{E_\xi;\ \xi\geqq 0\}$ of functions $E_\xi\in\mathfrak{G}(\Sigma)$ is called a *semigroup of $L_p(G)$-multipliers* [3] if

(i) for each $\xi\geqq 0$, E_ξ is an $L_p(G)$-multiplier;

(ii) $E_{\xi_1+\xi_2}=E_{\xi_1}\cdot E_{\xi_2}$ for all $\xi_1, \xi_2\geqq 0$.

Condition (ii) means that for each $\sigma\in\Sigma$, $\{E_\xi(\sigma);\ \xi\geqq 0\}$ is a semigroup of operators on the space H_σ and $\{E_\xi\}$ is called a strongly (uniformly) continuous semigroup of $L_p(G)$-multipliers if each semigroup $\{E_\xi(\sigma)\}$ is strongly (uniformly) continuous.

Throughout the rest of this paper, $\{T(\xi)\}$ will denote a semigroup of operators on $L_p(G)$ each of which commutes with right translations. Such a semigroup defines a semigroup $\{E_\xi\}$ of $L_p(G)$-multipliers, the functions E_ξ being defined by

$$(T(\xi)f)^\wedge(\sigma) = E_\xi(\sigma)\hat{f}(\sigma), \quad f\in L_p(G),\ \sigma\in\Sigma$$

(see [3]). The following lemma is contained in Theorem 28.39 of [1].

2.1. Lemma. *Let $\sigma\in\Sigma$ and for $U^{(\sigma)}$ in σ with representation space H_σ, let $\mathfrak{T}_\sigma(G)$ denote the set of all finite complex linear combinations of functions of the form $x\to\langle U_x^{(\sigma)}\xi, \eta\rangle$ as ξ, η vary over H_σ. Then $\{\hat{f}(\sigma): f\in\mathfrak{T}_\sigma(G)\}=B(H_\sigma)$.*

Following [2, 3.5.1], we shall say that $T(\xi)$ is *strongly measurable in* $(0, \infty)$ if for each $f\in L_p(G)$, there exists a sequence $\{u_n(\xi)\}$ of countably-valued functions (depending on f) from $(0, \infty)$ into $L_p(G)$ converging almost everywhere to $T(\xi)f$ in the topology of $L_p(G)$. For $\sigma\in\Sigma$, the semigroup $E_\xi(\sigma)$ is said to be *uniformly measurable in* $(0, \infty)$ if there exists a sequence of countably-valued functions $\{U_n(\xi)\}$ from $(0, \infty)$ into $B(H_\sigma)$ converging almost everywhere to $E_\xi(\sigma)$ in the uniform operator topology of $B(H_\sigma)$.

We can now state our first result.

2.2. Theorem. *Let $\{T(\xi)\}$ be a semigroup of operators on $L_p(G)$ each of which commutes with right translations and let $\{E_\xi\}$ be the associated semigroup of multipliers. Then $\{E_\xi\}$ is uniformly measurable if and only if $\{T(\xi)\}$ is strongly measurable.*

Proof. Suppose that $\{T(\xi)\}$ is strongly measurable and let σ be an arbitrary but fixed element of Σ. By Lemma 2.1, there exists $t\in\mathfrak{T}_\sigma(G)$ such that $\hat{t}(\sigma)=I_\sigma$, the identity operator on H_σ. The strong measurability of $\{T(\xi)\}$ implies that there exist a sequence $\{u_n\}$ of countably-valued functions on $(0, \infty)$ into $L_p(G)$ and a null set $E_0\subset(0, \infty)$ such that $\lim_{n\to\infty} \|T(\xi)t-u_n(\xi)\|_p=0$ for all $\xi\in(0, \infty)\sim E_0$. Then clearly $\{\hat{u}_n(\xi)(\sigma)\}$ is a sequence of countably-valued functions on $(0, \infty)$ into $B(H_\sigma)$. Moreover we have

$$\|E_\xi(\sigma)-\hat{u}_n(\xi)(\sigma)\|_{B(H_\sigma)} = \|E_\xi(\sigma)\hat{t}(\sigma)-\hat{u}_n(\xi)(\sigma)\|_{B(H_\sigma)} =$$
$$= \|[T(\xi)t-u_n(\xi)]\hat{\ }(\sigma)\|_{B(H_\sigma)} \leqq \|T(\xi)t-u_n(\xi)\|_p \to 0$$

as $n\to\infty$, for all $\xi\in(0, \infty)\sim E_0$. Hence $\{\hat{u}_n(\xi)(\sigma)\}$ converges almost everywhere on $(0, \infty)$ to $E_\xi(\sigma)$ in the uniform norm and so $E_\xi(\sigma)$ is uniformly measurable on $(0, \infty)$. Since σ was arbitrary, $\{E_\xi\}$ is uniformly measurable.

Conversely, let $\{E_\xi\}$ be uniformly measurable for $\sigma\in\Sigma$; there exist a sequence $\{U_n^\sigma\}$ of countably-valued functions on $(0, \infty)$ into $B(H_\sigma)$ and a null set $E_0^\sigma\subset(0, \infty)$ such that

$$\lim_{n\to\infty} \|E_\xi(\sigma)-U_n^\sigma(\xi)\|_{B(H_\sigma)} = 0 \quad \text{for all} \quad \xi\in(0, \infty) \sim E_0^\sigma.$$

By Lemma 2.1, this means there exists a sequence $\{t_n\}$ of countably valued functions on $(0, \infty)$ to $\mathfrak{T}_\sigma(G)$ such that $\hat{t}_n(\xi)(\sigma)=U_n^\sigma(\xi)$ and

$$\lim_{n\to\infty} \|E_\xi(\sigma)-\hat{t}_n(\xi)(\sigma)\|_{B(H_\sigma)} = 0 \quad \text{for all} \quad \xi\in(0, \infty) \sim E_0^\sigma.$$

Then for any coordinate function $u_{jk}^{(\sigma)}$, using the notation in the proof Theorem 3.3 of [3], we have

$$\|T(\xi)u_{jk}^{(\sigma)}-t_n(\xi)*u_{jk}^{(\sigma)}\|_p \leqq d_\sigma\left\|(T(\xi)u_{jk}^{(\sigma)})\hat{\ }(\sigma)-(t_n(\xi)*u_{jk}^{(\sigma)})\hat{\ }(\sigma)\right\|_{\Phi_1} =$$
$$= d_\sigma\|E_\xi(\sigma)\hat{u}_{jk}^{(\sigma)}(\sigma)-\hat{t}_n(\xi)(\sigma)\hat{u}_{jk}^{(\sigma)}(\sigma)\|_{\Phi_1} \leqq d_\sigma\|E_\xi(\sigma)-\hat{t}_n(\xi)(\sigma)\|_{\Phi_\infty}\|\hat{u}_{jk}^{(\sigma)}(\sigma)\|_{\Phi_1} =$$
$$= d_\sigma\|E_\xi(\sigma)-\hat{t}_n(\xi)(\sigma)\|_{B(H_\sigma)}\|\hat{u}_{jk}^{(\sigma)}(\sigma)\|_{\Phi_1} \to 0 \quad \text{as } n\to\infty \text{ and for all } \quad \xi\in(0, \infty) \sim E_0.$$

Hence for every coordinate function u, the sequence $\{t_n(\xi)*u\}$ of countably-valued functions on $(0, \infty)$ converges almost everywhere to $T(\xi)u$ in the $L_p(G)$-norm. That $\{t_n(\xi)*f\}$ converges almost everywhere to $T(\xi)f$ for each $f\in L_p(G)$

in the $L_p(G)$-norm now follows from the fact that the operators $T(\xi)$ are linear and continuous and the trigonometric polynomials are dense in $L_p(G)$. This concludes the proof.

2.3. Theorem. *Let $\{T(\xi)\}$ be a strongly measurable semigroup of operators on $L_p(G)$ each of which commutes with right translations and let $\{E_\xi\}$ be the associated semigroup of $L_p(G)$-multipliers. Suppose that $\{T(\xi)\}$ is of type ω_0 and that for each $f\in L_p(G)$ the integral $R(\lambda)f=\int\limits_0^\infty e^{-\lambda\xi}T(\xi)f d\xi$ exists for all λ with* re $(\lambda)>\omega_0$. *Then for each $\sigma\in\Sigma$, the integral $P(\lambda)(\sigma)=\int\limits_0^\infty e^{-\lambda\xi}E_\xi(\sigma)d\xi$ exists as an element of $B(H_\sigma)$ for all λ with* re $(\lambda)>\omega_0$. *Moreover, if $\{T(\xi)\}$ is strongly Abel-ergodic at zero with the identity operator as its Abel limit, then for each $\sigma\in\Sigma$, $\{E_\xi(\sigma)\}$ is uniformly Abel-ergodic at zero with the identity operator as its Abel limit.*

Note. Here and throughout this paper, the integrals are in the sense of Bochner [2, 3.7].

Proof. Since $\{T(\xi)\}$ is strongly measurable, $\{E_\xi(\sigma)\}$ is uniformly measurable for each $\sigma\in\Sigma$, by Theorem 2.2. If t is chosen as in the proof of Theorem 2.2, we have for all λ with re $(\lambda)>\omega_0$,

$$\int\limits_0^\infty \|e^{-\lambda\xi}E_\xi(\sigma)\|_{B(H_\sigma)}\,d\xi=\int\limits_0^\infty \left\|e^{-\lambda\xi}(T(\xi)t)\hat{\ }(\sigma)\right\|_{B(H_\sigma)}d\xi\leqq\int\limits_0^\infty \|e^{-\lambda\xi}T(\xi)t\|_p\,d\xi<\infty.$$

Hence by [2, Theorem 3.7.4], the Bochner integral $\int\limits_0^\infty e^{-\lambda\xi}E_\xi(\sigma)\,d\xi$ exists as an element of $B(H_\sigma)$ for each λ with re $(\lambda)>\omega_0$. Moreover, for all such λ, we have

$$\left\|\lambda\int\limits_0^\infty e^{-\lambda\xi}E_\xi(\sigma)\,d\xi-E_0(\sigma)\right\|_{B(H_\sigma)}=\left\|\lambda\int\limits_0^\infty e^{-\lambda\xi}(T(\xi)t)\hat{\ }(\sigma)\,d\xi-(T(0)t)\hat{\ }(\sigma)\right\|_{B(H_\sigma)}=$$

$$=\left\|\left[\lambda\int\limits_0^\infty e^{-\lambda\xi}T(\xi)t\,d\xi-T(0)t\right]\hat{\ }(\sigma)\right\|_{B(H_\sigma)}\leqq\left\|\lambda\int\limits_0^\infty e^{-\lambda\xi}T(\xi)t\,d\xi-t\right\|_p\to 0$$

as $\lambda\to\infty$, which completes the proof of the theorem.

The proof of our main result depends on the following very striking ergodic theorem which holds for a much wider class of semigroups than needed here [2, 18.8.3].

2.4. Theorem. *Let $\{S(\xi)\}$ be a semigroup of class* (0, A) *on a Banach space X and suppose that $\{S(\xi)\}$ is uniformly Abel-ergodic at zero with J as its Abel limit. Then $S(\xi)=J\exp(\xi A)$ where $J^2=J$, $A\in B(X)$; $AJ=JA=A$ and* uniform $\lim\limits_{\xi\to 0^+}S(\xi)=J$, i.e., *$S(\xi)$ converges uniformly to J in the sense of Cauchy.*

2.5. Theorem. *Let* $\{T(\xi)\}$ *be a semigroup of class* $(0, A)$ *on* $L_p(G)$ *each of which commutes with right translations. Then* $\{T(\xi)\}$ *is a semigroup of class* (C_0).

Proof. Let $\{E_\xi\}$, as before, denote the associated semigroup of $L_p(G)$-multipliers. Then for each $\sigma\in\Sigma$, $\{E_\xi(\sigma)\}$ is, by Theorem 2.3, uniformly Abel-ergodic with the identity operator as its Abel limit.

Since $\{E_\xi(\sigma)\}$ is clearly of class $(0, A)$, it follows from Theorem 2.4 that $\lim_{\xi\to 0^+} \|E_\xi(\sigma)-E_0(\sigma)\|_{B(H_\sigma)}=0$, $E_0(\sigma)=I_\sigma$, the identity operator on H_σ. Thus $E_\xi(\sigma)$ is uniformly continuous for all $\xi\geqq 0$ and the same is true for each $\sigma\in\Sigma$. Now $\{T(\xi);\ \xi\geqq 0\}$ is, in the terminology of [3], the semigroup of operators on $L_p(G)$ defined by the semigroup of $L_p(G)$-multipliers $\{E_\xi(\sigma);\ \xi\geqq 0,\ \sigma\in\Sigma\}$. Hence by Theorem 1.1, $\{T(\xi);\ \xi\geqq 0\}$ is strongly continuous for all $\xi\geqq 0$ and is therefore of class (C_0). This concludes the proof.

As stated in the Introduction, there are a number of classes between (C_0) and $(0, A)$. Theorem 2.5 shows that if $T(\xi)$ commutes with right translations, then all these classes collapse into (C_0).

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Ergodic theorems in von Neumann algebras

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0. Introduction. The classical individual ergodic theorem claims that if $(X, \mathfrak{S}, \mu)$ is a measure space, α is an invertible measure-preserving transformation of X then for every integrable complex function f on X the averages

$$(1) \qquad s_n(f) = \frac{1}{n}(f+\alpha f+\ldots+\alpha^{n-1}f)$$

converge μ-almost everywhere to an α-invariant function (where αf is defined by $(\alpha f)(x)=f(\alpha(x))$). In a von Neumann algebra setting one may investigate the convergence of averages of type (1), when f is an element of a von Neumann algebra $\mathfrak{A}$ and α is an automorphism of $\mathfrak{A}$. The first ergodic theorems for automorphisms of von Neumann algebras were established by KOVÁCS and SZŰCS [7], [8] and give that the averages (1) converge strongly provided that $\mathfrak{A}$ has a faithful normal α-invariant state φ. Later LANCE [10] proved an almost uniform ergodic theorem. Namely, if $A\in\mathfrak{A}$ then there exists an element $\hat{A}\in\mathfrak{A}$ such that for every $\varepsilon>0$ there is a projection E in $\mathfrak{A}$ with the property

$$(2) \qquad \varphi(I-E)<\varepsilon, \quad s_n(A)E \to \hat{A}E$$

in norm (shortly $s_n(A)\to\hat{A}$ φ-almost uniformly). A similar theorem was obtained by SINAĬ and ANŠELEVIČ [12] in special circumstances (for quantum lattice systems), for several parameters. The crucial point of Lance's proof is a maximal ergodic theorem: if $A\in\mathfrak{A}^+$ and $\varepsilon=\varphi(A)^{1/2}$ then there is an operator $C\in\mathfrak{A}$ such that $s_n(A)\leqq C$ for every $n\in\mathrm{N}$ and $\|C\|\leqq 2\|A\|$, $\varphi(C)\leqq 4\sqrt{\varepsilon}$. This does not have an analogue in the commutative ergodic theory but (and because) it is a simple consequence of Hopf's maximal ergodic theorem.

Further extension of the almost uniform theory has appeared in [2], [4], [14] and [15]. The main objective of this paper is to replace the invariant state with an invariant weight and to obtain a slightly weaker almost uniform convergence. In

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fact, instead of (2) we can prove

$$Es_n(A)E \to E\hat{A}E \tag{3}$$

in norm. YEADON [16] proved a similar convergence under the condition that there exists a faithful normal semifinite trace. We treat continuous flows and the case of several parameters, as well.

Let $\mathfrak{A}$ be a von Neumann algebra and φ a faithful semifinite normal weight on $\mathfrak{A}^+$. Then $\mathfrak{A}_0=\{A\in\mathfrak{A}: \varphi(A^*A)<+\infty$ and $\varphi(AA^*)<+\infty\}$ becomes a full left Hilbert algebra with *-algebra structure induced by $\mathfrak{A}$ and with inner-product $\langle A, B\rangle_\varphi=\varphi(B^*A)$ $(A, B\in\mathfrak{A}_0)$. Our main reference on this subject is the monograph [13], whose notation we shall follow. Denote by $\mathscr{H}$ the Hilbert space completion of $\mathfrak{A}_0$. For $B\in\mathfrak{A}_0$ one defines an $L_B\in\mathscr{B}(\mathscr{H})$ by the formula $L_BA=BA$ $(A\in\mathfrak{A}_0)$. $\mathscr{L}(\mathfrak{A}_0)=\{L_B: B\in\mathfrak{A}_0\}''$ is called the left von Neumann algebra of $\mathfrak{A}_0$. There is a faithful representation $\pi: \mathfrak{A}\to\mathscr{L}(\mathfrak{A}_0)$ defined by $\pi(A)B=AB$ $(A\in\mathfrak{A}, B\in\mathfrak{A}_0)$ such that for $A\in\mathfrak{A}^+$

$$\varphi(A)=\begin{cases}\|B\|_\varphi^2, & \text{if there exists } B\in\mathfrak{A}_0 \text{ such that } \pi(A)^{1/2}=L_B\\ +\infty, & \text{otherwise.}\end{cases}$$

Here $\|B\|_\varphi^2=\langle B, B\rangle_\varphi$. (See [13], p. 276 or [1].) So we may assume that $\mathfrak{A}$ is the left von Neumann algebra of a full (i.e., achieved) left Hilbert algebra $\mathfrak{A}_0$ and φ is the canonical weight on $\mathscr{L}(\mathfrak{A}_0)^+$.

Suppose that $\mathfrak{A}$ and φ are fixed. A linear mapping $\alpha: \mathfrak{A}\to\mathfrak{A}$ will be called a kernel provided that the following conditions hold:

(i) for $0\leq A\leq I$ and $A\in\mathfrak{A}$ we have $0\leq\alpha(A)\leq I$ and $\varphi(\alpha(A))\leq\varphi(A)$,

(ii) for every $A\in\mathfrak{A}$ the inequality $\varphi(\alpha(A)^*\alpha(A))\leq\varphi(A^*A)$ is valid.

Kernels proved to be useful in ergodic theory. Every Schwarz map satisfying condition (i) is a kernel. In particular, endomorphisms and completely positive maps of norm one are kernels. We are going to see that kernels have some automatic continuity.

1. The maximal ergodic theorem. The proofs of individual ergodic theorems usually need a maximal ergodic theorem. Ours involves a series of operators.

Theorem 1. *Let φ be a faithful semifinite normal weight on a von Neumann algebra $\mathfrak{A}$ and α a linear mapping $\mathfrak{A}\to\mathfrak{A}$ satisfying condition* (i). *Assume that $A_m\in\mathfrak{A}^+$ and $\varepsilon_m>0$ $(m\in\mathbf{N})$. Then there is a projection $E\in\mathfrak{A}$ such that*

$$\|Es_r(A_m)E\|\leq 2\varepsilon_m \quad (r, m\in\mathbf{N}), \tag{4}$$

$$\varphi(I-E)\leq 2\sum_{m=1}^{\infty}\varepsilon_m^{-1}\varphi(A_m). \tag{5}$$

We divide the proof into lemmas. We always assume $\mathfrak{A}$ to be a left von Neumann algebra $\mathscr{L}(\mathfrak{A}_0)$ of a full Hilbert algebra $\mathfrak{A}_0$.

Lemma 1. *Under the hypotheses of Theorem 1, for any* $n\in\mathrm{N}$ *there is a projection* $E_n\in\mathfrak{A}$ *such that*

$$\|E_n s_r(A_m)E_n\| \leqq \varepsilon_m \quad (r,\, m \leqq n), \tag{6}$$

$$\varphi(I-E_n) \leqq \sum_{m=1}^{\infty} \varepsilon_m^{-1}\varphi(A_m). \tag{7}$$

Proof. Let $\mathfrak{A}_0'$ be the right Hilbert algebra associated with $\mathfrak{A}_0$. So $\mathfrak{A}_0'\subset\mathscr{H}$ and for $\eta\in\mathfrak{A}_0'$ the formula $R_\eta\xi=L_\xi\eta$ $(\xi\in\mathfrak{A}_0)$ defines a bounded operator $R_\xi\in\mathscr{B}(\mathscr{H})$. It is well-known that

$$\{R_\eta\colon\, \eta\in\mathfrak{A}_0'\}' = \mathscr{L}(\mathfrak{A}_0).$$

(See [13] or [14].) Let φ' be the canonical weight on the right von Neumann algebra $\mathscr{R}(\mathfrak{A}_0')=\{R_\eta\colon \eta\in\mathfrak{A}_0'\}''$, that is, for $T\in\mathscr{R}(\mathfrak{A}_0')^+$

$$\varphi'(T) = \begin{cases} \|\eta\|^2, & \text{if there is } \eta\in\mathfrak{A}_0' \text{ such that } R_\eta = T^{1/2} \\ +\infty, & \text{otherwise.}\end{cases}$$

For $A\in\mathscr{L}(\mathfrak{A}_0)^+$ and $T\in\mathscr{R}(\mathfrak{A}_0')^+$ we define $h(A,T)$ if $\varphi(A)<+\infty$ or $\varphi'(T)<+\infty$. Namely, let

$$h(A,T) = \langle T\xi,\xi\rangle \quad \text{if } A^{1/2}=L_\xi \text{ for some } \xi\in\mathfrak{A}_0,$$

$$h(A,T) = \langle A\eta,\eta\rangle \quad \text{if } T^{1/2}=R_\eta \text{ for some } \eta\in\mathfrak{A}_0'.$$

h is *wo*-continuous and additive in each variable separately, and

$$\varphi(A) = \sup\{h(A,T)\colon\, 0\leqq T\leqq I,\ T\in\mathscr{R}(\mathfrak{A}_0'),\ \varphi'(T)<+\infty\},$$

$$\varphi'(T) = \sup\{h(A,T)\colon\, 0\leqq A\leqq I,\ A\in\mathscr{L}(\mathfrak{A}_0),\ \varphi(A)<+\infty\}.$$

Let $\eta\in\mathfrak{A}_0'$. Then the formula

$$(\xi_1,\xi_2)\mapsto\langle\alpha(L_{\xi_2^\#\xi_1})\eta,\eta\rangle \quad (\xi_1,\xi_2\in\mathfrak{A}_0)$$

defines a bounded sesquilinear form on $\mathfrak{A}_0$. Since $\varphi\big(\alpha(L_{\xi_i^\#\xi_i^\#})\big)\leqq\varphi(L_{\xi_i^\#\xi_i})=$ $=\|\xi_i\|^2<+\infty$ there is $\mu_i\in\mathfrak{A}_0$ such that $\alpha(L_{\xi_i^\#\xi_i})^{1/2}=\mu_i$ and we have the following estimation.

$$|\langle\alpha(L_{\xi_2^\#\xi_1})\eta,\eta\rangle| \leqq \langle\alpha(L_{\xi_1^\#\xi_1})\eta,\eta\rangle^{1/2}\langle\alpha(L_{\xi_2^\#\xi_2})\eta,\eta\rangle^{1/2} =$$

$$= \langle R_{\eta^b\eta}\mu_1,\mu_1\rangle^{1/2}\langle R_{\eta^b\eta}\mu_2,\mu_2\rangle^{1/2} \leqq \|R_{\eta^b\eta}\|\|\mu_1\|\|\mu_2\| \leqq \|R_{\eta^b\eta}\|\|\xi_1\|\|\xi_2\|.$$

Consequently, there is a bounded operator $\bar{\alpha}(R_{\eta b_\eta})\in\mathscr{B}(\mathscr{H})$ such that

(8) $$\langle\bar{\alpha}(R_{\eta b_\eta})\xi_1, \xi_2\rangle = \langle\alpha(L_{\xi_2^\# \xi_1})\eta, \eta\rangle.$$

If $T\in\mathscr{R}(\mathfrak{A}_0')^+$ and $\varphi'(T)<+\infty$ then $\bar{\alpha}(T)\in\mathscr{B}(\mathscr{H})^+$ and $\|\bar{\alpha}(T)\|\leqq\|T\|$. Since $\bar{\alpha}(T)$ commutes with L_ξ for every $\xi\in\mathfrak{A}_0$ we have $\bar{\alpha}(T)\in\mathscr{R}(\mathfrak{A}_0')^+$. Taking $A=L_{\xi^\#\xi}$ and $T=R_{\eta b_\eta}$ ($\xi\in\mathfrak{A}_0$, $\eta\in\mathfrak{A}_0'$) we obtain $h(A, \bar{\alpha}(T))=h(\alpha(A), T)$ from (8). We can use this to show that $\varphi'(\bar{\alpha}(T))\leqq\varphi'(T)$ for $T\in\mathscr{R}(\mathfrak{A}_0')^+$. Namely,

$$\varphi'(\bar{\alpha}(T)) = \sup\{h(A, \bar{\alpha}(T)) \colon A \leqq I,\ A\in\mathscr{L}(\mathfrak{A}_0)^+,\ \varphi(A) < +\infty\} =$$
$$= \sup\{h(\alpha(A), T) \colon A \leqq I,\ A\in\mathscr{L}(\mathfrak{A}_0)^+,\ \varphi(A) < +\infty\} \leqq \varphi'(T).$$

$\mathscr{R}(\mathfrak{A}_0')\overline{\otimes}M_n$ is the von Neumann algebra of $n\times n$ matrices with entries from $\mathscr{R}(\mathfrak{A}_0')$. Its elements will be denoted by $(X_{r,m})$, where $X_{r,m}\in\mathscr{R}(\mathfrak{A}_0')$ $(r, m\leqq n)$.

$$K = \{(X_{r,m})\in\mathscr{R}(\mathfrak{A}_0')\overline{\otimes}M_n \colon X_{r,m} \geqq 0,\ \Sigma X_{r,m} \leqq I\}$$

is an ultraweakly compact convex set. We define a real function on K in the following fashion:

$$g((X_{r,m})) = \sum_{r=1}^{n}\sum_{m=1}^{n} r\left[h(S_r(B_m), X_{r,m}) - \varphi'(X_{r,m})\right]$$

where $B_m\in\mathscr{L}(\mathfrak{A}_0)^+$ is fixed and $\varphi(B_m)<+\infty$ $(m\leqq n)$. The function g is ultraweakly upper semicontinuous and attains its finite maximum value for some choice $(X_{r,m})\in K$. If $I-\Sigma X_{r,m}=Z$, $X\in\mathscr{R}(\mathfrak{A}_0')$ and $0\leqq X\leqq Z$ then from the inequality

$$g((X_{r,m})) \geqq g((X_{r,m}+\delta(r, r_0)\delta(m, m_0)X))$$

we obtain

(9) $$h(s_{r_0}(B_{m_0}), X) \leqq \varphi'(X)$$

for every $r_0, m_0\leqq n$.

Now take

$$Y_{r,m} = \begin{cases} \bar{\alpha}(X_{r+1,m}) & \text{for } r \leqq n-1 \\ 0 & \text{for } r = n. \end{cases}$$

The properties of $\bar{\alpha}$ give that $(Y_{r,m})\in K$ and hence $g((X_{r,m}))\geqq g((Y_{r,m}))$. It follows that

$$\sum_{m=1}^{n}\sum_{r=1}^{n}[h(B_m, X_{r,m}) - \varphi'(X_{r,m})] \geqq \sum_{m=1}^{n}\sum_{r=1}^{n}(r-1)\left[\varphi'(X_{r,m}) - \varphi'(\bar{\alpha}(X_{r,m}))\right].$$

Replace B_m with $\varepsilon_m^{-1}A_m$. So

(10) $$\sum_{m=1}^{n}\sum_{r=1}^{n}\varepsilon_m^{-1}h(A_m, X_{r,m}) \geqq \sum_{m=1}^{n}\sum_{r=1}^{n}\varphi'(X_{r,m}),$$

and by (9)

(11) $$h(s_r(A_m), X) \leqq \varepsilon_m\varphi'(X) \quad (r, m \leqq n).$$

Let $E_0=\{\eta\in\mathfrak{A}_0'\colon R_\eta^*R_\eta\leqq\lambda Z$ for some $\lambda>0\}$. E_0 is a linear subspace of $\mathscr{H}$. If $\eta\in E_0$ and $\omega\in\mathfrak{A}_0'$ then $R_\omega\eta\in\mathfrak{A}_0'$ and by [13], p. 249, for $Z=R_{R_\omega\eta}$ we have $T^*T=R_\eta^*R_\omega^*R_\omega R_\eta\leqq\|R_\omega\|^2R_\eta^*R_\eta$. So E_0 is stable under the operators R_η $(\eta\in\mathfrak{A}_0')$ and if E_n denotes the orthogonal projection onto the closure of E_0, then $E_n\in\mathscr{L}(\mathfrak{A}_0)$.

If $\eta\in E_0$ then $\langle s_r(A_m)\eta,\eta\rangle=h(s_r(A_m),R_\eta^*R_\eta)\leqq\varepsilon_m\varphi'(R_\eta^*R_\eta)=\varepsilon_m\|\eta\|^2$ according to (11). Therefore we may conclude that $\|E_ns_r(A_m)E_n\|\leqq\varepsilon_m$.

Let F be a projection in $\mathscr{L}(\mathfrak{A}_0)$ such that $F\leqq I-E_n$ and $\varphi(F)<+\infty$. Then

$$\varphi(F)=\sup\{h(F,Z_1+\sum X_{r,m})\colon 0\leqq Z_1\leqq Z,\ \varphi'(Z_1)<+\infty\}\leqq$$
$$\leqq 0+h(F,\sum X_{r,m})\leqq\varphi'(\sum X_{r,m})\leqq\sum\varepsilon_m^{-1}h(A_m,X_{r,m})\leqq$$
$$\leqq\sum_m\varepsilon_m^{-1}h(A_m,\sum_m X_{r,m})\leqq\sum_{m=1}^n\varepsilon_m^{-1}\varphi(A_m).$$

Since φ is semifinite and lower w-semicontinuous we have $\varphi(I-E_n)=\sum_{m=1}^n\varepsilon_m^{-1}\varphi(A_m)$ and the proof is complete.

Lemma 2. *Under the conditions of Theorem* 1 *there is a* $C\in\mathscr{L}(\mathfrak{A}_0)^+$ *such that* $C\leqq I$ *and*

$$(12)\qquad Cs_n(A_m)C\leqq\varepsilon_mC\quad(n,m\in\mathbf{N}),$$

$$(13)\qquad \varphi(I-C)\leqq\sum_{m=1}^\infty\varepsilon_m^{-1}\varphi(A_m).$$

Proof. Let E_n be the projection guaranteed by Lemma 1. There is a convergent subsequence (E_{n_k}) of (E_n) and $E_{n_k}\xrightarrow{\text{wo}}C$ for some $C\in\mathscr{L}(\mathfrak{A}_0)$. Evidently $0\leqq C\leqq I$ and by the semicontinuity $\varphi(I-C)\leqq\sum_{m=1}^\infty\varepsilon_m^{-1}\varphi(A_m)$. From $E_ns_r(A_m)E_n\leqq\varepsilon_mE_n$ $(r,m\leqq n)$ a routine argument gives that $Cs_r(A_m)C\leqq\varepsilon_mC$ for every $r,m\in\mathbf{N}$.

Proof of Theorem 1. Take $C\in\mathfrak{A}^+$ with properties (12) and (13) in Lemma 2 and let $\int_0^1\lambda dP(\lambda)$ be the spectral resolution of C. For $E=I-P(1/2)$ we have $I-E=P(1/2)\leqq2(I-C)$ and (5) follows from (13). On the other hand,

$$Es_r(A_m)E=DCs_r(A_m)CD\leqq\varepsilon_mDCD\leqq2\varepsilon_mE$$

where $D=\int_{1/2}^1\lambda^{-1}dP(\lambda)$. This completes the proof.

The first maximal ergodic theorem similar to Theorem 1 was obtained by Yeadon [16] for a trace instead of a general weight and for a single operator instead of a sequence. A version for state and for a sequence appeared in Goldsteĭn's paper [4]. Here we utilized several of their ideas. If $A_m=0$ for $m>1$ then the theorem claims the existence of a projection possessing the properties $\|Es_r(A)E\| \leqq 2\varepsilon$ and $\varphi(I-E) \leqq 2\varepsilon^{-1}\varphi(A)$. In the commutative case this is equivalent to the inequality

$$\mu(\{x\colon \sup s_n(f)(x) > \lambda\}) \leqq \frac{c}{\lambda}\|f\|_1,$$

which frequently occurs in commutative ergodic theory (see, for example, [3], p. 705).

As a matter of fact, Theorem 1 implies Lance's maximal ergodic theorem. Namely,

$$s_r(A) \leqq 2Es_r(A)E+2(I-E)s_r(A)(I-E) \leqq 4\varepsilon E+2\|A\|(I-E) = C_\varepsilon.$$

If $\|A\| \leqq 1$ and $\varepsilon = \varphi(A)^{1/2}$ then $\|C_\varepsilon\| \leqq 2$ and $\varphi(C_\varepsilon) \leqq 8\varphi(A)^{1/2}$. We notice that if $\varphi(I) = +\infty$ then the assertion of Lance's maximal ergodic theorem is false even in the commutative case.

2. An individual ergodic theorem. In this paragraph we are going to use Theorem 1 to deduce the following

Theorem 2. *Let φ be a faithful semifinite normal weight on a von Neumann algebra $\mathfrak{A}$ and α a kernel on $\mathfrak{A}$. Assume that $A \in \mathfrak{A}$ and $\varphi(A^*A) < +\infty$, $\varphi(AA^*) < +\infty$. Then there is $\hat{A} \in \mathfrak{A}$ such that for every $\varepsilon > 0$ there exists a projection E in $\mathfrak{A}$ satisfying the following conditions*

$$\varphi(I-E) < \varepsilon \quad \textit{and} \quad \left\|E\big(s_n(A)-\hat{A}\big)E\right\| \to 0.$$

Moreover $\varphi(\hat{A}^*\hat{A}) < +\infty$ and $\varphi(\hat{A}\hat{A}^*) < +\infty$.

We notice that Lance proved $s_n(A) \to \hat{A}$ ultrastrongly in [10].

Lemma 3. *Suppose that $B \in \mathfrak{A}^{sa}$ and $\varphi(B^2) < +\infty$. Then there is a decomposition $B = C+D-E$ where $C \in \mathfrak{A}^{sa}$, $D \in \mathfrak{A}^+$, $E \in \mathfrak{A}^+$, $\|C\| \leqq \varphi(B^2)^{1/2}$, $\varphi(D) \leqq \varphi(B^2)^{1/2}$, $\varphi(E) \leqq \varphi(B^2)^{1/2}$ and $\|C\|$, $\|D\|$, $\|E\| \leqq \|B\|$.*

Proof. Let $\int_{-\infty}^{\infty} \lambda dP(\lambda)$ be the spectral resolution of B. Take $C = \int_{-\varepsilon}^{\varepsilon} \lambda dP(\lambda)$, $D = \int_{\varepsilon}^{\infty} \lambda dP(\lambda)$ and $E = -\int_{-\infty}^{-\varepsilon} \lambda dP(\lambda)$ where $\varepsilon = \varphi(B^2)^{1/2}$. Then $D, E \leqq \varepsilon^{-1}B^2$ and all the requirements are fulfilled.

