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Composite control Lyapunov functions for robust stabilization of constrained uncertain dynamical systems

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Nonlinear is better than linear

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B.1 For linear systems of dimension \( n = 2 \), two control Lyapunov functions necessarily share a common control law.
Abstract

This work presents innovative scientific results on the robust stabilization of constrained uncertain dynamical systems via Lyapunov-based state feedback control.

Given two control Lyapunov functions, a novel class of smooth composite control Lyapunov functions is presented. This class, which is based on the R-functions theory, is universal for the stabilizability of linear differential inclusions and has the following property. Once a desired controlled invariant set is fixed, the shape of the inner level sets can be made arbitrary close to any given ones, in a smooth and non-homothetic way. This procedure is an example of “merging” two control Lyapunov functions.

In general, a merging function consists in a control Lyapunov function whose gradient is a continuous combination of the gradients of the two parents control Lyapunov functions. The problem of merging two control Lyapunov functions, for instance a global control Lyapunov function with a large controlled domain of attraction and a local one with a guaranteed local performance, is considered important for several control applications. The main reason is that when simultaneously concerning constraints, robustness and opti-
mality, a single Lyapunov function is usually suitable for just one of these goals, but ineffective for the others.

For nonlinear control-affine systems, both equations and inclusions, some equivalence properties are shown between the control-sharing property, namely the existence of a single control law which makes simultaneously negative the Lyapunov derivatives of the two given control Lyapunov functions, and the existence of merging control Lyapunov functions.

Even for linear systems, the control-sharing property does not always hold, with the remarkable exception of planar systems.

For the class of linear differential inclusions, linear programs and linear matrix inequalities conditions are given for the the control-sharing property to hold.

The proposed Lyapunov-based control laws are illustrated and simulated on benchmark case studies, with positive numerical results.
Introduction

Control design must quite often compromise among performance, robustness and constraints, and Lyapunov theory offers suitable tools in this regard. The essential goals of constrained robust control design are assuring stability, fulfilling constraints and facing uncertainties.

Literature review

Lyapunov-based techniques for constrained robust control trace back to the '70s [74].

The solutions originally proposed where based on quadratic Lyapunov functions [42] and linear, possibly saturated, controllers. However it became immediately clear that quadratic functions are quite conservative in terms of both domain of attraction [41, 49] and robustness margin [21].

Solutions based on non-quadratic Lyapunov functions have been suggested for constrained control, initially based on the polyhedral ones [41, 49] or smoothed-polyhedral functions [23, 24]. An intensive research activity has then been devoted in discovering suitable
classes of Lyapunov functions, including composite-quadratic Lyapunov functions [47, 44, 45], truncated-quadratic functions [58, 79] and polynomial homogeneous functions [29, 30]. Surveys can be found in [19, 46, 26].

It turns out that polyhedral, smoothed-polyhedral and composite-quadratic control Lyapunov functions (namely, convex hull and max of quadratics) are universal classes for uncertain linear systems, i.e. Lyapunov stabilizability of an uncertain linear system is equivalent to the existence of a control Lyapunov function in these classes [18, 24, 47, 45]. We notice that all these classes consists of homothetic functions.

Among the mentioned classes of functions, only the smoothed-polyhedral functions and the convex hull of quadratics are smooth functions. Therefore only these latter functions can be employed to derive an explicit, continuous, stabilizing control law [24, 48]. However, the fact that smoothed-polyhedral functions and the convex hull of quadratics are homothetic functions implies that, in general, any explicit, continuous, stabilizing control law associated with them is conservative with respect to typical requirements of “good” closed-loop performance. In other words, both smoothed-polyhedral functions and the convex hull of quadratics have been introduced in the literature in order to achieve a large controlled domain of attraction with a continuous control law, but independently to “optimizing” closed-loop performances. This is one substantial reason for considering important the design of composite control Lyapunov functions.
INTRODUCTION

Composite control Lyapunov functions

There is a fundamental issue in the Lyapunov-based approach for control in which constraints, robustness and optimality are of concern: it turns out that a single Lyapunov function is typically suitable for one of these goals, but often ineffective for the others. For instance the size of the “safe set”, namely the domain of initial conditions for which the constraints are not violated, can be quite large if we consider a particular Lyapunov function. On the contrary, a different Lyapunov function based on some “optimal” cost function and assuring local “optimality”, may provide a significantly smaller domain of attraction.

The established solution to this trade-off problem is the control switching strategy. Two controllers are designed, each associated with one of these functions, whose domains of attractions are typically (not necessarily) nested. The control system switches from the “external” to the locally-optimal gain, or locally-optimal control Lyapunov function, as long as the state reaches the “smaller” region of attraction. Obviously, several control gains can be considered with several controlled-invariant regions \[82, 14\]. The drawback of the scheme is the discontinuity which can be “dangerous”, since the system state and the control could be subject to jumps which can be even persistent in the presence of noise.

Therefore it is of interest to find ways to “merge” two control Lyapunov functions in order to have a “smooth” transient from the level set of the “external” one to the “internal” one. A procedure of this kind is an example of what we refer as merging.

Recently, Andrieu and Prieur \[2\] proved that it is possible to
merge two control Lyapunov functions, in a setting actually related to the problem of uniting local and global controllers [64, 63, 3]. Their technique works under the assumption that there exists a suitable domain in which the two control Lyapunov functions share a common control law [2, Proposition 2.2]. There also has been recent interest in the topic for the class of nonlinear output feedback, global and local, control systems [73, 65].

More recently, Clarke [32] showed how to solve the problem of merging two semiconcave (continuous, locally Lipschitz but not everywhere differentiable) control Lyapunov functions, deriving a semiconcave non-smooth function based on the min operator.

**Manuscript organization**

This manuscript presents the innovative scientific results established by the doctoral candidate on the topic of robust stabilization of constrained uncertain dynamical systems, via composite control Lyapunov functions. For the first three chapters, the presentation will follow the chronological order according to which the results have been published. The fourth chapter presents the application of the proposed control techniques to a benchmark case study.

The motivation for this research activity first came from the problem to control a constrained linear system by guaranteeing a large controlled domain of attraction, together with close-to-optimal, or at least locally-optimal, closed-loop performances. Whenever admissible, a continuous control law is desired for the stabilization. A solution to this problem is in fact given from the design of an
opportune smooth control Lyapunov function.

A first tentative to address the problem for constrained linear systems is presented in Chapter 1, which groups together the results of the conference papers


that are references [7, 12, 9], respectively.

We take advantage of the theory of “R-functions” proposed in [69, 70] and later presented in the seminal papers [75, 76], which has been also exploited for the problem of estimating the region of asymptotic stability of nonlinear dynamical systems [5, 6, 4].

Namely, we present a constructive procedure, due to the technical properties of “R-functions”, to compose two Lyapunov functions. We show that the composition of two Lyapunov functions is a Lyapunov function as well. Therefore, in the case of Lyapunov-based
stabilization, assuming the existence of a common control law between the two parents control Lyapunov functions is sufficient for the composite function to be a control Lyapunov function as well. Technically speaking, this assumption has been also made in [2] for the class of nonlinear control-affine systems.

Chapter 2 contains the results of the journal paper


which is reference [11], and extends substantially the results of Chapter 1.

We indeed present a novel, more general (because it includes the basic composition of [7, 12, 9] as special case), composition rule for the merging of two control Lyapunov functions. We prove many interesting properties for the novel composition rule, which generates smooth, non-homogeneous control Lyapunov functions under the control-sharing assumption, which can be easily checked via the linear matrix inequality conditions we provide.

Our class of Lyapunov functions is universal for the stabilizability of uncertain linear systems, namely Lyapunov stabilizability is equivalent to the existence of a smooth, non-homogeneous control Lyapunov function in that class [39]. Moreover, under the control-sharing assumption, unlike smoothed-polyhedral functions and the convex hull of quadratics presented in the literature, our class allows to guarantee a large controlled domain of attraction together with locally-optimal closed-loop performances. We indeed present an
explicit, continuous, stabilizing control law which allows to simultaneously achieve these goals when associated with a control Lyapunov function in our class.

So far nothing is stated about the conservativism of assuming the control-sharing property. And in fact this was an open point of the literature on uniting control Lyapunov functions.

We indeed address the mentioned open problem in Chapter 3, which is based on


that are references 38, 39, respectively.

We establish general results on the possibility of taking merging procedures for the class of nonlinear control-affine systems and nonlinear control-affine differential inclusions.

The main results are that, for such classes of systems, the control-sharing property is equivalent to any merging function being an admissible control Lyapunov function. We show that the control-sharing property does not always hold, even for linear systems, with the exception of two-dimensional linear systems. We also prove through a counterexample that when considering continuous control laws, the blending procedure of 32 cannot be used for smooth merging control Lyapunov functions.
For the class of linear differential inclusions, we provide necessary and sufficient linear programs and linear matrix inequality conditions as efficient feasibility tests for the control-sharing property to hold. It is also given an illustrative example on the robust stabilization of a nonlinear inverted pendulum [38, Section V.B].

The proposed control laws are simulated on an industrial case study in Chapter 4, which mainly presents the arguments of the papers


that are references [8, 10], respectively.

We investigate the robust stabilization of a simplified model of a chemical reaction taking place in a continuous stirred tank reactor [35, 36, 53, 54]. We prove that, unlike the control approaches taken in the literature, a large controlled domain of attraction may be obtained together with a locally optimal closed-loop performance. In this section, we also provide two algorithms for the tuning of the free design parameter that comes out of our novel merging procedure proposed in [11].

We finally conclude the manuscript.

All the proofs are given in Appendix for ease of presentation.
INTRODUCTION

Notation

The notation used is adopted from [38].

We denote the Boolean set by $\mathbb{B} := \{0, 1\}$. The Heaviside function $h : \mathbb{R} \rightarrow \mathbb{B}$ is defined as $h(x) = 1$ for all $x \geq 0$ and $h(x) = 0$ for all $x < 0$.

$I_n$ denotes the $n \times n$ identity matrix. $1_s := (1, 1, ..., 1)^\top \in \mathbb{R}^s$. The notation $\text{co}(\cdot)$ denotes the convex hull [45]. $\text{int}S$ denotes the interior of a set $S$ and $\partial S$ denotes its boundary.

For any positive (semi)definite function $V : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$, $\mathcal{L}_V$ denotes its 1-level set, i.e. $\mathcal{L}_V := \{x \in \mathbb{R}^n \mid V(x) \leq 1\}$. Hence, for $\sigma \in \mathbb{R}_{\geq 0}$, $\mathcal{L}_{(V/\sigma)} := \{x \in \mathbb{R}^n \mid V(x) \leq \sigma\}$.

A square matrix $W \in \mathbb{R}^{s \times s}$ is an $\mathcal{M}$-matrix if $W_{i,j} \geq 0 \forall i \neq j$. 
Chapter 1
Composition of Lyapunov functions via R-functions

In this chapter we present a control strategy for the robust stabilization of constrained Linear Differential Inclusions (LDIs) via a novel, smooth, composite Control Lyapunov Function (CLF) associated with continuous, gradient-based, state feedback control laws, e.g. the minimum effort control \[62\].

Like (smoothed) polyhedral CLFs \[18, 24\], an arbitrary close approximation of the maximal controlled invariant set can be achieved. The advantage of the proposed composite CLF is that the inner sub-level sets are smooth. This allows the use of explicit gradient-based control laws.

Our technique is very general and it can be used to smooth both polyhedral and truncated ellipsoidal CLFs, in order to improve closed-loop performances, as shown in many benchmark examples.

The proposed smoothing technique follows from the interpreta-
Composition of Lyapunov functions via R-functions

The intersection of polyhedral and ellipsoidal regions in the framework of R-functions [69, 70, 71, 72], referred in the next section, which are real-valued functions associated with the basic Boolean operators.

1.1 R-functions for stability analysis and control applications

1.1.1 Basic definitions

The use of R-functions for the state feedback constrained stabilization of uncertain systems has been firstly proposed in [7].

The general definition of R-function is adopted from [75] as follows.

**Definition 1.1** A function \( r : \mathbb{D}^n \to \mathbb{R} \) is an R-function if there exists a Boolean function \( R : \mathbb{B}^n \to \mathbb{B} \) such that

\[
h ( r ( x_1, x_2, \ldots, x_n) ) = R ( h ( x_1), h ( x_2), \ldots, h ( x_n)),
\]

for all \( ( x_1, \ldots, x_n) \in \mathbb{D}^n \subseteq \mathbb{R}^n \), where \( h : \mathbb{R} \to \mathbb{B} \) is the Heaviside step function.

Informally, a real function \( r \) is an R-function if it can change its sign only when some of its arguments change the sign [7].

In the following, we will focus on the case \( n = 2 \), namely we will consider the composition of two functions \( r_1 \) and \( r_2 \).

There exists a parallelism between logic functions and R-functions whenever we consider classic Boolean operators, as described in Table 1.1 [12].
1.1 R-functions for stability analysis and control applications

Table 1.1: Correspondence between logic functions and R-functions. The parameter $\alpha \in [0, 1]$ is a free design parameter.

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<tr>
<th>BOOLEAN</th>
<th>R-COMPOSITION</th>
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<tr>
<td>NOT ( \neg )</td>
<td>$r_{\neg} := -r$</td>
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<tr>
<td>AND ( \land )</td>
<td>$r_{\land} := \frac{r_1 + r_2 - \sqrt{r_1^2 + r_2^2 - 2\alpha r_1 r_2}}{2 - \sqrt{2 - 2\alpha}}$</td>
</tr>
<tr>
<td>OR ( \lor )</td>
<td>$r_{\lor} := \frac{r_1 + r_2 + \sqrt{r_1^2 + r_2^2 - 2\alpha r_1 r_2}}{2 + \sqrt{2 - 2\alpha}}$</td>
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For instance, according to Table 1.1, the meaning of the “AND composition” is that the composed function $r_{\land} := r_1 \land r_2$ is positive if and only if both $r_1$ and $r_2$ are positive. The result can be obtained by exploiting the triangle inequality and the law of cosines, and it holds for all values of $\alpha \in [0, 1]$. The terms at the denominator in Table 1.1 are (positive) normalizing factors, so that $r_{\land} = 1$ whenever $r_1 = r_2 = 1$.

**Remark 1.1** For $\alpha := 1$, we have $r_1 \land r_2 = \min \{r_1, r_2\}$ and $r_1 \lor r_2 = \max \{r_1, r_2\}$.

In the following, we consider only the AND composition because, since we are concerned with convex (controlled) sets, we consider the intersection of convex sets, which is convex as well.
1.1.2 Construction of control Lyapunov function candidates based on the R-composition

In this subsection it is shown how to design a suitable CLF candidate $V_\lambda$ corresponding to the R-composition of two CLFs $V_1$ and $V_2$. We focus on the compact sets $L_{V_1}$ and $L_{V_2}$, defined as $L_{V_i} := \{ x \in \mathbb{R}^n \mid V_i(x) \leq 1 \}$, $i = 1, 2$. According to Lemma (A.3) in Appendix A, we have that $L_{V_\lambda} = L_{V_1} \cap L_{V_2}$, independently from the parameter $\alpha$.

For all $x \in \mathbb{R}^n$ we hence define

$$
R_1(x) := 1 - V_1(x), \quad R_2(x) := 1 - V_2(x), \quad (1.1)
$$

and the R-composition $R_\lambda$ for fixed $\alpha \in [0, 1]$:

$$
R_\lambda(x) := \frac{R_1(x) + R_2(x) - \sqrt{R_1(x)^2 + R_2(x)^2 - 2\alpha R_1(x)R_2(x)}}{2 - \sqrt{2 - 2\alpha}}. \quad (1.2)
$$

Finally, we define the CLF candidate $V_\lambda : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ as

$$
V_\lambda(x) := 1 - R_\lambda(x). \quad (1.3)
$$

According to Lemmas (A.1), (A.2), $V_\lambda$ is positive definite and differentiable in $\text{int}L_{V_\lambda} = \text{int}(L_{V_1} \cap L_{V_2})$.

A geometric interpretation of R-functions is now provided.

**Example 1.1** Consider the polyhedral function

$V_1(x) := \max\{ x^\top F_1^\top F_1 x, x^\top F_2^\top F_2 x \}$ and the quadratic function

$V_2(x) := x^\top Px$ where

$$
F = \begin{bmatrix}
1.50 & -0.50 \\
-0.50 & 1.50
\end{bmatrix}, \quad P = \begin{bmatrix}
2.07 & 0.66 \\
0.66 & 2.07
\end{bmatrix},
$$
1.1 R-functions for stability analysis and control applications

being $F_i$, the $i$th row of matrix $F$, $i = 1, 2$.

We compose the positive definite functions $V_1$ and $V_2$ in their 1-level sets, respectively $\mathcal{L}_{V_1}$ and $\mathcal{L}_{V_2}$; we define the functions $R_1(x) := 1 - V_1(x)$ and $R_2(x) := 1 - V_2(x)$. Without loss of generality, these functions have been normalized so that their maximum value is 1.

Then we compute the R-composition ($\text{AND} \wedge$) $R_\wedge := R_1 \wedge R_2$, according to the equation of Table 1.1, for an arbitrary value of $\alpha \in [0,1]$. Namely, the function $R_\wedge$ is defined as in (1.2).

The composite function $R_\wedge$ is the “smoothed intersection” between the polyhedral function $V_1$ and the quadratic one $V_2$, in the sense that $R_\wedge$ is positive inside the intersection region $\mathcal{L}_{V_1} \cap \mathcal{L}_{V_2} = \mathcal{L}_{V_\wedge}$, it is zero on the boundary $\partial \mathcal{L}_{V_\wedge}$, negative outside, and its maximum value is 1 at the origin.

The positive definite function associated with $R_\wedge$ is $V_\wedge(x) := 1 - R_\wedge(x)$. The sublevel sets of the function $V_\wedge$ are shown in Figure 1.1 for the two limit cases of $\alpha = 1$ (truncated ellipsoid [58], [79]) and $\alpha = 0$.

Remark 1.2 In [79] the term “truncated ellipsoid” has been introduced to define a candidate LF which shapes the intersection of a polyhedral region with an ellipsoidal one. Within the framework of R-functions, the truncated ellipsoid is recovered as a special case ($\alpha = 1$) of the R-composition between a polyhedral function and a quadratic one, see Figure 1.1.

The parameter $\alpha$ affects the smoothness of the inner sublevel sets of the composite function, while it does not affect the shape of the overall region $\mathcal{L}_{V_\wedge}$. For $\alpha < 1$ such smoothing technique yields non-homothetic sublevel sets and a differentiable function $V_\wedge$ on $\text{int} \mathcal{L}_{V_\wedge}$. 
1.1.3 Stability analysis of nonlinear systems via composite Lyapunov functions

In this subsection, the intersection function $V_\lambda$ (1.3) is used as candidate LF for stability analysis. We consider a locally bounded function $f : \mathbb{R}^n \to \mathbb{R}^n$, a compact set $X \subset \mathbb{R}^n$ and a constrained
1.1 R-functions for stability analysis and control applications

autonomous system

\[ \dot{x} = f(x), \quad x \in \mathbb{X}. \]  \hspace{1cm} (1.4)

In the following result we consider the R-composition of two LFs \( V_1 \) and \( V_2 \) for the system (1.4), respectively in \( \mathcal{L}_{V_1} \) and \( \mathcal{L}_{V_2} \). We avoid the lack of differentiability of \( V_\wedge \) on \( \partial \mathcal{L}_{V_\wedge} \), by considering the set \( \text{int} \mathcal{L}_{V_\wedge} \). Alternatively, we could consider the domain \( \mathcal{L}_{V_\wedge/(1-\epsilon)} = \{ x \in \mathbb{R}^n \mid V_\wedge(x) \leq 1 - \epsilon \} \), for any arbitrary \( \epsilon \in (0,1) \).

Theorem 1.1 Consider two LFs \( V_1, V_2 : \mathbb{R}^n \to \mathbb{R}_{\geq 0} \) such that for all \( x \in \mathcal{L}_{V_1} \cap \mathcal{L}_{V_2} = \mathcal{L}_{V_\wedge} =: \mathbb{X} \) and \( i = 1, 2 \) we have

\[ \nabla V_i(x)f(x) \leq -\eta V_i(x) \]

for some \( \eta > 0 \).

Then for all \( x \in \text{int} \mathbb{X} \) we have \( \nabla V_\wedge(x)f(x) \leq -\eta V_\wedge(x) \), with \( V_\wedge \) defined as in (1.3).

The result of Theorem 1.1 means that given two Lyapunov functions \( V_1 \) and \( V_2 \), respectively with domain of attraction \( \mathcal{L}_{V_1} \) and \( \mathcal{L}_{V_2} \), their associated R-composed function \( V_\wedge \) is a Lyapunov function as well independently from the choice of \( \alpha \in [0,1] \), basically with domain of attraction \( \mathcal{L}_{V_\wedge} = \mathcal{L}_{V_1} \cap \mathcal{L}_{V_2} \).

Formally, in the limit case of \( \alpha = 1 \), the requirement of differentiability for a candidate LF is violated, therefore, the above result yield differentiable LFs for \( \alpha < 1 \). In fact, according to Remark 1.1 for \( \alpha = 1 \) the non-smooth operator \( \max \) is recovered.
1.2 Robust stabilization of constrained linear differential inclusions

Let us consider the robust stabilization of constrained uncertain linear systems, namely constrained linear differential inclusions of the kind

\[ \dot{x} \in \text{conv} \{ A_i x + Bu \mid i \in [1, N] \} \]  

(1.5)

where \( A_i \in \mathbb{R}^{n \times n} \) for all \( i \in [1, N] \), \( B \in \mathbb{R}^{n \times m} \).

We consider linear state constraints \( X := \{ x \in \mathbb{R}^n \mid Fx \leq 1 \} \), for some given \( F \in \mathbb{R}^{s \times n} \), and input constraints \( U := \{ u \in \mathbb{R}^m \mid \|u\|_\infty \leq 1 \} \).

The objective is to design a state feedback control law \( u : X \to U \) such that \( x(t) \) asymptotically converges to the origin, in accordance to the state and input constraints.

