Acta Sci. Math., 40 (1978), 121—128

On a special decomposition of regular semigroups

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In [1] a general disjoint decomposition of semigroups was given, which can be applied for the case of regular semigroups. The aim of the present paper is to obtain a characteristic decomposition of regular semigroups based on the decomposition studied in [1]. We shall investigate the components of this decomposition and the interrelations between them. By making use of [2] we study the cases of regular semigroups with or without a left or right identity element.

Notation. For two sets A, B we write $A\subset B$ if A is a proper subset of B. By a magnifying element we mean a left magnifying element.

1. Let S be a semigroup without nonzero annihilator. This is not a proper restriction because every semigroup can be reduced to this case. Then S has the following disjoint decomposition:

$$S=\bigcup_{i=0}^{5} S_i, \tag{1}$$

where

$$S_0=\{a\in S|aS\subset S \text{ and } \exists x\in S;\ x\neq 0 \text{ and } ax=0\},$$
$$S_1=\{a\in S|aS=S \text{ and } \exists y\in S,\ y\neq 0 \text{ and } ay=0\},$$
$$S_2=\{a\in S\setminus(S_0\cup S_1)|aS\subset S \text{ and } \exists x_1, x_2\in S,\ x_1\neq x_2 \text{ and } ax_1=ax_2\},$$
$$S_3=\{a\in S\setminus(S_0\cup S_1)|aS=S \text{ and } \exists y_1, y_2\in S,\ y_1\neq y_2 \text{ and } ay_1=ay_2\},$$
$$S_4=\{a\in S\setminus(S_0\cup S_1\cup S_2\cup S_3)|aS\subset S\},$$
$$S_5=\{a\in S\setminus(S_0\cup S_1\cup S_2\cup S_3)|aS=S\}.$$

It is easy to see that the components S_i $(i=0, 1, \dots, 5)$ are semigroups, $S_i\cap S_j=\emptyset$ $(i\neq j)$ and the following relations hold:

$$\begin{gathered} S_5S_i\subseteqq S_i,\quad S_iS_5\subseteqq S_i\quad (0\leqq i\leqq 5),\\ S_4S_3\subseteqq S_2,\quad S_4S_2\subseteqq S_2,\quad S_4S_1\subseteqq S_0,\quad S_4S_0\subseteqq S_0,\\ S_2S_3\subseteqq S_2,\quad S_0S_1\subseteqq S_0. \end{gathered} \tag{2}$$

Received May 27, 1977.

It is obvious that there exists an analogous decomposition

$$(1') \qquad S = \bigcup_{i=0}^{5} T_i,$$

where T_i $(0 \leqq i \leqq 5)$ is the dual of S_i.

Remark. The above decomposition is in fact "group oriented". That is, we select consecutively the elements of S having a property that is very far from that of an element of a group. So we consecutively select the annihilators, the (left) zero divisors, the elements for which the products are not left cancellative, and what remains is a right group.

Our theorems concern the decomposition (1), but analogous results can be formulated for the decomposition (1').

Theorem 1.1. *S_5 is a right group.*

Proof. It is easy to see that S_5 is right simple and left cancellative, whence the assertion follows.

Set $S_0 \cup S_2 = \bar{S}_2$ and $S_1 \cup S_3 = \bar{S}_3$.

Theorem 1.2. *$\bar{S}_2$ is a subsemigroup of S.*

Proof. If $s_0 \in S_0$ and $s_2 \in S_2$, then $s_0 s_2 \in \bar{S}_2$. There are elements $x, y \in S$, $x \neq y$ such that $s_2 x = s_2 y$. We have $s_0 s_2 \notin \bar{S}_2$ and $s_0 s_2 \notin S_5$ because $s_0 s_2 S = s_0(s_2 S) \subset S$. If $s_0 s_2 \neq 0$, then $(s_0 s_2)x = (s_0 s_2)y$ $(x \neq y)$, whence $s_0 s_2 \in S_2 \subseteq \bar{S}_2$. Similarly, $s_2 s_0 \in \bar{S}_2$. If $s_0 \neq 0$ then $s_2 s_0 \neq 0$ because $s_2 \in S_2$. Since $s_0 \in S_0$, there is an element $z \neq 0$ such that $s_0 z = 0$, hence $(s_2 s_0) z = 0$. Therefore $s_2 s_0 \in S_0$. Q.E.D.