Lemma 4. *For $B \in \mathfrak{A}$ we have $\left\|s_n(B-s_k(B))\right\| \leqq kn^{-1}\|B\|$ if $n > k$.*

Proof. It is straightforward from the identity

$$ns_n(B-s_k(B)) = \sum_{i=0}^{k-2} \alpha^i(B) - \sum_{i=0}^{k-2} \frac{i+1}{k}\alpha^i(B) - \sum_{i=0}^{k-2} \frac{k-1-i}{k}\alpha^{n+i}(B) \quad (k > 1).$$

Proof of Theorem 2. For $\xi\in\mathfrak{A}_0$ let $V\xi=\mu$ where $\mu\in\mathfrak{A}_0$ and $\alpha(L_\xi)=L_\mu$. Since $\varphi(\alpha(L_\xi)^*\alpha(L_\xi))\leqq\varphi(L_{\xi^\#\xi})=\|\xi\|^2$ such a μ exists and V can be extended to a contraction on $\mathscr{H}$. By the mean ergodic theorem for a contraction ([11], p. 144) there is a projection $P\in\mathscr{B}(\mathscr{H})$ such that $n^{-1}\sum_{i=0}^{n-1}V^i\xi\to P\xi$ for every $\xi\in\mathfrak{A}_0$. If $f\in\mathscr{L}(\mathfrak{A}_0)_*$ and $f(B)=\langle B\eta_1,\eta_2\rangle$ for some $\eta_1,\eta_2\in\mathfrak{A}_0'$ then

$$\Phi(f)=\lim\langle s_n(A)\eta_1,\eta_2\rangle=\lim\left\langle R_{\eta_1}n^{-1}\sum_{i=0}^{n-1}V^i\xi_0,\eta_2\right\rangle=$$

$$=\langle R_{\eta_1}P\xi_0,\eta_2\rangle=\langle P\xi_0,\eta_2\eta_1^b\rangle$$

where $A=L_{\xi_0}$. Since $|\Phi(f)|\leqq\|A\|\,\|\eta_1\|\,\|\eta_2\|=\|S\|\,\|f\|$ there is an element $\hat{A}\in\mathscr{L}(\mathfrak{A}_0)$ with the property

$$\langle\hat{A}\eta_1,\eta_2\rangle=\langle R_{\eta_1}P\xi_0,\eta_2\rangle$$

for every $\eta_1,\eta_2\in\mathfrak{A}_0'$. Similarly,

$$\langle(\hat{A})^*\eta_1,\eta_2\rangle=\langle R_{\eta_1}P\xi_0^\#,\eta_2\rangle.$$

Hence $\hat{A}\eta=R_\eta P\xi_0$ and $(\hat{A})^*\eta=R_\eta P\xi_0^\#$ for every $\eta\in\mathfrak{A}_0'$. By [13], p. 252, we may conclude $P\xi_0\in\mathfrak{A}_0''=\mathfrak{A}_0$ and $\hat{A}=L_{P\xi_0}$. Consequently, $s_n(A)\xrightarrow{\text{w}}\hat{A}$ and $\alpha(\hat{A})=\hat{A}$. According to the mean ergodic theorem,

$$\xi_0-P\xi_0=\xi_0-k^{-1}\sum_{i=0}^{k-1}V^i\xi_0+\xi_k\quad(k\in\mathrm{N}),$$

where $\|\xi_k\|=\delta_k$ and $\delta_k\to 0$. By the left representation L we have

$$A-\hat{A}=A-s_k(A)+B_k$$

where $\|B_k\|_2=\varphi(B_k^*B_k)^{1/2}=\|\xi_k\|=\delta_k$. If $A=A^*$ then $B_k=B_k^*$, and by splitting into selfadjoint and skewadjoint parts we arrive at the decomposition

$$A-\hat{A}=A-s_k(A)+B_k^1+iB_k^2\tag{14}$$

and here $B_k^1,B_k^2\in\mathscr{L}(\mathfrak{A}_0)^{\text{sa}}$ and $\|B_k^1\|_2,\|B_k\|_2\leqq\delta_k$. Apply Lemma 3 for B_k^1 and B_k^2. So

$$A-\hat{A}=A-s_k(A)+C_k^1+D_k^1+E_k^1+i(C_k^2+D_k^2-E_k^2)\tag{15}$$

and $\|C_k^i\|\leqq\delta_k$, $\varphi(D_k^i)\leqq\delta_k$, $\varphi(E_k^i)\leqq\delta_k$ $(i=1,2)$. Choose a subsequence (δ_{m_k}) of (δ_k) such that $\delta_{m_k}<16\cdot k^{-1}2^{-k}\cdot\varepsilon$ and use Theorem 1. Taking $\{A_m\}=\bigcup_{k=1}^\infty\{D_{m_k}^1,E_{m_k}^1,D_{m_k}^2,E_{m_k}^2\}$ and putting $1/k$ in the role of ε corresponding to $D_{m_k}^1,E_{m_k}^1,D_{m_k}^2$ and $E_{m_k}^2$, we obtain a projection E such that

$$\|Es_n(D^i_{m_k})E\| \leqq 2k^{-1} \quad (i = 1, 2, k\in\mathbf{N}, n\in\mathbf{N}),$$
$$\|Es_n(E^i_{m_k})E\| \leqq 2k^{-1} \quad (i = 1, 2, k\in\mathbf{N}, n\in\mathbf{N}),$$

$$\varphi(I-E) \leqq 4\cdot 2 \sum_{k=1}^{\infty} k\delta_{m_k} \leqq \varepsilon.$$

In order to prove $\|E(s_n(A)-\hat{A})E\| \to 0$ we can estimate in the following way:

$$\|Es_n(A-\hat{A})E\| \leqq \big\|s_n(A-s_{n_k}(A))\big\| + 2(\delta_{m_k}+4k^{-1}) \leqq 2n^{-1}m_k\|A\| + 10k^{-1}.$$

(Lemma 4 was used to estimate the first term.) This inequality shows the required result.

We notice that the proof has given a little more than what was formulated in the theorem. Since the mean ergodic theorem is valid even for power-bounded operators instead of property (ii) of kernels, the weaker condition

(ii$_0$) there is a $C>0$ such that for every $n\in\mathbf{N}$ and $A\in\mathfrak{A}$, $\varphi(\alpha^n(A^*)\alpha^n(A))\leqq \leqq C\varphi(A^*A)$ fulfils

would have been sufficient. However, in the really interesting cases, when α is an automorphism or a completely positive map, condition (i) implies condition (ii).

3. Results on several kernels. Let $\mathfrak{A}$ be a von Neumann algebra and φ a faithful semifinite normal weight on $\mathfrak{A}^+$. If $\alpha_i\colon \mathfrak{A}\to\mathfrak{A}$ is a kernel for $i\leqq k$ then

$$s^i_n(A) = \frac{1}{n}\sum_{l=0}^{n-1} \alpha^l_i(A)$$

converges in some sense to a limit $\Phi^i(A)\in\mathfrak{A}$ provided that $\varphi(A^*A)$ and $\varphi(AA^*)$ are finite. The joint behaviour of several kernels in von Neumann algebras was investigated by Conze and Dang-Ngoc [2]. This paragraph generalizes some results from [2], where φ is assumed to be a state.

Theorem 3. *Let $\mathfrak{A}, \varphi, \alpha_i, \Phi^i, s^i_n$ $(i\leqq k)$ be as above. If $A_m\in\mathfrak{A}^+$ and $\varepsilon_m>0$ $(m\in\mathbf{N})$ then there is a projection E in $\mathfrak{A}$ such that*

$$\|Es^k_{n_k}\dots s^1_{n_1}(A_m)E\| \leqq C(k, A_m)\varepsilon_m, \tag{16}$$

$$\varphi(I-E) \leqq 2^{k+1}\sum_{m=1}^{\infty} \varepsilon_m^{-k}\varphi(A_m), \tag{17}$$

where $C(1, A_m)=2$ and $C(k+1, A_m)=2C(k, A_m)+4\|A_m\|$.

Proof. For $k=1$ this is Theorem 1. By induction there is a projection E_m such that

$$\|E_m s^{k-1}_{n_{k-1}}\dots s^1_{n_1}(A_m)E_m\| \leqq C(k-1, A_m)\varepsilon_m,\ \varphi(I-E_m) \leqq 2^k\varepsilon_m^{-k+1}\varphi(A_m).$$

Apply Theorem 1 with α_k, $I-E_m$ and ε_m ($m\in\mathrm{N}$). We obtain a projection E with the following properties:

$$\|Es^k_{n_k}(I-E_m)E\| \leqq 2\varepsilon_m,$$

$$\varphi(I-E) \leqq 2\sum_{m=1}^{\infty}\varepsilon_m^{-1}\varphi(I-E_m) \leqq 2^{k+1}\sum_{m=1}^{\infty}\varepsilon_m^{-k}\varphi(A_m).$$

Hence E satisfies (17), and we verify (16).

$$Es^k_{n_k}\dots s^k_{n_1}(A_m)E \leqq$$
$$\leqq Es^k_{n_k}\big(2E_m s^{k-1}_{n_{k-1}}\dots s^1_{n_1}(A_m)E_m + 2(I-E_m)s^{k-1}_{n_{k-1}}\dots s^1_{n_1}(A_m)(I-E_m)\big) \leqq$$
$$\leqq 2C(k-1, A_m)\varepsilon_m I + 2\|A_m\|\, Es^k_{n_k}(I-E_m)E \leqq 2C(k-1, A_m)\varepsilon_m I + 4\varepsilon_m\|A_m\|\, I.$$

Corollary. *Let $\mathfrak{A}, \varphi, \alpha_i, \Phi^i, s^i_n\ (i\leqq k)$ be the same as above. Suppose that $\varphi(I)=1$ and $A\in\mathfrak{A}^+$. Then there exists an operator $C\in\mathfrak{A}$ such that*

$$s^k_{n_k}\dots s^1_{n_1}(A) \leqq C$$

for every $n_1, n_2, \dots, n_k\in\mathrm{N}$. Moreover,

$$\|C\| \leqq \delta\|A\|, \quad \varphi(C) \leqq \varepsilon\varphi(A)^{1/k+1}\|A\|^{k/k+1}$$

where ε and δ are constant (depending only on k).

Proof. Assume that $\|A\|=1$ and apply Theorem 3 in the case $A_1=A$, $\varepsilon_1=$ $=\varphi(A)^{1/k+1}$ and $A_m=0$ for $m>1$. Then

$$Es^k_{n_k}\dots s^1_{n_1}(A)E \leqq C(k, A)\varphi(A)^{1/k+1},$$

$$\varphi(I-E) \leqq 2^{k+1}\varphi(A)\varphi(A)^{-k/k+1} = 2^{k+1}\varphi(A)^{1/k+1}.$$

Therefore $s^k_{n_k}\dots s^1_{n_1}(A)\leqq 2C(k, A)\varphi(A)^{1/k+1}+2(I-E)=C_1$ and

$$\|C_1\| \leqq 2C(k, A)\varphi(A)^{1/k+1}+2 \leqq 2C(k, A)+2,$$

$$\varphi(C_1) \leqq 2C(k, A)\varphi(A)^{1/k+1}+2^{k+2}\varphi(A)^{1/k+1}.$$

Now we have $\|C_1\|\leqq\delta$ and $\varphi(C_1)\leqq\varepsilon\varphi(A)^{1/k+1}$.

In the general case one can obtain an operator C_1 for $A\|A\|^{-1}$ as above and take $C=\|A\|C_1$. So C satisfies both requirements.

Theorem 4. *Let $\mathfrak{A}, \varphi, \alpha_i, \Phi^i, s^i_n\ (i\leqq k)$ be as above. Suppose that $A\in\mathfrak{A}$ and $\varphi(A^*A), \varphi(AA^*)$ are finite. Then for every $\varepsilon>0$ there is a projection $E\in\mathfrak{A}$ such that $\varphi(I-E)<\varepsilon$ and*

$$\big\|E\big(s^k_{n_k}\dots s^1_{n_1}(A)-\Phi^k\dots\Phi^1(A)\big)E\big\| \to 0$$

if $n_1\to\infty, \dots, n_k\to\infty$ independently.

Proof. We follow the lines of the proof of Theorem 2 but use Theorem 3 instead of Theorem 1. For the sake of simplicity we assume that $k=2$.

Similarly to (15) we have the following decompositions on the basis of Lemma 3:

$$A = \big(A - s_l^1(A)\big) + \Phi^1(A) + C_l^1 + D_l^1 - E_l^1 + i(C_l^2 + D_l^2 - E_l^2),$$

$$\Phi^1(A) = \big(\Phi^1(A) - s_l^2\Phi^1(A)\big) + \Phi^2\Phi^1(A) + C_l^3 + D_l^3 + E_l^3 + i(C_l^4 + D_l^4 - E_l^4).$$

Here $0 \leqq D_l^i, E_l^i, \|C_l^i\| \leqq \delta_l$, $\varphi(D_l^i) \leqq \delta_l$, $\varphi(E_l^i) \leqq \delta_l\, \|D_l^i\| \leqq 2\|A\|$, $\|E_l^i\| \leqq 2\|A\|$ $(i=1, 2, 3, 4)$ and $\delta_l \to 0$. For every $l \in \mathbf{N}$,

$$s_{n_2}^2 s_{n_1}^1(A) - \Phi^2\Phi^1(A) = s_{n_2}^2 s_{n_1}^1\big(A - s_l^1(A)\big) + s_{n_2}^2\big(\Phi^1(A) - s_l^2\Phi^1(A)\big) +$$

$$+ s_{n_2}^1 s_{n_1}^1 (C_l^1 + D_l^1 - E_l^1 + iC_l^2 + iD_l^2 - iE_l^2) + s_{n_2}^2 (C_l^3 + D_l^3 - E_l^3 + iC_l^4 + iD_l^4 - iE_l^4).$$

Choose a subsequence (δ_{m_l}) of (δ_l) such that $\delta_{m_l} < l^{-k} 2^{-l} 2^{-k-5}\varepsilon$ and apply Theorem 3 to the elements $D_{m_l}^i, E_{m_l}^i$ with $1/l$ in the role of ε $(l \in \mathbf{N},\ i \leqq 4)$. So we have a projection E such that

$$\varphi(I-E) \leqq 2^{k+1} 8 \sum_{l=1}^{\infty} l^k l^{-k} 2^{-l} 2^{-k+5} \varepsilon \leqq \varepsilon$$

and

$$\|E s_n^2 s_{n_4}^1 (X_{m_l}^i) E\| \leqq \frac{2}{l}$$

where $n_1, n_2, l \in \mathbf{N}$, $i \leqq 4$ and $X = D, E$. Use Lemma 4 and the inequalities above to obtain the estimate

$$\big\|E\big(s_{n_2}^2 s_{n_1}^1(A) - \Phi^2\Phi^1(A)\big)E\big\| \leqq \frac{2m_l\|A\|}{n_1} + \frac{2m_l\|A\|}{n_2} + 4\delta_{m_l} + \frac{16}{l}$$

which concludes our proof.

Theorem 4 is a discrete Dunford—Schwartz—Zygmund type ergodic theorem for non-commuting kernels (cf. [17]). A continuous version will be contained in the next paragraph.

4. Continuous flows. First we establish an automatic continuity of kernels.

Lemma 5. *If* $\alpha\colon \mathfrak{A} \to \mathfrak{A}$ *is a kernel then there is a w-continuous kernel* $\alpha^c\colon \mathfrak{A} \to \mathfrak{A}$ *such that* $\alpha(A) = \alpha^c(A)$ *if* $\varphi(A^*A)$ *and* $\varphi(AA^*)$ *are finite.*

Proof. Let $\mathfrak{A}_0 = \{A \in \mathfrak{A}\colon \varphi(A^*A), \varphi(AA^*) < +\infty\}$. We show that α is weakly continuous on the unit ball of $\mathfrak{A}_0$. By Remark 2.2.3 in [6] it follows that $\alpha|\mathfrak{A}_0$ extends to a w-continuous mapping of $\mathfrak{A}$, which is obviously a kernel.

First we prove that if $V\colon \mathscr{H} \to \mathscr{H}$ is defined by $\alpha(L_\xi) = L_{V\xi}$ $(\xi \in \mathfrak{A}_0)$ and $\eta \in \mathfrak{A}_0'$

then $V^*(\eta\eta^b)\in\mathfrak{A}_0'$. Take $\bar\alpha(R_{\eta^b\eta})$ from (8). Then

$$\langle\xi_1,\bar\alpha(R_{\eta^b\eta})\xi_2\rangle=\langle\bar\alpha(R_{\eta^b\eta})\xi_1,\xi_2\rangle=\langle\alpha(L_{\xi_2^\#\xi_1})\eta,\eta\rangle=$$

$$=\langle\xi_2^\#\xi_1,V^*(\xi\xi^b)\rangle=\langle\xi_1,L_{\xi_1}V^*(\xi\xi^b)\rangle.$$

So $L_\xi V^*(\eta\eta^b)=\bar\alpha(R_{\eta^b\eta})\xi=\bar\alpha(R_{\eta^b\eta})^*\xi$ for every $\xi\in\mathfrak{A}_0$. According to [13], p. 248, $V^*(\eta\eta^b)\in\mathfrak{A}_0'$ and $R_{V^*(\eta^b\eta)}=\bar\alpha(R_{\eta\eta^b})$. Moreover, since $\varphi'(\bar\alpha(R_{\eta\eta^b}))<+\infty$, there is $\eta_1\in\mathfrak{A}_0'$ such that $V^*(\eta\eta^b)=\eta_1\eta_1^b$.

Let (L_{ξ_γ}) be a directed net in the unit ball of $\{L_\xi:\xi\in\mathfrak{A}_0\}$ converging weakly to 0. We have

$$\langle\alpha(L_{\xi_\gamma})\eta,\eta\rangle=\langle\xi_\gamma,V^*(\eta\eta^b)\rangle=\langle L_{\xi_\gamma}\eta_1,\eta_1\rangle\to 0.$$

By polarization $\langle\alpha(L_{\xi_\gamma})\eta,\mu\rangle\to 0$ for every $\eta,\mu\in\mathfrak{A}_0'$ and we have obtained $\alpha(L_{\xi_\gamma})\to 0$ weakly.

In this paragraph we deal with one-parameter semigroups of kernels. Namely, for $t\in\mathbf{R}^+$ let $\alpha_t:\mathfrak{A}\to\mathfrak{A}$ be a kernel so that $\alpha_0=$identity and $\alpha_t\circ\alpha_s=\alpha_{t+s}$ $(t,s\in\mathbf{R}^+)$. We assume the following continuity property:

(iii) $t\mapsto\varphi(\alpha_t(A)^*\alpha_t(A))$ is continuous if $\varphi(A^*A)$ and $\varphi(AA^*)$ are finite.

If α_t's are endomorphisms and φ is α_t-invariant for every $t\in\mathbf{R}^+$ then (iii) is always fulfilled.

Define $V_t\in\mathscr{B}(\mathscr{H})$ by $\alpha_t(L_\xi)=L_{V_t\xi}$ $(\xi\in\mathfrak{A}_0)$. Then (V_t) is a one-parameter semigroup of contractions, $t\mapsto V_t\xi$ is continuous for every $\xi\in\mathfrak{A}_0$. We need the following technical lemma.

Lemma 6. *Let (α_t) be a one-parameter semigroup of kernels with property* (iii). *Then for $\xi\in\mathfrak{A}_0$ the integral*

$$\sigma_T(L_\xi)=\frac{1}{T}\int_0^T\alpha_t(L_\xi)\,dt\quad(T>0)$$

exists in weak sense. In addition, $\mu=\frac{1}{T}\int_0^T V_t\xi\,dt\in\mathfrak{A}_0$ and $L_\mu\sigma_T(L_\xi)$.*

Proof. Let $\zeta_T=\frac{1}{T}\int_0^T V_t\xi\,dt$ for $\xi\in\mathfrak{A}_0$. If $\eta\in\mathfrak{A}_0'$ then

$$R_\eta\zeta_T(\xi)=\frac{1}{T}\int_0^T R_\eta V_t\xi\,dt=\frac{1}{T}\int_0^T\alpha_t(L_\xi)\eta\,dt.$$

There is a unique operator $\sigma_T(L_\xi)\in\mathscr{B}(\mathscr{H})$ such that

$$\sigma_T(L_\xi)\eta=\frac{1}{T}\int_0^T\alpha_t(L_\xi)\,dt\quad(\eta\in\mathfrak{A}_0'). \tag{18}$$

Similarly $\sigma_T(L_{\xi^\#})\eta = \frac{1}{T}\int\limits_0^T \alpha_t(L_{\xi^\#})\eta\, dt$ and it is easy to see that $\sigma_T(L_{\xi^\#}) = \sigma_T(L_\xi)^*$. Using [13], p. 252, we may conclude $\zeta_T(\xi) \in \mathfrak{A}_0'' = \mathfrak{A}_0$ and $L_{\zeta_T(\xi)} = \sigma_T(L_\xi)$. The rest of the assertion is given by (18).

An important consequence of the above lemmas is that for any kernel α we have

$$\alpha(\sigma_T(A)) = \frac{1}{T}\int\limits_0^T \alpha(\alpha_t(A))\, dt \tag{19}$$

if $\varphi(A^*A)$ and $\varphi(AA^*)$ are finite.

Now we are in a position to prove the maximal ergodic theorem for a one-parameter semigroup of kernels.

Theorem 5. *Let* $\mathfrak{A}, \varphi, (\alpha_t), \sigma_T$ *be as above. If* $A_m \in \mathfrak{A}^+$ *and* $\varepsilon_m > 0$ $(m \in \mathrm{N})$ *then there is a projection* E *in* $\mathfrak{A}$ *such that*

$$\varphi(I-E) \leqq 2 \sum_{m=1}^{\infty} \varepsilon_m^{-1} \varphi(A_m), \quad \|E\sigma_T(A_m)E\| \leqq \varepsilon_m \quad (m \in \mathrm{N}, T \in R^+).$$

Proof. For $\delta > 0$ we define $A_m^\delta = \frac{1}{\delta}\int\limits_0^\infty \alpha_t(A_m)\, dt$ and $s_n^\delta(A) = \frac{1}{n}\sum_{i=0}^{n-1} \alpha_{i\delta}(A)$. Then $s_n^\delta(A_m^\delta) = \sigma_{n\delta}(A_m)$ according to (19) and $\varphi(A_m^\delta) \leqq \varphi(A_m)$. Now apply Lemma 2 to $A_m^\delta, \varepsilon_m, \alpha_\delta$. So we obtain $C_\delta \in \mathfrak{A}_1^+$ with the properties

$$C_\delta \sigma_{n\delta}(A_m) C_\delta = C_\delta s_n^\delta(A_m^\delta) C_\delta \leqq \varepsilon_m C_\delta, \ \varphi(I-C) \leqq \sum_{m=1}^{\infty} \varepsilon_m^{-1} \varphi(A_m).$$

Choose a sequence (δ_k) such that $\delta_k \searrow 0$ and $C_{\delta_k} \to C$ weakly for some $C \in \mathfrak{A}_1^+$. Then

$$\varphi(I-C) \leqq \sum_{m=1}^{\infty} \varepsilon_m^{-1} \varphi(A_m), \quad C\sigma_{n\delta_k}(A_m) C \leqq \varepsilon_m C \quad (k, m, n \in \mathrm{N}).$$

By straightforward estimation,

$$\begin{aligned} C\sigma_T(A_m) C &\leqq C(\sigma_T(A_m) - \sigma_{n\delta_k}(A_m))C + C\sigma_{n\delta_k}(A_m) C \leqq \\ &\leqq 2T^{-1}|T-n\delta_k|\,\|A_m\|\, C^2 + \varepsilon_m C. \end{aligned}$$

Since $|T-n\delta_k|$ can be chosen arbitrary small we infer $C\sigma_T(A_m)C \leqq \varepsilon_m C$ $(m \in \mathrm{N})$.

If $\int\limits_0^1 \lambda dP(\lambda)$ is the spectral resolution of C then take $E = I - P(1/2)$ again as in the proof of Theorem 1.

Theorem 6. *Let* $\mathfrak{A}, \varphi, (\alpha_t), \sigma_T$ *be as above. If* $\varphi(A^*A)$ *and* $\varphi(AA^*)$ *are finite then there exists an operator* $\Phi(A) \in \mathfrak{A}$ *with the following property. For* $\varepsilon > 0$ *there is a projection* $E \in \mathfrak{A}$ *such that* $\varphi(I-E) < \varepsilon$ *and*

$$\|E(\sigma_T(A) - \Phi(A))E\| \to 0 \quad T \to +\infty.$$

Proof. By Lemma 6, $\sigma_T(A)$ exists as a weak* integral and $t\mapsto V_t\xi$ is continuous for $\xi\in\mathfrak{A}_0$. Since $\mathfrak{A}_0$ is dense in $\mathscr{H}$, now $t\mapsto V_t x$ is strongly integrable over every finite interval (cf. [3], p. 685). Hence $\zeta_T(x)=\frac{1}{T}\int\limits_0^T V_t x\,dt$ is defined for every $x\in\mathscr{H}$. Now we apply Theorem 1 from [3], p. 687, and obtain that $\zeta_T x\to Px$ for every $x\in\mathscr{H}$ as $T\to+\infty$. From this point on one can follow the lines of the proof of Theorem 2. One can show that if $\xi\in\mathfrak{A}_0$ then $P\xi\in\mathfrak{A}_0$ and $\sigma_T(L_\xi)\eta\to L_{P\xi}\eta$ for every $\eta\in\mathfrak{A}_0'$. Let $\Phi(L_\xi)=L_{P\xi}$. From the equality $\xi-P\xi=\xi-\zeta_k(\xi)+\xi_k$ defining ξ_k by left representation we have

$$A-\Phi(A) = A-\sigma_k(A)+A_k \quad (k\in\mathbf{N})$$

where $A=L_\xi$, $A_k=L_{\xi_k}$ and $\|A_k\|_2=\varphi(A_k^*A_k)^{1/2}=\|\xi_k\|=\delta_k\to 0$ as $k\to+\infty$. Hence we can write

$$\sigma_T(A)-\Phi(A) = [\sigma_T(A)-\sigma_{n\delta}(A)]+[\sigma_{n\delta}\sigma_k(A)-\sigma_T\sigma_k(A)]+ \tag{20}$$

$$+[s_n^\delta\sigma_\delta(A)-s_n^\delta s_l^\delta\sigma_\delta(A)]+\sigma_{l\delta}\left(\frac{1}{n}\sum_{i=0}^{n-1}\alpha_{i\delta}(A)-\sigma_{n\delta}(A)\right)+\sigma_T(A_k)$$

where $s_m^\delta(B)=\frac{1}{m}\sum_{i=1}^{m}\alpha_{i\delta}(B)$ and we assume that l, n are integers, $k=l\delta$, $[T+1]=n\delta$.

For the sake of notational simplicity we denote by $D_j(T, k, \delta)$ the jth term on the right hand side in (20) ($j\leqq 3$). Then

$$\|D_j(T, k, \delta)\| \leqq 2T^{-1}(n\delta-T)\,\|A\| \leqq 2T^{-1}\|A\| \quad (j = 1, 2),$$

and by Lemma 4

$$\|D_3(T, k, \delta)\| \leqq 2ln^{-1}\|\sigma_\delta(A)\| \leqq 2kT^{-1}\|A\|.$$

On the other hand, taking

$$D_4(T, k, \delta) = \frac{1}{n}\sum_{i=1}^{n-1}\alpha_{i\delta}(A)-\sigma_{n\delta}(A)$$

we have

$$\|D_4(T, k, \delta)\|_2 = \left\|\frac{1}{n}\sum_{i=0}^{n-1}V_{i\delta}(\xi)-\zeta_{n\delta}(\xi)\right\| \to 0$$

if $[T+1]=n\delta$ is fixed and $\delta\to 0$. For every integer $[T]$ we choose $\delta>0$ such that $\|D_4(T, k, \delta)\|_2\leqq T^{-1}2^{-T}\varepsilon$. Splitting A into selfadjoint and skewadjoint part, taking a subsequence (δ_{m_k}) of (δ_k) with the requirement $\delta_{m_k}<k^{-1}2^{-k}\varepsilon$, we obtain

$$\sigma_T(A)-\Phi(A) = \sigma_T\big(B^1(k)\big)+i\sigma_T\big(B^2(k)\big)+\sigma_{m_k}\big(B^3([T])\big)+$$

$$+i\sigma_{m_k}\big(B^4([T])\big)+\sum_{j=1}^{3}D_j(T, m_k).$$

Here $B^i(l)$ is selfadjoint and $\|B^i(l)\|_2 \leqq l^{-1}2^{-l}\varepsilon$. Now split all the $B^j(l)$'s into 3 summands by Lemma 3. So $B^j(l)=C^j(l)+D^j(l)-iE^j(l)$ and $\|C^j(l)\|$, $\|D^j(l)\|_1$, $\|E^j(l)\|_1 \leqq l^{-1}2^{-l}\varepsilon$. Apply Theorem 5 to $D^j(l)$ and $E^j(l)$ with the constant l^{-1} ($l \in \mathbf{N}, j \leqq 3$) and get a projection E. Then on the one hand,

$$\varphi(I-E) \leqq 2\cdot 8 \sum_{l=1}^{\infty} l l^{-1}2^{-l}\varepsilon = 16\varepsilon,$$

and on the other hand, we estimate in the following fashion:

$$\left\|E\big(\sigma_T(A)-\Phi(A)\big)E\right\| \leqq 2k^{-1}\cdot 2^{-k}\varepsilon + 8k^{-1} + 2[T]^{-1}2^{-[T]} + 8[T]^{-1} + \\ + 4T^{-1}\|A\| + 2m_k T^{-1}\|A\|.$$

Therefore $\|E\big(\sigma_T(A)-\Phi(A)\big)E\| \to 0$ as $T \to +\infty$ and the proof is complete.

Finally we formulate a continuous form of Theorem 4, which is a Dunford—Schwartz—Zygmund type theorem (cf. [17]). Let $\mathfrak{A}$ be a von Neumann algebra and φ a semifinite faithful normal weight on it and for $i \leqq k$ let (α_t^i) be a one-parameter semigroup of kernels possessing the continuity requirement (iii). Define

$$\sigma_T^i(A) = \frac{1}{T}\int_0^T \alpha_t^i(A)\,dt$$

and we know that $\sigma_T^i(A) \to \Phi^i(A)$ under the conditions and in the sense of Theorem 6, under the hypotheses of Theorem 6.

Theorem 7. *Let $\mathfrak{A}$, φ, (α_t^i), Φ^i, σ^i be as above and $A \in \mathfrak{A}$ such that $\varphi(A^*A)$, $\varphi(AA^*)$ are finite. Then for $\varepsilon > 0$ there is a projection $E \in \mathfrak{A}$ such that*

$$\left\|E\big(\sigma_{T_k}^k \dots \sigma_{T_1}^1(A) - \Phi^k \dots \Phi^1(A)E\right\| \to 0$$

if $T_1 \to +\infty, \dots, T_k \to +\infty$ independently and $\varphi(I-E) < \varepsilon$.

Since the proof is very similar to that of Theorem 4, we omit it. We only note that instead of Theorem 3, one has to use the continuous form of it.

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Contractions and unilateral shifts

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A contraction T on a separable Hilbert space is said to be a weak contraction if $I-T^*T\in(\tau, C)$ which denotes the trace class, and $\sigma(T)\neq\bar{D}$, where D is the open unit disk. It is well known that there is a C_0-C_{11} decomposition for a weak contraction ([3]). Therefore we can easily show that if T is of class C_{10} (about $C_{10}, C_{\cdot 0}$, etc., see p. 72 of [3]) and if $I-T^*T\in(\tau, C)$, then

$$\sigma_p(T^*) = D \quad \text{and} \quad \sigma_p(T)\cap D = \emptyset.$$

In this note, we shall investigate a contraction T such that $I-T^*T\in(\tau, C)$ and $\sigma(T)=\bar{D}$.

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1. Operator valued functions

For $T\in I+(\tau, C)$, Bercovici and Voiculescu defined the algebraic adjoint T^{a}, which satisfies

$$T^{\mathrm{a}}T = TT^{\mathrm{a}} = (\det T)I.$$

They showed that if $\Theta(\lambda)$ is a contractive holomorphic function and if $\Theta(\lambda)\in I+ +(\tau, C)$ for every $\lambda\in D$, then $\Theta(\lambda)^{\mathrm{a}}$ is a contractive holomorphic function. In this case, if $\det\Theta(e^{it})\neq 0$ a.e., then $\Theta(e^{it})$ is invertible and its inverse is $\Theta(e^{it})^{\mathrm{a}}/\det\Theta(e^{it})$ a.e.

Theorem 1. *Let $\Theta(\lambda)$ be an inner function (that is, $\Theta(\lambda)$ is a contractive holomorphic function defined on D and $\Theta(e^{it})$ is isometric a.e.) with values in $\mathcal{L}(E, E')$, where E, E' are separable Hilbert spaces. If there is an isometry V in $\mathcal{L}(E, E')$ such that for every $\lambda\in D$*

(1.1) $$I_E-V^*\Theta(\lambda)\in(\tau, C),$$

(1.2) $$\det V^*\Theta(\lambda)\not\equiv 0,$$

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then there is a bounded holomorphic function $\Delta(\lambda)$ *with values in* $\mathscr{L}(E', F)$ *for a suitable Hilbert space* F *such that*

$$\Theta(e^{it})E \oplus \Delta(e^{it})^* F = E' \quad a.e. \tag{1.3}$$

Proof. If V is unitary, then $\Theta(e^{it})$ is invertible a.e. Hence we may assume that V is not unitary. Set $F=E' \ominus VE$. Let $E_0=E\oplus F$ be the direct sum of E and F. For $\lambda \in D$, define $\Theta'(\lambda)\in\mathscr{L}(E_0, E')$ by

$$\Theta'(\lambda)|_E = \Theta(\lambda) \quad \text{and} \quad \Theta'(\lambda)|_F = I_F.$$

For simplicity, set $d(\lambda)=\det V^*\Theta(\lambda)$ and $A(\lambda)=(V^*\Theta(\lambda))^a$. Determine $\Delta(\lambda)\in \in\mathscr{L}(E', F)$ by

$$\Delta(\lambda) = -P_F\Theta(\lambda)A(\lambda)V^* + d(\lambda)P_F \tag{1.4}$$

and $\Delta'(\lambda)\in\mathscr{L}(E', E_0)$ by

$$\Delta'(\lambda) = A(\lambda)V^* + \Delta(\lambda).$$

Then we have

$$\begin{aligned}\Delta'(\lambda)\Theta'(\lambda)|_E &= \Delta'(\lambda)\Theta(\lambda) = A(\lambda)V^*\Theta(\lambda) + \Delta(\lambda)\Theta(\lambda) = \\ &= d(\lambda)I_E - P_F\Theta(\lambda)\,d(\lambda)I_E + d(\lambda)P_F\Theta(\lambda) = d(\lambda)I_E, \\ \Delta'(\lambda)\Theta'(\lambda)|_F &= A(\lambda)V^*I_F + \Delta(\lambda)I_F = d(\lambda)I_F,\end{aligned}$$

and

$$\begin{aligned}\Theta'(\lambda)\Delta'(\lambda) &= \Theta(\lambda)A(\lambda)V^* + \Delta(\lambda) = (I-P_F)\Theta(\lambda)A(\lambda)V^* + d(\lambda)P_F = \\ &= VV^*\Theta(\lambda)A(\lambda)V^* + d(\lambda)P_F = V\,d(\lambda)V^* + d(\lambda)P_F = d(\lambda)I_{E'}.\end{aligned}$$

Thus we have

$$\Delta'(\lambda)\Theta'(\lambda) = d(\lambda)I_{E_0}, \quad \Theta'(\lambda)\Delta'(\lambda) = d(\lambda)I_{E'}.$$

Since the inverse of $\Theta'(e^{it})$ is $\Delta'(e^{it})/d(e^{it})$ a.e., the orthogonal complement of $\Theta(e^{it})E=\Theta'(e^{it})E$ is

$$\frac{\Delta'(e^{it})^*}{\overline{d(e^{it})}}(E_0\ominus E) = \Delta(e^{it})^*F.$$

It is clear that $\Delta(\lambda)$ is a bounded holomorphic function.

Cambern showed that the orthogonal complement of a finite dimensional holomorphic range function is conjugate holomorphic (c.f. p. 94 of [2]). Now, we can show this result as a corollary.

Corollary 1. *Let* $\Theta(\lambda)$ *be an inner function with values in* $\mathscr{L}(E, E')$. *Suppose* $\dim E=m<\infty$. *Then there is a bounded holomorphic function* $\Delta(\lambda)$ *satisfying* (1.3).

Proof. We may assume that $E\subset E'$ and $\Theta(e^{it})$ is a matrix. Since

$$1=\det\left(\Theta(e^{it})^*\Theta(e^{it})\right)=\sum_\sigma |\det\Theta_\sigma(e^{it})|^2,$$

a.e., where $\sum_\sigma$ is taken over all $m\times m$ submatrices of $\Theta(e^{it})$, there is at least one σ such that $\det\Theta_\sigma(e^{it})\neq 0$ a.e. Thus there is an isometry V such that

$$\det V^*\Theta(e^{it}) = \det\Theta_\sigma(e^{it}) \neq 0 \quad \text{a.e.}$$

(see [4]). Hence V and $\Theta(\lambda)$ satisfy (1.1), (1.2).

2. Quasi unilateral shifts

We begin with a short review about the canonical model theory of B. Sz.-Nagy and C. Foiaş. Let T be a contraction of class $C_{\cdot 0}$ on a separable Hilbert space H. Set $D_T=(I-T^*T)^{1/2}$, and let E and E' be the closures of D_TH and $D_{T^*}H$, respectively. Then the characteristic function $\Theta(\lambda)$ of T determined by

$$\Theta(\lambda) = \{-T+\lambda D_{T^*}(I-\lambda T^*)^{-1}D_T\}|_E \quad \text{for} \quad \lambda\in D \tag{2.1}$$

is an inner function with values in $\mathcal{L}(E, E')$. Therefore

$$\dim E \leqq \dim E'.$$

Moreover T is unitary equivalent to $S(\Theta)$ on $H(\Theta)$ defined by

$$H(\Theta) = H^2(E')\ominus\Theta H^2(E), \quad S(\Theta)^*h = \frac{1}{\lambda}(h(\lambda)-h(0)) \quad \text{for } h \text{ in } H(\Theta). \tag{2.2}$$

T is of class $C_{1\cdot}$ if and only if $\Theta(\bar\lambda)^*H^2(E')$ is dense in $H^2(E)$ (that is, Θ is $*$-outer).