1.2.1 Lyapunov-based control laws

A positive definite, smooth away from zero, radially unbounded function \( V : \mathbb{R}^n \to \mathbb{R}_{\geq 0} \) is a suitable CLF for (1.5) if the condition

\[ \{ x \in X \mid \nabla V(x) B = 0^\top \} \subseteq \{ x \in X \mid \max_{i \in [1, N]} \nabla V(x) A_i x < 0 \} \]

(1.6)

is satisfied, or equivalently if

\[ \{ x \in X \mid \nabla V(x) B = 0^\top \text{ and } \max_{i \in [1, N]} \nabla V(x) A_i x \geq 0 \} = \emptyset \]  

(1.7)

Roughly speaking, conditions (1.6) and (1.7) mean that whenever the control action is ineffective (\( \nabla V(x) B = 0^\top \)), the function \( V \) should “decrease” just the same (\( \max_{i \in [1, N]} \nabla V(x) A_i x < 0 \)).
1.2 Robust stabilization of constrained linear differential inclusions

Now, the “Lyapunov derivative” of the CLF $V$, 
$$\dot{V}(x, u) := \max_{i \in [1, N]} \nabla V(x)^T A_i x + \nabla V(x)^T B u$$
is minimized by the constrained control law $\kappa : X \rightarrow U$ defined as 
$$\kappa(x) := -\text{sign} \left( B^T \nabla V(x)^T \right), \quad (1.8)$$
so that we get 
$$\dot{V}(x, \kappa(x)) = \max_{i \in [1, N]} \nabla V(x)^T A_i x - \sum_{j=1}^m |(\nabla V(x)^T B)_i| = \max_{i \in [1, N]} \nabla V(x)^T A_i x - \|\nabla V(x)^T B\|_1. \quad (1.9)$$

Therefore it is possible to derive a Petersen-like condition which guarantees that $V$ is a suitable CLF by means of a constrained state feedback control: 
$$\left\{ x \in X \mid \max_{i \in [1, N]} \nabla V(x)^T A_i x - \|\nabla V(x)^T B\|_1 \geq 0 \right\} = \emptyset. \quad (1.9)$$

A possible way of checking condition (1.9) is by considering the following optimization problem.
$$\max_{x \in X} \left\{ \max_{i \in [1, N]} \nabla V(x)^T A_i x - \|\nabla V(x)^T B\|_1 \right\} < 0 \quad (1.10)$$
Then condition (1.9) is satisfied if and only if the solution of (1.10) is negative.

The drawback of the control law (1.8) is that it is highly discontinuous and often not implementable on real actuators. The discontinuity caused by the \text{sign}$(\cdot)$ function can be avoided by approximating the control law (1.8) with arbitrary precision as follows. 
$$\kappa(x) := -\text{sat} \left( kB^T \nabla V(x)^T \right), \quad (1.11)$$
for $k > 0$ sufficiently large \(^\text{[20]}\), where \(\text{sat}(\cdot)\) is the component-wise vector saturation function.

Another admissible “gradient-based” state feedback control law is the minimum effort control \(^\text{[62]}\):

$$\kappa(x) := \arg\min_{u \in U} \|u\| \quad \text{subject to:}$$

$$\max_{i \in [1, N]} \nabla V(x)A_i x - \nabla V(x)Bu + \eta \|x\| \leq 0. \quad (1.12)$$

The minimum effort control is continuous if the CLF \(V\) is differentiable everywhere (except at the origin) \(^\text{[61]}\). For this reason, it is convenient to set $\alpha < 1$ so that differentiability is gained for \(V_\lambda\), and a continuous gradient-based control law is hence obtained.

### 1.3 Illustrative examples: stabilization with control Lyapunov functions composed via R-functions

In this section some benchmark examples are provided to show the effectiveness of a gradient-based control together with an everywhere differentiable CLF. The smoothing has been performed by setting the parameter $\alpha = 0$.

We compare the performances of the gradient-based control law \((1.11)\) associated with our CLF and also to the standard smoothed PCLF. We consider typical control indices: “ISE” is the Integral of the Squared Error values and it should be small to avoid large state errors; “ISTE” is the Integral Square Time Error and it also should be small to avoid large state errors or slow convergence; “T”
1.3 Illustrative examples: stabilization with control Lyapunov functions composed via R-functions

represents the time of convergence (inside a given, small threshold); finally “IADU” is the Integral of the Absolute value of the time Derivative of the control signal $u$.

For ease of visualization, the examples presented here are two-dimensional. Higher-dimensional examples are shown in [7, 12, 9].

### 1.3.1 Example with a smoothed polyhedral function

In this subsection, we consider the example proposed in [23], where the control law is a gradient-based control associated with a polyhedral function, smoothed with standard Minkowski $2^p$-norms.

**Example 1.2**

\[
\dot{x} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u,
\]

with $x \in X = \{ \xi \in \mathbb{R}^2 \mid \|\xi\|_\infty \leq 1 \}, \ u \in U = \{ v \in \mathbb{R}^2 \mid \|v\|_\infty \leq 1 \}.$

The CLF proposed in [23] is the smoothed polyhedral function $V_1(x) = \|Fx\|_{2^p}$, with $p = 3$, where

\[
F = \begin{bmatrix} 1.5 & -0.5 \\ -0.5 & 1.5 \end{bmatrix}.
\]

Here we use the framework of R-functions to smooth the inner sublevel sets (with $\alpha = 0$) of the polyhedral function $\|Fx\|_\infty$. The smoothed function can be proved to be an admissible CLF by solving problem (1.10). The controlled region is $\{ x \in \mathbb{R}^2 \mid \|Fx\|_\infty \leq 1 \}$, that is a bit larger than the one, $\{ x \in \mathbb{R}^2 \mid \|Fx\|_{2^p} \leq 1 \}$, provided in [23], because also the “corners” of the polyhedral region are included.
With the use of R-functions all performance indices are improved, see [12, Section 4.1], since the non-homothetic sublevel sets provide a smoother state convergence with respect to high-order $2p$-norms.

### 1.3.2 Example with a truncated ellipsoid

We consider the constrained double integrator as addressed in [43, 58].

**Example 1.3**

\[
\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u,
\]

with \( x \in X = \{ \xi \in \mathbb{R}^2 \mid |\xi_1| \leq 25, |\xi_2| \leq 5 \}, \) \( u \in U = \{ \nu \in \mathbb{R} \mid |\nu| \leq 1 \}. \)

The truncated ellipsoid and the linear control law \( \kappa(x) = Kx \) designed in [58] are characterized by the matrices

\[
P = \begin{bmatrix} 0.0016 & 0.0027 \\ 0.0027 & 0.0243 \end{bmatrix}, \quad K = \begin{bmatrix} 0.0281 & 0.1475 \end{bmatrix}.
\]

We compose the functions \( V_1(x) = \|Fx\|_{\infty} \), with

\[
F = \begin{bmatrix} 1/25 & 0 \\ 0 & 1/5 \end{bmatrix},
\]

and \( V_2(x) = x^\top Px. \)

The static state feedback control is compared with the gradient-based control associated with the truncated ellipsoid smoothed via R-functions. The gradient-based control is smoother and it yields
1.3 Illustrative examples: stabilization with control Lyapunov functions composed via R-functions

much faster convergence, see [12, Section 4.2]. Moreover, the use of a linear control law yields an “undesirable” oscillating behavior of the state trajectory, as it is shown in Figure 1.2.

1.3.3 Example with a smoothed polyhedral function together with minimum effort control

This section compares our proposed smoothed CLF with the classic smoothing method based on Minkowski 2p-norms. We consider a linear uncertain system from [22].

Example 1.4

\[
\dot{x} = \begin{bmatrix} 0 & -1.5 + w \\ -2 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 10 \end{bmatrix} u \quad (1.13)
\]

with bounded uncertainty \( w \in [-0.5, 0.5] \).

Similarly to [22], we consider the PCLF \( V_1(x) = \|Fx\|_\infty \) with

\[
F = \begin{bmatrix} 0 & 1 \\ 4.97 & -0.497 \\ 4.97 & -0.2485 \end{bmatrix}.
\]

We first define \( R_i(x) = 1 - x^TF_i^TF_ix \), for \( i = 1, 2, 3 \). We consider the following two-steps composition via R-functions. We define \( R^{(1,2)}_\lambda := R_1 \land R_2 \) and then \( R_\lambda := R^{(1,2)}_\lambda \land R_3 \), so that \( V_\lambda(x) := 1 - R_\lambda(x) \). All R-compositions are made with \( \alpha := 0 \).

The minimum effort control (1.12) is used for comparisons between our CLF \( V_\lambda \) and the smoothed CLF proposed in [22], namely \( V_p(x) = \|Fx\|_{2p} \), with \( p = 6 \).
Figure 1.3 shows some state trajectories of the controlled system starting from randomly-taken initial states. Averaging over many simulations starting from randomly-taken initial conditions, we get that, also with the minimum effort control, our CLF yields better performances as reported in [7, Section 4].
1.3 Illustrative examples: stabilization with control Lyapunov functions composed via R-functions

Figure 1.2: Controlled state trajectories starting from randomly taken initial conditions and converging to the origin. On top the system evolves under a linear control law, which is associated with a truncated ellipsoid control Lyapunov function. The bottom plot show the system evolution under a gradient-based control law associated with our proposed smooth control Lyapunov function, composed via R-functions.
Figure 1.3: Controlled state trajectories starting from randomly taken initial conditions and converging to the origin. The system evolves under a gradient-based control law associated with a our proposed smooth control Lyapunov function, composed via R-functions, which smooths a polyhedral control Lyapunov function.
Chapter 2

A novel composition of control Lyapunov functions for constrained uncertain linear systems

In this chapter, we investigate the robust stabilization of constrained uncertain linear systems via the set-theoretic framework of Lyapunov functions induced by R-functions.

The main contribution is the definition of a novel composition rule to merge two different Control Lyapunov Functions (CLFs), allowing the design of a non-homogeneous smooth CLF with the following properties: the external level set exactly shapes an arbitrarily-close approximation of the maximal controllable invariant set; the inner sublevel sets can be made arbitrarily close to any given choice of smooth ones. These properties allow to define a stabilizing gradient-
A novel composition of control Lyapunov functions for constrained uncertain linear systems based control law which is continuous everywhere inside the controlled invariant set.

The results of [7, 12, 9], where a basic composition rule is used, are extended to the class of constrained uncertain linear systems. Here we propose a Linear Matrix Inequality (LMI) feasibility test for the candidate composite CLF. As in [31, 45], the control-design condition is obtained via Bilinear Matrix Inequalities. Our composite CLFs can smooth both Polyhedral CLFs (PCLFs) [18, 24, 27] and truncated ellipsoids [79] in a non-homothetic way, and can be made everywhere differentiable.

The constrained linear quadratic control is addressed as an illustrative application, in order to show the benefits of the proposed Lyapunov-based stabilization technique.

2.1 A novel composition rule related to the framework of R-functions

According to the idea beyond the framework of R-functions, presented in Chapter [1], in the following we introduce a novel composition rule which is associated with the Boolean AND

\[
r_\land = r_1 \land r_2 := \rho(\phi) \left( \phi r_1 + r_2 - \sqrt{(\phi r_1)^2 + r_2^2} \right),
\]

(2.1)

where \( \phi > 0 \) and \( \rho(\phi) := (\phi + 1 - \sqrt{\phi^2 + 1})^{-1} \) is the normalizing factor such that, for all \( \phi > 0 \), we have \( r_\land = 1 \) whenever \( r_1 = r_2 = 1 \).

The corresponding OR composition rule, which we do not employ here, is

\[
r_\lor = r_1 \lor r_2 := \rho_\lor(\phi) \left( \phi r_1 + r_2 + \sqrt{(\phi r_1)^2 + r_2^2} \right),
\]

with normalization factor \( \rho_\lor(\phi) := (\phi + 1 + \sqrt{\phi^2 + 1})^{-1} \).
2.1 A novel composition rule related to the framework of R-functions

We report that all the results of this chapter can be also stated by adopting the AND composition

\[ \rho(\phi, \alpha) \left( \phi r_1 + r_2 - \sqrt{(\phi r_1)^2 + r_2^2 - 2\alpha \phi r_1 r_2} \right), \]

with \( \rho(\phi, \alpha) := (\phi + 1 - \sqrt{\phi^2 + 1 - 2\alpha \phi})^{-1} \), where \( \alpha \in [0, 1] \) is the free parameter used in \([7, 9, 12]\). But for ease of presentation, we basically set \( \alpha := 0 \).

The following result shows that the basic property of R-functions is valid also for the novel composition rule (2.1) introduced above.

**Lemma 2.1** For all \( \phi > 0 \) we have

\[ r_\wedge > 0 \iff \{ r_1 > 0 \text{ and } r_2 > 0 \}. \]

In the following we consider only controlled sets that are convex and 0-symmetric, as in \([24, 45]\). The following technical properties will be further exploited in this chapter.

**Proposition 2.2** For all \( r_1, r_2 \geq 0 \) and \( \phi > 0 \), the function \( r_\wedge := r_1 \wedge r_2 \) defined in (2.1) satisfies

\[ \min \{ r_1, r_2 \} \leq r_\wedge \leq \max \{ r_1, r_2 \}. \]

**Proposition 2.3** For all \( r_1, r_2 > 0 \), the function \( r_\wedge := r_1 \wedge r_2 \) converges pointwise to \( r_2 \) (\( r_1 \)) as \( \phi \) goes to infinity (zero), namely:

\[ r_\wedge_{\phi \to \infty} r_2, \quad r_\wedge_{\phi \to 0^+} r_1. \]

As in Section \([1.1.2]\) we design a suitable CLF candidate \( V_\wedge \) corresponding to the R-composition of two CLFs \( V_1 \) and \( V_2 \). We focus
A novel composition of control Lyapunov functions for constrained uncertain linear systems

on the compact sets $L_{V_1}$ and $L_{V_2}$, so that $L_{V_\wedge} = L_{V_1} \cap L_{V_2}$ (Lemma [A.3], Appendix A). For all $x \in \mathbb{R}^n$ we hence define

$$R_1(x) := 1 - V_1(x), \quad R_2(x) := 1 - V_2(x), \quad (2.2)$$

and the R-composition for fixed $\phi > 0$:

$$R_\wedge(x) := \frac{\phi R_1(x) + R_2(x) - \sqrt{(\phi R_1(x))^2 + R_2(x)^2}}{\phi + 1 - \sqrt{\phi^2 + 1}}. \quad (2.3)$$

Finally, we define the CLF candidate $V_\wedge : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ as

$$V_\wedge(x) := 1 - R_\wedge(x). \quad (2.4)$$

According to Lemmas [A.1], [A.2], $V_\wedge$ is positive definite and differentiable in $\text{int}L_{V_\wedge} = \text{int}(L_{V_1} \cap L_{V_2})$.

**Example 2.1** We consider Example 1.1 in $\mathbb{R}^2$, with a polyhedral function of the second order $F_{i}^\top F_i x$, being $F_i$ the $i$th row of matrix $F$, and a quadratic function $V_2(x) := x^\top P x$.

We compose the positive definite functions $V_1$ and $V_2$ in their 1-level sets, respectively $L_{V_1}$ and $L_{V_2}$; we define the functions $R_1(x) := 1 - V_1(x)$ and $R_2(x) := 1 - V_2(x)$. Without loss of generality, these functions have been normalized so that their maximum value is 1.

Then we compute the R-composition $R_\wedge := R_1 \wedge R_2$, according to (2.3), for arbitrary values of $\phi > 0$.

The composite function $R_\wedge$ is the “smoothed intersection” between the polyhedral function $V_1$ and the quadratic one $V_2$ in the sense that, for all $\phi > 0$, $R_\wedge$ is positive inside the intersection region $L_{V_1} \cap L_{V_2} = L_{V_\wedge}$, it is zero on the boundary $\partial L_{V_\wedge}$, negative outside, and its maximum value is 1 at the origin.
2.1 A novel composition rule related to the framework of R-functions

Figure 2.1: On the left side the sublevel sets of the composed function \( V_\wedge \), for \( \phi = 1/4 \), are shown on the domain \( L_{V_\wedge} \). On the right the sublevel sets of the composed function \( V_\wedge \), for \( \phi = 4 \), are shown again on the domain \( L_{V_\wedge} \). We remark that \( L_{V_\wedge} = L_{V_1} \cap L_{V_2} \) independently from the values of the shape parameter \( \phi > 0 \).

The positive definite function associated with \( R_\wedge \) is \( V_\wedge = 1 - R_\wedge(x) \). The sublevel sets of the function \( V_\wedge \) are shown in Figure 2.1.

The novelty of the proposed composition consists in the fact that, unlike all compositions proposed in the literature, a parameter, i.e. \( \phi \), can be used to trade-off the shape of the sublevel sets of the composite function \( V_\wedge \) between the ones of the two generating functions \( V_1 \) and \( V_2 \), still preserving the overall domain \( L_{V_\wedge} = L_{V_1} \cap L_{V_2} \). We hence notice that the trade-off parameter \( \phi \) provides an additional degree of freedom that could be exploited to improve the closed-loop performances with respect to the use of homothetic functions recovered in the two limit cases presented in Proposition 2.3.
A novel composition of control Lyapunov functions for constrained uncertain linear systems

2.1.1 Stability analysis of nonlinear systems via composite Lyapunov functions

In this section we extend the results of Section 1.1.3. The composite function $V^\wedge$ (2.4) is used as candidate LF for stability analysis. We consider a locally bounded function $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$, a compact set $X \subset \mathbb{R}^n$ and a constrained autonomous system

$$\dot{x} = f(x), \quad x \in X. \quad (2.5)$$

In the following result we consider the R-composition of two LFs $V_1$ and $V_2$ for the system (2.5), respectively in $L_{V_1}$ and $L_{V_2}$. We avoid the lack of differentiability of $V^\wedge$ on $\partial L_{V^\wedge}$, by considering the set $\text{int}L_{V^\wedge}$.

**Theorem 2.4** Consider two LFs $V_1, V_2 : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ such that for all $x \in L_{V_1} \cap L_{V_2} = L_{V^\wedge} =: X$ and $i = 1, 2$ we have

$$\nabla V_i(x)f(x) \leq -\eta V_i(x)$$

for some $\eta > 0$.

Then for all $x \in \text{int}X$ we have $\nabla V^\wedge(x)f(x) \leq -\eta V^\wedge(x)$, with $V^\wedge$ defined in (2.4).

According to the previous result, given two Lyapunov functions $V_1$ and $V_2$, respectively with domain of attraction $L_{V_1}$ and $L_{V_2}$, their associated R-composed function $V^\wedge$ is a Lyapunov function as well independently from the choice of $\phi > 0$, basically with domain of attraction $L_{V^\wedge} = L_{V_1} \cap L_{V_2}$. Therefore all the trade-off functions $V^\wedge$ (for all the values of $\phi$) are suitable Lyapunov functions.
2.2 Linear Matrix Inequality conditions for composite control Lyapunov functions

2.2.1 Statement of the robust control problem

As in Section 1.2, we consider the robust stabilization of constrained uncertain linear systems, namely constrained linear differential inclusions.

\[ \dot{x} \in \text{conv} \{ A_i x + B_i u \mid i \in [1, N] \} \tag{2.6} \]

where \( A_i \in \mathbb{R}^{n \times n}, B_i \in \mathbb{R}^{n \times m} \) for all \( i \in [1, N] \), \( B \in \mathbb{R}^{n \times m} \), with linear 0-symmetric constraints:

\[ X := \{ x \in \mathbb{R}^n \mid \| L x \|_\infty \leq 1 \}, \quad U := \{ u \in \mathbb{R}^m \mid \| u \|_\infty \leq 1 \}. \tag{2.7} \]

The objective is to design a state feedback control law \( u : X \to U \) such that \( x(t) \) asymptotically converges to the origin, in accordance to the state and input constraints.

A polyhedral approximation (with arbitrary precision) \( \mathcal{L}_{V_i} \) of the maximal controllable set for the system (2.6)–(2.7) can be explicitly computed via sequential linear programming [17, 18], obtaining a controlled set \( \{ x \in \mathbb{R}^n : \| F x \|_\infty^2 \leq 1 \} \subseteq X \) described by a full column-rank matrix \( F \in \mathbb{R}^{s \times n} \). We hence address a “large” controlled domain \( \mathcal{L}_{V_i} \) and we define the PCLF \( V_1(x) := \| F x \|_\infty = \max_{i \in [1, s]} \{ x^\top F_i^\top F_i x \} \), which is the one that shapes the considered polyhedral domain.
2.2.2 Linear Matrix Inequality conditions for robust stabilizability

We here focus on the R-composition $V \wedge (2.4)$ of the PCLF $V_1$ and a certain QCLF $V_2(x) := x^\top Px$, $P > 0$. Let us assume that $V_1$ has a large controlled domain of attraction, namely $\mathcal{L}_{V_1}$, which is recovered also for $V_\wedge$ by a-priori scaling $V_2$, and hence scaling $P$, such that $\mathcal{L}_{V_2} \supset \mathcal{L}_{V_1}$, so that $\mathcal{L}_{V_\wedge} = \mathcal{L}_{V_1} \cap \mathcal{L}_{V_2} = \mathcal{L}_{V_1}$.

The composite function $V_\wedge$ is used as candidate CLF for (2.6) on the domain $\mathcal{L}_{V_\wedge}$.

The following theorem provides a sufficient LMI condition for the robust stabilizability of (2.6)–(2.7) on the domain $\mathcal{L}_{V_1}$, having vertices $v(l)$’s.

**Theorem 2.5** Assume there exist $K \in \mathbb{R}^{m \times n}$, $P \in \mathbb{R}^{n \times n}$, $P > 0$, $\eta > 0$ and $\gamma_{ijk} \geq 0$, $i = 1, ..., N$, $j, k = 1, ..., s$, such that

$$(A_i + B_iK)^\top F_k^\top F_k + F_k^\top F_k (A_i + B_iK) \preceq -\eta F_k^\top F_k + \sum_{j=1}^{s} \gamma_{ijk} (F_j^\top F_j - F_k^\top F_k) \quad (2.8a)$$

$$(A_i + B_iK)^\top P + P(A_i + B_iK) \preceq -\eta P \quad (2.8b)$$

$$-1 \leq K v(l) \leq 1, \quad (2.8c)$$

for all $i \in [1, N]$, $k \in [1, s]$, and for all $l$.