Theorem 1.3. *$\bar{S}_3$ contains all the magnifying elements of S and only them.*

Proof. Let $a \in S_1 \cup S_3$. If $a \in S$ and $aS = S$, and if furthermore, there is an $y \neq 0$ such that $ay = 0$, then $S' = S \setminus \{0\} \subset S$ and $aS' = S$, whence a is a magnifying element. If $a \in S_3$, $aS = S$ and if, furthermore, there exist $x, y \in S$ $(x \neq y)$ such that $ax = ay$, then $a(S - \{x\}) = S$ and a is a magnifying element.

Conversely, if $a \in S$ is a magnifying element, then $a \notin S_0 \cup S_2 \cup S_4$ and $aM = S$ $(M \subset S)$. Thus there exist $m \in M$ and $s \in S \setminus M$ such that $am = as$. Hence it follows that $a \in S_1 \cup S_3$. Q.E.D.

Remark. Theorems 1.2 and 1.3 imply

$$(3) \qquad S_0 S_2 \subseteqq S_0 \cup S_2, \quad S_2 S_0 \subseteqq S_0 \cup S_2, \quad S_1 S_3 \subseteqq S_1 \cup S_3, \quad S_3 S_1 \subseteqq S_1 \cup S_3.$$

In what follows we assume that S is a regular semigroup, i.e. for every $a \in S$ there is an $x \in S$ such that $a = axa$ and $x = xax$ (x is an inverse of a). The elements

ax, xa are idempotent and $aS \supseteqq axS \supseteqq axaS = aS$ implies $axS = aS$, and similarly, $xaS = xS$. The regular semigroup S can contain a zero element hence the components S_0 and S_1 can exist in the decomposition (1).

Theorem 1.4. *The inverses of the elements of $\bar{S}_3$ are in S_4 and the inverses of the elements of S_4 are in $\bar{S}_3$.*

Proof. Let $a \in \bar{S}_3$ and let $x \in S$ be an inverse of a, that is, let $axa = a$ and $xax = x$. First we show that $xS \subset S$. Suppose that $xS = S$, then there is a subset $S' \subset S$ such that $aS' = S$ because a is a magnifying element. Hence it follows that $xaS' = xS = S$. But we have $(xa)S = xS = S$ and xa is idempotent, that is, xa is a left identity of S. Therefore, $(xa)S' = S' \neq S$, which is a contradiction. Thus $xS \subset S$, whence x is contained in S_0, S_2 or S_4. If $x \in S_2$, then $xs_1 = xs_2$ $(s_1 \neq s_2)$ and $(ax)s_1 = (ax)s_2$. Since $(ax)S = aS = S$ and ax is idempotent we obtain that ax is a left identity of S, i.e. $(ax)s_1 = (ax)s_2$ implies $s_1 = s_2$, which is a contradiction. It can be proved similarly that $x \notin S_0$. It remains the case $x \in S_4$.

Conversely, let $b \in S_4$, that is, $bS = S' \subset S$. Let y be an inverse of b in S. Hence $byS = bS = S'$. Suppose that $yS \subset S$. Let $yS = S''$ $(\neq S)$. Hence $bS'' = byS = bS$. Thus there are elements $s \notin S''$, and $s'' \in S''$ such that $bs'' = bs$. But every element a of S for which $ax_1 = ax_2$ $(x_1 \neq x_2)$, is contained in $S_0 \cup S_1$ or $S_2 \cup S_3$, which contradicts the fact that $b \in S_4$. Thus necessarily $yS = S$, that is, $y \notin S_0 \cup S_2 \cup S_4$. If $y \in S_5$, then $(yb)S = yS = y(bS) = yS' = S$ $(S' \neq S)$, i.e. $y \in S_1 \cup S_3$, which is a contradiction. It remains the only case $y \in S_1 \cup S_3 = \bar{S}_3$. Q.E.D.

It is easy to see that the set of inverses of the elements of $\bar{S}_3$ is equal to S_4 and the set of inverses of the elements of S_4 is equal to $\bar{S}_3$.

Corollary 1.5. *If a regular semigroup S does not contain a magnifying element ($\bar{S}_3 = \emptyset$), then $S_4 = \emptyset$ and conversely, $S_4 = \emptyset$ implies $\bar{S}_3 = \emptyset$.*

Corollary 1.6. *If a regular semigroup S does not contain a left identity, then $\bar{S}_3 = \emptyset$; and hence $S_4 = \emptyset$.*

For if $a \in \bar{S}_3$ and $x \in S_4$ is an inverse of a, then ax is a left identity of S.