In this note, for simplicity, we call T a *quasi unilateral shift* if T is a contraction of class $C_{\cdot 0}$ such that

$$I-T^*T\in(\tau, C), \quad \mathcal{K}(T) = \{0\} \quad \text{and} \quad \mathcal{K}(T^*) \neq \{0\},$$

where $\mathcal{K}(T)$ denotes the kernel of T.

Theorem 2. *If T is a quasi unilateral shift on H, then there is a bounded operator X with dense range satisfying*

$$XT = SX, \tag{2.3}$$

where S is a unilateral shift satisfying

$$0>\text{index } S = \text{index } T \geqq -\infty,$$

where $\text{index } T=\dim\mathcal{K}(T)-\dim\mathcal{K}(T^*)$.

Proof. We may assume $I-T^*T\neq 0$. From $T(I-T^*T)=(I-TT^*)T$, it follows that $TE\subset E'$, $T(H\ominus E)=H\ominus E'$, where E and E' are the spaces defined above. Thus we have

(2.4) $$H \ominus TH = E' \ominus TE \neq \{0\}.$$

Let $\{e_1, e_2, \dots, e_n, \dots\}$ be the C.O.N.B. of E such that $(I-T^*T)e_n = \mu_n e_n$, $\mu_n > 0$. Then $f_n = (1-\mu_n)^{-1/2} Te_n$ $(n=1, 2, \dots)$ is a C.O.N.B. of TE and $T^* f_n = (1-\mu_n)^{1/2} e_n$ (see p. 324 of [3]). Setting $Ve_n = -f_n$ $(n=1, 2, \dots)$, V is an isometry from E to E', and

(2.5) $$V + T|_E \in (\tau, C) \quad \text{(see [1])}.$$

Setting $F = E' \ominus VE$, from (2.4) it follows that

(2.6) $$\dim F = -\text{ index } T.$$

$I - T^*T \in (\tau, C)$ implies $D_T \in (\sigma, C)$ which denotes the Hilbert—Schmidt class. Since $(I-TT^*)|_{TE}$ is unitarily equivalent to $I - T^*T$, we have $D_{T^*}|_{TE} \in (\sigma, C)$. Thus

$$\lambda V^* D_{T^*}(I - \lambda T^*)^{-1} D_T = \lambda V^*(D_{T^*}|_{TE})(I - \lambda T^*)^{-1} D_T \quad (\lambda \in D)$$

belongs to (τ, C). Thus, from (2.1), (2.5), we have

$$I - V^* \Theta(\lambda) \in (\tau, C) \quad \text{for each } \lambda.$$

Since

$$|\det(V^* \Theta(0))|^2 = \det(\Theta(0)^* VV^* \Theta(0)) = \det(T^* VV^* T|_E) = \det(T^* T|_E) \neq 0,$$

we have $\det V^* \Theta(\lambda) \not\equiv 0$. Thus V and $\Theta(\lambda)$ satisfy the conditions of Theorem 1. Hence $\Delta(\lambda)$ defined by (1.4) satisfies (1.3). Since $\Delta(\lambda)\Theta(\lambda) = 0$, setting

(2.7) $$X_0 h = \Delta h \quad \text{for } h \text{ in } H(\Theta),$$

we have $X_0 \in \mathscr{L}(H(\Theta), H^2(F))$ and $X_0 S(\Theta) = S_0 X_0$, where S_0 is the unilateral shift on $H^2(F)$. Since

$$H^2(F) \supset X_0 H(\Theta) = \Delta H^2(E') \supset \Delta H^2(F) = (\det V^* \Theta(\lambda))\, H^2(F),$$

it follows that $S = S_0|_{\overline{X_0 H(\Theta)}}$ is unitarily equivalent to S_0. Thus, from (2.6), we have

$$\text{index } S = \text{index } S_0 = -\dim F = \text{index } T.$$

Consequently an operator X from $H(\Theta)$ to $\overline{X_0 H(\Theta)}$ defined by

(2.8) $$Xh = X_0 h \quad \text{for } h \text{ in } H(\Theta)$$

satisfies (2.3).

Corollary 1. *Let T be a contraction of class C_{00} such that $I - T^*T$ and $I - TT^*$ belong to (τ, C). Then, for $a \in D$, $\mathscr{K}(T - aI) = \{0\}$ if and only if $\mathscr{K}(T^* - \bar{a}I) = \{0\}$.*

Proof. Set $T_a=(T-aI)(I-\bar{a}T)^{-1}$ and $A=(1-|a|^2)^{1/2}(I-\bar{a}T)^{-1}$. Then we have $I-T_a^*T_a=A^*(I-T^*T)A$, $I-T_aT_a^*=A(I-TT^*)A^*$, and T_a is of class C_{00} (see p. 240 and p. 257 of [3]). Suppose $\mathscr{K}(T-aI)=\{0\}$ and $\mathscr{K}(T^*-\bar{a}I)\neq\{0\}$. Then T_a is a quasi unilateral shift. Therefore, there is an X satisfying $XT_a=SX$, which implies that T is not of class C_{00}. This is a contradiction. Thus $\mathscr{K}(T-aI)=\{0\}$ implies $\mathscr{K}(T^*-\bar{a}I)=\{0\}$. Similarly we can prove the converse assertion.

For a contraction T on H, we have

$$\|I-T^*T\|_p^p+\dim\mathscr{K}(T^*)=\|I-TT^*\|_p^p+\dim\mathscr{K}(T), \tag{2.9}$$

where $\|\ \|_p$ denotes the p-Schatten norm. Indeed, from $T(I-T^*T)=(I-TT^*)T$, $(I-T^*T)|_{\overline{T^*H}}$ and $(I-TT^*)|_{\overline{TH}}$ are unitarily equivalent. $(I-T^*T)|_{\mathscr{K}(T)}=I_{\mathscr{K}(T)}$ and $(I-TT^*)|_{\mathscr{K}(T^*)}=I_{\mathscr{K}(T^*)}$ imply that

$$\|I-T^*T\|_p^p=\|(I-T^*T)|_{\overline{T^*H}}\|_p^p+\dim\mathscr{K}(T),$$

$$\|I-TT^*\|_p^p=\|(I-TT^*)|_{\overline{TH}}\|_p^p+\dim\mathscr{K}(T^*).$$

Thus we have (2.9). Similarly we have

$$\operatorname{rank}(I-T^*T)+\dim\mathscr{K}(T^*)=\operatorname{rank}(I-TT^*)+\dim\mathscr{K}(T). \tag{2.9$'$}$$

Proposition 1. *Let T be a Fredholm quasi unilateral shift. Suppose X with dense range satisfies $XT=SX$, where S is a unilateral shift with* index $S=$ index T. *Then $T|_{\mathscr{K}(X)}$ is of class C_0.*

Proof. Let $T=\begin{bmatrix} T_1 & T_{12} \\ 0 & T_2 \end{bmatrix}$ be a decomposition of T corresponding to $H=\mathscr{K}(X)\oplus\mathscr{K}(X)^\perp$. Then T_1 is injective and, from (2.3), also T_2 is injective. From the assumption and (2.9), it follows that $I-T^*T\in(\tau, C)$ and $I-TT^*\in(\tau, C)$, which implies

$$I-T_1^*T_1\in(\tau, C), \tag{2.10}$$

$$I-(T_1T_1^*+T_{12}T_{12}^*)\in(\tau, C), \tag{2.11}$$

$$I-(T_{12}^*T_{12}+T_2^*T_2)\in(\tau, C), \tag{2.12}$$

$$I-T_2T_2^*\in(\tau, C). \tag{2.13}$$

From $\mathscr{K}(T_2^*)\subset\mathscr{K}(T^*)$, it follows that

$$\operatorname{index} T=-\dim\mathscr{K}(T^*)\leqq-\dim\mathscr{K}(T_2^*)\leqq-\dim\mathscr{K}(S^*)=\operatorname{index} T,$$

which implies index $T=$ index T_2. From (2.9) and (2.13), we have $I-T_2^*T_2\in(\tau, C)$, which, by (2.12), implies $T_{12}\in(\sigma, C)$. Therefore, from (2.10) and (2.11), T_1 is a Fredholm operator. Since

$$\operatorname{index} T=\operatorname{index}\begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix}=\operatorname{index} T_1+\operatorname{index} T_2,$$

we have $\operatorname{index} T_1=0$. Thus T_1 is invertible. Hence T_1 is a weak contraction of class $C_{\cdot 0}$. Consequently T_1 is of class C_0.

Corollary 2. *Let T be a Fredholm quasi unilateral shift of class C_{10}. Then $\mathscr{K}(A)=\{0\}$ provided $AT=TA$ and $\mathscr{K}(A^*)=\{0\}$* (cf. [6]).

Proof. For X defined in Theorem 2, we have $(XA)T=S(XA)$. From Proposition 1, we have $\mathscr{K}(XA)=\{0\}$.

Proposition 2. *Let T be of class $C_{\cdot 0}$. Then T is of class C_{10} if and only if*

$$\Theta L^2(E)\cap H^2(E') = \Theta H^2(E). \tag{2.14}$$

Proof. Since, for h in $H^2(E')$ and f in $H^2(E)$, we have

$$(\Theta(\bar{\lambda})^* h(\lambda), f(\lambda))_{H^2(E)} = \frac{1}{2\pi}\int_0^{2\pi} (\Theta(e^{-it})^* h(e^{it}), f(e^{it}))_E\, dt =$$

$$= -\frac{1}{2\pi}\int_0^{-2\pi} (\Theta(e^{it})^* h(e^{-it}), f(e^{-it}))_E\, dt = \frac{1}{2\pi}\int_0^{2\pi} (\Theta(e^{it})^* h(e^{-it}), f(e^{-it}))_E\, dt =$$

$$= \frac{1}{2\pi}\int_0^{2\pi} (\Theta(e^{it})^* e^{-it} h(e^{-it}), e^{-it} f(e^{-it}))_E\, dt = (\Theta(\lambda)^* \bar{\lambda} h(\bar{\lambda}), \bar{\lambda} f(\bar{\lambda}))_{L^2(E)},$$

$\Theta(\bar{\lambda})^* H^2(E')$ is dense in $H^2(E)$ if and only if $\Theta(\lambda)^*(H^2(E'))^\perp$ is dense in $(H^2(E))^\perp$, where $\perp$ denotes the orthogonal complement. We have always

$$\Theta L^2(E)\cap H^2(E') \supset \Theta H^2(E).$$

At first, assume that T is of class C_{10}. Suppose

$$\Theta g \in \{\Theta L^2(E)\cap H^2(E')\} \ominus \Theta H^2(E).$$

Then $\Theta g\in H^2(E')$ and $g\perp H^2(E)$, because Θ is an isometry from $L^2(E)$ to $L^2(E')$. Thus $g\perp \Theta^*(H^2(E'))^\perp$ and $g\in (H^2(E))^\perp$. Since $\Theta(\lambda)$ is $*$-outer, we have $g=0$. Consequently (2.14) follows.

Conversely assume (2.14). Suppose $f\perp \Theta(\lambda)^*(H^2(E'))^\perp$ and $f\in(H^2(E))^\perp$. Then $\Theta f\in H^2(E')$ and $\Theta f\perp \Theta H^2(E)$. Thus from (2.14), we have $\Theta f=0$ and hence $f=0$. Consequently $\Theta(\lambda)$ is $*$-outer.

Theorem 3. *Let T be a quasi unilateral shift. Then $T\prec S$ (that is, there is an X such that $\mathscr{K}(X)=\mathscr{K}(X^*)=\{0\}, XT=SX$), where S is a unilateral shift with* $\operatorname{index} S=\operatorname{index} T$, *if and only if T is of class C_{10}.*

Proof. Assume that T is of class C_{10}. From Theorem 2, there is an X with dense range satisfying (2.3). If $Xh=0$ for h in $H(\Theta)$, then, from (2.7) and (2.8), $\Delta(e^{it})h(e^{it})=0$ a.e. Thus, from (1.3), $h\in\Theta L^2(E)$, so that, from (2.14), $h\in\Theta H^2(E)$. Consequently $h=0$. Thus we have $T\prec S$.

Conversely, assume $XT=SX$ and $\mathcal{K}(X)=\mathcal{K}(X^*)=\{0\}$. From $XT^n=S^nX$ $(n=1, 2, \dots)$ it follows that T is of class C_{10}.

Remark 1. If T is a Fredholm operator, then, from Theorem 2 and Proposition 1, it is clear that $T\prec S$ if T is of class C_{10}.

Remark 2. Theorem 3 implies that the Jordan model of a quasi unilateral shift of class C_{10} is a unilateral shift.

Corollary 3. *Let T be a quasi unilateral shift of class C_{10}. Then T^* has a cyclic vector.*

Proof. $T\prec S$ implies that $S^*\prec T^*$. Since S^* has a cyclic vector, also T^* does.

Proposition 3. *Let T be a quasi unilateral shift. Then there is an injection Y such that*

$$YS = TY, \tag{2.15}$$

where S is a unilateral shift with index $S=$index T.

Proof. Consider $S(\Theta)$ defined by (2.2) instead of T. Let V be an isometry defined in the proof of Theorem 2. Then

$$E' = VE \oplus F \quad \text{and} \quad \det V^*\Theta(e^{it}) \neq 0 \quad \text{a.e.}.$$

Define an operator Y from $H^2(F)$ to $H(\Theta)$ by

$$Yh = P_{H(\Theta)}h \quad \text{for } h \text{ in } \quad H^2(F).$$

Then we have

$$YSh = P_{H(\Theta)}Sh = P_{H(\Theta)}SP_{H(\Theta)}h = S(\theta)Yh,$$

which implies (2.15). Suppose $Yh=0$. Then $h=\Theta f$ for some $f\in H^2(E)$. Thus $0=V^*h(e^{it})=V^*\Theta(e^{it})f(e^{it})$ a.e. Since $V^*\Theta(e^{it})$ is invertible a.e., $f(e^{it})=0$ a.e. Consequently Y is injective.

Proposition 4. *Let T be a quasi unilateral shift of class C_{10}. Then, if $T\prec S'$, where S' is a unilateral shift, then* index $S'=$index T.

Proof. From $S'^*\prec T^*$, $\dim \mathcal{K}(S'^*)\leqq \dim \mathcal{K}(T^*)$. The proposition above implies that there is an injection Y' such that

$$Y'S = S'Y', \text{ index } S = \text{index } T,$$

which implies that $0>$index $S\geqq$index S' (cf. [4]). Thus we have

$$\text{index } T = \text{index } S \geqq \text{index } S' \geqq \text{index } T,$$

from which index $T=$index S' follows.

Remark 3. P. Y. Wu [6] showed that if $I-T^*T$ is a finite rank operator, and if $T\prec S'$, then

$$\operatorname{rank}(I-TT^*)-\operatorname{rank}(I-T^*T)=-\operatorname{index} S'.$$

From (2.9)$'$, our proposition is an extension of this result.

3. Cyclic vector

In this section, we consider a quasi unilateral shift of class C_{10} which has a cyclic vector. The next proposition is a partial extension of Proposition 2 of [4] and Theorem 3.1 of [5].

Proposition 5. *Let T be a quasi unilateral shift of class C_{10}. Then next conditions are equivalent:*

(a) *T has a cyclic vector;*

(b) *there is a bounded operator Y satisfying*

$$YS_1=TY,\quad \mathscr{K}(Y^*)=\{0\}, \tag{3.1}$$

where S_1 is a unilateral shift with $\operatorname{index} S_1=-1$;

(c) $S_1\prec T$;

(d) $S_1\prec T$ *and* $T\prec S_1$;

(e) $\|I-TT^*\|_1-\|I-T^*T\|_1=1$, *and there is a bounded holomorphic function Γ with values in $\mathscr{L}(\mathbf{C},E')$ satisfying*

$$\|\Gamma(e^{it})\|\leqq 1\quad a.e., \tag{3.2}$$

$$\Gamma H^2(\mathbf{C})\vee\Theta H^2(E)=H^2(E'), \tag{3.3}$$

where Θ is the characteristic function of T defined by (2.1).

Proof. (a)$\to$(e). From Theorem 3, for a unilateral shift S with $\operatorname{index} S=$ $=\operatorname{index} T$, we have $T\prec S$. That T has a cyclic vector implies that also S does. Thus $\operatorname{index} S=-1$. Consequently, from (2.9), we have

$$\|I-TT^*\|_1-\|I-T^*T\|_1=1.$$

We can construct a function Γ in the same way as in [4].

(e)$\to$(b). The contraction Y defined by $Yh=P_{H(\Theta)}\Gamma h$ for h in $H^2(\mathbf{C})$ satisfies (3.1).

(b)$\to$(c). Suppose $\mathscr{K}(Y)\neq\{0\}$. Since $S_1\mathscr{K}(Y)\subset\mathscr{K}(Y)$, there is a scalar inner function ψ such that $\mathscr{K}(Y)=\psi H^2(\mathbf{C})$. Thus

$$\mathscr{K}(Y)^\perp=H(\psi)\quad(=H^2(\mathbf{C})\ominus\psi H^2(\mathbf{C})),\quad Y|_{H(\psi)}S(\psi)=TY|_{H(\psi)},$$

where $S(\psi)=P_{H(\psi)}S\,|_{H(\psi)}$. Since $S(\psi)$ is of class C_0, T must be of class C_0. This is a contradiction. Consequently $\mathcal{K}(Y)=\{0\}$.

(c)→(d). $S_1\prec T$ implies $T^*\prec S_1^*$, from which it follows that $\dim\mathcal{K}(T^*)\leqq\leqq\dim\mathcal{K}(S_1^*)=1$. That T is a quasi unilateral shift, implies index $T<0$. Thus index $T=-1$. By Theorem 3, we have $T\prec S_1$.

(d)→(a). This is obvious.

(3.3) implies that $[\Gamma,\Theta]$ is an outer function from $H^2(\mathbf{C})\oplus H^2(E)$ to $H^2(E')$. Generally $[\Gamma,\Theta]$ is not contractive. Therefore $d(\lambda)=\det[\Gamma(\lambda),\Theta(\lambda)]\in H^\infty$ and $d(\lambda)\leqq 1$ are not obvious. We shall show these results.

Let $A\in\mathcal{L}(E,E')$ be a contraction and $V\in\mathcal{L}(E,E')$ an isometry with index $V==-1$. Let $\{e_1,e_2,\ldots,e_n,\ldots\}$ be a C.O.N.B. in E. Then, setting $d_n=Ve_n$ $(n=1,2,\ldots)$, $\{d_0,d_1,\ldots,d_n,\ldots\}$ is a C.O.N.B. in E', where d_0 is a unit vector in $\mathcal{K}(V^*)$. For $i=1,2,\ldots,$ define an isometry $V_i\in\mathcal{L}(E,E')$ by

$$V_ie_1=d_0,\ldots,V_ie_i=d_{i-1},\ V_ie_{i+1}=d_{i+1},\ V_ie_{i+2}=d_{i+2},\ldots.$$

Let $a_{ij}=(Ae_j,d_i)$ $(i\geqq 0,j\geqq 1)$. Then, in the base $\{e_1,e_2,\ldots\}$, we have

$$V_i^*A=\begin{bmatrix} a_{01} & ,\ldots, & a_{0j} & ,\ldots \\ \vdots & & \vdots & \\ a_{i-1\,1}, & \ldots, & a_{i-1\,j}, & \ldots \\ a_{i+1\,1}, & \ldots, & a_{i+1\,j}, & \ldots \\ \vdots & & \vdots & \end{bmatrix}\quad (i=1,2,\ldots).$$

Let $E_0=\mathbf{C}\oplus E$ be a direct sum of $\mathbf{C}$ and E, and e_0 a unit vector in $\mathbf{C}$. Let x_n $(n=0,1,2,\ldots)$ be a scalar number such that $\sum_{n=0}^{\infty}|x_n|^2\leqq 1$. Let $B\in\mathcal{L}(E_0,E')$ be an operator defined by

$$(Be_0,d_i)=x_i,\ (Be_j,d_i)=a_{ij}\quad (i\geqq 0,j\geqq 1).$$

Determine a unitary $U\in\mathcal{L}(E_0,E')$ by $Ue_i=d_i$ $(i\geqq 0)$. Then in the base $\{e_0,e_1,\ldots,\ldots,e_i,\ldots\}$ of E_0 we have

$$U^*B=\begin{bmatrix} x_0, & a_{01}, & \ldots, & a_{0j}, & \ldots \\ x_1, & a_{11}, & \ldots, & a_{1j}, & \ldots \\ \vdots & & & \vdots & \\ x_i, & a_{i1}, & \ldots, & a_{ij}, & \ldots \\ \vdots & & & \vdots & \end{bmatrix}.$$

Let $I_E-V^*A\in(\tau,C)$. Then, since $(V_i^*Ae_j,e_k)=(V^*Ae_j,e_k)$ for $j\geqq 1$ and $k\geqq i+1$, $I_E-V_i^*A\in(\tau,C)$ for every i.

$$P_E(I_{E_0}-U^*B)|_E=I_E-V^*A$$

implies $I_{E_0}-U^*B\in(\tau,C)$.

Lemma. *Let* $I_E - V^*A \in (\tau, C)$. *Set* $V_0 = V$. *Then*

$$\det U^*B = \sum_{i=0}^{\infty} x_i(-1)^i \det(V_i^*A),$$

and

$$\sum_{i=1}^{\infty} |x_i(-1)^i \det(V_i^*A)| \leqq 1.$$

Proof. For simplicity, let $[A]_n$ denote the first $n\times n$ submatrix of A, and write A_n for $A|_{E_n}$, where $E_n = \langle e_1, \dots, e_n\rangle$. For any k and n as $n \geqq k$, we have

$$\sum_{i=0}^{k} |\det[V_i^*A]_n|^2 \leqq \det(A_n^* A_n) = \det[A^*A]_n \leqq 1, \tag{3.4}$$

because A is a contraction. Since for each i

$$\det[V_i^*A]_n \to \det(V_i^*A) \quad (n \to \infty),$$

we have $\sum_{i=0}^{k} |\det(V_i^*A)|^2 \leqq 1$, which implies

$$\sum_{i=0}^{\infty} |\det(V_i^*A)|^2 \leqq 1. \tag{3.5}$$

Consequently $\sum_{i=0}^{\infty} |x_i(-1)^i \det(V_i^*A)| \leqq 1$. For any $\varepsilon > 0$, take an m such that

$$\sum_{i=m+1}^{\infty} |x_i|^2 < \varepsilon^2. \tag{3.6}$$

Since $\det[U^*B]_n \to \det(U^*B)$, and $\det[V_i^*A]_n \to \det(V_i^*A)$ as $n \to \infty$, we can take an N such that

$$n \geqq N \to |\det[U^*B]_n - \det(U^*B)| < \varepsilon, \tag{3.7}$$

and

$$n \geqq N \to \sum_{i=0}^{m} |\det[V_i^*A]_n - \det(V_i^*A)|^2 < \varepsilon^2. \tag{3.8}$$

Fix a k as $k \geqq N+1$ and $k \geqq m+1$. Then it follows that

$$\left|\det(U^*B) - \sum_{i=0}^{\infty} x_i(-1)^i \det(V_i^*A)\right| \leqq$$

$$\leqq |\det(U^*B) - \det[U^*B]_k| + \left|\det[U^*B]_k - \sum_{i=0}^{m} x_i(-1)^i \det[V_i^*A]_{k-1}\right| +$$

$$+ \left|\sum_{i=0}^{m} x_i(-1)^i \{\det[V_i^*A]_{k-1} - \det(V_i^*A)\}\right| + \left|\sum_{i=m+1}^{\infty} x_i(-1)^i \det(V_i^*A)\right|.$$

From (3.7) $|\det(U^*B)-\det[U^*B]_k|<\varepsilon$, and from (3.8)

$$\left|\sum_{i=0}^{m} x_i(-1)^i\{\det[V_i^*A]_{k-1}-\det(V_i^*A)\}\right| \leqq$$

$$\leqq \left(\sum_{i=0}^{m}|x_i|^2\right)^{1/2}\left(\sum_{i=0}^{m}|\det[V_i^*A]_{k-1}-\det(V_i^*A)|^2\right)^{1/2} < \varepsilon.$$

(3.5) and (3.6) implies that

$$\left|\sum_{i=m+1}^{\infty} x_i(-1)^i\det(V_i^*A)\right| < \varepsilon.$$

By finite matrix theory

$$\left|\det[U^*B]_k-\sum_{i=0}^{m} x_i(-1)^i\det[V_i^*A]_{k-1}\right| = \left|\sum_{i=m+1}^{k-1} x_i(-1)^i\det[V_i^*A]_{k-1}\right| < \varepsilon,$$

because the last inequality follows from (3.4), (3.6). Consequently, for any $\varepsilon>0$ we have

$$\left|\det(U^*B)-\sum_{i=0}^{\infty} x_i(-1)^i\det(V_i^*A)\right| < 4\varepsilon.$$

In (e) of Proposition 5, set $(\Gamma(\lambda)e_0, d_i)=h_i(\lambda)$ for $i\geqq 0$. Then we have:

Proposition 6. $|\det(U^*[\Gamma(\lambda), \Theta(\lambda)])|\leqq 1$, *and*

(3.9) $$\det(U^*[\Gamma(\lambda), \Theta(\lambda)]) = \sum_{i=0}^{\infty} h_i(\lambda)(-1)^i\det(V_i^*\Theta(\lambda))$$

is holomorphic on D.

Proof. From (3.2), we have $\sum_{i=0}^{\infty}|h_i(\lambda)|^2\leqq 1$. Since $V_i^*\Theta(\lambda)$ is a contractive holomorphic function, $\det(V_i^*\Theta(\lambda))\in H^\infty$. Since $\Theta(\lambda)$ is a contraction for every $\lambda\in D$, it follows that

$$\sum_{i=1}^{\infty}\left|h_i(\lambda)(-1)^i\det(V_i^*\Theta(\lambda))\right| \leqq 1,$$

which implies that $\sum_{i=0}^{\infty} h_i(\lambda)(-1)^i\det(V_i^*\Theta(\lambda))$ is holomorphic. Equality (3.9) follows from Lemma.

Problem. Is $\det(U^*[\Gamma(\lambda), \Theta(\lambda)])$ outer?

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On the equiconvergence of expansions by Riesz bases formed by eigenfunctions of the Schrödinger operator

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The equiconvergence theorems play an important role in the theory of expansions. One of the first results of this type was proved by A. HAAR [1] in 1910—11. Later on general equiconvergence theorems were proved for the self-adjoint Schrödinger operator. However, many problems of practical interest require the investigation of the non-selfadjoint case. Under general conditions there do not exist complete orthonormal systems of eigenfunctions. However, introducing the eigenfunctions of higher order (a notion similar to what is known from the Jordan theorem in linear algebra), the existence of a Riesz basis consisting of eigenfunctions of higher order was proved for such cases, too [4], [5], [6]. During the investigation of a non-classical heat transfer problem a concrete Riesz basis consisting of eigenfunctions of higher order with infinitely many eigenfunctions of order $\geqq 1$ was found by A. A. SAMARSKIĬ and N. I. IONKIN [12].

The aim of the present paper is to prove a general equiconvergence theorem with respect to Riesz bases, which extends the previous results for the case of discrete spectrum in several directions. Namely we consider a complex potential function from the class $L^1_{\text{loc}}(G)$ where G is an arbitrary interval and the eigenvalues may be arbitrary complex numbers. This theorem was first obtained by the authors independently. The present proof is a synthesis which is based on a fruitful method of V. A. Il'in [11] and uses also some new ideas of the papers [7]—[10].

1. Bessel-systems of eigenfunctions

Let G be an arbitrary open interval on the real line, $q\in L^1_{\text{loc}}(G)$ an arbitrary complex function and consider the formal Schrödinger operator $Lu=-u''+qu$. Given a complex number λ, the function $u: G\to\mathbf{C}$, $u\not\equiv 0$ is called an eigenfunction

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of order -1 of the operator L with the eigenvalue λ. Furthermore, a function $u: G \to \mathbf{C}$, $u \not\equiv 0$ is called an eigenfunction of order m $(m=0, 1, \ldots)$ of the operator L with the eigenvalue λ if u and its derivative u' are locally absolutely continuous on G and $Lu = \lambda u - u^*$ almost everywhere on G, where u^* is an eigenfunction of order $m-1$ of the operator L with the same eigenvalue λ.

Let us introduce for any $\mu \in \mathbf{C}$, $t > 0$ the functions

$$f_1(\mu, t) := t\frac{\sin \mu t}{\mu},$$

$$f_i(\mu, t) := \int_0^t \frac{\sin \mu(t-t_{i-1})}{\mu} \int_0^{t_{i-1}} \ldots \int_0^{t_2} \frac{\sin \mu(t_2-t_1)}{\mu} t_1 \frac{\sin \mu t_1}{\mu} dt_1 \ldots dt_{i-1} \quad (i = 2, 3, \ldots)$$

and for any $u \in L^\infty_{\mathrm{loc}}(G)$, $\mu \in \mathbf{C}$, $x \pm t \in G$, $t > 0$ the functions

$$g_0(u, \mu, x, t) := \int_{x-t}^{x+t} \frac{\sin \mu(t-|x-\xi|)}{\mu} q(\xi) u(\xi)\, d\xi,$$

$$g_i(u, \mu, x, t) := \int_0^t \frac{\sin \mu(t-t_i)}{\mu} \int_0^{t_i} \ldots \int_0^{t_2} \frac{\sin \mu(t_2-t_1)}{\mu} \times$$

$$\times \int_{x-t_1}^{x+t_1} \frac{\sin \mu(t_1-|x-\xi|)}{\mu} q(\xi) u(\xi)\, d\xi\, dt_1 \ldots dt_i \quad (i = 1, 2, \ldots).$$

Lemma 1. *Let u_m be an eigenfunction of order $\leqq m$ of the operator L with the eigenvalue $\lambda = \mu^2$ and put $u_{j-1} := \lambda u_j - L u_j$, $j = 0, 1, \ldots, m$. Then*

$$(1) \quad u_m(x+t) + u_m(x-t) - 2u_m(x) \cos \mu t = \sum_{i=1}^m f_i(\mu, t) u_{m-i}(x) + \sum_{i=0}^m g_i(u_{m-i}, \mu, x, t)$$

whenever $x \pm t \in G$, $t > 0$. Moreover, putting $\nu := \operatorname{Im} \mu$, the following estimates are valid:

$$(2) \quad \begin{aligned} |f_i(\mu, t) u_{m-i}(x)| &\leqq \left|\frac{t}{\mu}\right|^i |u_{m-i}(x) \operatorname{ch} \nu t|, \\ |g_i(u_{m-i}, \mu, x, t)| &\leqq \frac{\|q\|_{L^1(x-t, x+t)}}{|\mu|} \left|\frac{t}{\mu}\right|^i \sup_{|x-\xi| \leqq t} |u_{m-i}(\xi) \operatorname{ch} \nu(t-|x-\xi|)|. \end{aligned}$$

Proof. We recall the generalized Titchmarsh formula of Joó [7]:

$$(3) \quad \begin{aligned} &u_m(x+t) + u_m(x-t) - 2u_m(x) \cos \mu t = \\ &= \int_{x-t}^{x+t} \frac{\sin \mu(t-|x-\xi|)}{\mu} [q(\xi) u_m(\xi) + u_{m-1}(\xi)]\, d\xi. \end{aligned}$$

One can easily see that

$$\tag{4} \begin{aligned} &\int_{x-t}^{x+t} \frac{\sin\mu(t-|x-\xi|)}{\mu} u_{m-1}(\xi)\,d\xi = t\frac{\sin\mu t}{\mu} u_{m-1}(x)+ \\ &+\int_0^t \frac{\sin\mu(t-\tau)}{\mu}[u_{m-1}(x+\tau)+u_{m-1}(x-\tau)-2u_{m-1}(x)\cos\mu\tau]\,d\tau. \end{aligned}$$

Combining (3) and (4) we obtain

$$\tag{5} \begin{aligned} &u_m(x+t)+u_m(x-t)-2u_m(x)\cos\mu t = \\ &= \int_{x-t}^{x+t} \frac{\sin\mu(t-|x-\xi|)}{\mu} q(\xi)u_m(\xi)\,d\xi + t\frac{\sin\mu t}{\mu} u_{m-1}(x)+ \\ &+\int_0^t \frac{\sin\mu(t-\tau)}{\mu}[u_{m-1}(x+\tau)+u_{m-1}(x-\tau)-2u_{m-1}(x)\cos\mu\tau]\,d\tau. \end{aligned}$$

Now the formula (1) can be proved by an easy induction on m. Indeed, for $m=0$ both formulas (3) and (5) coincide with (1). Assume (1) is true for $m-1$ instead of m. Then applying this formula in the last integral of (5), we obtain (1) for m.

The estimates (2) follow immediately from the definition of f_i and g_i. The lemma is proved.

Let us now given a system $(u_k)\subset L^2(G)$ of eigenfunctions of the operator L. Let λ_k (resp. o_k) denote the eigenvalue (resp., the order) of u_k and assume that the following conditions are satisfied:

$$\tag{6} \sup o_k < \infty,$$

$$\tag{7} \text{in case } o_k>0,\ \lambda_k u_k - Lu_k = u_{k-1},$$

(8) (u_k) is a Bessel system, i.e., for any $w\in L^2(G)$, $\sum_k |\langle u_k, w\rangle|^2 \leqq C_0\|w\|^2_{L^2(G)}$ where C_0 is a constant independent of w.

The purpose of this section is to prove the following

Proposition 1. *Given any compact interval $K\subset G$, there exists an $R>0$ with*

$$\sup_{\mu>0} \sum_{|\mu-|\mathrm{Re}\sqrt{\lambda_k}||\leqq 1} (\|u_k\|_{L^\infty(K)} \operatorname{ch}(R\,\mathrm{Im}\sqrt{\lambda_k}))^2 < \infty.$$

We need some preliminary lemmas. For brevity, let us denote by μ_k a square root of λ_k for which $\mathrm{Re}\,\mu_k\geqq 0$, and put $\varrho_k:=\mathrm{Re}\,\mu_k$, $\nu_k:=\mathrm{Im}\,\mu_k$. We shall repeatedly deal with compact intervals $K=[a,b]$ having the property

$$\tag{9} K_R := [a-R, b+R]\subset G \quad \text{for} \quad R:=(b-a)/4.$$

We introduce in this case the functions $d, v_k\colon K_R\to\mathbf{C}$ defined by

$$d(\xi):=\min\{\xi-(a-R),\ (b+R)-\xi\},\ v_k(\xi):=u_k(\xi)\operatorname{ch}(\nu_k d(\xi)).$$

Lemma 2. *Given any compact interval* $K=[a,b]$ *having property* (9),

$$\|v_k\|_{L^\infty(K_R)}\leqq[3+o(1)]R^{-0.5}\|v_k\|_{L^2(K)}+o(1)\sum_{i=1}^{\sigma_k}\|v_{k-i}\|_{L^2(K)}\quad(|\nu_k|\to\infty)\tag{10}$$

uniformly in k.