Then, for all $\phi > 0$, the composite function $V_\wedge (2.4)$ is a CLF for (2.6) with domain of attraction $\mathcal{L}_{V_\wedge}$.

According to Theorem 2.5 in the interior of $\mathcal{L}_{V_\wedge}$, any trade-off shape obtained by varying $\phi$ is suitable. However, no explicit rule
for selecting $\phi$ (and hence a particular shape) is here presented. This opens up the possibility of defining some criterion for the choice of $\phi$. This subject is not addressed here and it is hence left for future investigations (see [10, Section 5] for possible choices of $\phi$).

In view of Proposition 2.2, we have $\min\{V_1(x), V_2(x)\} \leq V_\wedge(x) \leq \max\{V_1(x), V_2(x)\}$ for all $x \in \mathcal{L}_{V_\wedge}$, therefore $V_\wedge$ grows quadratically as well: there exist constants $c_1, c_2 > 0$ such that $c_1 x^T x \leq V_\wedge(x) \leq c_2 x^T x$. If the decreasing rate of $V_\wedge$ is $\eta$, i.e. there exists $c > 0$ such that $V_\wedge(x(t)) \leq c \cdot e^{-\eta t} V_\wedge(x(0))$, then the convergence rate in terms of the 2-norm is $\eta/2$. In fact, we have $c_1 \|x(t)\|_2^2 \leq V_\wedge(x(t)) \leq c_2 \cdot e^{-\eta t} \|x(0)\|_2^2$, which implies $\|x(t)\|_2 \leq \sqrt{cc_1 \cdot e^{-\eta t} \cdot c_2 \|x(0)\|_2^2}$ for all $t \geq 0$.

**Remark 2.1** The first inequality in (2.8) is a BMI in the variables $K$, $P$, $\eta$, $\gamma_{ijk}$'s. While if $P$ is fixed, then (2.8) becomes an LMI. Also in [79] a BMI problem has to be solved for the synthesis of an unsmooth truncated ellipsoidal CLF together with a linear state feedback control law. The advantage of the proposed approach with respect to [79] is that if the BMI is feasible, then a smooth CLF is obtained. This implies that explicit nonlinear gradient-based control laws can be used [62, 24], improving the closed-loop performances.

**Remark 2.2** The assumption of Theorem 2.5 on the existence of a linear control is adopted in the earlier works on stabilization of constrained linear systems by means of PCLFs [81, 80, 16, 15], where the Linear Constrained Regulator Problem (LCRP) has been first addressed. More recently, the same assumption is required for the feasibility of the BMI problems proposed in [58] for semi-ellipsoidal...
sets, in [79] for truncated ellipsoids, in [2] for the problem of unifying two different CLFs.

The choice of $P := \frac{1}{2} \sum_{i=1}^{s} F_i^T F_i$ (which makes the shape of $V_2$ “close” to the one of the PCLF $V_1$) yields $\mathcal{L}_{V_2} \supset \mathcal{L}_{V_1}$. We notice that this particular choice corresponds to imposing $\gamma_{ijk} = \gamma_{ikj}$ in (2.8), and that in this way we actually recover an LMI feasibility problem. Although conservative, LMI (2.8) with such particular choice of $P$ is feasible for all benchmark examples in [80, 16, 23, 24, 58, 79].

We also emphasize that, without loss of generality, Theorem 2.5 is also valid for $V_\wedge$ composed starting from the smoothed PCLF $V_1(x) := \|Fx\|_{2p}^2$, for $p$ sufficiently large [24], and $V_2(x) = x^T P x$. Unlike the standard $2p$-norm of [24], we can provide a trade-off composition between (smoothed) polyhedral and quadratic functions.

### 2.3 Explicit Lyapunov-based state feedback control law

Given a differentiable CLF $V$, a known continuous control law is the minimum effort control [62]:

$$\kappa(x) := \arg \min_{v \in U} \|v\| \text{ subject to:} \max_{i \in [1,N]} \nabla V(x) (A_i x + B v) + \eta V(x) \leq 0.$$  \hspace{1cm} (2.9)

Note that the minimum effort control may be not continuous if applied to a polyhedral function since differentiability fails [24]. The explicit formulation of (2.9) and the general case of uncertain matrix $B$ are addressed in [24, Section 5].
2.4 Application to approximate constrained linear quadratic optimal control

Since $\nabla V_\lambda(x) = -\nabla R_\lambda(x)$ and $\nabla V_i(x) = -\nabla R_i(x)$, the gradient $\nabla V_\lambda(x)$ is a nonlinear, positive combination of $\nabla V_1(x)$ and $\nabla V_2(x)$, see [B.2], which can be computed explicitly.

2.4 Application to approximate constrained linear quadratic optimal control

Designing the shape of the candidate CLF, via our novel R-composition, suggests the application to the constrained LQ optimal control problem. In fact, while the external set can be designed in accordance to the shape of a large controllable set, the inner sublevel sets can be (independently) made close to the locally-optimal quadratic ones.

Consider constrained linear systems

$$\dot{x} = Ax + Bu, \ x \in \mathbb{X}, \ u \in \mathbb{U},$$

with $\mathbb{X}$ and $\mathbb{U}$ as in (2.7) and with standard quadratic performance cost

$$J(x, u) := \int_0^\infty (\|x(t)\|^2_Q + \|u(t)\|^2_R)dt,$$

where $Q, R > 0$. Let $P^* > 0$ be the solution of the Algebraic Riccati Equation (ARE)

$$A^TP + PA + Q - PBR^{-1}B^TP < 0.$$

For the unconstrained LQ optimal control problem it is possible to scale matrix $P^*$ without loss of generality, because if $P^*$ is the solution of the ARE, then for any $\delta > 0$, $\tilde{P}^* = \delta P^*$ is the solution associated with $Q \mapsto \tilde{Q} = \delta Q$, $R \mapsto \tilde{R} = \delta R$, so that minimizing $\tilde{J}(x, u) = \delta J(x, u)$. Therefore we assume that $\mathcal{L}_{V_i} \subset \mathcal{L}_{x^TP^*x}$. 
A novel composition of control Lyapunov functions for constrained uncertain linear systems

A “good” control solution can be obtained by fixing $P = P^*$ in the inequality (2.8) of Theorem 2.5. The proposed composite CLF has large controlled invariant set and inner sublevel sets close to the quadratic optimal ones, as shown in the example of Section 2.4.1.

Considering the approximate Hamilton–Jacobi–Bellman equation, an explicit state-feedback control law is here proposed.

$$\kappa(x) := \arg\min_{\nu \in U} \nabla V_\lambda(x) (Ax + Bu) + x^T Qx + u^T Ru \quad \text{subject to:}$$

$$\nabla V_\lambda(x) (Ax + Bu) + \eta V_\lambda(x) \leq 0. \quad (2.12)$$

It can be proved that control (2.12) follows from the minimal selection control [37, Section 2.4] and therefore it is continuous [37, Section 4.2], as $\nabla V_\lambda$ is continuous in the interior of the domain $L_{V_\lambda}$.

The control law (2.12) requires the on-line solution of a Quadratic Program (QP) in $\mathbb{R}^m$. Namely, the computational effort required to be performed online is quite low, especially if compared with typical optimization-based control laws as, for instance, receding horizon control laws.

This kind of approach is “memoryless” and therefore differs from explicit model predictive control [13] where the state space is off-line partitioned in a certain (often huge) number of polyhedral regions, whose number grows exponentially with the prediction horizon, leading to huge requirements of memory to be checked in the on-line search of the “current region”. However, while [50] provides a sequence of sub-optimal QP solutions converging to the optimal one, no theoretical bounds of sub-optimality are discussed here.
2.4 Application to approximate constrained linear quadratic optimal control

2.4.1 Illustrative example: robust stabilization with large domain of attraction and locally-optimal performance via composite control Lyapunov functions

We consider the constrained linear system $\dot{x} = Ax + Bu$ (2.10) from [23], with

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$  

The performance cost $J$ (2.11) has $Q = 0.15I_2$ and $R = 0.3I_2$.

Let us consider the controlled invariant set $\mathcal{L}_{V_1}$ of [23], where $V_1(x) := \|Fx\|_\infty$. We fix $P$ to be the solution of the ARE and we scale $V_2$ so that $\mathcal{L}_{V_1} \subseteq \mathcal{L}_{V_2}$. The same controlled invariant set is also recovered for the composite CLF $V_\lambda$ (2.4) since the LMI problem (2.8) is feasible.

For comparisons, we use the control law (2.12) associated with some differentiable CLFs: the smoothed PCLF [24], the smoothed truncated ellipsoid, i.e. the composite CLF with $\phi = 1$ of [12], besides our novel composite CLF $V_\lambda$ with $\phi = 100$. Note that a non-differentiable CLF, for instance a PCLF or a truncated ellipsoid, yields considerable control chattering [24] and hence much worse closed-loop performances. The constrained optimal control [50] is also used for comparisons.

In [11, Section 5.1] it is shown that our novel composite CLF yields less stress on the control actuators and “good” closed-loop performances. From our numerical experience, as one would expect when gradient-based control laws are used, the closed-loop behavior...
A novel composition of control Lyapunov functions for constrained uncertain linear systems is “close” to the one induced by the CLF $V_1$ ($V_2$) if $\phi \ll 1$ ($\phi \gg 1$). The level sets of $V_\wedge$ are shown in Figure 2.2.

Figure 2.2: The level sets of the novel composite control Lyapunov function $V_\wedge$, which is the composition with $\phi := 100$ of a smoothed polyhedral control Lyapunov function and the Riccati-optimal quadratic control Lyapunov function, are shown on the controlled domain $\mathcal{L}_{V_\wedge}$.
Chapter 3

Control-sharing and merging control Lyapunov functions

Given two control Lyapunov functions (CLFs), a “merging” is a new CLF whose gradient is a positive combination of the gradients of the two parents CLFs. The merging function is an important trade-off since this new smooth function may, for instance, approximate one of the two parents functions close to the origin while being close to the other far away.

Recently, Andrieu and Prieur [1, 2] proved that it is possible to merge two CLFs, in a setting actually related to the problem of uniting local and global controllers [64, 63]. Their technique works under the assumption that there exists a suitable domain in which the two control Lyapunov function share a common control [2, Proposition 2.2]. More recently, Clarke [32] showed how to solve the prob-
lem of merging two semiconcave (continuous, locally Lipschitz but not everywhere-differentiable) CLFs, deriving a semiconcave function based on the min operator.

In this chapter we investigate the control-sharing property, namely the existence of a single control law which makes simultaneously negative the Lyapunov derivatives of two given Lyapunov functions. We show some equivalence properties about the control sharing and the possibility of adopting a merging procedure.

The control-sharing property is not necessarily satisfied even for linear systems, with the remarkable exception of the planar case (i.e. with two-dimensional state space). Therefore, for the class of constrained uncertain linear systems, we provide efficient, Linear Matrix Inequalities (LMIs) based, computational tests to check the control-sharing property for some special classes of functions including polyhedral, quadratic, piecewise quadratic and truncated ellipsoids.

Finally we provide as merging example the technique based on the theory of “R-functions”, and we show how local optimality can be compromised with a large controlled Domain of Attraction (DoA), under constraints, adopting a single smooth CLF.

The essential results of the chapter are summarized next.

• For planar linear time-invariant systems two convex CLFs always share a control. A third-order counterexample shows that this is not true in general.

• Given two CLFs $V_1, V_2$, a merging function $V$ is defined as any positive definite function whose gradient has the form $\nabla V(x) = \gamma_1(x)\nabla V_1(x) + \gamma_2(x)\nabla V_2(x)$, where $\gamma_1, \gamma_2 : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ are continuous functions. For the class of control-affine non-
3.1 Technical background and negative results

linear systems, it is shown that any merging function $V$ (i.e. for any possible $\gamma_1$ and $\gamma_2$) is also a CLF if and only if $V_1$ and $V_2$ share a stabilizing control.

• For the class of linear systems, the above statements are also equivalent to the existence of a “regular” type merging, namely, the case in which $\nabla V$ is “close” to $\nabla V_1$ far from the state-space origin and $\nabla V$ is “close” to $\nabla V_2$ in a neighborhood of the origin.

• Several conditions are provided to check the control-sharing property. These are based on Linear Programming (LP) in the case of piecewise-linear functions, and on LMIs in the case of piecewise-quadratic and truncated-ellipsoidal functions.

• The “R-composition” merging technique presented in [11] is considered to solve the problem of preserving the large DoA under constraints of one Lyapunov function and assuring local optimality guaranteed by the other at the same time.

3.1 Technical background and negative results

Let us consider nonlinear control-affine systems

$$\dot{x} = f(x) + g(x)u,$$  \hspace{1cm} (3.1)

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $f : \mathbb{R}^n \to \mathbb{R}^n$, with $f(0) = 0$, $g : \mathbb{R}^n \to \mathbb{R}^{n \times m}$ are locally-bounded functions. We also consider the following notion of control Lyapunov function.
Definition 3.1 (Control Lyapunov Function) A positive definite, radially unbounded, smooth away from zero, function $V : \mathbb{R}^n \to \mathbb{R}_\geq 0$ is a control Lyapunov function for (3.1) if there exists a locally-bounded control law $u : \mathbb{R}^n \to \mathbb{R}^m$ such that for all $x \in \mathbb{R}^n$ we have

$$\nabla V(x)(f(x) + g(x)u(x)) < 0. \quad (3.2)$$

$V$ is a control Lyapunov function with domain $L_{(V/\sigma)}$, for $\sigma > 0$, if (3.2) holds for all $x \in L_{(V/\sigma)}$.

The following definition is fundamental in the sequel.

Definition 3.2 (Control-Sharing Property) Two control Lyapunov functions $V_1$ and $V_2$ for (3.1) have the control-sharing property if there exists a locally-bounded control law $u : \mathbb{R}^n \to \mathbb{R}^m$ such that for all $x \in \mathbb{R}^n$ we have the following inequalities simultaneously satisfied.

$$\nabla V_1(x)(f(x) + g(x)u(x)) < 0 \quad (3.3a)$$

$$\nabla V_2(x)(f(x) + g(x)u(x)) < 0 \quad (3.3b)$$

$V_1$ and $V_2$ have the control-sharing property under constraints $x \in X \subseteq \mathbb{R}^n$, $u \in U \subseteq \mathbb{R}^m$ if (3.3) holds for all $x \in X$ with a constrained control law $u : X \to U$.

For the class of control-affine differential inclusions

$$\dot{x} \in F(x) + G(x)u, \quad (3.4)$$

where $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ and $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^{n \times m}$ are compact-valued mappings, the previous definitions hold unchanged provided that conditions (3.2) and (3.3) holds with $\dot{x} = \varphi + \Gamma u$, for all $(\varphi, \Gamma) \in (F(x), G(x))$. 

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3.1 Technical background and negative results

3.1.1 Negative results on control sharing, even for linear systems

Let us also consider linear time-invariant (LTI) systems

\[ \dot{x} = Ax + Bu, \quad (3.5) \]

where \( x \in \mathbb{R}^n, u \in \mathbb{R}^m, A \in \mathbb{R}^{n \times n} \) and \( B \in \mathbb{R}^{n \times m} \).

For second-order systems, we have the following result on the control-sharing property.

**Theorem 3.1** Two convex CLFs for (3.5) do necessarily have the control-sharing property if \( n \leq 2 \).

**Remark 3.1** The previous result extends that provided in [1, Proposition 2], where it is shown that for planar linear systems there always exists a common control law between two quadratic CLFs. Here we show that such a property is valid for convex CLFs of any class.

However, even for second-order systems, the previous result is not “robust”. Consider the class of Linear Differential Inclusions (LDIs)

\[ \dot{x} \in \text{co} \{ A_i x + B_i u \mid i \in [1, N] \}, \quad (3.6) \]

for some \( N > 0, A_i \in \mathbb{R}^{n \times n} \) and \( B_i \in \mathbb{R}^{n \times m} \) for all \( i \in [1, N] \).

The result of Theorem 3.1 does not hold for this class of systems according to the following result.

**Proposition 3.2** Two CLFs for (3.6) do not necessarily have the control-sharing property.
In general, for \( n > 2 \), the control-sharing property does not hold even for LTI systems.

**Proposition 3.3** Two CLFs for (3.5) do not necessarily have the control-sharing property if \( n > 2 \).

### 3.2 Gradient-type merging control Lyapunov functions

#### 3.2.1 Merging homogeneous control Lyapunov functions

In the sequel, all the results refer to \( V_1 \) and \( V_2 \) being given CLFs.

**Standing Assumption 3.1** Functions \( V_1, V_2 : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0} \) are two CLFs.

**Definition 3.3 (Gradient-type merging CLF)** Let \( V : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0} \) be positive definite and smooth away from zero. \( V \) is a gradient-type merging candidate if there exist two continuous functions \( \gamma_1, \gamma_2 : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0} \) such that \((\gamma_1(x), \gamma_2(x)) \neq (0, 0)\) and

\[
\nabla V(x) = \gamma_1(x) \nabla V_1(x) + \gamma_2(x) \nabla V_2(x). 
\]

(3.7)

\( V \) is a gradient-type merging CLF if, in addition, it is a CLF.

**Remark 3.2** The blending CLF \( V(x) = \min\{V_2(x), c \cdot V_1(x) + d\} \) [32, Section 9], for opportune constants \( c, d > 0 \), does not fall into the class of gradient-type merging because it is not a differentiable
3.2 Gradient-type merging control Lyapunov functions

function. However it can be approximated with arbitrary precision by the “smoothed min” $V = \sqrt[2p]{V_1^p + (c \cdot V_1 + d)^p}$ for $p < 0$ small enough.

All the possible merging functions form a class much wider of those considered specifically later (based on “R-compositions”). For instance, the “smoothed max” $V = \sqrt[2p]{V_1^p + V_2^p}$, for $p > 0$, or $V = \gamma_1(V_1, V_2)V_1 + \gamma_2(V_1, V_2)V_2$ are possible merging candidates.

3.2.2 Gradient-type merging for nonlinear control-affine differential equations

For nonlinear systems (3.1), we show that any gradient-type merging candidate is a CLF if and only if there exists a common stabilizing control law between the CLFs $V_1$ and $V_2$.

Theorem 3.4 The following statements are equivalent for (3.1).

1. Any gradient-type merging of $V_1$ and $V_2$ is a CLF.

2. $V_1$ and $V_2$ have the control-sharing property.

Remark 3.3 The main contribution of Theorem 3.4 relies on the necessity of the existence of a common control law, i.e. implication 1) $\implies$ 2); conversely, the sufficient part, i.e. 2) $\implies$ 1) may follow from the results in [2, Theorem 1, Proposition 1]. We also notice that since the system (3.1) is control-affine, the existence of a stabilizing common control law is equivalent to the existence of a continuous stabilizer, see [32, Theorem 1.5], [77, Section 5.9].
3.2.3 Regular gradient-type merging

The property that any gradient-type merging of two CLFs is a CLF is quite strong. In practice we will be interested in the case in which the gradient-type merging candidate \( V \) has the same domain of \( V_1 \), namely \( \mathcal{L}_V = \mathcal{L}_{V_1} \); \( V \) has its gradient \( \nabla V(x) \) aligned with \( \nabla V_1(x) \) whenever \( x \in \partial \mathcal{L}_V \), while (“almost”) aligned with \( \nabla V_2(x) \) whenever \( x \) is “close” to the origin.

**Definition 3.4 (Regular gradient-type merging CLF)** A gradient-type merging candidate \( V \) is regular with tolerance \( \varepsilon \geq 0 \) if \( \mathcal{L}_V = \mathcal{L}_{V_1} \) and the associated functions \( \gamma_1, \gamma_2 \) satisfy the following conditions.

\[
\{ \gamma_1(x) = 1 \text{ and } \gamma_2(x) = 0 \} \iff x \in \partial \mathcal{L}_{V_1};
\]

\[
0 \leq \gamma_1(0) \leq \varepsilon \text{ and } 1 - \varepsilon \leq \gamma_2(0) \leq 1.
\]

A gradient-type merging candidate \( V \) is regular if it is regular with tolerance \( \varepsilon = 0 \). \( V \) is a regular gradient-type merging CLF if, in addition, it is a CLF.

We then consider regular control laws \( u(\cdot) \), namely we consider a “small control property” meaning that \( u(x) \) goes to 0 at least linearly as \( x \) goes to 0.

**Definition 3.5** A control law \( u : \mathbb{R}^n \rightarrow \mathbb{R}^m \) is regular if it is continuous and for any given \( x \in \mathbb{R}^n \) the limit

\[
\bar{u}_x := \lim_{{\lambda \to 0^+}} \frac{u(\lambda x)}{\lambda}
\]

exists and satisfies \( \|\bar{u}_x\| < \infty \).
3.2 Gradient-type merging control Lyapunov functions

For instance, for linear control laws \( u(x) = Kx \), we just have \( \bar{u}_x = Kx \).

For linear systems \( (3.5) \), we have the following result for the regular gradient-type merging.

**Theorem 3.5** Assume that \( V_1 \) and \( V_2 \) are positively homogeneous CLF of the same degree, each associated with a regular control. Then, the following statements are equivalent for \( (3.5) \).

1. There exists a regular gradient-type merging CLF associated with a regular control.

2. Any gradient-type merging is a CLF associated with a regular control.

3. \( V_1 \) and \( V_2 \) share a regular control.

**Remark 3.4** Assuming positively homogeneous CLFs is a limitation. Choosing the same degree of homogeneity is without loss of generality because, if \( \dot{V} \leq -\eta V \), for some \( \eta > 0 \), then \( \dot{(V^p)} \leq -\eta p V^p \) for any real \( p > 0 \).

We can relate our “merging” CLFs to the literature on “blending” CLFs \([32]\) and “uniting” CLFs \([2, 63]\) as follows. In \([32]\) Theorem 9.1, it is shown that from the knowledge of two CLFs \( V_1, V_2 \), it is possible to build up a “blending” CLF of the form \( V(x) = \min\{V_1(x), cV_2(x) + d\} \), for appropriate \( c, d \geq 0 \), so that \( V \) necessarily admits a stabilizing control law \( \kappa : \mathbb{R}^n \to \mathbb{R}^m \) of the form \( \kappa(x) \in \{\kappa_1(x), \kappa_2(x)\} \). We show that even for linear systems \( (3.5) \), the result does not necessarily hold for gradient-type merging CLFs,
Control-sharing and merging control Lyapunov functions

namely because of the differentiability property of gradient-type merging candidates.