Theorem 1.7. *$\bar{S}_2$ is a regular semigroup and the inverses of an element of $\bar{S}_2$ are contained in $\bar{S}_2$.*

Proof. Let $a \in \bar{S}_2$ and x an inverse of a in S. Since $a \in S_0 \cup S_2$, we have $aS \subset S$. Assume that $xS = S$. Then $(xa)S = x(aS) = xS = S$, whence x is a magnifying element, i.e., $x \in \bar{S}_3$. But every inverse of an element of $\bar{S}_3$ is (by Theorem 1.4) in S_4, thus $a \in S_4$, which is a contradiction. Therefore $xS \subset S$. But $x \notin S_4$ because $a \in \bar{S}_2$. We conclude that $x \in S_0 \cup S_2 = \bar{S}_2$. Q.E.D.

The above results yield:

Theorem 1.8. *A semigroup S is regular if and only if it has a decomposition* (1) *where*

a) $\bar{S}_2 = S_0 \cup S_2$ *is regular;*

b) *the inverses of the elements of* $\bar{S}_3 = S_1 \cup S_3$ *are contained in* S_4 *and conversely;*

c) S_5 *is a right group.*

Proof. Necessity follows from Theorems 1.1, 1.4, 1.7. Sufficiency follows from the fact that a right group is regular.

2. In this section we shall deepen our kowledge concerning the decomposition (1) of a regular semigroup S as well as the components $\bar{S}_2, \bar{S}_3$ and S_4.

Theorem 2.1. *Let S be a regular semigroup without (left) magnifying elements. Using the notations* $\bar{S}_2 = \bar{S}_2^1$, $S_5 = S_5^1$ *we obtain the following decompositions:*

$S = \bar{S}_2^1 \cup S_5^1$ *and if* $\bar{S}_2^1$ *has no magnifying element,*

$\bar{S}_2^1 = \bar{S}_2^2 \cup S_5^2$ *and if* $\bar{S}_2^1$ *has no magnifying element,*

$\vdots$

$\bar{S}_2^k = \bar{S}_2^{k+1} \cup S_5^{k+1}$,

$\vdots$

where every $\bar{S}_2^k$ *is a regular semigroup, every* S_5^k *is a right group and the following inclusions hold:*

$$(4) \qquad \begin{aligned} S_5^k S_5^j &\subseteq S_5^k, & S_5^j S_5^k &= S_5^k \quad \textit{for} \quad k \geqq j, \\ S_5^k \bar{S}_2^j &= \bar{S}_2^j, & \bar{S}_2^j S_5^k &\subseteq \bar{S}_2^j \quad \textit{for} \quad k \leqq j. \end{aligned}$$

Proof. It is enough to give a proof for the cases:

$$S_5^1 S_5^k, \quad S_5^k S_5^1, \quad S_5^1 \bar{S}_2^j, \quad \bar{S}_2^j S_5^1$$

because the proof for the semigroups $\bar{S}_2^i$ is similar.

The proof is by induction on k and j. It is trivial that

$$S_5^1 S_5^1 = S_5^1, \quad s_5^1 \bar{S}_2^1 = \bar{S}_2^1, \quad s_5^2 \bar{S}_2^1 = \bar{S}_2^1 \quad (s_5^k \in S_5^k).$$

Hence, $s_5^1 s_5^2 \bar{S}_2^1 = \bar{S}_2^1$, i.e., $s_5^1 s_5^2 \in S_5^2$ for all $s_5^1 \in S_5^1$ and $s_5^2 \in S_5^2$. Since $s_5^1 \bar{S}_2^1 = \bar{S}_2^1$ and, furthermore, $s_5^1 S_5^2 \subseteq S_5^2$ and $s_5^1 (s_2^2 \bar{S}_2^1) \subset \bar{S}_2^1$, that is, $s_5^1 s_2^2 \in \bar{S}_2^2$, we conclude that $s_5^1 S_5^2 = S_5^2$ and $s_5^1 \bar{S}_2^2 = \bar{S}_2^2$, whence $S_5^1 S_5^2 = S_5^2$, $S_5^1 \bar{S}_2^2 = \bar{S}_2^2$. Thus we have $S_5^1 S_5^1 = S_5^1$, $S_5^1 \bar{S}_2^1 = \bar{S}_2^1$, $S_5^1 S_5^2 = S_5^2$, $S_5^1 \bar{S}_2^2 = \bar{S}_2^2$, $S_5^2 S_5^1 \subseteq S_5^2$ because $s_5^2 s_5^1 S_5^2 = s_5^2 S_5^2 = S_5^2$, and thus $s_5^2 s_5^1 \in S_5^2$. The first step of the proof is complete.