Proof. Using the inequalities $|\operatorname{sh}(\operatorname{Im} z)|\leqq|\cos z|$ $(z\in\mathbf{C})$, $R\leqq d(x)\leqq 3R$ $(x\in K)$, and applying Lemma 1 for any $x\in K$ with $t:=d(x)$, we get

$$|v_k(x)|\leqq|0.5\operatorname{cth}\nu_k t|\,|2u_k(x)\cos\mu_k t|\leqq$$

$$\leqq[0.5+o(1)]\Big\{|u_k(x-t)|+|u_k(x+t)|+$$

$$+\sum_{i=0}^{\sigma_k}|\mu_k|^{-i-1}\|q\|_{L^1(K_R)}t^i\|v_{k-i}\|_{L^\infty(K_R)}+\sum_{i=1}^{\sigma_k}|\mu_k|^{-i}t^i|v_{k-i}(x)|\Big\}\leqq$$

$$\leqq[0.5+o(1)]\Big\{2\|v_k\|_{L^\infty(K_R\setminus K)}+(\operatorname{ch}\nu_k R)^{-1}\|v_k\|_{L^\infty(K)}+o(1)\sum_{i=0}^{\sigma_k}\|v_{k-i}\|_{L^\infty(K_R)}\Big\}\leqq$$

$$\leqq[1+o(1)]\|v_k\|_{L^\infty(K_R\setminus K)}+o(1)\sum_{i=0}^{\sigma_k}\|v_{k-i}\|_{L^\infty(K_R)}.$$

This is obviously true for all $x\in K_R\setminus K$, too, therefore

$$\|v_k\|_{L^\infty(K_R)}\leqq[1+o(1)]\|v_k\|_{L^\infty(K_R\setminus K)}+o(1)\sum_{i=0}^{\sigma_k}\|v_{k-i}\|_{L^\infty(K_R)}.\tag{11}$$

Now for any $a-R\leqq x\leqq a$ and $R\leqq t\leqq 2R$, we apply Lemma 1 with $x+t$ instead of x, and multiply by $\operatorname{ch}\nu_k d(x)$, to derive

$$|v_k(x)|\leqq|v_k(x+2t)|+2|v_k(x+t)|+o(1)\sum_{i=0}^{\sigma_k}\|v_{k-i}\|_{L^\infty(K_R)}.$$

Applying the transformation $R^{-1}\int_R^{2R}\cdot\,dt$ and using in the first two integrals of the right side the Hölder inequality, we have

$$|v_k(x)|\leqq 3R^{-0.5}\|v_k\|_{L^2(K)}+o(1)\sum_{i=0}^{\sigma_k}\|v_{k-i}\|_{L^\infty(K_R)}.$$

This is true analogously for all $b\leqq x\leqq b+R$, too, therefore

$$\|v_k\|_{L^\infty(K_R\setminus K)}\leqq 3R^{-0.5}\|v_k\|_{L^2(K)}+o(1)\sum_{i=0}^{\sigma_k}\|v_{k-i}\|_{L^\infty(K_R)}.$$

Substituting this into the right side of (11), we obtain

$$\|v_k\|_{L^\infty(K_R)} \leqq [3+o(1)]R^{-0.5}\|v_k\|_{L^2(K)}+o(1)\sum_{i=0}^{\sigma_k}\|v_{k-i}\|_{L^\infty(K_R)},$$

whence

$$\|v_k\|_{L^\infty(K_R)} \leqq [3+o(1)]R^{-0.5}\|v_k\|_{L^2(K)}+o(1)\sum_{i=1}^{\sigma_k}\|v_{k-i}\|_{L^\infty(K_R)}. \tag{12}$$

Now we prove (10) by induction on σ_k. For $\sigma_k=-1$ (10) is trivial because $v_k\equiv 0$. Suppose (10) is true for $\sigma_k<m$ $(m\geqq 0)$. Then it is true also for $\sigma_k=m$. Indeed, using (12) and the induction hypothesis,

$$\|v_k\|_{L^\infty(K_R)} \leqq [3+o(1)]R^{-0.5}\|v_k\|_{L^2(K)}+$$

$$+o(1)\sum_{i=1}^{\sigma_k}\Big\{[3+o(1)]R^{-0.5}\|v_{k-i}\|_{L^2(K)}+o(1)\sum_{j=1}^{\sigma_{k-i}}\|v_{k-i-j}\|_{L^2(K)}\Big\}=$$

$$=[3+o(1)]R^{-0.5}\|v_k\|_{L^2(K)}+o(1)\sum_{i=1}^{\sigma_k}\|v_{k-i}\|_{L^2(K)}$$

(in the last step we used that $\sigma_{k-i}=\sigma_k-i$). The lemma is proved.

Lemma 3. *Each point of* G *has a neighbourhood* K *having property* (9) *such that*

$$\sup_{\mu>0}\sum_{\substack{|\mu-\varrho_k|\leqq 1\\ \varrho_k\geqq B|\nu_k|\\ |\nu_k|\geqq B}}(\|u_k\|_{L^\infty(K)}\operatorname{ch}0.5\nu_k R)^2=O(1)\quad(B\to\infty). \tag{13}$$

Proof. Putting for brevity

$$I_\mu=I_\mu(B):=\{k\colon\ |\mu-\varrho_k|\leqq 1,\ \varrho_k\geqq B|\nu_k|,\ |\nu_k|\geqq B\}, \tag{14}$$

it suffices to show in view of (6) that to any $m\in\{-1,0,1,\ldots\}$, each point $y\in G$ has a neighbourhood $K_{y,m}$ with property (9) such that

$$\sup_{\mu>0}\sum_{\substack{k\in I_\mu\\ \sigma_k\leqq m}}(\|u_k\|_{L^\infty(K_{y,m})}\operatorname{ch}0.5\nu_k R_{y,m})^2=O(1)\quad(B\to\infty). \tag{15}$$

This is obvious for $m=-1$: each point of G has a neighbourhood having property (9). Let now $m\geqq 0$ and assume (15) is true for $m-1$. Let now $K=K_{y,m}$ be an arbitrary compact subinterval of $K_{y,m-1}$ which is 6 times shorter than $K_{y,m-1}$ and contains y. It follows then from the inductive hypothesis that

$$\sup_{\mu>0}\sum_{\substack{k\in I_\mu\\ \sigma_k<m}}\|v_k\|^2_{L^2(K)}=O(1)\quad(B\to\infty). \tag{16}$$

Indeed, for any k,

$$\|v_k\|_{L^2(K)} \leqq (\|u_k\|_{L^\infty(K)} \operatorname{ch} 3\nu_k R)(4R)^{0.5} =$$
$$= (4R)^{0.5}\|u_k\|_{L^\infty(K)} \operatorname{ch}(0.5\nu_k R_{y,m-1}) \leqq (4R)^{0.5}\|u_k\|_{L^\infty(K_{y,m-1})} \operatorname{ch}(0.5\nu_k R_{y,m-1}).$$

Let us now fix $\mu>0$ and $x\in K$ arbitrarily and introduce the function $w: G\to\mathbf{R}$:

$$w(x+t) := \begin{cases} \cos\mu t & \text{if } |t| \leqq d(x), \\ 0 & \text{otherwise.} \end{cases}$$

In the sequel we shall consider always $k\in I_\mu$ and the estimates o, O $(B\to\infty)$ will be uniform in $\mu>0, x\in K$ and $k\in I_\mu$. Obviously,

$$\|w\|^2_{L^2(G)} = O(1). \tag{17}$$

For any $k\in I_\mu$, multiplying (1) by $\cos\mu t$ and integrating by t from 0 to $d(x)$, we obtain

$$\int_0^{d(x)} 2\cos\mu t\cos\mu_k t\,dt\cdot u_k(x) = \langle u_k, w\rangle - \tag{18}$$
$$-\sum_{i=1}^{\sigma_k}\int_0^{d(x)} \cos\mu t f_i(\mu_k, t)\,dt\cdot u_{k-i}(x) - \sum_{i=0}^{\sigma_k}\cos\mu t\, g_i(u_{k-i}, \mu_k, x, t)\,dt.$$

Here

$$\int_0^{d(x)} 2\cos\mu t\cos\mu_k t\,dt = \frac{\sin(\mu-\mu_k)d(x)}{\mu-\mu_k} + \frac{\sin(\mu+\mu_k)d(x)}{\mu+\mu_k};$$

using $\varrho_k\geqq 0$, $d(x)\geqq R$, the definition (14) of I_μ and the inequalities $|\operatorname{sh}\operatorname{Im} z|\leqq$ $\leqq|\sin z|\leqq\operatorname{ch}\operatorname{Im} z$ $(z\in\mathbf{C})$, we get that

$$\left|\frac{\sin(\mu-\mu_k)d(x)}{\mu-\mu_k}\right| \geqq \frac{\operatorname{ch}\nu_k d(x)}{|\nu_k|}|\operatorname{th}\nu_k d(x)|\sqrt{\frac{\nu_k^2}{1+\nu_k^2}} = [1-o(1)]\frac{\operatorname{ch}\nu_k d(x)}{|\nu_k|},$$
$$\left|\frac{\sin(\mu+\mu_k)d(x)}{\mu+\mu_k}\right| \leqq \frac{\operatorname{ch}\nu_k d(x)}{|\nu_k|}\sqrt{\frac{\nu_k^2}{B^2\nu_k^2+\nu_k^2}} = o(1)\frac{\operatorname{ch}\nu_k d(x)}{|\nu_k|},$$

whence

$$\left|\int_0^{d(x)} 2\cos\mu t\cos\mu_k t\,dt\right| \geqq [1-o(1)]\frac{\operatorname{ch}\nu_k d(x)}{|\nu_k|}. \tag{19}$$

(18), (19), (2) and (14) imply

$$[1-o(1)]|\nu_k|^{-1}|v_k(x)| \leqq |\langle u_k, w\rangle| +$$
$$+\|q\|_{L^1(K_R)}d(x)\left|\frac{\nu_k}{\mu_k}\right|\left\|\frac{v_k}{\nu_k}\right\|_{L^\infty(K_R)} + o(1)\sum_{i=1}^{\sigma_k}\|v_{k-i}\|_{L^\infty(K_R)},$$
$$\left|\frac{v_k(x)}{\nu_k}\right| \leqq O(1)|\langle u_k, w\rangle| + o(1)\left\|\frac{v_k}{\nu_k}\right\|_{L^\infty(K_R)} + o(1)\sum_{i=1}^{\sigma_k}\|v_{k-i}\|_{L^\infty(K_R)}.$$

Applying (10), we obtain

$$\left|\frac{v_k(x)}{\nu_k}\right| \leqq O(1)\,|\langle u_k, w\rangle| + o(1)\left\|\frac{v_k}{\nu_k}\right\|_{L^2(K)} + o(1)\sum_{i=1}^{\sigma_k}\|v_{k-i}\|_{L^2(K)}.$$

Summing up the square of this inequality for an arbitrary finite index set $I\subset\subset\{k\in I_\mu\colon o_k\leqq m\}$, then applying (17) and the Bessel inequality (8) to the first sum on the right side, we obtain

$$\sum_{k\in I}\left|\frac{v_k(x)}{\nu_k}\right|^2 \leqq O(1)+o(1)\sum_{k\in I}\left\|\frac{v_k}{\nu_k}\right\|^2_{L^2(K)} + o(1)\sum_{k\in I}\sum_{i=1}^{\sigma_k}\|v_{k-i}\|^2_{L^2(K)} \leqq$$

$$\leqq O(1)+o(1)\sum_{k\in I}\left\|\frac{v_k}{\nu_k}\right\|^2_{L^2(K)} + o(1)\sum_{\substack{k\in I_\mu\\ \sigma_k<m}}\|v_k\|^2_{L^2(K)} \leqq O(1)+o(1)\sum_{k\in I}\left\|\frac{v_k}{\nu_k}\right\|^2_{L^2(K)}$$

(we used (16) in the last step), whence, integrating by x on K, we get

$$\sum_{k\in I}\left\|\frac{v_k}{\nu_k}\right\|^2_{L^2(K)} \leqq O(1)+o(1)\sum_{k\in I}\left\|\frac{v_k}{\nu_k}\right\|^2_{L^2(K)},$$

$$\sum_{k\in I}\left\|\frac{v_k}{\nu_k}\right\|^2_{L^2(K)} \leqq O(1).$$

Since I was chosen arbitrarily,

$$\sum_{\substack{k\in I_\mu\\ \sigma_k\leqq m}}\left\|\frac{v_k}{\nu_k}\right\|^2_{L^2(K)} = O(1).$$

Using (10) and (16) again, we see that

$$\sum_{\substack{k\in I_\mu\\ \sigma_k\leqq m}}\left\|\frac{v_k}{\nu_k}\right\|^2_{L^\infty(K_R)} \leqq O(1)\sum_{\substack{k\in I_\mu\\ \sigma_k\leqq m}}\left\|\frac{v_k}{\nu_k}\right\|^2_{L^2(K)} + o(1)\sum_{\substack{k\in I_\mu\\ \sigma_k<m}}\|v_k\|^2_{L^2(K)} =$$

$$= O(1)+o(1)\sum_{\substack{k\in I_\mu\\ \sigma_k<m}}\|v_k\|^2_{L^2(K)} = O(1),$$

and hence (15) follows with $K=K_{y,m}$ because for any k,

$$\|u_k\|_{L^\infty(K)}\,\mathrm{ch}\,(0.5\,\nu_k R) \leqq \|u_k\|_{L^\infty(K)}\frac{\mathrm{ch}\,\nu_k R}{|\nu_k R|} \leqq R^{-1}\left\|\frac{v_k}{\nu_k}\right\|_{L^\infty(K_R)}.$$

The lemma is proved.

Lemma 4. *Each point of G has a neighbourhood K having property* (9) *such that for any fixed $B>0$,*

$$\sum_{\substack{\varrho_k\leqq B|\nu_k|\\ |\nu_k|\geqq C}} (\|u_k\|_{L^\infty(K)}\,\mathrm{ch}\,0.5\nu_k R)^2 = O(1) \quad (C\to\infty). \tag{20}$$

Proof. Setting

$$J = J(C) := \{k\colon \varrho_k \leqq B|\nu_k|,\ |\nu_k| \geqq C\}, \tag{21}$$

it suffices to show in view of (6) that for any $m \in \{-1, 0, \ldots\}$ and $y \in G$, there exists a neighbourhood $K_{y,m}$ of y having property (9) such that

$$\sum_{\substack{k\in J\\ \sigma_k \leqq m}} (\|u_k\|_{L^\infty(K_{y,m})} \operatorname{ch} 0.5\nu_k R_{y,m})^2 = O(1) \quad (C \to \infty). \tag{22}$$

This is obvious for $m=-1$. Assume $m \geqq 0$ and (22) is true for $m-1$. Let $K=K_{y,m}$ be an arbitrary compact subinterval of $K_{y,m-1}$ containing y which is at least 6 times shorter than $K_{y,m-1}$ and which satisfies the following condition:

$$81(m+2)R^2\|q\|^2_{L^1(K_R)} < 8^{-1}. \tag{23}$$

As in the preceding lemma, we have

$$\sum_{\substack{k\in J\\ \sigma_k < m}} \|v_k\|^2_{L^2(K)} = O(1) \quad (C \to \infty). \tag{24}$$

Let us fix $x \in K$ arbitrarily and define $w\colon G \to \mathbf{R}$ by

$$w(x+t) := \begin{cases} 1 & \text{if } |t| \leqq d(x), \\ 0 & \text{otherwise.} \end{cases}$$

In the following considerations the estimates o, O will be uniform in $x \in K$ and $k \in J$ $(C \to \infty)$. Obviously,

$$\|w\|^2_{L^2(G)} = O(1). \tag{25}$$

For any $k \in J$, $o_k \leqq m$, integrating (1) by t from 0 to $d(x)$, we get

$$\begin{aligned} \int_0^{d(x)} 2\cos\mu_k t\, dt\, u_k(x) &= \langle u_k, w\rangle - \\ - \sum_{i=1}^{\sigma_k} \int_0^{d(x)} f_i(\mu_k, t)\, dt\, u_{k-i}(x) &- \sum_{i=0}^{\sigma_k} g_i(u_{k-i}, \mu_k, x, t)\, dt. \end{aligned} \tag{26}$$

Here, by the inequality $d(x) \geqq R$,

$$\left|\int_0^{d(x)} 2\cos\mu_k t\, dt\right| = \left|\frac{\sin\mu_k d(x)}{\mu_k}\right| \geqq \left|\frac{\operatorname{sh}\nu_k d(x)}{\mu_k}\right| = [1-o(1)]\frac{\operatorname{ch}\nu_k d(x)}{|\mu_k|},$$

and therefore (26), (2), and (21) imply

$$[1-o(1)]\left|\frac{v_k(x)}{\mu_k}\right| \leqq |\langle u_k, w\rangle| + 3R\|q\|_{L^1(K_R)}\left\|\frac{v_k}{\mu_k}\right\|_{L^\infty(K_R)} + \\ + o(1)\sum_{i=1}^{\sigma_k}\|v_{k-i}\|_{L^\infty(K_R)}$$

(we used that $d(x) \leqq 3R$), whence in view of (10),

$$\left|\frac{v_k(x)}{\mu_k}\right| \leqq O(1)|\langle u_k, w\rangle| + [9+o(1)]R^{0.5}\|q\|_{L^1(K_R)}\left\|\frac{v_k}{\mu_k}\right\|_{L^2(K)} +$$

$$+o(1)\sum_{i=1}^{\sigma_k}\|v_{k-i}\|_{L^2(K)},$$

and

$$\left|\frac{v_k(x)}{\mu_k}\right|^2 \leqq O(1)\,|\langle u_k, w\rangle|^2 +$$

$$+(2+o_k)[81+o(1)]R\|q\|^2_{L^1(K_R)}\left\|\frac{v_k}{\mu_k}\right\|^2_{L^2(K)} + o(1)\sum_{i=1}^{\sigma_k}\|v_{k-i}\|^2_{L^2(K)}.$$

$$\leqq O(1)\,|\langle u_k, w\rangle|^2 + (8R)^{-1}[1+o(1)]\left\|\frac{v_k}{\mu_k}\right\|^2_{L^2(K)} + o(1)\sum_{i=1}^{\sigma_k}\|v_{k-i}\|^2_{L^2(K)}$$

(we used (23) and $o_k \leqq m$). Summing up this inequality for an arbitrary finite index set $I \subset \{k \in J;\ o_k \leqq m\}$, then applying (8) and (25) on the right side, and finally integrating by x on K (the length of which is $4R$), we obtain

$$\sum_{k\in I}\left\|\frac{v_k}{\mu_k}\right\|^2_{L^2(K)} \leqq O(1) + [0.5+o(1)]\sum_{k\in I}\left\|\frac{v_k}{\mu_k}\right\|^2_{L^2(K)} + o(1)\sum_{\substack{k\in J\\ \sigma_k<m}}\|v_k\|^2_{L^2(K)}.$$

Hence, using (24) and the choice of I, we have

$$\sum_{\substack{k\in J\\ \sigma_k\leqq m}}\left\|\frac{v_k}{\mu_k}\right\|^2_{L^2(K)} = O(1),$$

and taking into account the estimate $|\mu_k| \leqq (1+B)|\nu_k|$, we get

$$\sum_{\substack{k\in J\\ \sigma_k\leqq m}}\left\|\frac{v_k}{\nu_k}\right\|^2_{L^2(K)} = O(1).$$

Now the proof can be finished exactly as in the preceeding lemma, using (24) instead of (16). The lemma is proved.

Lemma 5. *Given any compact interval $K=[a, b]\subset G$ and any number $D>0$, we have*

(27) $$\sup_{\mu>0}\sum_{\substack{|\mu-\varrho_k|\leqq 1\\ |\nu_k|\leqq D}}\|u_k\|^2_{L^\infty(K)} < \infty.$$

Proof. Putting $I_\mu := \{k\colon |\mu-\varrho_k|\leqq 1, |\nu_k|\leqq D\}$, we will show by induction on m that

$$\sup_{\mu>0} \sum_{\substack{k\in I_\mu \\ \sigma_k\leqq m}} \|u_k\|^2_{L^2(K)} < \infty \quad (m=-1,0,\ldots). \tag{28}$$

Hence (27) will follow in view of (6) because by a result of Joó [7] there exists a constant $C_{m,D}$ such that

$$\|u_k\|_{L^\infty(K)} \leqq C_{m,D}\|u_k\|_{L^2(K)} \quad \text{whenewer} \quad o_k\leqq m \text{ and } |\nu_k|\leqq D. \tag{29}$$

(28) is true for $m=-1$. Let now $m\geqq 0$ and assume (28) is true for $m-1$, i.e.,

$$\sup_{\mu>0} \sum_{\substack{k\in I_\mu \\ \sigma_k< m}} \|u_k\|^2_{L^2(K)} < \infty. \tag{30}$$

In the following arguments the estimates O, o $(R\to 0)$ will be uniform in $\mu>0$, $k\in I_\mu$, and $x\in K$.

For any $\mu>0$ and $a\leqq x\leqq 2^{-1}(a+b)$ define $w\colon G\to\mathbf{R}$ by

$$w(y) := \begin{cases} 2\cos\mu(y-x)-0.5 & \text{if } x\leqq y\leqq x+R,\\ -0.5 & \text{if } x+R<y\leqq x+2R,\\ 0 & \text{otherwise.}\end{cases}$$

Obviously,

$$\|w\|^2_{L^2(G)} = O(1) \quad (R\to 0). \tag{31}$$

Applying (1) for any $k\in I_\mu$, $o_k\leqq m$ with $0\leqq t\leqq R\leqq 4^{-1}(b-a)$ and with $x+t$ instead of x, we obtain that

$$u_k(x) = 2u_k(x+t)\cos\mu t - u_k(x+2t) + 2u_k(x+t)[\cos\mu_k t-\cos\mu t] +$$

$$+\sum_{i=1}^{\sigma_k} f_i(\mu_k, t)u_{k-i}(x+t) + \sum_{i=0}^{\sigma_k} g_i(u_{k-i}, \mu_k, x+t, t);$$

integrating by t from 0 to R, we see that

$$Ru_k(x) = \langle u_k, w\rangle + \int_0^R 2u_k(x+t)[\cos\mu_k t-\cos\mu t]\,dt +$$

$$+\sum_{i=1}^{\sigma_k}\int_0^R f_i(\mu_k, t)\,u_{k-i}(x+t)\,dt + \sum_{i=0}^{\sigma_k}\int_0^R g_i(u_{k-i}, \mu_k, x+t, t)\,dt.$$

Taking into account that $\mu\in\mathbf{R}$ and $|\mu-\mu_k|\leqq D+1$, we get

$$|\cos\mu_k t-\cos\mu t| = O(t) \quad (t\to 0).$$

Furthermore using instead of (2) the estimates

$$\tag{32} \begin{aligned} |f_i(\mu_k, t)\, u_{k-i}(x)| &= O(t^{2i})\,|u_{k-i}(x)| \quad (t \to 0),\\ |g_i(u_{k-i}, \mu_k, x, t)| &= O(t^{2i+1})\,\|u_{k-i}\|_{L^\infty(x-t,\,x+t)} \quad (t \to 0), \end{aligned}$$

which follow from $|v_k| \leqq D$ and from the definition of f_i, g_i, we obtain that

$$R\,|u_k(x)| \leqq |\langle u_k, w\rangle| + O(R^2)\,\|u_k\|_{L^\infty(K)} +$$

$$+\sum_{i=1}^{\sigma_k} O(R^{2i+1})\,\|u_{k-i}\|_{L^\infty(K)} + \sum_{i=1}^{\sigma_k} O(R^{2i+2})\,\|u_{k-i}\|_{L^\infty(K)} \leqq$$

$$\leqq |\langle u_k, w\rangle| + O(R^2)\,\|u_k\|_{L^\infty(K)} + O(1)\sum_{i=1}^{\sigma_k} \|u_{k-i}\|_{L^\infty(K)} \quad (R \to 0),$$

yielding

$$R^2\,|u_k(x)|^2 \leqq O(1)\,|\langle u_k, w\rangle|^2 + o(R^2)\,\|u_k\|^2_{L^\infty(K)} + O(1)\sum_{i=1}^{\sigma_k} \|u_{k-i}\|^2_{L^\infty(K)} \quad (R \to 0)$$

and in view of (29),

$$R^2\,|u_k(x)|^2 \leqq O(1)\,|\langle u_k, w\rangle|^2 + o(R^2)\,\|u_j\|^2_{L^2(K)} + O(1)\sum_{i=1}^{\sigma_k} \|u_{k-i}\|^2_{L^2(K)} \quad (R \to 0).$$

Summing up this inequality for any finite set $I \subset \{k \in I_\mu \colon o_k \leqq m\}$, and applying (8), (31) and (30), we have

$$\sum_{k\in I} R^2\,|u_k(x)|^2 \leqq O(1) + o(R^2)\sum_{k\in I} \|u_k\|^2_{L^2(K)}.$$

By a similar argument, this inequality is true for all $2^{-1}(a+b) \leqq x \leqq b$, too. Thus, integrating by x on K we get that

$$\sum_{k\in I} R^2\,\|u_k\|^2_{L^2(K)} \leqq O(1) + o(R^2)\sum_{k\in I} \|u_k\|^2_{L^2(K)},$$

$$\sum_{k\in I} \|u_k\|^2_{L^2(K)} = O(R^{-2})$$

and $I \subset \{k \in I_\mu \colon o_k \leqq m\}$ being arbitrary,

$$\sum_{\substack{k\in I_\mu\\ \sigma_k \leqq m}} \|u_k\|^2_{L^2(K)} = O(R^{-2}) \quad (R \to 0).$$

Hence (28) follows and the lemma is proved.

Now we can prove the proposition formulated after Lemma 1. Given a point $y \in G$ arbitrarily, there exists by Lemma 3 a neighbourhood K_1 of y and two numbers $R_1, B > 0$ with

$$\tag{33} \sup_{\mu>0} \sum_{\substack{|\mu-\varrho_k|\leqq 1\\ \varrho_k \geqq B|v_k|\\ |v_k|\geqq B}} (\|u_k\|_{L^\infty(K_1)}\operatorname{ch} v_k R_1)^2 < \infty.$$

Fixing B, there exists by Lemma 4 another neighbourhood K_2 of y and two numbers $R_2, C>0$ such that

$$(34) \qquad \sum_{\substack{\varrho_k \leqq B|\nu_k| \\ |\nu_k| \geqq C}} (\|u_k\|_{L^\infty(K_2)} \operatorname{ch} \nu_k R_2)^2 < \infty.$$

Finally, for

$$(35) \qquad K := K_1 \cap K_2, \quad R := \min\{R_1, R_2\} \quad \text{and} \quad D := \max\{B, C\},$$

it follows from Lemma 5 that

$$(36) \qquad \sup_{\mu>0} \sum_{\substack{|\mu-\varrho_k| \leqq 1 \\ |\nu_k| \leqq D}} (\|u_k\|_{L^\infty(K)} \operatorname{ch} \nu_k R)^2 < \infty.$$

(33)—(36) imply

$$\sup_{\mu>0} \sum_{|\mu-\varrho_k| \leqq 1} (\|u_k\|_{L^\infty(K)} \operatorname{ch} \nu_k R)^2 < \infty,$$

i.e., each point $y \in G$ has a neighbourhood K_y such that for some $R_y > 0$,

$$(37) \qquad \sup_{\mu>0} \sum_{|\mu-\varrho_k| \leqq 1} (\|u_k\|_{L^\infty(K_y)} \operatorname{ch} \nu_k R_y)^2 < \infty.$$

Hence the proposition follows by an elementary compactness argument. Indeed, given any compact interval $K \subset G$, it can be covered by a finite system $\{K_{y_i}: i=1, 2, \ldots, N\}$ of intervals having property (37) with some $R_i > 0$, $i=1, \ldots, N$. Setting $R := \min\{R_1, \ldots, R_N\}$, we have obviously

$$\sup_{\mu>0} \sum_{|\mu-\varrho_k| \leqq 1} (\|u_k\|_{L^2(K)} \operatorname{ch} \nu_k R)^2 < \infty,$$

completing the proof of Proposition 1.

2. An equiconvergence theorem

Let G be an arbitrary open interval on the real line, $q, \hat{q} \in L^1_{\text{loc}}(G)$ arbitrary complex functions. Let (u_k) (resp. $(\hat{u}_k)$) be a Riesz basis in $L^2(G)$ consisting of eigenfunctions of the operator $Lu = -u'' + qu$ (resp. $\hat{L}u = -u'' + \hat{q}u$) and having the following properties:

$$(38) \qquad \sup o_k < \infty, \ \sup \hat{o}_k < \infty,$$

$$(39) \qquad \text{in case } o_k > 0 \quad (\text{resp. } \hat{o}_k > 0)$$

$$\lambda_k u_k - L u_k = u_{k-1} \quad (\text{resp. } \hat{\lambda}_k \hat{u}_k - \hat{L} \hat{u}_k = \hat{u}_{k-1}).$$

where λ_k and o_k (resp. $\hat{\lambda}_k$ and $\hat{o}_k$) are the eigenvalue and the order of u_k (resp. $\hat{u}_k$).

Now let us introduce some notations:

$$\sigma_\mu(f, x) := \sum_{|\mathrm{Re}\sqrt{\lambda_k}|<\mu} \langle f, v_k\rangle u_k(x),$$
(40)
$$\hat{\sigma}_\mu(f, x) := \sum_{|\mathrm{Re}\sqrt{\lambda_k}|<\mu} \langle f, \hat{v}_k\rangle \hat{u}_k(x) \quad (f\in L^2(G),\ x\in G,\ \mu > 0)$$

where (v_k) (resp. $(\hat{v}_k)$) is the dual system of (u_k) (resp. $(\hat{u}_k)$), i.e., (v_k), $(\hat{v}_k)\subset L^2(G)$ and $\langle v_k, u_j\rangle = \langle \hat{v}_k, \hat{u}_j\rangle = \delta_{kj}$. The following result holds:

Theorem. *Given any compact interval $K\subset G$, for all $f\in L^2(G)$*

$$\lim_{\mu\to\infty} \sup_{x\in K} |\sigma_\mu(f, x) - \hat{\sigma}_\mu(f, x)| = 0. \tag{41}$$

For $f\in L^2(G)$, $\mu>0$ and $x\pm R\in G$, define

$$S_\mu(f, x) = S_\mu(f, x, R) := \int_{x-R}^{x+R} \frac{\sin\mu(y-x)}{\pi(y-x)} f(y)\,dy. \tag{42}$$

The theorem will follow obviously from the following assertion:

Proposition 2. *Given any compact interval $K\subset G$, for any sufficiently small $R>0$, and for all $f\in L^2(G)$, we have*

$$\lim_{\mu\to\infty} \sup_{x\in K} |S_\mu(f, x) - \sigma_\mu(f, x)| = 0. \tag{43}$$

Indeed, an analogous result holds for $\hat{\sigma}_\mu(f, x)$, too, and it remains only to apply the triangle inequality.

For the sake of brevity, from now on we shall denote by μ_k a square root of λ_k with $\mathrm{Re}\,\mu_k\geqq 0$ and we set $\varrho_k := \mathrm{Re}\,\mu_k$, $\nu_k := \mathrm{Im}\,\mu_k$. For the proof of Proposition 2 we shall need two preliminary lemmas.

Lemma 6. *Given any $R>0$, there exists a constant $C=C(R)$ such that with the notation*

$$\delta(\mu, \varrho_k) := \begin{cases} 1 & \text{if } \mu > \varrho_k, \\ 1/2 & \text{if } \mu = \varrho_k, \\ 0 & \text{if } \mu < \varrho_k, \end{cases} \tag{44}$$

for any $\mu>0$ and k, we have

$$\left| \frac{2}{\pi} \int_0^R \frac{\sin\mu t \cos\mu_k t}{t}\,dt - \delta(\mu, \varrho_k) \right| \leqq C(R)\frac{\mathrm{ch}\,\nu_k R}{2+|\mu-\varrho_j|}. \tag{45}$$

Proof. We recall that

$$\int_0^\infty \frac{\sin t}{t}\,dt = 2^{-1}\pi \tag{46}$$

and

$$\left|\int_x^\infty \frac{\sin t}{t}\,dt\right| < 6(1+x)^{-1} \quad \text{for all} \quad x \geqq 0 \tag{47}$$

((47) can be seen by integrating by parts). Setting for brevity $\varrho_k=\varrho$ and $\nu_k=\nu$, we can write

$$(2/\pi)\int_0^R \frac{\sin \mu t \cos \mu_k t}{t}\,dt =$$

$$= (2/\pi)\int_0^R \frac{\sin \mu t}{t}(\cos \varrho t \operatorname{ch} \nu t - i \sin \varrho t \operatorname{sh} \nu t)\,dt =$$

$$= (2/\pi)\int_0^\infty \frac{\sin \mu t \cos \varrho t}{t}\,dt - (2/\pi)\int_R^\infty \frac{\sin \mu t \cos \varrho t}{t}\,dt + \tag{48}$$

$$+(2/\pi)\int_0^R \sin \mu t \cos \varrho t \frac{\operatorname{ch}(\nu t)-1}{t}\,dt - (2i/\pi)\int_0^R \sin \mu t \sin \varrho t \frac{\operatorname{sh}(\nu t)}{t}\,dt \equiv$$

$$\equiv I_1 + I_2 + I_3 + I_4.$$

Here, by (46),

$$I_1 = \pi^{-1}\int_0^\infty \frac{\sin(\mu+\varrho)t + \sin(\mu-\varrho)t}{t}\,dt =$$

$$= \pi^{-1}\big(\operatorname{sgn}(\mu+\varrho) + \operatorname{sgn}(\mu-\varrho)\big)\int_0^\infty \frac{\sin t}{t}\,dt = \delta(\mu, |\varrho|) \tag{49}$$

and

$$I_2 = -\pi^{-1}\int_R^\infty \frac{\sin(\mu+\varrho)t + \sin(\mu-\varrho)t}{t}\,dt =$$

$$= -\pi^{-1}\left((\operatorname{sgn}\mu+\varrho)\int_{|\mu+\varrho|R}^\infty \frac{\sin t}{t}\,dt + \operatorname{sgn}(\mu-\varrho)\int_{|\mu-\varrho|R}^\infty \frac{\sin t}{t}\,dt\right),$$

whence, in view of (47),

$$|I_2| \leqq \pi^{-1}\left(\frac{6}{1+|\mu+\varrho|R} + \frac{6}{1+|\mu-\varrho|R}\right) \leqq \frac{4}{1+|\mu-\varrho|R}. \tag{50}$$

Considering now the quantities I_3, I_4, we obviously have

$$|I_3| \leqq (2/\pi)\int_0^R \frac{\operatorname{ch}(\nu t)-1}{t}\,dt \leqq (2/\pi)(\operatorname{ch}\nu R - 1) \tag{51}$$

and

$$|I_4| \leqq (2/\pi)\int_0^R \frac{\operatorname{sh}(|\nu| t)}{t}\,dt \leqq (2/\pi)\operatorname{sh}\nu R \tag{52}$$

On the other hand, in case $\mu\neq\varrho$ we can write

$$I_3=\pi^{-1}\int_0^R(\sin(\mu+\varrho)t+\sin(\mu-\varrho)t)\frac{\operatorname{ch}\nu t-1}{t}\,dt=$$

$$=\pi^{-1}\left[\left(\frac{-\cos(\mu+\varrho)t}{\mu+\varrho}+\frac{-\cos(\mu-\varrho)t}{\mu-\varrho}\right)\frac{\operatorname{ch}\nu t-1}{t}\right]_0^R+$$

$$+\pi^{-1}\int_0^R\left(\frac{\cos(\mu+\varrho)t}{\mu+\varrho}+\frac{\cos(\mu-\varrho)t}{\mu-\varrho}\right)\left(\frac{\operatorname{ch}\nu t-1}{t}\right)'dt$$

and

$$I_4=(i/\pi)\int_0^R(\cos(\mu+\varrho)t-\cos(\mu-\varrho)t)\frac{\operatorname{sh}\nu t}{t}\,dt=$$

$$=(i/\pi)\left[\left(\frac{\sin(\mu+\varrho)t}{\mu+\varrho}-\frac{\sin(\mu-\varrho)t}{\mu-\varrho}\right)\frac{\operatorname{sh}\nu t}{t}\right]_0^R-$$

$$-(i/\pi)\int_0^R\left(\frac{\sin(\mu+\varrho)t}{\mu+\varrho}-\frac{\sin(\mu-\varrho)t}{\mu-\varrho}\right)\left(\frac{\operatorname{sh}\nu t}{t}\right)'dt,$$

Hence, taking into account that the functions $\left(\frac{\operatorname{ch}\nu t-1}{t}\right)'$, $\left(\frac{\operatorname{sh}\nu t}{t}\right)'$ do not change sign, we obtain that

$$|I_3|\leqq\frac{2}{\pi|\mu-\varrho|}\left(\frac{\operatorname{ch}\nu R-1}{R}+\left|\int_0^R\left(\frac{\operatorname{ch}\nu t-1}{t}\right)'dt\right|\right)=\frac{4\operatorname{ch}\nu R}{\pi R|\mu-\varrho|}. \tag{53}$$

and

$$|I_4|\leqq\frac{2}{\pi|\mu-\varrho|}\left(\left|\frac{\operatorname{sh}\nu R}{R}-\nu\right|+\left|\int_0^R\left(\frac{\operatorname{sh}\nu t}{t}\right)'dt\right|\right)\leqq\frac{4\,|\operatorname{sh}\nu R|}{\pi R\,|\mu-\varrho|}. \tag{54}$$

Now the lemma follows from the relations (48)—(54).