**Proposition 3.6** Assume $\kappa_1, \kappa_2 : \mathbb{R}^n \to \mathbb{R}^m$ are control laws respectively associated with $V_1$ and $V_2$. Then, even for linear systems (3.5), a regular gradient-type merging CLF $V$ does not necessarily admit a control law of the kind $\kappa(x) \in \{\kappa_1(x), \kappa_2(x)\}$.

**Remark 3.5** For nonlinear control-affine systems, [63, Section 2.2] shows that there exists a topological obstruction in uniting a local and a global control law by means of a static time-invariant continuous control law. It follows from the proof of Proposition 3.6, see Appendix B.3.6, that such a obstruction is also valid for the class of linear systems whenever we look for a control law of the kind used in the proof of [32, Theorem 9.1].

### 3.2.4 Gradient-type merging for nonlinear control-affine differential inclusions

We now consider nonlinear differential inclusions (3.4) and we provide the following results.

**Proposition 3.7** If $V_1$ and $V_2$ have the control-sharing property for (3.4), then any gradient-type merging is a CLF.

**Theorem 3.8** Assume that, in (3.4), the mapping $G$ is single-valued. Then the following statements are equivalent for (3.4).

1. Any gradient-type merging is a CLF.
3.3 Linear differential inclusions: conditions for the existence of a common control law

2. \( V_1 \) and \( V_2 \) have the control-sharing property.

The result of Theorem 3.8 does also apply to LDIs (3.6) having \( B_i = B \) for all \( i \in [1, N] \).

3.3 Linear differential inclusions: conditions for the existence of a common control law

In this section we consider the class of LDIs (3.6) and we propose several matrix inequality conditions for the existence of a common control law between the CLFs \( V_1 \) and \( V_2 \). For ease of presentation, the matrix conditions presented next do not include the control constraints; however, it is worth mentioning that they can be considered without conceptual difficulties. We address the following classes of homogeneous functions: (symmetric) polyhedral, quadratic, max of quadratics and truncated ellipsoid.

Remark 3.6 Note that some of the mentioned functions are non-smooth. However, we can apply the smoothing procedure in [24]. For instance, if \( \|Fx\|_\infty^2 \) is a polyhedral CLF (PCLF) with a certain control law \( \kappa \) for an LDI (3.6), the same control law \( \kappa \) assures that \( \|Fx\|_{2p}^2 \) is a Lyapunov function if \( p > 0 \) is taken large enough [24]. Therefore if the CLF \( V_1(x) = \|Fx\|_\infty^2 \) shares a control with the CLF \( V_2 \), then also \( \|Fx\|_{2p}^2 \) does for \( p \) sufficiently large.

Let \( V_p : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0} \) be a positive definite polyhedral function and let \( X = [x_1 | x_2 | ... | x_s] \in \mathbb{R}^{n \times s} \) be the matrix whose columns are
the vertices of $\mathcal{L}_{V_p}$, i.e. \cite[Equation (4.28)]{26}

$$V_p(x) := \min \left\{ \sum_{j=1}^{s} \alpha_j : x = X\alpha, \ \alpha \geq 0 \right\} = \min \left\{ \sum_{j=1}^{s} \alpha_j : x = \sum_{j=1}^{s} \alpha_j x_j, \ \alpha_j \geq 0 \right\}. \quad (3.8)$$

Then $V_p$ is a PCLF for (3.6) if and only if there exist $\eta > 0$, $M$-matrices $W_1, W_2, ..., W_N \in \mathbb{R}^{s \times s}$ and $U \in \mathbb{R}^{m \times s}$ such that for all $i \in [1, N]$ we have \cite[Proposition 7.19]{26}

$$A_i X + B_i U = XW_i, \quad \Gamma_s W_i \leq -\eta \Gamma_s. \quad (3.9)$$

The following result is technical, and it will be exploited later. It states that given a PCLF $V_p$ represented by a matrix $X$, we can always add a “redundant vertex”, either in the interior $\text{int} \mathcal{L}_{V_p}$ or on the boundary $\partial \mathcal{L}_{V_p}$, achieving a feasibility condition similar to (3.9).

**Lemma 3.9** Assume (3.9) is feasible. Given $\alpha_1, \alpha_2, ..., \alpha_s \geq 0$ such that $\sum_{j=1}^{s} \alpha_j = 1$, consider $\bar{x} = \sum_{j=1}^{s} \alpha_j x_j$ and let $\bar{X} := [X | \bar{x}] \in \mathbb{R}^{n \times (s+1)}$. Then there exist $M$-matrices $\bar{W}_1, \bar{W}_2, ..., \bar{W}_N \in \mathbb{R}^{(s+1) \times (s+1)}$ such that for all $i \in [1, N]$ we have

$$A_i \bar{X} + B_i \bar{U} = \bar{X} \bar{W}_i, \quad \Gamma_{s+1} \bar{W}_i \leq -\eta \Gamma_{s+1}, \quad (3.10)$$

where $\bar{U} := [U | \bar{u}]$ with $U = [u_1 | u_2 | ... | u_s]$ and $\bar{u} = \sum_{j=1}^{s} \alpha_j u_j$.

Let us consider the case of two PCLFs. In view of Lemma 3.9, according to the construction of Figure 3.1, for any vertex $x_k^1$ of $\mathcal{L}_{V_1}$ we add a “fictitious” redundant vertex $\tilde{x_k}^1$ on the boundary of $\mathcal{L}_{V_2}$ aligned with $x_k^1$ and vice-versa, so augmenting both the describing matrices $X_1$ and $X_2$. We have the following result.
3.3 Linear differential inclusions: conditions for the existence of a common control law

Figure 3.1: In the construction of Theorem 3.10 we add redundant vertices to $L_{V_1}$ and $L_{V_2}$: the black points are “true vertices”, while the white points are “fictitious vertices”.

---

55
Theorem 3.10 Assume that $V_1$ and $V_2$ are two PCLFs of the form (3.8), with $X_1 = [x_1^1|...|x_{s_1}^1]$ and $X_2 = [x_1^2|...|x_{s_2}^2]$, respectively. For each column of $X_1$, namely each vertex $x_1^k$, take point $\tilde{x}_1^k := cx_1^k \in \partial \mathcal{L}_{V_2}$, for some $c > 0$ (see Fig. 3.1). Analogously, take $\tilde{x}_2^k := cx_2^k \in \partial \mathcal{L}_{V_1}$, for some $c > 0$. Define $\bar{X}_1 := [X_1|\tilde{x}_1^1|...|\tilde{x}_1^{s_1}] \in \mathbb{R}^{n \times (s_1 + s_2)}$ and $\bar{X}_2 := [\tilde{x}_1^1|...|\tilde{x}_1^{s_1}X_2] \in \mathbb{R}^{n \times (s_1 + s_2)}$.

Then $V_1$ and $V_2$ have the control-sharing property if there exist $\eta > 0$, $\mathcal{M}$-matrices $\bar{W}_1^1, ..., \bar{W}_N^1 \in \mathbb{R}^{(s_1 + s_2) \times (s_1 + s_2)}$, $\bar{W}_1^2, ..., \bar{W}_N^2 \in \mathbb{R}^{(s_1 + s_2) \times (s_1 + s_2)}$, and $\bar{U} \in \mathbb{R}^{m \times (s_1 + s_2)}$ such that for all $i \in [1, N]$ we have

\[
A_i \bar{X}_1 + B_i \bar{U} = \bar{X}_1 \bar{W}_i^1 \quad \bar{1}^\top \bar{W}_i^1 \leq -\eta \bar{1}^\top \tag{3.11a}
\]

\[
A_i \bar{X}_2 + B_i \bar{U} = \bar{X}_2 \bar{W}_i^2 \quad \bar{1}^\top \bar{W}_i^2 \leq -\eta \bar{1}^\top \tag{3.11b}
\]

simultaneously satisfied.

We now consider the control-sharing between polyhedral and quadratic CLF (QCLF) for (3.6).

Theorem 3.11 Assume that $V_1 = V_p$ as in (3.8) and $V_2(x) = x^\top P x$ respectively are PCLF and QCLF for (3.6). Let $r$ be the number of facets of $\mathcal{L}_{V_1}$ and let $\mathcal{V}_k$ be the set of the vertices belonging to the $k^{th}$ facet, whose cardinality is $s_k \in [1, s]$. For all $k \in [1, r]$ and $i \in [1, N]$, define the matrices $S_k,i(\eta, U) \in \mathbb{R}^{s_k \times s_k}$ componentwise as

\[
[S_k,i(\eta, U)]_{h,j} := x_h^\top P ((A_i + \eta I_n)x_j + B_i u_j) + x_j^\top P ((A_i + \eta I_n)x_h + B_i u_h), \tag{3.12}
\]
3.3 Linear differential inclusions: conditions for the existence of a common control law

where \( x_h, x_j \in V_k \). Then \( V_1 \) and \( V_2 \) have the control-sharing property if there exist \( \eta > 0 \), \( M \)-matrices \( W_1, W_2, \ldots, W_N \in \mathbb{R}^{s \times s} \) and \( U = [u_1 \ldots u_s] \in \mathbb{R}^{m \times s} \) such that (3.9) holds and the matrices \( -S_{k,i}(\eta, U) \) are copositive for all \( k \in [1,r] \) and \( i \in [1,N] \).

The condition proposed in Theorem 3.11 requires the solution of a copositive programming problem. This problem is convex, but still hard to solve. A sufficient condition which can be checked via LP is that the matrices \( S_{k,i}(\eta, U) \) have non-positive elements.

**Corollary 3.12** Under the assumptions of Theorem 3.11, \( V_1 \) and \( V_2 \) have the control-sharing property if there exist \( \eta > 0 \), \( M \)-matrices \( W_1, W_2, \ldots, W_N \in \mathbb{R}^{s \times s} \) and \( U \in \mathbb{R}^{m \times s} \) such that (3.9) holds and the elements (3.12) of \( S_{k,i}(\eta, U) \) are non-positive for all \( k \in [1,r] \) and \( i \in [1,N] \).

Then, we consider positive definite 0-symmetric functions \( V_s : \mathbb{R}^n \to \mathbb{R}_{\geq 0} \) defined as

\[
V_s(x) := \max \{ x^\top Q_k x \mid k \in [1,s] \}
\]

for some \( Q_1, Q_2, \ldots, Q_s \succ 0 \), hence covering the case of symmetric polyhedral functions, truncated ellipsoids and max of quadratics.

**Theorem 3.13** Assume that \( V_1 = V_s \) (3.13) and \( V_2(x) = x^\top Px \) respectively are CLF and QCLF for (3.6). Then \( V_1 \) and \( V_2 \) have the control-sharing property if there exist \( \eta > 0 \), \( \lambda_{i,j,k} \geq 0 \), \( K_k \in \mathbb{R}^{m \times n} \), for \( i = 1, 2, \ldots, N \), and \( j, k = 1, 2, \ldots, s \), such that

\[
(A_i + B_i K_k)^\top Q_k + Q_k (A_i + B_i K_k) \preceq -2\eta Q_k + \sum_{i=1}^s \lambda_{i,j,k} (Q_j - Q_k)
\]

\[
(A_i + B_i K_k)^\top Q_k + Q_k (A_i + B_i K_k) \preceq -2\eta Q_k + \sum_{i=1}^s \lambda_{i,j,k} (Q_j - Q_k)
\]

\[\text{(3.14a)}\]

\(^1M \text{ is copositive if } x^\top M x \geq 0 \text{ for all nonnegative vectors } x.\]
Control-sharing and merging control Lyapunov functions

\[(A_i + B_i K_k)^\top P + P(A_i + B_i K_k) \preceq -2\eta P + \sum_{i=1}^s \lambda_{i,j,k} (Q_j - Q_k)\]

(3.14b)

for all \(i \in [1, N], k \in [1, s]\).

Remark 3.7 Theorem 3.13 is more general than [11, Theorem 2], because condition (3.14) relies on a piecewise-linear common control law, rather than a linear common control law as in [11, matrix conditions (11)].

3.4 The composition via R-functions as an example of regular gradient-type merging

We start by considering as an example the double integrator system

\[\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u\]

with constraints \(\|x\|_\infty \leq 1, \|u\|_\infty \leq 1\).

A typical problem is to choose between a CLF \(V_1(x)\) assuring a “large” domain of attraction, see Figure 3.2 (top), or a function which is “locally optimal” in some sense, such as \(V_2(x) = x^\top P x\).

The main idea is compromising the two given functions by a non-homogeneous one which looks like \(V_2(x)\) close to 0 and like \(V_1(x)\) far from 0 as in Fig. 3.2 (bottom). A CLF with such characteristics is an example of what we call (regular) gradient-type merging CLF.

In this section, we indeed investigate the “R-composition” between two homogeneous CLFs proposed in [11] [10], which is shown
3.4 The composition via R-functions as an example of regular gradient-type merging

Figure 3.2: Level sets of the smoothed-polyhedral function $V_1$ (top) and of a merging function $V_\land$ (bottom), composed from $V_1$ and $V_2$. 
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to be a regular gradient-type merging CLF in the sequel. The composition consists of the following steps.

1) Define \( R_1, R_2 : \mathbb{R}^n \to \mathbb{R} \) as \( R_i(x) = 1 - V_i(x) \), \( i = 1, 2 \).

2) For fixed \( \phi > 0 \), define \( R^\wedge : \mathbb{R}^n \to \mathbb{R} \) as

\[
R^\wedge(x) := \rho(\phi) \left( \phi R_1(x) + R_2(x) - \sqrt{\phi^2 R_1(x)^2 + R_2(x)^2} \right),
\]

where \( \rho(\phi) := \left( \phi + 1 - \sqrt{\phi^2 + 1} \right)^{-1} \) is the normalization factor \[^{11}\] \( \text{Section 2} \).

3) Define the “R-composition” \( V^\wedge : \mathbb{R}^n \to \mathbb{R}_{\geq 0} \) as

\[
V^\wedge(x) := 1 - R^\wedge(x).
\]

It turns out that \[^{11}\] \text{Proof of Theorem 1} \]

\[
\nabla V^\wedge(x) = \rho(\phi) \left[ \phi c_1(\phi, x) \nabla V_1(x) + c_2(\phi, x) \nabla V_2(x) \right],
\]

where \( c_1, c_2 : \mathbb{R}_{>0} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0} \) are defined as

\[
c_1(\phi, x) := 1 + \frac{-\phi R_1(x)}{\sqrt{\phi^2 R_1(x)^2 + R_2(x)^2}},
\]

\[
c_2(\phi, x) := 1 + \frac{-R_2(x)}{\sqrt{\phi^2 R_1(x)^2 + R_2(x)^2}}.
\]

It follows from the properties of the “R-functions”, see Appendix \[^{A}\] \text{that} \( V^\wedge \) is positive definite (Lemma \[^{A.1}\] \text{Lemma A.1} \), differentiable in \( \text{int} L_{V^\wedge} \) (Lemma \[^{A.2}\] \text{Lemma A.2} \), and that \( L_{V^\wedge} = L_{V_1} \cap L_{V_2} \) (Lemma \[^{A.3}\] \text{Lemma A.3} \).

The function \( V^\wedge \), namely the merging of \( V_1 \) and \( V_2 \) from Standing Assumption \[^{3.1}\] \text{will be used as a candidate CLF later on.}

\[^2\]The level set 1 is taken without loss of generality. With this choice we have \( R_i(x) \geq 0 \iff x \in L_{V_i} \).

\[^3\]For ease of reading, the dependence of \( R^\wedge \) from \( \phi \) is not made explicit in the notation.
3.4 The composition via R-functions as an example of regular gradient-type merging

**Proposition 3.14** $V_\wedge$ is a gradient-type merging candidate.

We can now show that $V_\wedge$ is a regular merging-type candidate with arbitrarily small tolerance.

**Proposition 3.15** Let $L_{V_2} \supset L_{V_1}$. Then for any $\varepsilon > 0$ and $\delta \in (0,1)$ there exists $\bar{\phi} > 0$ such that for all $\phi \geq \bar{\phi}$ we have that $V_\wedge$, with domain $L_{V_\wedge/\delta}$, is a regular gradient-type merging candidate with tolerance $\varepsilon$.

According to Theorem 3.4 and Proposition 3.7, if $V_1$ and $V_2$ are CLFs for (3.4) and share a constrained control law $\kappa$, then $\kappa$ is admissible as well for $V_\wedge$, which turns out to be a CLF for (3.4) under constraints.

It follows from the proof of Lemma A.1 that, independently from $\phi > 0$, the unit level set of $V_\wedge$ is $\partial L_{V_\wedge} = \{ x \in \mathbb{R}^n | \max\{V_1(x), V_2(x)\} = 1 \}$. Conversely, in $\text{int} L_{V_\wedge}$, $\phi$ imposes a trade-off between the shape of the level sets of $V_1$ and of $V_2$. Namely, in light of [11, Proposition 2], we have $V_\wedge(x) \xrightarrow{\phi \to \infty} V_2(x)$ and $V_\wedge(x) \xrightarrow{\phi \to 0^+} V_1(x)$, point-wise in $\text{int} L_{V_\wedge}$. Moreover, according to Lemmas A.4, A.5, A.6, we have $\nabla V_\wedge(x) \xrightarrow{\phi \to \infty} \nabla V_2(x)$ and $\nabla V_\wedge(x) \xrightarrow{\phi \to 0^+} \nabla V_1(x)$ uniformly on compact subsets of $\text{int} L_{V_\wedge}$.

This particular property of fixing the “external” shape, while making the “inner” one “close” to any given choice can be exploited to fix a “large” DoA while achieving “locally-optimal” closed-loop performances.

**Remark 3.8** We remind that the (smoothed) polyhedral functions of the kind [28, 57, 18, 24], composite quadratics [45] and the convex...
Control-sharing and merging control Lyapunov functions

The hull of quadratics [47] are universal classes of homogeneous functions for the stability/stabilizability of LDIs (3.6). Exploiting Lemma A.6, we can merge one of them with any $V_2$ (homogeneous of degree 2) to indeed achieve a new class of universal non-homogeneous Lyapunov functions [39, Section IV].

3.4.1 Explicit Lyapunov-based control design under state and input linear constraints

We now investigate the existence of a continuous locally-optimal control under constraints $x \in L_{V_1}$ and $u \in U \subseteq \mathbb{R}^m$ which is closed (possibly compact) and convex. For simplicity, we consider (3.6) with $B_i = B$ for all $i \in [1, N]$. Since the CLF $V_\lambda$ is differentiable, in principle, the existence of a stabilizing control law $\kappa$ continuous with the exception of the origin, or including $x = 0$ if $V_\lambda$ satisfies the small control property\footnote{A CLF $V$ satisfies the small control property if, for $u := \kappa(x)$, we have that for all $v \in \mathbb{R}_{>0}$ there exists $\epsilon \in \mathbb{R}_{>0}$ so that, whenever $\|x\| < \epsilon$ we have $\|u\| < v$ [77].} could be proved by using the arguments in [37, Chapters 2–4].

To have $L_{V_\lambda} = L_{V_1}$, we preliminary scale $V_2$ so that $L_{V_2} \supset L_{V_1}$. In light of Proposition 3.7, we formulate the control-sharing assumption, which can be checked using the results in Section 3.3.

**Assumption 3.1** Functions $V_1$ and $V_2$ have the control-sharing property under constraints $x \in L_{V_1}$, $u \in U$. Associated with $V_2$ there is an “optimal” continuous control law $\kappa_2 : \mathbb{R}^n \to \mathbb{R}^m$ such that $\kappa_2(x) \in U$ for all $x$ in a neighborhood of the origin.
3.4 The composition via R-functions as an example of regular gradient-type merging

It follows from the proof of Proposition 3.7 that, under Assumption 3.1, $V_{\lambda}$ is a CLF for (3.7) under constraints. Namely, since for all $x \in \mathbb{R}^n$ we have $\min \{ V_1(x), V_2(x) \} \leq V_{\lambda}(x) \leq \max \{ V_1(x), V_2(x) \}$ \[11, Proposition 1, Section 4.2\], there exists $\eta > 0$ such that the following convex-valued mapping of admissible (constrained) controls is non-empty for all $x \in \mathcal{L}_{V_{\lambda}}$.

$$\mathcal{U}(x) := \left\{ u \in \mathbb{U} \mid \max_{i \in \{1, N\}} \nabla V_{\lambda}(x)(A_i x + B u) + \eta x^\top x \leq 0 \right\}. \quad (3.19)$$

We indeed propose the control law

$$\kappa(x) := \arg \min_{v \in \mathcal{U}(x)} \|v - \kappa_2(x)\|. \quad (3.20)$$

**Theorem 3.16** Suppose Assumption 3.1 holds. Then the control law $\kappa (3.20)$ associated with $V_{\lambda} (3.16)$ is continuous, satisfies the constraints in $\mathcal{L}_{V_1}$, and is locally optimal.

**Remark 3.9** In the case of constrained “linear-quadratic” (LQ) stabilization, the approximate Hamilton–Jacobi–Bellman control $\tilde{\kappa}(x) := \arg \min_{v \in \mathcal{U}(x)} \{ \nabla V_{\lambda}(x)(A x + B v) + x^\top Q x + v^\top R v \}$ has been proposed in \[11, Section 5\]. An advantage of $\kappa (3.20)$ over $\tilde{\kappa}$ is that, according to Theorem 3.16, local optimality is here guaranteed.

### 3.4.2 Illustrative example: stabilization of an inverted pendulum via merging control Lyapunov functions

We address the constrained stabilization of a simplified inverted pendulum, whose dynamics is given by the nonlinear differential equation $I \ddot{\theta}(t) = mgl \sin(\theta(t)) + \tau(t)$. The goal is the stabilization of $(\theta, \dot{\theta})$...
Control-sharing and merging control Lyapunov functions

to the origin, under the constraints $|\theta| \leq \pi/4$, $|\dot{\theta}| \leq \pi/4$ and $|\tau| \leq 2$.