Now suppose that the following conditions hold:

$$S_5^1 S_5^k = S_5^k, \quad S_5^k S_5^1 \subseteq S_5^k, \quad S_5^1 \bar{S}_2^j = \bar{S}_2^j, \quad \bar{S}_2^j S_5^1 \subseteq \bar{S}_2^j.$$

By definition, we have $s_5^{k+1} \bar{S}_2^k = \bar{S}_2^k$. Hence, $(s_5^1 s_5^{k+1}) \bar{S}_2^k = s_5^1 \bar{S}_2^k = \bar{S}_2^k$, whence $s_5^1 s_5^{k+1} \in S_5^{k+1}$. Thus we obtain $S_5^{k+1} = (s_5^1 s_5^{k+1}) S_5^{k+1} = s_5^1 S_5^{k+1}$, whence $S_5^1 S_5^{k+1} = S_5^{k+1}$.

We have $(s_5^{k+1} s_5^1) S_5^{k+1} = S_5^{k+1}$ and, furthermore, $s_5^{k+1} s_5^1 \in \bar{S}_2^k$; thus $s_5^{k+1} s_5^1 \in S_5^{k+1}$ implies $S_5^{k+1} S_5^1 \subseteq S_5^{k+1}$. We also have $(s_5^1 s_2^{j+1}) \bar{S}_2^j \subset s_5^1 \bar{S}_2^j = \bar{S}_2^j$, whence $s_5^1 s_2^{j+1} \in$

$\in \bar{S}_2^{j+1}$, and $s_5^1 \bar{S}_2^{j+1} = \bar{S}_2^{j+1}$ implies $S_5^1 \bar{S}_2^{j+1} = \bar{S}_2^{j+1}$. Finally, we have $s_2^{j+1} s_5^1 \in \bar{S}_2^j$ and $s_2^{j+1} s_5^1 \bar{S}_2^j = s_2^{j+1} S_2^j \subset S_2^j$, whence it follows that $s_2^{j+1} s_5^1 \in \bar{S}_2^{j+1}$ and $\bar{S}_2^{j+1} S_5^1 \subseteqq \subseteqq \bar{S}_2^{j+1}$. Q.E.D.

Corollary 2.2. *If S and $\bar{S}_2^k$ $(k \geqq 1)$ are regular semigroups without magnifying elements, then S has one of the following four types of decompositions:*

a) $S = ((((\dots) \cup S_5^4) \cup S_5^3) \cup S_5^2) \cup S_5^1$, *with an infinite number of components;*

b) $S = \bar{S}_2 \cup ((((\dots) \cup S_5^4) \cup S_5^3) \cup S_5^2) \cup S_5^1$, *where $\bar{S}_2$ is a semigroup of type $\bar{S}_2$ and there are infinitely many components;*

c) $S = (((S_5^n \cup \dots) \cup S_5^3) \cup S_5^2) \cup S_5^1$, *where the number of components equals n;*

d) $S = ((((\bar{S}_2^m \cup S_5^m) \cup \dots) \cup S_5^3) \cup S_5^2) \cup S_5^1$, *where the number of components is* $m+1$.

We shall treat some properties of the semigroups $\bar{S}_3$ and S_4.

Theorem 2.3. *Let $a, b \in \bar{S}_3$, and let x be an inverse of a, and y an inverse of b $(x, y \in S_4)$. Then xy is an inverse of ba.*

Proof. Since ax and by are left identities of S, we have $baxyba = b(axy)ba = byba = ba$, and $xybaxy = xyb(axy) = xyby = xy$. Q.E.D.

Theorem 2.4. *If $a, b \in S_4$ and if x is an inverse of a and y is an inverse of b, then yx and ab are inverses of each other.*

Proof. By Theorem 2.3, $(yby)(xax)$ is an inverse of ab. Then we get $ab = ab(yby)(xax)ab = a(byb)yx(axa)b = abyxab$, $yxabyx = ybyx = yx$, since xa, yb are left identities of S. Q.E.D.

By Theorem 1.4, $\bar{S}_3 \cup S_4$ is a regular subset of S, but it fails to be a subsemigroup, because, e.g., $S_4 S_3 \subseteqq S_2$ (cf. (2)). Set

$$X_1 = \{x \in S_4 \mid x \text{ is an inverse of some } a \in S_1\},$$

$$X_3 = \{y \in S_4 \mid y \text{ is an inverse of some } b \in S_3\}.$$

Then $S_4 = X_1 \cup X_3$.