Lemma 7. *Given any $R>0$, there exist constants $C_i=C_i(R)$ such that for the functions f_i, g_i of Lemma* 1,

$$\int_0^R\left|\frac{f_i(\mu,t)}{t}\right|dt\leqq C_i\operatorname{ch}(R\operatorname{Im}\mu)(1+|\mu|)^{-i}\quad(\mu\in\mathbf{C},\ i=1,2,\ldots); \tag{55}$$

furthermore, for any $u\in L^\infty_{\mathrm{loc}}(G)$, $\mu\in\mathbf{C}$ and $x\pm R\in G$,

$$\int_0^R\left|\frac{g_i(u,\mu,x,t)}{t}\right|dt\leqq C_i\|q\|_{L^1(x-R,x+R)}\|u\|_{L^\infty(x-R,x+R)}\times \tag{56}$$

$$\times\operatorname{ch}(R\operatorname{Im}\mu)(1+|\mu|)^{-i-1}\quad(i=1,2,\ldots),$$

$$\int_0^R\left|\frac{g_0(u,\mu,x,t)}{t}\right|dt\leqq C_0\|q\|_{L^1(x-R,x+R)}\|u\|_{L^\infty(x-R,x+R)}\times \tag{57}$$

$$\times\operatorname{ch}(R\operatorname{Im}\mu)\,(1+\ln(1+|\mu|))\,(1+|\mu|)^{-1}.$$

Proof. Using the inequalities

$$\left|\frac{\sin z}{z}\right| \leqq 2\,\mathrm{ch}\,(\mathrm{Im}\, z), \quad |\sin z| \leqq \mathrm{ch}\,(\mathrm{Im}\, z) \quad (z\in\mathbf{C}),$$

$$\mathrm{ch}\,\alpha\,\mathrm{ch}\,\beta \leqq \mathrm{ch}\,(\alpha+\beta), \quad \min\{\alpha, \beta\} \leqq \frac{1+\alpha}{1+1/\beta} \quad (\alpha, \beta > 0),$$

and the notations

$$v := \mathrm{Im}\,\mu, \quad M := \|q\|_{L^1(x-R,x+R)}\|u\|_{L^\infty(x-R,x+R)}$$

we see from the definition of f_i, g_i that

$$|f_i(\mu, t)| \leqq t^i\,\mathrm{ch}\,vt \min\{2t, |1/\mu|\}^i \leqq t^i(1+2t)^i\,\mathrm{ch}\,vt(1+|\mu|)^{-i},$$

$$|g_i(u, \mu, x, t)| \leqq t^i\,\mathrm{ch}\,vt \min\{2t, |1/\mu|\}^{i+1} M \leqq$$

$$\leqq t^i(1+2t)^{i+1}\,\mathrm{ch}\,vt\,(1+|\mu|)^{-i-1}M.$$

Hence (55), (56) and the case $R \leqq |1/\mu|$ of (57) follows at once:

$$\int_0^R \left|\frac{f_i(\mu, t)}{t}\right| dt \leqq RR^{i-1}(1+2R)^i\,\mathrm{ch}\,vR\,(1+|\mu|^{-i},$$

$$\int_0^R \left|\frac{g_i(u, \mu, x, t)}{t}\right| dt \leqq RR^{i-1}(1+2R)^{i+1}\,\mathrm{ch}\,vR(1+|\mu|)^{-i-1}M \quad (i = 1, 2, \ldots),$$

$$\int_0^R \left|\frac{g_0(u, \mu, x, t)}{t}\right| dt \leqq \int_0^R t^{-1}\,\mathrm{ch}\,vt\,2t\,M\,dt \leqq$$

$$\leqq 2RM\,\mathrm{ch}\,vR \leqq 2M(R+1)\,\mathrm{ch}\,vR\,(1+|\mu|)^{-1}.$$

In view of this last estimate, for the case $R > |1/\mu|$ it remains to remark that

$$\int_{|1/\mu|}^R \left|\frac{g_0(u, \mu, x, t)}{t}\right| dt \leqq \int_{|1/\mu|}^R t^{-1}(1+2t)\,\mathrm{ch}\,vt\,(1+|\mu|)^{-1}M\,dt \leqq$$

$$\leqq (\ln R - \ln|1/\mu|)(1+2R)\,\mathrm{ch}\,vR\,(1+|\mu|)^{-1}M =$$

$$= (1+2R)\,M\,\mathrm{ch}\,vR\,(\ln R + \ln|\mu|)(1+|\mu|)^{-1},$$

and the lemma is proved.

Let us now turn to the proof of Proposition 2. Since $(u_k)\subset L^2(G)$ is a Riesz basis and $(v_k)\subset L^2(G)$ is the dual system of (u_k), there exists a constant C_0 such that for all $f, w\in L^2(G)$,

$$\sum_k |\langle u_k, w\rangle|^2 \leqq C_0\|w\|^2_{L^2(G)}, \tag{58}$$

$$\sum_k |\langle f, v_k\rangle|^2 \leqq C_0\|f\|^2_{L^2(G)}, \tag{59}$$

$$\langle f, w\rangle = \sum_k \langle f, v_k\rangle\langle u_k, w\rangle. \tag{60}$$

Given a compact interval $K=[a,b]\subset G$ arbitrarily, we can fix by Proposition 1 an $R>0$ such that $K_R:=[a-R,b+R]\subset G$ and

$$\sup_{\mu>0}\sum_{|\mu-\varrho_k|\le 1}(\|u_k\|_{L^\infty(K_R)}\operatorname{ch}\nu_k R)^2<A<\infty. \tag{61}$$

Applying Lemmas 6, 7 and using $\|q\|_{L^1(K_R)}<\infty$ and (38), we can fix a constant $C=C(R)$ such that

$$\left|\frac{2}{\pi}\int_0^R\frac{\sin\mu t\cos\mu_k t}{t}\,dt-\delta(\mu,\varrho_k)\right|\le C\operatorname{ch}\nu_k R(2+|\mu-\varrho_k|)^{-1}, \tag{62}$$

$$\left|\int_0^R\frac{\sin\mu t}{\pi t}f_i(\mu_k,t)\,dt\right|\le C\operatorname{ch}\nu_k R(1+|\mu_k|)^{-1}, \tag{63}$$

$$\left|\int_0^R\frac{\sin\mu t}{\pi t}g_i(u_{k-i},\mu_k,x,t)\,dt\right|\le C\|u_{k-i}\|_{L^\infty(K_R)}\operatorname{ch}\nu_k R(1+|\mu_k|)^{-3/4} \tag{64}$$

for all $k,\mu>0$ and $i=1,2,\dots,o_k$ in (63), $i=0,1,\dots,o_k$ in (64).

Fixing $x\in K$ and $\mu>0$ arbitrarily, define $w\colon G\to\mathbf{R}$ by

$$w(x+t):=\begin{cases}\dfrac{\sin\mu t}{\pi t} & \text{if}\quad |t|\le R,\\ 0 & \text{otherwise.}\end{cases}$$

Then, for any $f\in L^2(G)$, (42) and (60) imply that

$$S_\mu(f,x)=\sum_k\langle f,v_k\rangle\langle u_k,w\rangle. \tag{65}$$

Applying Lemma 1 with $m:=\sup o_k\ (<\infty)$ and using (40), (62)—(65), we obtain the inequality

$$|S_\mu(f,x)-\sigma_\mu(f,x)|\le 2^{-1}\sum_{\varrho_k=\mu}|\langle f,v_k\rangle u_k(x)|+$$
$$+\sum_k|\langle f,v_k\rangle u_k(x)|\,C\operatorname{ch}\nu_k R(2+|\mu-\varrho_k|)^{-1}+$$
$$+\sum_k\sum_{i=1}^{\sigma_k}|\langle f,v_k\rangle u_{k-i}(x)|\,C\operatorname{ch}\nu_k R(1+|\mu_k|)^{-1}+$$
$$+\sum_k\sum_{i=0}^{\sigma_k}|\langle f,v_k\rangle|\,\|u_{k-i}\|_{L^\infty(K_R)}\,C\operatorname{ch}\nu_k R(1+|\mu_k|)^{-3/4}.$$

Using for each sum the Cauchy—Schwarz inequality, (50) and $o_k \leqq m$, we get that

$$|S_\mu(f, x) - \sigma_\mu(f, x)| \leqq \sqrt{C_0}\, \|f\|_{L^2(G)} \{2^{-1}(\sum_{\varrho_k=\mu} |u_k(x)|^2)^{1/2} +$$
$$+ C(\sum_k (u_k(x) \operatorname{ch} \nu_k R)^2 (2+|\mu-\varrho_k|)^{-2})^{1/2} +$$
$$+ mC(\sum_k (u_k(x) \operatorname{ch} \nu_k R)^2 (1+|\mu_k|)^{-2})^{1/2} +$$
$$+ mC(\sum_k \|u_k\|_{L^\infty(K_R)} \operatorname{ch} \nu_k R)^2 (1+|\mu_k|)^{-3/2})^{1/2}\}.$$

Applying to these expressions the estimate (52), we have

$$\sum_{\varrho_k=\mu} |u_k(x)|^2 < A,$$

$$\sum_k (u_k(x) \operatorname{ch} \nu_k R)^2 (2+|\mu-\varrho_k|)^{-2} \leqq$$

$$\leqq \sum_{i=0}^{\infty} (1+|\mu-i|)^{-2} \sum_{i \leqq \varrho_k < i+1} (u_k(x) \operatorname{ch} \nu_k R)^2 \leqq 2A \sum_{j=1}^{\infty} j^{-2},$$

$$\sum_k (u_k(x) \operatorname{ch} \nu_k R)^2 (1+|\mu_k|)^{-2} \leqq$$

$$\leqq \sum_{i=0}^{\infty} (1+i)^{-2} \sum_{i \leqq \varrho_k < i+1} (u_k(x) \operatorname{ch} \nu_k R)^2 \leqq A \sum_{j=1}^{\infty} j^{-2},$$

and similarly

$$\sum_k (\|u_k\|_{L^\infty(K_R)} \operatorname{ch} \nu_k R)^2 (1+|\mu_k|)^{-3/2} \leqq A \sum_{j=1}^{\infty} j^{-3/2}.$$

Therefore there exists a constant $D>0$ such that

(66) $$\sup_{\mu>0} \sup_{x \in K} |S_\mu(f, x) - \sigma_\mu(f, x)| \leqq D \|f\|_{L^2(G)} \quad \text{for all } f \in L^2(G).$$

Given now $f \in L^2(G)$ and $\varepsilon > 0$ arbitrarily, let us choose a finite linear combination $P := \sum_{k=1}^{n} c_k u_k$ with

(67) $$\|f-P\|_{L^2(G)} < \varepsilon/2D.$$

P being continuously differentiable, it is well known [3] that

$$\lim_{\mu\to\infty} \sup_{x \in K} |S_\mu(P, x) - P(x)| = 0.$$

Thus we can fix $N>0$ so that

(68) $$\sup_{x \in K} |S_\mu(P, x) - P(x)| < \varepsilon/2 \quad \text{whenever} \quad \mu > N.$$

Let now $\mu > \max\{N, \varrho_1, ..., \varrho_n\}$ be arbitrary; then $\sigma_\mu(P, x) \equiv P(x)$ and therefore (66)—(68) imply that

$$\sup_{x \in K} |S_\mu(f, x) - \sigma_\mu(f, x)| \leqq \sup_{x \in K} |S_\mu(f-P, x) - \sigma_\mu(f-P, x)| +$$

$$+ \sup_{x \in K} |S_\mu(P, x) - P(x)| \leqq D\frac{\varepsilon}{2D} + \varepsilon/2 = \varepsilon;$$

this finishes the proof of Proposition 2 and also that of the Theorem.

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Two remarks on pointwise periodic topological mappings

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In his investigations [2] concerning the fixed points of biholomorphic automorphisms of the closed unit ball in $C(\Omega)$ spaces, the second author proved, that a pointwise periodic automorphism $T: \Omega \to \Omega$ of a topological F-space Ω is necessarily periodic. (I.e., if for every $x \in \Omega$ there exists a natural number $n = n(x) > 1$ for which $T^n x = x$, then there exists $n_0 > 1$ such that $T^{n_0} x = x$ for every $x \in \Omega$.) His proof made essential use of the abstract properties of the function space $C(\Omega)$ and a lemma stating that the linear operator $\hat{T}: f \mapsto f \circ T$ on $C(\Omega)$ is periodic whenever it is pointwise periodic.

In this note we present a simple elementary generalization of the mentioned theorem about Ω-automorphisms. This may have interest even in itself since so far we have very lacunary information about the structure of automorphisms in abstract topological spaces. Furthermore, we also investigate some extensions of the lemma concerning $\hat{T}$.

1. Quasi F-spaces

Definition. Let Ω be a topological space. We say that Ω is a *quasi F-space* if for every pair of sequences $x_1, x_2, \dots; y_1, y_2, \dots$ in Ω such that $\{x_n : n \in N\} \cap \cap \{y_n : n \in N\} = \emptyset$ there exists an infinite index set $I \subset N$ with $\{x_n : n \in I\}^- \cap \cap \{y_n : n \in I\}^- = \emptyset$ (here $^-$ stands for the closure operation in Ω).

Remark. If Ω is a totally regular F-space (for the definition see [1]) then, by a theorem of Henricksen (see [1]), every countable subset is C^*-imbedded in Ω. Hence totally regular F-spaces are all quasi F-spaces. On the other hand, the real line equipped with the topology where the family τ of open sets is given by $\tau = \{G \setminus S$: where G is open in the usual sense, S is countable$\}$ is obviously a quasi F-space but not an F-space.

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The definition of the quasi-F property can be stated equivalently in the following slightly sharper form.

Lemma. *If Ω is a quasi F-space then for every family of pairwise disjoint sequences $[x_n^{(k)}: n\in N]$ $(k=1,2,\dots)$ there exists an infinite index set $I\subset N$ such that $\{x_n^{(k)}: n\in I\}^- \cap \{y_n^{(l)}: n\in I\}^- = \emptyset$ whenever $k\neq l$.*

Proof. By hypothesis, given any J infinite $\subset N$ and $k\neq l$, we may fix $H_{k,l}(J)$ infinite $\subset J$ such that $\{x_n^{(k)}: n\in H_{k,l}(J)\}^- \cap \{x_n^{(l)}: n\in H_{k,l}(J)\}^- = \emptyset$. Now we can define $I_1\supset I_2\supset I_3\supset\dots$ recursively by $I_1=\mathbf{N}$, $I_{n+1}=H_{n+1,1}\,H_{n+1,2}\dots H_{n+1,n}(I_n)$.

Clearly we have $\{x_n^{(k)}: n\in J_N\}^- \cap \{x_n^{(l)}: n\in J_N\}^- = \emptyset$ whenever $0<k<l\leqq M$ Therefore the choice

$$I = \{\min\{k\in I_M:\ k\geqq M\}:\ M\in N\}$$

suits the requirements of the lemma.

Theorem. *Let Ω be a countably compact quasi F-space and let T denote a pointwise periodic continuous mapping of Ω onto itself. Then T is necessarily periodic.*

Proof. Suppose T is not periodic. Then there exists a sequence $x_1, x_2, \dots\in\Omega$ such that the sequence $p_k=\min\{n>0: T^n x_k = x_k\}$ strictly monotonically tends to ∞ (as $k\to\infty$). Observe that $T^n x_k\neq T^m x_l$ if $k\neq l$ and $0\leqq n<p_k$, $0\leqq m<p_l$. Hence, applying the lemma to the sequence $[x_n^{(k)}: n\in N]$ with

$$x_n^{(k)} = \begin{cases} T^k x_n & \text{if } 0\leqq k<p_n \\ x_n & \text{otherwise}\end{cases}$$

we can find I infinite $\subset N$ such that $\{T^k x_n: n\in I\}^- \cap \{T^l x_n: n\in I\}^- = \emptyset$ for all $k\neq l$. By the countably compactness of Ω there exists an accumulation point $x\in\Omega$ of the sequence $\{x_n: n\in I\}$. But then we have $T^k x\neq T^l x$ whenever $k\neq l$, contradicting the pointwise periodicity of T.

Corollary. *T is a topological automorphism of Ω.*

2. Baire group homomorphisms

In [2] it is shown that a pointwise periodic bounded linear operator on a Banach space is necessarily periodic. The proof of this fact is straightforward if we make full use of the vector structure of the underlying space. However, one can raise the question, what the deeper role of the algebraic considerations here is. The answer is contained in the following substantially sharper result whose proof is, however, also very short.

Theorem. *Let G be a connected topological group endowed with a Baire topology and let U be a pointwise periodic group homomorphism of G into itself. Then U is necessarily periodic.*

Proof. Set $G_n=\{x\in G\colon U^n x=x\}$ $(n=1, 2, \ldots)$. Since U^n is also a continuous group endomorphism of G, G_n is a closed subgroup of G for each n. From the pointwise periodicity of U we obtain $G=\bigcup_{n>0} G_n$. Thus, by the Baire category theorem there exists $n_0>0$ such that the interior of G is not empty. Since G is a subgroup of G, this means that G_{n_0} is also open in G. Therefore, by the connectedness of G, we have $G_{n_0}=G$. That is, $U^{n_0}x=x$ for all $x\in G$.

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A general ordering and fixed-point principle in complete metric space

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1. In the proof of the celebrated theorem of BISHOP and PHELPS [1] on the density of the set of support points of a bounded closed convex set in a Banach space, a lemma [1, Lemma 1], which can be considered as an ordering principle using essentially the completeness of the space [3], played a central role. The lemma has many generalizations, a maximal one being perhaps the one which is due to EKELAND [6]. The generalizations of the lemma have surprisingly many applications in various branches of mathematics as a survey paper of EKELAND [7] and, without any completness, the papers of BRØNDSTED [3], KIRK [8] and SULLIVAN [9] show.

The purpose of this paper is to show that the different generalizations of the lemma can be considered fundamentally as different forms of a general ordering, fixed point or inductive principle based on the completeness of the metric space. The importance of the different forms are essential from a very pragmatic (and, of course, very significant) point of view: which form fits better the considered problem (see other principles of analysis like e.g. the Hahn—Banach theorem which has many equivalent forms, too).

In the second section of this paper we deal with the equivalence of some well-known forms of the principle, in the third one we give two other forms and a very simple new proof of the principle. In section 4 we show that our new forms seem to fit better the proof of Menger's Theorem than the form of Caristi's fixed point theorem. In section 5 we give an application in measure theory which illustrates the fact that the principle could be a central tool in the theory of measure spaces.

2. Four equivalent forms of the principle. Throughout this section (X, d) will denote a complete metric space, and $\varphi: X \to R \cup \{+\infty\}$ a l.s.c. function, $\not\equiv +\infty$, bounded from below. Firstly we recall four theorems.

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Theorem 2.1. *If $f: X \to X$ is a map satisfying the inequality*

$$d(x, f(x)) \leqq \varphi(x) - \varphi(f(x)) \quad \textit{for all} \quad x \in X, \tag{2.1}$$

then f has a fixed point in X.

Theorem 2.2. *There is a point $\bar{x}$ in the space X, for which the inequality*

$$d(\bar{x}, x) > \varphi(\bar{x}) - \varphi(x) \tag{2.2}$$

holds for all $x \in X \setminus \{\bar{x}\}$.

Theorem 2.3. *If $\tilde{x}$ is an arbitrary point of the space X, then there exists a point $\bar{x}$ in X, such that the inequalities*

$$d(\tilde{x}, \bar{x}) \leqq \varphi(\tilde{x}) - \varphi(\bar{x}), \tag{2.3}$$

$$d(\bar{x}, x) > \varphi(\bar{x}) - \varphi(x) \quad \textit{for all} \quad x \in X \setminus \{\bar{x}\} \tag{2.4}$$

hold.

Theorem 2.4. *Let ε be an arbitrary positive number and u a point in X such that*

$$\varphi(u) \leqq \inf_{x \in X} \varphi(x) + \varepsilon. \tag{2.5}$$

Then for arbitrary $\lambda > 0$ there exists a point v in X such that the following inequalities hold:

$$\varphi(v) \leqq \varphi(u), \tag{2.6}$$

$$d(u, v) \leqq \lambda, \tag{2.7}$$

$$\varphi(x) > \varphi(v) - (\varepsilon/\lambda)\, d(v, x) \quad \textit{for all} \quad x \in X \setminus \{v\}. \tag{2.8}$$

Theorems 2.3 and 2.4 are due to Ekeland [6, 7].

Theorem 2.1 appeared firstly in the paper of Caristi and Kirk [8] as a theorem of Caristi. A slightly different form of Theorem 2.2 is a corollary of Theorem 2.4 in the paper of Ekeland [7], and is called a weak statement contrary to the strong statement of his Theorem 2.4. The weakness of Theorem 2.2 is, of course, illusory according to the equivalence of the statements. The equivalence (or one or another part of the implications) of the above mentioned theorems are contained, explicitly or implicitly, in Ekeland [7], Sullivan [9], Brøndsted [3], and so our very simple proofs can be found partly in these papers.

Next we turn to the proof of the equivalences of the above theorems. The logical scheme of our proof is as follows:

$$\begin{array}{ccccc} \text{Theorem 2.1} & \Leftrightarrow & \text{Theorem 2.2} & \Rightarrow & \text{Theorem 2.3} \\ & & \Uparrow & & \Downarrow \\ & & & \text{Theorem 2.4.} & \end{array}$$

Theorem 2.1 $\Rightarrow$ Theorem 2.2. If there would not exist an $\bar{x}$ satisfying (2.2), then for all $x\in X$ there would be a point $f(x)\neq x$ in the space X such that $d(x, f(x))\leqq\varphi(x)-\varphi(f(x))$, contrary to Theorem 2.1.

Theorem 2.2 $\Rightarrow$ Theorem 2.1. If a point $\bar{x}$ satisfies (2.2), then $\bar{x}$ is a fixed point of each self-map f satisfying (2.1) since otherwise the inequality $d(\bar{x}, f(\bar{x}))>$ $>\varphi(\bar{x})-\varphi(f(\bar{x}))$ would hold, contradicting (2.1).

Theorem 2.2 $\Rightarrow$ Theorem 2.3. The lower semicontinuity of φ implies that the set $S=\{x\in X \mid d(\tilde{x}, x)\leqq\varphi(\tilde{x})-\varphi(x)\}$ is closed, hence the metric space (S, d) is complete. Applying Theorem 2.2 for the space S we get a point $\bar{x}$ with $d(\tilde{x}, \bar{x})\leqq$ $\leqq\varphi(\tilde{x})-\varphi(\bar{x})$ and $d(\bar{x}, x)>\varphi(\bar{x})-\varphi(x)$, for all $x\in S\setminus\{\bar{x}\}$. For Theorem 2.3 we have to show that the last inequality holds in $X\setminus S$, as well. If for $x\in X\setminus S$ the inequality $d(\bar{x}, x)\leqq\varphi(\bar{x})-\varphi(x)$ would be true, then adding it to the inequality $d(\tilde{x}, \bar{x})\leqq\varphi(\tilde{x})-\varphi(\bar{x})$ we would get $d(\tilde{x}, x)\leqq\varphi(\tilde{x})-\varphi(x)$, contrary to $x\notin S$.

Theorem 2.3 $\Rightarrow$ Theorem 2.4. Applying Theorem 2.3 with the metric $(\varepsilon/\lambda)d$ and $\tilde{x}=u$, we have a point $v=\bar{x}$ such that $(\varepsilon/\lambda)d(v, x)>\varphi(v)-\varphi(x)$ for all $x\in X\setminus\{v\}$, and $(\varepsilon/\lambda)d(u, v)\leqq\varphi(u)-\varphi(v)$. Hence we immediately get (2.6) and (2.8). The inequality $\varphi(u)\leqq\inf\varphi(x)+\varepsilon$ implies $\varphi(u)-\varphi(v)\leqq\varepsilon$; thus $(\varepsilon/\lambda)d(u, v)\leqq\varepsilon$, which gives (2.7), too.

Theorem 2.4 $\Rightarrow$ Theorem 2.2. Taking $\varepsilon=\lambda$ the implication is evident from (2.8).

Remarks. From the proof of the first equivalence one may observe, that the set of the fixed points of the selfmaps satisfying the assumption of Theorem 2.1 coincides with the set of points $\bar{x}$ satisfying (2.2) in Theorem 2.2. This obvious observation shows that all f in Theorem 2.1 have common fixed points.

It is interesting, that the fixed points in Theorem 2.1 can be localized similarly like in Theorem 2.3 or 2.4.

3. Two new forms of the principle. Firstly we will state two equivalent theorems which can be considered as new versions of the principle. We shall prove the first proposition directly, and this proof of the principle seems to be the simplest we have learned till now.

Theorem 3.1. *Let (X, d) be a complete metric space and Φ be a map $X\to 2^X$, which satisfies the following conditions:*

(3.1) *$\Phi(x)$ is a closed set for all $x\in X$.*

(3.2) *$x\in\Phi(x)$ for all $x\in X$.*

(3.3) *$x_2\in\Phi(x_1)\Rightarrow\Phi(x_2)\subseteq\Phi(x_1)$ for all $x_1, x_2\in X$.*

(3.4) *For all sequences $x_1, x_2, \dots, x_n, \dots$ in X, that are generalized Picard-iterations, i.e. fulfil*

$$x_2\in\Phi(x_1),\ x_3\in\Phi(x_2),\ \dots,\ x_n\in\Phi(x_{n-1}),\ \dots$$

the distances $d(x_n, x_{n+1})$ tend to zero as $n\to+\infty$.

Then the map Φ has a fixed point $\bar{x}$ in X in the sense $\Phi(\bar{x})=\{\bar{x}\}$. (In localized version: For arbitrary $\tilde{x}\in X$ there is a fixed point in $\Phi(\tilde{x})$.)

Theorem 3.2. *Let (X, d) be a complete metric space with a continuous partial ordering $\preceq$. If for each increasing sequence $x_1\preceq x_2\preceq \dots \preceq x_n\preceq \dots$ in X the distances $d(x_n, x_{n+1})$ tend to zero, then there is a maximal element in X. (In localized version: For all $\tilde{x}\in X$ there is a maximal element in the set $\{x\in X \mid \tilde{x}\preceq x\}$.)*

Direct proof of Theorem 3.1. If the distance d satisfies condition (3.4) then the equivalent distance $d'=d/(1+d)$ also does, so we can suppose d is bounded on X. Let us denote the diameter of a subset $A\subset X$ by $\delta(A)$. Because of (3.2) $\Phi(x)\neq\emptyset$ for all $x\in X$, and we can construct a generalized Picard-iteration such that $x_1=\tilde{x}$, $x_n\in\Phi(x_{n-1})$ and

$$d(x_n, x_{n-1}) \geqq \delta(\Phi(x_{n-1}))/2-1/2^{n-1}.$$

Hence from conditions (3.3) and (3.4) we have

$$\Phi(x_{n-1}) \supseteqq \Phi(x_n) \quad \text{and} \quad \delta(\Phi(x_n)) \to 0 \quad \text{as} \quad n\to+\infty.$$

Using the completeness of the space, the non-empty closed sets $\Phi(x_n)$ $(n=1, 2, \dots)$ have a unique common point $\bar{x}$, i.e. $\bigcap_{n=1}^{\infty}\Phi(x_n)=\{\bar{x}\}$. The point $\bar{x}$ is fixed, since on the one hand $\bar{x}\in\Phi(x_n)$ and (3.3) imply $\Phi(\bar{x})\subseteqq\bigcap_{n=1}^{\infty}\Phi(x_n)=\{\bar{x}\}$, and on the other hand from (3.2) we have $\{\bar{x}\}\subseteqq\Phi(\bar{x})$. The localization is trivial from $x_1=\tilde{x}$.

Theorem 3.1 $\Rightarrow$ Theorem 3.2. Let $\Phi(x)=\{y \mid x\preceq y\}$. The relation $y\in\Phi(x)$ is equivalent to $x\preceq y$, hence the reflexivity and the transitivity of the ordering imply (3.2) and (3.3), respectively. From the continuity of the ordering we can conclude that the set $\Phi(x)$ is closed. If $x_1\preceq x_2\preceq \dots \preceq x_n\preceq \dots$, then $x_1, x_2, \dots, x_n, \dots$ is a generalized Picard-iteration, hence all the conditions of Theorem 3.1 are fulfilled, therefore there is a fixed point $\bar{x}$ of Φ, which is obviously maximal in X.

Theorem 3.2 $\Rightarrow$ Theorem 3.1. Define now an ordering $\preceq$ by, $x\preceq y$, iff $y\in\Phi(x)$. From this step on the proof is entirely analogous to the previous one.

Theorem 3.1 $\Rightarrow$ Theorem 2.3. Let $\Phi(x)=\{y \mid d(x, y)\leqq\varphi(x)-\varphi(y)\}$. Since φ is l.s.c., $\Phi(x)$ is closed. Condition (3.2) is satisfied evidently. The summing up of the inequalities $d(x_1, x_2)\leqq\varphi(x_1)-\varphi(x_2)$ and $d(x_2, x_3)\leqq\varphi(x_2)-\varphi(x_1)$ gives (3.3) at once. Similarly, taking the inequalities $d(x_{n-1}, x_n)\leqq\varphi(x_{n-1})-\varphi(x_n)$ $(n=2, 3, \dots)$ and summing them up we have $\sum_{n=2}^{\infty}d(x_{n-1}, x_n)<+\infty$, using the boundedness of φ from below. Applying Theorem 3.1 we have a fixed point $\bar{x}$, and by the definition of Φ the point $\bar{x}$ satisfies (2.2).

The localized version of Theorem 3.1 implies that of Theorem 2.3 in a similar way.

KIRK [8] and SZILÁGYI [10] observed, that forms 2.1 and 2.4 of the principle (Theorems 2.1, 2.4) characterize the completeness of the metric space in some sense. Similarly, we shall prove an analogous result for our forms of the principle.

Theorem 3.3. *If the metric space* (X, d) *is noncomplete, then there is a* Φ *which satisfies conditions* (3.1)—(3.4) *but has no fixed point.*

Proof. From the assumption there is a sequence $X = N_0 \supseteq N_1 \supseteq \ldots \supseteq N_n \supseteq \ldots$ of non-empty closed sets in X so that $\delta(N_n) \to 0$, but $\bigcap_{n=1}^{\infty} N_n = \emptyset$. Define the map Φ in the following way:

$$\Phi(x) = N_{i+1} \cup \{x\}, \quad \text{if} \quad x \in N_i \quad \text{and} \quad x \notin N_{i+1}.$$

The map Φ satisfies the assumptions of Theorem 3.1, but has no fixed point, since if $\bar{x}$ were a fixed point of Φ, we would have $\Phi(\bar{x}) = \{\bar{x}\} = N_{i_0} \cup \{\bar{x}\}$, implying $N_{i_0} = \emptyset$, contrary to the assumption.

4. Application in metric convexity. KIRK [8] observed that using the fixed point theorem of Caristi (Theorem 2.1.) it is possible to give a simple proof for Menger's Theorem, a famous theorem on metric convexity. Here we show, that other versions, namely Theorems 3.1 and 3.2 seem to fit even better to prove Menger's Theorem.

Firstly we introduce some notions and notations from distance geometry [2]. Let (Y, d) be a metric space. If for some point $a, b, c \in Y$ we have $d(a, b) = d(a, c) + d(c, b)$, then we say the point c is between the points a and b and use the notation $a\,c\,b$. Similarly the symbol $a_1\,a_2 \ldots a_s$ means that $d(a_1, a_s) = d(a_1, a_2) + \ldots + d(a_{s-1}, a_s)$. It is evident, that the set $\{x\colon a\,x\,b\}$ is closed and it easy to see that the betweenness relation is transitive: $a\,c\,b$ and $a\,d\,c$ imply $a\,d\,b$ (or $a\,d\,c\,b$), more generally $a_i\,b\,a_{i+1}$ and $a_1\,a_2 \ldots a_s$ imply $a_1 \ldots a_i\,b\,a_{i+1} \ldots a_s$ and obviously $a_1\,a_2 \ldots a_s$ implies $a_{i_1}\,a_{i_2} \ldots a_{i_j}$ $(1 \leqq i_1 \leqq i_2 \leqq \ldots \leqq i_j \leqq s)$.

The metric space (Y, d) is called convex if for any two points $a, b \in Y$ there is a point c different from a and b such that $a\,c\,b$. The space is called a metric segment space if for any two points $a, b \in Y$ there is an isometric map

$$\varphi\colon [0, d(a, b)] \to \{x\colon a\,x\,b\} \quad \text{for which} \quad \varphi(o) = a \quad \text{and} \quad \varphi(d(a, b)) = b.$$

It is obvious that if the space Y is a metric segment space then it is convex. The converse statement is not generally true but it is true if the space is complete, as it is stated by the following theorem.

Theorem (Menger). *If the metric space* (Y, d) *is complete and convex then it is a metric segment space.*

First proof. It is sufficient to show, that for all $\lambda\in(0, d(a, b))$ there exists an x_λ such that $a\, x_\lambda\, b$ and $d(a, x_\lambda)=\lambda$, since the map $\lambda\to x_\lambda$ is isometric in this case.

Let $\lambda\in(0, d(a, b))$ be a fixed number and put

$$Y_1=\{y\in Y\,|\,d(a, y)\leqq\lambda\} \quad\text{and}\quad Y_2=\{y\in Y\,|\,d(b, y)\leqq d(a, b)-\lambda\}.$$

We shall apply Theorem 3.1 for the complete metric space $(Y_1\times Y_2, \varrho)$, where

$$\varrho((u_1, u_2), (v_1, v_2))=d(u_2, v_2)+d(u_1, v_1).$$

Define the map Φ in the following way:

$$\Phi(y_1, y_2)=\{(u_1, u_2)\in Y_1\times Y_2\,|\,a\, y_1\, u_1\, u_2\, y_2\, b\}.$$

The map Φ satisfies the conditions of Theorem 3.1. The assumptions (3.1) and (3.2) are obviously fulfilled, while (3.3) follows from the transitivity of betweenness: $(u_1, u_2)\in\Phi((y_1, y_2))$ and $(v_1, v_2)\in\Phi((u_1, u_2))$ mean that $a\, y_1\, u_1\, u_2\, y_2\, b$ and $a\, u_1\, v_1\, v_2\, u_2\, b$, thus from transitivity $a\, y_1\, u_1\, v_1\, v_2\, u_2\, y_2$, hence $a\, y_1\, v_1\, v_2\, y_2\, b$, i.e. $(v_1, v_2)\in\Phi((y_1, y_2))$.

If $(y_1^{(n)} y_2^{(n)})\in\Phi((y_1^{(n-1)}, y_2^{(n-1)}))$ is a generalized Picard-iteration then from the transitivity we have $a\, y_1^{(1)}\, y_1^{(2)}\dots y_1^{(n)}\, y_2^{(n)}\, y_2^{(n-1)}\dots y_2^{(1)}\, b$ as before. Hence

$$d(a, y_1^{(1)})+\dots+d(y_1^{(n)}, y_2^{(n)})+\dots+d(y_2^{(1)}, b)=d(a, b),$$

i.e.

$$\varrho((a, b), (y_1^{(1)}, y_2^{(1)}))+\dots+$$
$$+\varrho((y_1^{(n-1)}, y_2^{(n-1)}), (y_1^{(n)}, y_2^{(n)}))+d(y_1^{(n)}, y_2^{(n)})=d(a, b),$$

which yields at once that $\varrho[(y_1^{(n-1)}, y_2^{(n-1)}), (y_1^{(n)}, y_2^{(n)})]$ as $n\to\infty$, i.e. the assumption (3.4) is also satisfied.

According to the theorem we have a fixed point $(\bar y_1, \bar y_2)$, i.e. $\Phi(\bar y_1, \bar y_2)=\{(\bar y_1, \bar y_2)\}$. Now we use the convexity of the space Y to prove that $\bar y_1=\bar y_2$. Assume $\bar y_1\neq\bar y_2$; then there is a w such that $w\neq\bar y_1$, $w\neq\bar y_2$ and $\bar y_1\, w\, \bar y_2$, hence from transitivity we have $a\, \bar y_1\, w\, \bar y_2\, b$ and since $d(a, w)\leqq\lambda$ or $d(w, b)\leqq d(a, b)-\lambda$ holds, $(w, \bar y_2)$ or $(\bar y_1, w)$ is an element of $\Phi(\bar y_1, \bar y_2)$, contradicting the fixed point property. Finally, we get $\bar y_1=\bar y_2=y$. Since $y\in Y_1$, $y\in Y_2$ and $a\, y\, b$, we have $d(a, y)=\lambda$.

Second proof (Sketch). Let $\mathscr{H}$ be the set of isometric maps f to $\{x\,|\,a\,x\,b\}$ having closed domains in $[0, d(a, b)]$ and with $f(0)=a, f(d(a, b))=b$. The set $\mathscr{H}$ is not empty, since it contains the map f_0, for which $\operatorname{dom}(f_0)=\{0, d(a, b)\}$ and $f_0(0)=a, f_0(d(a, b))=b$. Each element of $\mathscr{H}$ can be identified with its domain or range. Let us denote by $\mathscr{K}$ the set of closed subsets of the interval $[0, d(a, b)]$, and introduce the Hausdorff-metric h on $\mathscr{K}$. It is well known that the space $(\mathscr{K}, h)$ is complete. From the properties of the Hausdorff-metric one can prove that $\mathscr{H}$ is a closed subset of $\mathscr{K}$. Let us order the elements of $\mathscr{H}$ (or equivalently, the adequate elements of $\mathscr{K}$) according to the set inclusion of the domain of maps. It is easy to see that this ordering is continuous for the metric h in $\mathscr{K}$ and also that it satisfies the last assumption of Theorem 3.2, since if $\operatorname{dom}(f_n)$ $(n=1, 2, \ldots)$ is an increasing sequence, then $\sum_{n=1}^{\infty} h(\operatorname{dom}(f_n), \operatorname{dom}(f_{n+1})) \leqq d(a, b)$. The theorem gives a maximal element $\bar{f}$ in $\mathscr{H}$ (with maximal domain in $\mathscr{K}$). If $\operatorname{dom}(\bar{f}) = [0, d(a, b)]$, then Menger's Theorem is proved, otherwise $[0, d(a, b)] \setminus \operatorname{dom}(\bar{f})$ is an open set and contains an open interval (z_1, z_2), $z_1, z_2 \in \operatorname{dom}(\bar{f})$. Now using the convexity we have a point w with $w \neq z_1$, $w \neq z_2$ and $z_1 w z_2$, and so the map $\tilde{f}: \tilde{f} = \bar{f}$ on $\operatorname{dom}(\bar{f})$ and $f(w) = d(a, w)$ is isometric, contrary to the maximality of $\bar{f}$.