With notation $x_1 = \theta$, $x_2 = \dot{\theta} = \dot{x}_1$, $u = \tau$ and
\[
w(x) := \left\{ \frac{\sin(x_1)}{x_1} \mid |x_1| \leq \frac{\pi}{4} \right\},
\]

the following constrained uncertain linear model can be derived.
\[
\dot{x} \in \begin{bmatrix} 0 & 1 \\ aw(x) & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ b \end{bmatrix} u,
\]
(3.21)

where $a = (mgl/I)$, $b = (1/I)$; $w(x) \simeq [0.89, 1]$, $w(0) = 1$; $|x_1| \leq \pi/4$, $|x_2| \leq \pi/4$, $|u| \leq 2$. The numerical parameters used in the simulation are $I = 0.05$, $m = 0.5$, $g = 9.81$, $l = 0.3$.

We adopt the infinite-horizon quadratic performance cost
\[
J(x, u) := \int_0^{\infty} (\|x(t)\|_Q^2 + \|u(t)\|_R^2) dt,
\]
with weight matrices $Q = I_2$, $R = 10$. Let us indeed define the locally-optimal (i.e. for $w \equiv 1$) cost function $\bar{V}_2(x) = x^\top P x$, where $P$ is the unique solution of the Algebraic Riccati Equation. It can be shown that function $\bar{V}_1(x) = \|Fx\|_\infty^2$, with
\[
F = \begin{bmatrix} 0 & 1.53 & 4/\pi \\ 4/\pi & 0.51 & 0 \end{bmatrix}^\top,
\]
is a PCLF for the constrained LDI (3.21) and therefore also for the constrained nonlinear system. Then we define the smoothed PCLF $V_1(x) = \|Fx\|_d^2$ and we indeed focus on the DoA $\mathcal{L}_{V_1}$. Let us also define $V_2$ scaling $\bar{V}_2$, so that $\mathcal{L}_{V_2} \supset \mathcal{L}_{V_1}$. Since the LMI condition (3.13) is satisfied under constraints, $V_1$ and $V_2$ share a constrained
3.4 The composition via R-functions as an example of regular gradient-type merging

control law in $\mathcal{L}_{V_1}$, therefore any gradient-type merging is a CLF. We indeed construct the composite CLF $V_\wedge$ with $\phi = 20$.

Now, $V_1$ has a “large” DoA but it induces a “poor” performance when used with gradient-based control laws of the kind (3.20) (Figure 3.3 in fact shows that the constraint $u \in \mathcal{U}(x)$ (3.19) with $V_1$ in place of $V_\wedge$ may be “too restrictive”). On the other hand, $V_2$ is locally optimal, but both gradient-based control laws, for instance (3.20) with $V_2$ in place of $V_\wedge$, and the standard LQ regulator yield constraint violations, even in the case $w \equiv 1$. We notice that $V_\wedge$, see Figure 3.4, with control law (3.20), inherits the benefits of $V_1$ ("large" DoA under constraints) and $V_2$ (local optimality). For the linearized system (i.e. for $w \equiv 1$), our extensive Monte Carlo numerical experiments show that the closed-loop performance is “quite close” to the constrained “global optimal” (obtained via a receding “long”-horizon control law, under a “fine” system discretization).
Figure 3.3: A controlled state trajectory starting from $x_0 = (0.6, 0.1)^\top$ and converging to the origin. The state is actually “forced” to always “enter” the level sets of the smoothed PCLF $V_1$. This actually introduces some conservativism with respect to the performance to be optimized.
3.4 The composition via R-functions as an example of regular gradient-type merging

![Figure 3.4: A controlled state trajectory starting from \( x_0 = (0.6, 0.1) \) and converging to the origin “in accordance” to the level sets of the composite CLF \( V_\lambda \). We notice that the shape of the inner level curves of \( V_\lambda \) is locally optimal and so does its associated Lyapunov-based control law.](image-url)
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Chapter 4

Robust control of a chemical process via composite control Lyapunov functions

In this chapter, we apply the control-theoretic, Lyapunov-based, tools developed in the previous chapters to multivariable constrained chemical process control.

Since modern chemical processes are continuously faced with the requirements of becoming safer, more reliable, and more economical in operation, the need for a rigorous and practical approach for the design of effective chemical process control systems, inherently Multi-Input Multi-Output (MIMO), becomes increasingly apparent [34]. Moreover, the unavoidable presence of physical constraints on the process variables and in the capacity of control actuators not
only limit the nominal performance of the controlled system, but also can affect the stability of the overall system. Therefore, the stabilization of such processes is one of the most attractive research areas for the chemical and control engineering community [33].

Model Predictive Control (MPC) [52, 67], also known as Receding Horizon Control (RHC) [66, 56, 51], can handle both state and control input constraints within an optimal control setting [78], [54]. Since also the Explicit MPC [13, 50], can be quite computationally-demanding, a large literature has been developed for fast computation of sub-optimal (robust) MPC solutions, see [59], [60] among others in recent literature.

In [54], an interesting Lyapunov-based MPC approach has been proposed for the control of an exothermic chemical reaction, taking place in a Continuous Stirred Tank Reactor (CSTR). In particular, a quadratic Control Lyapunov Function (CLF) is used together with an horizon-1 MPC. However, an ellipsoidal set can not accurately fit the typically-polyhedral state constraints describing the limits on the admissible concentration of the chemical reactant and on the reactor temperature. Therefore a Quadratic CLF (QCLF) is actually not conclusive for a large part of the controllable invariant set [8]. On the other side, the estimate of the controlled invariant state-space region can be enlarged via the design of Polyhedral CLFs (PCLFs) [18], also in an MPC setting [40], composite-quadratic CLFs [47], [45], smoothed PCLFs [24], Truncated Ellipsoid (TE) CLFs [58], [79], and smoothed TE CLFs [7], [12].

Focusing on constrained uncertain linear systems, we consider the class of composite CLFs introduced in [11], characterized by a controlled invariant set which is an arbitrarily-close approximation
4.1 Simplified model of an irreversible, exothermic first-order chemical reaction taking place in a continuous stirred tank reactor

of the maximal one, eventually asymmetric, and “close-to-optimal” shape for the inner sublevel sets.

In particular, as [8, 11] do not discuss the problem of tuning the free design parameters, constructive tuning algorithms are here proposed. Unlike switching control strategies [36], within this novel approach both constraints and optimality arguments can be handled by a unique smooth CLF, together with a continuous control law [9, 8, 11, 38].

A benchmark case study, namely the simplified model of a continuous stirred tank reactor, is simulated to show the benefits of the proposed control technique.

4.1 Simplified model of an irreversible, exothermic first-order chemical reaction taking place in a continuous stirred tank reactor

Consider an irreversible, exothermic first-order reaction of the form $A \xrightarrow{k} B$, taking place in a CSTR. The inlet stream consists of pure $A$ at flow rate $F$, concentration $C_{A0}$ and temperature $T_{A0}$ [33]. The dynamic model of the process is of the form

$$\begin{align*}
\dot{C}_A &= \frac{F}{V} (C_{A0} - C_A) - k_0 \exp \left( -\frac{E}{RT_R} \right) C_A \\
\dot{T}_R &= \frac{F}{V} (T_{A0} - T_R) - \frac{\Delta H}{\rho c_p} k_0 \exp \left( -\frac{E}{RT_R} \right) C_A + \frac{Q}{\rho c_p V},
\end{align*}$$

(4.1)
Table 4.1: Process parameters and steady state values

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V$</td>
<td>0.1 m$^3$</td>
</tr>
<tr>
<td>$R$</td>
<td>8.314 kJ/(kmol °K)</td>
</tr>
<tr>
<td>$C_{A0_s}$</td>
<td>1 kmol/m$^3$</td>
</tr>
<tr>
<td>$T_{A0_s}$</td>
<td>310 °K</td>
</tr>
<tr>
<td>$Q_s$</td>
<td>0 kJ/min</td>
</tr>
<tr>
<td>$\Delta H$</td>
<td>$-4.78 \cdot 10^4$ kJ/kmol</td>
</tr>
<tr>
<td>$k_0$</td>
<td>$72 \cdot 10^9$ 1/min</td>
</tr>
<tr>
<td>$E$</td>
<td>$8.314 \cdot 10^4$ kJ/kmol</td>
</tr>
<tr>
<td>$c_p$</td>
<td>0.239 kJ/(kg °K)</td>
</tr>
<tr>
<td>$\rho$</td>
<td>1000 kg/m$^3$</td>
</tr>
<tr>
<td>$F$</td>
<td>0.1 m$^3$/min</td>
</tr>
<tr>
<td>$T_{R_s}$</td>
<td>395.3 °K</td>
</tr>
<tr>
<td>$C_{A_s}$</td>
<td>0.57 kmol/m$^3$</td>
</tr>
</tbody>
</table>

where $C_A$ denotes the concentration of the species $A$, $T_R$ denotes the temperature of the reactor, $Q$ is the heat input to the reactor, $V$ is the volume of the reactor, $k_0$, $E$, $\Delta H$ are, respectively, the pre-exponential constant, the activation energy and the enthalpy of the reaction, $c_p$ and $\rho$ are, respectively, the heat capacity and the fluid density in the reactor. The numerical values of the process parameters, taken from [54], are shown in Table 4.1.

The nonlinear model (4.1) of the reactor has to be stabilized at the unstable equilibrium point

$$\bar{x} := [C_{A_s}, T_{A_s}]^\top = [0.57 \ kmol/m^3, \ 395.3 \ °K]^\top, \ \bar{u} := 0,$$
4.1 Simplified model of an irreversible, exothermic first-order chemical reaction taking place in a continuous stirred tank reactor according to the state constraints

\[ |C_A - C_{A_0}| \leq 0.16 \text{ kmol/m}^3, \quad |T_R - T_{R_s}| \leq 3 \text{ } ^\circ \text{K}. \]

The control variables \( \nu \) are the variation of the inlet concentration of species \( A \), \( \nu_1 := \Delta C_{A0} = C_{A0} - C_{A0_s} \), and the heat input to the reactor \( \nu_2 := Q \). These manipulated control inputs are constrained as follows:

\[ |\Delta C_{A0}| \leq 1 \text{ kmol/m}^3, \quad |Q| \leq 1 \text{ kJ/h}. \]

It is indeed desired to find a continuous constrained control law \( \nu : \mathbb{R}^2 \to \mathbb{R}^2 \) that drives the state \( \xi = [C_A, T_R]^\top \) to \( \bar{x} \), also in accordance to the state constraints. In fact, discontinuous and/or chattering control laws, such as the ones usually obtained by switching controllers, are actually not well implementable on real actuators [5].

As in [33], we focus on system (4.1) under constraints, linearized in a neighborhood of the equilibrium point \((\bar{x}, \bar{u})\), namely we consider

\[
\dot{x} = \begin{bmatrix} -1.7428 & -0.0271 \\ 148.5626 & 4.4191 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 0.0418 \end{bmatrix} u, \quad (4.2)
\]

where \( x := \xi - \bar{x} \) and \( u := \nu - \bar{u} \).
4.2 Problem statement and control-theoretical technical background

4.2.1 Static state feedback control based on the Riccati-optimal quadratic control Lyapunov function

An optimization problem is usually considered in many control approaches for multivariable chemical processes [35], [53], [54], and, typically, a (piecewise) QCLF is designed. This particular choice is motivated by the fact that the gradient-based control \( u(x) = -R^{-1}B^TP^*x \), being \( P^* \succ 0 \) the (unique) solution of the Algebraic Riccati Equation (ARE)

\[
A^TP + PA + Q - PBR^{-1}B^TP = 0,
\]
asymptotically stabilizes the unconstrained linearized system (4.2), by minimizing the quadratic performance cost

\[
J(x,u) = \int_0^{+\infty} (x(\tau)^TQx(\tau) + u(\tau)^TRu(\tau)) \, d\tau. \tag{4.3}
\]

In the case of constrained systems, both linear and nonlinear ones, the particular choice of the CLF is a critical point in the control design, since the largest (indeed non conservative) estimate for the controllable state space set should be provided. Considering the weight matrices \( Q = R = I_2 \), the candidate QCLF deriving from the solution of the corresponding ARE for system (4.2) is \( x^TP^*x \), with

\[
P^* = \begin{bmatrix}
20.3640 & 1.6312 \\
1.6312 & 0.1979
\end{bmatrix}.
\]
4.2 Problem statement and control-theoretical technical background

Figure 4.1: Level sets of the Riccati-optimal quadratic control Lyapunov function. We notice that the largest admissible controlled set with Riccati-optimal shape is quite smaller compared to the polyhedral admissible state space. This is an example in which a quadratic function is not suited to shape a relatively-large controlled invariant set.

As shown in Figure 4.1 since a quadratic function can not fit well the polyhedral state constraints, only a shrunk QCLF can be used for any static control design, in order to guarantee the fulfillment of the state constraints.

4.2.2 Enlarging the controlled invariant state-space region

Considering an uncertainty of ±25% on the influence of the input flow $F$ to the autonomous dynamics, we have dynamics (4.2) with
uncertain matrix $A$ of the kind

$$A(w) := \begin{bmatrix} -1.7428(1 + w) & -0.0271 \\ 148.5626 & 4.4191(1 + w) \end{bmatrix},$$

where $w \in [-0.25, 0.25]$.

In general, for constrained (uncertain) linear systems

$$\dot{x} \in \text{conv} \{ A_ix + Bu \mid i \in [1, N] \}, \quad (4.4)$$

an arbitrary close approximation of the maximal controlled set can be computed via PCLFs, for instance by using the procedure proposed in [55]. Another way to obtain a larger estimate of the controllable set is the use of a truncated ellipsoid [58] [79] CLF, because the shape of its level sets takes into account the presence of the state constraints. However, since the standard truncated ellipsoid is not differentiable and thus (optimal) nonlinear gradient-based controllers can lead to a non-continuous control signal [62], a smoothing technique has been proposed in [12] via smooth composite CLFs.

Note that in order to handle both constraints and local optimality, it should be designed a CLF with external level sets in accordance to a very-large controllable set, provided by a PCLF, and with internal level sets close to the quadratic optimal ones, provided by the solution of the ARE. This can be done with the tool of R-functions, in order to obtain such kind of smooth CLF having non-homothetic level sets of the chosen shape (see Chapter 2).
4.3 Explicit Lyapunov-based state feedback control

In this section we exploit the composite control Lyapunov function of Chapter 2 together with the approximate Hamilton–Jacobi–Bellman control law.

We build-up a composite CLF $\mathcal{V}^\wedge$, which is the R-composition of a CLF $\mathcal{V}_1$ with a “very-large” domain of attraction $\mathcal{L}_{\mathcal{V}_1}$, and the Riccati-optimal CLF $\mathcal{V}_2$, according to the following assumption.

**Assumption 4.1** The function $\mathcal{V}_1(x) := \|Fx\|_2^p$, for some $F \in \mathbb{R}^{s \times n}$ and $p > 0$, is a smooth PCLF for (4.4), with “large” controlled set $\mathcal{L}_{\mathcal{V}_1}$. The system (4.4) is quadrically stabilizable with “optimal” QCLF $\mathcal{V}_2(x) := x^\top Px$, with domain $\mathcal{L}_{\mathcal{V}_2}$.

To guarantee that $\mathcal{V}^\wedge$ is a CLF by means of a constrained state feedback control law, we can use the same Petersen-like condition defined in 1.9, namely

$$\max_{x \in \mathcal{X}} \left\{ \max_{i \in [1,N]} \nabla \mathcal{V}_\wedge(x) A_i x - \|\nabla \mathcal{V}_\wedge(x) B\|_1 \right\} < 0 \quad (4.5)$$

or the LMI condition (3.14) in Theorem 3.13.

As the composite function $\mathcal{V}_\wedge$ guarantees both constraints fulfillment and locally-optimal shape, the following horizon-1 control law can be derived from an approximate Hamilton–Jacobi–Bellman (HJB) approach.

$$\kappa(x) := \arg \min_{\nu \in \mathcal{U}(x)} \nabla \mathcal{V}_\wedge(x) (A(0)x + B\nu) + x^\top Qx + \nu^\top R\nu, \quad (4.6)$$
Robust control of a chemical process via composite control

Lyapunov functions

where

\[ U(x) := \left\{ v \in \mathbb{R} \ | \ \max_{w \in W} \nabla V_\lambda(x) (A(w)x + Bv) + \eta \|x\|^2 \leq 0 \right\}. \]

The control law (4.6) is continuous because, after the change of variable \( u \mapsto R^{1/2}u + \frac{1}{2}R^{-1/2}B^T \nabla V_\lambda(x) \), it comes from the minimal selection control \([37\), Sections 2, 4], which is known to be continuous.

4.4 Constructive algorithms for the choice of the free parameters of the R-composition

We now propose two algorithms to tune the shape parameter \( \phi \). The constructive algorithms we propose represent the trade-off between the volume of the achieved controlled domain of attraction versus “optimality”. In both algorithms, the CLF \( V_1 \) is fixed. The parameters used are the desired decreasing rate \( \eta > 0 \), the R-functions parameters \( \alpha \in [0, 1) \), \( \phi > 0 \), a step tolerance \( \epsilon > 0 \) and a scaling factor \( \delta > 0 \) for \( V_2 \).

Algorithm 1 starts from the composition \( V_\lambda \) of \( V_1 \) and \( V_2 \) with \( \phi \gg 1 \). Function \( V_2 \) is initially scaled such that \( \mathcal{L}_{V_1} \subset \mathcal{L}_{V_2} \), so that \( \mathcal{L}_{V_\lambda} = \mathcal{L}_{V_1} \cap \mathcal{L}_{V_2} = \mathcal{L}_{V_1} \). Then \( \mathcal{L}_{V_2} \) is progressively reduced, until \( V_\lambda \) is a valid CLF (in the progressively reduced domain \( \mathcal{L}_{V_\lambda} \)).
4.4 Constructive algorithms for the choice of the free parameters of the R-composition

Algorithm 1. (addressing optimality)

1. Initialization
Define parameters $\eta, \alpha, \phi \gg 1, \epsilon$; $i = 0$; $\delta^{(0)}$ such that $\mathcal{L}_{V_1} \subset \mathcal{L}_{\delta^{(0)}V_2}$.

2. Iteration
\[ V^{(i)} \wedge := 1 \wedge ((1 - V_1) \wedge (1 - \delta^{(i)}V_2)) \]
according to the R-composition “$\wedge$” (2.3).

3. Feasibility test
if condition (4.5) is true for $V^{(i)} \wedge$
then STOP:
$V^{(i)} \wedge$ is a CLF in $\mathcal{L}_{V^{(i)} \wedge}$, with decreasing rate $\eta$.

else $i \mapsto i + 1$; $\delta^{(i)} = \delta^{(i-1)} + \epsilon$;
Go to Step 2.

Clearly $\mathcal{L}_{V^{(i)} \wedge} \supseteq \mathcal{L}_{V^{(i+1)} \wedge}$, therefore, in the worst-case, Algorithm 1 ends when $\mathcal{L}_{V_{\wedge}} = \mathcal{L}_{V_2}$ which is a robust controlled set in light of Assumption 4.1.

On the contrary, Algorithm 2 starts from the composition of $V_1$ and $V_2$, again initially scaled such that $\mathcal{L}_{V_1} \subset \mathcal{L}_{V_2}$, but with shape parameter $\phi = 0$. Therefore, as initial guess we take $V_{\wedge} = V_1$ according to Lemma A.3. Then $\phi$ is increased so that the level curves of the CLF $V_{\wedge}$ become closer to the optimal ones of $V_2$, as long as $V_{\wedge}$ remains a CLF, while still guaranteeing the maximal domain of attraction.
Algorithm 2. (fixing the maximal controlled invariant set)
1. Initialization
Define parameters \( \eta, \alpha, \epsilon; \delta \) such that \( \mathcal{L}_{V_1} \subset \mathcal{L}_{\delta V_2} \); \( i = 0; \phi^{(0)} = 0. \)
2. Iteration
\[
V^{(i)}_\wedge := 1 - ((1 - V_1) \wedge (1 - \delta V_2)),
\]
according to the R-composition “\( \wedge \)” \(2.3\), with \( \phi := \phi^{(i)}. \)
3. Feasibility test
\[
\text{if condition (4.5) is false for } V^{(i)}_\wedge \\
\text{then STOP:} \\
V^{(i-1)}_\wedge \text{ is a valid CLF in } \mathcal{L}_{V^{(i-1)}_\wedge}, \text{ with decreasing rate } \eta. \\
\text{else } i \mapsto i + 1; \phi^{(i)} = \phi^{(i-1)} + \epsilon; \\
\text{GO TO Step 2.}
\]

4.5 Numerical simulation of the controlled chemical reaction taking place in a continuous stirred tank reactor

In this section, we simulate the controlled chemical reaction taking place in the considered CSTR \(4.2\), with both nominal conditions and model uncertainties \( w. \)

Five control algorithms are tested: the standard LQR and the linear RHC (with a very-long prediction and control horizon) both based on the nominal system; the horizon-1 Lyapunov-based \(4.6\) control with three different CLFs, namely the Riccati-optimal QCLF.
4.5 Numerical simulation of the controlled chemical reaction taking place in a continuous stirred tank reactor

for the nominal system, the robust smoothed PCLF with very-large controlled DoA and the composite CLF which merges the above two CLFs.

In the design of the composite CLF $\mathcal{V}$, the function $\mathcal{V}_1$, shaping a very-large robust controlled set $\mathcal{L}_{\mathcal{V}_1}$, is computed as in [55]. Function $\mathcal{V}_1$ is composed with the Riccati-optimal QCLF $\mathcal{V}_2$ scaled such that $\mathcal{L}_{\mathcal{V}_2} \supset \mathcal{L}_{\mathcal{V}_1}$. We consider the quadratic performance cost $J \ (4.3)$. Parameters $\eta = 10^{-4}$, $\alpha = 0.1$, $p = 10$ are used in the R-composition procedures.

Algorithm 2 is used to tune parameter $\phi$, still preserving the maximal controlled set. Setting a step $\epsilon = 1$, the algorithm returns $\phi = 51$. In fact, for $\phi = 52$ the state $x_* = (-0.0469, -0.9438)^T$ leads to a violation of the feasibility (necessary and sufficient) condition in (4.5).