Corollary 2.5. *X_1 and X_3 are subsemigroups of S_4. In general, if $A \subseteqq \bar{S}_3$ is a subsemigroup, then the inverses of the elements of A form a subsemigroup in S_4.*

Proof. This is an easy consequence of Theorem 2.3.

Corollary 2.6. *$\bar{S}_3$ and S_4 have no idempotent elements.*

Proof. Every element of $\bar{S}_3$ is magnifying, thus $a \neq a^2$ $(a \in \bar{S}_3)$. Assume that $e \in S_4$ is idempotent. Since e is an inverse of e, $e \in \bar{S}_3$ (by Theorem 1.4), which is a contradiction.

Theorem 2.7. *Every element of* $\bar{S}_3$ *and* S_4 *generates an infinite cyclic semigroup.*

Proof. In the opposite case, $\bar{S}_3$ or S_4 contains an idempotent element which contradicts Corollary 2.6.

Theorem 2.8. 1) $\bar{S}_3$ *has no (proper) right magnifying element.* 2) S_4 *has no left magnifying element.* 3) *If* $1 \in S$ *(i.e. S is a monoid), then* $S_0 \cup S_2 \cup S_5$ *has no left or right magnifying element.* 4) S_5 *has no left magnifying element.*

Proof. 1) is a consequence of [4], Chap. III. 5.6 (β). Since in the product $s_4 S$ ($s_4 \in S_4$) the representation of each element is unique, thus the same holds for $s_4 S_4$, and 2) is true. 3) follows from [4], Chap. III. 5.6 (γ), because the union $S_0 \cup S_2 \cup S_5$ does not contain left or right magnifying element of S. Finally, S_5 is a right group, and hence has no left magnifying element, cf. [4], Chap. III. 5.3 (γ).

3. In this section the results of [2] will be applied to the decomposition (1) of regular semigroups. For a regular semigroup S we shall investigate the following cases based on Theorem 4 in [2]:

1) S has neither a left nor a right identity element;

2) S has an identity element;

3) S has either a single left or a single right identity element.

In the case 3) we may assume that S has only a left identity element. In the opposite case we have to study the decomposition (1′) instead of (1). As it is well known, an idempotent element e is $\mathcal{D}$-primitive if it is minimal among the idempotents D_e, where D_e is the $\mathcal{D}$-class of e ($\mathcal{D}$ is one of Green's relations).

In the case 1) S has no left magnifying element (cf. Corollary 1.6), that is, $S_1 \cup S_3 = \emptyset$ and $S_4 = \emptyset$, furthermore, $S_5 = \emptyset$, because in the opposite case S would have a left identity element. Hence $S = S_0 \cup S_2 = \bar{S}_2$.

In the case 2) suppose that $1 \in S$ is the identity element. If 1 is $\mathcal{D}$-primitive then we have $S_1 \cup S_3 = \emptyset$, $S_4 = \emptyset$, while $S_5 \neq \emptyset$ (e.g. $1 \in S_5$). In this subcase we obtain that $S = S_0 \cup S_2 \cup S_5$. If 1 is not $\mathcal{D}$-primitive, then there are magnifying elements, that is, $S_1 \cup S_3 \neq \emptyset$, $S_4 \neq \emptyset$, S_5 is equal to the subsemigroup of all invertable elements and thus it is nonempty. Since $S_4 S_3 \subseteq S_2$ and $S_4 S_1 \subseteq S_0$, at least one of the subsemigroups S_0, S_2 is nonempty. Hence we obtain $S = \bar{S}_2 \cup \cup \bar{S}_3 \cup S_4 \cup S_5$, where all the components are nonvoid.

In the case 3) suppose that e is the only left identity element of S. If e is $\mathcal{D}$-primitive, then $S_1 \cup S_3 = \emptyset$, $S_4 = \emptyset$, while $S_5 \neq \emptyset$ (for example, $e \in S_5$). Therefore $S = S_0 \cup S_2 \cup S_5$. If e fails to be $\mathcal{D}$-primitive, then there are magnifying elements, that is, $S_1 \cup S_3 \neq \emptyset$, $S_4 \neq \emptyset$, $S_5 \neq \emptyset$ and, similarly to the second subcase of 2), we have $S_0 \cup S_2 \neq \emptyset$. Hence $S = \bar{S}_2 \cup \bar{S}_3 \cup S_4 \cup S_5$, where all the components are nonempty.