5. Application in measure theory. In the theory of measure and integral there are a lot of ordered complete metric spaces, which satisfy the assumptions of Theorem 3.2. So it is easy to show applications, and therefore our application can only be considered as an illustrative example, but it is worth noting that our proof is easier than the proof of [5] (p. I. 335.).

Firstly we mention some well-known facts from measure theory. Let $(X, \mathscr{M}, \mu)$ be a measure space and let $\mathrm{M}(X, \mathscr{M}, \mu)$ be the space of classes of μ-equivalent real functions on X. Ordering the space $M(X, \mathscr{M}, \mu)$ by

$$f \leqq g \quad \text{iff} \quad f(x) \leqq g(x) \ \mu\text{-a.e.}$$

one may ask whether the lattice $(M, \leqq)$ is complete, i.e. whether all subsets $B \subseteqq M$ having an upper bound in the ordering have a least upper bound $f_0 = \sup B \in M(X, \mathscr{M}, \mu)$. The following famous theorem answers the question affirmatively. We shall deal with a finite measure, and the σ-finite case can be derived from this by standard arguments.

Theorem. *If $(X, \mathscr{M}, \mu)$ is a finite measure space, then $M(X, \mathscr{M}, \mu)$ is a complete lattice.*

Proof. The set M is Frechet-space with the quasi-norm

$$\|f\| = \int_X \frac{|f(x)|}{1+|f(x)|}\, d\mu.$$

A crucial property of this space is, that whenever f_n converges to f_0 then it has a subsequence f_{n_k} $(k=1, \ldots)$, which converges μ-a.e. to f_0, and the ordering is continuous.

Let $B \subseteqq M$ be an order-bounded set, and let g be an upper bound of B. If C is the set of the least upper bounds of the finite subsets of B, then $\sup B = \sup C$ obviously, so we can assume that whenever $f_1, f_2 \in B$ then $f_3 = \sup(f_1, f_2)$ is also in B. Let $\bar{B}$ denote the closure of B in the Frechet-space M. We shall prove, that $\bar{B}$ has a maximal element f_0, and f_0 is the least upper bound of B. The metric space $(\bar{B}, \|\cdot\|)$ is complete and the ordering introduced before is continuous.

If f_n $(n=1, 2, \ldots)$ is an increasing sequence in the order bounded set $\bar{B}$, then f_n is convergent a.e., consequently it converges in the quasi-norm, too. According to the above, Theorem 3.2 is applicable and we have a maximal element f_0. Now we shall prove that f_0 is an upper bound for B. Since $f_0 \in \bar{B}$, we have a sequence $f_n \in B$ such that $f_n \to f_0$ both in the quasi-norm and a.e. Hence if $f \in B$, we have $\sup(f_0, f) = \sup(\lim_n f_n, f) = \lim_n [\sup(f_n, f)] \in \bar{B}$. As f_0 is maximal in $\bar{B}$, $f \leqq \sup(f_0, f) = f_0$ holds. Finally let f be an upper bound for B, i.e. $\tilde{f} \geqq f$ for all $f \in B$. Since $f_0 = \lim f_n (f_n \in B), \tilde{f} \geqq \lim_n f_n = f_0$, i.e. f_0 is the least upper bound of B.

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Bibliographie

Joseph Bak and Donald J. Newman, Complex Analysis. Undergraduate Texts in Mathematics, X+244 pages with 69 illustrations, Springer-Verlag, New York—Heidelberg—Berlin, 1982

This is a fascinating little book. It provides an efficient and clear introduction into the theory of complex functions. The arrangement of the material is unusual, the reader need not be familiar even with the definition of complex numbers or of convergence and yet, starting at the very beginning, the authors were able to achieve a level in complex function theory where the proof of the prime number theorem is possible. Of course the book contains the standard topics such as the Cauchy—Riemann equations, line integrals, entire and meromorphic functions, singularities, Laurent series, residues, conformal mappings etc., but besides them some other less elementary results are incorporated, as well, e.g. the Phragmén—Lindelöf-method, natural boundaries, open mapping theorem etc. The last, 19th chapter illustrates the wide range of applicability of complex methods. The first question here is if the set of the positive integers can be partitioned into a finite number of artihmetic progressions such that these have no common differences (try it!). The second problem asserts the unicity of the solution to the system of equations

$$a_n+\binom{n}{1}a_{n-1}b_1+\binom{n}{2}a_{n-2}b_2+\ldots+b_n = 2^n \quad (n = 1, 2, \ldots)$$

$a_n, b_n \geqq 0$. In section 3 it is shown that the total variation of $\sin^2 x/x^2$ over $(-\infty, \infty)$ is e^2-5; in section 4 the Fourier unequeness theorem and, finally, in section 5 the prime number theorem is treated.

The book contains a lot of exercises together with hints for the hardest ones. Sometimes, e.g. at the Riemann-mapping-theorem, physical analogues illustrate the main ideas. Index and 69 illustrations help reading the book. We recommend Bak and Newman's "Complex Analysis" to lecturers and to every student with or without any skill in complex methods.

V. Totik (Szeged)

H. J. Baues, Commutator Calculus and Groups of Homotopy Classes (London Mathematical Society Lecture Note Series 50), 226 pages, Cambridge University Press, Cambridge—London, New York—New Rochelle—Melbourne—Sidney, 1981.

This book is divided into two parts consisting of four and three chapters, respectively. Part A is devoted to homotopy operations, nilpotent group theory and nilpotent Lie algebra theory. Starting with commutator calculus, the text contains a study of distributivity laws in homotopy theory, homotopy operations on spheres and concludes in an investigation into higher order Hopf invariants on spheres. Part B deals with homotopy theory over a subring of rationals. In this part the theory of the homotopy Lie algebra and spherical cohomotopy algebra, theory of groups of homotopy classes and finally the Hilton—Milnor theorem and its dual can be found.

László Gehér (Szeged)

Aldo Bressan, Relativistic Theories of Materials, (Springer Tracts in Natural Philosophy, Vol. 29), XIV+290 pages, Springer-Verlag, New York—Heidelberg—Berlin, 1978.

After Einstein's fundamental work in 1905 on special relativity the reletivistic developments of thermodynamics and elasticity were created as early as in 1911. When trying to include gravitation in relativity Einstein was forced to develop radical changes in his earlier spacetime concept in 1916. These new ideas of general relativity had given a new aspect for the study of the material. In spite of this fact the first general relativistic theory of thermodynamics and fluids and of finitely deformed materials were published only after 1955. Even more recently relativistic theories incorporating finite deformations for polarizable and magnetizable materials and those in which couple stresses are considered have been formulated.

The present book describes the foundation of this theory of general relativistic material. Furthermore it containes some applications of this theory, mainly to elastic waves.

After an introductory chapter the book is divided into two parts. The first part deals with the basis equation of gravitation, thermodinamics and electromagnetism, and constitutive equations from the Eulerian point of view. The second part contains the theory of material from the Lagrangian point of view. In this part the reader can find chapters on subjects such as kinematics and stresses, elasticity, accelaration waves, pieso-elasticity and magnetoelastic waves, couple stresses and more general stresses.

The book is not of an introductory character. It is assumed that the reader is familiar with the classical continuum mechanics and with the general relativity. The main definitions and theorems of these subjects are collected — without proofs — in Appendix A.

Z. I. Szabó (Szeged)

A. J. Chorin and J. E. Marsden, A Mathematical Introduction to Fluid Mechanics (Universitext), V+205 pages, Springer-Verlag, New York—Heidelberg—Berlin, 1979.

A good introduction to fluid mechanics is given in this book. The reader can get acquainted with the principal ideas, the equations of fluid motion under several hypotheses (e.g. the fluid is ideal, homogeneous, isentropic, stationary, viscous, compressible), the discussion of potential flows, vortex motions, boundary layers, one-dimensional gas flows. The starting principles of equations, demonstrations are the physical laws. The material in the book can be read easily, the proofs are written with mathematical exaction. The results derived from the models mathematically are always interpreted. There are many illustrations in the book making the material clear.

The book gives a good base to continue the study of fluid mechanics. We recommend it to mathematicians, engineers and students who want to know the basic ideas of this subject in a mathematically attractive manner.

J. Terjéki (Szeged)

Shui-Nee Chow—Jack K. Hale, Methods of Bifurcation Theory (Grundlehren der matematischen Wissenschaften, 251), XV+515 pages, Springer-Verlag, New York—Heidelberg—Berlin, 1982.

As experience shows, many physical phenomena, biological processes etc. can be modelled by differential equations containing parameters. As we change these parameters, the behaviour of the flow changes. This change is essential when the structure of the phase portrait is modified. This happens when the topology of the non-wandering set changes. Each time this occurs, we say there is a bifurcation.

The readers following the current papers and monographs in the field of the theory and applications of differential equations with attention can experience that nowadays the bifurcation is one of the fastest developing central topics in the field. There are a great number of difficult problems and, accordingly, a great number of publications in this topic with respect to several types of differential equations such as finite and infinite dimensional systems, ordinary and partial differential equations, functional differential equations etc. Now we get an excellent comprehensive handbook which helps us to inquire in several branches of this theory of wide range.

In the first chapter the authors give a flavour of the problems that occur in bifurcation theory by presenting some examples from the applications. Chapter 2 meets a long felt need in the literature: it gives a systematic, self-contained introduction to Nonlinear Analysis. One can find here in detail much of the relevant background material to the modern theory of bifurcation and stability from nonlinear functional analysis and the qualitative theory of differential equations (e.g. local and global implicit function theorem, Malgrange preparation theorem, manifolds and transversality, Sard's theorem, topological degree, Ljusternik—Schnirelman theory). The third chapter contains some applications of the implicit function theorem.

The authors distinguish two aspects of bifurcation theory: static and dynamic. The first one investigates the change of the structure of the set of zeros of a function as parameters in the function are varied. Dynamic bifurcation theory is concerned with the changes that occur in the qualitative behaviour of solutions of differential equations as parameters of the vector field are varied.

Chapter 4—8 (entitled "Variational Method"; "The Linear Approximation and Bifurcation"; "Bifurcation with One Dimensional Null Space"; "Bifurcation with Higher Dimensional Null Spaces"; "Some Applications") deal with static bifurcation theory. The results of the fourth chapter are applied to Hamiltonian systems, elliptic an hyperbolic problems.

Chapters 9—13 (entitled "Bifurcation Near Equilibrium"; "Bifurcation of Autonomous Planar Equations"; Bifurcation of Periodic Planar Equations"; "Normal Forms and Invariant Manifolds"; "Higher Order Bifurcation Near Equilibrium") are devoted to dynamic bifurcation theory.

The chapters are followed by bibliographical notes with informations and references for the history of the problems and the further study.

This well-written excellent book will be undoubtedly the standard reference in nonlinear analysis and bifurcation theory. It can serve also as a text-book (the authors give suggestions for adapting the material to several types of one semester courses). We recommend it for every mathematician, user and student interested in differential equations and their application.

L. Hatvani (Szeged)

James A. Cohran, Applied Mathematics: Principles, Techniques, and Applications, X+399 pages, Wadsworth International Group, Belmont, California, 1982.

The book is designated to be used as a second course in applied mathematics. The prerequisite knowledge includes a thorough grounding in the calculus through ordinary differential equations. A certain acquaintance with vector analysis, elementary complex variables, Fourier series, Laplace/Fourier transforms and partial differential equations is also taken for granted. (The most important definitions and theorems are collected in appendices at the end of the book.)

The topics presented are selected for their relevance to nonroutine applications encountered in today's and hopefully tomorrow's world. For this reason, advanced topics such as stability theory, conformal mapping, generalized functions (distributions) and integral equations are included as are seemingly more elementary topics such as linear algebra, differential equations and special functions.

The discussion of each major mathematical topic in this book is preceded by consideration of a relevant practical application. Indeed, the modelling of difficult "real-world" problems serves to motivate the mathematics that follows. Occasionally the book contains rather detailed analysis of various computational procedures and techniques of obtaining the "results".

The book consists of eleven chapters, six appendices and a long (author and subject) index. There are references and problems at the end of each chapter. Many of the proofs of theorems can be skipped in the text at first reading, solving an appropriate number of the problems, however, is a must. These problems are carefully chosen to illustrate or amplify various portions of the text and constitute an extremely important component of the learning process.

The table of contents (in the parantheses we pick up a theoretical result and/or a practical application characteristic to the chapter in question): 1. Linear algebra and computation (Ill-conditioning, LR and QR), 2. Eigenvalue problems for differential equation (Sturm—Liouville problems), 3. The special functions of applied mathematics (More on Bessel functions), 4. Optimization and the calculus of variations (Least action and Hamilton's principle in mechanics), 5. Analytic function theory and system stability (A satellite attitude-control system, The Cauchy—Goursat theorem), 6. Conformal mapping (Cavity and jet flows), 7. Integral transforms (The Mellin transform), 8. Green's functions (and partial differential equations, The Dirichlet problem for the n-ball), 9. Generalized functions (Delta functions in optics and electrostatics), 10. Linear integral equations (The Fredholm alternative, The Rayleigh—Ritz procedure), 11. Asymptotics (Order relations O and o, The method of steepest descent.)

The book is warmly recommended to engineering and applied mathematics students who will pursue industrial or business careers. But it will be undoubtedly useful to those who are interested in solving diverse physical problems at research laboratories.

F. Móricz (Szeged)

Combinatorial Mathematics IX, Proceedings of the Ninth Australian Conference on Combinatorial Mathematics Held at the University of Queensland, Brisbane, Australia, August 24—28, 1981. Edited by Elizabeth J. Billington, Sheila Oates-Williams, and Anne Penfold Street (Lecture Notes in Mathematics, Vol. 952), XI+443 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1982.

This volume contains seven invited papers and twenty contributed papers. A number of them is concerned with symmetric combinatorial structures (finite projective and affine planes, block design, perfect covering). The reader can find papers close to applied mathematics in the following topics; economic let scheduling, matroid algorithms, heuristics for determining a maximum weight planar subgraph of a given edge weighted graph, mathematical description of woven structures.

The titles of invited papers are: D. R. Breach, Star gazing in affine planes; P. J. Cameron, Orbits, enumeration and colouring; A. Gardiner, Classifying distance-transitive graphs; W. L. Kocay, Some new methods in reconstruction theory; V. Pless, On the uses of contracted codes; Ch. E. Praeger, When are symmetric graphs characterised by their local properties? and R. G. Stanton, Old and new results on perfect coverings.

L. A. Székely (Szeged)

Combinatorics, Prooceedings of the Eighth British Combinatorial Conference, University College, Swansea 1981. Edited by H.N.V. Temperley (London Mathematical Society Lecture Note Series, Vol. 52) 190 pages, Cambridge University Press, 1981.

This book contains the texts of nine invited lectures held at the Eighth British Combinatorial Conference. The list of these papers is as follows.

L. Babai, On the abstract group of automorphisms. Babai surveys results about the existence and non-existence of graphs with prescribed properties having a prescribed abstract group of automorphisms. Similar results and problems concerning other algebraic and combinatorial structures instead of graphs are mentioned.

L. W. Beineke, A tour through tournaments or bipartite and ordinary tournaments: a comparative survey. A bipartite tournament is an oriented complete bipartite graph. This paper seems to be a germ of the theory of bipartite tournaments. All results are is comparison with similar results concerning ordinary tournaments.

H. Baker and F. Piper, Shift register sequences. Linear and non-linear feedback shift registers are treated. Shift registers can be applied in cryptography so as to mix the statistics of letter frequencies (what may obstruct to discover the 1—1 function between letters and their codes).

B. Bollobás, Random graphs. The chapters of this paper are: the automorphism group, sparse graphs, threshold functions, graphs with many edges, and random regular graphs. The last one contains new results of great importance and includes sketches of the proofs.

F. R. K. Chung and R. L. Graham, Recent results in graph decompositions. This report gives a brief overall view of decomposition problems and treats some topics in which significant progress has been made recently, e.g. decomposition into complete bipartite graphs.

B. Grünbaum and G. C. Shephard, The geometry of planar graphs. This paper surveys the theory of infinite planar graphs. These graphs may occur as edge-graphs of tilings. Euler's Theorem and Kotzig's Theorem are generalized by the authors.

F. J. MacWilliams, Some connections between designs and codes. Author's introduction is: "This paper describes how to get designs from codes".

R. W. Robinson, Counting graphs with a duality property. Robinson surveys the enumeration of graphs and other structures satisfying a duality condition. The main tool is a modification of the Burnside lemma due to de Bruijn. The notion of duality used here includes self-complementarity.

J. G. Thompson, Ovals in a projective plane of order 10. The author investigates the following problem: "does there exist a set S of 99 fixed point free involution on 12 points such that for each involution $(ab)(cd)$ which moves just 4 points, there is a unique s in S which has $\{a, b\}$ and $\{c, d\}$ as orbits?"

L. A. Székely (Szeged)

Constructive Mathematics, Proceedings, New Mexico, 1980, edited by F. Richman, Lecture Notes in Mathematics, 873, VIII+347 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1981.

In the last decade or so one could observe increased interest in constructive mathematics. Surely this has to do with the developing computation/computer science but Bishop's book "Foundations of constructive analysis" also influenced a number of mathematicians, probably because it was able to take off the "dogmatism" of the earlier theories. Several branches (or degrees?) of constructivism can be found in current mathematics from Markov's school admitting only finite strings of symbols to those who accept classical mathematics, only they are interested in effective algorithms rather than just computation in principle. In 1980 at Las Cruces a conference was organized with the intention that the representatives of the several schools should exchange their ideas and thoughts concerning constructive mathematics. These proceedings contain all but five lectures delivered at this conference.

Since the main treads and ideas of constructive mathematics are unknown for many mathematicians even today, let us present here the table of contents: F. Richman: Seidenberg's condition P; W. Ruitenburg: Field extensions; R. Milnes and F. Richman: Dedekind domains; J. H. Davenport:

Effective mathematics — the computer algebra viewpoint; Y. K. Chan: On some open problems in constructive probability theory; A. Scedrov: Consistency and independence results in intuitionistic set theory; W. A. Howard: Computability of ordinal recursion of type level two; J. P. Seldin: A constructive approach to classical mathematics; D. Isles: On the notion of standard non-isomorphic natural number series; N. D. Goodman: Reflections on Bishop's philosophy of mathematics; M. Beeson: Formalizing constructive mathematics: why and how? J. Lambek and P. J. Scott: Independence of premisses and the free topos; R. Vesley: An intuitionistic infinitesimal calculus; N. Greenleaf: Liberal constructive set theory; D. S. Bridges, A. Calder, W. Julian, R. Mines and F. Richman: Locating metric complements in Euclidean space; J. R. Moschovakis: A disjunctive decomposition theorem for classical theories; D. S. Bridges: Towards a constructive foundation for quantum mechanics; A. S. Yessenin—Volpin: About infinity, finiteness and finitization; M. Gelfond: A class of theorems with valid constructive counterparts; J. R. Geiser: Rational constructive analysis.

Anynone who feels inclined to get acquainted with this "new world" (where it may happen e.g. that every real function is uniformly continuous) is recommended to consult these proceedings since several of their papers are expository or contain the philosophy of the subject.

V. Totik (Szeged)

I. P. Cornfeld, S. V. Fomin and Ya. G. Sinai, Ergodic Theory. Grundlehren der mathematischen Wissenschaften 245, X+482, Springer-Verlag, New York—Heidelberg—Berlin.

At the beginning ergodic theory dealt mainly with averaging problems but now, due to the readical changes in it during the last two decades, "it is a powerful amalgam of methods used for the analysis of statistical properties of dynamical systems". This book is an up-to-date development of the theory written by three outstanding scholars of the discipline. Since the authors' aim was to create a monograph focusing on applications, "Ergodic Theory" deserves the attention of research workers in other sciences as well, such as physics, biology, chemistry etc.

The book consists of four parts. Part I contains the description of several classes of dynamical systems. It begins with the basic definitions: ergodicity, mixing, operators adjoint to dynamical systems etc. and proceeds on to many classical constructions: dynamical systems on smooth manifolds, on torus, on homogeneous spaces; billiard type systems, systems in number theory and probability theory etc. In Part II the authors construct the direct and skew product of DS-s, introduce the important concept of entropy and give a detailed proof for the celebrated theorem of Ornstein on the existence of a stationary code. Part III is devoted to the spectral theory of DS-s. This is the shortest part of the book. Nevertheless, it contains von Neumann's theory of dynamical systems with discrete spectrum and the spectral analysis of DS-s associated to Gaussian stationary random processes. Finally, in Part IV the authors consider the possibility of approximation of dynamical systems by periodic DS-s and give some applications of the theory such as an example of an ergodic automorphism with a spectrum without the group property.

The authors pay much attention to illustrating the general concepts and theorems through concrete examples and these examples help very much in understanding the main ideas perhaps because they arise in very natural context. The bibliography contains more than 150 items. The publisher also did his best, the text is arranged in an especially legible form.

V. Totik (Szeged)

H. S. M. Coxeter, P. Du Val, H. T. Flather, J. F. Petrie, The Fifty-Nine Icosahedra, XX+26 pages with 20 plates and 9 figures, Springer-Verlag, New York—Heidelberg—Berlin, 1982.

An exciting problem of classical geometry was to enumerate and to describe the polyhedra that can be derived from the five Platonic solids by stellation, i.e., by extending or producing the faces until they meet again, always preserving the rotational symmetry of the original solid. The complete enumeration was first published by the above authors in 1938 by the University of Toronto Press. The present booklet is the reprint of this first edition with a new preface by P. Du Val. The text is a classical work of geometry which containes the mathematical explanation of the stellations and plates with pictures of all 59 variations descibing also the transformations between these stellations.

Z. I. Szabó (Szeged)

A. J. Dodd, The Core Model (London Mathematical Society Lecture Note Series 61), XXXVIII+229 pages, Cambridge University Press, Cambridge—London—New York—New Rochelle—Melbourne—Sydney, 1982.

This book is the first systematic study of the simplest core model K of set theory.

An arbitrary model M of ZFC is called an inner model iff M is transitive and $\mathrm{On}\in M$ (where On is the class of ordinals). An inner model M is said to have the covering principle, CP (M), for short, if for any uncountable $X\subseteq\mathrm{On}$, there exists $Y\in M$, such that $X\subseteq Y\subseteq\mathrm{On}$ and $\overline{\overline{X}}=\overline{\overline{Y}}$ (for a set $A, \overline{\overline{A}}$ denotes the cardinality of A). M is rigid if no elementary embedding of M into M other than identity exists.

The main aim of developing the core models is to obtain a generalization of the covering lemma due to Jensen: if L (the constructible universe of Gödel) is rigid, then CP (L). Core models are such models of set theory which subsumes L, the GCH is true in them and have the covering property under some inner model assumption. It is known, that if $M\models$ GCH and CP (M), then $M\models$ SCH, where SCH stands for the singular cardinal hypothesis, i.e., SCH denotes the assumption:

"for all singular cardinals $\alpha, 2^{cf(\alpha)}<\alpha$ implies $\alpha^{cf(\alpha)}=\alpha^{+}$"; hence in any core model SCH is true.

K is the simplest core model which is constructed by using two basic set theoretical tools: the fine structural investigations of the constructible universe developed mostly by Jensen, and the method of iterated ultrapowers due to Kunen, used in the theory of measurable cardinals.

The text is devided into six main parts followed by two appendices and a collection of historical notes (and, of course, a list of references).

The second part treats normal measures and iterated ultrapowers. The concept of a normal measure is used in the following form: U is a normal measure on $\varkappa$, if there are M and $j\colon V\to M$ such that M is an inner model, j is an elementary embedding with $j\restriction\varkappa=\mathrm{id}\restriction\varkappa$, $j(\varkappa)>\varkappa$, and M is the smallest model with $X\subseteq M$, where X is an elementary submodel of M such that $\mathrm{range}\,(j)\cup\varkappa\subseteq X$ and $X\in U$ iff $\varkappa\in j(X)$. If such a model M exists, then it is unique, and is called the ultrapower of V by the normal measure U. Albeit quantifiers range over proper classes, this definition can be formulated in the language of set theory: let $L[U]$ denote the universe of sets constructible from the normal measure U and suppose $L[U]\models$ "U is a normal measure", moreover, let M be the ultrapower of $L[U]$ by U; then $M\models L[\mathscr{V}]$, where $\mathscr{V}=j(U)$. By a result due to Scott, $L[\mathscr{V}]$ is a proper subclass of $L[U]$. It was shown by Kunen, that $\mathscr{V}$ is the only normal measure in $L[\mathscr{V}]$. The iteration of this construction, also defined by Kunen, taking the ultrapower of $L[\mathscr{V}]$ by $\mathscr{V}$ and using direct limits to get through limit stages yields to the recurrence:

$$M_0 = M, \quad U_0 = U;$$

M_{i+1}=the ultrapower of M_i by U_i with j; $M_i \to M_{i+1}$ and $j_{i+1}=j \cdot j_i$;
$U_{i+1}=j_{i+1}(U)$ for successors and
(M_λ, j_λ) is the direct limit of the sequence $(M_i, j_i)_{i<\lambda}$ for limit λ,
and $U_\lambda = j_\lambda(U)$.

Then, for all $\alpha \in \mathrm{On}$, $M_\alpha = L[U_\alpha]$.

Part II is a detailed exposition of this ultrapower construction.

The first part of the volume is devoted to developing the fine structural apparatus needed in constructing K. This is based mostly on the J hierarchy due to Jensen. The J hierarchy, as a whole is the same as the constructible universe L, the levels, however, are rearranged in the following way. Rudimentary functions, generalizing primitive recursive functions to arbitrary sets, are finitely generated by the initial functions consisting exhaustively of all projections, complementations, pairings, compositions and recursions of the form

$$f(y, \vec{x}) = \bigcup_{z \in y} g(z, \vec{x})$$

where $\vec{x}$ stands for a list of arguments. Let X be a set and put

$$R(X) = \{f(\vec{x}) | f \text{ is rudimentary and } \vec{x} \in X\}.$$

Then let $J_0 = \emptyset$, $J_{i+1} = R(J_i \cup \{J_i\})$ for successor i and $J_\lambda = \bigcup_{i<\lambda} J_i$ for limit λ. Clearly, $\bigcup_{\alpha \in \mathrm{On}} J_\alpha$ is the constructible universe of Gödel. Let U be a normal measure in an inner model M and suppose that for some α, $M = J_\alpha^U$. The basic fine structural tool, the projectum ϱ_M of M is the least γ such that there exists $A \subseteq \gamma$ such that A is Σ_1-definable over M and $A \notin M$. The main fine structural result, central for the construction of K states that if $\varrho_M \leqq \varkappa$, then M can be "coded" by a subset A_M of $\varkappa$; more precisely, if $\varrho_M \leqq \varkappa$, where U is a normal measure on $\varkappa$, then there is a surjective function from a subset A_M of $\varkappa$ onto J_α^U, such that A_M is Σ_1-definable over J_α^U. If $L[\mathscr{V}]$ is the ultrapower of $L[U]$ by U and j is an elementary embedding of $L[U]$ in $L[\mathscr{V}]$ such that $j \restriction \varkappa = \mathrm{id} \restriction \varkappa$, then $j(A_M) \cap \varkappa = A_M$, so $A_M \in L[\mathscr{V}]$ and hence $M \in L[\mathscr{V}]$.

Let $T = \{M \mid M = J_\alpha^{U_i}$ for some normal measure U_i on $\varkappa_i$ and $\varrho_M \leqq \varkappa_i\}$. Then the core model K is defined by $K = \bigcup T$. By a mouse, an element of T is meant. Mice are studied in details in Part III. It is shown for example, that the dependence of the definition of T on the normal measures U_i can be eliminated by allowing any normal measure instead of U_i. This process yields to the concept of a premouse: M is a premouse at $\varkappa$ if $M = J_\alpha^U$ for some U and $M \models$ "U is a normal measure on $\varkappa$". A premouse M is iterable iff the model $(M, \in_M)$ defined just as in the definition of a normal measure (ultrapower), with the only difference: "elementary" is replaced by "Σ_1-definable", is well-founded. Indeed, the well-foundedness property of M_i is inherited by the iteration of ultrapowers. Let $T' = \{M \mid M$ is an iterable premouse at some $\varkappa$ and $\varrho_M \leqq \varkappa\}$.

Then $K = \bigcup T' \cup L$. Part IV is devoted to the investigation of K. In particular, an important internal characterization of K is proved. If there exists an inner model $L[U]$ with $L[U] \models$ "U is a normal measure", then $K = \bigcap_{i<\infty} L[U_i]$. It is also shown, that $K \models \mathrm{ZFC}$ and $K \models \mathrm{GCH}$. Moreover, in Part V, a generalization of the covering lemma is obtained: if there is no inner model with a measurable cardinal, then CP(K). As an application, one has: if there is no inner model with a measurable cardinal, then $K \models \mathrm{SCH}$. It is also shown that several combinatorial principles such as $\diamondsuit$ and $\square$ hold in K. Part VI collects some recent results on core models larger than K. In particular, a few properties of supercompact and superstrong cardinals are established. Appendices relate core models to the forcing construction and to some absoluteness results for models of ZFC.

The volume is selfcontained, clearly written and gives a full exposition of the state of the art concerning core models. It is sure that this book will become a basic reference for researchers in the fields of large cardinals as well as for graduate students.

P. Ecsedi-Tóth (Szeged)

Burton Dreben and Warren D. Goldfarb, the Decison Problem, Solvable Classes of Quantificational Formulas, XII+271 pages, Addison-Wesley Publ. Comp. Inc., Advanced Book Program, Reading, Mass., 1979.

The classical decision problem (called "the fundamental problem of mathematical logic" by Hilbert) asked for an algorithm to decide for any formula if it is satisfiable. Since the work of Gödel and Church it is well known that there can be no such algorithm. Special eases with restricted classes of formulae having a decision procedure have been investigated intensively during the past decades. These classes are defined by syntactic restrictions e.g. on the form of quantifiers, a basic decidable case being the so-called Gödel—Kalmár—Schütte class of formulae with quantifiers $\exists \ldots \exists \forall \forall \exists \ldots \exists$.

This book gives a comprehensive description of the known solvable classes including the complete list of solvable prefix classes. A unified treatment is given to the subject by the use of the Herbrand expansion method.

The book is written in a very clear style and gives a good picture of the current state of this side of the decision problem, providing a deep knowledge of the important Herbrand expansion method and indicating some interesting open problems as well. It can be recommended to logicians and computer scientists. (A good complementary reading is given by the book Unsolvable Classes of Quantificational Formulas of H. R. Lewis [Addison—Wesley Publ. Comp. Inc. Adwanced Book Program, Reading, Mass., 1979, 214 pages], and a recent branch of the topic is described in the paper Complexity Results for Classes of Quantificational Formulas by H. R. Lewis [J. of Computer and System Sciences 21, No. 3. Dec. 1980, pp. 317—353].)

György Turán (Szeged)

C. H. Edwards, Jr., The Historical Development of the Calculus, XII+351 pages, Springer-Verlag, New York—Heidelberg—Berlin, 1979.

A scientific concept cannot be understood completely without knowing its development. Calculus has become the language of Western science for three centuries, so all the students in science have to know something about its history. Lecturers are to take a general view of this subject and they need a handbook in this topic.

Edwards' book is suitable for the above mentioned purpose. It begins calculus with Eudoxus' definition on proportionality of ratios and the method of exhaustion based on that definition. The method culminates in Archimedes' works to whom a chapter is devoted in the book. He used a double reductio ad absurdum rather than limits and his "geometric calculus" could not have been continued.

Edwards emphasizes the influence of medieval speculations on motion, variability and infinity to medieval mathematics to break up the Greek horror of infinity. A number of early tangent constructions are shown and the difference between them and the calculus according to Newton and Leibniz is elucidated.

The classicals of the calculus are treated circumstantially until Weierstrass.

A short postcript is devoted to two results of the twentieth century: the Lebesgue integral and the non-standard analysis. Edwards does not think non-standard analysis to be a correct reformulation of infinitesimals, he states "Leibniz seems not to have committed himself on the question of actual existence of infinitesimals, and he certainly expressed doubts on occasion".

The author does prove the importance of adequate concepts and notations in mathematics. It is clear all over the book that calculus is for calculations.

The reader can take part in the work of the classicals: there are exercises interspersed throughout the text and the reader is invited to solve them using the tools of that time.

The book is offered to lecturers, students and to the wide mathematical community.

L. A. Székely (Szeged)

Functional Differential Equations and Bifurcation, Proceedings of a Conference Held at São Carlos, Brazil, July 2—7, 1979, edited by A. F. Izé (Lecture Notes in Mathematics, 799), XXII+409 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1980.

This book contains contributions presented at the conference by authors from Brasil, Iceland, Italy, Japan, U.S.A. and South Africa. It gives a good survey on some present topics of the theory of differential equations and some related fields. The reader can find papers on subjects such as control theory, boundary value problems, periodic solutions, stability theory, structural stability, bifurcation theory, dissipative processes, existence type results, asymptotic equivalence of solutions, linear difference equations, Volterra—Stieltjes-integral equations, Hartree type equation, Levin—Nohel equation on the torus, almost periodic functional differential equations.

J. Terjéki (Szeged)

A. Gardiner, Infinite Processes (Background to Analysis), IX+306 pages with 182 illustrations, Springer-Verlag, New York—Heidelberg—Berlin, 1982.

The book provides a well-rounded picture of the basic material of Analysis. Its main goal is to show why the concepts such as infinite decimals, length, area, volume, functions are handled as they are in mathematics.

The text is divided into four parts. Part I is short and largely descriptive. It indicates how, around 1800, mathematicians began to realize that the lack of precision in their manipulation of the infinite processes involved in the naive calculus was a source of error and confusion.

Part II is the longest one of the book. It examines in detail infinite processes arising in arithmetic of the real numbers. Most of the text is devoted to the analysis of specific examples.

Part III explores that any attempt to invest the familiar geometric notions of length, area and volume with precision involves the fundamental properties of real numbers. It also points out to the fact that modern mathematics is not so much the study of numbers and space as the study of functions.

In Part IV the author outlines some of the basic questions which result from the differential and integral calculus. In particular, the following crucial question is considered: What exactly is a function?

A lot of exercises are included, which constitute an integral part of the text. They arise directly out of the text and need to be understood in context.

The book is described as a stimulus for thinking about the role of infinite processes in mathematics. The presentation is clear and precise, the ideas are illuminated by consideration of historical developments. The better understanding is helped by 182 figures. The book ends with an (author and subject) index.

This good overview is especially suited to mathematical history and review courses, as well as for math teachers and for nonspecialists who have mastered the calculus.

F. Móricz (Szeged)

Bernard Gelbaum, Problems in Analysis. Problem Books in Mathematics, VII+228 pages with 9 illustrations, Springer-Verlag, New York—Heidelberg—Berlin, 1982.

"The major part of every meaningful life is the solution of problems; a considerable part of the professional life of technicians, engineers, scientists etc. is the solution of mathematical problems. It is the duty of all techers, and of teachers of mathematics in particular, to expose their students to problems much more than to facts." These are the words of Paul Halmos about mathematical problems and he adds that we have to "...train our students to be better problem — posers and problem — solvers than we are". Surely it is not accidental that P. R. Halmos is the series editor of the new Springer-series "Problem Books in Mathematics". Anyone agreeing with Halmos on the role of problems will greet this fascinating idea: supply the students, teachers and mathematicians with books focusing on problems which may range from elementary exercises to unsolved research problems. As prototypes Pólya and Szegő's "Problems and Theorems in Analysis" and Hilbert's famous 23 problems are marked. However, it is an almost impossible task to give such a comprehensive selection as Pólya and Szegő's in any branch of mathematics, therefore the author's taste and "intelligence" play enormous role in writing these problem books. For a newly launched series the succesful start is vital and "Problem Books in Mathematics" accomplished this task excellently: the first two exemplares: Gelbaum's book and Kirillov and Gvishiani's "Theorems and Problems in Functional Analysis" are really worth for beginning the series with them.