The LQR and the Lyapunov-based control associated to the Riccati-optimal QCLF are control strategies focused on optimality (of the nominal model). The simulation results of these control strategies are not used for comparisons since in both cases the constraints are often violated: this is the case of the non-admissible controlled state trajectory in Figure 4.2. On the contrary, the use of the PCLF is suited for robust stabilization.

Simulating the nominal dynamics, the RHC (requiring the highest computational effort to be performed in “real-time”) provides the best performances, while in the perturbed case, the presence of a model uncertainty $w = w(t)$ yields to constraint violations. Therefore a non-standard, and hence computationally demanding, MPC should be used in order to robustly take in account model uncertainties.
Figure 4.2: The use of the Riccati-optimal quadratic control Lyapunov function yields violations on the state constraints, even in the case of nominal dynamics. This is due to the fact that the controlled domain of attraction associated with the mentioned quadratic function is sensibly smaller than the maximal controlled invariant set.
4.5 Numerical simulation of the controlled chemical reaction taking place in a continuous stirred tank reactor

As expected, only the Lyapunov-based control law associated to the smoothed PCLF $V_1$ and to the composite CLF $V_\Lambda$ yield to the constraints fulfillment even in the perturbed case. The main benefit of the proposed approach is that we achieve a “very-large” robust controlled DoA together with close-to-optimal closed-loop performances.

In [10, Table 2] it is shown that, in the nominal case, the proposed horizon-1 control law is very close to the closed-loop performance induced by the long-horizon MPC. Figure 4.3 shows some controlled state trajectories, of the uncertain dynamics, starting from initial states close to the boundary of the controlled polyhedron $L_{V_1}$. 
Figure 4.3: Controlled state trajectories starting from randomly taken initial conditions and converging to the origin. The perturbed system evolves under the approximate Hamilton–Jacobi–Bellman control law, according to the level sets of the control Lyapunov function $V_{\lambda}$ composed via R-functions. We notice that our smooth composite control Lyapunov function is characterized by a close approximation of the largest domain of attraction and also by locally-optimal closed-loop performances.
Conclusion

The problem of merging two control Lyapunov functions, for instance a global control Lyapunov function with a large controlled domain of attraction and a local one with a guaranteed local performance, is considered important for several control applications. The main reason is that when simultaneously concerning constraints, robustness and optimality simultaneously, a single Lyapunov function is usually suitable for just one of these goals, but ineffective for the others.

Previous results show how to combine control Lyapunov functions if these share a common control in a suitable region of the state space. For the class of nonlinear control-affine systems, both differential equations and inclusions, it has been shown the equivalence between the control-sharing property, namely the existence of a single control law which makes simultaneously negative the Lyapunov derivatives of two given Lyapunov functions, and the fact that merging control Lyapunov functions is always possible.

It has been shown that the control-sharing property does not always hold, even for linear systems, with the exception of planar, or scalar, linear systems.

For the class of uncertain linear systems, namely linear differen-
tial inclusions, linear programs and linear matrix inequalities conditions have been presented in order to guarantee the existence of a common control law. These conditions consider typical classes of Lyapunov functions, namely polyhedral, smoothed-polyhedral, quadratic, composite-quadratic and truncated-quadratic functions.

As an example of merging procedure, a constructive technique based on a novel composition, due to R-functions, has been given. It has been shown that such a novel class of non-homogeneous, smooth control Lyapunov functions is universal for constrained uncertain linear systems, namely that Lyapunov stabilizability is equivalent to the existence of a non-homogeneous, smooth control Lyapunov functions in this class.

Associated with this class of novel smooth control Lyapunov functions, it has been presented a stabilizing constraint-admissible continuous control law characterized by the large controlled domain of attraction of one parent control Lyapunov function and the locally-optimal performance of the second one.

The presented control strategy has been heuristically shown to be quite close to the constrained global optimality by many numerical simulations. However, no theoretical bounds of sub-optimality have been given.

One considered case study is the control of a constrained uncertain chemical process, namely the model of a chemical process taking place in a continuous stirred tank reactor.

We believe that this work may give insights on the field of optimal control for constrained dynamical systems, under some basic regularity assumptions, whenever a large controlled domain of attraction has to be guaranteed.
Moreover, we believe that adopting merging control Lyapunov functions may provide likely control solutions also for the problem of tracking a reference signal [25], not just for stabilization.
Appendix A

Technical properties of the composition via R-functions

Lemma A.1 $V_\wedge$ is positive definite.

Proof. At the origin we have $V_1(0) = V_2(0) = 0 \iff R_1(0) = 1$. Therefore, from (3.15), $R_\wedge(0) = 1$ and hence $V_\wedge(0) = 1 - R_\wedge(0) = 0$. Conversely, $V_\wedge(\bar{x}) = 0 \iff R_\wedge(\bar{x}) = 1$. From [11 Proposition 1], we have $1 = R_\wedge(\bar{x}) \leq \max\{R_1(\bar{x}), R_2(\bar{x})\}$. Since $R_1(x) \leq 1$ and $R_2(x) \leq 1$ by construction, we have that $R_1(\bar{x}) = 1$ or $R_2(\bar{x}) = 1$ (or both). Say $R_1(\bar{x}) = 1$. Therefore $R_1(\bar{x}) = 1 \iff V_1(\bar{x}) = 0 \iff \bar{x} = 0$.

Lemma A.2 Assume that $V_1$ and $V_2$ are differentiable respectively in $\mathcal{L}_{V_1}$ and $\mathcal{L}_{V_2}$. Then $V_\wedge$ is differentiable in int$\mathcal{L}_{V_\wedge}$.

Proof. The proof immediately follows from (3.17) since $\phi > 0$ is fixed and functions $c_i(\phi, x)$, $i = 1, 2$, are continuous whenever $R_i(x)$ and $R_2(x)$ are not simultaneously 0, i.e. in int$\mathcal{L}_{V_\wedge}$.
For ease of notation, in the following proofs, let us denote $V_1(x), V_2(x), R_1(x), R_2(x), c_1(\phi, x), c_2(\phi, x)$ without the explicit dependence on their arguments.

**Lemma A.3** \( \mathcal{L}_{V_\wedge} = \mathcal{L}_{V_1} \cap \mathcal{L}_{V_2} \).

**Proof.** According to [11, Lemma 1], we have \( R_\wedge > 0 \iff \{ R_1 > 0 \text{ and } R_2 > 0 \} \); moreover, from (3.15), \( R_\wedge = 0 \iff \{ R_1 = 0 \text{ or } R_2 = 0 \} \). Now by construction \( V_i = 1 - R_i, \ i \in \{1, 2\} \), and \( V_\wedge = 1 - R_\wedge \), therefore \( V_\wedge < 1 \iff \{ V_1 < 1 \text{ and } V_2 < 1 \} \), and \( V_\wedge = 1 \iff \{ V_1 = 1 \text{ or } V_2 = 1 \} \), i.e. \( \mathcal{L}_{V_\wedge} = \mathcal{L}_{V_1} \cap \mathcal{L}_{V_2} \). \( \blacksquare \)

**Lemma A.4** \( \nabla V_\wedge \) converges to \( \nabla V_2 \) uniformly on compact subsets of \( \text{int} \mathcal{L}_{V_\wedge} \), as \( \phi \to \infty \). Namely, for any \( \delta \in (0, 1) \) we have

\[
\lim_{\phi \to \infty} \max_{x \in \mathcal{L}_{V_\wedge}^{(\phi)}} \| \nabla V_\wedge(x) - \nabla V_2(x) \| = 0.
\]

**Proof.** First we have

\[
\lim_{\phi \to \infty} \rho(\phi) = \lim_{\phi \to \infty} \frac{1}{\phi + 1 - \sqrt{\phi^2 + 1}} = \lim_{\phi \to \infty} \frac{\phi + 1 + \sqrt{\phi^2 + 1}}{2\phi} = 1. \quad (A.1)
\]

Then

\[
\lim_{\phi \to \infty} \phi c_1 = \lim_{\phi \to \infty} \phi \left( 1 + \frac{-\phi R_1}{\sqrt{\phi^2 R_1^2 + R_2^2}} \right) =
\lim_{\phi \to \infty} \frac{R_2^2}{\phi R_1^2 + R_2^2 / \phi + R_1 \sqrt{\phi^2 R_1^2 + R_2^2}} \leq
\lim_{\phi \to \infty} \frac{1}{2\phi R_1^2} \leq \lim_{\phi \to \infty} \frac{1}{2\phi(1 - \delta)^2} = 0. \quad (A.2)
\]
The last inequality holds uniformly as \(R_1(x) \geq 1 - \delta > 0\) whenever \(x \in \mathcal{L}_{(V, \delta)} = \{y \in \mathbb{R}^n \mid V(y) \leq \delta\}\). Then we can also write

\[
\lim_{\phi \to \infty} c_2 = \lim_{\phi \to \infty} \left(1 + \frac{-R_2}{\sqrt{\phi^2 R_1^2 + R_2^2}}\right) = \lim_{\phi \to \infty} \frac{\sqrt{\phi^2 R_1^2 + R_2^2} - R_2}{\sqrt{\phi^2 R_1^2 + R_2^2}} = \lim_{\phi \to \infty} \frac{R_1^2}{R_1^2 + R_2^2/\phi + (R_2/\phi)\sqrt{R_1^2 + R_2^2/\phi^2}} = 1. \quad (A.3)
\]

Therefore, combining (A.1), (A.2) and (A.3), we get

\[
\lim_{\phi \to \infty} \nabla V_{\lambda}(x) = \lim_{\phi \to \infty} \rho(\phi)(\phi c_1(\phi, x) \nabla V_1(x) + c_2(\phi, x) \nabla V_2(x)) = \nabla V_2(x) \text{ uniformly on compact subsets of the kind } \mathcal{L}_{(V, \delta)}.
\]

**Lemma A.5** \(\nabla V_{\lambda}\) converges to \(\nabla V_1\) uniformly on compact subsets of \(\text{int} \mathcal{L}_{V_{\Lambda}}\), as \(\phi \to 0^+\). Namely, for any \(\delta \in (0, 1)\) we have

\[
\lim_{\phi \to 0^+} \max_{x \in \mathcal{L}_{(V_{\Lambda}, \delta)}} \|\nabla V_{\lambda}(x) - \nabla V_1(x)\| = 0.
\]

**Proof.** Since \(\nabla V_{\lambda} = \rho(\phi)[\phi c_1 \nabla V_1 + c_2 \nabla V_2]\), we have to prove that for any \(\delta \in (0, 1)\) we have \(\lim_{\phi \to 0^+} \rho(\phi)\phi c_1(x) = 1\) and \(\lim_{\phi \to 0^+} \rho(\phi)c_2(\phi, x) = 0\) for all \(x \in \mathcal{L}_{(V_{\Lambda}, \delta)}\).

Similarly to (A.1) and (A.2) we have that

\[
\lim_{\phi \to 0^+} \rho(\phi)\phi c_1 = \lim_{\phi \to 0^+} \frac{\phi}{\phi + 1 - \sqrt{\phi^2 + 1}} \left(1 + \frac{-\phi R_1}{\sqrt{\phi^2 R_1^2 + R_2^2}}\right) = \lim_{\phi \to 0^+} \frac{\phi + 1 - \sqrt{\phi^2 + 1}}{2\phi} \cdot \frac{\phi R_2}{\phi^2 R_1^2 + R_2^2 + \phi R_1 \sqrt{\phi^2 R_1^2 + R_2^2}} = 1. \quad (A.4)
\]

The last equality holds uniformly as \(R_1(x) \geq 1 - \delta > 0\) and \(R_2(x) \geq 1 - \delta > 0\) (both the numerator and the denominator are indeed
strictly positive) whenever \( x \in \mathcal{L}_{(V, \delta)} = \{ y \in \mathbb{R}^n \mid V(y) \leq \delta \} \).

Then we can also write

\[
\lim_{\phi \to 0^+} \rho(\phi)c_2 = \lim_{\phi \to 0^+} \frac{1}{\phi + 1 - \sqrt{\phi^2 + 1}} \left( 1 + \frac{-R_2}{\sqrt{\phi^2 R_1^2 + R_2^2}} \right) = \lim_{\phi \to 0^+} \frac{\phi + 1 + \sqrt{\phi^2 + 1}}{2\phi} \cdot \frac{\sqrt{\phi^2 R_1^2 + R_2^2} - R_2}{\sqrt{\phi^2 R_1^2 + R_2^2}} = \lim_{\phi \to 0^+} \frac{\phi + 1 + \sqrt{\phi^2 + 1}}{2} \cdot \frac{\phi R_1^2}{\left( \sqrt{\phi^2 R_1^2 + R_2^2} + R_2 \right) \sqrt{\phi^2 R_1^2 + R_2^2}} = 0.
\]

(A.5)

Since \( R_1(x), R_2(x) \geq 1 - \delta > 0 \), the denominator is strictly positive and hence the last equality holds uniformly. Therefore, from (A.4) and (A.5) we get \( \lim_{\phi \to 0^+} \rho(\phi)c_1(\phi, x) \nabla V_1(x) + c_2(\phi, x) \nabla V_2(x) \) = \( \nabla V_1(x) \) uniformly on compact subsets of the kind \( \mathcal{L}_{(V, \delta)} \).

Lemma A.6 Assume \( \mathcal{L}_{V_2} \supset \mathcal{L}_{V_1} \). Then \( \nabla V_\lambda \) converges to \( \nabla V_1 \) uniformly on \( \mathcal{L}_{V_1} \) as \( \phi \to 0^+ \), i.e.

\[
\lim_{\phi \to 0^+} \max_{x \in \mathcal{L}_{V_\lambda}} \| \nabla V_\lambda(x) - \nabla V_1(x) \| = 0.
\]

Proof. We first notice that, as \( \mathcal{L}_{V_2} \supset \mathcal{L}_{V_1} \), we have \( \mathcal{L}_{V_\lambda} = \mathcal{L}_{V_1} \) in view of Lemma A.3. Then we can use the same proof of Lemma A.5 if we notice that \( R_2(x) \) is strictly positive in \( \mathcal{L}_{V_\lambda} \) because \( \mathcal{L}_{V_2} \supset \mathcal{L}_{V_1} = \mathcal{L}_{V_\lambda} \). In fact, \( R_2(x) > 0 \) implies that both the numerator and the denominator of (A.4), and also the denominator of (A.5), are strictly positive for all \( x \in \mathcal{L}_{V_\lambda} \).
Appendix B

Technical proofs

B.1 Proofs of Chapter 1

B.1.1 Proof of Theorem 1.1

Proof. The assumption is equivalent to \( \dot{R}_i(x) \geq \eta(1 - R_i(x)) \), \( i = 1, 2 \). Since we have

\[
\dot{R}_\lambda(x) = \frac{1}{2 - \sqrt{2 - 2\alpha}} \left[ \dot{R}_1 \left( 1 + \frac{-R_1 + \alpha R_2}{\sqrt{R_1^2 + R_2^2 - 2\alpha R_1 R_2}} \right) + \right. \\
\left. \dot{R}_2 \left( 1 + \frac{-R_2 + \alpha R_1}{\sqrt{R_1^2 + R_2^2 - 2\alpha R_1 R_2}} \right) \right],
\]

the following inequality for the R-composition holds.
\[ \dot{R}_\alpha(x) \geq \frac{1}{2 - \sqrt{2} - 2\alpha} \left[ \eta(1 - R) \left( 1 + \frac{-R_1 + \alpha R_2}{\sqrt{R_1^2 + R_2^2 - 2\alpha R_1 R_2}} \right) + \right. \\
\quad \eta(1 - R_2) \left( 1 + \frac{-R_2 + \alpha R_1}{\sqrt{R_1^2 + R_2^2 - 2\alpha R_1 R_2}} \right) \right] = \\
\left. \frac{\eta}{2 - \sqrt{2} - 2\alpha} \left[ \left( \frac{(\alpha - 1)(R_1 + R_2)}{\sqrt{R_1^2 + R_2^2 - 2\alpha R_1 R_2}} + 2 \right) + \right. \right. \\
\quad -R_1 \left( -R_1 + \alpha R_2 + \sqrt{R_1^2 + R_2^2 - 2\alpha R_1 R_2} \right) + \\
\quad \left. \left. \frac{\sqrt{R_1^2 + R_2^2 - 2\alpha R_1 R_2}}{\sqrt{R_1^2 + R_2^2 - 2\alpha R_1 R_2}} \right] \right] = \\
\eta \left[ \frac{1}{2 - \sqrt{2} - 2\alpha} \left( \frac{(\alpha - 1)(R_1 + R_2)}{\sqrt{R_1^2 + R_2^2 - 2\alpha R_1 R_2}} + 2 \right) - R_\wedge \right]. \]

Finally, for all \( \alpha \in [0, 1] \) and all \( R_1, R_2 \in [0, 1] \), we prove that

\[ \frac{1}{2 - \sqrt{2} - 2\alpha} \left( \frac{(\alpha - 1)(R_1 + R_2)}{\sqrt{R_1^2 + R_2^2 - 2\alpha R_1 R_2}} + 2 \right) \geq 1. \]

In fact, equivalently, we have

\[ \frac{(1 - \alpha)(R_1 + R_2)}{\sqrt{R_1^2 + R_2^2 - 2\alpha R_1 R_2}} \leq \sqrt{2 - 2\alpha} \]

and by taking the square we get

\[ (1 - \alpha)^2 \left( R_1^2 + R_2^2 + 2R_1 R_2 \right) \leq 2(1 - \alpha) \left( R_1^2 + R_2^2 - 2\alpha R_1 R_2 \right). \]

(B.1)
For $\alpha = 1$ the previous inequality (B.1) is verified as equality. Considering the case of $0 \leq \alpha < 1$, we can divide both sides of (B.1) by $(1 - \alpha)$. Then we obtain

$$-\alpha R_1^2 - \alpha R_2^2 + 2 R_1 R_2 \leq R_1^2 + R_2^2 - 2\alpha R_1 R_2 \iff (1 + \alpha) (R_1 - R_2)^2 \geq 0.$$ 

Therefore we get $\dot{R}_\alpha(x) \geq \eta (1 - R_\alpha(x))$, which is equivalent to $\dot{V}_\alpha(x) \leq -\eta V_\alpha(x)$. This concludes the proof.

### B.2 Proofs of Chapter 2

#### B.2.1 Proof of Lemma 2.1

**Proof.** If $r_1 > 0$ and $r_2 > 0$, then $r_\wedge > 0$ because $\phi r_1 + r_2 > \sqrt{(\phi r_1)^2 + r_2^2}$ for all $\phi > 0$. Conversely, assume $r_\wedge > 0$. Trivially, $r_1$ and $r_2$ cannot both be negative. Squaring both sides of the previous inequality, we obtain $r_1 r_2 > 0$, that leads to a contradiction if one between $r_1$ and $r_2$ would be negative.

#### B.2.2 Proof of Proposition 2.2

**Proof.** We first notice that

$$\frac{\partial r_\wedge}{\partial r_1} = \phi \left( 1 + \frac{-\phi r_1}{\sqrt{\phi^2 r_1^2 + r_2^2}} \right) \geq 0,$$

and analogously

$$\frac{\partial r_\wedge}{\partial r_2} = \left( 1 + \frac{-r_2}{\sqrt{\phi^2 r_1^2 + r_2^2}} \right) \geq 0.$$
Therefore, let $\underline{r} := \min\{r_1, r_2\}$. Then

$$r_\wedge = \rho(\phi) \left( \phi r_1 + r_2 - \sqrt{\phi^2 r_1^2 + r_2^2} \right) \geq \rho(\phi) \left( \phi \underline{r} + \underline{r} - \sqrt{\phi^2 \underline{r}^2 + \underline{r}^2} \right) = \underline{r}.$$ 

Analogously, it can be proved that $r_\wedge \leq \overline{r} := \max\{r_1, r_2\}$ using similar arguments as above.

**B.2.3 Proof of Proposition 2.3**

**Proof.** The assumption that $r_1, r_2 > 0$ allows the division by $r_1$ and/or $r_2$.

$$\lim_{\phi \to \infty} r_\wedge = \lim_{\phi \to \infty} \frac{(\phi r_1 + r_2 - \sqrt{(\phi r_1)^2 + r_2^2})}{\phi + 1 - \sqrt{\phi^2 + 1}} = \lim_{\phi \to \infty} \frac{2\phi r_1 r_2}{\phi + 1 - \sqrt{\phi^2 + 1}} = r_2.$$ 

Analogously, it can be proved $\lim_{\phi \to 0^+} r_\wedge = r_1$ using similar arguments.

**B.2.4 Proof of Theorem 2.4**

**Proof.** We recall the R-composition, which is defined as follows.

$$R_\wedge := \frac{\phi R_1 + R_2 - \sqrt{(\phi R_1)^2 + R_2^2}}{\phi + 1 - \sqrt{\phi^2 + 1}}.$$ 

The candidate LF $V_\wedge$ is positive definite (see Appendix A.1) and as $V_1$ and $V_2$ are differentiable, $V_\wedge$ is everywhere differentiable as well in $\text{int} L_{V_\wedge}$ (see Appendix A.2).
B.2 Proofs of Chapter 2

Consider the Lyapunov derivative

\[
\dot{R}_\Lambda = \rho \left[ \phi R_1 \left( 1 + \frac{-\phi R_1}{\sqrt{(\phi R_1)^2 + R_2^2}} \right) + R_2 \left( 1 + \frac{-R_2}{\sqrt{(\phi R_1)^2 + R_2^2}} \right) \right].
\]

The assumption on the decreasing rate is equivalent to \( \dot{R}_i(x) \geq \eta (1 - R_i(x)), \ i = 1, 2, \) we have

\[
\dot{R}_\Lambda \geq \eta \rho [\phi c_1 + c_2 - (\phi c_1 R_1 + c_2 R_2)].
\]

As \( \rho (\phi c_1 R_1 + c_2 R_2) = R_\Lambda, \) and \( \dot{R}_\Lambda \geq \eta [\rho (\phi c_1 + c_2) - R_\Lambda], \) we need to prove \( \rho (\phi c_1 + c_2) \geq 1. \)

\[
\rho (\phi c_1 + c_2) \geq 1 \iff \frac{1}{\phi + 1 - \sqrt{\phi^2 + 1}} \left[ \phi + \phi - \frac{-\phi R_1}{\sqrt{(\phi R_1)^2 + R_2^2}} \right] + 1 + \frac{-R_2}{\sqrt{(\phi R_1)^2 + R_2^2}} \geq 1 \iff \sqrt{\phi^2 + 1} \geq \frac{\phi^2 R_1 + R_2}{\sqrt{(\phi R_1)^2 + R_2^2}}. \quad \text{(B.3)}
\]

Finally we square both sides of the latter inequality in (B.3) to get

\[
(\phi^2 + 1)(\phi^2 R_1^2 + R_2^2) \geq (\phi^2 R_1 + R_2)^2 \iff (R_1 - R_2)^2 \geq 0,
\]

which concludes the proof.