Summing up:

Theorem 3.1. *Let S be a regular semigroup. Then:*
1) *If S has no left identity element then* $S_1 \cup S_3 = \emptyset$, $S_4 = \emptyset$, $S_5 = \emptyset$.
2) *If S has an identity element and*
   a) *if 1 is $\mathcal{D}$-primitive then* $S_1 \cup S_3 = \emptyset$, $S_4 = \emptyset$, $S_5 \neq \emptyset$,
   b) *if 1 is not $\mathcal{D}$-primitive then* $S_1 \cup S_3 \neq \emptyset$, $S_4 \neq \emptyset$, $S_5 \neq \emptyset$, $S_0 \cup S_2 \neq \emptyset$.
3) *If e is the unique left identity of S and*
   a) *if e is $\mathcal{D}$-primitive then* $S_1 \cup S_3 = \emptyset$, $S_4 = \emptyset$, $S_5 \neq \emptyset$,
   b) *if e is not $\mathcal{D}$-primitive then* $S_1 \cup S_3 \neq \emptyset$, $S_4 \neq \emptyset$, $S_5 \neq \emptyset$, $S_0 \cup S_2 \neq \emptyset$.

4. Finally, we make some remarks concerning the decomposition (1). For $x \in S_4$, $a \in \bar{S}_3$ let $B_x = \{b \in S \mid b$ is an inverse of $x\}$, $C_a = \{y \in S \mid y$ is an inverse of $a\}$. If $x \in S_4$ and $b \in B_x$ $(b \in \bar{S}_3)$, then bx is a left identity of S. Analogonsly, ay $(a \in \bar{S}_3,\ y \in C_a)$ is also a left identity of S.

Theorem 4.1. *If $x \in S_4$ then B_x fails to be a subsemigroup. If $a \in \bar{S}_3$, then C_a fails to be a subsemigroup.*

Proof. Suppose that B_x is a semigroup and $a, b \in B_x$. Then $axa = a$, $bxb = b$ and $ba \in B_x$. Hence $baxba = ba$. Since ax is a left identity element, hence $b(ba) = ba$. On the other hand, $ba \in \bar{S}_3$, thus $baS = S$, whence $bs = s$ for all $s \in S$, which is a contradiction (b is a left magnifying element!).

Let $x, y \in C_a$. If C_a is a semigroup, then $a(xy)a = (ax)ya = ya$. But $ya \neq a$, because ya is idempotent, while the element $a \in \bar{S}_3$ is not. Thus $xy \notin C_a$. Q.E.D.

Let $M \subset S$ be a subset of S such that $aM = S$. Then the set M is *left increasable* by a. Such a set M *is not* uniquely determined by a.

Theorem 4.2. *If $a \in \bar{S}_3$ then* $a(S_0 \cup S_2 \cup S_4) = S$.

Proof. Let $a \in \bar{S}_3$ and $x \in S_4$ an inverse of a. Then we have $axS = aS = S$ and $xS \subset S$. On the other hand, $xS \subseteq S_4 S$, furthermore, by making use of the relations (2) we get

$$S_4 S = S_4(S_0 \cup S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5) \subseteq S_0 \cup S_2 \cup S_4.$$

Hence $xS \subseteq S_0 \cup S_2 \cup S_4$ and thus $a(S_0 \cup S_2 \cup S_4) = S$. Q.E.D.

Theorem 4.2 implies for every $a \in \bar{S}_3$ the existence of an element $y_a \in S_0 \cup S_2 \cup S_4$ such that $ay_a = a$.

Theorem 4.3. a) *If $a \in S_3$, then $y_a \notin S_0$.* b) *The elements $a \in \bar{S}_3$ for which $y_a \in S_4$ $(ay_a = a)$, have a two-sided identity element in S.*

Proof. a) If $y_a \in S_0$, then there is an $x \neq 0$ such that $y_a x = 0$. Thus $ax = (ay_a)x = a(y_a x) = a0 = 0$, whence $a \in S_0 \cup S_1$, which is a contradiction.

b) If $y_a \in S_4$, then there exists $b \in \bar{S}_3$, such that $by_a b = b$ and $y_a b y_a = y_a$. Then $ay_a b = ab$, $ay_a b y_a = aby_a$, that is, $ay_a = aby_a$, whence it follows that $a = a(by_a)$. On the other hand, by_a is a left identity element of S, whence $by_a a = a = aby_a$. Q.E.D.

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