Gelbaum's book contains 518 problems and their solutions. The topic is real analysis and the elements of functional analysis. The standard exercises of this theme are mostly left out, almost every problem requires some thinking — some of them may be very puzzling for a beginner. The proofs are short but sometimes incaccurate or lenghty, e.g. the solution of problem 248: "$\sum_{1}^{\infty} k\lambda(G_k)=\Sigma_n\lambda(A_n)$ where $G_k=\{x \mid x\in A_n$ for exactly k distinct values of $n\}$ $(A_n\neq A_m)$" presented in the book is not complete and at the same time the problem itself is easy if we use characteristic functions. Unfortunately there are false problems and solutions (!). For example for problem 126: "Let f be in $C(R, R)$ and assume $\lim\sup_{h\to 0+0} (f(x+h)-f(x))/h\geqq 0$ a.e. Show f is monotone increasing" any descreasing continuous singular function provides a counterexample. Problem 175" Give an example of a measure space (X, S, μ), a sequence $\{E_n\}$ of measurable sets of finite measure, and a sequence $\{f_n\}$ of functions such that f_n and $1-f_n$ are integrable, $0\leqq f_n\leqq 1$, $f_n=1$ on E_n, $\lim_{n\to\infty} f_n(x)=1$ a.e. and $\int_X (1-f)\,d\mu \not\to$ as $n\to\infty$ "asks for a non-existing construction. In some cases the formulation of the problem is clumsy since by the same method a much nicer problem could be solved, e.g. in problem 57 if $f_0\in C([0, 1], R)$ and $f_n(x)=\int_0^x f_{n-1}(t)\,dt$ then $f_0\equiv 0$ provided for every $x\in[0, 1]$ there is an n with $f_n(x)=0$. Nevertheless these faults must not be exaggerated since the majority of the problems and solutions are indeed very nice.

An undergraduate or graduate student should have enough knowledge to solve most of the problems although the author freely uses harder results from real analysis. For example, to solving problem 316 "If $f\in L^\infty(R, \lambda)$ and $\int_k \exp(-(x-y)^2)f(y)\,dy=0$ for all $x\in R$ then $f=0$ a.e." one has to know Wiener's tauberian theorem. A special merit of the book is that besides general abstract results and theorems it contains several "concrete" problems. For instance problem 473 states that if $\nu(n)=2^{-n!}$ $(n\geqq 2)$ and $\nu(1)=1-\sum_{1}^{\infty} 2^{-n!}$ then there are no nonconstant functions f and g independent with respect to ν.

Gelbaum's book may be recommended to students, teachers and research workers, as well, who may get fun and make progress while reading and solving these non-trivial excellent problems.

V. Totik (Szeged)

G. Gierz, K. H. Hoffmann, K. Keimel, J. D. Lawson, M. Mislove, D. S. Scott, A Compendium on Continuous Lattices, XX+371 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1980.

In lattice theory, the seventies brought a considerable development of (and interest in) the study of continuous lattices. Historically, the most important stimulation was Dana Scott's research on the problems of syntax and semantics of computer languages and his interpretation of computer programs (including cyclic ones) by means of continuous lattices, which led, among other things, to Scott's model in set theory of the type free λ-calculus. These results became at once well-known also outside algebra, among specialists of computer science and logic. But, at about the same time, important contributions were made by others, most of them on the list of authors of the book under review. Especially, the results of Hoffmann, Lawson, Mislove and Stralka on compact semilattices and those of Keimel and Gierz on the topological representation theory and spectral theory of non-distributive lattices are to be mentioned. This led to a collaboration of the authors of the book, resulting in what now may be called the theory of continuous lattices. The present book is the first monograph on the subject addressed to the general mathematical public.

To describe the subject, one first has to define the "way-below" relation, which is basic for the entire theory. We say that, for elements x, y of the complete lattice L, x is way below y, in symbols, $x \ll y$ iff for directed subsets $D \subseteq L$ the relation $y \leqq \sup D$ always implies the existence of a $d \in D$ with $x \leqq d$. (Elements satisfying $x \ll x$ are exactly the compact elements.) A lattice L is called a continuous lattice if L is complete and satisfies the axiom of approximation: $x= =\sup \{u \in L \colon u \ll x\}$ for all $x \in L$. (In particular, all algebraic lattices are continuous.) Intuitively, thinking of the realization in computability theory, x way below y can be interpreted as x is a "(finite) approximation" of y. Then the axiom of approximation says that each element is the limit of its finite approximations. The motivation for the study of continuous lattices comes not only from computer science and logic. Other fields where such lattices appear quite naturally (sometimes in disguised forms) are, for example, general topology, functional analysis, category theory, and, of course, algebra.

Chapter I introduces continuous lattices from an order theoretic point of view. In Section 1 the way-below relation is discussed. Section 2 gives an equational chracterization of continuous lattices. Section 3 deals with irreducible and prime elements. Section 4 considers the important special case of algebraic lattices. Chapter II defines the Scott topology and develops its applications to continuous lattices. The second important topology for continuous lattices, the Lawson topology, is discussed in Chapter III. Chapter IV considers various important categories of continuous lattices together with certain categorical constructions. Section 1 presents important duality theorems for the study of continuous lattices. The last two sections give general categorical constructions for obtaining continuous lattices which are fixed points with respect to some self-functor of the category. This process is needed for the construction of set-theoretic models of the λ-calculus. Chapter V deals with spectral theory. The most important result of Chapter VI is the Fundamental Theorem of Compact Semilattices, establishing the equivalence between the category of compact semilattices with small semilattices and the category of continuous lattices. Chapter VII completes the study of connections between topological algebra and continuous lattice theory with methods coming from topological algebra rather than from lattice theory.

This book is warmly recommended to anyone wishing to become acquainted with the subject.

A. P. Huhn (Szeged)

Franklin A. Graybill, Matrices with Applications in Statistics, Second edition (Wadsworth Statistics/Pribability Series), XII+461 pages, Wadsworth International Group, Belmont, California, 1983.

Matrices are used so extensively in the theory and applications of statistics that a firm knowledge of matrix and linear algebra is required from a student who wants to study the theory of linear statistical models. A number of topics in matrix algebra that are useful in a study of multivariate analysis is generally not available in an elementary course or in textbooks in linear algebra. On the other hand, the most part of monographs and advanced textbooks in matrix algebra has a too hard algebraical emphasis and some topics that are important for a statistician are mentioned only briefly. Graybill wanted to write a book which is "useful for any-one who takes courses in regression and correlation, analysis of variance, least squares, linear statistical models, multivariate analysis, or econometrics; and it could serve as a resource book for many other subjects". As the second edition of his book shows, the author achieved his purpose.

The book assumes that the reader has had a course that includes the most important and fundamental theorems in linear algebra. An introduction and summary are given in the first three chapters, the theorems are stated without proofs. Some geometric interpretations of vectors and the elementary theory of analytic geometry are discussed briefly in Chapters 4 and 5. Chapter 6 is devoted to the general inverse and conditional inverse. The author states some additional theorems on the inverses of special matrices and proves some theorems that can be used to compute the generalized inverse of a matrix. Chapter 7 deals with the existence and number of solutions of systems of linear equations. Approximate solutions of inconsistent systems including least squares are in the focus of this chapter. Chapter 8 contains theorems on the patterned and other special matrices (partitioned, triangular, dominant diagonal, Vandermonde, Fourier, permutation and Toeplietz matrices). The following chapter treats the many applications in which the sum of diagonal elements (trace) of a matrix plays an important role. Chapter 10 demonstrates how matrices and vectors can be used in transforming random variables, in evaluating multiple integrals and in differentation. These methods are useful in the study of multivariate normal distributions. The author briefly discusses some important general types of matrices (positive, non-negative, idempotent, tripotent matrices) in the last two chapters. Each chapter contains a lot of examples and exercises that can help the reader in understanding the presented material.

The book is very elegently and clearly written, it can be recommended to all students and statisticians interested in linear algebra from a statistical point of view.

Lajos Horváth (Szeged)

Richard K. Guy, Unsolved Problems in Number Theory (Unsolved Problems in Intuitive Mathematics, Vol. I), XVIII+161 pages with 17 figures, Springer-Verlag, New York—Heidelberg—Berlin, 1981.

This book lists 178 challenging open problems (or group of problems) to stimulate beginning researchers. No matter how easily one can understand them, none of us lives as long as to see the proof or counterexample for all the listed problems. The touched topics are prime numbers, divisibility, additive number theory, diophantine equations, sequences of integers, and others.

This volume is dedicated to Erdős Pál, whose influence to number theory can be observed everywhere in the book, as follows: "Among his several greatnesses are an ability to ask the right questions and to ask it of the right person." The reader is supplied with plentiful references. Many prizes are set by Erdős and some by Graham.

L. A. Székely (Szeged)

Frank C. Hoppensteadt, Mathematical Methods of Population Biology (Cambridge Studies in Mathematical Biology, 4), VIIII+149 pages, Cambridge University Press, Cambridge—London—New York—New Rochelle—Melbourne—Sydney, 1982.

According to the publisher, "this introduction to mathematical methods that are useful for studying population phenomena is intended for advanced undergraduate and graduate students, and will be accessible to scientists who do not have a strong mathematics background." The first two chapters introduce the usual deterministic models of total population and population age structure (Malthus, Verhulst, the predator pit, chaos, synchronisation, fisheries, Fibonacci's reproduction, McKendrick's model etc.), the third chapter deals with random models of bacterial and human genetics (urn, Fisher—Wright, and branching process models) and of epidemics (Reed—Frost model) based on Markov chains, and the last two chapters describe very shortly perturbation methods and diffusion approximations.

Matheatical notions are used without definitions. It is not that the reviewer would like to seee mention of the Radon—Nikodym theorem on p. 63, for example, where conditional expectations are used (and an embarrassing misprint is left in line 6 from bottom), but he feels that students and scientists "who do not have a strong mathematical background" will not learn the "mathematical methods" from this book. The reviewer agrees that "mathematical details" should not "obscure biological relevance" but such non-technical and non-sensical descriptions of the central limit theorem as the one on p. 97, that "(it) states that *any* random variable, discrete or continuous (?!), is in a definite sense approximated by a normally distributed random variable", will not help anybody to understand neither population biology, nor mathematics.

It is not a contradiction in terms, however, that this is a good book. Good for those who do have a stronger mathematics background and are interested in applications. These people will enjoy the numerous interesting examples and exercises from population biology.

Sándor Csörgő (Szeged)

B. Huppert—N. Blackburn, Finite Groups II—III (Grundlehren der mathematischen Wissenschaften, Band 242—243), XIII+531 and IX+454 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1982.

This is the continuation, awaited for 15 years, of the classic book on finite groups:
B. Huppert, Endliche Gruppen I (Grundlehren der mathematischen Wissenschaften, Band 134), XII+796 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1967; reprinted 1979.

While Volume I presents a large part of what was known about the structure of finite groups at the time of its writing (started in 1958), the same goal is evidently not attainable now. During the past two decades the subject has made a tremendeous progress, with a lot of new branches and powerful methods coming into existence which, combined together, produced a number of extremely deep results. Few accomplishments ever reached in mathematics are comparable with the recent completion of the classification of finite simple groups. In view of this great development it is no surprise that the authors of Volumes II and III had to be contended with selecting several important topics and even within those topics no attempt on completeness was made.

All three chapters of Volume II are devoted to discussing the role of linear methods in finite group theory. Representation theory is presented first (Chapter VII: Elements of General Representation Theory), the emphasis being put on the modular case, as the classical one is studied in Volume I, Chapter V. Next (Chapter VIII: Linear Methods in Nilpotent Groups) some ways of "translating" commutator calculations into calculations with linear structures are shown along with several theorems illustrating the power of these methods. For example, bilinear forms are used to determine the Suzuki 2-groups, and the Lie-ring method is applied to prove that for prime exponent the answer to the restricted Burnside problem is affirmative. The last chapter (IX: Linear Methods in Soluble Groups) gives an introduction to the Hall—Highman methods and numerous applications to obtain upper bounds for the p-length of a p-soluble group in terms of various invariants of its Sylow p-subgroups.

Volume III also consists of three chapters. The first one (Chapter X: Local Finite Group Theory) is concerned with deriving properties of the whole group from hypotheses involving only its p-subgroups and their normalizers (which are regarded as local properties of the group). Such results turned out to be important for example in proving the solubility of groups of odd order. The book ends with two chapters on permutation groups, including also several important characterization theorems, i.e., descriptions of specific groups solely in terms of group-theoretical properties. One of the earliest instances of such results was given for Zassenhaus groups, which is presented in full detail (Chapter XI: Zassenhaus Groups). The last chapter (XII: Multiply Transitive Permutation Groups) is a collection of some of the most interesting investigations on multiply transitive and sharply multiply transitive permutation groups.

No doubt, these volumes will soon become as indispensable reference books for group theorists, as Volume I. Besides, by giving a systematic treatment of a number of results which, up to now, were available in research papers only, they will be an immense help for those wishing to specialize in the subject.

Ágnes Szendrei (Szeged)

Ching-Lai Hwang, Abu Syed Md. Masud in collaboration with **Sudhakar R. Paidy and Kuangsun Yoon, Multiple Objective Decision Making—Methods and Applications,** A State-of-the-Art Survey, (Lecture Notes in Economics and Mathematical Systems, 164) XII+351 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1979.

This is a good guide through the literature of the Multiple Objective Decision Making (MODM) methods. The authors present the existing methods, their characteristics, and their applicability to analysis of MODM problems. The book contains a good classification of about two dozen MODM methods. The first level of the classification is the information available for the decision maker. The second level is the type of information, and the lowest level contains the major classes of methods. Most of these methods have been proposed by various researchers in the last few years, and the main usefulness of this work is the unified discussion. This is the first time they are presented together. The literature of these methods is identified and classified systematically. All procedures of each method have been illustrated by a simple numerical example in detail. This helps the reader understand the basic concept and the characteristic of each method.

Since most methods have not been tested by real-world problems yet, the authors cannot discuss in depth the advantages, disadvantages, computational complexity and difficulty of each method. The appendix contains a bibliography of more than 400 books, journal articles, technical reports and theses on this field of mathematics.

G. Galambos (Szeged)

D. L. Iglehart and G. S. Shedler, Regenerative Simulation of Response Times in Networks of Queues (Lecture Notes in Control and Information Sciences 26). XII+204 pages. Springer—Verlag. New York—Heidelberg—Berlin. 1980.

Discrete event digital simulation of stochastic models is one of the most important practical tools of systems analysis. The real systems are so complex that we are unable to study them analytically and we must, therefore, use computer simulation. This monograph deals with probabilistic and statistical methods for discrete event simulation of networks of queues.

The initial section provides some motivation for study of simulation methods for passage times in networks of queues. Section 2 gives a review of the regenerative method. The authors deal with a specification of the class of closed networks of queues in Section 3 and describe the marked job method in Section 4. Applications of the marked job method can be found in the next section and an extension of this method is the subject of Section 6. Further estimations for the first passage times are described in Sections 7 and 8. The statistical efficiency of the marked job and decomposition methods are studied in the next section. The estimation of passage times in closed networks of queues is the focus of Section 10. The last section is devoted to the algorithms for random number generation.

The presentation is selfcontained, some knowledge of elementary probability theory and stochastic processes is the only requirement from the reader.

Lajos Horváth (Szeged)

Kenneth Ireland and Michael Rosen, A Classical Introduction to Modern Number Theory, Graduate Texts in Mathematics Vol. 84, XIII+341 pages, Springer-Verlag, New York—Heidelberg—Berlin, 1982.

This book is a revised and greatly expanded version of the authors' Elements of Number Theory published in 1972 by Bogden and Quigley. The well selected topics and treatments bridge the gap between elementary number theory and the systematic study of advanced topics. The reader must be familiar with the material in a standard undergraduate course in abstract algebra, but a large portion of the first eleven chapters is understandable with a small amount of suplementary reading. The later chapters assume some knowledge of Galois theory and in the last ones an acquaintance with the theory of complex variables is necessary.

The authors' focus is on topics which point in the direction of algebraic number theory and arithmetic algebraic geometry, without requiring very much technical background. The major themes are the following: Unique factorizations and its applications; reciprocity laws which lead from the quadratic reciprocity to the Artin reciprocity law, one of the major achievements of algebraic number theory; the theory of Gauss and Jacobi sums and its generalizations; diophantic equations over finite fields and over the rational numbers; the Riemann zeta function.

There are also several hundreds of exercises, some routine, some challenging. Some of them supplement the text. In the last chapters a number of exercises is adopted from the recent research literature. Throughout the book there are considerable emphasis on the history of the subject.

This book with its particulary extensive bibliography is highly recommended to research students and to anyone who wants to be familiar with some of the themes and subjects currently under investigations in algebraic number theory and arithmetic algebraic geometry.

Lajos Klukovits (Szeged)

Thierry Jeulin, Semi-Martingales et Groississement d'une Filtration (Lecture Notes in Mathematics, 833), IX+142 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1980.

As the author writes, "probabilists have now fully accepted Doob's idea that the adequate structure for the study of a stochastic process is that of a probability space $(\Omega, \mathcal{A}, P)$ filtered by an increasing family $\mathcal{F}=\{\mathcal{F}_t, t\geqq 0\}$ of σ-fields. While $\mathcal{A}$ represents the whole universe, $\mathcal{F}_t$ consists of events whose outcome is known to the observer at time t, and predictions at time t are conditional expectations $E(\cdot \mid \mathcal{F}_t)$". However, the description of a partially observable random system requires a pair $\mathcal{F}=\{\mathcal{F}_t, t\geqq 0\}$, $\mathcal{G}=\{\mathcal{G}_t, t\geqq 0\}$ of filtrations such that $\mathcal{F}_t\subset\mathcal{G}_t$, $t\geqq 0$. The purpose of this monograph is to construct $\mathcal{G}$ from $\mathcal{F}$ "by forcing information into $\mathcal{F}$" and then to measure "how much the prediction processes relative to $\mathcal{F}$ have been distorted by the new information". Thus the two basic problems dealt with are: 1) Does an $\mathcal{F}$-martingale X remain a $\mathcal{G}$-semi-martingale? 2) If yes then give an explicit decomposition of X into a $\mathcal{G}$-local martingale and a process of bounded variation. After giving the necessary preliminaries in the short first chapter, Chapter 2 is devoted to the discussion of the most general results concerning the first problem. The next three chapters deal, respectively, with initial enlargement of $\mathcal{F}$ (at time 0), with progressive enlargement (when additional random variables are added to the set of stopping times as the time goes on), and with enlargement by adding a single "honest" random variable to $\mathcal{F}$ as an extra stopping time. The sixth chapter deals with concrete applications to Markov processes in general and to the Brownian notion, Brownian excursions and Bessel processes in particular. The book can be recommended to martingale theorists and perhaps also to experts in advanced engineering applications of filtration theory.

Sándor Csörgő (Szeged)

Ole G. J^rsboe and Leif Mejlbro: The Carleson-Hunt Theorem on Fourier Series, Lecture Notes in Mathematics 911, IV+123, pages, Springer-Verlag, Berlin—Heidelberg—New York, 1982.

The Carleson—Hunt theorem: The Fourier series of every function $f\in L^p[-\pi, \pi]$, $1<p\leqq\infty$ converges almost everywhere.

This is the book that should have been published many years ago. Besides the importance of the Carleson—Hunt theorem there are at least two reasons for publishing a book that contains nothing else but the proof of the above theorem. First of all the recent books on Fourier series only quote the theorem but leave out its proof. Since the original articles of Carleson and Hunt were written for specialists, it is necessary to have a treatment available also for mathematicians and students without much knowledge in harmonic analysis. According to this Jørsboe and Mejlbro's book assumes only some rudiments of measure theory and every other concept such as maximal function, Hilbert transform, interpolation of operators etc. and their properties needed in the proof is given in full detail. The second reason is connected with the first one, namely the Carleson—Hunt theorem can hardly be the topic of a regural or special course because of the fine and often very technical details of its proof, therefore it is desirable to have a work which may substitute these courses.

These lecture notes realize the above aims in a perfect way. The presentation is extraordinarily clear, the proof is built up from small steps each of which is proved very carefully. These small steps are united into four main chapters each preceded by a short but very useful description of the chapter's content. At almost every delicate step one can find a remark enlightening the necessity of the given concept or consideration. Nevertheless a warning is in order at this point: the Carleson—Hunt theorem is far from being trivial, so its proof is very hard, especially the material in Chapter 4 is very difficult to read.

In Chapter 1 the authors introduce the Hardy—Littlewood maximal operator and prove a special form of the Marcinkiewicz interpolation theorem. Finally, they prove the Carleson—Hunt theorem under the assumption

$$(*)\qquad \|(\sup_n |S_n(f)|)\|_{L^p} \leqq K_p \|f\|_{L^p} \quad (1<p<\infty,\ f\in L^p[-\pi,\pi])$$

($S_n(f)$ denotes the n th partial sum of the Fourier series of f). The rest one hundred pages is devoted to the proof of $(*)$. Chapter 2 contains some basic facts about the Hilbert transform. In Chapter 3 the necessary technique is introduced: dyadic intervals, modified Hilbert transform, generalized Fourier coefficients. The proof is completed in Chapter 4 by constructing in several steps a set of measure zero such that on the complement the Fourier series is "not very large"

We recommend the book to everybody working in related fields of mathematics as well as to students interested in the subject.

V. Totik (Szeged)

Hua Loo Keng, Introduction to Number Theory, XVII+572 pages with 14 figures, Springer-Verlag, Berlin—Heidelberg—New York, 1982.

This is the English edition of the famous Chinese original first published in 1957. The book is an excellent and broad introduction to the subject and will soon prove itself a very good successor of the classical introductory texbook "An Introduction to the Theory of Numbers" by G. M. Hardy and E. M. Wright. Several recent results in number theory appear in such a form as to make this textbook suitable for teaching purposes. This English edition contains additional notes compiled by Wang Yuan and Peter Shiu (the translator) at the end of nearly all chapters. These enable the reader to acquint himself with the current research literature. In the running text there are several examples and exercises to help the deeper understanding.

A great value of this book is that the author tries to highlight certain connections of elemantary number theory to other branches of mathematics. For example: the relationship between the prime number theorem and Fourier series; the partition problem, the four squares problem and their relationship to modular functions; the theory of quadratic forms, modular transformations and their connections to Lobachevskian geometry, etc.

The book, which serve not only as a textbook but a fundamental reference work, contains the following main topics: The elementary proof of the prime number theorem due to Erdős and Selberg; Roth's theorem; Gelfond's solution to Hilbert's seventh problem; Siegel's theorem on the class number of binary quadratic forms; Linnik's proof of the Hilbert—Waring theorem; Selberg's sieve method and Schnirelman's theorem on the Goldbach problem; Vinogradov's result concerning least quadratic non-residues. In addition, some of the author's own work is represented, too.

Lajos Klukovits (Szeged)

S. M. Khaleelulla, Counterexamples in Topological Vector Spaces, Lecture Notes in Mathematics 936, XXI+179 pp, Springer-Verlag, Berlin—Heidelberg—New York, 1982

"During the last three decades much progress has been made in the field of topological vector spaces. Many generalizations have been introduced... *To justify that a class C_1 of topological* vector spaces is a proper generalization of another class C_2..., it is necessary to construct an example of a topological vector space belonging to C_1 but not to C_2; such an example is called a counterexample". The book contains more than two hundred examples of this kind. Very often the same one (e.g. L^p ($0<p<1$ or $p>1$), $C[0, 1], l^\infty, l^1, C_0$ etc.) works in different situations by which several interesting properties are displayed for the most frequently used Banach spaces and topological linear spaces. The examples treated in the book range from perfectly trivial ones (e.g. "A bounded sequence in a topological vector space which is not convergent") to more sophisticated constructions. The hardest counterexamples are only recorded (with a reference) without proof or construction (e.g. Enflo's separable Banach space without basis).

The material is arranged in a clear way. It was a good idea to name the examples fully in the "Contents"; this helps in finding the needed constructions. The book is divided into eight chapters. Each chapter begins with definitions and some basic theorems, there is always a reference pointing to the source of the quoted results. The examples themselves are presented in a legible form, although the author very often leaves out the verification that they do work, and in many cases this constitutes the hardest part of the job. Detailed index and bibliography help the reader in remembering the concepts and in further study. The content of the chapters are as follows:

1) Topological vector spaces (general properties), 2) Locally convex spaces, 3) Special classes of locally convex spaces, 4) Special classes of topological vector spaces, 5) Ordered topological vector spaces, 6) Hereditary properties, 7) Topological bases, 8) Topological algebras.

These lecture notes should be used as a reference book but it may also be useful for anyone who is searching for the definition of a concept or even for a beginner who, while reading it, may get a quick glance of the most important facts of the topic.

V. Totik (Szeged)

A. A. Kirillov and A. D. Gvishiani, Theorems and Problems in Functional Analysis. Problem Books in Mathematics, IX+347 pages with 6 illustrations, Springer-Verlag, New York—Heidelberg—Berlin, 1982.

This is a translation of a Russian edition (1979). Its aim is to give a self-contained introduction to modern branches of functional analysis. It is a combination of a textbook and a problem book with detailed hints for solving the problems.

The book is divided into three parts: Theory, Problems and Hints. The chapters are subdivided into sections and the sections into subsections each containing 23 exerciеses, so altogether 828 problems are posed. Many of these require only minimal skill but there are a lot of harder problems that may be nontrivial even for an expert in the field. A rough table of contents: Set theory and topology, measures and integrals, linear topological spaces and linear operaors, elements of harmonic analysis and the spectral theory of operators. In the first part — on more than 130 pages — a brief account of the most important aspects of the theory is given with complete proofs. This part may be used in a one year course, although the presentation is very concise and brief. The problem part begins with a simple exercise about equivalence relations and ends with the spectral decomposition of the selfadjoint extension of the operator $A=-(d^2/dx^2)+x^2$ with initial domain $D(A)=S(R)$. Many standard results are incorporated as exercises such as Lebesgue's density theorem, Hölder's inequality, the Stone—Weierstrass theorem etc. The problems concerning up-to-date topics such as category theory, generalized functions, characters on Abelian groups etc. provide a smooth path to these advanced matters. The hints are sufficient for working out complete solutions.

List of notation and detailed index increase the utility of the book which should be on the bookshelf of every lecturer on functional analysis and surely will enjoy great success among students, as well.

V. Totik (Szeged)

Anders Kock, Synthetic Differential Geometry (London Mathematical Society Lecture Note Series, 51), VI+311 pages, Cambridge University Press, 1981.

Synthetic differential geometry, in the sense of the book, is the theory of general differential manifolds based on the assumption of sufficiently many nilpotent elements on the "real line". The first part of the book containes a detailed exposition of the differential and integral calculus on these manifolds such as directional derivatives, Lie derivation, forms and currents, Stokes' theorem etc. In the following part categorial logic is introduced into the exposition, and in the last part several models are presented in order to compare the synthetic theory with the analytic one.

The book assumes some knowledge on abstract algebra and category theory. It is recommended to graduate students and professionals who are interested in algebraic or differential geometry or category theory.

Z. I. Szabó (Szeged)

A. I. Kostrikin, Introduction to Algebra, Universitext, XIII+575 pages, Springer-Verlag, New York—Heidelberg—Berlin, 1982.

This is the translation of a textbook of the present undergraduate algebra course at Moscow State University. The book reflects the Soviet approach to teaching mathematics with its emphasis on applications and problem-solving. In the first place, Kostrikin's textbook motivates many of the algebraic concepts by practical examples. For instance, the heated plate problem, coding information and the states of a molecule are used to introduce linear equations and finite fields, systems of equations over finite fields and groups and group representations, respectively. In the second place, there are a large number of exercises so that the reader can convert a vague passive understanding to active mastery of the new ideas. The harder problems have hints at the end of the book. This helps those who learn algebra outside of the framework of an organized course. In the third place, there are topics in it which are usually not part of an elementary course but which are fundamental in applications.

The book consists of two parts (Sources of algebra and Groups, Rings, Modules) and nine chapters. The first three chapters constitute an introduction to elementary linear algebra: sets, mappings, integer arithmetic, vector spaces and matrices over the field of real numbers, linear maps, systems of linear equations, determinants. The later three chapters of part one deal with groups, rings and fields, complex numbers and polynomials and roots of polynomials. In part two the reader can find more about groups (classical groups of low dimensions, group theoretical constructions, the Sylow theorems and the fundamental theorem for finite abelian groups), the elements of the group representation theory (unitary, reducible, linear and irreducible representations) as well as more about fields, rings and modules, including a section on algebras over a field.

This valuable textbook is warmly recomended to undergraduate students, as well as to anyone who wants to be familiar with basic abstract algebra and certain applications of it.

Lajos Klukovits (Szeged)

Logic Year 1979—80. The University of Connecticut, Proceedings, edited by M. Lerman, J. H. Schmerl and R. I. Soare, Lecture Notes in Mathematics 859, VIII+326 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1981.

In 1979—80 the Mathematics Department of the University of Connecticut spensored a special year devoted to Mathematical Logic with emphasis on recursion theory and model theory. During this year a conference took place, November 11—13, 1979, with 80 participants. The papers in this volume have been based on talks presented at the conference or on seminar presentations held during the course of the year.

The majority of the 21 papers in this volume is devoted to various problems of degrees and hierarchy of recursivity which seems to be one of the blossoming branches of mathematical logic. Authors and titles from this topic: R. L. Epstein, R. Haas and R. L. Kramer: Hierarchies of sets of degrees below $0'$; P. A. Fejer and R. I. Soare: The plus-cupping theorem for the recursively enumerable degrees; S. D. Friedman: Natural α-RE degrees; C. G. Jockush: Three easy constructions of recursively enumerable sets; P. G. Kolaitis: Model theoretical characterizations in generalized recursion theory; M. Lerman: On recursive linear orderings; A. Macintyre: The complexity of types in field theory; D. P. Miller: High recursively enumerable degrees and the anti-cupping property; Y. N. Moschovakis: On the Grilliot—Harrington—MacQueen theorem; R. A. Shore: The degrees of unsolvability: global results. Without striving for completeness let us mention two further authors. M. Makkai writes about a construction that associates a certain new topos, the *prime completion*, with any coherent topos. T. Millar gives a necessary and sufficient condition for a universal theory to have a complete, decidable model completion and applies this result to an example concerning recursively saturated models.

V. Totik (Szeged)

Robert Lutz and Michel Goze, Nonstandard Analysis. A Practical Guide with Applications, Lecture Notes in Mathematics 881, XIV+261, Springer-Verlag, Berlin—Heidelberg—New York, 1981.

"This book is intended to enable the reader to use Non Standard Analysis by himself without fear, at any level of mathematical practice, from undergraduate analysis to important research areas." The necessity of an introductory work with this scope is obvious: if Nonstandard Analysis wants to be a useful tool in proving theorems or in applications then it must not assume the user to be familiar with all the model theory necessary for its rigorous foundation. As a matter of fact engineers, physicists etc. have been constantly using infinitesimals even before — remember e.g. the tricks which were applied during many university lectures on theoretical physics — nevertheless an "easy" "how to do" treatment would attract many mathematicians, since for most of them Nonstandard Analysis is rather mistery than part of mathematics. Unfortunately this book seems to have failed to accomphish its goals. While reading these notes a beginner would probably feel having got lost in the "swindles" (Lutz—Goze's terminology) of NSA. Instead of keeping a strict distinction between "real" and "extension" the authors quickly drop the stars of the transferred objects and after 10—15 pages the reader is completely ignorant of what may and may not be done in NSA. Detailed proofs and simple remarks concerning them would have helped much in understanding the material. Nevertheless Lutz and Goze's book may be a great help for those who are familiar with the elements of the nonstandard method but are unaware of the many possibilities it can grant in applications.

The lecture notes consist of four chapters. In chapter one the "elementary practice of Non-Standard Analysis" is introduced, many classical results are reviewed in the nonstandard framework. The second chapter is devoted to the logical foundations of NSA, however this introduction is far from being complete and hardly enlightens the mind of the confused inexperienced novice. The last two chapters have already the flavor of genuine applications. In Chapter III some classical topics such as integral curves of vector fields, compactness, holomorphic functions etc. are treated from a nonstandard point of view, while in the fourth chapter NSA methods are applied to perturbation problems in algebra and differential equations. Author index, glossary and the authors' good sense of humor help in reading the book.

V. Totik (Szeged)

George E. Martin, The Foundations of Geometry and the Non-Euclidean Plane, (Undergraduate Texts in Mathematics) XVI+509 pages, Springer-Verlag, New York—Heidelberg—Berlin, 1982.

The book is the second edition [originally published: New York: Intext Educational Publishers, 1975] of a text written for juniour, senior, and first-year graduate courses. After four introductory chapters the book starts with an axiomatic development of absolute geometry, which is a common ground between non-Euclidean and Euclidean geometries as it is independent of any assumption about parallel lines. Many models, including the Cartesian plane, are used to illustrate this system of axioms, and it is shown that this system together with one of the equivalents of Euclid's parallel postulate forms a categorical system. Part two is a very elegant development of the Bolyai-Lobachevsky geometry using many results of this theory for the study of euclidean geometry.

The text is self-contained and it is written in a very clear, enjoyable style. Beside historical materials it containes over 650 exercises, 30 of which are true-or-false questions.

Z. I. Szabó (Szeged)

J. Martinet, Singularities of Smooth Functions and Maps, (London Mathematical Society Lecture Note Series 58), XIV+256 pages, Cambridge University Press, Cambridge—London—New York—New Rochelle—Melbourne—Sidney, 1982.

The book consist of seventeen chapters which are ordered into four parts. The material of the book is based on a seminar held at the University of Michigan and a course at the Pontificia Universidade Catolica (Rio de Janeiro). The text gives a very good and significant choice from the rich subject covering the most important results and problems of the singularity theory of differentiable functions. Part one introduces the main idea by means of detailed exposition of numerous examples. Part two is devoted to the differentiable preparation theorem. In part three the preparation theorem is applied to the theory of universal deformations of function germs, by the aid of which the classification of Thom's "elementary catastrophes" is presented. Part four deals with singularities of differentiable mappings. In this part most of Mather's results are stated in their local version. For understanding the text familiarity in the basic ideas about Lie groups, modules over commutative rings and existence and uniqueness theorems for solutions of differential equations are needed.

László Gehér (Szeged)

Martingale Theory in Harmonic Analysis and Banach Spaces (Proceedings, Cleveland, Ohio, 1981), Edited by J.-A. Chao and W. A. Woyczyński (Lecture Notes in Mathematics, 939), VIII+225 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1982.

The conference indicated in the title was held at Cleveland State University between July 13—17, 1981. Professor D. L. Burkholder was the principal speaker at the meeting and delivered a series of ten lectures. His lecture notes will appear separately in the NSF—CBMS Conference Series of the American Mathematical Society. The present volume contains papers submitted by other conference participants.

The table of contents: 1. D. Allinger: A note on strong, non-anticipating solutions for stochastic differential equations: when is path-wise uniqueness necessary? 2. K. Bichteler and D. Fonken: A simple version of the Malliavin calculus in dimension one. — Both papers are devoted to the study of stochastic differential equations. 3. H. Byczkowska and A. Hulanicki: On the support of the measures in a semigroup of probability measures on a locally compact group. 4. J.-A. Chao: Hardy spaces on regular martingales. — This paper mainly treats the generalized Walsh—Fourier series. 5. B. Davis and J. L. Lewis: The harmonic measure of porous membranes in $\mathbf{R}^3$. 6. G. A. Edgar, A. Millet and L. Sucheston: On compactness and optimality of stopping lines. — This is a survey type paper containing results both to discrete and continuous parameter processes. 7. N. A. Ghoussoub: Martingales of increasing functions. 8. J. A. Guttierez and H. E. Lacey: On the Hilbert transform for Banach space valued functions. — The authors extend some results of C. Fefferman and E. M. Stein, and G. Pisier. 9. A. T. Ławniczak: Gaussian measures on Orlicz spaces and abstract Wiener spaces. 10. C. Mueller: Exit times of diffusions. 11. C.W. Onneweer: Generalized Lipschitz spaces and Herz spaces on certain totally disconnected groups. — The absolute convergence of Fourier series of functions belonging to a generalized Lipschitz (=Besov) space and embedding theorems for Herz — and Lorentz spaces are studied. 12. C. Park: Stochastic barriers for the Wiener process and a mathematical model. 13. G. Pisier: On the duality between type and cotype. — Those X Banach spaces are studied, for which X is of type p iff X^* is of cotype p' with $1/p+1/p'=1$. 14. L. H. Riddle and J. J. Uhl: Martingales and the fine line between Asplund spaces and spaces not containing a copy of l_1. — The following theorem of Rosenthal is the starting point: A Banach space X contains no copy of l_1 iff every bounded sequence in X has a weakly Cauchy subsequence. 15. J. Rosiński: Central limit theorems for dependent random vectors in Banach spaces. — This is a relatively large survey paper. 16. J. Rosiński and J. Szulga: Product random measures and double stochastic integrals. 17. W. H. Ruckle: Absolutely divergent series and Banach operator ideals. 18. G. Schechtman: Lévy type inequality for a class of finite metric spaces. — This short note is a variation on the theme of B. Maurey, but the proof is somewhat simpler and more general. 19. W. A. Woyczyński: Asymptotic behavior of martingales in Banach spaces II. — The pesent note is a continuation of a work by the same author, and concentrates on the Marczinkiewicz—Zygmund and Brunk's type strong laws of large numbers for martingales.