**B.2.5 Proof of Theorem 2.5**

**Proof.** The proof immediately follows from the one in Appendix 3.11. \( \blacksquare \)
Proof. We have to show that given $\kappa_1, \kappa_2 : \mathbb{R}^2 \to \mathbb{R}^m$ such that for all $x \in \mathbb{R}^2$ we have $\nabla V_i(x)(Ax + B\kappa_i(x)) < 0$, for $i = 1, 2$, then for all $x \in \mathbb{R}^2$ there exists $u \in \mathbb{R}^m$ such that the two inequalities $\nabla V_i(x)(Ax + Bu) < 0$ and $\nabla V_2(x)(Ax + Bu) < 0$ can be simultaneously satisfied.

Without any restriction, we assume $m = 1$, so that $B \in \mathbb{R}^{2 \times 1}$, otherwise the proof would be trivial. Assume by contradiction that $V_1$ and $V_2$ do not share a common control, i.e. there exists a point $z \neq 0$ such that the two inequalities (3.3a)-(3.3b) are not simultaneously satisfied.

If $z$ and $B$ are aligned, namely $z = \lambda B$ for some $\lambda \neq 0$, we can take $u = -c/\lambda$, for some $c > 0$, so that we simultaneously get

$$\nabla V_1(z)(Az + Bu) = \nabla V_1(z)Az - c\nabla V_1(z)z < 0 \quad \text{(B.4)}$$
$$\nabla V_2(z)(Az + Bu) = \nabla V_2(z)Az - c\nabla V_2(z)z < 0. \quad \text{(B.5)}$$

Since $V_1$ and $V_2$ are convex and positive definite, we have $\nabla V_1(z)z > 0$ and $\nabla V_2(z)z > 0$, therefore for $c$ large enough we have (B.4)-(B.5) simultaneously satisfied.

Let $z$ and $B$ be not aligned and hence consider the state transformation $\hat{x} := [B|z]^{-1}x$, so that $\hat{B} := [B|z]^{-1}B = (1, 0)^\top$ and $\hat{z} := [B|z]^{-1}z = (0, 1)^\top$ as in Figure B.1. We make this transformation for ease of understanding, so that in the sequel we consider $z = (0, 1)^\top$ and $B = (1, 0)^\top$.

Then consider the equation $\dot{z} = (Az + Bu) = -\omega z$ in the un-
known $u$ and $\omega$, or equivalently $[B|z](\begin{array}{c}u \\ z \end{array}) = -Az$, which has unique solution as $[B|z] = I_2$. Multiplying both sides by $z^\top$ we get $z^\top Az + z^\top Bu = z^\top Az = -\omega z^\top z$, hence $\omega$ has opposite sign to $z^\top Az$.

Therefore if $\omega > 0$ then we have $\dot{z} = Az + Bu = -\omega z$ so that we simultaneously get $\nabla V_1(z)(Az + Bu) = -\omega \nabla V_1(z)z < 0$ and $\nabla V_2(z)(Az + Bu) = -\omega \nabla V_2(z)z < 0$.

Let $\omega < 0$. Then the vector $Az$ must be directed upwards, see Figure B.1, so that $z^\top Az \geq 0$. 

Figure B.1: For linear systems of dimension $n = 2$, two control Lyapunov functions necessarily share a common control law.
Notice that $\nabla V_i(z)B \neq 0$, for $i = 1, 2$. In fact, let, by contradiction, $\nabla V_1(z)B = 0$. Then $\nabla V_1(z)$ is aligned to $z$ and points upwards, i.e. $\nabla V_1(z) = cz$ for some $c > 0$. But then $\nabla V_1(z)(Az + Bu_1) = cz^TAz \geq 0$ $\forall u_1 \in \mathbb{R}$, contradicting the assumption that $V_1$ is a CLF. Similarly, also $\nabla V_2(z)B = 0$ would contradict the fact that $V_2$ is a CLF.

If $\nabla V_1(x)B$ and $\nabla V_2(x)B$ have the same sign, then $(3.3a)$ and $(3.3b)$ can be simultaneously satisfied for negative $u$ with $|u|$ large enough.

Let $\nabla V_1(x)B$ and $\nabla V_2(x)B$ have opposite sign. Consider the compact sets $S_1 = \{x \in \mathbb{R}^2 \mid V_1(x) \leq V_1(z)\}$ and $S_2 = \{x \in \mathbb{R}^2 \mid V_2(x) \leq V_2(z)\}$. The tangent lines to $S_1$ and $S_2$ in $z$ (which is on the boundary of both sets, see lines $P - z$ and $Q - z$ in Figure B.1) respectively have positive and negative slope, as an immediate consequence that $\nabla V_1(z)B$ and $\nabla V_2(z)B$ have opposite signs.

Now let $v$ and $y$ be the “highest” points respectively inside $S_1$ and $S_2$, namely the solutions of the following convex optimization problems: $v := \text{arg max}\{z^T \mid x \in S_1\}$ and $y := \text{arg max}\{z^T \mid x \in S_2\}$. Note that $v$ and $y$ are necessarily in the second and in the first quadrant respectively, since the tangent lines in $z$ have opposite slopes. In view of the optimality conditions, we must have that the two gradients are vertical, then aligned with $z$: $\nabla V_1(v) = c_1z^T$, $\nabla V_2(y) = c_2z^T$, for some $c_1, c_2 > 0$. Therefore they are orthogonal to $B$: $\nabla V_1(v)B = \nabla V_2(y)B = 0$.

On the other hand, we assumed that $V_1$ and $V_2$ are CLFs, i.e. in
\[ \nabla V_1(v)(Av + B\kappa_1(v)) = \nabla V_1(v)Av = c_1 z^T Av < 0 \]
\[ \nabla V_2(y)(Ay + B\kappa_2(y)) = \nabla V_2(y)Ay = c_2 z^T Ay < 0, \]
so \( z^T Av < 0 \) and \( z^T Ay < 0. \)

We finally get a contradiction because \( z \) is in the cone generated by \( v \) and \( y \), therefore \( z = \alpha v + \beta y \) for some \( \alpha, \beta > 0 \), and \( z^T Az = \alpha z^T Av + \beta z^T Ay < 0 \), contradicting the fact that \( z^T Az \geq 0. \)

**B.3.2 Proof of Proposition 3.2**

**Proof.** We show a numerical example for \( n = 2, m = 1, N = 2 \), in which two QCLFs \( V_1(x) = x^T P_1 x \) and \( V_2(x) = x^T P_2 x \) do not share a common control law.

Consider (3.6) with
\[
A_1 = \begin{bmatrix} -1.408 & -0.476 \\ 0.819 & -1.694 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -0.357 & 1.196 \\ -1.428 & 1.721 \end{bmatrix}, \quad B_1 = B_2 = B = \begin{bmatrix} -1.981 \\ 0.600 \end{bmatrix}.
\]
The eigenvalues of \( A_1 \), \( A_2 \) respectively are \( \{-1.55 \pm i0.61\} \) and \( \{0.68 \pm i0.79\} \).

Let us consider
\[
P_1 = \begin{bmatrix} 3.478 & -3.988 \\ -3.988 & 7.825 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 4.610 & -18.53 \\ -18.53 & 96.40 \end{bmatrix}.
\]

With the linear control laws \( \kappa_1(x) = K_1 x \) and \( \kappa_2(x) = K_2 x \), being \( K_1 = (0.4815, -0.6934) \) and \( K_2 = (8.310, -42.17) \), we have
\[
(A_j + BK_i)^T P_i + P_i(A_j + BK_i) \preceq -\epsilon_i I_n \quad \text{for all } i, j \in \{1, 2\},
\]
with \( \epsilon_1, \epsilon_2 \geq 10^{-3} \). Therefore \( V_1 \) and \( V_2 \) are two CLFs for \((3.6)\).

Then, we show that for the state \( \bar{x} = (-1.813, -0.404)^\top \), there cannot exists a common control \( u \in \mathbb{R} \), i.e. the following system of equations is not admissible.

\[
\begin{align*}
\nabla V_1(\bar{x})(A_1\bar{x} + Bu) &< 0 \\
\nabla V_1(\bar{x})(A_2\bar{x} + Bu) &< 0 \\
\nabla V_2(\bar{x})(A_1\bar{x} + Bu) &< 0 \\
\nabla V_2(\bar{x})(A_2\bar{x} + Bu) &< 0
\end{align*}
\]  
\[\text{(B.6)}\]

In fact, we have \( \frac{1}{2} \nabla V_1(\bar{x})A_2\bar{x} = \bar{x}^\top P_1 A_2 \bar{x} = 6.94 \), \( \frac{1}{2} \nabla V_1(\bar{x})B = \bar{x}^\top P_1 B = 11.74 \), therefore \( u < -0.59 < 0 \); however \( \frac{1}{2} \nabla V_2(\bar{x})A_1\bar{x} = \bar{x}^\top P_2 A_1 \bar{x} = 1.89 \) and \( \frac{1}{2} \nabla V_2(\bar{x})B = \bar{x}^\top P_2 B = -1.48 \), therefore \( u > 1.28 > 0 \). \[\blacksquare\]

**B.3.3 Proof of Proposition 3.3**

**Proof.** We show a numerical example for \( n = 3 \), in which two QCLFs \( V_1(x) = x^\top P_1 x \) and \( V_2(x) = x^\top P_2 x \) do not share a common control law.

Consider \((3.5)\) with

\[
A = \begin{bmatrix} -1.990 & -1.135 & -1.063 \\
1.745 & 0.536 & -0.429 \\
-0.794 & -1.243 & -1.813 \end{bmatrix}, \quad B = \begin{bmatrix} -1.925 \\
-0.342 \\
0.257 \end{bmatrix}.
\]

Note that the eigenvalues of \( A \) are \( \{0.276, -1.772 \pm i0.114\} \).
Let us consider
\[
P_1 = \begin{bmatrix}
35.3372 & 27.5098 & -39.0922 \\
27.5098 & 21.4164 & -30.4326 \\
-39.0922 & -30.4326 & 43.2484
\end{bmatrix},
\]
\[
P_2 = \begin{bmatrix}
0.00031 & 0.04321 & -0.01465 \\
0.04321 & 80.5695 & -39.5654 \\
-0.01465 & -39.5654 & 19.6646
\end{bmatrix}.
\]

With the linear control laws \( \kappa_1(x) = K_1 x \) and \( \kappa_2(x) = K_2 x \), being \( K_1 = (0.5037, 0.5799, -0.2013) \) and \( K_2 = (4.5451, 4.5697, -0.0669) \), we have \((A + BK_i)^\top P_i + P_i (A + BK_i) \preceq -\epsilon_i I_n\), for \( i = 1, 2 \), with \( \epsilon_1, \epsilon_2 \geq 10^{-4} \). Therefore \( x^\top P_1 x \) and \( x^\top P_2 x \) are CLFs.

Then, we show that for the state \( \bar{x} = (-0.329, -1.094, -1.537)^\top \), there cannot exists a common control \( u \in \mathbb{R} \), i.e. the following system of equations is not admissible.

\[
\begin{cases}
\nabla V_1(\bar{x})(A\bar{x} + Bu) < 0 \\
\nabla V_2(\bar{x})(A\bar{x} + Bu) < 0
\end{cases}
\]

In fact, \( \frac{1}{2} \nabla V_1(\bar{x}) A\bar{x} = \bar{x}^\top P_1 A\bar{x} = -31.89 \), \( \frac{1}{2} \nabla V_2(\bar{x}) A\bar{x} = \bar{x}^\top P_2 A\bar{x} = 71.07 \), \( \frac{1}{2} \nabla V_1(\bar{x}) B = \bar{x}^\top P_1 B = -45.46 \), \( \frac{1}{2} \nabla V_2(\bar{x}) B = \bar{x}^\top P_2 B = 12.76 \) therefore we get

\[
\begin{cases}
-31.91 - 45.46u < 0 \quad \Leftrightarrow \quad u > -0.70 \\
71.07 + 12.76u < 0 \quad \Leftrightarrow \quad u < -5.57
\end{cases}
\]

that clearly is not feasible.

**Remark B.1** The sets of equations (B.6) and (B.7) are not influenced by any scaling of the matrices \( P_i \), meaning that the set of admissible solutions remains the same for \( P_i \mapsto \delta_i P_i, \delta_i > 0, i = 1, 2 \).
Such a scaling would influence $\epsilon_1, \epsilon_2$ in $(A_j + BK_i)^\top P_i + P_i(A_j + BK_i) \preceq -\epsilon_i I_n$ in the following sense. For any $\tilde{\epsilon}_1, \tilde{\epsilon}_2 > 0$, there exist $\delta_1, \delta_2 > 0$ such that $(A_j + BK_i)^\top \delta_i P_i + \delta_i P_i(A_j + BK_i) \preceq -\tilde{\epsilon}_i I_n$ for $i = 1, 2$. That is to say that we cannot run into numerical problems caused by “too small” $\epsilon_1, \epsilon_2$.

### B.3.4 Proof of Theorem 3.4

**Proof.** $V$ is a CLF if and only if for any $x \in \mathbb{R}^n$ there exists $u \in \mathbb{R}^m$ such that $\nabla V(x)(f(x) + g(x)u) < 0$. Assume that $V$ is a CLF and let $x$ be fixed. By definition, for any $\gamma_1, \gamma_2 \geq 0$ with $(\gamma_1, \gamma_2) \neq (0, 0)$, there exists $u \in \mathbb{R}^m$ such that $(\gamma_1 \nabla V_1(x) + \gamma_2 \nabla V_2(x))(f(x) + g(x)u) < 0$, or equivalently for any $(\alpha_1, \alpha_2) \in \mathcal{A} := \{(a, b) \in (\mathbb{R}_\geq 0)^2 \mid a + b = 1\}$ there exists $u \in \mathbb{R}^m$ such that

$$
(\alpha_1 \nabla V_1(x) + \alpha_2 \nabla V_2(x))(f(x) + g(x)u) < 0.
$$

Therefore we have

$$
\max_{(\alpha_1, \alpha_2) \in \mathcal{A}} \inf_{u \in \mathbb{R}^m} (\alpha_1 \nabla V_1(x) + \alpha_2 \nabla V_2(x))(f(x) + g(x)u) < 0. \tag{B.8}
$$

Since $\mathcal{A}$ is compact and $\mathbb{R}^m$ is closed, and the function in (B.8) is linear in both $(\alpha_1, \alpha_2)$ and $u$, we can exchange “max” and “min” [68, Corollary 37.3.2] to get the following equivalent condition.

$$
\max_{(\alpha_1, \alpha_2) \in \mathcal{A}} \inf_{u \in \mathbb{R}^m} (\alpha_1 \nabla V_1(x) + \alpha_2 \nabla V_2(x))(f(x) + g(x)u) = \\
\inf_{u \in \mathbb{R}^m} \max_{(\alpha_1, \alpha_2) \in \mathcal{A}} (\alpha_1 \nabla V_1(x) + \alpha_2 \nabla V_2(x))(f(x) + g(x)u) = \\
\inf_{u \in \mathbb{R}^m} \max_{(\alpha_1, \alpha_2) \in \mathcal{A}} \{\alpha_1 \nabla V_1(x)(f(x) + g(x)u) + \alpha_2 \nabla V_2(x)(f(x) + g(x)u)\} < 0.
$$
This is equivalent to

\[ \inf_{u \in \mathbb{R}^m} \max \{ \nabla V_1(x)(f(x) + g(x)u), \nabla V_2(x)(f(x) + g(x)u) \} < 0. \]

The last inequality is equivalent to the existence of a common control law. The result follows as all the considered inequalities are equivalent.

B.3.5 Proof of Theorem 3.5

Proof. In view of Theorem 3.4, we need only to prove that if there exists a regular gradient-type merging CLF, then the two functions have the control-sharing property.

Therefore, by assumption we have

\[ (\gamma_1(x)\nabla V_1(x) + \gamma_2(x)\nabla V_2(x))(Ax + Bu(x)) < 0 \]

for some locally bounded \( u: \mathbb{R}^n \to \mathbb{R}^m \).

Given a unit vector \( v \in \mathbb{R}^n, \|v\| = 1 \), consider the ray \( \mathcal{R} := \{ \lambda v \in \mathbb{R}^n | \lambda > 0 \} \). Since the functions are homogeneous, their gradients along \( \mathcal{R} \) are aligned, namely for all \( x = \lambda v \) we have \( \nabla V_1(x) = \lambda^p \nabla V_1(v) \) and \( \nabla V_2(x) = \lambda^q \nabla V_2(v) \) for some \( p, q > 0 \). Therefore we have

\[ (\gamma_1(\lambda v)\lambda^p \nabla V_1(v) + \gamma_2(\lambda v)\lambda^q \nabla V_2(v))(\lambda Av + Bu(\lambda v)) < 0, \]

or equivalently (divide by \( \gamma_1(\lambda v)\lambda^p + \gamma_2(\lambda v)\lambda^q > 0 \) and by \( \lambda > 0 \)) to
get
\[
\begin{bmatrix}
\frac{\gamma_1(\lambda v)\lambda^p}{\gamma_1(\lambda v)\lambda^p + \gamma_2(\lambda v)\lambda^q} \nabla V_1(v) + \\
\frac{\gamma_2(\lambda v)\lambda^q}{\gamma_1(\lambda v)\lambda^p + \gamma_2(\lambda v)\lambda^q} \nabla V_2(v)
\end{bmatrix}
\]
\[
= \alpha_1(\lambda)
\]
\[
(\lambda v + B\omega) < 0,
\]
where we define
\[
\omega := \frac{u(\lambda v)}{\lambda}.
\]
Denote by \(\bar{\lambda}\) the value of \(\lambda\) such that \(\bar{\lambda}v \in \partial L\), i.e. \(V(\bar{\lambda}v) = 1\). For all \(\lambda \in [0, \bar{\lambda}]\), we have \((\alpha_1(\lambda), \alpha_2(\lambda)) \in \mathcal{A} := \{ (\alpha, \beta) \in (\mathbb{R}_{\geq 0})^2 \mid \alpha + \beta = 1 \}\). Moreover as \(\lambda\) goes from 0 to \(\bar{\lambda}\), both \(\alpha_1(\lambda)\) and \(\alpha_2(\lambda) = 1 - \alpha_1(\lambda)\) assume all values from 0 to 1. This means that for all \((\alpha_1, \alpha_2) \in \mathcal{A}\) there exists \(\omega \in \mathbb{R}^m\) such that
\[
\max_{(\alpha_1, \alpha_2) \in \mathcal{A}} \inf_{\omega \in \mathbb{R}^m} \left( \alpha_1 \nabla V_1(v) + \alpha_2 \nabla V_2(v) \right) (Av + B\omega) < 0,
\]
i.e.
\[
\max_{(\alpha_1, \alpha_2) \in \mathcal{A}} \inf_{\omega \in \mathbb{R}^m} \left( \alpha_1 \nabla V_1(v) + \alpha_2 \nabla V_2(v) \right) (Av + B\omega) < 0.
\]
To complete the proof we just need to apply the same min-max argument of the proof of Theorem 3.4.

B.3.6 Proof of Proposition 3.6

**Proof.** We prove the claim by means of an example with \(n = m = 2\). Consider the linear system \(\dot{x} = u\), along with the linear control laws
\[
\kappa_1(x) = K_1 x = \begin{bmatrix} -\epsilon & 1/a \\ -a & -\epsilon \end{bmatrix}, \quad \kappa_2(x) = K_2 = K_1^T = \begin{bmatrix} -\epsilon & -a \\ 1/a & -\epsilon \end{bmatrix},
\]
B.3 Proofs of Chapter 3

for some $a, \epsilon > 0$. The functions

$$V_1(x) = \frac{1}{2} \left( ax_1^2 + \frac{1}{a} x_2^2 \right), \quad V_2(x) = \frac{1}{2} \left( \frac{1}{a} x_1^2 + ax_2^2 \right)$$

are two QCLFs, respectively with control laws $\kappa_1$ and $\kappa_2$. In fact, since

$$\nabla V_1(x) = \left( ax_1, \frac{1}{a} x_2 \right), \quad \nabla V_2(x) = \left( \frac{1}{a} x_1, ax_2 \right),$$

we have $\nabla V_i(x)(Ax + Bu) = -\epsilon V_i(x)$, for $i = 1, 2$.

Take any regular gradient-type merging candidate so that $\nabla V(x) = (\gamma_1(x)\nabla V_1(x) + \gamma_2(x)\nabla V_2(x))$ and $\{\gamma_1(x) = 1, \gamma_2(x) = 0\}$ “far” from the state-space origin and, vice-versa, $\{\gamma_1(x) = 0, \gamma_2(x) = 1\}$ “close” to the origin. Therefore $\nabla V(x)$ is continuous, there exists a point $R$ on the bisector in which $\nabla V$ is aligned to the bisector itself, i.e. there exist $\lambda, \xi \geq 0$ such that $\nabla V(\xi) = \lambda (\xi, \xi)$. In such a point, with both $\kappa_1(\xi)$ and $\kappa_2(\xi)$, we have

$$\nabla V(\xi)(A\xi + B\kappa_i(\xi)) = \lambda \left( -2\epsilon + \left( \frac{1}{a} - a \right) \right) \xi^2$$

is negative definite for all $\epsilon, a > 0$.