The book gives a good account of the present stage of the subject. It will certainly stimulate some of the readers to make research in this interesting field. We warmly recommend it to everybody who works either in Martingale Theory and/or in Abstract Harmonic Analysis.

F. Móricz (Szeged)

William S. Massey, Singular Homology Theory (Graduate Texts in Mathematics, Vol. 70) XII+265 pages, Springer-Verlag New York—Heidelberg—Berlin, 1980.

This book gives a systematic treatment of singular homology and cohomology theory. The author has tried to show all the standard results without unnecessary technical details and difficulties as long as it is possible. His program has been crowned with success.

Clear geometric motivation is given in and out of the first chapter devoted to the background for homology theory. Singular cubes are used rather than singular simplexes. It simplifies the proof of the invariance of the induced homomorphisms under homotopies since the product of a cube with the unit interval is a cube. Furthermore, the subdivision of a cube is simpler than the barycentric subdivision of a simplex. An appendix contains De Rham's theorem.

Massey considers his book as a sequel to his previous book in this Springer-Verlag series (Vol. 56) entitled "Algebraic Topology: An Introduction". Although the book does not really require any knowledge given in the previous one, it seems to be a textbook for a second course in algebraic topology rather than a first course since many technical details are left to the reader.

Prerequisite mathematical knowledge is as follows: minima of general topolgy and the theory of abelian groups, something about manifolds and fundamental groups, tensor product, Tor and Ext functors.

A last argument to prove that the author is right in his program is that his book is essentially shorter than other ones treating the same subject.

L. A. Székely (Szeged)

Richard S. Millman and George D. Parker, Geometry. A metric Approach with Models (Undergraduate Texts in Mathematics) VIII+355 pages, Springer-Verlag, New York—Heidelberg—Berlin 1981.

Birkhoff's metric approach to classical geometries means the use of real numbers at the building of several theories. The book develops the theory of neutral (absolute) geometry, hyperbolic geometry and of Euclidean geometry by this method. The various axioms are introduced slowly and the definitions and theorems with models, ranging from the Cartesian plane to the Poincaré upper half plane, the Taxicab plane and the Moulton plane, illustrate further these axioms. The last two chapters develop the concept of area resp. the theory of isometries in neutral geometry. Bolyai's beautiful theorem, asserting that if two poligonal regions have the same area then one can be cut up into a finite number of pieces and reassembled to form the other, is also proved here.

The book contains more than 700 problems in the exercise sets. It is an excellent introduction. It is addressed to undergraduate students and is warmly recommended to everyone who wants to make a quick acquaintance with classical geometries.

Z. I. Szabó (Szeged)

P. G. Moore, Principles of Statistical Techniques. Second Edition, VIII+288 pages, Cambridge, University Press, Cambridge—London—New York—Melbourne, 1979.

The first edition of this book was published in 1958, and it was reprinted in 1964. The second edition appeared in 1969, it was reprinted in 1974, and the nice pocket-size version under review is the first paperback edition. Such a story reflects a considerable success, and the book appears deserving it. Its longer subtitle describes it rather completely: "A first course, from the beginnings, for schools and universities, with many examples and solutions". It does not really require any mathematical prerequisites. Anybody graduating from a secondary school could, or should understand it. Nevertheless, the author provides a rather wide selection of effective tools of statistics so the the reader can tackle a whole variaty of concrete situations. The basic techniques of collection, tabulation and pictorial representation of data, of sampling and averaging; dispersion measures, tests of significance and time series are all explained through numerical examples.

The style is very nice. It sometimes represents an old world. The reviewer, for example, would find it difficult to ask his students to "catch a large number of specimens of a common species of butterfly and measure the length of the right wing of the butterflies. Do this on a number of occasions over the season...". However, not all the data is required to collect from the reader. The author has many interesting data sets of his own to work with. An earlier reviewer, cited on the cover, was right to write that "the book should prove useful to all who read it".

Sándor Csörgő (Szeged)

Jacob Palis, Jr. and Welington de Melo, Geometric Theory of Dynamical Systems. An Introduction, XII+198 pages, Springer-Verlag, New York—Heidelberg—Berlin, 1982.

This book is concerned with differential equations on manifolds from a global or topological point of view. Its purpose is to aquaint the reader with two central topics of the modern period of the geometric theory of dynamical systems: structural stability and genericity.

If a differential equation describes the evolution of a system, then, obviously, it cannot be supposed to be an absolutely correct model, e.g. the parameters of the system appearing in the differential equation cannot be given exactly. However, the user wishes to ensure the qualitative conclusion he draws from the equation at hand to be valid for the equation really describing his world. Probably this inspired Andronov and Pontrjagin (first of them was an engineer!) to introduce the concept of structural stability in 1937. Roughly speaking, they called a differential equation structurally stable if the differential equations near to it in a suitable metric on the space of all differential equations have the same phase portrait. 20 years later M. Peixoto proved that structurally stable differential equations form an open and dense set in the space of differential equations whose right-hand sides are defined on a compact 2-dimensional manifold, i.e. here almost all differential equations are structurally stable.

A property is said to be generic if it is satisfied by almost all differential equations. As it was defined by S. Smale, the main objective in the geometric theory of differential equations is the search for generic and stable properties.

The book gives the reader the flavour of this theory on an introductory level. The first chapter establishes the concepts and basic facts on differentiable manifolds and vector fields needed for understanding later chapters. The second chapter gives a systematic proof of the Hartman—Grobman Theorem, which says that local stability is a generic property. The same problem for periodic orbits is considered by the Kupka—Smale Theorem in Chapter 3. The last chapter is devoted to the proof of Peixoto's Theorem. There are a great number of interesting exercises of various difficulty.

We recommend this excellent book for mathematicians and students who want to get aquainted with this modern and fast developing branch of mathematics.

L. Hatvani (Szeged)

Steve Smale, The Mathematics of Time (Essyas on Dynamical Systems, Economic Processes, and Related Topics), VI+151 pages, Springer-Verlag, New York—Heidelberg—Berlin, 1980.

In the preceding review on Palis and Melo's book we tried to sketch what is geometrical theory and global analysis of dynamical systems. Undoubtedly, one of the most important people of this theory is Stephen Smale. For his outstanding research in differential topology and in global analysis he was awarded the Fields Medal of the International Mathematical Union in 1966.

The first piece in this collection of his earlier papers and addresses is his celebrated paper on "Differentiable dynamical systems" published originally in *Bulletin of the American Mathematical Society* in 1967. The Notes following the body of its new edition are of the greatest interest, where the author completes his "classical" work with up-to-date results and gives a riport on the history of the problems and conjectures he proposed in his original paper. In the second half of the book one can find some expository essays and addresses: What is global analysis?; Stability and genericity in dynamical systems; Personal perspectives on mathematics and mechanics; Dynamics in general equilibrium; Some dynamical questions in mathematical economics; On the problem of reviving the ergodic hypotheses of Boltzmann and Birkhoff. The book is concluded by personal confessions: "On how I got started in dynamical systems". The reader can get acquainted with such "intimacies" from the author's life as how he, as a topologist, entered into the mathematical world of ordinary differential equations; how "extraordinarily" he was impressed to meet the "powerful group of four young mathematicians: Anosov, Arnold, Novikov and Sinai in Moscow".

This nice book is of interest not only to topologist and global analysists, but also to those whose primary fields are applied mathematics, differential equations, physics, or mathematical economics.

L. Hatvani (Szeged)

Sudhakar G. Pandit—Sadashiv G. Deo, Differential Systems Involving Impulses (Lecture Notes in Mathematics, 954), VII+102 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1982.

In the classical analysis of differential systems the right-hand side is assumed to be continuous or integrable, so the solutions are continuous functions. However, in many physical problems the right-hand side of the modelling differential equation involves some perturbations of discontinuous behaviour. For example, the bang-bang principle in the optimal control theory shows that the parameters often have to change in an impulsive manner. Biological systems (heart beats, models for biological neural nets) exhibit an impulsive behaviour, too. These systems are described by so called "measure differential equations". The derivative involved in these equations is the distributional derivative, the solutions are functions of bounded variations. Consequently, the methods of classical analysis are not sufficient to describe the impulsive behaviour of systems.

These notes give a good unified survey on the results from several research papers published during the last fifteen years dealing with the basic problems such as existence, uniqueness, stability, boundedness and asymptotic equivalence associated with measure differential equations.

L. Hatvani (Szeged)

Probability Measures on Groups. Proceedings, Oberwolfach, Germany, 1981. Edited by H. Heyer (Lecture Notes in Mathematics, 928), X+477 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1982.

This collection contains the text of 22 lectures presented at the Sixth Conference in the series "Probability Measures on Groups" held at the Mathematisches Forschunginstitut, Oberwolfach, Germany, June 28—July 4, 1981. The subjects of this meeting cover various areas of stochastics and analysis including probability theory and potential theory on algebraic-topological structures as well as their interrelations with the structure theory of locally compact groups, Banach spaces and Banach lattices. The Editor of this volume classified the papers into four groups: (i) Probability measures on groups, semigroups and hypergroups, (ii) Stochastic processes with values in groups, (iii) Connections between probability theory on groups and abstract harmonic analysis, (iv) Applications of probability theory on algebraic-topological structures to quantum physics.

Lajos Horváth (Szeged)

Processes Aléatoires à Deux Indices (Colloque E. N. S. T. et C. N. E. T., Paris 1980), Edité par H. Korezlioglu, G. Mazziotto et J. Szpirglas (Lecture Notes in Mathematics, 863), IV+274 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1981.

This volume contains the texts of the talks made at the Conference "Two-parameter Stochastic Processes" held in Paris on June 30 and July 1, 1980, under the support of "l'École Nationale Supérieure des Télécommunications" (E.N.S.T.) and "Centre National d'Études des Télécommunications" (C.N.E.T.).

The table of contents: 1. P. A. Meyer; Théorie élémentaire des processus à deux indices. — This is a nice introduction into the subject, embracing the main notions and results up to the decomposition theorems of martingales with indices in $\mathbf{R}_+\times\mathbf{R}_+$ (continuous case) and $\mathbf{N}\times\mathbf{N}$ (discrete case) and the stochastic integrals. 2. D. Bakry: Limites "quadrantales" des martingales. — This talk closely attaches to the previous one by Meyer. 3. A. Millet: Convergence and regularity of stong submartingales. 4. G. Mazziotto, E. Merzbach et J. Szpirglas: Discontinuités des processus croissants et martingales à variation intégrable. 5. G. Mazziotto et J. Szpirglas: Sur les discontinuités d'un processus cad-lag à deux indices. 6. J. Brossard: Régularité des martingales à deux indices et inégalités de normes. — This is a good summarization of the methods how to obtain moment inequalities for the maximum partial sum and the martingale square function. 7. M. Ledoux: Inégalités de Burkholder pour martingales indexées par $\mathbf{N}\times\mathbf{N}$. 8. D. Nualart: Martingales à variation indépendante du chemin. 9. M. Zakai: Some remarks on integration with respect to weak martingales. — This gives interesting contributions to stochastic integration in the plane. 10. M. Dozzi: On the decomposition and integration of two-parameter stochastic processes. — While the previous talk treats weak martingales, this one does strong martingales. 11. J. B. Walsh: Optimal increasing paths. — Among other things, the author proves some Fatou type theorems concerning fine and nontangential limits of biharmonic functions at the distinguished boundary of a bicylinder. 12. D. Nualart and M. Sanz: The conditional independence property in filtrations associated to stopping lines. 13. X. Guyon et B. Prum: Identification et estimation de semi-martingales représentables par rapport à un Brownien à un indice double. 14. A. Al-Hussaini and R. J. Elliott: Stochastic calculus for a two parameter jump process. — The authors obtain some new formulae, which cannot be written as special cases of those for the two parameter Wiener or Poisson processes. 15. H. Korezlioglu, P. Lefort et G. Mazziotto: Une propriété markovienne et diffusions associées.

The book collects together materials that have been widely scattered in the literature, and is likely to be of special interest to those who work on the field of Stochastic Processes endowed with a partially ordered index set.

F. Móricz (Szeged)

Elmer G. Rees, Notes on Geometry (Universitext), VIII+109 pages with 99 figures, Springer-Verlgag, New York—Heidelberg—Berlin 1983.

There are several ways to introduce the classical geometries into university syllabuses. The most general method is the so-called axiomatic method which is in some cases rather cumbersome and not very informative. The present book shows how to give an introduction to geometries that is short and nevertheless is of rich content, taking a concrete viewpoint rather than an axiomatic one.

In the first part the Euclidean geometry is considered with a detailed examination of isometries and crystals. Projective geometry (Part II) and hyperbolic geometry (Part III) are treated from the point of view of Felix Klein's Erlanger Programme, supplemented with some topological aspects.

There is a large number of exercises throughout the notes, many of these are staightforward and are meant to test the reader's understanding.

The book is recommended to undergraduate studens and to teachers of elementary geometry.

Z. I. Szabó (Szeged)

G. F. Roach, Green's Functions, XIV+325 pages, Cambridge University Press, Cambridge—London—New York—New Rochelle—Melbourne—Sidney, 1982.

The first edition of this book was published in 1970. The author's aim was to give a self-contained and systematic introduction to the theory of Green's functions. The success of the book is shown by this new edition.

In my opinion the advantage of this work is that it gives for scientists a mathematically thoroughly developed tool to the investigation of boundary value problems associated with either ordinary or partial differential equations, and, at the same time, for postgraduate students a clear and well-motivated exposition of the problem showing also the necessity of the generalization of some notions (e.g. Riemann integral-Lebesgue integral).

Two new chapters (Calculations of particular Green's functions and approximate Green's functions) and four appendices (Summary of the Green's function method, Operators and expressions, The Lebesgue integral, Distributions) have been added in this edition. Especially the new appendices are very useful for those readers who lack the necessary mathematical background to understand more advanced accounts. (The other chapters are: The concept of a Green's function, Vector spaces and linear transformations, Systems of finite dimension, Continuous functions, Integral operators, Generalized Fourier series and complete vector spaces, Differential operators, Integral equations, Green's functions in higher-dimensional spaces.)

Summarizing, this is a well-written text giving the reader a picture how notions, proofs, and applications arise in this field.

L. Pintér (Szeged)

Derek J. S. Robinson, A Course in the Theory of Groups, Graduate Texts in Mathematics 80, XVIII+481 pages, Springer-Verlag, New York—Heidelberg—Berlin, 1982.

This book is an excellent up-to-date introduction to the theory of groups. It is general yet comprehensive, covering various branches of group theory. The fifteen chapters contain the following main topics: free groups and presentations, free products, decompositions, Abelian groups, finite permutation groups (including the Mathieu groups), representations of groups, finite and infinite soluble groups, group extensions, generalizations of nilpotent and soluble groups, finiteness properties.

The reader is expected to have at least the knowledge and maturity of a graduate student who has completed the first year of study at a North American university or of a first year research student in the U.K. He or she should be familiar with the more elementary facts about rings, fields and modules, possess a sound knowledge of linear algebra and be able to use Zorn's Lemma and transfinite induction. However, no knowledge of homological algebra is assumed. There are some 650 exercises, found at the end of each section. They must be regarded as an integral part of the text.

This book is highly recommended everybody who wants to read research texts in more specialized areas of groups theory.

Lajos Klukovits (Szeged)

Klaus Schittkowski, Nonlinear Programming Codes, (Lecture Notes in Economics and Mathematical Systems, 183) VIII+242 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1980.

In recent years a lot of effort has been made to implement efficient and reliable optimization programs for the solution of complex nonlinear systems. The author undertook collect many programs developed by various researchers, and so gives a designer the possibility to decide which optimization program could solve his problem in the most desirable way.

A reader who is interested in selecting a program for the numerical solution of his problem should start with Chapter I where the problem is formulated. Chapter II gives the mathematical background of the different methods. Chapter III is divided into two sections. The first one contains a table with different technical details (program length, the original precision of the program etc.). More detailed information is contained in the second section where all the programs are described individually. Chapter IV shows how test problems with predetermined solutions are generated. In Chapter V the reader finds different performance criteria (efficiency, global convergence, and the performance examinations). Final conclusions and some technical remarks are gathered in Chapter VI.

The two appendices contain the numerical data for constructing test problems and a sensitivity analysis.

G. Galambos (Szeged)

Laurent Schwartz, Geometry and Probability in Banach Spaces, Notes by Paul R. Chernoff (Lecture Notes in Mathematics, 852,) X+101 pages, Spinger Verlag, Berlin—Heidelberg—New York, 1981.

These Notes correspond to a course of lectures which was given by Prof. Laurent Schwartz at the University of California, Berkeley, in April-May 1978. It is Prof. Paul Chernoff who gives here a good account of these lectures.

The book summarizes a great number of new results, many of them found by mathematicians of the French school, in particular by Laurent Schwartz, Bernard Maurey, and Gilles Pisier. These results cover relationships between geometrical properties, properties of functional analysis, and probabilistic properties in Banach spaces. The present subject turns around the L^r spaces, $1\leqq r\leqq +\infty$.

The book contains 19 lectures arranged into four chapters and a new result of Pisier.

Ch. 1 gives a rapid account of the main ideas of the book, presents the Pietsch factorization theorem with applications, etc.

Ch. 2 is devoted to the study of cylindrical probabilities and radonifying maps, in particular, to P. Levy's p-stable laws, p-Pietsch spaces, the continuity and Hölder properties of the Brownian motion.

Ch. 3 is entitled by "Types and Cotypes". Let $\{\varepsilon_n\colon n\in\mathrm{N}\}$ be independent random variables, with values ± 1 with probability 1/2. A Banach space E is said to be of type p, $1\leqq p\leqq 2$, it $\sum_n |x_n|^p<\infty$ implies that $\sum_n \varepsilon_n x_n$ is almost surely convergent; of cotype q, $2\leqq q\leqq +\infty$, if the almost sure convergence of $\sum_n \varepsilon_n x_n$ implies $\sum_n |x_n| <+\infty$; where $x_n\in E$, $n\in\mathrm{N}$. It is remarkable that, for $1\leqq r\leqq 2$, L^r is of type r and cotype 2, and nothing better; for $2\leqq r<+\infty$, type 2 and cotype r, and nothing better; while L^∞ is very bad, it is of only type 1 and cotype $+\infty$.

Ch. 4 is the longest part of the book, mainly dealing with the questions in connection with ultrapowers and superproperties. Given a property P of Banach spaces, P is said to be a superproperty if two Banach spaces E and F are such that E has P and F is finitely representable in E, then F also has P. A number of interesting results are treated in this chapter: the Maurey and the Grothendieck factorization theorems, the nonexistence of $(2+\varepsilon)$-Pietsch spaces, the results of Pisier on martingale type and cotype, etc.

The presentation is rather tight. Some proofs are omitted, some are merely outlined. Practically there is no bibliography in the text. The "Séminaires de l'École Polytechnique" are indicated as general references.

This thin book is the first attempt to collect the main ideas of the new branch of mathematics which deals with the functional-analytic, geometric and probabilistic properties of Banach spaces. We warmly recommend it firstly to those who have some acquaintance with this heavy but fascinating subject.

F. Móricz (Szeged)

Zbigniew Semadeni, Schauder Bases in Banach Spaces of Continuous Functions, Lecture Notes in Mathematics 918, IV+136 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1982.

Zbigniew Semadeni is the author of a successful monograph "Banach Spaces of Continuous Functions". This fact and the booming interest in bases of Banach spaces guarantee the success of these lecture notes. To tell the truth, the author might have written a more complete, more attractive monograph by including the proofs of the most important "hard theorems" on bases in $C(X)$ (e.g. Karlin's theorem about the non-existence of unconditional bases in $C(0, 1]$ or Pelczynski's related theorem; Olveskii's result that a uniformly bounded orthonormal system cannot be a basis in $C[0, 1]$ etc.), but the book, as it stands, is a good introduction into the topic (concerning bases in more general Banach spaces we mention the books "Bases in Banach spaces I—II" by I. Singer (Springer) and "An Introduction to Nonharmonic Fourier Sieres" by R. M. Young (Academic Press)). Many references and a detailed bibliography help the interested reader to proceed on to finer topics. The style is that of a text-book and the book is more or less self-contained (it is a bit embrassing that the definition of "uniform cross-norm" and "tensor product" is not given). It contains several exercises, although these require mostly standard manipulations. The proofs are clearly presented but the nature of the material does not allow quick reading: it is not at all an easy task to catch up with a consideration about or construction of a pyramidal basis in higher dimension.

The lecture notes are divided into four chapters. The first one is a general introduction into the properties of bases in Banach spaces. Only the most important topics are treated: duality, stability and some properties of unconditional bases.

Chapter 2 contains the "classical part" of the book: the broken-line construction in one variable. The most frequently investigated systems of Haar, Faber—Schauder, Walsh and Franklin are introduced here. The definition of the Haar—system is somewhat misleading since the values at jump points are indifferent only if we consider the Haar functions as the elements of L^∞ and the author mentiones also the uniform convergence of the Haar expansion to a continuous function.

Chapter 3 is devoted to the multidimensional case. Everybody reading the previous chapter will feel that the same might be done in higher dimension but to write the construction down is another thing. Although this chapter is not very attractive, the author has good reasons for dealing with the higher dimensional case so lengthy: "For two (or perhaps even four) decades it has been known how to construct... bases, consisting of certain spline functions... In spite of the regularity of the construction of these bases and their nice properties, they have not yet attracted people working in numerical methods. A reason... may be that in the existing literature the descriptions... are geometrical, ..., without explicit formulas...". In Chapter 3 pictures help to follow the construction, and formulas are given for the coefficients etc. In the last paragraph the celebrated results of Ciesielski—Shoenfield and Bockariev concerning bases in $C^k[0, 1]^d$ and A are sketched.

The material in Chapter 4 is quite new. A detailed proof is given for the existence of monotone bases in separable spaces $C(X)$ (the solution of the basis problem in these spaces) and several interesting extensions of the mentioned Olevskii result are listed. It is a pity that the proofs of these last theorems are left out.

V. Totik (Szeged)

Set Theory and Model Theory, Proceedings, Bonn 1979, edited by R. B. Jensen and A. Prestel, Lecture Notes in Mathematics 872, IV+174 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1981.

On the occasion of Gisbert Hasenjaeger's 60-th birthday a symposium on set theory and model theory was held at Bonn, 1979 June 1—3. All of the contributors to these proceedings are former students and co-workers of Professor Hasenjaeger, and the papers are all dedicated to him. K. J. Devlin presents a new morass construction which leads to Souslin and Kurepa K_2-trees as limits of directed systems of countable trees. H. D. Donder shows how coarse morasses in L can be used to answer combinatorial questions in L, e.g. how Kurepa trees with additional properties can be obtained using the "natural" global coarse morass in L. S. Koppelberg reveals several properties of the partially ordered set of isomorphism type structures of complete Boolean algebras, such as their being distributive lattices with Stone and Heyting algebras as duals. A. Prestel introduces a suitable definition of pseudo real closed (prc)-fields and shows, among others, that with this definition every algebraic extension of a prc-field is again a prc-field. Finally, T. von der Twer simplifies Paris and Harrington' famous proof concerning the incompleteness of Peano's arithmetic by avoiding probabilities in PA.

V. Totik (Szeged)

Statistique non Parametrique Asymptotique, Proceedings, Rouen, France 1979. Edité par J. P. Raoult (Lecture Notes in Mathematics, 821), VIII+175 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1980.

This volume contains seven papers presented at the meeting "Journées Statistiques" Rouen, June 13—14, 1979. Three papers by Balascheff and Dupont, Harel, and Rüschendorf deal with the asymptotic behaviour of multivariate empirical processes. Using the weak convergence of the multivariate emprical process, Deheuvels presents nonparametric tests of independence. Adaptive rank tests and midrank statistics are considered in the papers of Albers and Ruymgaart. Collomb gives results on convergence in probability, with probability one, and in L_q, $1<q<\infty$, of the $k-NN$ estimator of a multivariate regression function.

Lajos Horváth (Szeged)

Ottó Steinfeld, Quasi-ideals in Rings and Semigroups, Disquisitiones Mathematicae Hungaricae 10., Akadémiai Kiadó, Budapest, 1978.

Quasi-ideals were introduced by the author of the book in 1956. This is the first monograph in the field, and it gives a fairly complete discussion of the results attained in these two decades. The book is completely self-contained — perhaps even too much so, as it gives definitions of literally all notions and rather meticulous proofs. It is very clearly written and well readable.

The first four chapters contain basic notions, examples and bits of general ring and semigroup theory used throughout the book. In §§ 5—7 the basic facts concerning minimal quasi-ideals of rings and minimal and 0-minimal ideals of semigroups are given, with a particular stress (in § 7) for semiprime rings and semigroups. § 8 deals with decomposition theorems for some classes of semiprime rings. An interesting result here is Theorem 8.1 which provides different decompositions for semiprime rings satisfying the minimum condition for principal left ideals. The next paragraph throws light on the behaviour of quasi-ideals in regular rings and semigroups, and § 10 contains analoga of the results of § 8 for regular semigroups (as for semiprime semigroups they don't hold in general). The last two chapters of the main part deal with the characterization of regular duo elements, rings and semigroups, and with different ways of generalization.

There is an Appendix on quasi-absorbents in so-called groupoid-lattices. This notion, too, was introduced by the author in 1970. It seems to be a good tool for finding the common roots of some properties of rings, groups and semigroups, and is far from being completely exhausted. Its possibilities are shown in this Appendix by an abstract version of Theorem 8.1 and its analoga.

There are over 20 problems in the text, collected also in a list at the end of the book. An important role in the book is played by examples, in particular counter-examples showing the limits of parallelism between rings and semigroups.

G. Pollák (Szeged)

Stochastic Integrals, Proceedings, LMS Durham Symposium, 1980. Edited by D. Williams (Lecture Notes in Mathematics, 851), IX+540 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1981.

The volume is devided into three parts. The first part contains three introductory articles to help make some of the later material accessible to a wider audience. Williams gives a self-contained introduction to some important concepts such as continuous martingales and the associated martingale representation, the Stroock—Varadhan theorem and its consequences for martingale representation, the Girsanov theorem. The main theme of this survey is the modern theory of the Kolmogorov forward (or Fokker—Planck) equation. Roger's paper provides a brief summary of the construction and the fundamental properties of stochastic integrals. Various kinds of integration are described by Elliott.

Longer research and survey papers are in the second part of the book. These 13 surveys, written by excellent probabilists, cover a wide part of stochastics. We mention only the following topics: Markov processes in quantum theory, Malliavin calculus, set-parametered martingales, Bessel processes and infinitely divisible laws, probability functionals of diffusion processes. The book ends with five shorter papers presented at the London Mathematical Society Durham Symposium.

Lajos Horváth (Szeged)

Árpád Szabó, The Beginings of Greek Mathematics, 358 pages, Akadémiai Kiadó Budapest, 1978.

This is the English edition of the German original, published by the same publishing house in 1969. The carefully written book is not intended to be an introduction to Greek mathematics (for this purpose the reader can consult the book of van der Waerden, Science Awakening). Its aim is to bring the problems associated with the early history of deductive science to the attention of classical scholars, and historians and philosophers of science.

The method used undoubtedly distinguishes this book from most of its prodecessors. It is based on a very careful investigation of original texts (the author is a classical philologist). Using this the author reconstructs the history of the early Greek mathematics, the origin of the axiomatic method.

The axiomatic method must have been existed before Euclid, but previous historians credited it to Aristotle or Plato. The author's idea is that the founders of this method were the Pythagoreans under the influence of the Eleatic philosophy.

In the first two parts of the book we can read the early history of the theory of irrationals and the pre-Euklidean theory of proportions. Part 3, the main part of the book, deals with the construction of mathematics within a deductive framework. The appendix "How the Pythagoreans discovered Proposition II. 5 of the Elements" serves to illustrate the kind of research which needs to be undertaken if we are to acquire a new understanding of the historical development of Greek mathematics.

Lajos Klukovits (Szeged)

Allen Tannenbaum, Invariance and System Theory: Algebric and Geometric Aspects (Lecture Notes in Mathematics, 845), X+161 pages, Springer-Verlag, Berlin-Heidelberg—New York, 1981.

These lecture notes are based on a series of lectures given by the author at the Mathematical System Theory Institute of the ETH, Zürich in 1980. The autor's purpose is to draw the attention of theoretical mathematicians and convince them that there are, in system theory, some interesting and deep problems from pure mathematics to be solved, and to introduce people working in system theory to the ideas of algebraic geometry, differential geometry, algebraic topology and invariant theory.

In the first three parts the author gives a good survey on some topics of algebraic geometry, system theory and invariant theory. Parts IV and V are devoted to the global and local moduli of linear time-invariant dynamical systems, respectively.

In Part VI the "system realization problem" is discussed which concerns the construction of a state space model of a system from its input/output behaviour. Part VII is concerned with the geometry of rational transfer functions. Finally, in Part VIII the stabilization through feedback is treated.

We can recommend these lecture notes to both theoretical mathematicians and system theory people interested in theoretical approaches in system theory.

L. Hatvani (Szeged)

The Correspondence Between A. A. Markov and A. A. Chuprov on the Theory of Probability and Mathematical Statistics. Edited by Kh. O. Ondar, Translated from the Russian by Charles and Margaret Stein, XVII+181 pages, Springer-Verlag, New York—Heidelberg—Berlin, 1981.

"I note with astonishment that in the book of A. A. Chuprov, Essays on the Theory of Statistics, on page 195, P. A. Nekrasov, whose work in recent years represents an abouse of mathematics, is mentioned next to Chebyshev. A. Markov". The story begins with this postcard, of 2 November 1910, from Markov (1856—1922) to Chuprov (1874—1926), and this of course settles the tone of the lively correspondence that followed in the next seven years between the two of them. We all know that Markov is an outstanding figure in the history of mathematics in general, and of the theory of probability and mathematical statistics in particular. Chuprov, not to be measured to Markov, was a fairly good statistician of his time under the influence of Lexis, Bortkiewicz, Quetelet and the emerging school of Pearson from one side, and of the inheritance of the Russian school of probability from the other. The latter, owing especially to Chebishev, Liapunov and Markov, was way ahead of the West at the time. The topics of the exchanges range wide: Lexis's coefficient of dispersion, the notorious "law of small numbers" of Bortkiewicz, the law of large numbers (including, at the time, what is nowadays the central limit theorem), Slutsky's work, Pearson's curves, the expectation of a ratio of dependent variables, Markov's linguistic statistics, etc.

Markov, known as "Neistovyi Andrei" to his contemporaries, throws himself vehemently into all the issues raised. In a firm consciousness of his authority he is not afaid to go into rather technical computations feverishly (on the three days 18—20 November 1910 he mailed not less then ten letters and postcards to Chuprov; they both lived in Petersburg), and to comment quite coarsely on what he feels a poor work. This is in accord with the contemporary discription above, which, in Jerzy Neyman's translation, is "Andrew the irrepressible, who does not pull any punches". Chuprov's role is seen, by the reviewer, as that of a prudent stimulator. We must be grateful to him that he was able to bring this out of Markov. Many of Chuprov's letters have not been found. The book contains 25 letters from him and 80 letters or postcards from Markov.

The idea of the translation has been put forward by the late Jerzy Neyman who wrote a charming introduction to the book. This is followed by the editor's preface outlining the life and work of Markov and Chuprov. Following the letters we find the editor's explanatory review of the correspondence. The first two of the four appendices are Markov's (negative) review of Chuprov's book and Chuprov's (positive) review of a posthumus edition of Markov's book on probability. The book is ended by Markov's and Chuprov's addresses at the bicentenary celebration of Bernoulli's law of large numbers in 1913, held by the Russian Academy upon the initiative of Markov. (This celebration must have been unique in the whole world.) The correspondence is really very exciting, one can hardly put the book down before finishing. The translation is excellent. I recommend reading it to every probabilist and statistician. It is a pity that I can probably never learn Neistovyi Andrei's limerick, mentioned in Professor Neyman's introduction, "not suited for the ears of ladies".

Sándor Csörgő (Szeged)

The Geometric Vein. The Coxeter Festschrift, edited by Chandler Davis, Branko Grünbaum and F. A. Sherk, VIII+598 pages with 5 color plates, 6 halftones and 211 line illustrations, Springer-Verlag, New York—Heidelberg—Berlin 1981.

H. S. M. Coxeter is one of the most inspiring geometers in the present century. Close to a hundred mathematicians from eight countries gathered on the Coxeter Symposium (held at the University of Toronto, 21—25 May 1979) testifying the deep influence of Coxeter's works in several fields of geometry such as the theory of polytopes and honeycombs, geometric transformations, groups and presentations of groups, extremal problems and combinatorial geometry.

The Geometric Vein is the collection of the lectures given at this Symposium, containing altogether 41 papers. Thus it would be impossible to give a detailed survey in this review. The reader can read papers among others by J. H. Conway, E. Ellers, G. Ewald, L. Fejes Tóth, B. Grünbaum, W. Kantor, P. McMullen, C. A. Rogers, B. A. Rosenfeld, J. J. Seidel, G. C. Shephard, J. Tits, W. T. Tutte, I. M. Yaglom.

The book is arranged very carefully and the papers are written in a brilliant style. Most of the papers are understandable also for the undergraduate studens. So they are warmly recommended to everyone who wants an insight into a very geometric geometry.

Z. I. Szabó (Szeged)

Arthur T. Winfree, The Geometry of Biological Time (Biomathematics, Volume 8), XIV+530 pages, 290 illustrations, Springer-Verlag, New York—Heidelberg—Berlin, 1980.

From opening and closing of flowers to heartbeat, from cell division to pupal eclosion of insects, from pattern formation of mashrooms to the migration of fishes, from bird navigation to the female cycle, from spatial wave organisation of catalytic oxidation to sleeping, from pacemaker neurons in the optic nerve of crayfish and in the brain of various flies to mating habits, and in fact in the whole life outside and inside us, we encounter all kinds of periodic patterns with circadian, seasonal and various other rythmicities, exhibited collectively or individually, and with various organising phase singularities. The encyclopaedic masterpiece under review visualizes Nature for us as mutually synchronized communities of chemical, physical and, as a main topic, biological clocks. The first ten chapters [Circular logic — Phase singularities (Screwy results of circular logic) — The rules of the ring— Ring popolations — Getting off the ring — Attracting cycles and isochrons — Measuring trajectories of a circadian clock — Populations of attractor cycle oscillators — Excitable kinetics and excitable media — The varieties of phaseless experience; in which the geometrical orderliness of rhythmic organization breaks down in diverse ways] constitute the more theoretical first part of the book, with elementary topological facts and little catastrophe theory, and a few differential equations here and there. Fortunately, Life is not raped by mathematics at all. Only the flavour is mathematical and the reviewer, being very far from what he thinks to be a biologist, has the feeling that this is modern biology of the highest quality. The second half of the book, the author calls it the Bestiary, with 13 chapters [The firefly machine — Energy metabolism in cells — The malonic acid reagent ("Sodium Geometrate") — Electrical rhythmicity and excitability in cell membranes — The aggregation of slime mould amoebae — Growth and regeneration – Arthropod cuticle — Pattern formation in the fungi — Circadian rythms in general — The circadian clooks of insect eclosion — The flower of Kalanchoe — The cell mitotic cycle — The female cycle], is a collection of extraordinary wealth of particular experimental systems of living organisms about which the first part theorises.

It is almost unbelievable that such a book could have been written. No doubt, its intrinsic theoretical value, its wealth, the fantastically easy-flowing style, and its flexible, broad and open view will make it a classic. The bibliography contains more than 1300 items, only the author index fills 14 pages. Some 290 illustrations, sometimes really beautiful, invite the attention of the reader, but the book is also very cheap (US $ 33). Biologists, physiologists, physicists, chemists and perhaps also applied mathematicians will find it a good and rewarding reading. Why then advertising it in these *Acta*, for readers almost exclusively in pure mathematics? The point is the recreational. Many of us, exhausted by daily abstraction or sophisticated calculation, would still like to have something slightly mathemtical but "real" around in the evening. This book is an ideal choice for such a purpose. It is partly dedicated "to those readers who, expecting wonders to follow so grand a title as it flaunts, may feel cheated by its actual content". I expected nothing to follow so grand a title as it flaunts, and I found wonders.

Sándor Csörgő (Szeged)

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