Note that for $a \gg 1$ the vector $\nabla V_1$ is almost “horizontal”, while the vector $\nabla V_2$ is almost “vertical”. Consider the ray (bisector) $R = \{ x = (\xi, \xi) \mid \xi \geq 0 \}$. Since $\nabla V$ is continuous, there exists a point $R$ on the bisector in which $\nabla V$ is aligned to the bisector itself, i.e. there exist $\lambda, \xi \geq 0$ such that $\nabla V(\xi) = \lambda (\xi, \xi)$. In such a point, with both $\kappa_1(\xi)$ and $\kappa_2(\xi)$, we have

$$\nabla V(\xi)(A\xi + B\kappa_i(\xi)) = \lambda \left( -2\epsilon + \left( \frac{1}{a} - a \right) \right) \xi^2$$
which is strictly positive for $\epsilon \ll 1$, $a \gg 1$.

B.3.7 Proof of Proposition 3.7

**Proof.** By assumption, for all $x \in \mathbb{R}^n$ there exists $u \in \mathbb{R}^m$ such that the inequalities

\[
\max_{(\varphi, \Gamma) \in (F(x), G(x))} \nabla V_1(x) (\varphi + \Gamma u) < 0 \iff \\
\max_{(\varphi, \Gamma) \in (F(x), G(x))} \gamma_1 \nabla V_1(x) (\varphi + \Gamma u) < 0 \quad (B.9a)
\]

\[
\max_{(\varphi, \Gamma) \in (F(x), G(x))} \nabla V_2(x) (\varphi + \Gamma u) < 0 \iff \\
\max_{(\varphi, \Gamma) \in (F(x), G(x))} \gamma_2 \nabla V_2(x) (\varphi + \Gamma u) < 0 \quad (B.9b)
\]

holds simultaneously for any $\gamma_1, \gamma_2 \geq 0$. Therefore also the following sum is negative:

\[
\max_{(\varphi, \Gamma) \in (F(x), G(x))} \{ \gamma_1 \nabla V_1(x) (\varphi + \Gamma u) \} + \\
\max_{(\varphi, \Gamma) \in (F(x), G(x))} \{ \gamma_2 \nabla V_2(x) (\varphi + \Gamma u) \} < 0,
\]

which immediately implies

\[
\max_{(\varphi, \Gamma) \in (F(x), G(x))} (\gamma_1 \nabla V_1(x) + \gamma_2 \nabla V_2(x)) (\varphi + \Gamma u) < 0.
\]

Having the left-hand side strictly less than zero is equivalent to claim that the gradient-type merging is a CLF. The proof is complete if we notice that $\gamma_1$ and $\gamma_2$ have been chosen arbitrarily. ■
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B.3.8 Proof of Theorem 3.8

Proof. The implication (2) ⇒ (1) follows from Theorem 3.7. To prove the claim (1) ⇒ (2) we write \( G(x) = g(x) \) to mean that \( G \) is single-valued. Fix arbitrary \( \gamma_1, \gamma_2 \geq 0 \) and define

\[
\tilde{f}(x) := \arg \max_{\varphi \in F(x)} (\gamma_1 \nabla V_1(x) + \gamma_2 \nabla V_2(x)) \varphi.
\]

Now, by assumption we have that

\[
\max_{\varphi \in F(x)} (\gamma_1 \nabla V_1(x) + \gamma_2 \nabla V_2(x)) (\varphi + g(x)u) < 0,
\]

namely

\[
\max_{\varphi \in F(x)} \{ (\gamma_1 \nabla V_1(x) + \gamma_2 \nabla V_2(x)) \varphi \} +
\]

\[
(\gamma_1 \nabla V_1(x) + \gamma_2 \nabla V_2(x)) g(x)u < 0.
\]

According to the definition of \( \tilde{f} \), the first term can be written as \( (\gamma_1 \nabla V_1(x) + \gamma_2 \nabla V_2(x)) \tilde{f}(x) \). Finally, we can just follow the proof of Theorem 3.4 for the nonlinear system \( \dot{x} = \tilde{f}(x) + g(x)u \).

B.3.9 Proof of Lemma 3.9

Proof. For each \( i \in [1, N] \) we explicitly construct \( \tilde{W}_i \) from \( W_i \). Therefore, for ease of notation, let us consider the case of \( N = 1 \). The general case easily follows as each \( \tilde{W}_i \) here constructed will depend exclusively on \( W_i \).

Let us denote \( \tilde{W} = [\tilde{w}_1 | \tilde{w}_2 | \ldots | \tilde{w}_{s+1}] \), where \( \tilde{w}_i \in \mathbb{R}^{s+1} \). Moreover we use the notation \( w_i(p) \) to denote the \( p \)th component of a column.
vector $w_i$ and $w_i([p, q])$ to denote its $p^{th}$, $(p + 1)^{th}$, ..., $(q - 1)^{th}$, $q^{th}$ components.

Then we have $A[x_1|...|x_s|x] + B[u_1|...|u_s|u] = [X|x][\bar{w}_1|...|\bar{w}_s|\bar{w}_{s+1}]$.

The first $s$ equations are

$$Ax_i + Bu_i = [X|x]\bar{w}_i, \ i \in [1, s],$$

therefore we can take $\bar{w}_i([1, s]) := w_i([1, s])$ and $\bar{w}_i(s + 1) := 0$. Note that this definition respects the fact that $\bar{W}$ has to be an $\mathcal{M}$-matrix.

The last equation is $A\bar{x} = B\bar{u} = [X|x]\bar{w}_{s+1}$, i.e.

$$A \left( \sum_{i=1}^{s} \alpha_i x_i \right) + B \left( \sum_{i=1}^{s} \alpha_i u_i \right) = X\bar{w}_{s+1}([1, s]) + \left( \sum_{i=1}^{s} \alpha_i x_i \right) \bar{w}_{s+1}(s + 1).$$

Now the left-hand side can be written as

$$A \left( \sum_{i=1}^{s} \alpha_i x_i \right) + B \left( \sum_{i=1}^{s} \alpha_i x_i \right) = \sum_{i=1}^{s} \alpha_i \underbrace{(Ax_i + Bu_i)}_{=Xw_i} = X \sum_{i=1}^{s} \alpha_i w_i$$

while the right-hand side can be written as

$$X\bar{w}_{s+1}([1, s]) + \left( \sum_{i=1}^{s} \alpha_i x_i \right) \bar{w}_{s+1}(s + 1) = X \bar{w}_{s+1}([1, s]) + \bar{w}_{s+1}(s + 1)\alpha$$

where $\alpha := (\alpha_1, \alpha_1, ..., \alpha_s)^\top$.

Therefore, from (B.10) and (B.11), it is sufficient to show that, for any given $\alpha \in (\mathbb{R}_{\geq 0})^s$, there exists $\bar{w}_{s+1}$ such that

$$\sum_{i=1}^{s} \alpha_i w_i = \bar{w}_{s+1}([1, s]) + \bar{w}_{s+1}(s + 1)\alpha. \quad \text{(B.12)}$$
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For ease of notation, we rename the free variables as

\[ \xi := \bar{w}_{s+1}([1, s]) \in (\mathbb{R}_{\geq 0})^s \quad \text{and} \quad \zeta := \bar{w}_{s+1}(s + 1) \in \mathbb{R}. \]

Then, we define

\[
\tilde{W} := \begin{bmatrix}
    w_1(1) & \cdots & w_s(1) \\
    \vdots & \ddots & \vdots \\
    w_s(1) & \cdots & w_s(s)
\end{bmatrix}.
\] (B.13)

Note that, as \( W \) is an \( \mathcal{M} \)-matrix by assumption, we have \( w_i(j) \geq 0 \iff j \neq i \). Therefore, according to \( (B.12) \) and \( (B.13) \), we have to find \( \xi \) and \( \zeta \) such that \( \tilde{W} \alpha = \xi + \zeta \alpha \), or equivalently:

\[
\left( \tilde{W} - \zeta I_s \right) \alpha = \xi.
\] (B.14)

Now defining \( \zeta := -\left( \max_{i \in [1, s]} |w_i(i)| \right) \), we get that the matrix \( \left( \tilde{W}_r - \zeta I \right) \) has all non-negative entries, therefore \( \left( \tilde{W} - \zeta I \right) \alpha \) becomes a vector of all non-negative components. This means that \( (B.14) \) can be satisfied by choosing \( \xi := (\tilde{W} - \zeta I) \alpha \in (\mathbb{R}_{\geq 0})^s \).

Summarizing, we found an admissible \( \mathcal{M} \)-matrix \( \bar{W} \in \mathbb{R}^{(s+1) \times (s+1)} \) of the kind

\[
\bar{W} := \begin{bmatrix}
    W & \xi \\
    0_s & \zeta
\end{bmatrix}.
\] (B.15)

To conclude the proof, we have to show that \( \bar{I}_{s+1}^T \bar{W} \leq -\eta \bar{I}_{s+1}, \)

i.e. that all the columns \( \bar{w}_i \) are such that \( \sum_{j=1}^{s+1} \bar{w}_i(j) \leq -\eta \). This is immediately true for the first \( s \) columns, as \( \bar{I}_s^T W \leq -\eta \bar{I}_s^T \) by assumption. In fact, see \( (B.15) \), we have \( \bar{w}_i(j) = w_i(j) \) if \( 1 \leq j \leq s \), 0 otherwise.
Finally, for the last column $\bar{w}_{s+1}$, from (B.14) we have that

$$
\left( \sum_{i=1}^{s} \xi_i \right) + \zeta = \alpha_1 \left( \sum_{i=1}^{s} w_1(i) \right) + \ldots + \alpha_s \left( \sum_{i=1}^{s} w_s(i) \right) \leq \eta + \ldots + \eta \leq -\eta \leq -\eta \left( \sum_{i=1}^{s} \alpha_i \right) = -\eta.
$$

\[\square\]

**B.3.10 Proof of Theorem 3.10**

**Proof.** For each vertex $x_k^j$ of $L_{V_j}$, say $j = 1$, (3.11a) is equivalent to the Lyapunov condition for $V_1$ in $x_k^1$, namely to $\dot{V}_1(x_k^1, \bar{U}_k) := \max_{i \in [1,N]} \nabla V_1(x_k^1)(A_i x_k^1 + B_i \bar{U}_k) < 0$. Analogously, (3.11b) is equivalent to the Lyapunov condition for $V_2$ in $\tilde{x}_k^1$, namely to $\dot{V}_2(\tilde{x}_k^1, \bar{U}_k) := \max_{i \in [1,N]} \nabla V_2(\tilde{x}_k^1)(A_i \tilde{x}_k^1 + B_i \bar{U}_k) < 0$. Since $V_2$ is homogeneous and the condition holds for all $i \in [1,N]$, this is equivalent to the above Lyapunov condition for $V_2$ in $x_k^1$ itself.

The common control law follows from $\bar{U}$ and therefore it is piece-wise linear. The proof immediately follows since the choice of $j \in \{1, 2\}$ and $k \in [1, s_j]$ have been made arbitrarily. \[\square\]

**B.3.11 Proof of Theorem 3.11**

**Proof.** The assumption that $V_1$ is a PCLF is equivalent to the existence of a piecewise-linear control law that follows from the control vectors $u_1, u_2, \ldots, u_s$ (respectively associated with the vertices $x_1, x_2, \ldots, x_s$), namely the columns of $U$, which shows up in
According to Lemma 3.9, if \(\{x_1, x_2, \ldots, x_r\}\) are the vertices of a given facet of the polyhedron \(L_{V_1}\), together with control vectors \(\{u_1, u_2, \ldots, u_r\}\), then the control vector \(\bar{u}(\alpha) := \sum_{h=1}^{r} \alpha_h u_h\), for \(\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_r) \in A := \{a \in (\mathbb{R}_{\geq 0})^r \mid \sum_{h=1}^{r} a_h = 1\}\), is an admissible control for \(V_1\) in the state point \(\bar{x}(\alpha) := \sum_{h=1}^{r} \alpha_h x_h\). Therefore it is sufficient to prove that for each facet of the polyhedron \(L_{V_1}\), the control \(\bar{u}(\alpha)\), parameterized by \(\alpha \in A\), is admissible also for \(V_2\), i.e. there exists \(\eta > 0\) such that

\[
\bar{x}(\alpha)^\top P \left[(A_i + \eta I_n)\bar{x}(\alpha) + B_i \bar{u}(\alpha)\right] \leq 0 \quad \forall i \in [1, N].
\]

Then we can write

\[
\left(\sum_{h=1}^{r} \alpha_h x_h\right)^\top P \left[(A_i + \eta I_n)\left(\sum_{h=1}^{r} \alpha_h x_h\right) + B_i \left(\sum_{h=1}^{r} \alpha_h u_h\right)\right] \leq 0
\]

for all \(i \in [1, N]\) \(\iff\)

\[
\sum_{h,j=1}^{r} \alpha_h \alpha_j \left(x_h^\top P [(A_i + \eta I_n)x_j + B_i u_j] + x_j^\top P [(A_i + \eta I_n)x_h + B_i u_h]\right) = (S,\cdot,\eta, U)_{h,j} \leq 0 \quad \forall i \in [1, N].
\]

We get that the left-hand side of the last inequality, namely the quadratic expression \(\alpha^\top S,\cdot,\eta, U, \alpha\), has to be non-positive for \(\alpha \in (\mathbb{R}_{\geq 0})^r\). Therefore the matrices \(-S_{k,\cdot,\eta, U}\), where the subscript \(k\) indicates the \(k^{th}\) facet, have to be copositive. This is equivalent to the assumption made.

\[\square\]

\section{Proof of Corollary 3.12}

\textbf{Proof.} The proof follows from the one in Appendix B.3.11.
B.3.13 Proof of Theorem 3.13

For all \( k \in [1, s] \), define the sectors

\[
S_k := \{ x \in \mathbb{R}^n \mid x^T P_k x \geq \max_j x^T P_j x \},
\]

so that we have

\[
x \in S_k \implies x^T P_k x \geq x^T P_j x \ \forall j \in [1, s]
\implies x^T \left( \sum_{j=1}^s \lambda_{i,j,k} (P_j - P_k) \right) x \leq 0 \quad \text{(B.16)}
\]

for any \( \lambda_{i,j,k} \geq 0 \), where \( i \in [1, N], j, k \in [1, s] \).

The matrix inequality condition (3.14a) is necessary and sufficient for \( V_1 \) to be a CLF for (3.6) [45], with piecewise-linear control law \( \kappa(x) := K(x)x \), where \( K(x) := \{ K_k \text{ if } x \in S_k \} \). Then we show that (3.14b) is sufficient for \( \kappa \) to be a valid control law also for \( V_2 \).

Consider \( x \in S_k \) and multiply (3.14b) by \( x^T \) on the left and by \( x \) on the right, so that

\[
2 \nabla V_2(x)(A_i + B_i K_k)x = x^T \left[ (A_i + B_i K_k)^T P + P(A_i + B_i K_k) \right] x \leq
- 2 \eta x^T P_x + x^T \left( \sum_{j=1}^s \lambda_{i,j,k} (P_j - P_k) \right) x \quad \text{for all } i \in [1, N].
\]

Therefore, in view of (B.16), we finally get to

\[
\nabla V_2(x)(A_i + B_i K_k)x \leq - \eta V_2(x) \quad \text{for all } i \in [1, N].
\]

The proof follows since the choice of the sector \( S_k \ni x \) has been made arbitrarily.

\( \blacksquare \)
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B.3.14 Proof of Proposition 3.14

Proof. The proof follows from Lemma A.2 as
\[ \nabla V(x) \wedge (x) = \rho(\phi) (c_1(\phi, x) \nabla V_1(x) + c_2(\phi, x) \nabla V_2(x)) \]
where the functions \( c_1, c_2 : \mathbb{R}_{>0} \times \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0} \) defined in (3.18) are continuous.

B.3.15 Proof of Proposition 3.15

As \( V_2 \) has been scaled so that \( \mathcal{L}_{V_2} \supset \mathcal{L}_{V_1} \), we have \( \mathcal{L}_{V_r} = \mathcal{L}_{V_1} \) from Lemma A.3. Let us use the notation \( \gamma_1(x) := \rho(\phi)c_1(\phi, x) \) and \( \gamma_2(x) := \rho(\phi)c_2(\phi, x) \), so that \( \nabla V_r(x) = \gamma_1(x) \nabla V_1(x) + \gamma_2(x) \nabla V_2(x) \), where \( \gamma_1 \) and \( \gamma_2 \) also depend on the parameter \( \phi \).

According to the proof of Lemma A.4, we have the following fact. For any \( \varepsilon > 0 \) and \( \delta > 0 \) there exists \( \phi_1 > 0 \) such that for all \( \phi \geq \phi_1 \) we have \( \max_{x \in \mathcal{L}_{V_r/\delta}} \gamma_1(x) \leq \varepsilon \). Analogously, for any \( \varepsilon > 0 \) and \( \delta > 0 \) there exists \( \phi_2 > 0 \) such that for all \( \phi \geq \phi_2 \) we have \( 1 - \varepsilon \leq \max_{x \in \mathcal{L}_{V_r/\delta}} \gamma_1(x) \leq 1 \). The proof is complete if we take \( \bar{\phi} := \max\{\phi_1, \phi_2\} \), so that for all \( \phi \geq \bar{\phi} \) and \( x \in \mathcal{L}_{V_r/\delta} \) we have \( \gamma_1(x) \in [0, \varepsilon] \) and \( \gamma_2(x) \in [1 - \varepsilon, 1] \).

B.3.16 Proof of Theorem 3.16

According to Theorem 3.7, \( V_\lambda \) is a CLF in \( \mathcal{L}_{V_r} \). Moreover, it follows from [11, Proposition 1] that \( V_\lambda \) grows quadratically, i.e.
\[ \min\{\bar{V}_1(x), \bar{V}_2(x)\} \leq V_\lambda(x) \leq \max\{\bar{V}_1(x), \bar{V}_2(x)\} \forall x \in \mathbb{R}^n. \]
Therefore for some \( \eta > 0 \), we have that for all \( x \in \mathcal{L}_{V_r} \) there exists \( u \in \mathbb{R}^m \) such that
\[ \max_{i \in \{1, N\}} \nabla V_\lambda(x)(A_i x + Bu) \leq -\eta x^T x. \] Analogously, for some \( \eta > 0 \), which we do not relabel, we also have that for all \( x \in \mathcal{L}_{V_1} \) there
exists $u \in \mathbb{R}^m$ such that $\max_{i \in [1,N]} \nabla V(x)(A_i x + Bu) \leq -\eta x^\top x$.

Let $\tilde{\kappa}$ be the piecewise-linear control law shared by $\tilde{V}_1$ and $\tilde{V}_2$. It follows from the proof of Theorem 3.7 that $V_\wedge$ is a Lyapunov function for the closed-loop differential inclusion

$$\dot{x} \in \operatorname{co}\{A_i x + B\tilde{\kappa}(x) \mid i \in [1,N]\},$$

therefore $V_\wedge$, and hence also $V$, satisfies the small control property. The optimization problem (3.20) follows from the minimal selection control

$$m(x) := \arg \min_{v \in U} \|v\| \quad \text{subject to:} \quad \max_{i \in [1,N]} \nabla V_\wedge(x)A_i x + \nabla V_\wedge(x)B(v + \kappa_2(x)) + \eta \|x\|^2 \leq 0,$$

which is known to be continuous [37, Section 4.2]. Hence the optimal solution of (3.20) can be written as $\kappa(x) := m(x) + \kappa_2(x)$, which is the sum of two continuous functions.

In the following, we prove that $\kappa_2$ is an admissible control for $V_\wedge$ in a neighborhood of the origin. This will also imply that $V_\wedge$ satisfies the small control property.

According to Lemma A.4, we have the following property. For any $\epsilon > 0$ and $\sigma \in (0,1)$ there exists $\tilde{\phi} > 0$ such that $\phi \geq \tilde{\phi}$ implies that $\nabla V_\wedge(x) = \nabla \tilde{V}_2(x) + v(x)^\top$, with $\max_{x \in \mathcal{L}(V_\wedge,\sigma)} \|v(x)\| \leq$
\[ \max_{i \in [1,N]} \nabla V(x)(A_i x + B \kappa_2(x)) = \]
\[ \max_{i \in [1,N]} \nabla V_2(x)(A_i x + B \kappa_2(x)) + v(x)^\top (A_i x + B \kappa_2(x)) \leq \]
\[ \max_{i \in [1,N]} \nabla V_2(x)(A_i x + B \kappa_2(x)) + \max_{i \in [1,N]} v(x)^\top (A_i x + B \kappa_2(x)). \]  

(B.17)

We notice that there exist \( \eta, \sigma > 0 \) such that
\[ \max_{i \in [1,N]} \nabla V(x)(A_i x + B \kappa_2(x)) \leq -2\eta x^\top x \] for all \( x \) in the compact set \( \mathcal{L}_{(V_2/\sigma_2)} \). Therefore we choose \( \sigma \) so that \( \{ x \in \mathbb{R}^n \mid V_\wedge(x) \leq \sigma \} \subseteq \{ x \in \mathbb{R}^n \mid V_2(x) \leq \sigma_2 \} \), namely as \( \sigma := \max \{ c \in [0,1] \mid \mathcal{L}_{(V_\wedge/c)} \subseteq \mathcal{L}_{(V_2/\sigma_2)} \} \).

We can now choose \( \epsilon \geq \|v(x)\| \) such that
\[ \max_{x \in \mathcal{L}_{(V_\wedge/\sigma)}} \left\{ \max_{i \in [1,N]} v(x)^\top (A_i x + B \kappa_2(x)) - \eta x^\top x \right\} \leq 0. \]  

(B.18)

Therefore, using (B.18) in (B.17), we get that \( \kappa_2 \) is an admissible control for \( V_\wedge \) in a neighborhood of the origin, i.e.
\[ \max_{i \in [1,N]} \nabla V(x)(A_i x + B \kappa_2(x)) \leq -\eta x^\top x. \] This means that for all \( x \in \mathcal{L}_{(V_\wedge/\sigma)} \), the constraint \( v \in \mathcal{U}(x) \) in (3.20) is not active and therefore \( \kappa(x) = \kappa_2(x) \) is locally optimal. Moreover, we also get that the control law \( \kappa \) is continuous also at the origin.
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